– 2 –

Oscillations and Waves

2.1 DEFINITIONS

Along with translational and rotational motions, oscillations play an important role in the macro- and micro-world. We can distinguish both chaotic and periodic oscillations. Periodic oscillations are characterized by the time of repetitions: through certain periods of time a system passes one and the same position, running in one and the same direction. An example is given in Figure 2.1, which shows the man's cardiogram in the form of a graph of electrical signals of the heart's oscillations. It is possible to select the period of time of one complete oscillation T, thus presenting a periodical process.

However, periodicity is not a unique feature of oscillations: a rotational movement can also be characterized by periodicity. The presence of an equilibrium position is a second particularity of oscillatory motion (rotation is characterized only by the so-called indifferent balance: a well-balanced raised car wheel, being rotated, stops in any position with equal probability). Thirdly, any deflection force tends to return an oscillatory system to its initial equilibrium position; i.e., the restoring force is always directed to the position of equilibrium. The presence of all these three signs distinguishes oscillations from other types of motion.

Consider several specific examples of oscillatory motion. Clamp one end of a steel straightedge in a vice and let the other end move freely. The returning force will try to draw the free end of the straightedge toward the equilibrium position. Passing by this position, the straightedge will have a certain velocity and a certain stock of kinetic energy. The inertia forces will not permit the straightedge to stop in the position of equilibrium and will work against the internal elastic force to decrease the kinetic energy. This will bring about an increase in potential energy. When the kinetic energy is completely exhausted, the potential energy will reach a maximum. The forces of elasticity will also reach a maximum and will be directed to the position of equilibrium. All these features were described in detail in Sections 1.3.5 (eq. (1.3.20)), 1.4.1 (eq. (1.4.9)) and 1.5.4 (Figure 1.33) in the language of potential curves. Oscillation will repeat until the total mechanical system energy disappears into the surrounding space.

Another well-known example is that of pendulum oscillation. We have seen this example in Chapter 1 (refer to Figure 1.33) and will often come across it in different aspects.

Figure 2.1 An example of a periodic oscillation process: the cardiogram of a human being.

Oscillation can be not just mechanical. So, for instance, one can consider the oscillations of an electric current in an oscillatory circuit or a magnetic field strength in a dynamo, etc. These can be described by an equation similar to the one that defines mechanical displacements from a position of equilibrium. In spite of this fact, we will mostly analyze mechanical oscillations, keeping in mind their applicability to other types of oscillation.

The time in which a system accomplishes one complete oscillation is called the *oscillation period* T. A value inverse to the period expressing a number of full oscillations in the time unit is referred to as the oscillation frequency ν (i.e., completely identical to rotation)

$$\nu = \frac{1}{T}. \tag{2.1.1}$$

Let us begin the analysis of oscillatory processes with the simplest case of a one-dimensional oscillation, i.e., of a system with one degree of freedom. The *degree of freedom* described a number of independent variables required for the complete description of the positions of all parts of a given system. If, for instance, pendulum oscillations are limited by one plane and the thread of the pendulum is not stretched, it is sufficient to assign either an angle of deflection of the thread from a vertical line or any other value of displacements from the position of equilibrium. Each of them is enough to define the position of the pendulum in full. In this case the system considered possesses one degree of freedom. The same pendulum, if it can occupy any position on a section of a spherical surface, possesses two degrees of freedom.

2.2 KINEMATICS OF HARMONIC OSCILLATIONS

From the entire variety of periodic oscillations we will select first of all the so-called harmonic oscillations. Interest in harmonic oscillations is due to the following reasons: firstly, it is relatively simple to describe harmonic oscillations mathematically, and, secondly, any periodic oscillations can be presented as a superposition of harmonic oscillations. This latter circumstance is very important, and we will return to it in Section 2.3.2.

Harmonic oscillations are an abstraction, since they have to continue for an infinitely long period ($-\infty < t < +\infty$), according to certain laws, without any changes, which is not the case in the real macroscopic world. We consider harmonic oscillations to be one of the common physical models.

