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Mechanics

1.1 INTRODUCTION

An enormous number of physical events and phenomena are taking place around us all the time: the movement of all types of transport (bicycles, cars, trains, airplanes, etc.), building activity, athletes in competition, rain falling, wind blowing, water flowing, earthquakes and a wide range of other phenomena. All of these are performed at speeds much smaller than the velocity of light ($c \approx 3 \times 10^8$ m/sec) and at scales much greater than atomic scales ($\sim 10^{-10}$ m). All are described by classical mechanics, based mainly on Newton's laws.

This does not, of course, exclude the existence of other phenomena described by other physical branches. Quantum mechanics deals with the world of atoms and molecules, their transformations and accompanying changes in property. The overwhelming majority of them are invisible to the naked eye, but experience shows the following to be true: all materials, though differing in their characteristics, consist of a limited number of various particles—atoms and molecules. This is the world of so-called quantum mechanics. We can indirectly observe these phenomena manifest themselves, but for their investigation and understanding, a special knowledge is needed.

To continue this analysis, we can mention one more branch of phenomena that manifest themselves at velocities close to the velocity of light; this is the more exotic area of classical and quantum relativistic physics.

1.2 KINEMATICS

Kinematics is the branch of mechanics that explores the motion of material bodies from the standpoint of their space–time relationships, disregarding their masses and the forces acting on them.

1.2.1 Kinematics of a material point

For a description of a point's motion in space and time, a reference system should be chosen. The reference system is a collection of instruments: the time-measuring device (e.g., a watch) and the bodies conditionally considered as being fixed in space with respect to which the motion is considered. Time, a continuously changing scalar value, is measured by a watch, and cannot be negative. In problems of kinematics time is usually taken as an independent variable (or argument), the rest of the parameters being considered as functions of time.

For different problems the reference system can be chosen either in the form of Cartesian coordinates, or as a cylindrical or spherical coordinates system. A moving point describes a certain continuous line in space that is referred to as a trajectory. In a number of problems the path itself will define the motion (for instance, its rails will dictate the motion of a railway carriage). At a certain instant, corresponding to a certain body motion, tangent unit vectors—principle normal and binormal vectors—are taken as natural axes. In the following we will consider only plane motion, so there is no need for a binormal vector. The principle normal is perpendicular to the tangent and is directed to the center of curvature. The direction of the tangent and normal unit vectors will be denoted as $\boldsymbol{\tau}$ and $\mathbf{n}$.

Let us recall some information about the line curvature (trajectory). The tangent lines assigned by vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ at two adjoining points A and B of the plane form an angle (Figure 1.1) to be drawn, which is referred to as the angle of contingence. If we then make the distance AB shorter, an arc AB $= \Delta l$ aspires to zero. At the limit $\Delta\varphi/\Delta\ell$, it gives the trajectory curvature K in a given point:

$$\lim \frac{\Delta\varphi}{\Delta\ell} = \frac{\Delta\varphi}{\Delta\ell} = K \quad \text{at} \quad \text{AB} \to 0.$$

The reciprocal value $\rho = 1/K$ is the curvature radius in point A. In fact, a circle's curvature is equal to its radius; the curvature radius of a straight line is infinity.

The simplest object in mechanics is called a material point (MP); this implies a body whose size in the framework of a given problem can be considered to be negligibly small. Another definition of an MP is that it is a point that possesses a mass. Different objects in different problems can be considered differently: the molecules acting on a vessel's wall can be imagined as an MP, the earth moving around the sun may, in some instances, also be treated as an MP. However, the same objects in different problems cannot be considered

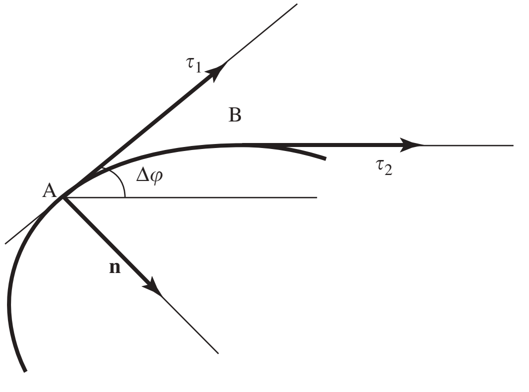


Figure 1.1. The trajectory curvature.

as MPs: e.g., molecules in molecular spectroscopy rotating around their center of mass (CM), the earth rotating around its geographical axis, etc.

An important task in kinematics is to assign an equation of motion, i.e., to construct the necessary mathematical equations that are sufficient to determine the MP's position in space at any instant of time. In the Cartesian coordinate system such an equation is the time dependence of the radius vector $\mathbf{r}(t)$; three scalar equations $x(t)$, $y(t)$ and $z(t)$ correspond to one vector equation.

If a point in a time interval Δt moves from point A to B along an arc l (Figure 1.2), the vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is referred to as the displacement, whereas the length of the arc AcB is the distance travelled. If one takes one's car in the morning, travels some distance during the day and then returns the car to the garage, the overall day displacement is equal to zero, whereas the distance travelled is the non-zero speedometer indication. The distance travelled and the displacement can coincide in two cases: when the movement occurs along a straight line or at $\Delta t \to 0$.

The equation

$$\langle \upsilon \rangle = \frac{\Delta r}{\Delta t}. \tag{1.2.1}$$

allows us to calculate the average speed at a time interval Δt. The instant velocity is given by the equation

$$\upsilon = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt} = \dot{r}(t). \tag{1.2.2}$$

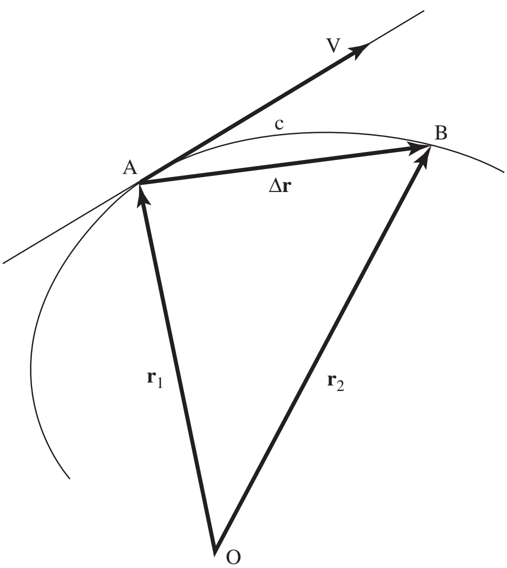


Figure 1.2. A displacement vector $\Delta\mathbf{r}$ and distance travelled AcB.

The velocity at a given point is a physical value, numerically equal to the time derivative from the radius vector of the MP in the reference system under consideration. Remember that for brevity of writing, the time derivative function is denoted by a point above the letter, expressing a given function.

Where the direction of the vector $\boldsymbol{\upsilon}$ is concerned, in the limit of the movement of point B to point A the secant will coincide with the tangent to the trajectory in point A. Consequently, an instant velocity vector is directed along the tangent to the trajectory, and the modulus is the time derivative from the function, expressing the law of point movement.

As usual, the point radius vector $\mathbf{r}(t)$ can be decomposed upon the orts

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \tag{1.2.3}$$

and therefore,

$$\boldsymbol{\upsilon}(t) = \dot{r}(t) = \mathbf{i}\dot{x}(t) + \mathbf{j}\dot{y}(t) + \mathbf{k}\dot{z}(t). \tag{1.2.4}$$

The velocity vector is

$$\boldsymbol{\upsilon} = \mathbf{i}\upsilon_x + \mathbf{j}\upsilon_y + \mathbf{k}\upsilon_z, \tag{1.2.5}$$

where υ_x, υ_y and υ_z are its projections onto the coordinate axes:

$$\upsilon_x = \dot{x}(t), \quad \upsilon_y = \dot{y}(t), \quad \upsilon_z = \dot{z}(t) \tag{1.2.6}$$

The modulus of the velocity vector is the square root sum of their projections' squares:

$$\upsilon = \sqrt{\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2} \tag{1.2.7}$$

Acceleration is the change of the velocity vector in time. If, in the time interval Δt, an MP displaced along the trajectory and a change in velocity and its direction had taken place then $\Delta\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_2 - \boldsymbol{\upsilon}_1$. The mean acceleration in the Δt interval is then

$$\langle \boldsymbol{a} \rangle = \frac{\Delta \boldsymbol{\upsilon}}{\Delta t}. \tag{1.2.8}$$

The acceleration at a given time instant (instantaneous acceleration) is the limit of the ratio $(\Delta\boldsymbol{\upsilon}/\Delta t)$ at $\Delta t \to 0$.

$$\boldsymbol{a} = \lim \frac{\Delta \boldsymbol{\upsilon}}{\Delta t} = \frac{d\boldsymbol{\upsilon}}{dt} = \dot{\boldsymbol{\upsilon}}(t). \tag{1.2.9}$$

or

$$\boldsymbol{a} = \mathbf{i}\dot{\upsilon}_x + \mathbf{j}\dot{\upsilon}_y + \mathbf{k}\dot{\upsilon}_z, \tag{1.2.10}$$

(because orts are in this case independent of time). In another form

$$\boldsymbol{a} = \mathbf{i}a_x + \mathbf{i}a_y + \mathbf{k}a_z, \tag{1.2.11}$$

where a_x, a_y and a_z are projections of the **a** vector onto the coordinate axes. Comparison of eqs. (1.2.6) and (1.2.11) gives

$$a_x = \dot{\upsilon}_x(t) = \ddot{x}(t); \quad a_y = \dot{\upsilon}_y(t) = \ddot{y}(t); \quad a_z = \dot{\upsilon}_z(t) = \ddot{z}(t) \tag{1.2.12}$$

and correspondingly

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{I.2.13}$$

At curvilinear movement the velocity vector is the product $\boldsymbol{\upsilon} = \upsilon\boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is a tangent ort. Because of the fact that the point is moving along a curvilinear trajectory and "draws" the unit vector $\boldsymbol{\tau}$ behind, its position is also dependent on time. In this case:

$$a = \frac{d\boldsymbol{\upsilon}}{dt} = \frac{d\upsilon}{dt}\boldsymbol{\tau} + \upsilon\frac{d\boldsymbol{\tau}}{dt}. \tag{1.2.14}$$

Expression (1.2.15) shows that acceleration is the sum of the two vectors: the first is directed along the tangent and is equal to the first derivative of velocity and the second term depends on the change of $\boldsymbol{\tau}$ in time. To determine the magnitude and direction of the second term, we need to find the meaning of the derivative $d\tau/dt$. Let the direction of the velocity vector at two adjacent positions separated by time interval Δt be specified by orts $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ (Figure 1.3). Then the change of the vector $\boldsymbol{\tau}$ in the time interval Δt can be expressed by vector $\Delta\boldsymbol{\tau} = \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1$. We shall consider the derivative $d\tau/dt$ as a limit of a ratio $\Delta\tau/\Delta\tau$ for $\Delta t \to 0$. We find the value of vector magnitude $\Delta\tau$ from the triangle ACD: $\frac{(\Delta\tau/2)}{\tau} = \sin\frac{\Delta\varphi}{2}$, then$= \frac{\Delta\tau}{2} = \sin\frac{\Delta\varphi}{2}$, at $\Delta t \to 0$ numerically $\Delta t \to \Delta\varphi$, since the unit-vector magnitude is unity and $\sin(\Delta\varphi/2) \approx \Delta\varphi/2$ at $\Delta\varphi \ll 1$. Then

$$\frac{d\tau}{dt} = \lim_{\Delta t \to 0}\frac{\Delta\tau}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta\varphi}{\Delta t} \tag{1.2.15}$$

Multiplying both the numerator and denominator of the function $\Delta\boldsymbol{\varphi}/\Delta t$ by the arc length Δl we obtain

$$\frac{d\boldsymbol{\tau}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} \times \frac{\Delta\ell}{\Delta\ell} = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta\ell} \times \lim_{\Delta t \to 0} \frac{\Delta\ell}{\Delta t}$$

Let us consider both these limits. Since an angle $\Delta\varphi$ is the angle of contiguity, the $\lim(\Delta\varphi/\Delta l) = K$ is equal to the curvature of a curve at a given point, i.e., to curvature radius ρ. The second limit is the velocity magnitude

$$\lim_{\Delta t \to 0} \frac{\Delta\ell}{\Delta t} = \frac{d\ell}{dt} = \boldsymbol{\upsilon}.$$

Thus,

$$\frac{d\tau}{dt} = \upsilon K = \frac{\upsilon}{\rho}. \tag{1.2.16}$$

To determine the direction of the vector $(d\boldsymbol{\tau}/dt)$ we shall draw a straight line from point A parallel to $\Delta\boldsymbol{\tau}$ and examine the value of an angle $\angle$CAE at the limit $\Delta t \to 0$. As can be seen from Figure 1.3, an angle $\angle\text{CAE} = \angle\text{CAF} + \Delta\varphi/2 = (\pi/2) + \Delta\varphi/2)$. At $\Delta t \to 0$ the contiguity angle $\Delta\boldsymbol{\varphi} \to 0$ whereas $\angle\text{CAE} \to (\boldsymbol{\pi}/2)$. Therefore, the vector $(d\boldsymbol{\tau}/dt) = \lim(\Delta\boldsymbol{\tau}/\Delta t)$ will be directed along the normal to the center of curvature at point A; it can be presented as

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{\upsilon}{\rho}\mathbf{n}. \tag{1.2.17}$$

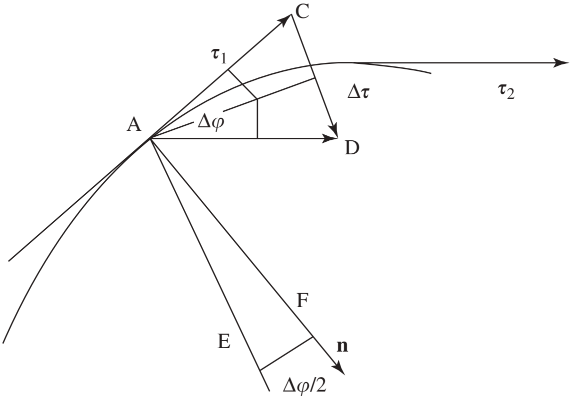


Figure 1.3. Calculation of the $\dfrac{d\boldsymbol{\tau}}{dt}$ derivative.

Returning to expression (1.2.14), we can write

$$\boldsymbol{a} = \frac{d\upsilon}{dt}\boldsymbol{\tau} + \frac{\upsilon^2}{\rho}\mathbf{n} \tag{1.2.18}$$

Therefore, the total acceleration **a** in curvilinear movement can be separated into two parts: the first is the tangent acceleration

$$a_\tau = \frac{d\upsilon}{dt} \tag{1.2.19}$$

and the second is

$$a_\mathrm{n} = \frac{\upsilon^2}{\rho}. \tag{1.2.20}$$

The tangent part influences the absolute velocity magnitude whereas the normal part changes the direction of the velocity vector. The square of the total acceleration can be written as

$$a^2 = a_\tau^2 + a_\mathrm{n}^2 = \left(\frac{d\upsilon}{dt}\right)^2 + \left(\frac{\upsilon^2}{\rho}\right)^2. \tag{1.2.21}$$

The expressions derived are valid for the general movement of the MP along the curve with an arbitrary regime of velocity change. Let us consider some particular cases:

Rectilinear movement: $\rho = \infty$, $a_\mathrm{n} = 0$, $\mathbf{a} = a\boldsymbol{\tau}$

Uniform movement along a circle: $\rho =$ const., $a_\tau = 0$, $\mathbf{a} = \mathbf{n}(\upsilon^2/R)$

Nonuniform movement along a circle: $\rho =$ const., $a_\tau \neq 0$, $a_\mathrm{n} \neq 0$.

In the case of uniformly alternating movement $a_\tau =$ const. $\neq 0$, $a_\mathrm{n} = 0$; it is possible to derive the general expressions. Integration of an expression $(d\upsilon/dt) = a_\tau$ over time gives

$$\upsilon(t) = \int a(t)dt + C = at + \upsilon_0, \tag{1.2.22}$$

where υ_0 is the initial speed (at $t = 0$). The distance travelled can be derived from (1.2.22) by repeated integration as

$$x(t) = x_0 + \upsilon_0 t + \frac{at^2}{2}, \tag{1.2.23}$$

where x_0 is the initial coordinate (at $t = 0$).

EXAMPLE E1.1

A car moving uniformly covers a distance of 100 km at a speed 60 km/h, although on the way back it travels at a speed of 40 km/h. Determine the average speed of the car.

Solution: At first glance the answer is simple: 50 km/h. However, this is incorrect. The average speed is the total distance travelled (200 km) divided by the total time spent (100/60) + (100/40) = 4.16 h. Therefore, the average speed is (200/4.16) $\approx$ 44 km/h.

EXAMPLE E1.2

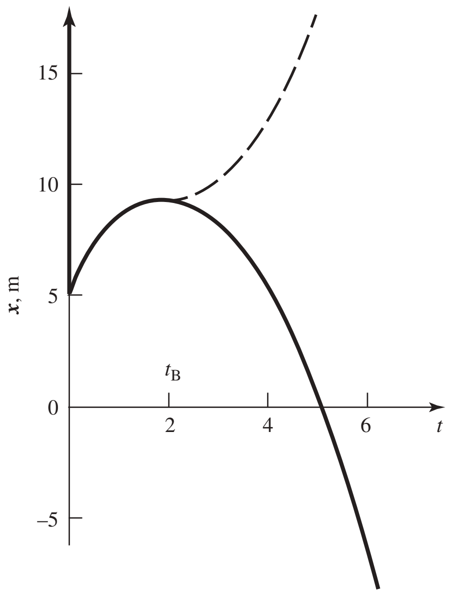
The movement of a MP along an x-axis is described by the equation $x = A + Bt + CT^3$, where $A = 4$ m, $B = 2$ m/sec, $C = –0.5$ m/sec^3. For the instance of time $t_1 = 2$ sec determine: (1) the MP coordinate x_1, (2) an instant velocity υ_1, and (3) an instant acceleration a_1.

Solution: (1) To find the point coordinate one should substitute time t for the instant time $t_1 x_1 = A + Bt_1 + Ct_1{}^3$. Inserting the given values we obtain: $x_1 = 4 + 2.2 - 0.5 \times 2^3 = 4$ m. (2) To find an instant speed at any time we should differentiate a coordinate on time: $\upsilon = (dx/dt) = B + 3Ct_1{}^2$. Introducing B, C and t_1 we obtain: $\upsilon_1 = - 4$ m/sec. The sign shows that at that very moment the point moves in a negative direction on the x-axis. (3) To find acceleration as a function of time we should take the second time derivative from coordinate: $a = (d^2x/dt^2) = (dx/dt) = 6Ct$. To find the instant acceleration at t_1 we should introduce the given data and obtain the result. $a_1 = -6.0 \times 5.2 = –6$ m/sec^2; the sign shows that the movement is decelerative.

EXAMPLE E1.3

The movement of an MP along an x-axis is described by equation $x = A + Bt + Ct^2$, where $A = 5$ m, $B = 4$m/sec, $C = –1.0$ m/sec^3. Draw a graph of $x(t)$ and the distance travelled $S(t)$.

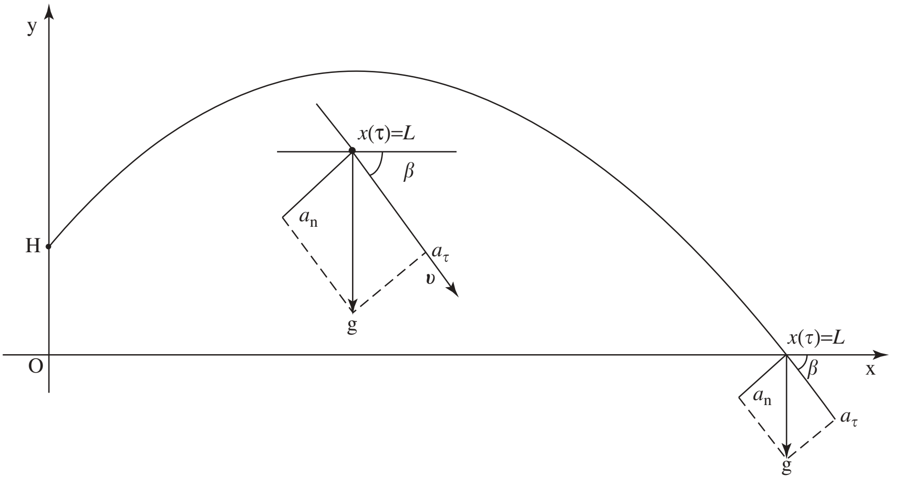
Solution: For the drawing of the graph of the point coordinate time dependence $x(t)$, we find characteristic values of movement: initial and maximum coordinates The initial coordinate corresponds to the moment $t = 0$, its value equals $x(0) = A = 5$ m. The point reaches maximum height corresponding to the moment when the point starts to move back (speed changes sign). We can find this moment having equated to zero the time first derivative from coordinate: $\upsilon = (dx/dt) = B + 2Ct = 0$, wherefrom $t = –(B/2C) = 2$ sec. The maximum coordinate $x_{max} = x(2) = 9$ m. The time instant t when $x = 0$ can be found from equation $x = A+Bt+Ct^2 = 0$. Solving the quadratic equation we obtain $t = (2 \pm 3)$ sec. The negative value does not satisfy the problem. Therefore $t = 5$ sec. Using the data obtained we can draw the graph of coordinates' dependence on time. The distance travelled and the coordinate coincide until the point stops; from this time the point goes in opposite direction and its coordinate diminishes; however the distance travelled continues to grow (Figure E1.3).



EXAMPLE E1.4

A mortar is installed on a hill at a height of $H = 60.0$ m above ground level. It fires a missile at an initial angle of $\alpha = 60°$ to the horizon. The missile's initial velocity is $\upsilon_0 = 80$ m/sec. Derive: (1) the kinematical equations of the missile's flight $x(t)$ and $y(t)$; (2) the equation of trajectory $y(x)$; (3) the expression for projections $\upsilon_x(t)$ and $\upsilon_y(t)$ on the coordinate axes x and y and the time velocity dependence $\upsilon(t)$; (4) the velocity dependence on time $\boldsymbol{\upsilon}(t)$ on absolute value and direction; (5) the absolute values of tangential $|a_\tau|$ and normal $|a_n|$ acceleration dependence on time (derive a corresponding formula and execute calculations): (6) the maximum height y_{max} of flight; (7) the time of the missile's flight τ; (8) the range L of missile; (9) the missile's velocity $\boldsymbol{\upsilon}$ (on modulus and direction) at the moment of falling on the ground; (10) the curvature radii trajectory ρ_1 and ρ_2 at the moment of falling and at the highest point of flight, respectively.

Solution: To solve this problem we have to begin with the choice of reference frame. The motion of the missile is subject to a constant acceleration **g** directed downward. Therefore, the flight trajectory is a plane (two-dimensional). Choose a Cartesian system xOy in such a way that the x-axis is horizontal and the y-axis is vertical; the flight will occur in this plane. The origin is superposed with the earth surface, axis x directed horizontally and y vertically upward. Accept the missile as an MP. The movement in this case can be separated into two independent components: along axes Ox and Oy. A movement along axis Ox is uniform with a speed $\upsilon_x = \upsilon_0 \cos\alpha$; however along axis Oy it uniformly accelerates with initial coordinate $y_0 = H$ and initial speed $\upsilon_{oy} = \upsilon_0 \sin\alpha$ and acceleration $a_y = g$ (see inset in Figure E1.4).



Thus,

(1) Kinematical equations for mine movement projected on axes x and y can be written:

$$x(t) = \upsilon_0 t \cos\alpha \quad \text{and} \quad y(t) = H + \upsilon_0 (\sin\alpha) t - \frac{gt^2}{2}.$$

(2) An equation of the trajectory can be obtained by excluding time from the kinematical equation for the missile's movement:

$$\text{since } t = \frac{x}{\upsilon_0 \cos\alpha}, \text{ then } y(x) = H + \frac{\upsilon_0 \sin\alpha}{\upsilon_0 \cos\alpha} x - \frac{g}{2\upsilon_0^2 \cos^2\alpha} x^2 = H + (\tan\alpha) x - \frac{g}{2\upsilon_0^2 \cos^2\alpha} x^2.$$

(3) The velocity $\upsilon(t)$ projection on coordinate axis can be found by time differentiation of $x(t)$ and $y(t)$ as

$$\upsilon_x(t) = \frac{dx}{dt} = \upsilon_{Ox} = \upsilon_0 \cos\alpha = \text{const.}, \quad \upsilon_y(t) = \frac{dy}{dt} = \upsilon_0 \sin\alpha - gt.$$

(4) The dependence of velocity $\boldsymbol{\upsilon}(t)$ on time, in vector form, can be obtained in the form $\boldsymbol{\upsilon}(t) = \mathbf{i}\upsilon_x(t) + \mathbf{j}\upsilon_y(t)$. Then the velocity modulus is

$$|\boldsymbol{\upsilon}(t)| = \sqrt{\boldsymbol{\upsilon}_x^2 + \boldsymbol{\upsilon}_y^2} = \sqrt{\boldsymbol{\upsilon}_0^2 - 2gt\boldsymbol{\upsilon}_0 \sin\alpha + g^2t^2}.$$

The velocity $\boldsymbol{\upsilon}(t)$ direction can be determined by an angle β between this vector and the axis Ox. It can be seen from a Figure E1.4 that

$$\tan\beta(t) = \frac{\upsilon_y(t)}{\upsilon_x(t)} = \frac{\upsilon_0 \sin\alpha - gt}{\upsilon_0 \cos\alpha}, \quad \text{therefore} \quad \beta(t) = \arctan\left(\frac{\upsilon_0 \sin\alpha - gt}{\upsilon_0 \cos\alpha}\right).$$

(5) Since the total acceleration is constant (i.e., $\mathbf{g}$) the moduli of tangential $|a_\tau|$ and normal $|a_n|$ components (as can be seen from Figure E1.3) will be equal to: $a_\tau = g \sin\beta$ and $a_n = g \cos\beta$, where

$\sin\beta = (\upsilon_y/\upsilon_x)$ and $\cos\beta = (\upsilon_x/\upsilon_y)$ or

$\boldsymbol{a}_\tau = \boldsymbol{\tau}(g(\upsilon_{oy} - gt) / (\upsilon_o^2 - 2gt\upsilon_{oy} + g^2t^2)^{1/2})$ and

$\boldsymbol{a}_n = \mathbf{n}(g\upsilon_{oy}/(\upsilon_o^2 - 2gt\upsilon_{oy} + g^2t^2))$.

(6) The highest point of the flight y_{max} can be found from the kinematical equations $\upsilon_y(t)$ and $y(t)$. From the first dependence one can find the time of ascent t_{asc}, from the second—the maximal ascent $y_{max} = y(t_{max})$. In the upper trajectory point $\upsilon_y = 0$, therefore $\upsilon_{Oy} - gt_{asc} = 0$, then $t_{asc} = (\upsilon_{Oy}/g)$. Therefore:

$$y_{max} = H + \frac{gt_{asc}^2}{2} = H + \upsilon_0 \sin\alpha \frac{\upsilon_{Oy}^2}{g} - \frac{g\upsilon_{Oy}^2}{2g^2} = H + \frac{\upsilon_{Oy}^2}{2g} = 245\,\text{m}$$

(7) The missile's total flight time τ can be found from the fact that in the moment of its drop $y(t) = y(\tau) = 0$, i.e., $H + \upsilon_0\tau \sin\alpha - (g\tau^2/2) = 0$ and $\tau^2 - \tau(2\upsilon_0 \sin\alpha/g) - (2/g) - H = 0$. Solving the quadratic equation regarding τ we can arrive at

$$\tau_{1,2} = \tfrac{1}{2}\frac{2\upsilon_0 \sin\alpha}{g} \pm \sqrt{\frac{\upsilon_0^2 \sin^2\alpha}{g^2} + \frac{2gH}{g^2}},$$
$$\tau = \tfrac{1}{g}\left(\upsilon_0 \sin\alpha + \sqrt{\upsilon_0^2 \sin^2\alpha + 2gH}\right)$$

(Since the time cannot be negative we should accept sign "+"). Executing calculations we obtain $\tau = 14.9$ sec.

(8) The missile's range L can be found by inserting τ into the $x(t)$: $L = x(\tau) = \upsilon_{Ox}\tau = 596$m.

(9) The modulus of velocity at the moment it hits the ground can be found using the equation given in point (4) substituting a running time on τ as

$$\upsilon = \upsilon(\tau) = \sqrt{\tau\upsilon_0^2\ 2\upsilon_0 g\tau\ \sin\alpha\ +\ g^2\tau^2} = 87\ \text{m/sec}$$

or, converting this expression to a form $\upsilon = \sqrt{\upsilon_0^2 + 2gH}$, one can obtain the same result. Since $\tan\beta = (\upsilon_y/\upsilon_x)$, then $\beta(\tau) = \arctan = -62.5°$; a minus sign shows that the velocity vector makes with the x-axis an angle β, counts off in a negative direction, i.e., clockwise (Figure E 1.4, insertion).

(10) To find curvature radius ρ one can use expressions $\alpha_n = (\upsilon^2/\rho)$ wherefrom $\rho = (\upsilon^2/a_n)$, and $a_n = g\cos\beta$, and υ is the speed at the moment of hitting the ground: $\upsilon = \upsilon(\tau)$. Therefore,

$$\rho_1 = \frac{\upsilon_0^2 - 2\upsilon_0 g\tau + g^2\tau^2}{g\cos\beta} = \frac{\upsilon_0^2 + 2gH}{g\cos\beta} = 1.67\,\mathrm{m},$$

however, at the highest point $\rho_2 = \upsilon_{0x}^2/g = 163$ m. Note that the curvature at the maximum height is approximately 100 times less than at point of hitting the ground.

By solving a problem in this way, we can then use all these particular equations in future.

1.2.2 Kinematics of translational movement of a rigid body

A body in which the distance between two arbitrary points remains at constant temperature, unchanged by any motions or interactions, is referred to as an ideally rigid body (IRB). During the translational motion of the IRB, any segment inside of it remains parallel to itself at any time. With such motion the displacement, velocity and acceleration of any point of the IRB are the same at any time. Therefore, many characteristics of the IRB's translational movement can be described by the motion of a single body's point with a mass equal to the mass of the whole body moving with velocity (acceleration) in any point of the body. The best point to choose is the centre of mass (CM) (see below).

1.2.3 Kinematics of the rotational motion

Rotational movement is widespread in nature, no less (but can be even more) than translation motion. Indeed, the motion of electrons around the nucleus (within the Bohr atomic model) and the earth around the sun, the rotation of a gyroscope, the rotation of numerous details and assemblies in technology and industry, the rotation of a wheel (this genius invention of mankind)—all of these are examples of rotational motion.

The rotational motion of the IRB around a motionless axis Oz in which all points of the body are moving in parallel planes, making circles with their centers lying on a single straight line coinciding with the z-axis, is referred to as the rotational motion of the IRB.

When rotating, all points of the IRB have linear velocities differing in size and direction, depending on the point distance from the axis of rotation. So, for a description of rotational motion we should introduce angular kinematic features unique to the whole body: angular displacement, angular velocity and angular acceleration.

Let us restrict ourselves to the case of IRB rotation around an axis whose space position does not change in time.

Angular displacement

Consider a body revolving around axis Oz. Select in the body a point, not lying on the axis of rotation (point A in Figure 1.4; the body itself is not shown in the figure). In accordance with the definition of rotational motion, this point while moving describes a circle with a radius R, the center of which (O) is lying on an axis Oz. While rotating, vertical planes drawn through the axis of rotation and any body point turn on the same angle. Let the plane (and the body) turn on an angle $d\varphi$. This angle is referred to as the angular displacement. The angular displacement is a vector, coinciding with the axis of rotation, whose direction is defined by the right-handed system. Remember that this rule concludes that if the right screw reconciles with the axis of rotation and turns it in a direction complying with the rotating body, the translational direction of the screw movement along the axis of rotation complies with the direction of the vector $d\boldsymbol{\varphi}$.

Vectors whose directions are aligned with the rotation direction are called axial vectors. Angular displacement $d\boldsymbol{\varphi}$ is an axial vector, the modulus of which is equal to the ratio of arc dS and radius R and the direction of which coincides with the rotation axis in accordance with the right screw rule.

$$d\boldsymbol{\varphi} = \frac{dS}{R}\mathbf{k}, \tag{1.2.24}$$

k being the ort of rotation axis Oz.

The value to which the $\lim_{\Delta t \to 0}$ tends is called the angular velocity ω:

$$\omega = \frac{d\boldsymbol{\varphi}}{dt} = \dot{\boldsymbol{\varphi}}(t). \tag{1.2.25}$$

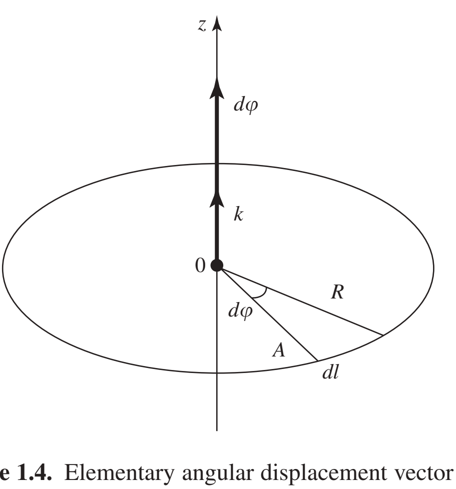


Figure 1.4. Elementary angular displacement vector $d\boldsymbol{\varphi}$.

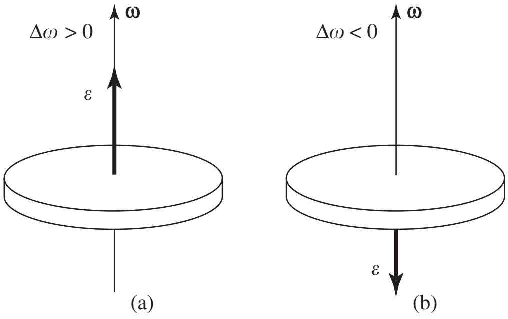


Figure 1.5. Relationship between angular velocity and angular acceleration.

Angular velocity is a first time derivative from the vector of angular displacement. It shows the speed of angular displacement changing with time. Angular velocity $\boldsymbol{\omega}$ is also an axial vector, which coincides in direction with the angular displacement vector $d\varphi$.

The value to which the limit $(\Delta\omega/\Delta t)$ tends is called angular acceleration:

$$\boldsymbol{\varepsilon} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}\boldsymbol{k} = \frac{d\omega}{dt}\boldsymbol{k}, \tag{1.2.26}$$

Angular acceleration is also an axial vector.

The different mutual orientations of angular velocity and angular acceleration are presented in Figure 1.5: when the angular velocity is rising ($d\boldsymbol{\omega} > 0$), then the direction of the angular acceleration vector coincides with the former (both are directed along the axis of rotation); if the angular velocity decreases ($d\boldsymbol{\omega} < 0$), then the direction of the angular acceleration vector is opposite to the angular velocity.

If the axis of a body rotation changes its orientation in the course of time, some interesting effects appear which are unfortunately beyond the scope of our consideration here.

There exists a linear relationship between angular and translation features. It can be seen in Figure 1.4 that $dl = Rd\varphi$, i.e. $d\varphi = (dl/\rho)$. Time derivation $(d\varphi/dt) = \omega = (1/R)(dl/dt) = (1/R)\upsilon$, i.e.,

$$\upsilon = \omega R. \tag{1.2.27}$$

The relationship between angular acceleration ε and linear tangent acceleration a_τ can also be obtained. The modulus of angular acceleration is $\varepsilon = (d\omega/dt)$, where $\omega = (\upsilon/R)$; then $\varepsilon = (d\upsilon/Rdt)$. Since $(d\upsilon/dt)$ is a point linear (tangential) acceleration a_τ then $\varepsilon = (a_\tau/R)$ or

$$a_\tau = \varepsilon R. \tag{1.2.28}$$

The last formula connect the linear and angular characteristics (Figure 1.6). They can be given in vector form:

$$\boldsymbol{\upsilon} = [\boldsymbol{\omega}\mathbf{r}], \tag{1.2.29}$$

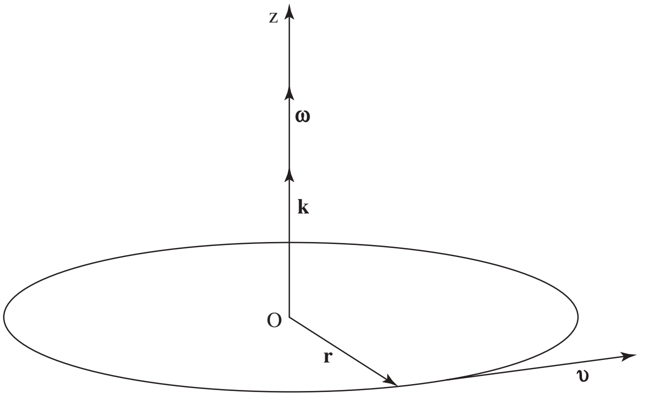


Figure 1.6. Relationship between vectors of angular **ω** and linear **υ** velocity.

and

$$\boldsymbol{\varepsilon} = [\boldsymbol{a}_\tau \mathbf{r}]. \tag{1.2.30}$$

Using eqs. (1.2.27) and (1.2.28) we can obtain by integration the dependencies of $\omega(t)$ and $\varphi(t)$ like (1.2.22) and (1.2.23) valid for uniformly accelerated rotation

$$\omega = \omega_0 + \varepsilon t, \tag{1.2.31}$$

and

$$\varphi = \varphi_0 + \omega_0 t + \varepsilon \frac{t^2}{2}. \tag{1.2.32}$$

where ω_0 and φ_0 are angular characteristics at the initial instant of time.

The structure of these expressions is equivalent to those obtained for linear motion (eqs. (1.2.22) and (1.2.23)).

However, rotational motion is distinct from linear because of the fact that it is periodical. In rotation, the values of φ are repeated in a certain time interval. If these intervals are constant (uniform rotation, ω = const.), the period T, a duration of one full turn (on 360°), and accordingly rotating frequencies, i.e., numbers of full repetition in the unit time ($v = n = 1/T$), can be used. Bearing in mind that one turn corresponds to an angular displacement equal to 2π radian, we can introduce an angular velocity $\omega = 2\pi v = 2\pi/T$.

The difference between angular vector features of revolution and the corresponding features of linear motion lies in the fact that angular vectors are directed not along the linear motion of each point of the IRB, but along the axis of rotation (perpendicular to their planes of motion). Many remarkable characteristics of rotational motion are bound up with this circumstance (refer to Section 1.3.9, Figure 1.17 and Appendix 2).

In general, the arbitrary motion of the IRB can be presented as a combination of translation motion of an MP with a mass equal to the mass of the whole IRB and located in the center of inertia (refer to Section 1.3.7, see below), and rotation of the body's points around the center of inertia.

EXAMPLE E1.5

A disc of a radius $R = 10$ cm starts to rotate with angular acceleration $\varepsilon = 0.1$ rad/sec^2 around a motionless axis, perpendicular to the disc's plane passing its geometrical center. Determine at the time instant $\tau = 12$ sec after the beginning of the disc rotation: (1) an angle of the disk turns φ (an angular displacement); (2) the number of complete revolution N_{tot}; (3) the net turn angle $\Delta\varphi$; (4) the distance traveled by any point A of the disk crown S along an arc; (5) the angular speed value ω and (6) a frequency of rotation n at this moment.

Solution: (1) For a uniformly accelerated rotation a kinematical equation is (1.2.32), where $\varphi(t)$ is a turning angle for time instance t, φ_0 and ω_0 are the initial angle and angular speed; in our case $\varphi_0 = 0$ and $\omega_0 = 0$. Therefore $\varphi(t) = (\varepsilon t^2/2)$. Introducing the numerical values and execute calculations for the time instant τ we obtain $\varphi(\tau) = (0.1\times 12^2)/2$ rad $= 7.20$ rad (413°). (2) We can find the number of revolutions N by dividing the previous result by 2π, i.e. $N = (\varphi/2\pi) = 7.20/(2 \times 3.14) = 1.15$ revolutions. Since the number of revolutions is an integer then $N_{\text{tot}} = 1$. (3) The net turn angle $\Delta\varphi$ can be found as a difference between the final turn angle minus the $2\pi \times$ integer value: $\Delta\varphi = (720 - 2\pi \times 1)$ rad $= 0.917$ rad (52.6°). (4) The total distance S traveled by point A along an arc can be found multiplying the turning angle by the radius R:$S = \varphi R = 7.20 \times 0.1$ m $= 0.72$ m. (5) To determine the disk angular speed at the time instance τ one first should take the time derivative of an angular displacement

$$\omega = \frac{d\varphi}{dt} = \frac{d}{dt}\left(\frac{\varepsilon t^2}{2}\right) = \varepsilon t.$$

For $t = \tau$ we obtain $\omega(\tau) = \varepsilon\tau$. Execute the calculations $\omega = 0.1 \times 12 = 1.20$ rad/sec. (6) The instant frequency of rotation $n(\tau)$ can be obtained as $n = (\omega/2\pi) = (1.20/2\pi) = 0.19$ sec^{-1}.

1.3 DYNAMICS

Dynamics deals with the study of a body's motion with definite mass under the action of applied forces.

1.3.1 Newton's first law of motion: inertial reference systems

Generally speaking, the same physical events can be described differently in different reference systems. Undoubtedly, we would wish to find a reference system in which the laws of different physical phenomena have the simplest expression. On the other hand, it would be interesting to find in the surrounding world a system that would be at absolute rest so that any motion could be considered with respect to this system. Is it possible to find such a system? To answer this question we shall analyze the simplest form of motion—the motion of a free body. A body is called free if it is at such a distance from all other bodies that their effect on it is negligible. (Such a body and such a motion is actually a physical abstraction since it cannot be fully realized. Nevertheless this model has played a very important role in the development of physics, from Aristotle to Galileo and Newton). So, experiencing no external effect, the free body must move rectilinearly and uniformly. Such a motion cannot be achieved in any reference system but only in the so-called inertial one. A reference system is referred to as inertial if the free body moves in relation to it with constant velocity—in magnitude and direction.

Besides, if one of the reference systems moves relative to another, additional effects can appear. All these questions are the subjects of different theories of relativity, realizing the relationships between physical laws in reference systems moving relative each other (refer to Section 1.6).

The classical theory of relativity is based on Galileo's and Newton's hypotheses. Their main feature is separation and independence the space and time and their independence. The laws of the classical theory of relativity appear from mankind's everyday experience of isotropic and uniform space (all directions are equivalent and space metric is constant everywhere) and independence of time intervals from the reference system (an interval in Moscow is the same as in London). These laws prove to be perfectly justified in the case of motion of a material body with velocities $\upsilon << c$.

Experiencing no external influences, a free body must, consequently, move rectilinearly and uniformly. However, this cannot be achieved in all reference systems, but only in those that are referred to as inertial systems. A reference system in which a free body of constant mass proceeds with constant velocity is called an inertial reference system.

The existence of an inertial reference system is a sequence of definite characteristics of space and time: the uniformity and isotropy of space and uniformity of time. The uniformity of space and time means the equivalence of all positions of free bodies in space at all instants of time, and space isotropy means the equivalence of different directions. Therefore, it is possible to give another definition of an inertial reference system: as a system relative to which space is homogeneous and isotropic and time is uniform.

The statement describing inertial reference systems as systems in which the motion of a free body is rectilinear and uniform, forms the essential part of inertia law: a free body preserves a state of the rest or the uniform rectilinear motion until another body makes it leave this state. The idea of an inertial system was incorporated by Newton into his system of the main laws of dynamics: it is referred as the first law of dynamics or Newton's first law.

Having found a single, accidental inertial system it would be imagined that the unique motionless system is found, relative to which any motion of bodies in the Universe should be considered. However, this is not so, because there exist countless inertial reference systems. We can illustrate this quite simply. Let there be two reference systems moving towards each other uniformly and rectilinearly; one of them is known to be inertial. Confirm that the other system will also be inertial. In fact, a body that is in a state of uniform and rectilinear motion with regard to the first reference system (which we know is inertial), will move uniformly and rectilinearly towards the second reference system (though, with different velocity); however this (second) reference system will then also be inertial. Thereby, any reference system, moving rectilinearly and uniformly relative to any system (which we know to be inertial), is also an inertial one. Hence, there are countless sets of equivalent inertial systems.

Equivalence of all inertial systems is equivalent to the statement that laws of mechanics are invariant with respect to the Galileo's transforms. The term "invariant" signifies that laws of mechanics and their mathematical writings are similar in all inertial systems.

1.3.2 Galileo's relativity principle: Galileo transformations

So, there are countless sets of equivalent inertial systems. Moreover, it has been proved to be physically impossible to distinguish one inertial system from another—all inertial systems are equivalent. This last statement is Galileo's mechanical relativity principle: there are no mechanical experiments that can be carried out within a given closed inertial reference system that can distinguish whether this reference system moves rectilinearly and uniformly or is at rest. Being, for instance, in the windowless sheep's hold uniformly and rectilinear moving without heaving (example have been taken from Galileo's book) or in the cabin of plane with closed windows, an explorer arranging any possible mechanical experiments (balls collision, throwing any subjects in different directions, free fall of bodies, etc), can not find, whether moves this system uniformly and rectilinear or is motionless. (We also refer the reader to Jules Verne's novel "From the Cannon to the Moon", in which passengers discuss a missile flying to the moon.)

The mechanical principle of relativity, coupled with the suggestion of uniformity of time flow in all inertial reference systems, is referred to as Galileo's principle of relativity.

Let us find a correlation that would allow us to transmit from one inertial system to another. Suppose that there are two inertial reference system $K(x,y,z,t)$ and $K'(x',y',z',t')$, a second system (K') being moved with regard to the first (K) with a constant velocity $\mathbf{V}_0$ so that axes x and x' coincide (Figure 1.7). If, at the initial instant both coordinate systems coincide, at moment t, the coordinates x and x' will be bonded by the correlation $x = x' + V_{0x}t$. For three-dimensional movement a similar correlation appears between all coordinates so the correlation system will look like

$$\begin{aligned} x &= x' + V_{0x}t, \\ y &= y' + V_{0y}t, \\ z &= z' + V_{0z}t. \end{aligned} \tag{1.3.1}$$

In general form this system can be written as

$$\mathbf{r} = \mathbf{r}' + \mathbf{V}_0 t. \tag{1.3.2}$$

Expressions (1.3.1 and 1.3.2) together with the independency of the time flowing in both systems ($t = t'$) are called Galileo transitions. They permit one to go from one inertia system to another.

The equivalence of all inertial reference systems is similar to the statement that the laws of mechanics are invariant with respect to Galileo's transforms. The phrase "are invariant"

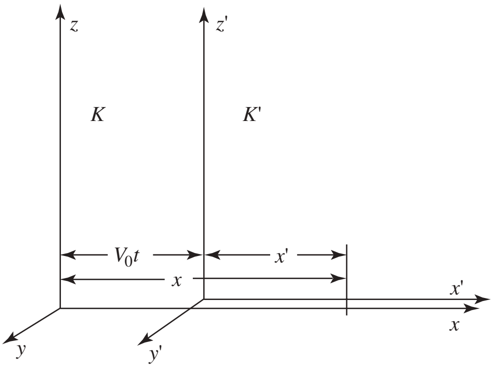


Figure 1.7. Galileo's transforms.

signifies that the laws of mechanics and their mathematical writings are similar in all inertial systems. However, each particular physical value can differ when turning from one system to another.

Having differentiated both right- and left-hand sides of expression (1.3.2) in respect to time, we can find a known law of velocities summation:

$$\frac{dr}{dt} = \frac{dr'}{dt} + V_0 \tag{1.3.3}$$

or

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{V}', \tag{1.3.4}$$

where $\mathbf{V}$ and $\mathbf{V}'$ are velocities of a MP in inertial systems K and K′. It can be seen that velocity is noninvariant regarding Galileo's transforms. The second derivative gives

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}'}{dt^2}, \tag{1.3.5}$$

or

$$\boldsymbol{a} = \boldsymbol{a}', \tag{1.3.6}$$

i.e., acceleration is invariant regarding Galileo's transforms. An important conclusion can be derived from this consideration: all the velocity's dependent physical values (for instance, momentum, kinetic energy, etc.) are noninvariant; however, all the acceleration-dependent physical values (for instance, force, etc.) are invariant regarding Galileo's transforms. Other examples of invariants are mass, distance, temperature, time, etc.

Einstein has generalized Galileo's principle of relativity. According to Einstein's principle of relativity, it is impossible by either mechanical or by physical experiment (in particular, electrical, magnetic or optical) conducted in an inertial system, to distinguish if this system is at rest or in rectilinear uniform motion. This statement is the basis of the special relativistic theory (see Section 1.6).

Now we have to solve a problem: of the many real reference systems we usually deal with, in practice, those that can be considered to be inertial. Many problems of mechanics are considered in a laboratory reference system, strictly bound to the surface of the earth. Is this reference system an inertial one? Strictly speaking, the answer is no, since the earth rotates daily, the points on the terrestrial surface (excluding the poles) possess different acceleration, perpendicular to the axis of the earth's rotation. However, in comparison with free-falling acceleration, this acceleration is very small, and for practical problems connected with the earth in a laboratory system, it is possible to consider it to be inertial.

1.3.3 Newton's second law of motion: Momentum

Any change in the movement of an MP or in its state is caused by the action of other bodies or force fields. The quantitative measure of mechanical action of bodies on another body is called a force. This action can be exhibited as a change in the velocity of a body, or as its deformation. It can be measured in both cases, so the force can be quantitatively evaluated experimentally.

Newton's second law gives a relationship between the change in motion velocity and the force F causing this change. Generalizing a large number of experimental facts, Newton suggested that acceleration is proportional to force. The law can be given as follows: the body's velocity change is proportional to the applied force and results in the direction of the line along which the force is acting.

With the action of the same force on different bodies, their accelerations can be different, depending upon the body's mass. In this instance, mass acts as a proportionality factor between the force, acting on the body and its acceleration. Such mass is identified as the inert mass (unlike the gravitational, which will be discussed below). The mass is a measure of the body's inertial property relative to its translational motion.

Accordingly, the mathematical expression of Newton's second law can be written in vector form:

$$\mathbf{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt}, \tag{1.3.7}$$

or in coordinate form:

$$m\ddot{x} = F_x, m\ddot{y} = F_z, m\ddot{z} = F_z. \tag{1.3.8}$$

Equations (1.3.8) are called the differential equations of an MP's movement.

If the mass is constant, eq. (1.3.7) will have the form

$$\frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt} = \mathbf{F}. \tag{1.3.9}$$

This equation is the most general mathematical expression for Newton's second law valid in many cases (for constant and varying masses, relativistic and quantum mechanics). Vector **p** here equals the product of the body's mass and its velocity and is referred to as the momentum:

$$\mathbf{p} = m\mathbf{v}. \tag{1.3.10}$$

Vectors of momentum and velocity coincide in direction.

In Cartesian reference frames the momentum vector can be expressed as:

$$\mathbf{p} = m\mathbf{v} = \mathbf{i}mv_x + \mathbf{j}mv_y + \mathbf{k}mv_z = \mathbf{i}p_x + \mathbf{j}p_y + \mathbf{k}p_z, \tag{1.3.11}$$

where p_x, p_y and p_z are projections of **p** on the coordinate axis.

Using expression (1.3.9), one can find the relation between the force and the momentum increment, which is produced by the force: the rate of change of the momentum of a particle is proportional to the net force acting on the particle and is in the direction of that force

$$d\mathbf{p} = \mathbf{F}dt. \tag{1.3.12}$$

It can be seen that the elementary body's momentum change is a product of the force acting on the body and the time of its action. The product $\mathbf{F}dt$ is called the elementary force impulse.

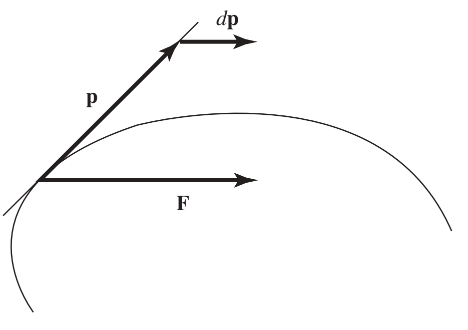


Figure 1.8. Second Newtonian law: relation between the direction of force and the direction of momentum increment. A material point trajectory is shown.

Hence, Newton's second law can be reformulated: the change of the body's momentum is proportional to the force applied and coincides with the direction of the force action (Figure 1.8).

Independence of force action principle

When discussing the force action we implied a single force only. In many cases, however, a body exerts the action of several forces. In this case the principle of independence of force action is taking place: if there are several forces acting on a body the acceleration exerted by the body under the action of each force is independent of whether or not other forces exist.

Let forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, … simultaneously act on a body. The i-th force imparts to a body an acceleration $\boldsymbol{a}_\mathrm{i} = \mathbf{F}_\mathrm{i}m$. The simultaneous action of all forces will impart to the body an acceleration equal to the sum of all accelerations:

$$\boldsymbol{a}_i = \frac{\sum_{i=1}^{N} \mathbf{F}_i}{m}. \tag{1.3.13}$$

This is the generalization of the second Newtonian law to the case of simultaneous action of N forces. The geometrical sum of forces is called the resultant force applied to a body. In the general case, the resultant force imparts acceleration to a body, its direction coinciding with the direction of the resultant force.

If N forces are applied, the body's motion can be written in the coordinate form as

$$\begin{aligned} m\frac{d^2\upsilon_x}{dt^2} &= \sum_{i=1}^{N} F_{xi}, \\ m\frac{d^2\upsilon_y}{dt^2} &= \sum_{i=1}^{N} F_{yi}, \\ m\frac{d^2\upsilon_z}{dt^2} &= \sum_{i=1}^{N} F_{zi}. \end{aligned} \tag{1.3.14}$$

EXAMPLE E1.6

An acting force is a function of a body displacement (deformation) $F(x) = -\beta x$ (such force is characteristic of elastic forces, e.g., stretching a string). Find the law of a body's motion.

Solution: Let the initial conditions be: at $t = 0$, $\upsilon = \upsilon_0$ and $x_0 = 0$. The differential equation of the motion has the form: $m\ddot{x} = -\beta x$. Reduce the derivation order $m(d\upsilon_x/dt) = -\beta x$. The number of variables is three; therefore one should try to go over from variable t to variable x. This can be done by multiplying and dividing the left-hand side of the equation by dx: $(d\upsilon_x/dx)\,(dx/dt) = d\upsilon_x/dx)$. Simple transformations give:

$$\upsilon_x d\upsilon_x = -\frac{\beta}{m}xdx; \quad \text{and then} \quad \frac{\upsilon_x^2}{2} = -\frac{\beta x^2}{2m} + c_1.$$

The initial conditions allow us to find a constant $c_1 : c_1 = \dfrac{\upsilon_0}{2}$. Then

$$\frac{\upsilon_x^2}{2} = -\frac{\beta}{m}\times\frac{x^2}{2} + \frac{\upsilon_0^2}{2} \quad \text{or} \quad \upsilon_x = \sqrt{\upsilon_0^2 - \frac{\beta}{m}x^2}.$$

Since $\upsilon_x = (dx/dt)$, we obtain

$$\frac{dx}{\sqrt{\upsilon_0^2 - \left(\frac{\beta}{\mathrm{m}}\right)x^2}} = dt.$$

Integration gives:

$$\arcsin\frac{\sqrt{(\beta/m)}\times x}{\upsilon_0} = \sqrt{\frac{\beta}{m}}t + c_2,$$

At $t = 0$ the coordinate is $x_0 = 0$, therefore $c_2 = 0$.
Then $x = \upsilon_0\sqrt{(m/\beta)}\times\sin\sqrt{(\beta/m)}\times t$; i.e., under an elastic force a system acquires periodical movement according to sin (or cos) law (this case will be considered in more detail in Section 2.4.2).

EXAMPLE E1.7

A body of mass $m = 80$ kg falls from a motionless airborne helicopter. Besides the gravity force $m\mathbf{g}$, the force of resistance of the air $\mathbf{F}_c = -k\boldsymbol{\upsilon}$ ($k = 10$ kg /sec) operates on the body. Find the time dependence of speed $\boldsymbol{\upsilon}(t)$. Find also the body's steady speed.

Solution: Direct an axis y vertically downwards. Two forces will act on a falling body: gravity $m\mathbf{g}$ and force of air resistance F_c (Figure E1.7a). According to the second Newtonian law the equation of movement in the vector form looks like $m\boldsymbol{a} = m\mathbf{g} + \mathbf{F}_c$. Since $a = d\upsilon/dt$ and $\mathrm{F}_c = -k\upsilon$, hence $m(d\upsilon/dt) = m\mathrm{g} - k\upsilon$. In projection onto axis Oy an equation of movement in a scalar form is $m(d\upsilon/dt) = mg - k\upsilon$. This is a differential equation of the first order with separatable variables. Therefore, $(d\upsilon/mg - k\upsilon) = (dt/m)$. Integration gives:

$$\int_0^{\upsilon}\frac{d\upsilon}{mg - k\upsilon} = \frac{1}{m}\int_0^{t}dt,$$

i.e.,

$$\frac{1}{k}\ln\left|mg-k\upsilon\right|_0^{\upsilon}=\frac{t}{m}$$

and then

$$-\frac{1}{k}\ln\frac{mg-k\upsilon}{mg}=\frac{t}{m}$$

or

$$\ln\frac{mg-k\upsilon}{mg}=-\frac{k}{m}t.$$

Exponentiation gives

$$\frac{mg-k\upsilon}{mg}=\mathrm{e}^{-(k/m)t}.$$

From this the dependence $\upsilon(t)$ can be derived:

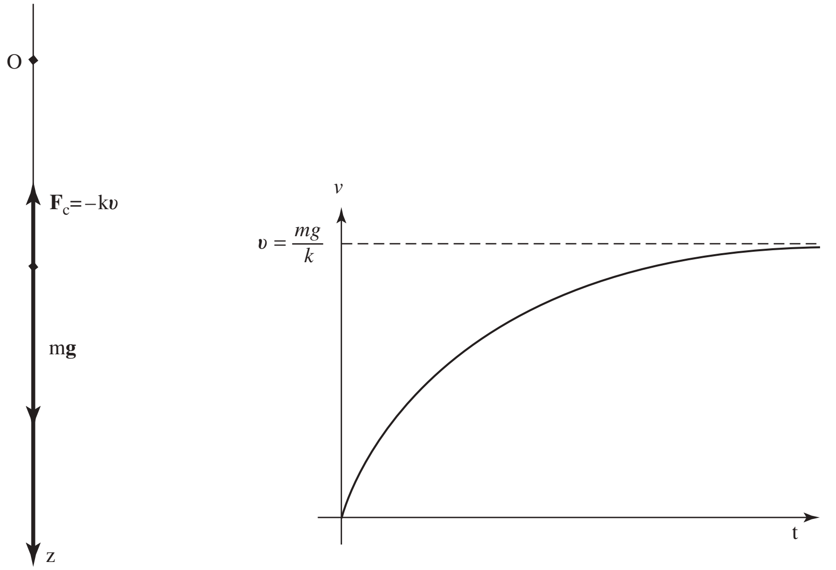
$$\upsilon(t)=\frac{mg}{k}(1-\mathrm{e}^{-(k/m)t}).$$

The graph of $\upsilon(t)$ is presented in Figure E1.7(b). One can see that the body's speed asymptotically reaches the speed of steady movement υ_{st}, i.e.:

$$\upsilon_{\mathrm{st}}=\lim\frac{mg}{k}(1-\mathrm{e}^{-(k/m)t}).$$

At $t\to\infty$, $\upsilon_{\mathrm{st}}\to mg/k$. After introduction of the figures one arrives at:

$$\upsilon_{\mathrm{st}}=\frac{80.9\times 81}{10}\,\mathrm{m/sec}=78.5\,\mathrm{m/sec}.$$



EXAMPLE E1.8.

A high-speed boat moving on calm water achieves a speed of $\upsilon = 90$ km/h. The total mass (m) of the boat with a man on board is 400 kg. The force of water resistance to the boat's movement changes under law $F_c = -k\upsilon\boldsymbol{\upsilon}$. Draw a graph of the time dependence of the speed $\upsilon(t)$ and calculate by how much the speed of the boat changes ($\Delta\upsilon$) in time $t = 1$ min after the man has switched off the engine. The resistance force constant k equals 0.8 kg/m.