Oscillations are referred to as harmonic if the changes in time of some physical values occur under the sine or cosine law

$$\begin{aligned} \xi(t) &= A\sin(\omega t + \varphi_1) \\ \xi(t) &= A\cos(\omega t + \varphi_2), \end{aligned} \tag{2.2.1}$$

where $\xi(t)$ is the time dependence of a displacement. By the term "displacement" we understand the magnitude of the displacement of any physical value at a given time instance t. In particular, when considering the simplest mechanical oscillations under displacement we shall understand a deflection of varying point from the position of equilibrium. The maximum value of the displacement is called the oscillation amplitude A; it is always taken as a positive value. The expression in parentheses is the phase of oscillations, φ is the initial phase of oscillations (i.e., the phase of oscillation at moment $t = 0$) and ω is the angular (or circular) oscillation frequency.

The choice of sine or cosine for describing the harmonic oscillations as well as the initial phase is rather arbitrary and is chosen for convenience. A cosine form is preferable in most cases as will be seen later. The transition from one form to another is easily realized by a corresponding change of the initial phase. So, for instance, if harmonic oscillation is described by an expression $\xi(t) = \mathrm{A}\sin(\omega t + \varphi_1)$, it is also possible to present it in the form $\xi(t) = \mathrm{A}\cos(\omega t + \varphi_1 + \pi/2) = \mathrm{A}\cos(\omega t + \varphi_2)$, where $\varphi_2 = \varphi_1 + \pi/2$.

Let us make an interconnection between the angular frequency ω, the frequency v and the period T. Sine and cosine are periodical functions with period 2π. This means that after the time interval T a system returns to its initial state and the phase is changed to 2π, i.e., $[\omega(t + T)\varphi] - [\omega t + \varphi] = 2\pi$. Thereby, the angular frequency is connected with period T and frequency v by the expressions

$$\omega = \frac{2\pi}{T} \tag{2.2.2}$$

or

$$\omega = 2\pi v. \tag{2.2.3}$$

Let us examine the change in the speed and acceleration of an oscillating point. If the displacement of an oscillating MP is expressed by eq. (2.2.1), its velocity and acceleration can be found by the first and second time derivative of displacement as

$$\upsilon(t) = \frac{d\xi(t)}{dt} = \dot{\xi}(t) = -A\omega\sin(\omega t + \varphi), \tag{2.2.4}$$

$$a(t) = \frac{d^2\xi(t)}{dt^2} = \ddot{\xi}(t) = -A\omega^2\cos(\omega t + \varphi). \tag{2.2.5}$$

Comparing these expressions, we arrive at

$$a(t) = -\omega^2 \xi(t). \tag{2.2.6}$$

This means that the acceleration of harmonic oscillations is always proportional to the displacement and is in the opposite direction. A graphical diagram of displacement (a), velocity (b) and acceleration (c) depending on the phase ωt is presented in Figure 2.2. It can be seen from these curves that the phase of velocity differs from that of displacement by $\pi/2$; however, the phase of acceleration shifted by π in comparison to the displacement phase. This can be summarized as follows: the velocity phase leaves behind the displacement phase by $\pi/2$; however the acceleration is in antiphase to the displacement. In other words, when the displacement is maximum ($\xi = A$) the velocity is equal to zero; however acceleration reaches its maximum value ($a_{\max} = \omega_0^2 A$). On the other hand, while the MP passes the equilibrium position the velocity is maximum ($\upsilon_{\max} = \omega_0 A$), and acceleration at this moment is zero.

It is instructive to consider the relationship between a simple harmonic motion along a line and uniform circular motion. In this respect, the harmonic oscillation can be presented by uniform rotation of the radius vector (or amplitude vector). Let us imagine a segment with a length numerically equal to the amplitude value A uniformly rotated around one of its ends (Figure 2.3) with an angular frequency ω. We denote φ as an angle with an abscissa axis ξ. At the instant of time t this angle is $\omega t + \varphi$. Projection of a point B onto the ξ-axis will increase in time and be described mathematically in just the same manner as the harmonic oscillation $\xi(t) = A\cos(\omega t + \varphi)$. Therefore, the radius vector accomplishes a rotational motion, whereas its projection on the ξ-axis oscillates according the harmonic law.