Solution: Two vertical forces operate on the boat in the lake: the gravitational (mg) and extruded Archimedes force F_a; both of them equalize each other, therefore we do not need to consider them. We shall connect the inertial system of reference with the earth. We shall direct the coordinate axes horizontal. The equation of the Newtonian second law in aprojection to an axis x will be written as

$$m\frac{d\upsilon}{dt} = F_c = -k\upsilon^2$$

Separating the variables we obtain $(d\upsilon/\upsilon^2) = -(k/m)dt$. In order to draw a graph it is more convenient to take an indefinite integral

$$-\int (d\upsilon/\upsilon^2) = (k/m)\int dt + C.$$

After taking an integral we obtain $-(1/\upsilon(t)) = (k/m)t + C$. We can find the constant using the initial conditions: at $t = 0$, $\upsilon(0) = \upsilon_0$ and then $C = -(1/\upsilon_0)$. Therefore,

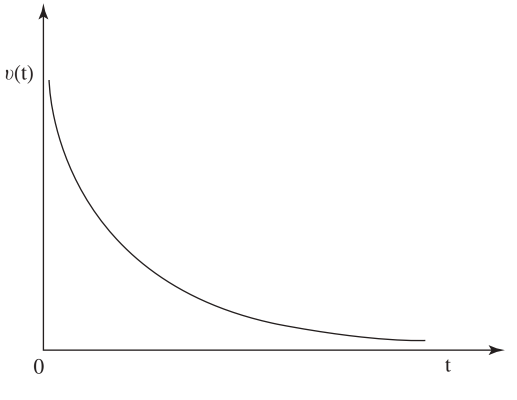
$$-\frac{1}{\upsilon} = \frac{k}{m}t + \frac{1}{\upsilon_0} = \frac{k\upsilon_0 t + m}{m\upsilon_0}.$$

We should solve an equation relative $\upsilon(t)$:

$$\upsilon(t) = \frac{m\upsilon_0}{k\upsilon_0 t + m}.$$

The graph is a hyperbola (Figure E1.8). To determine $\Delta\upsilon$ we should find a difference $\upsilon(t) - \upsilon_0 = \Delta\upsilon = -\upsilon_0(1-(m/kt\upsilon_0 + m))$. To obtain the answer we should express all values in brackets in terms of SI. The speed value before a bracket can be left in km/h (then the answer will turn out to be in the same unit):

$$\Delta\upsilon = 90\left[1 - \frac{400}{(400 + 400 \times 0.8 \times 25 \times 60)}\right]\text{km/h} = 22.5\,\text{km/h}.$$



Motion of a body with a variable mass

In some problems of mechanics a body's mass, in the process of motion, does not stay constant. Examples can be found in modern technology (e.g., refuelling airborne aircraft, etc.), particularly in systems whose movement is based on the combustion of fuel accumulated entirely in the moving system. Let us derive the main laws of such a motion using a space rocket as an example (Figure 1.9). As a simple example, we can consider a rocket fired in an open space where no gravitational force or air resistance exists. All of the fuel stored is eventually burned and ejected from the nozzle of the rocket's engine; the rocket mass is

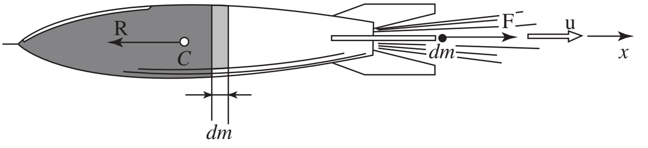


Figure 1.9. Principle of rocket propulsion.

time dependent in this case. It is impossible to apply Newton's second law to the rocket alone; however taking the rocket and its ejected combustion products together allows us to consider such a combined system to be a closed one and to apply this law.

During the flight time the mass of the rocket and its velocity become time-dependent, $m(t)$ and $\upsilon(t)$. For a time accretion dt the rocket mass and velocity increments are dm and $d\upsilon$, the value of dm being negative. The momentum of the rocket will become $(m + dm) \times (\upsilon + d\upsilon)$, and the momentum of gas exhausted will be $dm_g u_g$, where dm_g is the mass of the combustion gas and u_g is the exhausted gas speed. The momentum increment for the time dt can be obtained by subtracting from the written momentum its initial value. According to eq. (1.3.12), this difference is a forward impulse Fdt. In projections on an axis along which the motion proceeds it is

$$(m+dm)(\upsilon+d\upsilon)+dm_g\upsilon_g-m\upsilon=Fdt \tag{1.3.15}$$

Opening brackets and taking into account that $dm= -dm_g$ and $\upsilon_{rel} = \upsilon_g - \upsilon$ (υ_{rel} is the velocity of the gas exhausted speed relative to the rocket), neglecting the term of the lower infinitesimal order ($dm{\cdot}d\upsilon$) one can arrive at the expression: $md\upsilon = \upsilon_{rel}dm + Fdt$ or

$$m\frac{d\upsilon}{dt}=\upsilon_{rel}\frac{dm}{dt}+F$$

Since we consider the movement in open space without the influence of any gravitation or resistance, we assume $F = 0$. Therefore,

$$m\frac{d\upsilon}{dt}=\upsilon_{rel}\frac{dm}{dt}$$

The right-hand term υ_{rel} (dm/dt) is equidimensional to a force; it presents the reactive propulsive force or thrust of the rocket R

$$\upsilon_{rel}\frac{dm}{dt}=R \tag{1.3.16}$$

According to its form, eq. (1.3.16) corresponds to the second Newtonian law eq. (1.3.7). However, here the mass is not constant: it is decreasing with time because of the fuel combustion; the faster the combustion rate and exhausted gas speed, the larger the rocket engine thrust. This equation was derived for the first time by I.V. Mescherski, and is referred to as the first rocket equation. In the absence of external force, $F = 0$ (the motion in outer space), only the propulsive force is acting on the rocket. Finally, if no external force **F** acts on the rocket, with an initial condition $m(0) = m_0$ and under constant combustion rate ($(dm/dt) = \text{const.}$), the velocity $\upsilon(t)$ can be found:

$$\upsilon(t) = \upsilon_0 - \upsilon_r \ln \frac{m_0}{m(t)} \tag{1.3.17}$$

This equation is referred to as the second rocket equation. Actually, from the expression $m(d\upsilon/dt) = (dm/dt)\upsilon_{rel}$, it follows that $d\upsilon = (dm/m)\upsilon_{rel}$. By integration, the expression presented above can be derived.

The relation of the rocket velocity and its mass change can be derived from expression (1.3.17) (at initial velocity $\upsilon_0 = 0$):

$$m(t) = m_0 \exp\left(-\frac{\upsilon(t)}{\upsilon_{rel}}\right).$$

This expression is referred to as K.E. Tsiolkovsky's law (derived in the 1920s).

EXAMPLE E1.9.

A spacecraft of mass $m = 10$ T is at a great distance from the earth and the other planets of the solar system. To change its speed to $\Delta\upsilon = 0.8$ km/h a jet engine is turned on for a time τ. Determine this time if the fuel combustion rate μ is 100 kg/sec. The exhaust speed **u** is 2 km/sec.

Solution: Accept that the spacecraft moves with some speed $\boldsymbol{\upsilon}$ in an inertial reference system relative to the Sun (heliocentric reference system). Its great distance from all planets allows us to neglect the gravitational forces acting on it. Therefore, on turning on the jet engine the single force acting on the spacecraft is the jet thrust $\mathbf{R} = -\mu\mathbf{u}$. Using the first rocket equation for $\mu < 0$, we shall obtain $m(t)\, d\upsilon/dt = -\mu$u. The mass of the spacecraft is continuously decreasing (owing to the fuel burning and the exhaust from the engine nozzle) according to the equation $m(t) = (m - \mu t)$, where μt is the fuel mass burned up in time t. Let the coordinate axis Ox be codirected with vector $\boldsymbol{\upsilon}$. We shall write down the equation of movement in coordinate form as $(m - \mu t)\, d\upsilon/dt = \mu u$.

After variable separation the equation looks like

$$\frac{d\upsilon}{\mu u} = \frac{dt}{m_c - \mu t}.$$

Time integration in the limits from 0 to τ and, consequently, from υ to $\Delta\upsilon$ gives

$$\frac{1}{\mu u}\int_{v}^{v+\Delta v} dv = -\frac{1}{\mu}\ln\left|m-\mu t\right|_{0}^{\tau}.$$

Deleting on μ and introducing the limits we obtain

$$-\frac{\Delta v}{u} = \ln\frac{m-\mu\tau}{m} = \ln\left(1-\frac{\mu\tau}{m}\right).$$

After exponentiation, this equation reduces to $1 - (\mu\tau/m) = e^{(-\Delta v/u)}$.
The time sought can be extracted from this equation:

$$\tau = \frac{m}{\mu}\left[1-\exp\left(-\frac{\Delta v}{u}\right)\right].$$

Calculations give the time $\tau = (10^4/10^2)(1-e^{-0.8/2}) = 33$ sec.

1.3.4 The third Newtonian law

Till now we have discussed the question of how other bodies act on a given body. Quantitatively, this action is defined by force. Experience shows that such an action cannot be unilateral, any action has the nature of interaction. "Actions" and "counteraction" are equal and indistinguishable. They simultaneously appear and simultaneously disappear, but are attributed to different bodies. All these facts form the essence of the third dynamics law or Newton's third law: the forces with which bodies act on each other are always equal in magnitude and oppositely directed. This signifies that at the interaction of two bodies a force F_{12}, with which the first body acts on the second, is of the absolute value and oppositely directed to the force F_{21}, with which the second body acts on the first. That is,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{1.3.18}$$

1.3.5 Forces classification in physics

All the known forces of interaction existing in nature can be reduced to a small number of main types. Belonging to the first type are the gravitational and electromagnetic forces; belonging to the last type are the forces of interatomic and intermolecular interaction pertaining to which macroscopic manifestation are elasticity forces. (Outside the scope of this book are the short-range nuclear forces, bonded nucleons in nuclei, and weak interactions, revealed in the decay of elementary particles.)

The forces of gravitation are weaker than all the others. At the same time their action is realized through a gravitational field onto great distances. The expression for gravitational interaction between two point masses M and m is defined by the law of Newtonian attraction

$$\mathbf{F} = -G\frac{Mm}{r^2}\frac{\mathbf{r}}{r}, \tag{1.3.19}$$

where G is the gravitational constant, $\mathbf{r}$ is the radius vector drawn from one MP to another being equal in absolute value to the distance between them. These forces govern the interaction between the heavenly bodies.

A body on the Earth's surface ($r = R$, i.e., R is the distance between the center of the earth and the body) experiences the attraction $F =$ mg, or in another form (according to eq. (1.3.19)), $|F| = |G(mM/R^2)|$; it then follows that mg $= G(mM/R^2)$, or $g = G(M/R^2)$. (This relationship can be used in order to simplify the solution to some problems.)

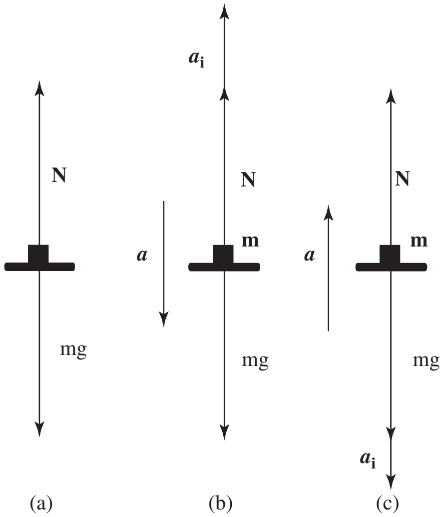
We meet here for the second time the notion of "mass." In this respect the mass is called gravitational. Generally speaking, this coefficient can be different from that appearing in the second Newtonian law. However, practice shows that, fortunately, inertial mass is just the same as gravitational mass; i.e., the mass is the objective characteristic of a body exhibiting equally both inertial and gravitational laws.

It is worth mentioning here the difference between the terms "weight" and "mass." Mass is an inherent property of a body, whereas weight is a measure of an action of the body on a support or suspension. A reaction from the support to the gravitational force exists. When a body lies on a motionless bench, two forces—gravitational and support reaction—compensate each other (according to the third Newtonian law); however, the first is applied to the body and the second to the supports. If the "bench" falls with an acceleration g no reaction appears at all, the weight diminishes to zero and a state of weightlessness occurs. Therefore, the mass is a property of the body, but the weight depends on its motion (on acceleration).

EXAMPLE E1.10A

A body of mass $m = 10$ kg is resting on spring scales in an elevator. The elevator moves with an acceleration of $a = 2$ m/sec^2. Determine the readings of the spring scales in two cases: when the elevator's acceleration is directed vertically upwards and then vertically downwards (Figure E1.10).

Solution: To determine scale readings means to find the weight of the body m**g** (a), i.e., the force with which the body acts on a spring. This force, under the third Newtonian law in the inertial system connected to the earth, is equal on the modulo and is opposite in direction to the force of elasticity (force of a support reaction) from which the spring cup of the scales operates on a body ***N***, P being the scale reading, that is m**g** $= -\boldsymbol{N}$ or in scalar form $P = N$.



Hence, the problem of the scale readings is reduced to finding the reaction of the support N. There are two forces acting on a body: the gravity force m**g** and the support reaction ***N***. Let first the acceleration a be directed upwards. (We can descend the index z because both forces are collinear; the direction will be marked by signs.) The second Newton's law equation can be written as $N - mg = ma$, whereas $N = mg + ma = m(g + a)$. Since $N = m(g + a)$. The sign of acceleration should be accounted: at $a > 0$ $N = 10(9.81 + 2)$ kg m/sec^2 = 118 N (b) whereas at $a < 0$ $N = 10(9.81 - 2)$ kg m/sec^2 = 78 N (c).

Electromagnetic forces retain electrons in atoms, keep atoms in molecules and crystals, and define the interaction of molecules between themselves, etc. Electromagnetic forces are long term (i.e., similarly to gravitational forces they decrease with distance as ~$(1/r^2)$).

In practice, one usually deals with gravity forces, elasticity and friction (resistances). These forces reduce to those already mentioned: gravity forces are a result of gravitational interactions, elasticity forces and friction are manifestations of electromagnetic interactions of atoms and molecules both inside the bodies and between them. Examples of macroscopic elastic forces are forces acting on the suspension to which a body is attached. Under gravitational attraction the mass m will act with the force mg to the support, and, as a reaction, elastic forces appear in the support. The body's weight is the force with which this body acts on the support or stretches a suspension, all of them being unmovable relative to the earth. The elasticity force appearing in suspension is called tension. The force acting on a body from the side of the support is also an elastic force. In this case elastic forces appear as a result of the support's deformation. Such forces are called a reaction. Under small deformations, the elastic strength linearly depends on the deformation value, i.e., it follows a linear Hooke's law: the value of deformation is proportional to the deforming force and is opposite in sign:

$$F_{\text{elast}} = -\beta x, \tag{1.3.20}$$

where β is the coefficient of proportionality (coefficient of elasticity or rigidity), x is a value of deformation. The term "quasi-elastic force" is often used in physics though the forces are not of mechanical tension; it implies that a force is proportional to deformation.

Under large deformations the elastic force can depend on deformation nonlinearly:

$$F = -\beta x + \gamma x^2 + \gamma' x^3 + \cdots \tag{1.3.21}$$

The coefficients γ are called unharmonicity coefficients.

Friction forces of sliding appear under the direct contact of surfaces at the relative motion of one body upon the other. They are stipulated by the phenomena occurring in the shallow layers of the surfaces of the contacting bodies. Such friction, which acts between surfaces of different bodies, is called external friction. The friction force P in this case is proportional to the normal pressure force N

$$P = fN, \tag{1.3.22}$$

where f is the friction factor. This factor depends on the material, conditions of the surfaces, the presence of lubrication and others. Friction force is always directed against the direction of motion and lies in planes of contiguity.

Friction can appear as a result of the interaction of different parts of one and the same material, for instance, between the different layers of a moving liquid or gas. Such friction is referred to as internal and the phenomena itself as viscosity. When a solid body moves in liquids or gas, internal friction can appear. This is a result of the molecules sticking to the solid body surface and moving together with it. The phenomena are referred to as internal (viscous) friction; under small velocities such friction force can have a linear dependence on velocity:

$$F = k\upsilon, \tag{1.3.23}$$

where k is a coefficient dependent on the medium property and the dimension and form of a body. The nonlinearity can appear as the velocity increase.

In some velocity interval the force of friction can appear to be proportional to the second power of velocity

$$\mathbf{F}_{\mathrm{fr}} = -k_1 \upsilon^2 \frac{\boldsymbol{\upsilon}}{\upsilon}. \tag{1.3.24}$$

The k_1 coefficient is also dependent on both the medium nature and the body size and form.

The information presented on the nature of forces in physics does not, however, give any hope of receiving an answer to the simple question: what keeps atoms in molecules. Moreover, it is proved that forces of only electrostatic nature are unable to ensure the molecules' stability. We should accept that within the framework of classical physics the answer to this question is absent in general. Quantum laws, namely the identity of electrons, bring about the so-called exchange interactions, creating additional forces and stabilizing molecules. The nature of these forces however still remains electrostatic. We can say that the exchange forces are the result of using usual classical presentations in the field of, and according to, quantum physics.

1.3.6 Noninertial reference systems. An inertia force: D'Alembert principle

So far we have considered the motion of a body in inertial reference systems. However, there exist many problems where it is necessary to use noninertial reference systems such as, for example, the motion of molecules in a centrifuge along a circular path or accelerating motion in the rocket. In noninertial reference systems expressions (1.3.14) are not fulfilled. Reference systems in which the motion of a free body is not rectilinear and uniform are referred to as noninertial systems. Consequently, any reference system moving with acceleration relative to any inertial reference system is a noninertial one. The acceleration can be both translational ($a_\tau \neq 0$) or rotational ($a_n \neq 0$). In the general case, the acceleration of different points of a moving body can be different. This means that the space connected with the noninertial reference systems is neither uniform, nor isotropic.

The equation of a MP motion regarding the noninertial system looks different from an inertial one. Consider the specific example of a noninertial reference system K' (x', y', z') moving with an acceleration $\boldsymbol{a}_0$ comparative to a certain inertial system $K(x, y, z)$. Suppose then that a MP in this inertial system moves with an acceleration $\boldsymbol{a}$, and this acceleration is caused by the action of forces $F_1, F_2, \ldots, F_N$. According to eq. (1.3.13) we can write

$$m\boldsymbol{a} = \sum_{i=1}^{N} F_i. \tag{1.3.25}$$

In the system K' the same MP will have an acceleration $\boldsymbol{a}'$ (relative acceleration), which is the sum of $\boldsymbol{a}$ and the acceleration of the noninertial reference system $\boldsymbol{a}_0$, i.e., $\boldsymbol{a}' = \boldsymbol{a} - \boldsymbol{a}_0$. We can multiply the right- and left-hand sides of this expression by the mass of an MP (m):

$$ma' = \sum_{i=1}^{N} F_i - ma_0 \tag{1.3.26}$$

The expression obtained differs from the equation of motion in the inertial reference system (1.3.7) by the term $-m\boldsymbol{a}_0$. The noncompliance with the second Newton law is caused by the appearance of that additional term. Moreover, if the geometric sum of acting forces is equal to zero, then $\boldsymbol{a}' = \boldsymbol{a}_0$, whereas according to the second Newton law it also has to be zero. The product of the body's mass and the acceleration of the noninertial reference system taken with the opposite sign is called the force of inertia.

$$\mathbf{F}_i = -m\boldsymbol{a}_0. \tag{1.3.27}$$

Inertia forces are the uncommon forces that disobey the laws of classical Newton mechanics. Indeed, in a noninertia reference system we are unable to indicate a body whose action can explain the appearance of inertia forces. This signifies that Newtonian laws are not executed in noninertial reference systems. Figuratively speaking, there exists a force of "actions" (the force of inertia), but no force of "counteraction." In noninertial reference systems, these particularities of inertia forces do not allow the selection of a closed system of bodies (refer to 1.3.7), since for any body in a noninertial system the inertia forces are the internal ones. Thus, in the noninertial reference system the conservation laws of energy and momentum, which will be considered below (see Section 1.5), are not valid.

The importance of introducing the forces of inertia consists of the fact that with the provision of these forces it is possible to use Newtonians laws, as they would occur in an inertial system. With provision for eq. (1.3.26) Newton's second law (1.3.7) takes the form:

$$m\boldsymbol{a} = \sum_{k=1}^{N} \mathbf{F}_i + \mathbf{F}_{\text{inert}}. \tag{1.3.28}$$

If a body is resting in the noninertial system ($a = 0$), eq. (1.3.28) simplifies to:

$$\sum_{i=1}^{N} \mathbf{F}_i + \mathbf{F}_{\text{inert}} = 0 \tag{1.3.29}$$

and the problem of dynamics (in a noninertial reference system) is reduced to the equality to zero as the result of all acting on the body forces and the force of inertia, i.e., to the problem of the static in the inertial reference system. The statement that problems of dynamics can be reduced to problems of static by the addition of the usual forces acting on the body, the force of inertia, is called the D'Alembert principle. It must be remembered that the tasks of static can be solved more easily than the problems of dynamics.

EXAMPLE E1.10B

This task can be solved in a noninertial system. In this case a reference system can be connected to the elevator. It means that to all forces the D'Alambert force ($F_i = -\mathrm{m}a_i$) should be added (a_i is the acceleration of the elevator movement relative to the earth). Three forces are acting on a body in this case: gravitational force $m\mathbf{g}$, elasticity force $\mathbf{N}$ and the inertia force F_i. In the reference system connected with the elevator the body is at rest. Therefore, the sum of the forces is zero: $m\mathbf{g} + \mathbf{N} + \mathbf{F}_i = 0$. After projection on the z-axis the equation is transformed to $N - g - ma = 0$, whereas the support reaction is $N = mg + ma = m(g + a)$; we arrive at the same equation as in the Part A of this example.

Of particular interest is a specific case of the manifestation of inertia forces. This case concerns the uniform motion of an MP along the circle trajectory. An MP participating in such a motion possesses normal (centripetal) acceleration a_n. Then, obviously, the reference system connected with this point (!) will be a noninertial one. In this noninertial reference system to the MP of mass m, aside from the others, an inertia force $F_i = -ma_0$ is applied; it is called the centrifugal inertia force. This force is attached not to the MP but to the bonds retaining this MP on the circle trajectory; it is directed from the center along the radius. It is important not to confuse them!

1.3.7 A system of material points: internal and external forces

Any set of MPs (or bodies) is called a material points system. Each system's points can interact both with bodies of the same system and with bodies not belonging to it. Forces acting between the system's MP (bodies) are referred to as internal forces. Forces acting on the system's points from outside are referred to as external forces. A system is called closed (or isolated) if it comprises all interacting bodies. Thus, in the closed system only internal forces are acting. They compensate each other according to the third Newtonian law.

Strictly speaking, there are no closed systems in nature. However, it is almost always possible to define a task neglecting external forces in the limit of accuracy. The choice of the border surface is a person's own prerogative and can be chosen by the researcher on the basis of an analysis of internal and external forces. One and the same system can be considered as closed or open in different situation, depending on the statement of the problem and pre-given accuracy. All processes are described more easily and decided more clearly in the closed system. This is what in physics is called the physical model.

The *these two notion are of the some importance and should be given accordingly* of a system plays an enormous role not only in art (architecture, painting, ornaments, composing, etc.), but to no less a degree in science. Let us first provide some definitions. Any operation superposing an object with itself is referred to as an operation of symmetry. Atoms and molecules, plants, animals and people, building materials, etc. are symmetrical. In crystallography, chemistry and quantum chemistry translations (t), mirror planes (m) and axes of symmetry (L_n) are mainly used. Endless checked paper (e.g., graph paper) allows one to illustrate an infinite translation: under certain translations (refer to Section 9.1), this sheet being shifted on a certain vector will coincide with itself. A plane perpendicular to the benzene ring going through two opposite atoms of carbon is a mirror symmetry plane. An axis going through the oxygen atom in the water molecule along the bisector of the valence angle is an axis of symmetry of the second order L_2. (The numeral 2 appears because of the fact that the symmetry rotation here is ($360°/2 = 180°$)). The combination of an L_2 axis and a perpendicular mirror plane generates the center of symmetry. There are also other combinations: $360°/n$. At $n = 3$ (L_3 is a designation of an axis of symmetry of the third order, for instance, a molecule of ozone O_3), at $n = 4$ (axis of symmetry is of the fourth order, square molecules), at $n = 6$ (axis of symmetry is of the sixth order, for instance, the benzene ring with the axis L_6, passing perpendicularly through the center of the carbon ring). A very important element is the center of symmetry, the "reflection in the point" (the benzene ring has a symmetry center, but the molecule of water does not).

In many cases, analysis of a problem's symmetry simplifies its solution. We use this in many sections of this book.

Center of mass

Let us introduce a very important notion, a centre of mass (CM) (or the center of inertia). The CM of a system is the point at which the system's mass can be assumed to be concentrated. It can be described with a radius vector

$$\mathbf{R}_c = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} \tag{1.3.30}$$

The total mass of the system is further denoted by m.

If one places the origin into the CM (point C) then $R_c = 0$ and

$$\sum_{i=1} m_i \mathbf{r}_i = 0. \tag{1.3.31}$$

Another definition can be drawn from this equation: the CM of a mechanical system is the point for which the sum of products of the masses of all MPs comprising the system, on their radius vectors, drawn from this point, are zero. In Figure 1.10 all these positions are illustrated using the example of a system consisting of two MPs.

In the Cartesian reference system, the radius vector $\mathbf{R}_c$ has coordinates X_C, Y_C, Z_C:

$$X_C = \frac{\sum_{i=1}^{N} m_i x_i}{m}, \quad Y_C = \frac{\sum_{i=1}^{N} m_i y_i}{m}, \quad Z_C = \frac{\sum_{i=1}^{N} m_i z_i}{m}. \tag{1.3.32}$$

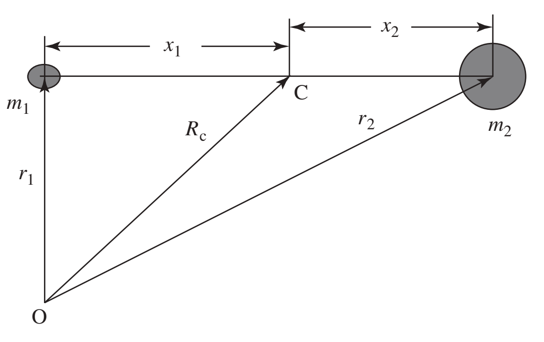


Figure 1.10. The center of a two-mass system.

So far we have considered the set of discrete MPs. The question is now: how to find the CM in a continuous body? It is quite natural to move from the sum to integrals:

$$\mathbf{R}_{\mathrm{C}} = \frac{1}{m}\int_V \mathbf{r}dm \tag{1.3.33}$$

and in coordinate form

$$X_{\mathrm{C}} = \frac{\int_V xdm}{m}; \quad Y_{\mathrm{C}} \frac{\int_V ydm}{m}; \quad Z_{\mathrm{C}} \frac{\int_V zdm}{m}. \tag{1.3.34}$$

It is easy to guess that for bodies having a plane of symmetry, the CM is located on this plane. If a body possesses a symmetry axis, the CM certainly must lie on this axis. If a body possesses a center of symmetry, it is not necessary to determine the CM position: they have to coincide with each other.

In order to show the importance of the CM point, let us determine how it moves. Let us write the expressions (1.3.32) in the form

$$\sum_{i=1}^{N} m_i x_i = mX_c, \quad \sum_{i=1}^{N} m_i y_i = mY_c, \quad \sum_{i=1}^{N} m_i y_i = mZ_c.$$

The time second derivation of them gives:

$$\sum_{i=1}^{N} m_i \ddot{x}_i = m\ddot{X}_{\mathrm{C}}, \quad \sum_{i=1}^{N} m_i \ddot{y}_i = m\ddot{Y}_{\mathrm{C}}, \quad \sum_{i=1}^{N} m_i \ddot{z}_i = m\ddot{Z}_c$$

Comparing these equalities with eq. (1.3.8) one can find that

$$m\ddot{X}_{\mathrm{C}} = \sum F_{xi}, \quad m\ddot{Y}_{\mathrm{C}} = \sum F_{yi}, \quad m\ddot{Z}_{\mathrm{C}} = \sum F_{zi}.$$

or

$$m\ddot{\mathbf{R}}_{\mathrm{C}} = \sum_{i=1}^{N} \mathbf{F}_{\mathrm{i}}. \tag{1.3.35}$$

These equations, called the differential equations of the CM movement, coincide in their structure with the differential equations of MP movement. Therefore, one can conclude: the CM of the mechanical system moves as MP, the mass of which is equal to the total system's mass and to which all the acting forces are applied.

If the system is free from the external forces (i.e., closed) ($\mathbf{F}_i = 0$ or $\Sigma\mathbf{F}_i = 0$), then

$$\dot{\mathbf{R}}_C = V_C = \text{const.}, \tag{1.3.36}$$

and, hence, the velocity of the CM is constant (i.e., is preserved). The internal forces do not influence its movement. If at some time in some reference system the CM's point of a closed system is at rest it means that it will rest further.

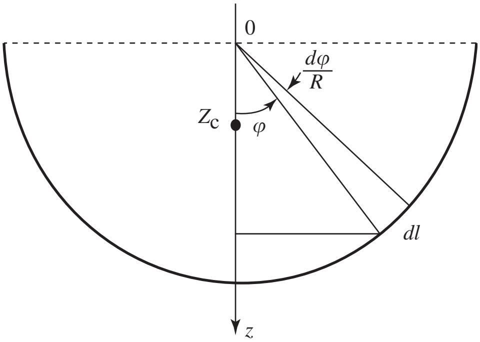
Many problems of mechanics can be solved in the easiest way in the coordinate system connected with the mass center.

EXAMPLE E1.11

The ends of a half hoop are connected by a straight weightless wire. The radius of the half hoop is R. Find the position of the CM of this figure.

Solution: The half hoop has a symmetry axis of a second order; it divides the figure into two equal parts (see Figure E1.11). The center of inertia should definitely lie on this axis, direct a z-axis along this axis; therefore we have to find only one coordinate. Use the formula for the Z_C coordinate (eq. (1.3.32)). In our case it is $Z_c = (\int dmz)/\text{M}$, where M is an unknown mass of a half hoop. Allocate a segment $\text{d}l = Rd\varphi$ (see figure), write down the evident relations $\text{d}l\rho = dm$, $(z/R) = \cos\varphi$, $M/\pi R = \rho$. Substituting these equations into the expression for Z_C we arrive at

$$\begin{aligned} Z_C &= \frac{1}{M}\int d\ell \rho R\cos\varphi = \frac{1}{M}\int d\varphi R\frac{M}{\pi R}R\cos\varphi \\ &= \frac{R}{\pi}\int_{-\pi/2}^{+\pi/2}\cos\varphi\, d\varphi = \frac{2R}{\pi}. \end{aligned}$$



1.3.8 Specification of a material points system

To assign a system state means that one can describe a system's configuration (i.e., to know the position of all the elements) at the initial instance of time, being able to calculate it at any other time.

The space position of an MP (Figure 1.11) can be assigned by radius vector **r**, which in turn is assigned by three coordinates (for instance, Cartesian coordinates x, y, z). How many numbers does one have to know to be able to find a system position at any following points of time $t + dt$? For the time dt a point displaces from position 1 to position 2:

$$\mathbf{r}(t+dt) = \mathbf{r}(t) + d\mathbf{r} = \mathbf{r}(t) + \mathbf{V}(t)dt. \tag{1.3.37}$$

This implies that for the determination $\mathbf{r}(t + dt)$ one needs to know $\mathbf{r}(t)$ and $\upsilon(t)$, i.e., six numbers in total. If a system consists of N particles, the number of parameters is $6N$.

For the assignment of a system state in modern physics, a so-called configuration space is introduced. The dimensionality of this space is the number of parameters defining the state of a system at a point of time, the change of this state being defined by the set of points, i.e., the line. The element of the configuration space volume $d\tau$ in analogy with a three-dimensional case is written as

$$d\tau = \prod_{i=1}^{N} dx_i dy_i dz_i dp_{xi} dp_{yi} dp_{zi}. \tag{1.3.38}$$

We will use this representation in Chapters 3 and 9.

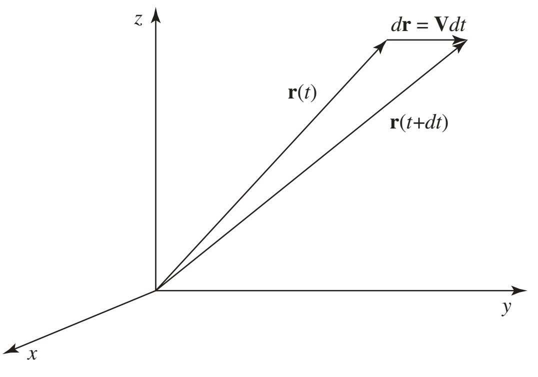


Figure 1.11. A material point movement description.

1.3.9 The dynamics of rotational motion

When considering the kinematics of the rotational movement of an IRB, we had to introduce some new characteristics: elemental angular displacement, angular velocity and angular acceleration. These values are identical to the whole rotating body, whereas the linear characteristics for all the body's MPs differ.

In rotational motion other dynamic characteristics are also required, such as a force moment (torque) with regard to a motionless axis, a moment of inertia (MI) and an angular momentum, being in some respect analogous to the characteristics of linear motion (mass, force, momentum).

These models do not exhaust the whole description of problems concerning rotational motion, but cover a wide range of phenomena with which the chemist may be confronted. Ignoring the logic of physics for the sake of simplification, we will start from the description of the rotation of an IRB relative to a motionless axis. We will then generalize the results obtained and apply them to the motion of an MP around a pole.

The dynamics of rotation of an IRB around a motionless axis

Suppose that force **F** is arbitrarily applied to a body's point (in Figure 1.12 the body itself is again not shown). Divide the force vector into two components: one parallel to the axis of rotation $F_{||}$, and the other lying in the plane perpendicular to the rotation axis $F_{\perp}$. Only one of them ($F_{\perp}$) influences the rotation, whereas $F_{||}$ exerts pressure on the bearings in which an axis is fixed. The force momentum (torque) M in respect to an axis Oz is the value

$$M = F_{\perp} R \sin\alpha = F_{\perp} h, \tag{1.3.39}$$

where α is an angle between the radius of circular point path (R) and the force arm (h). In turn, $R\sin\alpha = h$, h being the shortest distance from the point O to the line along which the force component $F_{\perp}$ is acting (the force arm). Note that the torque can be zero though the force itself is not, if the force line action crosses the axis of rotation or is parallel to it.

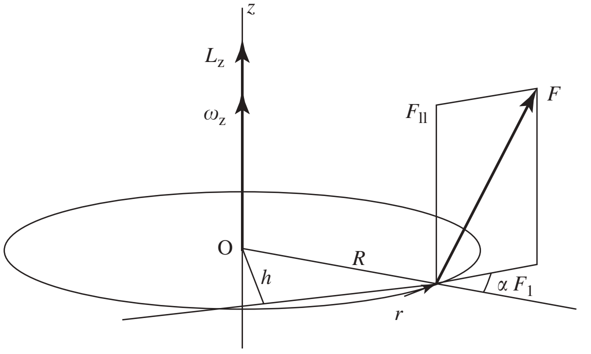


Figure 1.12. A force moment; a work of this moment at a body rotation.

Since all the body's points are at different distances from the axis of rotation their linear velocities and, correspondingly, their momentums are different. In order to find a body's rotation characteristics let us draw at a moving point, a mass element $dm = \rho dV$, where ρ is the body's density. The value

$$dL_z = R\,dp = \upsilon R\,dm, \tag{1.3.40}$$

is referred to as the angular momentum of a mass element dm relative to the axis Oz, R being the distance of the dm element from the axis and υ the linear velocity of this element. Referring to eq. (1.2.27) we can arrive at

$$dL_z = \omega_z R^2 dm = \omega_z\, dI_z. \tag{1.3.41}$$

A value

$$dI_z = R^2 dm \tag{1.3.42}$$

is referred to as a moment of inertia (MI) of the mass's element dm relative to the axis Oz. Integrating it over the body's volume V, one obtains

$$L_z = \omega_z \int_V dI_z = \omega_z \int_V R^2 dm.$$

In this expression

$$I_Z = \int_V R^2 dm \tag{1.3.43}$$

is referred to as the body's momentum of inertia relative to an axis Oz. Then one can arrive at

$$L_Z = I_Z \omega_Z. \tag{1.3.44}$$

That is, *the angular momentum of a body relative to a motionless axis is the product of the moment of inertia and the angular velocity of a body's rotation relative to the same axis*. (This definition according to its "structure" is equivalent to the definition "momentum of the translational movement is a product of its mass and their velocity.")

Moments of inertia of some symmetric bodies

Consider moments of inertia of some symmetric figures relative to their central axes, i.e., axes passing through their CM (i.e., the symmetry axes).

Thin rod. Let us derive the MI of a thin rod (i.e., a rod of mass m whose transverse linear dimensions are much less than its length l) with regard to the axis Oz passing perpendicularly to the rod passing through its CM (Figure 1.13). Choose an elementary fragment dx, remote from the axis at distance x from the axis Oz. The mass of this fragment is $dm = (m/l)dx$. According to the formula (1.3.43) we will arrive at

$$I_z = \frac{m}{l}\int_{-l/2}^{+l/2} x^2 dx = \frac{ml^2}{12}. \tag{1.3.45}$$

Hoop. Consider a hoop of mass m and radius R. The material cross-section is negligibly small (Figure 1.14). The ring MI regarding the axis Oz drawn through its center perpendicular to the ring plane is

$$I_z = \int_0^{2\pi R} R^2 \frac{m}{2\pi R} dl = mR^2 \tag{1.3.46}$$

where dl is the length of an arc of $dm = (m/2\pi R)dl$.

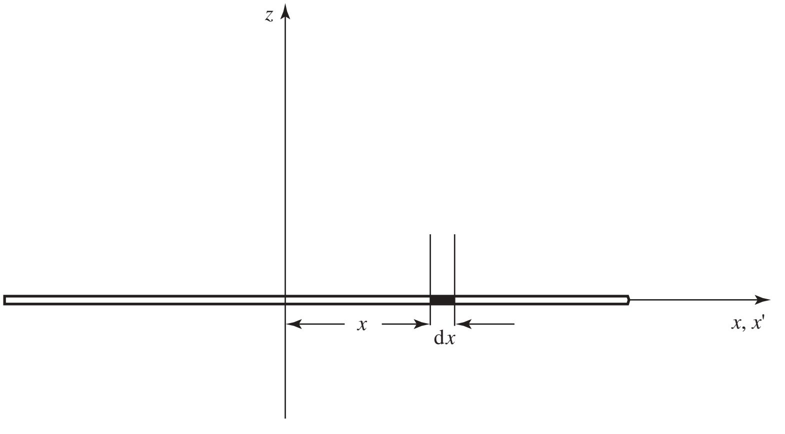


Figure 1.13. A moment of inertia of a thin rod.

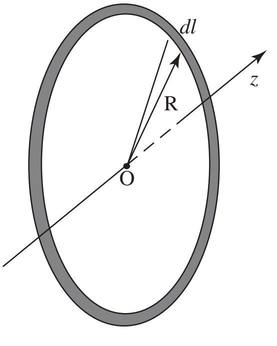


Figure 1.14. A moment of inertia of a thin hoop.

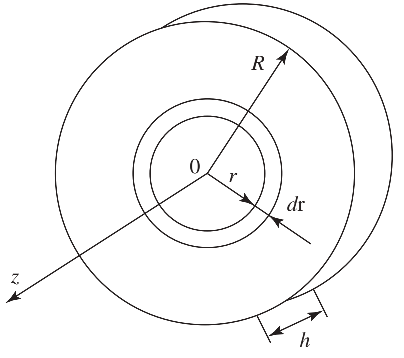


Figure 1.15. A moment of inertia of a disc (cylinder).

Disc (*cylinder*). Let us select in the solid-bulk disc of mass m and radius R (Figure 1.15) an elementary volume dV in the form of a coreless cylinder of radius r, height h and thickness of walls dr. Its mass dm is $dm = \rho dV$. As $\rho = m/V = m/\pi R^2 h$ then $dm = (m/\pi R^2\mathrm{h})2\pi r\,dr\,h = (2m/R^2)r\,dr$, h and I_z makes up $I_z = (2m/R^2)\int_0^R r^3 dr = (2mR^4 / R^2 4)$, or finally

$$I_z = \frac{mR^2}{2}. \tag{1.3.47}$$

In many cases it is necessary to calculate the MI with regard to an axis z' parallel to the symmetry axis z but shifted from it to the distance d remaining parallel to the first (Figure 1.16). The MI I_Z regarding the central axis z is taken as known. The $I_{z'}$ value should be calculated. In the shifted reference system the coordinate of mass element dm is x' the last being equal to $x' = d + x$. The MI $I_{z'}$ can be written as

$$I_{z'} = \int_{-l/2}^{+l/2} (d+x)^2 dm = \int_{-l/2}^{+l/2} d^2 dm + 2d\int_{-l/2}^{+l/2} x dm + \int_{-l/2}^{+l/2} x^2 dm$$

However,

$$\int_{-l/2}^{l/2} d^2 dm = d^2 \int_{-l/2}^{l/2} dm = d^2 m, \quad 2d\int_{-l/2}^{l/2} x dm = 0$$

(according the definition of the CM (1.3.31)). The term $d^2 \int_{-l/2}^{l/2} dm = d^2 m$ is I_Z; therefore we arrive at

$$I_{Z'} = I_Z + md^2. \tag{1.3.48}$$

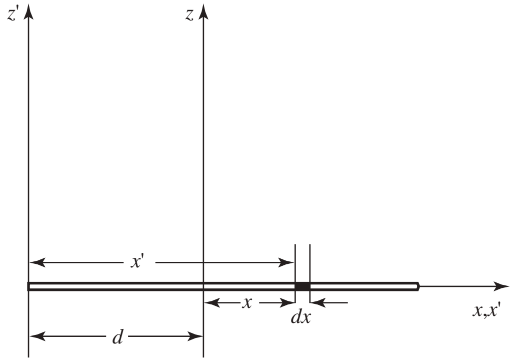


Figure 1.16. A parallel axis theorem.

This expression is referred to *the parallel axis theorem*. Notice that the value md^2 is always positive; it means that the MI relative to the symmetry axis I_Z has a minimum value.

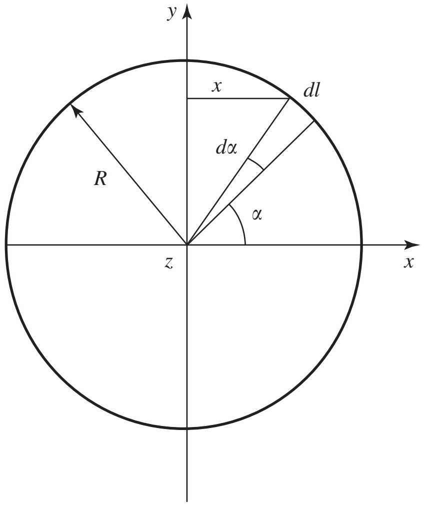
There exists one more method of simplifying the MI calculations. We mean calculations of the MIs of planar figures, i.e., figures that slightly differ from a two-dimensional form (have low thickness). A method is derived and presented in Example E1.12. The main essence of this method is the relationship between MI regarding three orthogonal axes

$$I_z = I_x + I_y. \tag{1.3.49}$$

Sometimes this method is referred to as the *theorem of orthogonal axis*.

EXAMPLE E1.12

Calculate the MI of a hoop relative to its diameter (a y-axis, Figure E1.12); R is its radius and m its mass (an axis y lies in the hoop's plane passing its center).



Solution: Let us solve this problem by the first method. On the hoop, allocate an elementary segment dl with mass dm = $(m/2\pi R)d\ell$. Find the MI of a chosen element considering it as MP: $dI_y = x^2dm = x^2(m/2\pi R)d\ell$. Note that x (the shortest distance to axis Oz) can be expressed through an angle α: $x = R\cos\alpha$. In its turn, $d\ell/R = d\alpha$ and a MI of a selected element is $dI_y = R^2\cos^2\alpha(m/2\pi)d\alpha$. By integration in the limits from 0 to 2π we arrive at:

$$I_y = \frac{mR^2}{2\pi}\int_0^{2\pi}\cos^2\alpha = \frac{mR^2}{2\pi}\times\frac{1}{2}\int_0^{2\pi}(1+\cos\alpha)\mathrm{d}\alpha = \frac{mR^2}{4\pi}2\pi = \frac{mR^2}{2}$$

Now consider the second method. Consider the same plain figure: a hoop R in diameter and mass M: the x- and y-axes coincide with the diameters; the z-axis is directed perpendicular to the plane. Allocate an elementary area (as MP) of a mass dm with coordinates x and y ($z = 0$). Then the I_y value relative to the y-axis is: $dI_y = x^2dm$; $dI_x = y^2dm$ and $dI_z = (x^2 + y^2)dm = x^2dm + y^2dm$; then integration over the whole figure gives

$$I_z = I_x + I_y$$

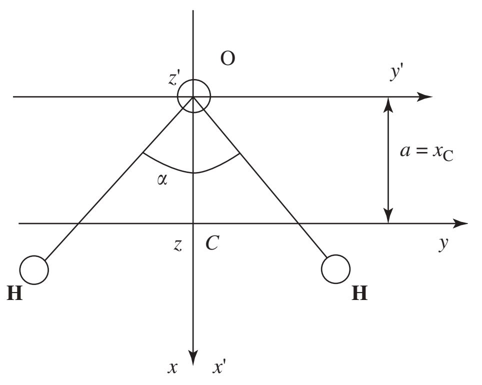
Note that we already know the I_z of a thin ring relative to an axis perpendicular to the ring's plane and passing through its center: $I_z = mR^2$ (see eq. (1.3.46)). Symmetry consideration gives $I_y = I_x$, then according to (1.3.49) we can write

$$I_z = I_x + I_y = 2I_y \text{ and } I_y = \frac{1}{2}I_z = \frac{mR^2}{2}$$

It can be seen that the second method in some cases is less troublesome than the direct solution. This approach is referred to as the *perpendicular (orthogonal) axis theorem.*

EXAMPLE E1.13

Calculate the MI of a H_2O molecule relative three mutually orthogonal axes x, y, z passing through the molecule's CM. Interatomic distances O–H are d = 95.7 pm, valence angle α = 104.5° (see Figure E1.13). Relative nuclear masses of atoms: $A_{r,1}$ = 1 and oxygen $A_{r,2}$ = 16.



Solution: Let's arrange a molecule as it is represented in Figure E1.13. A water molecule consists of three MPs (nuclei of atoms) with a total mass $m = 2m_1 + m_2$, where m_1 and m_2 are masses of hydrogen and oxygen atoms, respectively. Calculate first the MI I_z of the water molecule relative to an axis z' which passes through oxygen atom perpendicular to a molecule plane. The origin (intersection of three axes C) is superimposed with the molecule CM. We direct an axis z upwards perpendicular to a molecule planes. To find the molecule MI we shall take advantage of the theorem on parallel axis (see (1.3.48)): $I_{z'} = I_{z,C} + ma^2$, where $I_{z'}$ is the MI of the molecule relative axis z', passing through an oxygen atom and parallel to the axis z. The required MI is $I_{z,C} = I_{z'} - ma^{2*}$, where a is a distance between two parallel z axes. One can find the MI relative the axis z' as a sum of two MI of MP being at the distance d from the z' axis $I_{z'} = 2m_1d^2$. The distance a is just the coordinate x_C of the oxygen atom laying on y' axis; it can be found according eq. (1.3.34), i.e.,

$$a = x_C = \frac{\sum m_i x_i'}{m} = \frac{2m_1x_1' + m_2x_2'}{m}.$$

Therefore, $a = (2m_1x'_1 + m_2x'_2)/m$. Taking into account that $x'_1 = d\cos(\alpha/2)$ and $x'_2 = 0$, we can obtain $a = 2\,m_1 d\cos(\alpha/2)/m$. Substitute $I_{z'}$, m and a into the star* equation we obtain

$$I_{z,C} = 2m_1d^2\left(1 - \frac{2m_1}{2m_1 + m_2}\cos^2\frac{\alpha}{2}\right)$$

The $I_{y'}$ can be obtained accordingly: $I_{y'} = 2m_1d_2\sin^2(\alpha/2)$.

Now we can use the advantage of the previous correlation of MI's of the plain figures (see E1.12) ($I_z = I_x + I_y$) and found

$$I_x = 2m_1d^2\left(1 - \frac{2m_1}{m}\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) = 2m_1d^2\cos^2\frac{\alpha}{2}\left(1 - \frac{2m_1}{2m_1 + m_2}\right)$$

Calculations can be simplified a bit taking into account that expression in bracket are dimensionless. The mass m_1 and d should be translated into SI, keeping in mind that 1 a.m.u. $= 1.66 \times 10^{-27}$ kg and 1 pm $= 10^{-12}$ m, then $m_1 = A_{r,1} \times 1.66 \times 10^{-27}$ kg and $d = 95.7 \times 10^{-10}$ m. Substitute all these values into the final formula and executing calculation gives

$$\begin{aligned}
I_{z,C} &= 2\times 1.66\times 10^{-27}(0.957\times 10^{-10})^2\left[1 - \frac{2.1}{2.1+16}\cos^2\left(\frac{104.5^\circ}{2}\right)\right]\mathrm{kg\,m^2}\\
&= 2.91\times 10^{-47}\ \mathrm{kg\,m^2},\\
I_{y,C} &= 2\times 1.66\times 10^{-27}(0.957\times 10^{-10})^2 = 1.888\times 10^{-47}\ \mathrm{kg\,m^2},\\
I_{x,C} &= 2\times 1.66\times 10^{-27}(0.957\times 10^{-10})^2 = 1.02\times 10^{-47}\ \mathrm{kg\,m^2}
\end{aligned}$$

Such kinds of information can be found from the optical molecular spectroscopy and from experiments on inelastic scattering of neutrons.

Diatomic molecule as a rigid rotator

A system rotating around a motionless axis is referred to as a rotator. If the intermolecular distance is constant, the rotator is called a rigid one.

Consider a diatomic molecule, rotated around an axis passing through the CM (Figure 1.17). The problem is to express the MI of this molecule and the corresponding kinetic energy of rotation through its parameters, which are supposed to be known (for instance, from the reference literature). In the figure, the masses of two atoms of the molecule are marked by letters m_1 and m_2, and the letter d denotes an interatomic distance ($d = x_1 + x_2$).

The point C is a mass center. Bearing in mind the characteristic of the CM (1.3.31) we can derive $x_1m_1 = x_2m_2$. Therefore, $x_1 = x_2(m_1/m_2)$, $x_2 = x_1(m_2/m_1)$ further $x_1 = d\, m_2/(m_1 + m_2)$ and $x_2 = d\, m_1/(m_1 + m_2)$. It is no trouble to calculate the MI:

$$I_Z = m_1x_1^2 + m_2x_2^2 = \frac{m_1m_2}{m_1+m_2}d^2 = \mu d^2, \tag{1.3.50}$$

where $m_1m_2\; m_1\, m_2/(m_1+m_2) = \mu$ is referred to as the *reduced mass* of the molecule and I_z as the reduced molecular MI. It can be seen from the equation derived, that the rotation of diatomic molecule relative to the axis passing through the CM can be reduced to a single mass (μ) rotation around the same axis at the distance from axis being d. Note that eq. (1.3.50) contains only tabulated molecule characteristics which can easily be found in the reference books.

The kinetic energy of a rotating body can be given by the equation

$$K = \frac{I_z\omega^2}{2} = \frac{L_z^2}{2I_z} \tag{1.3.51}$$

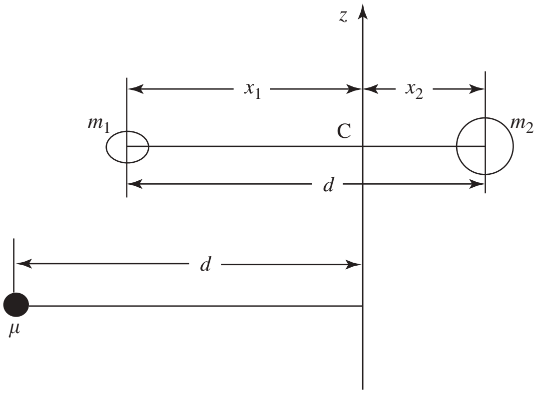


Figure 1.17. A diatomic molecule as a rigid rotator.

The equations derived are used in molecular spectroscopy.

Let us now analyze projections of the dynamic characteristics of a body rotating around the motionless axis. The time derivation equation (1.3.44) gives

$$\frac{dL_z}{dt} = \frac{d}{dt}(I_z\omega_z) = I_z\frac{d\omega_z}{dt} = I_z\varepsilon_z. \tag{1.3.52}$$

(with I_Z not changing in time). If one acts on the mass element dm by the force F along the tangent to the circular trajectory, the mass element will receive acceleration a_τ ($= \varepsilon_Z R$): $dF = a_\tau dm = R\varepsilon_Z dm$. Multiplying the left- and right-hand sides of this equation by R, we receive $dM_Z = dI_Z\varepsilon_Z$, where dM_Z is a force momentum with regard to the axis Oz. Integration over the whole body gives

$$M_Z = I_Z\varepsilon_Z. \tag{1.3.53}$$

(This equation is equivalent to the second Newtonian law for translation motion.) Comparing eqs. (1.3.52) and (1.3.53) we arrive at

$$\frac{dL_z}{dt} = M_z. \tag{1.3.54}$$

Applying this equation for the case of N forces (with corresponding momentums),

$$\frac{dL_z}{dt} = \sum_{i=1}^{N}(M_z)_i. \tag{1.3.55}$$

This equation presents another form of Newton's second law for body rotation relative to the axis Oz: the change of the angular momentum projection onto the axis Oz is equal to the sum of projections of all the force momentums applied to the body relative to the same axis.

The planar motion of the material point relative to a pole

Let an MP of mass m move along the planar orbit around the pole O with a velocity υ (and, consequently, with momentum $\mathbf{p} = m\boldsymbol{\upsilon}$). The angular momentum of a MP relative to the pole O is a vector product

$$L = [\mathbf{r} \cdot \mathbf{p}], \tag{1.3.56}$$

where **r** is a radius vector of the MP (Figure 1.18). According to the definition, the vector **L** is perpendicular to the plane in which both vectors **r** and **p** are lying (i.e. to the orbit plane) and is directed in such a manner that **r**, **p** and **L** produce the right-hand rule system.

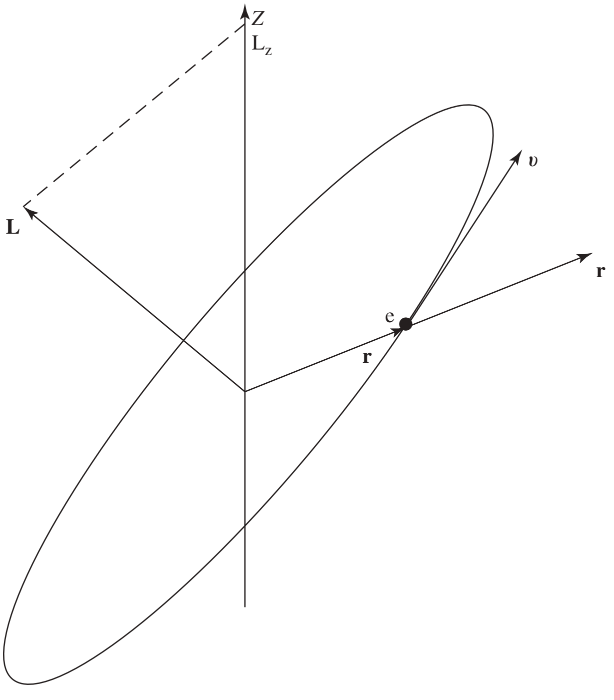


Figure 1.18. A force moment relative to a pole.

Calculating the time derivation of eq. (1.3.56) gives

$$\frac{d\mathbf{L}}{dt} = \left[\frac{d\mathbf{r}}{dt}\mathbf{p}\right] + \left[\mathbf{r}\frac{d\mathbf{p}}{dt}\right].$$

Hence $d\mathbf{r}/dt = \boldsymbol{\upsilon}$ and $\mathbf{p} = m\boldsymbol{\upsilon}$, the first term is then zero since both vectors are collinear. Taking into account that $d\mathbf{p}/dt = \mathbf{F}$, the second term can be written as $[\mathbf{r}{\cdot}\mathbf{F}]$. This vector is called the momentum of the force (torque) $\mathbf{F}$ with respect to pole O.

$$\mathbf{M} = \mathbf{M}_0(\mathbf{F}) = [\mathbf{r}\cdot\mathbf{F}]. \tag{1.3.57}$$

Therefore,

$$\frac{d\mathbf{L}}{dt} = \mathbf{M}, \tag{1.3.58}$$

i.e., the time derivative of the angular momentum of an MP is equal to the applied force momentum. This is the basic law of the dynamics of rotational motion or Newton's second law for planar MP revolution. The projection of the equation onto the axis Oz gives $dL_z/dt = M_Z$ and we arrive at the known eq. (1.3.54).

It is very useful to apply eq. (1.3.58) to the explanation of the effect known as a gyroscopic effect. Using this example, one can see how the rules of rotational motion are distinguished from our usual beliefs about the mechanics of motion and what effects they can bring about.

In Figure 1.19 a sufficiently sophisticated situation is shown. Imagine a body (in Figure 1.19 it is arbitrarily drawn in the form of a disc) which rotates around axis *z*. The angular momentum of this body **L** is directed along the same axis. Apply to the axis of the rotating body a force *F* acting along the *x* axis. The torque of this force **M** is a vector, directed along axis *y*. According to eq. (1.3.58) the increment $d\mathbf{L}$ is directed not along the line of the force action but along the torque mentioned, i.e., along axis *y*. (The indexes *x* and *y* in the figure indicate the fact that force **F** is directed along axis *x*, but vector $d\mathbf{L}$ is along axis *y*). Thereby, a force directed along axis *x* creates a rotation of vector **L** perpendicular to the force $\mathbf{F}_x$.

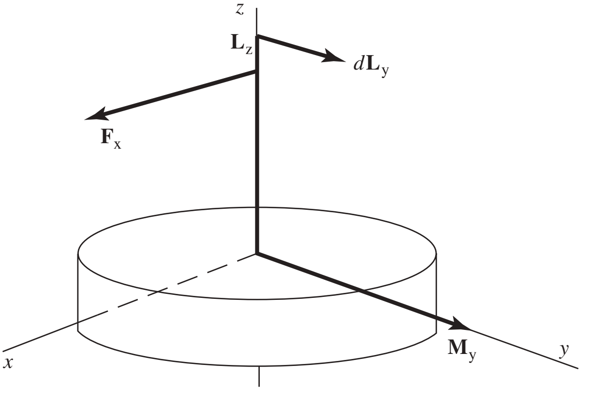


Figure 1.19. A gyroscopic effect.

Table 1.1

The relationships between the characteristics of translational and rotational motion

Translational motion (along *z* axis)		Rotational motion (relative to *z* axis)	
M	Body's mass	I_z	Moment of inertia relative the rotation axis
F_x	Force projection	M_z	Force moment relative the rotation axis
$p_x = m\upsilon_x$	Momentum projection	$L_z = I_z\omega$	Angular momentum relative the rotation axis
$dp_x = F_x dt$	Basic law of dynamics of the translational motion	$dL_x = M_z dt$	Basic law of rotational motion relative an axis
$dA = F_x dx$	Elementary force work	$dA = M_z d\varphi$	Elementary work of force momentum
$W = F_x \upsilon_x$	Power	$W = M_z \omega_z$	Power

It is reasonable here to look at Figure 1.8 where an increment $d\mathbf{p}$ in the translation motion, according to Newton's second law, is directed along the acting force line but is by no means perpendicular to it.

As an example, a precession of an unbalanced gyroscope (a top) in a uniform gravity field is considered in Appendix 2. Many physical events, such as diamagnetism, precession of magnetic moments (atomic and nuclear) in the external magnetic field and others are based on gyromagnetic effects (refer to Chapter 8 and Appendix 2).

In conclusion, it is useful to note that the structure of the formulas of the kinematics and dynamics of rotational motion relative to a fixed axis have the same "structure" as formulas of translation motion. One has only to substitute all translational characteristics with rotational ones. This analogy can be seen in Table 1.1.

1.4 WORK, ENERGY AND POWER

1.4.1 Elementary work of a force and a torque

Let an MP, under the action of a force F, undertake an elementary displacement dl (Figure 1.20). The elementary work dA of the force F is the scalar product of the force and the elementary displacement of a point of force application:

$$dA = (\mathbf{F}\, d\boldsymbol{l}) \tag{1.4.1}$$

or

$$dA = F\, dl \cos\alpha = F_l\, dl. \tag{1.4.2}$$

Depending on the magnitude of the angle α, the elementary work can be positive or negative: at $0 < \alpha < \pi/2$ the work is positive and at $\pi/2 < \alpha < \pi$ it is negative. When $\alpha = \pi/2$ the work is zero. The work on the finite displacement L is equal to

$$A = \int_L (\mathbf{F}\, d\boldsymbol{l}). \tag{1.4.3}$$

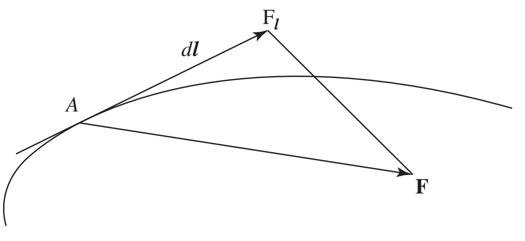


Figure 1.20. Force work **F** over a displacement d***l***.

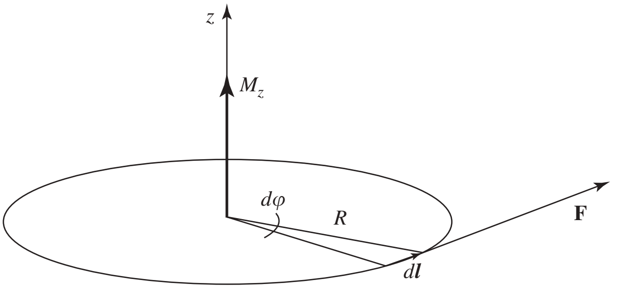


Figure 1.21. Torque work on an angular displacement dφ.