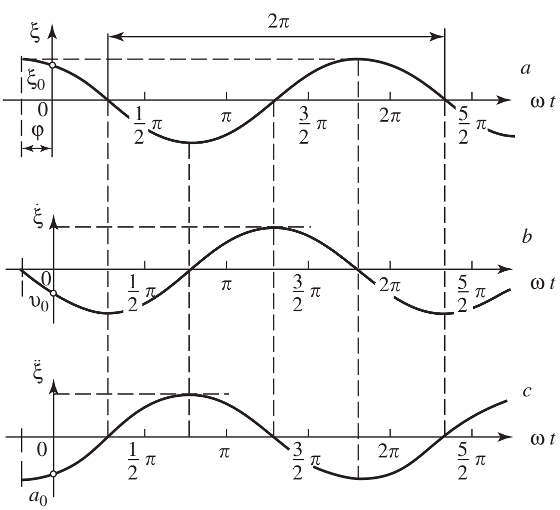


Figure 2.2 Kinematics of harmonic oscillations. Initial values of displacement, velocity and acceleration $\xi_0 = \xi(0)$, $\boldsymbol{\upsilon}_0 = \dot{\xi}(0)$ and $a_0 = \ddot{\xi}(0)$, respectively, are shown.

The angular velocity of rotation can be found by time derivation $[d(\omega t + \varphi)/dt] = \omega$. So, the harmonic oscillation can be formally presented by the rotating radius vector **A** with angular velocity ω, the phase being found by an angle of the radius vector OB with the ξ-axis. Here, the initial phase is an angle the radius vector forms with the ξ-axis in the instant of time $t = 0$.

By describing harmonic oscillation in this way, we can accept that the phase is a more exhausting characteristic of harmonic oscillations than displacement. This can be better seen in Figure 2.4. The advantage of phase to displacement is that the former uniquely

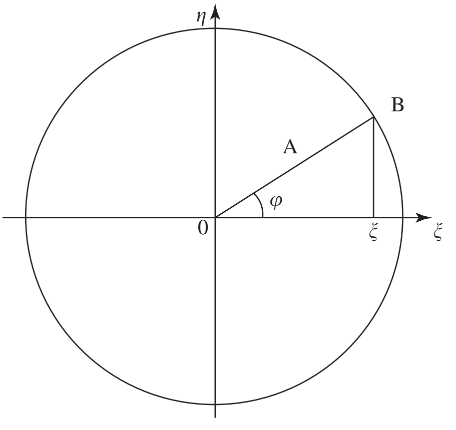


Figure 2.3 A vector diagram of harmonic oscillations.

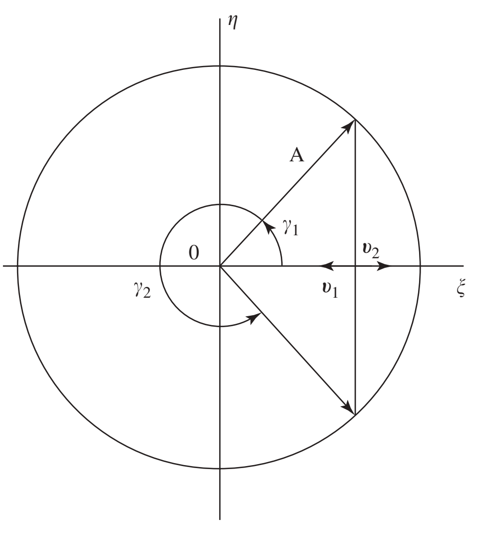


Figure 2.4 Ambiguity of oscillations with displacement: two phases γ_1 and γ_2 correspond to one and the same displacement ξ, whereas velocities $\boldsymbol{\upsilon}_1$ and $\boldsymbol{\upsilon}_2$ have opposite directions.

describes the oscillation, whereas in the latter case one should choose the real phase of two (two phases γ_1 and γ_2 lead to the same displacement; in the successive moments the oscillating points will scatter in different directions with velocities υ_1 and υ_2).