In a particular case in which the force is constant and the application point is moving along straight line l, eq. (1.4.3) simplifies to:

$$A = Fl\cos\alpha = F_l\,l \tag{1.4.4}$$

where F_1 is the force projection on the direction of motion.

A force **F** being applied to a point A of a rotating body performs a work $dA = Fdl$ (Figure 1.21). Since Rdl is an angle displacement $d\varphi$ then

$$dA = M_Z(F_\perp)d\varphi, \tag{1.4.5}$$

where $M_Z(F)$ is the torque of the force **F** regarding the Oz axis. The work done on a finite angle φ is

$$A = \int_\varphi M_Z(\mathbf{F})\,d\varphi. \tag{1.4.6}$$

If $M_Z(\mathbf{F}) = $ const. one has

$$A = M_Z(\mathbf{F})\varphi. \tag{1.4.7}$$

Let us consider some examples.

Elastic force work. Elastic force depends on deformation (displacement of the force application point at a body) according to the linear law $F(x) = -\beta x$ (refer to Section 1.3.5). The simplest example of elastic force is the small deformation of a spring. Let us superpose the origin with the point of the force application when the spring is in a nondeformed state and an external force is not acting (Figure 1.22, point O). Suppose that a body can move without friction (!) along the horizontal x-axis. After application of an external force, two forces will act on the body: the external force F_2 and the elastic one F_1. At any position they are in balance: $F_1 = -F_2$. When the body returns to the initial position both forces perform equal work but of opposite sign. The elementary work of the elastic force on a displacement dx is

$$dA = F(x)dx = -\beta x\,dx; \tag{1.4.8}$$

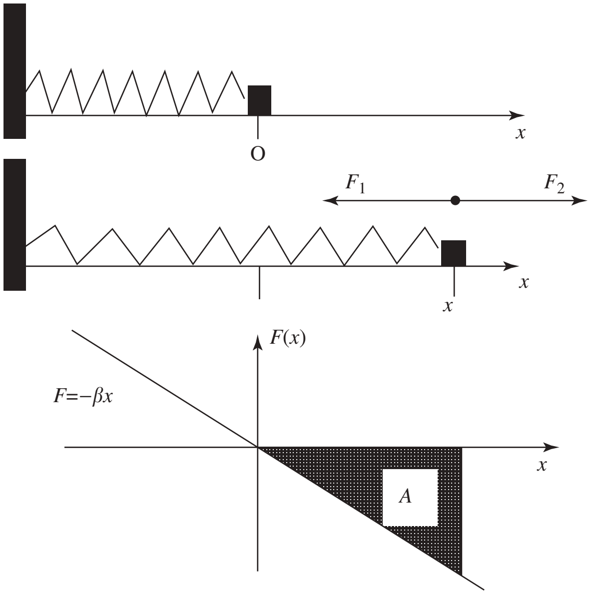


Figure 1.22. A work of elastic force F_1.

The total work of the external force on the displacement x is

$$A = -\int_0^x \beta x \, dx = -\frac{\beta x^2}{2}, \tag{1.4.9}$$

whereas the elastic (internal) force produces the positive work:

$$A = \frac{\beta x^2}{2}. \tag{1.4.10}$$

The external force work is equal to the internal one in absolute value but is opposite to it in sign.

1.4.2 Power

When work is produced during some time interval, the question arises of how fast the work is made. This leads to the notion of power. For time Δt, let work ΔA be accomplished. Then an averaged power P for a given interval Δt is

$$\langle P \rangle = \frac{\Delta A}{\Delta t}$$

Under $\Delta t \to 0$ one can obtain a power made by a force in a given time instant, i.e., the instant power

$$P = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}. \tag{1.4.11}$$

Keeping in mind eq. (1.4.1), one can write

$$P = \frac{\mathbf{F}d\boldsymbol{l}}{dt} = \mathbf{F}\boldsymbol{\upsilon} \tag{1.4.12}$$

The power is the scalar product of the force and the velocity of point of its application.

When rotational work is produced by a force moment $M_Z(F)$ applied to the body the power is

$$P = \frac{M_z d\varphi}{dt} = M_Z \times \omega. \tag{1.4.13}$$

1.4.3 Kinetic energy

The kinetic energy of an MP is a scalar measure of its mechanical motion, equal to half of the MP's mass and the square of its velocity

$$K = \frac{m\upsilon^2}{2}. \tag{1.4.14}$$

As can be seen, the kinetic energy is always positive.

Because of the motion, an MP possesses a certain stock of mechanical energy, referred to as kinetic energy. Reduction of velocity means a loss of kinetic energy; velocity increase leads to an accumulation of kinetic energy. Changing the energy goes to accomplishing a work.

The kinetic energy is assigned in a certain inertial reference system. Turning to another inertial system the kinetic energy value (in consequence of a velocity change) will be different; the kinetic energy (in the same way as the velocity) is noninvariant with respect to Galileo's transformations though the mathematic expressions are the same and the physical values differ only by a constant.

The total kinetic energy of a mechanical system consisting of a set of N material points is the sum of the kinetic energy of all the system's elements:

$$K = \sum_{i=1}^{N} \frac{m_i \upsilon_i^2}{2}. \tag{1.4.15}$$

Consider a value of kinetic energy for some types of motion.

Translational motion. Taking into account that under translational motion of an IRB the velocities of all the body's points are the same, we arrive at

$$K = \sum_{i=1}^{N} \frac{m_i \upsilon_i^2}{2} = \frac{\upsilon^2}{2} \sum_{i=1}^{N} m_i = \frac{m\upsilon^2}{2}, \tag{1.4.16}$$

where m is the body's total mass and υ is the velocity of any point of the body.

Rotational motion. For the rotational motion of a body around a fixed axis, the velocity of an arbitrary point is $\upsilon_{\mathrm{i}} = \omega R_i$ where ω is the angular velocity of a body, R_i is the distance of a corresponding point from the axis of rotation. Then

$$K = \sum_{i=1}^{N} m_i \frac{\omega^2 R_i^2}{2} = \frac{\omega^2}{2} \sum_{i=1}^{N} m_i R_i^2. \tag{1.4.17}$$

When the last sum is a moment of inertia I_{z} of a given body relative to the axis of rotation then

$$K = \frac{I_{\mathrm{z}}\omega^2}{2} = \frac{L_{\mathrm{z}}^2}{2I_{\mathrm{z}}}. \tag{1.4.18}$$

General case of a body's (a system's) energy. For an arbitrary system a theorem is equitable: the kinetic energy of a system of MPs is the sum of the kinetic energy of the system's mass as a whole, imaginary concentrated in the CM and moving together with it, and the kinetic energy of all the system's elements with respect to the CM.

Apply the theorem to an arbitrary body motion. In this case the motion can be divided into the translational part (at the velocity of the CM) and the rotational part relative to the axis, passing through the CM. According to the theorem, the total kinetic energy in this case is the sum of the kinetic energy of a translational motion (with the velocity of its CM, V_{C}) and the rotational motion relative to the axis, passing through the CM (with the angular velocity ω):

$$K = \frac{mV_{\mathrm{c}}^2}{2} + \frac{I_{\mathrm{c}}\omega^2}{2}. \tag{1.4.19}$$

Theorem on the kinetic energy change. Let an MP of a mass m under the action of a force F gain an increment dl. Projecting the vectors of equation $m\boldsymbol{a} = \mathbf{F}$ onto the tangent to the trajectory in the point of force application we obtain $ma_\tau = F_\tau$, where a_τ is the absolute value of the point's tangent acceleration of an MP, and F_τ is the corresponding force projection. Taking into account that $a_\tau = d\upsilon/dt = (d\upsilon/dl)\ (dl/dt) = \upsilon d\upsilon/dl$ we obtain: $m\upsilon d\upsilon = F_\tau dl$. Considering the right-hand side of this equation as the force's elementary work dA and the left-hand side as a differential of the body's kinetic energy we arrive at $\mathrm{d}(m\upsilon^2/2) = dA$, or

$$dK = dA, \tag{1.4.20}$$

i.e., the elementary change of the kinetic energy is equal to the elementary force work. In integral form, this equation can be presented as

$$K_2 - K_1 = A, \tag{1.4.21}$$

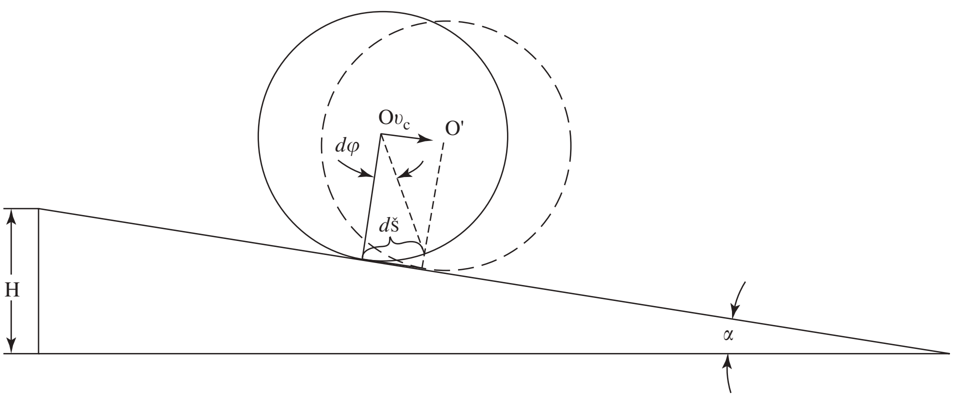
i.e., the change of the MP's kinetic energy is equal to the work of all forces applied to it.

No limitation has been applied to the nature of the force in eq. (1.4.21). Therefore it is valid for any system and has to take into account all kinds of forces: internal and external, potential and dissipative.

The theorem discussed is also valid for a system of MPs.

EXAMPLE E1.14

A uniform hoop and/or disc begins to roll, without either friction nor sliding, along an inclined plate with an angle of slope $\alpha = 10°$ from the height $h = 40$ cm (Figure E1.14). Determine: (1) linear velocities of CM V_{c1} (hoop) and V_{c2} (cylinder) of both bodies at the end of the plane; (2) time of their rolling τ_1 and $\tau_{2.}$



Solution: (1) In consequence of the absence of the dissipative forces we can take advantage of the law of the mechanical energy conservation $E_1 = E_2$, where E is the total mechanical energy at the top and bottom position. This energy can be written as the sum of kinetic and potential energies $K_1 + U_1 = K_2 + U_2$. In the initial and final positions both bodies have identical potential energies. Their kinetic energies at the initial position is zero, both bodies are at rest. In the final position, according to the energy conservation theorem and taking into account the theorem on total energy, the total energy will be equal to the sum of kinetic energies of the translational motion of the body's CM and rotational motions, $K_2 = (mv_C^2/2) + (I_z\omega^2/2)$, where v_C is the CM's speed of bodies, and MI is the MI concerning a horizontal axis z passing through the CM of bodies parallel to the inclined plane and ω is the angular speed of rotation of bodies concerning these axes.

The point of contact of rolled bodies and the inclined plane is the instant center of speeds, in all time instants it is at rest; movement can be considered as instant rotation of the bodies concerning this point. Therefore,

$$\omega = \frac{d\varphi}{dt} = \frac{dS}{R\,dt} = \frac{V_C dt}{R\,dt} = \frac{V_C}{R}.$$

In order not to solve the same problem twice we designate the MI of bodies $I_Z = \beta m R^2$ (for the hoop $\beta = 1$, for the disc $\beta = 1/2$). Therefore

$$K_2 = \frac{mV_C^2}{2} + \frac{\beta m R^2 (V_C/R)^2}{2} = \frac{mV_C^2}{2} + \beta\frac{mV_C^2}{2}$$

and finally, $K_2 = (mV_C^2/2)(1+\beta)$

In this case the rolling kinetic energy can be written down for both the bodies and for any other cylindrically symmetric body, one should substitute only a proper β value.

In order to find the velocities one should use the conservation law.

$$mgh = (mV_C^2/2)(1+\beta).$$

Reducing expression of masses m we can arrive at

$$V_C = \sqrt{\frac{2gh}{1+\beta}}.$$

Carrying out calculations for speed of the CM we shall obtain (see below).

We can find the time of the bodies' slope run by taking the formula of the kinematics of uniformly accelerating movements $\tau = l/\langle \upsilon \rangle$, where l is the inclined plane length and $\langle \upsilon \rangle$ is an average speed of uniformly accelerating motion $\langle \upsilon \rangle = V_C/2$ (because the initial speed was equal to 0). We can then arrive at:

$$\text{for the hoop } (\beta = 1)\, V_{C1} = \sqrt{gh} = 1.98\,\text{m/sec and } \tau_1 = \frac{2h}{V_{C1}\sin\alpha} = 2.33\,\text{sec}$$

$$\text{for the disk } (\beta = 1/2)\, V_{C2} = \sqrt{4/3\, gh} = 2.29\,\text{m/sec and } \tau_2 = \frac{2h}{V_{C2}\sin\alpha} = 2.01\,\text{sec}.$$

1.4.4 A force field

A physical field is a particular form of matter that links together material particles and transmits an influence from one body to another (with finite velocity). Each type of interaction has its own special corresponding force field. The force field is an area of space in which a force acts on any material particle placed in this space point, depending on coordinates and time. A force field is called a stationary one if the acting forces do not depend on time. A force field at any point of which the acting force has one and the same value (on modulus and direction) is referred to as a uniform one.

It is possible to characterize a force field by force lines. In this case, the tangent to force lines defines the direction of a force and the line's density is proportional to the force value.

The force field is referred to as a central one if every force line passes through one particular point, called the center of forces (Figure 1.23). The magnitude of force **F**, acting on a MP in such a field, depends only on distance r from the center of forces, i.e.,

$$\mathbf{F}(\mathbf{r}) = F(r)\frac{\mathbf{r}}{r}, \tag{1.4.22}$$

($\mathbf{r}/r$ is a unity vector in the direction of **r**). All the force lines pass through a single point (pole) O; the force momentum in this case is identically equal to zero ($\mathbf{M}_0(F) \equiv 0$). Gravitational and Coulomb forces are related to the central ones.

An example of a uniform force field is depicted in Figure 1.24: in every point the lines of force action are the same both in direction and magnitude, i.e.,

$$\mathbf{F}(\mathbf{r}) = \mathbf{F} = \text{const.}$$

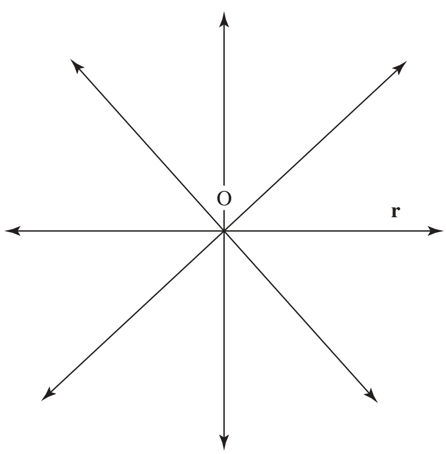


Figure 1.23. Force lines of a central force field lines.

In Figure 1.25 an example of a nonuniform force field is given. In this case

$$F(xyz) \neq \text{const.}$$

and

$$\frac{\partial F(xyz)}{\partial x}, \quad \frac{\partial F(xyz)}{\partial y}, \quad \frac{\partial F(xyz)}{\partial z}$$

(i.e., all partial derivatives) deviate from zero.

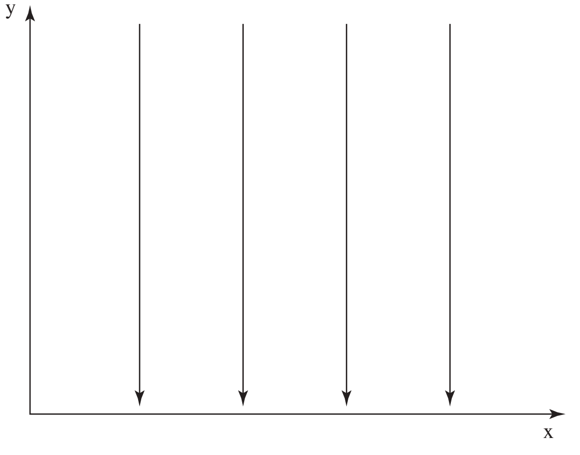


Figure 1.24. A schematic representation of a uniform force field lines.

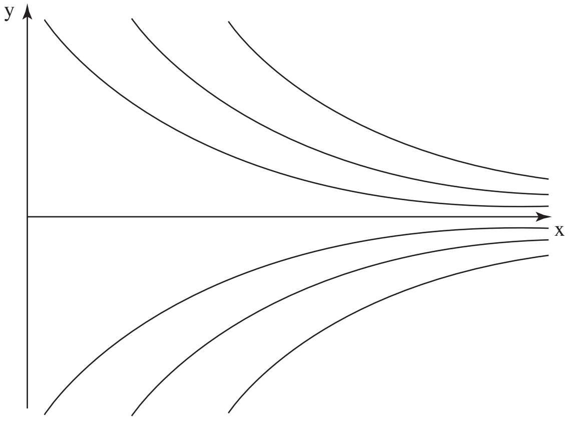


Figure 1.25. Force lines representation of a nonuniform force field.

All the mechanical forces can be divided into two groups: conservative forces (acting in potential fields) and nonconservative forces (or dissipative). The forces are referred to as conservative (or potential) ones if their work depends neither on the trajectory form, nor on the path length, and is defined only by the position of the point of the force application in the initial and final positions. The field of conservative forces is called the potential (conservative) one.

Let us see that the work of the conservative forces along a closed contour is zero (Figure 1.26). Arbitrarily divide the contour into two parts: 1a2 and 1b2. Since the force is conservative $A_{1a2} = A_{1b2}$. On the other hand it is obvious that $A_{1b2} = -A_{2b1}$. Then $A_{1a2b1} = A_{1a2} + A_{2b1} = A_{1a2} - A_{1b2} = 0$, which has to be proven.

The inverse statement is fair: if the work of a force on the closed contour is zero, the forces are conservative and the field is potential. This condition can be written as a contour integral (circulation of a vector along a closed contour):

$$\oint_L \mathbf{F}d\boldsymbol{l} = 0. \tag{1.4.23}$$

i.e., the circulation of vector **F** along the closed contour L is zero.

The work of the nonconservative (dissipative) forces in the general case depends both on the form of the paths travelled and the lengths of the way. Examples of nonconservative forces can be given by friction and resistance forces. In both cases the mechanical energy transforms into another type of energy.

The central forces are referred to as the conservative forces (Figure 1.27). In fact, if a force F is a central one, the work of this force dA can be presented as $dA = \boldsymbol{Fdl} = F(r)(\mathbf{r}/r)d\boldsymbol{l}$ and $dA = F(r)\cos\alpha\, dl = F(r)dr$ (because $dl \cos\alpha = dr$).

Then the work is

$$A_{12} = \int_{r_1}^{r_2} F(r)dr = f(r_2) - f(r_1), \tag{1.4.24}$$

where $f(r)$ are the antiderivative functions. It can be seen from this equation that the work A_{12} of the central force depends only on the form of the f-function and on the positions of the initial and final points (r_1 and r_2) but not on the path length. This statement is just the indication of a conservative field (conservative force).

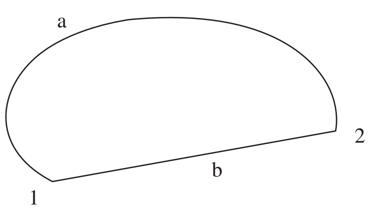


Figure 1.26. Proof of the work of a conservative force to be equal to zero.

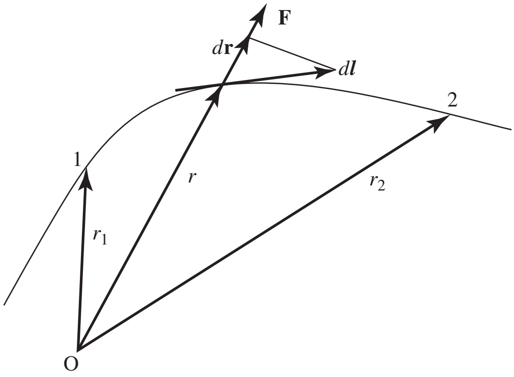


Figure 1.27. Proof of the conservative nature of a central force (a central field).

The proof given is common to any central force and, consequently, covers the above-mentioned types of forces—gravitational and electrostatic ones.

1.4.5 Potential energy

In a potential system, the notion of potential energy can be introduced as a function of a point coordinate.

In a system, first choose a state that we can arbitrarily admit as a point with zero potential energy (position $U_0 = 0$). Further, suppose that we need to find an MP potential energy in another point of the system, which we assign as position 1 (i.e., find the value U_1). The potential energy of a system in position 1 is taken to be numerically equal to the work of the field force on transferring the system from position 1 to that position where the potential energy is chosen as zero:

$$A_{10} = U_1 - U_0 = U_1. \tag{1.4.25}$$

If the field is potential, the work A_{10} does not depend on the pathway 1–0. It characterizes the system in point 1 with respect to point 0. If one needs to define the potential energy in position 2, the work of the field force should be measured. Obviously, $A_{20} = U_2$ and $A_{12} = U_2 - U_1$. Because $A_{21} = -A_{12}$, the work of the force is

$$A_{12} = U_1 - U_2 = -\Delta U, \tag{1.4.26}$$

i.e., the work of the internal force (force of the field) is equal to the decrease of the potential energy. On the other hand, the work of the external force, acting against the field force, brings about the potential energy growth.

$$A_{21} = \Delta U = U_2 - U_1. \tag{1.4.27}$$

Position 0 was chosen arbitrarily; any point of the system can be accepted as the zero point. This signifies that the defined potential energy is accurate to a constant value C. This "arbitrariness" is not essential, since in the calculations of the difference of energy (refer, for instance, to eq. (1.4.25–27)) constants C are mutually canceled out. Also, the presence of the constant in the equation does not affect the derivative of the potential energy function in respect to the coordinates.

The correlation obtained shows how one can determine the potential energy of a system at a certain position. There is no universal formula for such a calculation (as for the kinetic energy). Correlation (1.4.25) shows a way of determining the system's potential energy by calculating the force work which leads a system to the given zero point.

Below are some important examples.

EXAMPLE E1.15

Calculate the potential energy of a deformed spring using the potential energy definition.

Solution: In Figure 1.22 the scheme of the spring, originally in a nondeformed state, is presented: the left end of the spring is rigidly fixed, the other end, under the action of an external force, can move along an axis x. The spring is also stretched under the action of external force F_2. On the movable end of the spring in an arbitrary state two oppositely directed forces operate: external force F_2 and force of elasticity F_1: $\mathbf{F}_1 = -\mathbf{F}_2$. For the zero position (with zero potential energy) we choose the spring to be in a nondeformed state ($x = 0$). According to eq. (1.4.25) $dA = -\beta x dx$, i.e.,

$$U(x) = \int_X^0 dA = \beta \int_X^0 x dx = \frac{\beta x^2}{2}.$$

The straight line in Figure 1.22 represents Hooke's law whereas the potential energy is shown by a hatched area.

EXAMPLE E1.16

Determine the potential energy $U(r)$ of the body in the gravitational field of the earth. The earth's mass is M and the distance from the center (!) of the earth to the body of mass m is r.

Solution: According to Newton's gravitation law (1.3.19) we have $F(r) = -G(Mm/r^2)$. Assume as a position with zero potential energy an infinite body's remoteness from the earth ($U(\infty) = 0$). By definition, the potential energy of the body in the given (arbitrary) space point r is numerically equal to the work of the gravitational force when carrying the body from a position r to $r = \infty$

$$U(r) = -GMm\int_r^\infty \frac{r\,d\ell}{r^3} = -GMm\int_r^\infty \frac{dr}{r^2} = G\frac{Mm}{r}\Big|_r^\infty$$

Therefore, one arrives at $U(r) = -G(Mm/r)$

The potential of the gravitational field $\varphi = (U/m)$ is numerically equal to the potential energy of the MP of a unit mass placed into a given point r; therefore, $\varphi(r) = -G(M/r)$. Note that r is the distance to the body from the center of the earth; this formula is valid for any spherically symmetric force's field. The relationship is also useful: $U(r) = m\varphi(r)$. This is the most general expression.

Relationship between the force and potential (*work-energy theorem*). In order to establish a relationship between energy and work we shall consider a body in a stationary field. Acquire an elementary body's displacement dl. The internal forces of the field will produce the work dA equal to the potential energy change ($dA = -dU$). Therefore,

$$F_l = -\frac{dU}{dl}. \tag{1.4.28}$$

This is the relation sought. The dl quantity is a displacement.

In a one-dimensional problem, consider $dl \equiv dx$. Therefore:

$$F(x) = -\frac{dU(x)}{dx}. \tag{1.4.29}$$

Considering further a motion in the central force field $dl \equiv dr$

$$F(r) = -\frac{dU(r)}{dr}. \tag{1.4.30}$$

In a three-dimensional case

$$\mathbf{F}(\mathbf{r}) = -\left(\frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}\right) = -\mathrm{grad}\,U(xyz), \tag{1.4.31}$$

i.e., the force is equal to the gradient of the potential energy taken with an opposite sign.

That is, the potential energy is determined within an accuracy to a constant component; this does not influence the result: calculating the differences or derivations using these formulas (1.3.28–1.3.31) the arbitrary component makes no contribution to the result.

EXAMPLE E1.17

Calculate the potential energy $U(x)$ of an elasticity force $F(x)$ using the relation of force and potential energy.

Solution: Since $F(x) = -\beta x$ and $F(x) = -(dU/dx)$, then $\beta x = (dU(x)/dx)$; therefore $dU(x) = \beta x dx$. Integration gives $U(x) = (\beta x^2/2) + C$. Assigning $U(0) = 0$ we can obtain $C = 0$. Then $U(x) = (\beta x^2/2)$. The work to deform a spring from x_1 to x_2 can be expressed by the equation:

$$A_{21} = +\Delta U = U_2 - U_1 = \tfrac{1}{2}\beta(x_2^2 - x_1^2)$$

EXAMPLE E1.18

Derive an expression for the potential energy $U(h)$ for a body raised on h, not high above the Earth's surface.

Solution: We should start with the precise relation (see example E1.16). According to the definition, the potential energy is numerically equal to the gravitation force work

$$A(r) = U(r) = GMm\int \frac{dr}{r^2} \quad \text{or} \quad U(r) = -G\frac{Mm}{r} + C.$$

We can determine an integration constant C in two ways: firstly, having accepted the point where the potential energy of a body removed to an infinite distance has zero potential energy, i.e., $U(\infty) = 0$. Then $C = 0$. Hence, $U(r) = -G(Mm/r)$*.

Secondly, we can assign the potential energy to be equal to zero when the body is resting on the surface of the earth, i.e. $U(R) = 0$; that is to measure the potential energy from ground level. Then the equation * can be rewritten as

$$U(R) = -G\frac{Mm}{R} + C = 0. \quad \text{Therefore,} \quad C = G\frac{Mm}{R}$$

In this case the potential energy of a body at height h above ground level will be:

$$U(h) = -G\frac{Mm}{R+h} + G\frac{Mm}{R} = \frac{GMmh}{R(R+h)}.$$

At $h \ll R$ we have $U(h) \approx G(Mm/R^2)h$; since $G(M/R^2) = g$ and $U(h) = mgh$.

This is a well-known formula describing the potential energy of a body lifted above the Earth's surface to a moderate height ($h \ll R$) (a straight line $U(h)$ in Figure 1.28).

Generally, the gravitational field of the earth, as for any central field, is nonuniform (the force lines of the field are not parallel to each other). However, if we limit the volume of the field by a definite dimension h, we can always calculate an accuracy within which we can use either a precise eq. (1.3.19) or an approximate formula (1.4.37). We shall note that eq. (1.4.37) is only a special case, though very important in normal engineering practice. The precise formula is given in Example E1.16.

In Figure 1.28 the graph of the function $U(r)$ is presented: hyperbolic dependence $U(r) \sim(-1/r)$ lies in the areas of negative values of potential energy. We can see that $U(r) \to 0$ at $r \to \infty$. If we take another border condition, $U(R) = 0$, then the origin in the graph should be shifted by $-G(Mm/R)$ down the U-axis and by R along the r-axis. At ($h \ll R$) dependence $U(h)$ is represented by a straight line. The place where the curve $U(r)$ and linear line $U(h)$ at small h practically coincide can be seen in Figure 1.28.

In general, the gravitational field of the earth, as with any other central field, is nonuniform (force lines of a field are not parallel to each other). However, if a volume of the field is limited by a definite dimension, it is always possible to calculate to an accuracy within which either a precise or an approximate formula can be used.

Figure 1.28 presents a graph of the function $U(r)$: hyperbolic function $U(r) \sim(-1/r)$ lies in the areas of negative values of potential energy. It can be seen that U(r) $\to 0$ at $r \to \infty$. If we accept that $U(R) = 0$ then the origin in the graph should be shifted on $-G(Mm/R)$ down along the U-axis and on R along the r-axis. At $h \ll R$, dependence $U(h)$ is represented by a straight line. It can be seen in Figure 1.28 that the curve $U(r)$ and linear dependence $U(h)$ at small h practically coincide. (All these aspects are analyzed in Examples E1.16 and E1.18 in detail.) Simple calculations show that to within 1% the equation $U = mgh$ can be used up to a height of 64 km (see Example E1.19). In fact, the gravitational field of the earth up to a certain accuracy can be counted as homogeneous. The zero position of the potential energy is not necessarily connecting with a level of the Earth's surface; potential energy can be counted from any level near the earth.

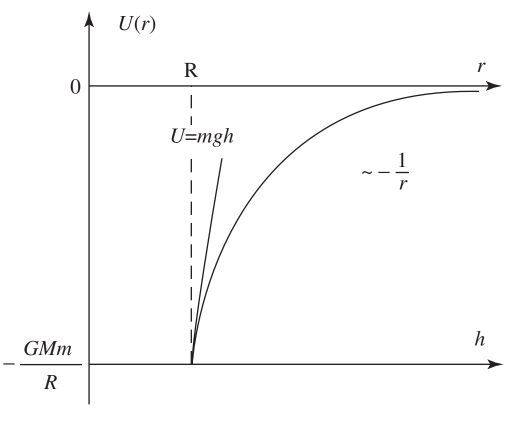


Figure 1.28. Work of a gravitational force.

EXAMPLE E1.19

Find in some respect a less-precise expression for the potential energy of a body in the Earth's gravitational field at large distances. Evaluate the approximations. The concept of potential energy of the gravitational interaction permits some arbitrariness: everything depends on the choice of from which level to count this energy. There are different options, among which are exact and approximate ones. In the latter case it is always possible to estimate an error arising when using the given approximations. Estimate this error using the formula (1.4.37): in what limits of a body lifting will the error not exceed 1% in comparison to a general approach (1.4.28).

Solution: The equation in Example E1.16 is definitely the most general. It has already been noticed that the most important thing is not the absolute value of the potential energy but the difference in their two states.

There is one more approach to the gravitational potential energy evaluation

$$U(h)=U(R+h)-U(R).$$

Then

$$\begin{aligned}U(h)&=-G\frac{mM}{R+h}-\left(-G\frac{mM}{R}\right)=GmM\left(\frac{1}{R}-\frac{1}{R+h}\right)\\&=Gh\frac{mM}{R(R+h)}\times\frac{R}{R}=mgh\frac{R}{R+h}.\end{aligned}$$

In a very good approximation we can consider the gravitational force on the earth level: $mg = G(mM/R^2)$. Then $U(h) = mgh(R/(R+h))$. This expression is precise enough and can be used sometimes at any h. In order to obtain an answer to our main question we should calculate the uncertainty ratio

$$\frac{\Delta U}{U(h)}=\frac{mgh-mgh\dfrac{R}{R+h}}{mgh\dfrac{R}{R+h}}=\frac{R+h}{R}-1=\frac{h}{R}.$$

To find the uncertainty 1% error we equate the last prescribed accuracy with $0.01R$: $h = 0.01\,R = 0.01 \times 6.4 \times 10^3$ m $= 64$ km. Therefore, we can use the mgh formula in the limit of 1% while climbing Everest, but not in planning the spacecraft flights.

Note that the potential energy of a body (e.g., a satellite) is negative; it is because of the fact that we chose the value $U = 0$ at infinite interval from the earth. The potential energy of a body lifted above the earth is always positive.

Because in the framework of Galileo's transforms the distance is invariant, the potential energy is also invariant (remember that kinetic energy is noninvariant in relation to Galileo's transformations because of the dependence of a body's velocity on the choice of the coordinate system).

The work of the gravitational forces at a body displacement from some point 1 to a point 2 will generally be expressed as follows:

$$A_{12} = -\Delta U = U_1 - U_2 = -GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = m(\varphi_1 - \varphi_2),$$

and in the case of a uniform gravitational field:

$$A_{12} = mg(h_2 - h_1).$$

1.5 CONSERVATION LAWS IN MECHANICS

Conservation laws are the most general, fundamental laws of nature. They have an enormous scientific value. Their importance is defined by the fact that the solution to many kinds of problem can be achieved with their help and without detailed analysis of specific circumstances and details.

1.5.1 Conservation law of mechanical energy

Let 1 and 2 be the two positions of an MP in a potential-force field; U_1 and U_2 are their potential energies at these points. According to the theorem of kinetic energy change (refer to (1.4.3))

$$K_2 - K_1 = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = A_{12}, \tag{1.5.1}$$

where A_{12} is the work of all forces, external and internal, conservative and dissipative applied to the MP on displacement from point 1 to point 2. If we restrict ourselves to conservative (potential) systems and remove the external forces (according to a criterion of the closed system), then according to (1.4.26)

$$A_{12} = U_1 - U_2.$$

Combining the two last equations, we obtain

$$K_2 + U_2 = K_1 + U_1. \tag{1.5.2}$$

Because two arbitrary positions were chosen, the general equation for the total mechanical energy can be written as:

$$K + U = E = \text{const.}, \tag{1.5.3}$$

which is valid for any body of the system and for the system as a whole. So, *the total mechanical energy of the closed potential system is conserved.*

The given determination comprises certain conditions, the removal of any of which breaks the law of conservation. In fact, if one takes into consideration the dissipate forces (for instance, friction), part of the mechanical energy transforms into heat, and the law of conservation of mechanical energy becomes invalid. On the other hand, if one takes into consideration other forms of energy (heat, electric and others), the condition of system conservation becomes an excessive condition: the total energy of any closed systems is preserved.

In differential form, the law of mechanical energy conservation can be denominated by equation, $dU + dK = 0$ or

$$dU = -dK. \tag{1.5.4}$$

This means that in the resting system the kinetic energy can appear only as a consequence of the reduction of potential energy. But if the potential energy in the given state has a minimal value, the motion simply cannot appear. Consequently, the system is in a state of equilibrium when the potential energy has a minimum value.

As we know, the potential energy is a function of a body's coordinate; the kinetic energy, however, depends on its velocity (or momentum).

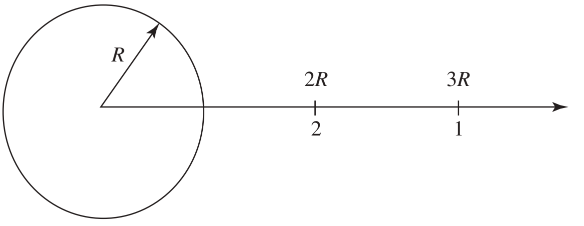
$$E(\mathbf{r}, \mathbf{p}) = U(\mathbf{r}) + K(\mathbf{p}) = U(xyz) + K(p_x, p_y, p_z)$$

As was mentioned in Section 1.3.8, in order to specify a system state one has to know the coordinates and momentums of all points, i.e., the same parameters that define the energy. In this sense it can be said that the total mechanical energy is a function of a system's state. When changing a system's state its energy changes as well. The work in this case is presented as a measure of changing a system's energy. This concerns the physical sense of work.

The law of total mechanical energy conservation is invariant with respect to Galileo's transforms. This does not mean, however, invariance of the total energy with respect to Galileo's transforms literally, since the kinetic energy with respect to different reference systems has different values. So, the constant characterizing the total energy in each case can be different, though the principle in general is the same.

EXAMPLE E1.20

In a gravitational field of the earth, a body with a weight m moves from a point 1 to a point 2 (Figure E1.20). Define the speed of the body at point 2 if its speed at point 1 was $\upsilon_1 = \sqrt{gR} = 7.9$ km/sec. Assume the acceleration of free fall at all points to be equal g.



Solution: An earth–body system is conservative and closed. Therefore, we can use the energy conservation law. We connect the origin to the earth's centre. We can write: E_1–E_2 i.e. $U_1 + K_1 = U_2 + K_2$, where U and K are the potential and kinetic energies at points 1 and 2. Because the reference frame is connected with the earth, the relative body energy does not depend on its movement. Then

$$K_1 = \frac{mv_1^2}{2}; \quad U_1 = -G\frac{Mm}{3R}; \quad K_2 = \frac{mv_2^2}{2}; \quad U_2 = -G\frac{Mm}{2R}.$$

Substituting these expressions in the energy conservation law we obtain:

$$\frac{mv_1^2}{2} - G\frac{Mm}{3R} = \frac{mv_2^2}{2} - G\frac{Mm}{2R}.$$

Substituting GM by gR^2 and executing elementary simplifications we obtain:

$$v_2^2 = v_1^2 + \frac{gR}{3} \text{ and then } v_2 = \sqrt{v_1^2 + \frac{gR}{3}}.$$

Because $v_1^2 = gR$ then $v_2 = \sqrt{4gR/3} = v_1\sqrt{4/3}$.
Executing calculations we arrive at $v_2 = 9.12$ km/sec.

1.5.2 Momentum conservation law

Returning to expression (1.3.9), consider a case when there are no forces acting on an MP, i.e., when the right-hand part of the equation is zero. Then $d(mv) = 0$ and, hence,

$$\mathbf{p} = m\mathbf{v} = \text{const.}, \tag{1.5.5}$$

i.e., in the absence of external forces the momentum of an MP remains constant.

We can apply the result obtained for a mechanical system. If the system is closed, external forces are absent, hence $v_C = \text{const.}$ (According to Newton's third law, the mutual action of all bodies of a given system is counterbalanced and when calculating the total momentum of the system, the internal forces make no contribution to the momentum of the system.) At constant system mass the total system momentum $m\boldsymbol{\upsilon}_C = \mathbf{p}_c$ also remains constant.

$$\mathbf{p}_c = \text{const.}, \tag{1.5.6}$$

That is, the total momentum of a closed system $\mathbf{p}_c$ is conserved. Note that the law of total momentum is conserved in any closed system regardless of whether it is conservative and/or dissipative.

Several consequences arising from the law of momentum conservation can be mentioned:

1. A reference system associated with the CM of a closed mechanical system is inertial. In fact, if $\mathbf{p}_c$ = const. the constant speed of the CM $\boldsymbol{\upsilon}_C$ should be constant too.
2. The center of inertia of the closed system is preserved as a resting state or a state of rectilinear uniform movement despite any mutual displacements (under the action of internal forces) of any of the system's elements. From this it is clear that it is impossible to change the position of the mass' center of the closed system by only internal forces.

Some special cases can be noted in this respect. A system is not closed, but the result of forces is zero. In this case the total momentum of the system is preserved. A system is not closed but one of the projections of external forces (e.g., F_X) is zero. Then $\Delta p_X = 0$ and p_x = const. Hence, the law of conservation is valid only to the given motion direction. This case corresponds, for example, to the motion without friction in a field of a gravity along a horizontal axis with non-zero initial velocity (refer to Example E1.4).

EXAMPLE E1.21

A carriage $L = 3$ m in length and mass $m_1 = 120$ kg rests on a smooth horizontal surface. At one end of the carriage is a man whose weight is $m_2 = 80$ kg. Define the displacement $\Delta\mathbf{L}$ (modulo and direction) if the man walks from one end of the carriage to another. Neglect friction.

Solution: Choose an inertial reference system, having connected with the earth. Since projections of the external forces acting on the system "man-carriage" from the earth are perpendicular to the Earth's surface this system can be counted as closed (regarding a horizontal axis); a momentum conservation law (see Section 1.5.2) can be used. In the initial state the man and the carriage are motionless and, hence, the system's total momentum is zero. We shall assume that the man moves along the carriage uniformly at a speed u, thus the carriage also begins to move uniformly at a speed v_2 relative to the earth. As the relative speed $u = v_1 - v_2$ (where v_1 is the speed of the man relative the earth),

$$v_1 = u + v_2$$

Then the momentum conservation law for the case considered can be written as $m_1v_1 + m_2v_2 = 0$ or $m_1(v_1 + v_2) + mv_2 = 0$, whence after disclosing brackets and regrouping terms we obtain $v_2(m_1 + m_2) + m_1u = 0$ or

$$\mathbf{v}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{u}.$$

Let us multiply this equality on both sides by the time τ taken by the man to move from one side of the carriage to the other; we obtain

$$v_2\tau = \left[\frac{m_1\tau}{m_1 + m_2}\right]u,$$

where $\mathbf{v}_2\tau = \Delta L$ is the carriage displacement relative to the Earth's surface and $\mathbf{u}\tau$ is the man's displacement relative to the carriage (i.e., L). Therefore,

$$\Delta L = -\frac{m_1}{m_1 + m_2}L$$

Introducing the values given above, we obtain the carriage displacement.

$$\Delta L = 2\,\text{m}.$$

The negative sign indicates that the man and carriage displacements are in opposite directions to each other.

EXAMPLE E1.22

A nucleus decays into two fractions $m_1 = 1.6 \times 10^{-25}$ kg and $m_2 = 2.4 = 10^{-25}$ kg. Determine the kinetic energy K_2 of the second fragment if the kinetic energy of the first is $K_1 = 18$ nJ.

Solution: According to the momentum conservation law, the momentums of fragments after decay are the same $p_1 = p_2$*. Express the momentums through their kinetic energies $p = mv$, $p^2 = m^2v^2$, $K = \mathrm{m}v^2/2$ and, $2mK = m^2v^2$. From these expressions find p: $\mathrm{p} = \sqrt{2mK}$ Substituting this equality into * we obtain $\sqrt{2m_1K_1} = \sqrt{2m_1K_2}$ whence we can find K_2:$K_2 = (m_1/m_2)\,K_1 = 1.2\times10^{-8}$ J $= 12$ nJ.

1.5.3 Angular momentum conservation law

We will start from the general law of the dynamics of a rotational motion (1.3.58). In a closed system (at $M_F(F) = 0$) no change of angular momentum is observed, $d\mathbf{L} = 0$ and, hence,

$$\mathbf{L} = \text{const.}, \tag{1.5.7}$$

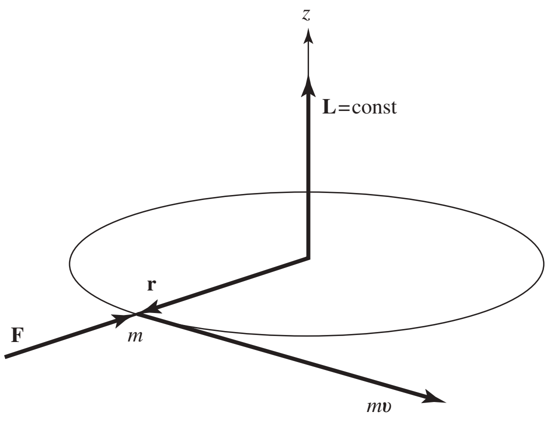


Figure 1.29. The movement of a body in a central field.

That is, in an absence of torques of external forces the angular momentum of a system remains constant. This statement concerns an MP, an MP system and an IRB. In other words, the angular momentum of a closed system is conserved.

Apply this law to the analysis of the motion of an MP under the action of a central force (Figure 1.29). Let an MP of mass m be under the action of an external force so that in all its positions the line of force action passes through one point (through the center of a circle). Then $\mathbf{M}_F(\mathbf{F}) \equiv 0$, accordingly $(d\mathbf{L}/dt) = 0$ and $\mathbf{L} = \text{const}$. It can be seen that if movement takes place under the action of the central force, vector $\mathbf{L}$ is fixed, therefore vectors $\mathbf{p}$ and $\mathbf{L}$ are fixed as well (as $[\mathbf{r} \cdot \mathbf{p}] = \mathbf{L}$). It, in turn, fixes a plane, in which vectors $\mathbf{r}$ and $\mathbf{p}$ lie. Hence, under the action of the central force the MP (a body) moves along a flat trajectory (circular, elliptic or hyperbolic) so that $[\mathbf{r} \cdot \mathbf{p}] = \text{const}$. (Again it is appropriate to recollect the conversations of Jules Verne's heroes in the projectile in which they tried to reach the moon.) Examples of such movement are the motion of the planets around the Sun (according Kepler's laws) and the electron motion in atoms (within the framework of the Bohr model, refer to Chapter 6, Section 6.7).

EXAMPLE E1.23

A person of mass $m_2 = 75$ kg stands at the edge of a platform which is in the form of a homogeneous disc of a radius R and mass $m_1 = 200$ kg. The platform can rotate freely around a vertical axis that passes through its center. Define the angle φ of turn of the platform if the person, moving along its edge, returns to the initial point of the platform.

Solution: (*Before solving this problem it is useful to remember the example on the law of momentum conservation, Example E1.21*).

An inertial reference system is useful to relate to the earth. Only gravitational forces and bearing reactions act on the system, all of them being parallel to the rotation axis; their torques being zero. It is therefore possible to take advantage of the angular momentum conservation law. Since, at the initial moment, the "person–platform" system is at rest, the total angular momentum is zero. We shall consider that the person starts to move uniformly on the platform. Since the total angular momentum should remain zero, the platform should move in the opposite direction with an angular speed axis ω_2. We shall denote the angular velocity of the person concerning the earth as ω_1. Then, according to the conservation law of angular momentum, in a projection to axis Oz: $I_{z1}\omega_1 + I_{z2}\omega_2 = 0^*$, where $I_{z,1}$ is the MI of the person relative to the Oz axis and $I_{z,2}$ is the MI of the platform relative to the same axis. Note that the angular velocity ω_{rel} of the person relative to the platform is determined by the equality $\omega_{rel} = \omega_1 - \omega_2$. Wherefrom $\omega_1 = \omega_{rel} + \omega_2$. Then the star equation becomes

$$I_{z,1}(\omega_{rel} + \omega_2) + I_{z,2}\omega_2 = 0$$

Having removed the brackets and rearranged the terms we obtain $(I_{z,1} + I_{z,2})\omega_2 = -I_{z,1}\omega_{rel}$. Let us multiply both parts of the equality by the time τ taken by the person to return to the initial point:

$$(I_{z,1} + I_{z,2})\omega_2\tau = -I_{z,1}\omega_{rel}\tau$$

Here $\omega_2\tau = \varphi$ is the turning angle of the platform relative to the earth and $\omega_{rel}\tau = 2\pi$ is the angle over which the person travels relative to the platform. If we consider the person as an MP, then $I_{z,1} = m_1R^2$.
The MI of a disc can be calculated according to eqn (1.3.47): $I_{z,2} = (1/2)\, m_2R^2$.

Having substituted these values in the formula obtained and made simple transformations, we arrive at $\varphi = -4\pi\, [m_1 /(m_2 + 2m_1)]$

Substitution of numerical values gives the final result:

$$\varphi = \frac{6\pi}{7} = 2.69\,\text{rad} = 154°$$

EXAMPLE E1.24

A disc-shaped wheel of mass $m = 50$ kg and radius $r = 20$ cm is twisted to promptness $n_1 = 480\ \text{min}^{-1}$ and released to rotate freely. It then stops because of friction. Find the torques for the following two cases: (1) the wheel stops in $t = 50$ sec; (2) the wheel makes $N = 200$ revolutions before stopping.

Solution: (1) According to Newton's second law (applied to rotation motion) we can write

$$M\,\Delta t = \mathrm{I}\omega_2 - \mathrm{I}\omega_1,$$

where I is the wheel's MI and ω is the angular speed. Since $\omega_2 = 0$ and Δt is t then $Mt = -\mathrm{I}\omega_1$ and therefore $M = -(\mathrm{I}\omega_1/t)$*. The disk's MI is $\mathrm{I} = (mr^2/2)$. Substituting this equation into * we obtain

$$M = -mr^2\omega_1/2t.$$

Since $\omega = 2\pi n$ we obtain $M = -1$ Nm.

(2) We can find the torques from its work $A = (J\omega_1^2/2) = M\varphi$ (see Section 1.4.1). Then

$$M = -\frac{mr^2\omega_1^2}{4\varphi}.$$

The total number of revolutions until disc shutdown (i.e., its total angular displacement) can be found: $\varphi = 2\pi N = 1256$ pad. Therefore, we arrive at the same torque $M = -1$ Nm. The sign shows that the torque has damped the movement.

1.5.4 Potential curves

We will consider two problems in the framework of the conservation laws: the potential curves principle and the theory of collisions in areas of science close to chemistry.

One of the main problems in chemistry is the investigation of interactions between particles. One of the most accepted methods for describing such interaction is the language of potential curves. Potential curve $U(r)$ is a graphic picture of the potential energy between interacting particles as dependent on the interparticle distance. If the origin is kept in one of the particles (for instance, in particle number 1), then r is the distance between particles 1 and 2. As was mentioned earlier, when describing the interaction due to the force field, the potential energy is usually taken to be zero when particles are at infinite distance (i.e., at $r = \infty$).

In spite of the fact that curve $U(r)$ reflects only the potential energy change, with its imaging we can also find the value of the kinetic energy in each interparticle point (at given total energy E).

The potential energy of the gravitational interaction

Let two MPs having mass m_1 and m_2 be at a distance r from each other; then the potential energy of their interaction, as was shown above, $(U/(r) = -G(m_1m_2/r))$, can have a hyperbolic form. In the case of attraction (Figure 1.30a) the potential energy increases as far as the points move further from each other. At infinity, the potential energy of the interaction reaches a maximum. On the accepted condition this maximum will correspond to zero potential energy. If the maximum is zero, all the potential energy values, in the case of attraction force on finite distances, will be negative.

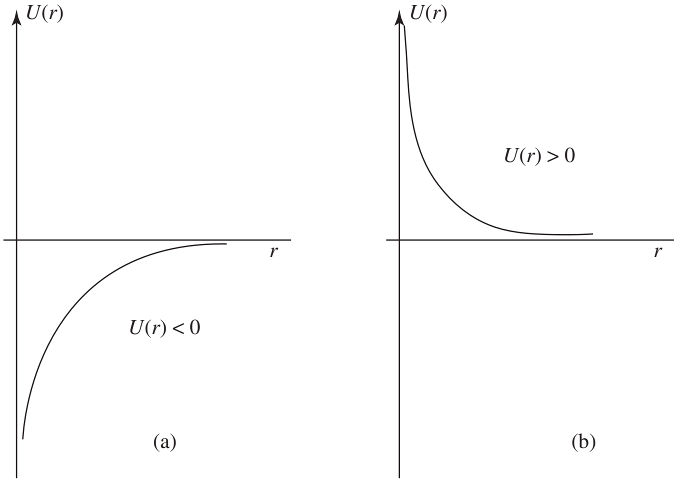


Figure 1.30. The potential curves for forces of (a) attraction and (b) repulsion.

Potential energy of electrostatic interaction

It describes the interaction between two resting charges depending on the intercharge distance r; it is expressed by the formula

$$U(r) = \frac{q_1 q_2}{4\pi\varepsilon_0 r},$$

where q_1 and q_2 are values of point charges, ε_0 is the electrical constant (refer to Section 4.1). The sign of potential energy is taken into account "automatically" due to signs of interacting charges. In the case of different charges, the potential energy of interaction (attractions) is negative. The form of the graph is qualitatively just the same as in the gravitational interaction provided the charges are of different sign. In the case of similar charge signs, the potential energy is positive going to zero in infinity (Figure 1.30b).

Potential energy of elastic interaction

In Section 1.4.5 the particularities of elastic interaction energy have been considered. Note that elastic forces manifest themselves equally successfully both in the macro- and the micro-world. When deforming a body or "stretching" an intermolecular bond, an external force produces work. The magnitude of this work is equal to the change of the potential energy of the elastic interaction. As mentioned in Section 1.4.5, the potential energy in this case is $U = (1/2)\beta x^2$, where x (in this instance) is the deformation magnitude. The graph of this function is described by a parabolic curve, symmetrical relative to the U-axis (Figure 1.31). Since the elastic forces are conservative, the total energy E is constant (according to the laws of conservation of mechanical energy). This is depicted in the graph

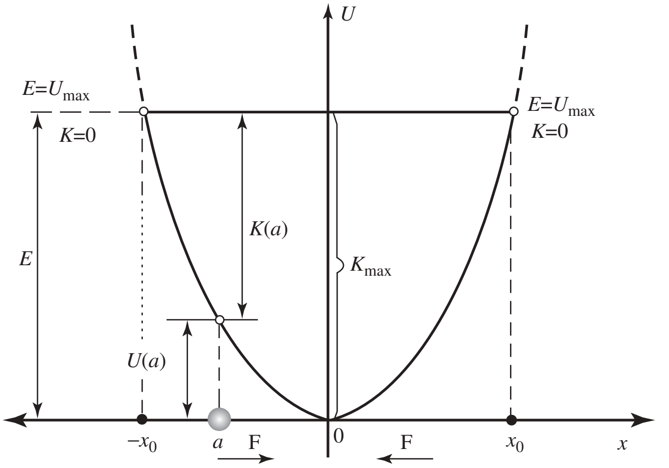


Figure 1.31. Potential curve for a harmonic oscillator.

by the straight line E = const. In the arbitrary point $x = a$ the potential and kinetic energy values are expressed by lengths of segments $U(a)$ and $K(a)$. The elastic force is always directed to the origin. Approaching the origin, the particle's potential energy U decreases. Conversely, at the origin ($x = 0$) kinetic energy reaches its maximum ($K_{\max} = \mathrm{E}$, since $U(0) = 0$). Continuing motion, a particle loses its kinetic energy until it vanishes to zero. At this point, potential energy reaches its maximum and is equal to the total energy,

$$U_{\max} = E = \frac{1}{2}\beta x_0^2,$$

where x_0 is a maximum displacement. Thereby, the maximum point displacement $x_0 = \pm\sqrt{2E/\beta}$ is defined by its total energy. The particle performs an oscillation relative to the origin (Chapter 2).

More complex type of interaction is the interplay of molecules

Without looking deeply into the nature of these interactions, we can say that both attraction and repulsion forces act simultaneously between molecules. Because attraction forces decrease with distance slower than repulsion forces, attraction forces dominate at longer distances and at shorter distances repulsion forces dominate.

One of the popular approximations is a power dependency of potential energy upon distance r in the form

$$U(r) = -\frac{a}{r^6} + \frac{b}{r^{12}}. \tag{1.5.8}$$

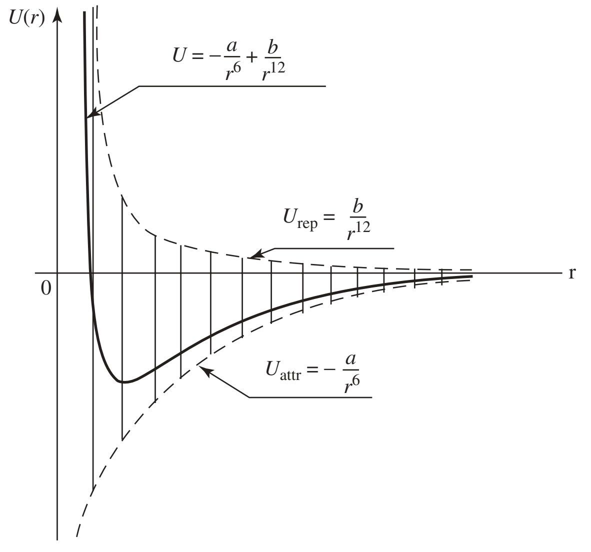


Figure 1.32. "6–12" (Lennard–Jones) potential.

This is the so-called Lennard–Jones potential "6–12," offered for the interaction description of nonpolar molecules. Values a and b for different molecules are different. The first term expresses the potential energy of attraction whereas the second term expresses the potential energy of repulsion. In Figure 1.32 both curves are represented by dotted lines and the solid line is a resulting curve (eq. (1.5.8)).

The formula of the Lennard–Jones potential can be written in another form as

$$U(r) = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma'}{r}\right)^{6}\right],$$

where ε and σ are again different for different molecules.

Let us look at the change of the force acting on particle 2 as it approaches the origin, where particle 1 is resting. For this purpose it is effective to use the connection of the central force and the potential energy in the form (1.4.30) and carry out a graphic derivation (Figure 1.33). It can be seen that at great distances the derivative is small (the magnitude of angle of the slope of the tangent line KK_1 to abscissa is small) and is positive (angle α is sharp). It means that the force F acting on particle 2 is negative. This signifies that force F is directed to the negative direction of the r axis, i.e., to the origin. As particle 2 approaches particle 1 the derivative increases and in the inflexion point a reaches to its maximum (angle of the tangent slope has at this point its highest value). At this point the force F is minimum (the maximum value of attraction force). Then the angle begins to decrease and at point r_0 vanishes to zero (the tangent line to the curve $U(r)$ is parallel to the abscissa). At this point

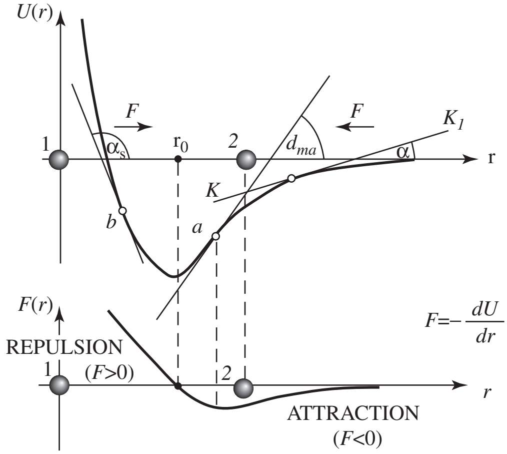


Figure 1.33. Potential energy and force as a function of interparticle distance.

the attraction and repulsion forces are in balance. Distance r_0 is referred to as the equilibrium distance. At distances $r < r_0$ an angle α is obtuse (for instance, at point b); in this area the tangent changes its sign ($dU/dr < 0$). This signifies that the force F changes its direction, becoming repulsive ($F > 0$) and then increasing rapidly as particle 2 approaches the origin. A graph of the dependence $F(r)$ is given in the lower part of Figure 1.33.

Considering curve $U(r)$ (Figures 1.30, 1.32 and 1.33) one can be assured that only by being at a point with the minimum potential energy, a particle does not feel any force action at all. When the particle displaces to the right or to the left from the position of equilibrium, in which the potential energy is minimal, the force appears, directed to the position of balance. Therefore, it is seen that the particle will always tend to occupy a position with minimum potential energy (however the force of inertia does not allow it, assuming that the dissipation of energy is absent).

Let us now trace the change of kinetic and potential energy, provided the total energy is known, when molecules approach each other. Let, at infinite distance, the molecules have a total energy $E > 0$. As we have already agreed, the potential energy of infinitely separated molecules is zero. Consequently, kinetic energy is equal to the total energy E (Figure 1.33). When molecules are approaching, their kinetic energy increases and at the point of balance (r_0, refer to Figure 1.34) reaches a maximum. While approaching further ($r < r_0$), the kinetic energy of interaction decreases rapidly until it equals zero. According to the conservation law of mechanical energy, a total energy at this point is admitted a maximum potential energy $E = U_{max}$. The closest approach σ, to which under a given total energy molecules can be reached, is referred to as an efficient molecule diameter. After an instant stop, all events will run in the inverse sequence and the molecules will return to the initial position (to infinity), possessing kinetic energy, which is the same as before the rapprochement. In this case, a moving particle is unlimited in space. Such spatially unrestricted motion is referred to as an infinite one.

Particles whose total energy turns out to be negative behave otherwise ($U < 0$, and $|U|>K$, then $E < 0$). In this case, a particle turns out to be "locked" in the potential well

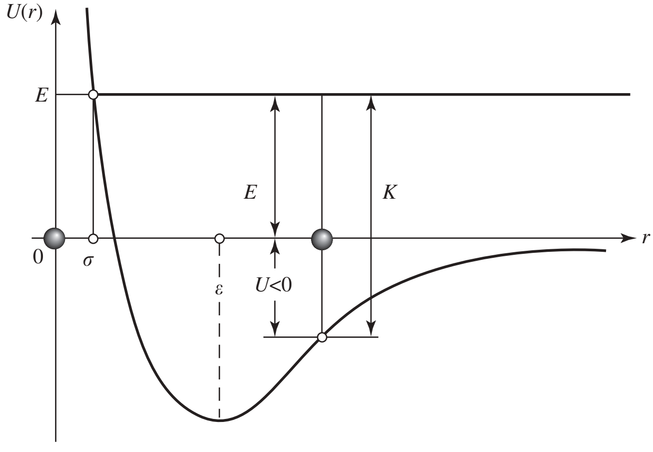


Figure 1.34. A particle with energy over a potential well ($E > 0$).