Besides comparing two harmonic oscillations with the same frequency but with different and unknown amplitudes, values of displacements cannot give a valuable result; whereas a knowledge of phases can provide a clear picture. For instance, if phases coincide, both harmonic oscillations reach maximum displacement simultaneously (oscillations are in-phase) and vice versa. When changing a phase by $2\pi n$ harmonic oscillation returns to its initial state ($n = 0, 1, 2, ...$). If phases differ by π, 3π, 5π, etc. (in the general case $(2n + 1)\pi$), such oscillations are accepted to be in opposite phases or in antiphase.

Figure 2.5 shows a vector diagram with displacement, velocity and acceleration. It can be seen that velocity is ahead of displacement by an angle $\pi/2$ and acceleration is antiphase to displacement.

Here, the vector diagram of the harmonic oscillation enables us to use representations of complex numbers. In some cases it allows us to avoid bulky trigonometric transformations and essentially simplifies mathematical calculations and physical view.

Let us consider a complex number plane. On the abscissa we shall put the real part of a complex number and on the ordinates its imaginary part (Figure 2.6). Then any complex number can be written as

$$Z = \xi + \mathrm{i}\eta. \tag{2.2.7}$$

Its modulus is equal to

$$A = |Z| = \sqrt{\xi^2 + \eta^2} \tag{2.2.8}$$

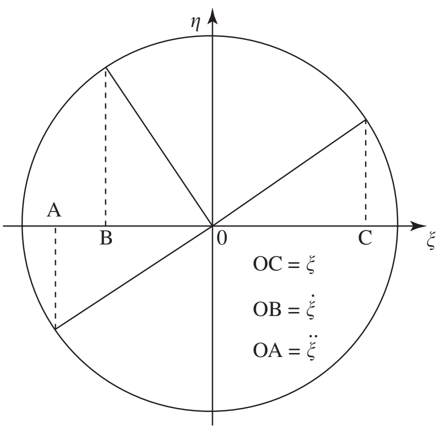


Figure 2.5 A vector diagram of displacements, velocities and acceleration. The frequency is chosen as unity.

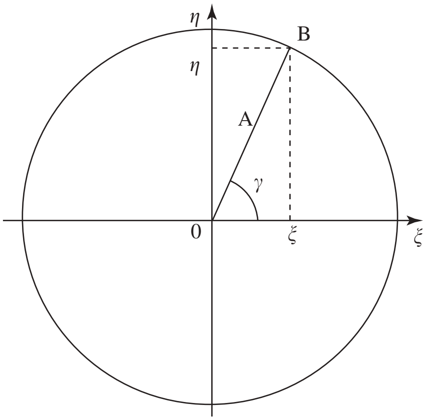


Figure 2.6 Graph of a complex number $Z = \xi + i\eta$.

Real (ξ) and imaginary (η) parts of complex number $\xi + i\eta$ can be expressed through the modulus A and argument γ by formulas

$$\xi = A\cos\gamma, \quad \eta = A\sin\gamma. \tag{2.2.9}$$

Therefore, any complex number can also be expressed as

$$Z = A(\cos\gamma + \mathrm{i}\sin\gamma). \tag{2.2.10}$$

At uniform rotation of a radius vector **A** as an argument we can take $\gamma = \omega t + \varphi$. Then the real part (Re) of a complex number Z will change under the harmonic law:

$$\mathrm{Re}(Z) = A[\cos(\omega t + \varphi)], \tag{2.2.11}$$

where the modulus of the complex number is equal to the amplitude of the harmonic oscillations represented by this complex number.