(Figure 1.34). Particles are not able to approach more than to distance r_{min}, and cannot retreat more than distance r_{max}. Such spatially restricted motion is referred to as a finite one. The particle will perform an oscillatory motion. The two-particle state is a bonded one (i.e., particles are not able to separate) (Figure 1.35).

From this qualitative analysis one can draw an important conclusion. If the total energy of the two-particle system is positive ($E > 0$), no bound state of particles occurs. If the total energy of the particles is negative ($E < 0$), the bound state is possible. We will meet examples of this in the book.

1.5.5 Particle collisions

In this section we will apply the laws of mechanical energy and the momentum conservation principle to processes of particle collisions. By collision, we mean any short interaction between particles.

Unlike the collision of macroscopic bodies, where the direct contact of bodies and their deformation on impact can be observed (e.g., the action of a boot-kicking a ball can be captured by a rapid photograph), interaction of atomic particles is realized by means of force fields and cannot be directly observed. In this respect, the collision of electrical charges, which interact through their electric fields or the collisions of neutrons (where the interaction takes place via nuclear field), etc. can be mentioned.

In principle, within the framework of classical macroscopic mechanics one could consider the detailed process of collision and draw a conclusion about a body's deformation and interaction forces (their changes during the process of collision), taking into account the results of collisions, analyzing direction and velocities of collided particles during and after the collision. However, such detailed consideration even for classical objects is very difficult, but for quantum objects it is absolutely impossible. So, for instance, it is impossible to

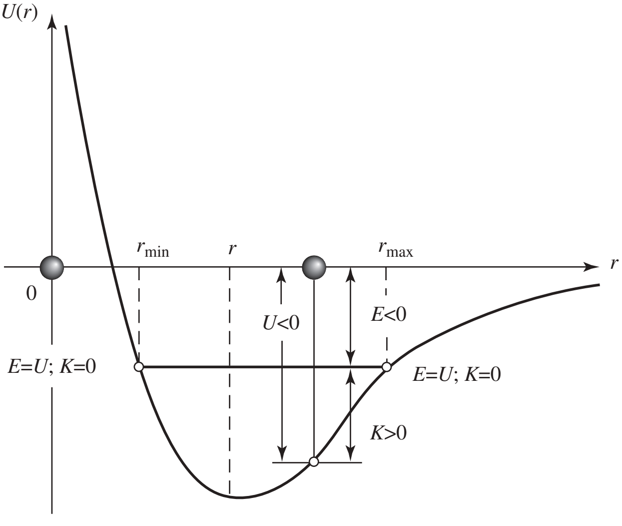


Figure 1.35. A particle in a potential well ($E < 0$).

describe in detail the process of the collision of neutrons because no one knows the exact laws of the nuclear forces' properties. Nothing can be said about the mechanism of the collision of neutrons with phonons (refer to Chapter 9), γ-quantum with an electron, etc. Moreover, a detailed description of the process of collision of microparticles in general is impossible because of the uncertainties principles (refer to Section 7.2).

Nevertheless much of our information about atomic and molecular particles is obtained experimentally by observing the effects of collisions between them. For instance, it was just such collision experiments that allowed Rutherford to suggest a planetary model of atoms.

Fortunately, it appears that in many cases such detailed consideration of interactions is not needed. For the determination of velocities of particles after the collision it is sufficient to know an initial state and use the conservation laws.

Collisions are not limited to cases in which two bodies come into contact in the usual sense. However, we will restrict ourselves to only pair collisions.

Collisions of particles can be of two types: elastic and inelastic.

Elastic particles collisions

A collision is called elastic if the total particle kinetic energy is preserved. Only the kinetic energy repartition between collided particles takes place, whereas the inner-particle state remains unchanged.

Consider in general form an elastic collision of two particles noninteracting at distance (i.e., their potential energy before and after collision is zero, $U = 0$). The collision is called a head-on impact (or frontal one), if vectors of their velocities (of the CMs of collided bodies) before the impact are directed along one and the same direction (say, axis x). If this condition is not valid, the collision is called a glancing one.

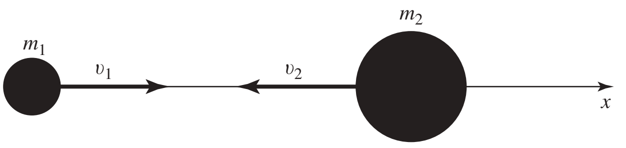


Figure 1.36. A head-on collision.

Consider a frontal collision of particles moving translationally (without rotation) (Figure 1.36) along the x-axis. Momentums of particles are directed along this line, and therefore we can change the vector writings to a scalar form (in projections on the axis x) (the sign x for simplicity is deleted). The velocities of the particles after collision are denoted by u.

Consider a system of two colliding particles being a closed and a conservative one; it is possible to apply to collision both conservation laws: the energy and momentum conservation laws in mechanics. The energy conservation law looks as follows:

$$\frac{m_1\upsilon_1^2}{2}+\frac{m_2\upsilon_2^2}{2}=\frac{m_1u_1^2}{2}+\frac{m_2u_2^2}{2}, \tag{1.5.9}$$

and the momentum conservation law

$$m_1\upsilon_1+m_2\upsilon_2=m_1u_1+m_2u_2. \tag{1.5.10}$$

Expressions (1.5.9) and (1.5.10) can be considered as a system of two equations with two unknowns u_1 and u_2. To solve this system one can transform it to the form:

$$m_1(\upsilon_1^2-u_1^2)=m_2(\upsilon_2^2-u_2^2)$$

and

$$m_1(\upsilon_1-u_1)=m_2(\upsilon_2-u_2),$$

and then divide the upper equation with the lower one. The result is:

$$\upsilon_1+u_1=\upsilon_2+u_2.$$

The first equation divided by m_1 gives

$$\upsilon_1-u_1=\frac{m_2}{m_1}u_2-\frac{m_2}{m_1}\upsilon_2.$$

We arrive at the new system consisting of two linear equations with two unknowns.

In order to find u_2 one can sum up these two equations:

$$2\upsilon_1 = u_2\left(1+\frac{m_2}{m_1}\right)+\upsilon_2\left(1-\frac{m_2}{m_1}\right),$$

from which it follows:

$$u_2 = \frac{2\upsilon_1 - \upsilon_2\left(1-\frac{m_2}{m_1}\right)}{1+\frac{m_2}{m_1}}. \tag{1.5.11}$$

The velocity of the first particle can be found analogously:

$$u_1 = \frac{2\upsilon_2 - \upsilon_2\left(1-\frac{m_2}{m_1}\right)}{1+\frac{m_2}{m_1}}. \tag{1.5.12}$$

(This equation can be obtained from (1.5.11) by changing all indexes $1 \leftrightarrow 2$ keeping the mathematical symmetry in mind.)

With a little effort, one can obtain an analogous expression in the form:

$$u_1 = -\upsilon_1 + 2\frac{m_1\upsilon_1 + m_2\upsilon_2}{m_1 + m_2} \tag{1.5.13}$$

and

$$u_2 = -\upsilon_2 + 2\frac{m_1\upsilon_1 + m_2\upsilon_2}{m_1 + m_2} \tag{1.5.14}$$

The fractions in (1.5.13) and (1.5.14) present the velocity of the two particles' mass center motion. Then

$$u_1 = -\upsilon_1 + 2V_c$$

and

$$u_2 = -\upsilon_2 + 2V_c,$$

where

$$V_c = \frac{m_1 \upsilon_1 + m_2 \upsilon_2}{m_1 + m_2},$$

underlining once more that υ_1 and υ_2 are the velocities of each particle and V_c is the CM velocity. (Knowing m_1, m_2, υ_1, υ_2, i.e., all the initial data, it is easy to find V_c; refer to eq. (1.3.32) and Figure 1.13). If one transfers now according to the Galileo principle to the coordinate system connected to the mass center (i.e., $V_c = 0$), the equations can be simplified significantly:

$$u_1 = -\upsilon_1, u_2 = -\upsilon_2 \tag{1.5.15}$$

After collision in this coordinate system, the colliding particles change their direction of motion in such a way that the absolute magnitudes remain unchanged.

Some particular cases are of general interests and importance.

1. If the second particle remains at rest in the laboratory system ($\upsilon_2 = 0$). Then

$$u_1 = \frac{-\upsilon_1\left(1 - \dfrac{m_2}{m_1}\right)}{1 + \dfrac{m_2}{m_1}} = -\frac{m_2 - m_1}{m_2 + m_1}\upsilon_1 \tag{1.5.16}$$

and

$$u_2 = \frac{2m_1}{m_1 + m_2}\upsilon_1. \tag{1.5.17}$$

It can be seen from eq. (1.5.17) that, after the collision, the initially resting particle 2 acquire the velocity u_2, the direction of which always coincides with the initial velocity of the first particle before collision. However, the velocity direction of the first particle after collision depends upon the mass ratio of both particles. If $m_1 < m_2$ the first particle changes the direction of flight to the opposite one (as can be seen from eq. (1.5.16)). If $m_1 > m_2$, the direction of motion of both particles after collision is the same.

If the masses of both particles are the same ($m_2/m_1 = 1$), then

$$u_1 = \frac{2\upsilon_2 - (1-1)}{1+1} = \upsilon_2, \quad u_2 = \frac{2\upsilon_1 - (1-1)}{1+1} = \upsilon_1,$$

i.e., in the case of the central elastic collision of particles of equal mass, a simple exchange of velocities and, consequently, kinetic energies occurs. The following takes place: the moving particle stops (after the collision $u_1 = \upsilon_2 = 0$); however, the resting particle begins to move with a velocity $u_2 = \upsilon_1$, i.e., at the velocity of the first particle. In this case, the moving particle will completely transfer all its kinetic energy to the resting particle.

2. Consider now the collision of particles that differ noticeably in their mass, say $m_1/m_2 \ll 1$. Let the second particle with the larger mass be at rest ($\upsilon_2 = 0$). One can neglect the fraction m_1/m_2 in eq. (1.5.16) in comparison with 1. One obtains

$$u_1 \approx -\upsilon_1$$

and

$$u_2 = \frac{2\upsilon_1}{1+\dfrac{m_2}{m_1}} \approx 0.$$

Thus, the particle with the smaller mass after collision changes its velocity to the opposite without loss of kinetic energy.

This effect appears to be very important in chemical kinetics. For example, in exothermic reactions with the production of atomic hydrogen as an intermediate product (which is very light in comparison with another reagents), its kinetic energy can significantly exceed the quasi-equilibrium one. This phenomenon in kinetics is called the "effect of hot atoms". The slow energy transfer from the small particle to the large one is also determined in the relaxation processes in lasers with compounds of halogens with hydrogen as a working medium.

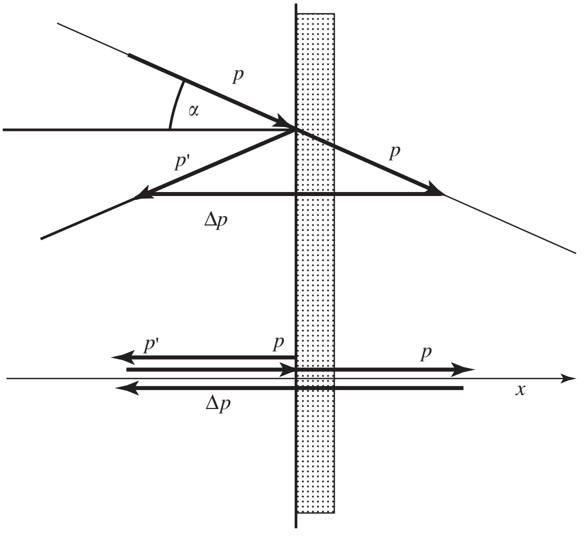


Figure 1.37. An impact of a particle onto a wall.

The result obtained is fulfilled with even greater accuracy when a particle collides with a wall (for instance, when a molecule collides with a vessel wall, Figure 1.37). In the latter case a molecule moving perpendicularly to the wall reflects from it and proceeds with the previous velocity backwards $u_1 = -\upsilon_1$ (Figure 1.37).

Determine the momentum change Δp in this case. According to the definition $\Delta p = p_1' - p_1$, where $p_1 = m_1\upsilon_1$ and $p_1' = m_1 u_1$, then $\Delta p = m_1(u_1 - \upsilon_1)$. Since $u_1 = -\upsilon_1$, then $\Delta p = -2m_1\upsilon_1$ or

$$\Delta p = -2p_1. \tag{1.5.18}$$

The negative sign shows the direction of momentum changing, which complies with the direction of "elastic force" acting on the particle from the wall during the collision.

If the particle falls to the wall under an angle α to a normal, the component of velocity parallel to the surface of the wall remains unchanged, and only the component normal to the wall (along the x axis) takes part in the momentum transfer to the wall (i.e., $\Delta p = -2p_1\cos\alpha$) (see Figure 1.37).

As it follows from the results obtained, the particle does not change its kinetic energy on collision with the wall, but the wall receives from the particle a momentum equal to $2p_1$. (Herewith the wall "velocity" is zero; however the value of transferred momentum is not zero).

One can calculate (this we will leave to the reader), that on direct central collision with a resting particle of mass m_2 the colliding particle of mass m_1 will transfer part of its kinetic energy:

$$K_2 = |\Delta K_1| = \frac{4m_1 m_2}{(m_1 + m_2)^2} K_1.$$

Then the relative loss of the particle kinetic energy will be:

$$\frac{|\Delta K_1|}{K_1} = \frac{4m_1 m_2}{(m_1 + m_2)^2}. \tag{1.5.19}$$

It is easy to see that at the equality of mass ($m_1 = m_2$) the first particle loses all its energy, i.e., $\Delta K_1/K_1 = 1$, and stops. On collision with a wall $m_1/\mathrm{m}_2 \to 0$ and then $\Delta K_1/K_1 \to 0$, its kinetic energy remains unchanged.

The process of deceleration of neutrons in atomic reactors is based on the phenomena of a "fast" particle transferring energy to a resting particle. This is because in the elementary act of chain nuclear reactions only fast neutrons are produced. For the realization of the next act of a chain reaction—capture of a neutron by a nucleus of a uranium-235 atom—it is necessary to slow neutrons down until their energy becomes commensurate with the energy of the thermal motion of the molecules. This occurs when neutrons collide with the atomic nuclei of the moderator material. Judging by the formulas given, the best moderator is a hydrogen-containing material: during a single front collision with protons, the neutrons immediately lose the whole of their kinetic energy ($u_1 = 0$). However, protons easily participate in the reaction with neutrons; protons capture a neutron (with a deuteron formation) and remove neutrons from the chain reaction process. Therefore, the heavy water D_2O is more often used (then $\Delta K_1/K_1 = 0.9$) or graphite ($\Delta K_1/K_1 = 0.28$). Note also, that in the use of tungsten nuclei this ratio is 0.02, the impact of the neutron to the tungsten nucleus is closer to the case of the collision with a wall, with a negligible loss of kinetic energy.

EXAMPLE E1.25

How many times k will the neutron kinetic energy decrease after N consecutive collisions with atomic nuclei which practically do not capture neutrons. Consider the collisions to be elastic and central with atoms: deuterium ^2H, carbon ^{13}C and tungsten ^{84}W. Let N be 3.

Solution: The problem with the single collision of a moving and motionless particle is solved in the text where formulas for the speed of each of the particles after collision (see formulas (1.5.16) and (1.5.17)) are given. Using these results it is possible to obtain expressions for neutron kinetic energy after collision K_1' in relation to its initial energy K_1:

$$K_1' = \frac{m_1 u_1^2}{2} = \frac{m_1}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 v_1^2 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 K_1$$

After the second collision the kinetic energy K_1' should be treated as initial. Then the neutron kinetic energy $K_1^{(2)}$ is

$$K_1^{(2)} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 K_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^{2\times 2} K_1$$

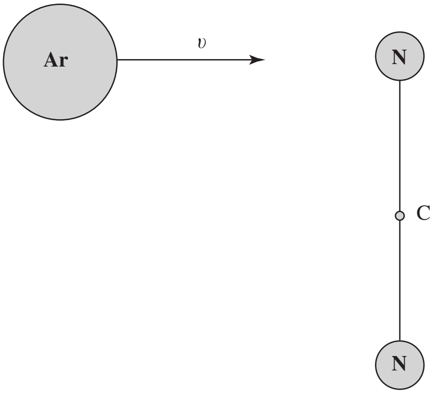
It is easy to guess that after N collisions

$$K_1^{(N)} = -(m_1 + m_2)/(m_1 - m_2)^{2N}\, K_1$$

Then the ratio after N collisions is $k = K_1/K_1^{(N)} = (m_1 + \mathrm{m}_2)/(\mathrm{m}_1 - m_2)^{2N}$.

EXAMPLE E1.26

An argon atom collides with one N atom of a resting molecule N_2 perpendicular to the N–N bond (Figure E1.26). The velocity of the argon atom velocity is $v_1 = 400$ m/sec. The impact is elastic. Atoms can be represented as MP and the N_2 molecule considered as a rigid rotator. Relative masses are $Ar_{\mathrm{A,r}} = 40$ and $A_{\mathrm{r,N}} = 14$, the interatomic distance N–N is 0.109 nm. For the nitrogen molecule after collision, determine: (1) the velocity V_c of its mass center; (2) the angular momentum L_z acquired by N_2 molecule relative to the z-axis (perpendicular to the drawing plane and passing through its CM, point C in the figure); (3) the angular velocity ω of the molecule rotation relative to the z-axis (Figure E1.26).



Solution: Assume that collision of Ar and N(1) atoms is head-on and elastic. Using the formulas of Section 1.5.5 (eq. (1.5.16)) we obtain

$$u_1 = \frac{m_1 - m_2}{m_1 + m_2}\upsilon_1.$$

To determine the CM's velocity Vc we can use only the momentum conservation law because part of the kinetic energy will go to the kinetic energy of the molecule rotation. Therefore, this part of the interaction should be considered as inelastic and the energy of rotation as internal. Then

$$m_1\upsilon_1 = 2m_2V_C + m_1u_1, \quad \text{therefore } V_C = \frac{m_1}{2m_2}(\upsilon_1 - u_1)$$

Substituting the velocity u_1 in this expression we obtain

$$V_C = \frac{m}{2m}\left(\upsilon_1 - \frac{m_1 - m_2}{m_1 + m_2}\upsilon_1\right) = \frac{m_1}{m_1 + m_2}\upsilon_1.$$

Executing calculations we obtain $V_C = (40/(40 + 14)) \times 400 = 296$ m/sec.

(2) Since there are no external force actions for angular momentum calculations we can use the angular momentum conservation law relative to the z-axis and passing the CM:

$$L_{z,1} = L'_{z,1} + L'_{z,C},$$

where $L_{z,1} = m_1\upsilon_1(d/2)$ is the angular momentum of the argon atom before the impact,

$L'_{z,1} = m_1u_1(d/2)$ after the impact and $L'_{z,C}$ is the angular momentum of the nitrogen molecule after the collision relative to the Oz axis, wherefrom

$$L'_{z,C} = m_1(\upsilon_1 - u_1)\frac{d}{2} = m_1\left(\upsilon_1 - \frac{m_1 - m_2}{m_1 + m_2}\upsilon_1\right)\frac{d}{2} \times \frac{m_1 m_2}{m_1 + m_2}\upsilon_1 d$$

Executing calculations we arrive at

$$L'_{z,C} = \frac{40 \times 14}{40 + 14} \times 1.66 \times 10^{-27} \times 400 \times 1.09 \times 10^{-10} = 7.51 \times 10^{-34}\ \text{kg}\,\text{m}^2/\text{sec}$$

(3) We can find the angular velocity of the molecule rotation knowing the angular momentum value after impact; writing the angular momentum according to eq. (1.3.44):

$$L'_{z,C} = I_z\omega_z = \mu d^2\omega_z,$$

where μ is a reduced molecular mass (eq. (1.3.50)) (in our case $m_1 = m_2$ and $\mu = m_2/2$). Then

$$\omega = \frac{2L_{z,C}}{m_2 d^2} = \frac{2}{m_2 d^2} \times \frac{m_1 m_2}{m_1 + m_2}\upsilon_1 d \text{ or } \omega = \frac{2m_1}{m_1 + m_2} \times \frac{\upsilon_1}{d}$$

Executing calculation we arrive at

$$\omega = \frac{2 \times 40}{40 + 14} \times \frac{400}{1.09 \times 10^{-10}} = 5.44 \times 10^{12}\ \text{rad/sec}$$

Inelastic collision of particles

As indicated above, the collision of particles is called inelastic if the total kinetic energy of the particles before the collision is not equal to the total kinetic energy after the collision. The kinetic energy partly or totally transforms to the internal energy (i.e., particles change their energy state, for instance, temperature). A limiting case of inelastic collision is absolute inelastic collision, under which a maximum loss of kinetic energy occurs. After the collision, both particles do not move away, but move together as a single particle. Under inelastic collision, the law of mechanical energy conservation is not executed, since the system is a dissipative one (mechanical energy transforms into another type of energy). The general law of total energy conservation is certainly executed, but we are unable to use it, since we do not know *a priori*, what part of the kinetic energy transforms into internal energy. It is therefore possible to use only the law of momentum conservation:

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}, \tag{1.5.20}$$

where p is the total momentum of particles after an absolute inelastic collision. Under frontal inelastic collision all velocity vectors are directed along a single line, passing through the center of the particle masses. In projections onto axis x, complying with the direction of motion, the law of momentum conservation can be written as (sign x is omitted)

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)u,$$

where u is the velocity of particles after the collision. From this expression, we can find the velocity of the particles after collision

$$u = \frac{m_1 \upsilon_1 + m_2 \upsilon_2}{m_1 + m_2}. \tag{1.5.21}$$

Now it is possible to define that part of the total kinetic energy, which at collision transforms into the internal energy of the particles. If we denote U as the internal energy of particles, then $U = K_1 + K_2 - K'$, where K' is the kinetic energy of both particles after collision. In the comprehensive form the above formula can be rewritten as

$$\Delta U = \frac{m_1 \upsilon_1^2}{2} + \frac{m_2 \upsilon_2^2}{2} - \frac{(m_1 + m_2)u^2}{2}. \tag{1.5.22}$$

Formula (1.5.22) can be written in another form:

$$\Delta U = \frac{1}{2}\mu(\upsilon_1 - \upsilon_2)^2, \tag{1.5.23}$$

where μ is the reduced mass of the system of two particles:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1.5.24}$$

We can see that if particles of similar masses and similar velocities move to meet each other, their velocity after absolutely inelastic collision is zero, and, consequently, the whole kinetic energy transforms into internal energy.

Chemistry students should note that the above consideration of collision processes cannot be directly extended to the case of atomic–molecular collisions even in classical physics. This is, in particular, because in a real atomic–molecular system a process of interaction significantly depends on the so-called adiabatic factor, i.e., on the correlation of the collision time and the intrinsic molecular frequencies.

1.6 EINSTEIN'S SPECIAL RELATIVISTIC THEORY (STR) (SHORT REVIEW)

Until now, our discussions have been based on the classical Galileo–Newton representations. Moving on to quantum-optical phenomena and others, these ideas appear to be insufficient and it is necessary to consider some more general notions, in particular, Einstein's special theory of the relativity (STR, 1905).

There are a number of theories on the relationships between descriptions of the same physical phenomena in various systems moving relative to each other. They are referred to as *theories of relativity* (e.g., Galileo's principle, refer to Section 1.3.2). Another more general theory is Einstein's STR. This theory is based on a reliably established experimental fact: *the speed of light propagation is independent of the speed of it source.* This fact has been proved in the well-known Mickelson–Morley experiments: they had coordinated the parts of their large-scale devices on the earth that could determine the speed of light sent along the trajectory of the earth's movement around the sun and opposite to it with very high accuracy. Experiment has shown that this speed does not change whether the movements are directed in parallel or oppositely. *The speed of light* (*i.e.*, *electromagnetic waves*) *appeared to be independent of the speed of a source.* This greatly contradicted Galileo's principle of speed additions (eq. (1.3.4)). Moreover, this result appeared contradictory to the theory of Maxwell electrodynamics.

In order to preserve the principles mentioned above, it was necessary to proceed further than Galileo's transformations (1.3.1) and (1.3.2). These transformations have been replaced by the mathematical Lorentz equations already known in physics.

Lorentz transformations replace Galileo's transforms. When a coordinate system $K'(x', y', z', t')$ moves relative to the system $K'(x, y, z, t)$ in x direction (this restriction is made for simplicity) with a speed u ($u_x = u$) they look as follows:

$$
\begin{aligned}
&x' = \frac{x-ut}{\sqrt{1-\beta^2}} \quad x = \frac{x'+ut}{\sqrt{1-\beta^2}}\\
&y' = y, \quad y = y',\\
&z' = z, \quad z = z',\\
&t' = \frac{t-\dfrac{ux}{c^2}}{\sqrt{1-\beta^2}} \quad t = \frac{t'+\dfrac{ux}{c^2}}{\sqrt{1-\beta^2}}
\end{aligned}
\tag{1.6.1}
$$

where $\beta = u/c$; an expression is often used $\gamma = 1/\sqrt{1-\beta^2}$.

These equations were derived by Lorenz in 1904 as mathematical expressions of coordinates and time transformation as an attempt to preserve the system of Maxwell's theory in all inertial coordinate systems. The same transformations were obtained by Einstein in 1905; he proceeded from Newton's postulate on the equality of all inertial coordinate systems and experimental fact on the independence of light speed upon the source speed. Analysis of eq. (1.6.1) shows that *time is incorporated in space and movement.* Thus, Einstein considered it necessary to change Newton's representation about time and space (refer to Section 1.3.1). Moreover, in order to make Newton's transformations be invariant to Lorentz's equations it was necessary to accept that the mass of a particle is speed-dependent, being described by the expression

$$m = \frac{m_0}{\sqrt{1-\beta^2}}, \tag{1.6.2}$$

where m_0 is the particle mass at rest (at $\upsilon = 0$).

Some conclusions follow from the STR:

1. Relativistic speed's summation law. If an MP m moves with a speed υ' (υ'_x, 0, 0) in an inertial system of coordinates K' which, in turn, moves relative to another inertial coordinate system K with a speed $\mathbf{u}$ (u_x, 0, 0), then the speed $\upsilon(\upsilon_x$, 0, 0) of that MP in a system K (according to Lorentz's transformation), can be presented as:

$$\upsilon = \frac{u+\upsilon'}{1+\frac{u\upsilon'}{c^2}}. \tag{1.6.3}$$

The formula (1.6.3) refers to *relativistic law of speed's summation*. Obviously, the resulting speed υ is less than the sum of the two speeds u and υ'. From (1.6.2) and (1.6.3) it follows that: firstly, the speed of light is the same in any inertial system as at $\upsilon = c$ for any $u \leq c$, and, secondly, the body with non-zero m_0 cannot reach in any inertial system the speed of light in vacuum or exceed it, as $\upsilon' < c$ and $u < c$ it follows that $\upsilon = c$.

In the theory of relativity the expressions ct and ut have the dimension of length; it behaves as the fourth spatial coordinate. From (1.6.3) it follows that values ct and x can mix depending on the speed of the observer.

2. Relativistic shortening of length. Let a rod of length L lie along the x axis and move with a speed u in an inertial system K′; its length L′ as measured from the resting system K will be shorter than the length L: L < L′

$$L = \sqrt{1-\beta^2}\, L'. \tag{1.6.4}$$

Various observers (being in different inertial systems) consider the same rod to have a different length. From a physical point of view this discrepancy can be explained by the concepts of simultaneity, i.e., events for one observer are not simultaneous with those for another.

EXAMPLE E1.27

Determine a relativistic electron's momentum p and its kinetic energy K if the electron moves at a speed $\upsilon = 0.9c$.

Solution: The relativistic momentum is $p = m_0c\ (\beta/\sqrt{1-\beta^2})$. Executing all calculations we arrive at $p = 5.6 \times 10^{-22}$ kg m/sec. The kinetic energy in the relativistic mechanics is the difference between total energy $E = mc^2$ and energy $E_0 = m_0c^2$ at rest. Therefore,

$$K = \frac{m_0c^2}{\sqrt{1-\beta 2}} - m_0c^2,$$

or finally

$$K = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right).$$

Executing calculations we arrive at
$K = 1.06 \times 10^{-13}$ J $= 0.66$ MeV

EXAMPLE E1.28

A spacecraft moves with a velocity $\upsilon = 0.9c$ to the center of the earth. What distance l does it cover in the earth reference framework K in a time interval $\Delta t_0 = 1$ sec, measured by the spacecraft clock (system K'). Ignore diurnal rotation and the earth's movement around the Sun.

Solution: We can determine the distance l that the spacecraft can cover in the K' system according to the formula $l = \upsilon\Delta t$, where Δt is the time interval measured in the K system. This interval in the K system and that measured in the K' system are related by the formula

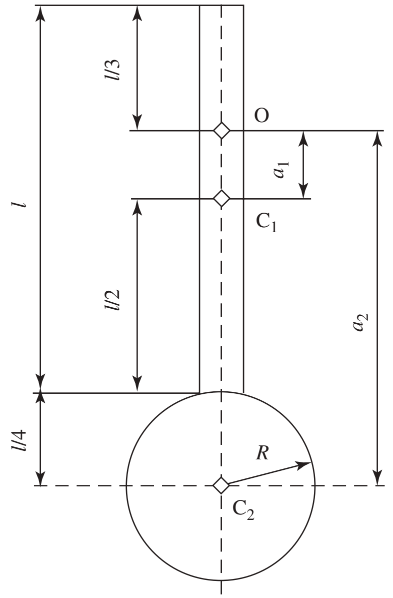
$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\beta^2}}.$$

We should substitute this value into the previous relation:

$$l = \frac{\upsilon\,\Delta\, t_0}{\sqrt{1-\beta^2}}.$$

Executing all calculations we arrive at $l = 6.19 \times 10^8$ m.

A body such as a physical pendulum consists of a rod of $l = 1$ m in length and of mass $m_1 = 1$ kg. A disc of mass $m_2 = 0.5m_1$ is fixed on one side of the rod. Find the MI of such a body relative to an axis passed perpendicularly to the picture through a point O (Figure E1.29).