The Euler formula plays an important role in interpretation of oscillations. Accordingly, the real part of the complex number (written in exponential form)

$$Z = A\exp[\mathrm{i}(\omega t + \varphi)] \tag{2.2.12}$$

changes in time under the harmonic law:

$$\mathrm{Re}[A\exp\mathrm{i}(\omega t + \varphi)] = A\cos(\omega t + \varphi).$$

In all these cases, displacement ξ of harmonic oscillations can be presented as

$$\xi = \mathrm{Re}(Z) \tag{2.2.13}$$

Henceforth, we shall designate complex number Z by the same letter as displacement (ξ), meaning every time that harmonic oscillation is described by the real part of this complex number. Hence, the time dependence of displacement at harmonic oscillations can be written as follows:

$$\xi = A\cos(\omega t + \varphi). \tag{2.2.14}$$

This expression can be rewritten as

$$\xi = A\exp(\mathrm{i}\varphi)\exp(\mathrm{i}\omega t). \tag{2.2.15}$$

A exp(iφ) defines the length and direction of the radius vector (**A**) at the initial instant of time and is referred to as complex amplitude (which we shall designate a)

$$a = A\exp(\mathrm{i}\varphi). \tag{2.2.16}$$

Then harmonious oscillations can be written even easier:

$$\xi = a\exp(\mathrm{i}\omega t). \tag{2.2.17}$$

The sign in an exponent shows the direction of the radius vector **A** rotation. In physics, factor $\mathrm{e}^{-\mathrm{i}\varphi}$ sometimes stands for an operator of rotation. In fact, multiplication by this factor is equivalent to turning the vector **A** counterclockwise at an angle φ.

EXAMPLE E2.1

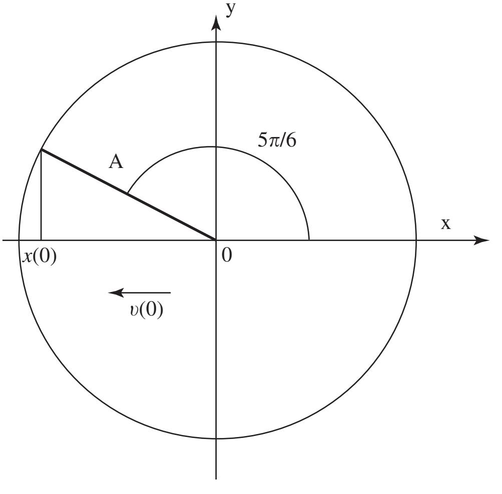
A small weight is suspended on a long, nonextendable string. Prior to oscillating, it was removed from the position of equilibrium to the utmost left-hand side and then set off. Write down the equation of oscillation and find the initial phase.

Solution: First of all, we have to choose the form of the answer (sin or cos, with their signs). Let it be cos. The sign depends on a positive direction of the axis chosen; let it be from left to right. If the weight was released at its negative utmost deviation the displacement at $t = 0$ should be $-A$. Therefore, the equation is $\xi(0) = A\cos(\omega t + \pi)$; π is just the initial phase. If we choose function (sin) the initial phase would be $3\pi/2$. It is expedient to draw the vector diagram for this case.

EXAMPLE E2.2

An MP oscillates with simple harmonic motion according to the equation $x(t) = A\cos(\omega t + \varphi)$, amplitude A being equal to 2 cm. Find the initial phase φ if $x(0) = -\sqrt{3}$ cm and $\dot{x}(0) < 0$. Draw a vector diagram for the zero instance of time ($t = 0$).

Solution: Express a displacement at $t = 0$ via initial phase: $x(0) = A\cos\varphi$. The initial phase is $\varphi = \arccos[x(0)/A]$ and further $\varphi = \arccos(-\sqrt{3}/2)$. Two angles correspond to these phases $\varphi_1 = (5\pi/6)$ and $\varphi_2 = (7\pi/6)$. To find for a certain phase we have to use the condition $\dot{x}(0)<0$. Keeping in mind that $\dot{x}(t) = -\omega A\sin(\omega t + \varphi)$ we can substitute numerical values φ_1 and φ_2 and find $\dot{x}_1(0) = -A\omega/2$ and $\dot{x}_2(0) = +A\omega/2$. Since $A > 0$ and $\omega > 0$, only the first value satisfies the condition $\dot{x}(0)<0$. Hence, the value sought is $\varphi_1 = (5\pi/6)$. The results can be seen in Figure E2.2.