Solution: The MI of the combined body can be calculated according to the formulas presented in Section 1.3. Choose an x-axis directed along the rod with its origin at point O. The overall MI of the whole body relative to the x-axis is the sum of the composite details: $I_z = I_{z1} + I_{z2}$, where part 1 is the rod and 2 is the disc. In order to find I_{z1} and I_{z2} we should use the theorem on parallel axis (1.3.48). The rod's MI can be given by the expression

$$I_{z1} = \frac{m_1 \ell^2}{12} + m_1 a_1^2 \quad (\text{where } a_1 = \mathrm{OC}_1).$$

We can see in Figure E1.29 that $a_1 = l/6$. Therefore,

$$I_{z1} = \frac{m_1 \ell^2}{12} + m_1 (l/6) a^2 = \frac{m_1 \ell^2}{9} = 0.111 m_1 l^2.$$

The disc's MI is $I_{z2} = (m_2 R^2/2) + m_2 a_2^2$, where $R = l/4$ is the disc radius and $m_2 a_2^2$ is the addition of parallel axis transfer. A distance $\mathrm{OC}_2 = a_2$ is equal to $l(2/3+1/4) = l(11/12)$. Therefore,

$$J_{z2} = \frac{m_2 \ell^2}{2}\left(\frac{1}{16}\right)^2 + m_2 \ell^2 \left(\frac{11}{12}\right)^2 = m_2 l^2 (0.0312 + 0.840) = 0.871 m_2 l^2.$$

Summing up the results for two parts, we arrive at $J_z = 0.111m_1l^2 + 0.871m_2l^2 = (0.111m_1+0.871m_2)l^2 = 0.547m_1l^2 = 0.547$ kgm^2. (We used here the condition that $m_2 = 0.5m_1$).

EXAMPLE E1.29

How many times does the density of a rod in the laboratory reference system (system K) change if its speed relative to this system equals $0.8c$ (c – speed of light in vacuum).

Solution: It is clear that

$$\rho = \frac{m}{\ell S} = \frac{m_0}{\sqrt{1-\beta^2}} \times \frac{1}{\ell_0\sqrt{1-\beta^2} \times S} = \frac{m_0}{\ell_0 S} \times \frac{1}{1-\beta^2}$$

Therefore

$$\frac{\rho}{\rho_0} = \frac{1}{1-\beta^2}.$$

Executing calculations we arrive at

$$\frac{\rho}{\rho_0} = \frac{1}{1-\beta^2} = \frac{1}{1-0.8^2} = \frac{1}{0.36} = 2.78$$

3. Dilation of time. Let the inertial system of coordinates K' move regarding another inertial coordinate system K with a speed $\mathbf{u}(u,0,0)$. If in a moving system K' at the origin two events A' and B' with a time interval $\Delta t'$ take place the observer in motionless system K will find that the time interval between these events Δt is shorter than $\Delta t'$:

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}} = \gamma\,\Delta \qquad (1.6.5)$$

That is, from the point of view of the motionless observer time in the moving system flows slower. For example, for $u = 0.5c$ the interval $\Delta t' = 1$ sec will correspond to an interval $\Delta t = 1.15$ sec.

As time dilation in moving systems is a property of time, not only the moving watch but all physical processes (including the ratio of chemical reactions) is taking place as well. This means that the ageing of organisms also slows down. However, the real speed of a spacecraft is still much less than the speed of light and the effect of dilation on the ageing of an astronaut is very small.

4. A relativity of simultaneity. If two events A and B occur in moving inertial system K' at different points of space at the same instant of time $t'_A = t'_B$, for example, in points with

$x' = 0.5a$ and $x'' = -0.5a$, the same events are not simultaneous for the observer in a motionless inertial system K:

$$\Delta t = \Delta t' = \frac{ua}{c^2}\frac{1}{\sqrt{1-\beta^2}} \tag{1.6.6}$$

The effect is very small: at $a = 1000$ km calculation gives $\Delta t = 0.023$ sec. In SRT the concept of simultaneity is meaningful for events in one and the same coordinate system only.

5. Relativistic speed and acceleration. According to Newton's physics, the constant force **F** acting on some constant mass m accelerates it. If $t = 0$, $\upsilon_0 = 0$, $m = m_0$, one can find speed $\boldsymbol{\upsilon}(t)$ and acceleration $\boldsymbol{a}(t)$ of a mass m. The expression for **F** according to Newton's second law has in SRT the form:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left[\frac{m_0\boldsymbol{\upsilon}}{\sqrt{1-\beta^2}}\right] = \frac{m_0\boldsymbol{a}}{(1-\beta^2)^{3/2}} \tag{1.6.7}$$

It can be seen from eq. (1.6.7) that it is impossible to accelerate a body with a non-zero resting mass by a finite force **F** to a speed equal to the light speed in vacuum. In order to explain this circumstance Einstein had to introduce the speed dependence of the particle mass (1.6.2).

From the expression (1.6.7) we can obtain:

$$\upsilon(t) = \frac{Ft}{m_0}\frac{1}{\sqrt{1+\left(\dfrac{Ft}{m_0c}\right)^2}}, \quad a(t) = \frac{F}{m_0}\frac{1}{\sqrt{\left[1+\left(\dfrac{Ft}{m_oc}\right)^2\right]^3}}. \tag{1.6.8}$$

In the motion beginning for small time intervals the speed is still small and corresponds to classic mechanics: $\upsilon(t) = Ft/m_0$, $a(t) = F/m_0$. If the time increases ($t \to \infty$), at constant acting force the speed υ asymptotically approaches the light speed value c and the acceleration decreases to zero:

$$\upsilon = c\left[1-\frac{1}{2}\left(\frac{m_0c}{Ft}\right)^2\right] \to c \text{ at } a = \left(\frac{m_0}{F}\right)^{1/2}\left(\frac{c}{t}\right)^{3/2} \to 0.$$

EXAMPLE E1.30

Determine the relativistic momentum of an electron and its relativistic kinetic energy if the electron is moving at a speed of $0.9c$.

Solution: The relativistic momentum can be determined according to equation $p = m_0c\ (\beta/\sqrt{1-\beta^2})$. Calculation gives $p = 5.6 \times 10^{-22}$ kgm/sec. The kinetic energy is determined as the difference between the total energy and the resting energy: $K = E - E_0$, where $E = mc^2$ and $E_0 = m_0c^2$. We then obtain

$$K = \frac{m_0c^2}{\sqrt{1-\beta^2}} - m_0c^2 \text{ and finally } K = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right).$$

Substituting all values we arrive at $K = 1.06 \times 10^{-13}$J. Sometimes the particle energy is expressed in terms of energy: the electron energy at rest is $m_0c^2 = 0.51$ MeV. Then $K = 0.66$ MeV.

6. Relativistic momentum. As with many other properties the relativistic momentum in STR is

$$p = \gamma m\upsilon,\ p = \frac{m_0\upsilon}{\sqrt{1-\beta^2}} = m_0c\frac{\beta}{\sqrt{1-\beta^2}}. \tag{1.6.9}$$

Unlike the classical definition, the relativistic expression permits the momentum p to approach infinitely large values as the particle speed υ approaches the speed of light.

7. Relativistic energy. The relativistic expression of the total energy E is

$$E = mc^2 = m_0c^2 + K; \tag{1.6.10}$$

$E_0 = m_0c^2$ is the total energy at rest. The kinetic energy is

$$K = E_0\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right). \tag{1.6.11}$$

The total energy E is therefore expressed as follows:

$$E = \gamma mc^2 = mc^2 + K \tag{1.6.12}$$

8. Relation of energy and momentum. From the above relations it follows that

$$E^2 = (pc)^2 + (mc^2)^2, \tag{1.6.13}$$

and

$$pc^2 = K(K + 2E_0). \tag{1.6.14}$$

Equation (1.6.10) allows a different treatment. There is a treatment that explains this equation as a measure of possible energy release confined in mass m. From the other point of view there are no equivalence of "mass" and "energy" in such a simplified statement. We can explain this expression as a kind of energy conservation law: at certain conditions a body can release some amount of energy in the form of γ-radiation which, being absorbed by other atoms, increases its kinetic energy, i.e., heating it; the energy of γ-quanta initiates the heat of the environment. A simple calculation shows that the general law of energy conservation is valid in this case too; however part of the inert mass transforms into the γ-ray's mass, i.e., into the mass of particles which do not possess the resting mass.

Both explanations are nearly equivalent; however, the latter accepts the physical meaning but not a scholastic statement on energy and mass equivalence.

From the equations presented it follows also that the photon energy $\varepsilon = h\upsilon$ corresponds to its mass $m = h\upsilon/c^2$.

It should be noted that in 1911 Einstein had expanded his theoretical consideration of noninertial systems and had suggested the *general relativistic theory of gravitation*. On the basis of this theory Einstein postulated the principle of equivalence: *the action of a gravitational field is equivalent to the action of accelerated motion of a system.* Corresponding mathematical expressions can be interpreted that any mass perturbs the environmental space; therefore all bodies will move on the trajectories curved in a vicinity of the disturbing mass while approaching it.

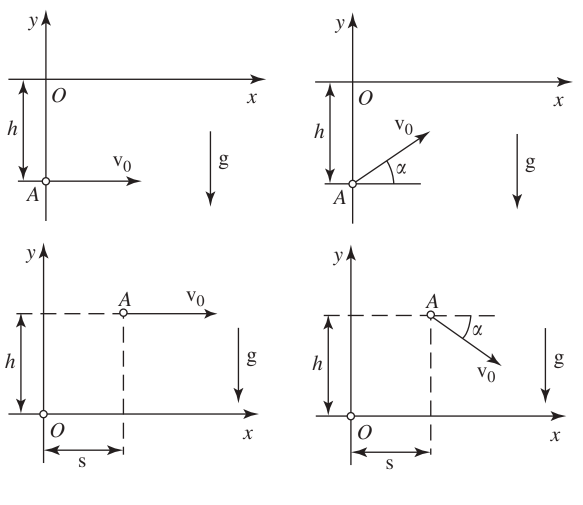
Obviously, all relativistic expressions transform into classic ones at speeds that are small in comparison with the speed of light in vacuum. Therefore the principles presented by Einstein do not contradict the general statements of the Galileo relativity. Chemists only meet the relativistic approach occasionally, e.g., energy of inner electrons of heavy atoms, some details of physical methods of investigations and some others.

For physical objects and reference frames moving with speeds $\upsilon \ll c$ or $u \ll c$, Einstein's theory led to the results of classical nonrelativistic theory: Lorentz transformations changed into Galileo's transforms and the Einstein relativity principle into Galileo's relativity principle.

PROBLEMS/TASKS

1.1. Two direct roads cross at a corner $\alpha = 60°$. From the crossroads two cars start simultaneously along these roads with speed $v_1 = 60$km/h and $v_2 = 80$ km/h. Determine the rate at which the cars move away from each other. Consider two variants.

1.2. For the four cases presented in Figure T1.2 calculate: (1) kinematical equations of movement $x(t)$ and $y(t)$; and (2) the trajectory equation y(x).



1.3. A body covers the first half of its journey at $t_1 = 2$ sec and the second part at $t_1 = 8$ sec. Determine the body's average speed $\langle \upsilon \rangle$ if the distance travelled is $s = 20$ m.
1.4. From what height (H) has a body fallen if it moves the last meter of its drop ($s = 1$ m) in time $t = 0.1$ sec?
1.5. A car covers three quarters of its journey at $\upsilon_1 = 60$ km/h and the remainder at a speed of $\upsilon_2 = 80$ km/h. What is the average speed $\langle \upsilon \rangle$ for the whole journey?
1.6. An MP moves along a circle with a radius $R = 4$ m. The initial point speed is $\upsilon_0 = 3$ m/sec, tangential acceleration $a_\tau = 1$ m/sec. For time interval $t = 2$ sec determine: (1) the distance travelled by the MP; (2) the displacement modulus $|\Delta\mathbf{r}|$; (3). the average speed $\langle \upsilon \rangle$, (4) the modulo of an average velocity $|\langle \boldsymbol{\upsilon} \rangle|$.
1.7. Determine the relationship of the free fall acceleration g with a distance from the Earth's center r. Assume that the earth density ρ is a constant independent of the earth point. Draw a graph $g(r)$. Assume the earth's radius R to be constant.
1.8. A thin stick with a length of $l = 25$ cm staying vertically on a horizontal surface begins to fall. Determine the angular velocity ω and linear speed v of (1) the middle point of stick; and (2) the upper end of the stick. The friction is so great that the stick's lower end does not slip.
1.9. A stone is thrown upwards at an initial speed of $\upsilon_0 = 20$ m/sec. In $\tau = 1$ sec another stone is thrown in the same direction at the same speed. At what height h do they meet?
1.10. The movement of an MP is set by equation $\mathbf{r}(t) = A(\mathbf{i}\cos\omega t + \mathbf{j}\sin\omega t)$ where $A = 0.5$ m/sec, $\omega = 5$ rad/sec. Draw the MP's trajectory. Determine the speed modulo of $|\mathbf{v}|$ and $|\boldsymbol{a}_n|$.
1.11. An aircraft flying at a height $H = 2940$ m at a speed of $v = 360$ km/h has to reset a bomb. At what time (t) before passing above the target and at what distance (s) from it should the plane reset the bomb to strike the target? Neglect any air resistance.
1.12. A projectile is fired at an angle $\alpha = 30°$ to the horizon. It twice reached the same height at $t_1 = 10$ sec and $t_2 = 50$ sec after the shot. Determine the initial speed v_0 and height h.
1.13. A tank $l = 4$ m in length filled with water moves with acceleration $a = 0.5$ m/sec^2. Find the difference of the water levels (h) in the front and the rear of the tank.
1.14. A helicopter with a mass $m = 3.5$ T and propeller vane length $d = 18$ m hangs motionless in the air. Determine the velocity v with which the propeller throws down the air-blast. Assume that the air-blast diameter is equal to the rotor diameter.
1.15. Initially at rest, a disc with a radius $r = 10$ cm begins to revolve with constant angular acceleration $\varepsilon = 0.5$ rad/sec^2. Find the tangent (a_τ), normal (a_n) and total (a) acceleration of the points on the wheel crown at the end of 2 sec of movement.
1.16. A spacecraft of mass $m = 3500$ kg begins to reorient in space. The exhausted gas speed is $\upsilon = 800$ m/sec; fuel consumption is $Q_m = 0.2$ kg/sec. Find the reactive trust (R) and acceleration of the craft.
1.17. A rocket of mass $M = 6$ T is launched upwards. The thrust of the engine is $F = 500$ kN. Determine the rocket's acceleration (a) and the tension force of a free hanging cable (T) in a section one quarter of the total cable length distant from the fixation point.
1.18. A disc-shaped wheel of mass $m_1 = 48$ kg and radius $R = 40$ cm can rotate freely around a horizontal axis. One end of a thin nonstretched rope is fixed to the rim of the wheel. A weight $m_2 = 0.2$ kg is fastened to the other end of the rope. The weight

is lifted to $h = 2$ m and then let drop. The rope tightens and the wheel begins to rotate. Calculate the angular velocity ω of the wheel.

1.19. Two skaters of mass $m_1 = 80$ kg and $m_2 = 50$ kg hold two ends of a stretched rope and stay motionless on the ice. One of them begins to extend the rope at a speed $\upsilon = 1$ m/sec. Find the speeds u_1 and u_2 at which the skaters move on the ice. Ignore any friction force.

1.20. Find the distance (x) of the CM of the system earth–moon ($M_{earth} = 6 \times 10^{24}$ kg, $m_{moon} = 7.33 \times 10^{22}$ kg, $d = 3.84 \times 10^8$ m) from the center of the earth.

1.21. A molecule decays into two atoms. The mass of one piece is $N = 3$ times bigger than the other. Neglecting the initial molecule kinetic energy, determine their kinetic energies K_1 and K_2 if the total kinetic energy is $K = 0.032$ nJ.

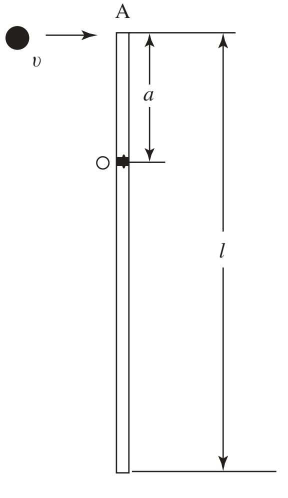
1.22. A projectile with a mass $m = 10$ kg has at the upper point of its motion a velocity of $\upsilon = 200$ m/sec. It explodes at this point into two parts. The smallest part $m_1 = 3$ kg acquires a velocity $u_1 = 400$ m/sec in the previous direction. Find the velocity of the second part after the explosion.

1.23. The movement of an MP along a curvilinear trajectory is given by equations $x = A_1 t^3$ and $y = A_2\, t$, where $A_1 = 1$ m/sec^3 $A_2 = 2$ m/sec. Find the MP trajectory equation, MP speed υ and the total acceleration a at the moment $t = 0.8$ sec.

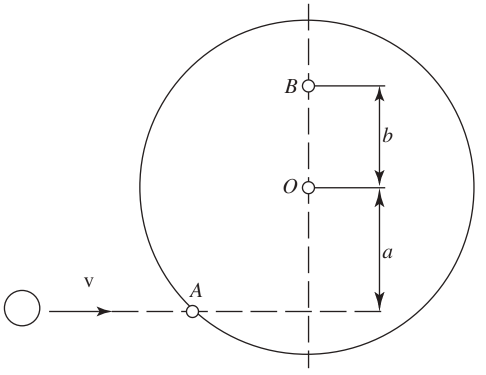
1.24. A bullet of mass $m = 10$ g moves horizontally at a speed $v = 800$ m/sec rotating around a longitudinal axis with a frequency $n = 3000$ sec^{-1}. Assuming the bullet to be a cylinder with diameter $d = 8$ mm determine the bullet's total kinetic energy K.

1.25. Two balls of masses $m_1 = 2$ kg and $m_2 = 3$ kg move at velocities $\upsilon_1 = 8$ m/sec and $\upsilon_2 = 4$ m/sec. Find the change of the inner energy ΔU after their inelastic collision in the following two cases: (1) when the smallest ball overtakes the other; and (2) when the balls move towards each other.

1.26. A uniform thin stick of mass $m_1 = 0.2$ kg and length $l = 1$ m can oscillate freely around an axis passing a point O (Figure T1.26). A sticky ball with a mass $m_2 = 10$ g, moves horizontally at a speed of $\upsilon = 10$ m/sec and get stuck at point A on the stick. Determine both angular ω and linear velocity u of the lower point of the stick in the initial instant of time. Carry out the calculation for the (1) $a = l/2$; (2) $l/3$; and (3) $l/4$.



1.27. A uniform disc of mass $m_1 = 0.2$ kg and radius $R = 20$ cm can rotate freely around a horizontal axis Oz, which passes through point O. Onto a point A on the disc rim a small sticky ball of mass $m_2 = 10$ g moving horizontally at a velocity $\upsilon = 10$ m/sec, strikes the disc and cleaves to it. Find the disc's angular velocity ω and linear velocity u of point B at the instant of the blow provided the ball hits the rod at point A on the disc rim. Carry out a calculation for (1) $a = b = R$; (2) $a = R/2$, $b = R$; (3) $a = 2R/3$, $b = R/2$; and (4) $a = R/3$, $b = 2R/3$ (Figure T1.27).



1.28. A block is suspended to spring scales. A cord is crossed over the block. Masses $m_1 = 1.5$ kg and $m_2 = 3$ kg are attached to the ends of the cord. Find the scale readings at which the weights will move. Ignore block mass, the cord and the friction in the block.

1.29. A hammer of mass $m = 1000$ kg falls from a height $h = 2$ m onto an anvil. Impact duration is $\tau = 0.01$ sec. Determine the average value of the impact $\langle F \rangle$ force.

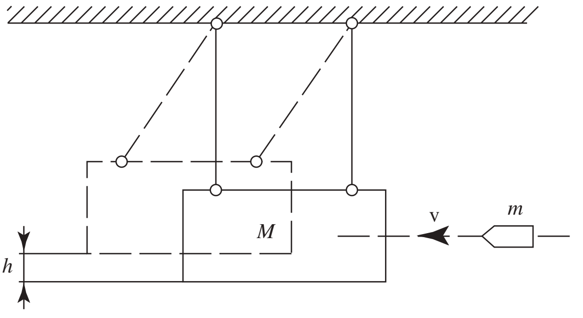
1.30. A ball of mass $m = 300$ g collides with a wall and rebounds from it. Find the momentum p_1 obtained by the wall if, in the last moment before the impact, the ball has a velocity of $\upsilon_0 = 10$ m/sec directed at an angle of $\alpha = 30°$ to the wall's surface. The impact is perfectly elastic.

1.31. A rocket of mass $m_c = 2$ T leaves surface of the moon. After time τ it reaches the first space (moon) velocity $\upsilon_1 = 1.68$ km/sec. Determine the mass fuel consumption μ if the nozzle is 4 km/h. Ignore the gravitation of the moon.

1.32. The ratio of a rocket's fuel mass to total starting rocket mass is ¾. Determine the velocity of the rocket after total consumption of the fuel if the fuel consumption from the nozzle u is 2 km/sec. Ignore the air resistance.

1.33. Determine the maximum part (w) of kinetic energy K_1 which a particle of mass $m_1 = 2 \times 10^{-22}$ g can transmit to particle $m_2 = 6 \times 10^{-22}$ g through an elastic collision. The second particle is at rest before the collision.

1.34. A bullet of mass $m = 10$ g moving at a speed $\upsilon = 600$ m/sec hits a ballistic pendulum of mass $M = 5$ kg and lodges in it. Determine the maximum height h of the pendulum's lift (Figure T1.34).



1.35. A ball of mass $m_1 = 2$ kg collides with another ball $m_2 = 8$ kg. The momentum p_1 of the first ball is 10 kg m/sec. The impact is direct and elastic. Determine just after collision: (1) the momentum of the first p_1 and of the second p_2 ball; (2) the change in momentum of the first ball Δp_1; (3) the kinetic energies of the first K_1' and the second K_2' ball; (4) the change in kinetic energy of the first ball $\Delta K_1'$; and (5) the portion w of the kinetic energy transferred from the first ball to the second.

1.36. A ball of mass $m_1 = 6$ kg collides with another, rested ball $m_2 = 4$ kg. The momentum p_1 of the first ball is 5 kg m/sec. The impact is direct and inelastic. Determine just after collision: (1) the momentum of the first p_1 and the second p_2 ball; (2) the change of momentum of the first ball Δp_1; (3) the kinetic energies of the first K_1' and the second K_2' ball; (4) the change in kinetic energy of the first ball ΔK_1; (5) the portion w_1 of the kinetic energy transferred from the first ball to the second, and the portion w_2 of the residual kinetic energy of the first ball; (6) the change of the inner energy ΔU of the balls; (7) the portion w of the kinetic energy of the first ball transferred into the inner energy of the balls.

1.37. The kinetic energy of a rotating wheel is $K = 1$ kJ. Under the action of a constant retarding torque it begins to rotate, uniformly retarded. After making $N = 80$ revolutions it stops. Find the retarding torque M.

ANSWERS

1.1. $v' = 122$ km/h; $v'' = 72$ km/h.

1.2. (a) $x = \upsilon_0 t$, $y = -h-(gt^2/2)$; $y = -h - gx^2/2\upsilon_0^2$.

(b) $x = \upsilon_0 t\cos\alpha$, $y = -h + \alpha_0 t \sin\alpha - (gt2/2)$; $y = -h + x tg\alpha - \dfrac{gx^2}{2\upsilon_0\cos^2\alpha}$.

(c) $x = s + \upsilon_0 t$, $y = h-(gt^2/2)$; $y = h \dfrac{g(x-s)^2}{2\upsilon_0}$.

(d) $x = s + \upsilon_0 t\cos\alpha$, $y = h - \upsilon_0 t\sin\alpha - (gt^2/2)$; $y = h - (x-s)tg\alpha - \dfrac{g(x-s)^2}{2\upsilon_0^2\cos^2\alpha}$.

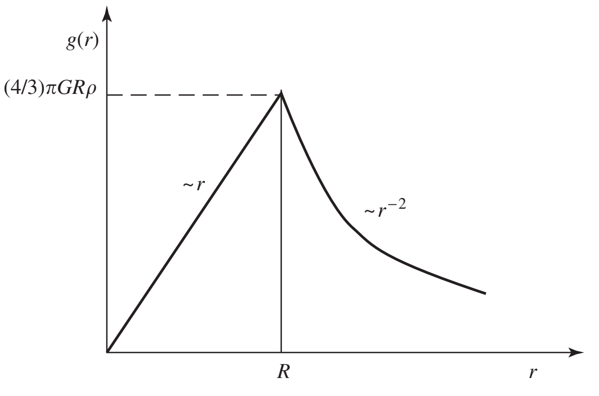
1.3. $\langle \upsilon \rangle = s/(t_1 + t_2) = 2$ m/sec.

1.4. $H = (2s + gt^2)/(8gt^2) = 5.61$ m.

1.5. $\langle \upsilon \rangle = 64$ km/h.

1.6. (1) $s = 8$ m; (2) $|\Delta\mathbf{r}| = 6.73$ m; (3) $\langle \upsilon \rangle = 4$ m/sec, (4) $|\langle \upsilon \rangle| = 3.36$ m/sec.

1.7. $g(r) = (4/3)\pi G\rho r$ at $r \ll R$ and $g(r) = 4\pi GR^3\rho/(3r^2)$ at $r \geq R$ (Figure T1.7).



1.8. (1) $\omega_1 = 14$ rad/sec, $u_1 = 1.05$ m/sec; (2) $\omega_2 = 14$ rad/sec, $u_2 = 2.1$ m/sec.

1.9. $h = 19.2$ m.

1.10. $|\mathbf{v}| = 2.5$ m/sec, $|\boldsymbol{a}_n| = 12.5$ m/sec^2.

1.11. $t = 24.5$ sec, $s = 2.45$ km.

1.12. $\upsilon_0 = \dfrac{g(t_1+t_2)}{2\sin\alpha} = 588$ m/sec; h $= g\, t_1 t_2 = 2.45$ km.

1.13. $h = 20.4$ cm.

1.14. $v = (1/d)[(4m\rho)/\pi\rho]^{1/2} = 10.2$ m/sec (ρ is the air density).

1.15. $a_\tau = 5$ cm/sec^2, $a_n = 10$ cm/sec^2 , $|\mathbf{a}| = 11$ cm/sec^2.

1.16. $R = -Q_m v = -160\,N$, $a = -Q_m v/m = 4.57$ cm/sec^2.

1.17. $a_0 = \dfrac{F}{M+m} - g = 73.5$ m/sec^2; $T = \dfrac{3}{4}\,\dfrac{m}{M+m}\,F = 625$ N.

1.18. $\omega = \dfrac{2m_2\sqrt{2gh}}{(m_1 + m_2)R} = 0.129$ rad/sec.

1.19. $u_1 = 0.385$ m/sec; $u_2 = -0.615$ m/sec.

1.20. $x = 4.69 \times 10^6$ m (radius of earth is $R = 6.4 \times 10^6$ m).

1.21. $K_1 = nK/(n + 1) = 24$ pJ.

1.22. $u_2 = 114$ m/sec.

1.23. $y^3 - 8x = 0$; $\upsilon = 2.77$ m/sec, $a = 4.8$ m/sec^2.

1.24. $K = (m/4)(2v^2 + \pi^2 n^2 d^2) = 3.21/$kJ.

1.25. $\Delta U = \dfrac{m_1 m_2\,(\upsilon_1 \pm \upsilon_2)^2}{2(m_1+m_2)} =$ 9.6 J and 86.4 J.

1.26. (1) $\omega_1 = \dfrac{6m_2 v}{(3m_2 + m_1)\ell} = 2.61$ rad/sec. $u_1 = \dfrac{3m_2 v}{(3m_2 + m_1)} = 1.30$ m/sec.

(2) $\omega_2 = \dfrac{3m_2 v}{(m_2 + m_1)\ell} = 1.43$ rad/sec. $u_2 = \dfrac{2m_2 v}{(m_2 + m_1)} = 0.952$ m/sec.

(3) $\omega_3 = \dfrac{4m_2 v}{(m_2 + (7/3)m_1)\ell} = 0.839$ rad/sec. $u_3 = \dfrac{3m_2 v}{m_2 + (7/3)m_1} = 0.629$ m/sec.

1.27. (1) $\omega_1 = 4.55$ rad/sec, $u_1 = 0.909$ m/sec.

(2) $\omega_2 = 2.27$ rad/sec, $u_2 = 0.454$ m/sec.

(3) $\omega_3 = 3.03$ rad/sec, $u_3 = 0.303$ m/sec.

(4) $\omega_4 = 1.52$ rad/sec, $u_4 = 0.202$ m/sec.

1.28. $F = \dfrac{4m_1 m_2}{m_1 + m_2} g.$

1.29. $\langle F \rangle = (m/\tau)\sqrt{2gh} = 626$ kN.

1.30. $p_1 = 2m\upsilon_0 \sin\alpha = 3$ N sec.

1.31. $\mu = \dfrac{m_c}{\tau}\left(1 - \exp\dfrac{\upsilon_1}{u}\right) = 1.68$ km/sec.

1.32. $\upsilon = u \ln \dfrac{1}{1 - \eta} = 2.77$ km/sec.

1.33. $w = \dfrac{4m_1m_2}{(m_1+m_2)^2} = 0.75.$

1.34. $h = m^2\upsilon^2/[2g(m + M)^2] = 7.32$ cm.

1.35. (1) $p_1' = \dfrac{m_1 - m_2}{m_1+m_2} = -6$ kg.m/sec, $p_2' = \dfrac{2m_2}{m_1 + m_2} = 16$ kg m/sec.

(2) $\Delta p_1 = -p_2' = -16$ kg m/sec.

(3) $K'_1 = \dfrac{p_1^2}{2m}\ \dfrac{m_1-m_2}{m_1+m_2} = 9$ J, $K_2' = p_1^2\ \dfrac{2m_2}{m_1 + m_2} = 16$ J.

(4) $|\Delta K'_1| = K_2' = 16$ J.

(5) $w = \dfrac{\Delta K'_1}{K'_1} = \dfrac{4m_1m_2}{(m_1 + m_2)^2} = 0.64.$

1.36. (1) $p_1' = \dfrac{m_1 p_2}{m_1 + m_2} = 3$ kg m/sec, $p_2' = \dfrac{m_2 p_1}{m_1 + m_2} = 2$ kg m/sec.

(2) $\Delta p_1' = p_2' = -\ 2$ kg m/sec.

(3) $K'_1 = \dfrac{m_1 p_1^2}{2(m_1 + m_2)^2} = 0.75$ J, $K_2' = \dfrac{m_2 p_1^2}{2(m_1 + m_2)^2} = 0.5$ J.

(4) $\Delta K_1 = \dfrac{m_2(2m_1+m_2)\, p_1^2}{2(m_1 + m_2)^2 m_1} = 1.33$ J.

(5) $w_1 = \dfrac{K'_2}{K'_1} = \dfrac{m_1m_2}{(m_1 + m_2)^2} = 0.24$, $w_2 = \dfrac{K'_1}{K'_1} = \dfrac{m_1^2}{(m_1 + m_2)^2} = 0.36.$

(6) $\Delta U = \dfrac{m_2 p_1^2}{2m_1(m_1 + m_2)} = 0.833$ J.

(7) $w = \dfrac{\Delta U}{K_1} = \dfrac{m_2}{m_1 + m_2} = 0.4.$

1.37. $M = \dfrac{K}{2\pi N} = 1.99$ Nm.