2.3 SUMMATION OF OSCILLATIONS

2.3.1 Summation of codirectional oscillations

Let us begin with the simplest case: summation of two oscillations with the same frequencies:

$$\xi_1(t) = A_1\cos(\omega t + \varphi_1)$$

and

$$\xi_2(t) = A_2\cos(\omega t + \varphi_2).$$

The resulting displacement $\xi(t)$ can be found as an algebraic sum of oscillations $\xi(t) = \xi_1(t) + \xi_2(t) = A\cos(\omega t + \varphi)$, where A and φ are the amplitude and initial phase sought of the final oscillation. Such a summation can be realized both graphically and analytically, although the graphical method is more visual. Each oscillation in the same time instance (say, $t = 0$) can be presented as vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ (Figure 2.7), plotted from the abscissa under angles φ_1 and φ_2, correspondingly. Since the frequencies of both oscillations are the same, the mutual positions of both vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ remain unchanged during their rotation with the same angular velocity. Consequently, the total oscillation can be represented by vector $\mathbf{A}$, which is the vector sum of $\mathbf{A}_1$ and $\mathbf{A}_2$. This is the oscillation accomplished with the same cyclic frequency ω (the complex parallelogram is rotated with this angular velocity). The amplitude A can be determined according to the cosine theorem taking into account that instead of angle $180° - (\varphi_2 - \varphi_1)$, in the following expression, we use angle $\varphi_2 - \varphi_1$ that influences the sign in radicand

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)}. \tag{2.3.1}$$

Let us analyze the result. The phase difference of the summing oscillation remains constant; at any time instance it is the difference of the initial phases, i.e.,

$$\Delta\varphi = \left[(\omega t + \varphi_2) - (\omega t + \varphi_1)\right] = \varphi_2 - \varphi_1.$$

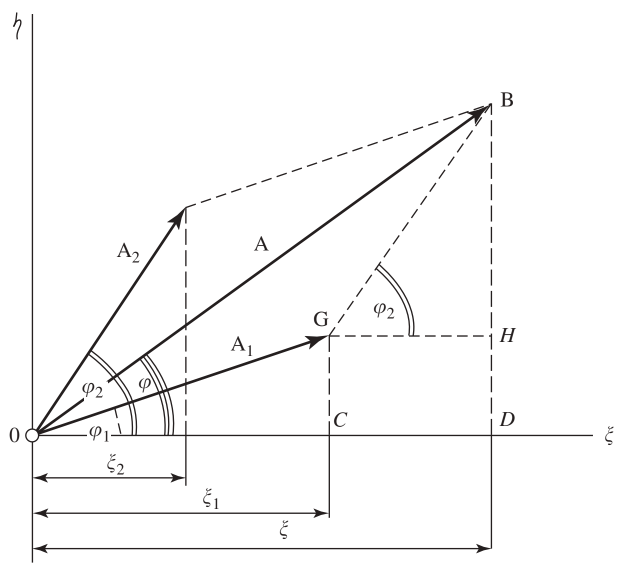


Figure 2.7 Summation of two codirectional oscillations with same frequency.

As can be seen from expression (2.3.1), the resulting amplitude depends on the phase difference of the summed oscillation. In particular, if the phase difference $\Delta\varphi$ satisfies the condition

$$\Delta\varphi = \pm 2\pi \mathrm{n}, \tag{2.3.2}$$

where $n = 0, 1, 2,...$, then $\cos \Delta\varphi = 1$, and the resulting amplitude will have the maximum value

$$A_{\max} = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2}$$

or

$$A_{\max} = A_1 + A_2. \tag{2.3.3}$$

If the phase difference is

$$\Delta\varphi = \pm(2n+1)\pi, \tag{2.3.4}$$

then $\cos \Delta\varphi = -1$ and the resulting amplitude will have the minimum value

$$A_{\min} = \sqrt{A_1^2 + A_2^2 - 2A_1 A_2}$$

or

$$A_{\min} = |A_1 - A_2|. \tag{2.3.5}$$

The modulus is used here because the amplitude must be positive.

It is clearly seen that oscillation will not take place at all in this case: having equal amplitudes and oscillating in antiphase, they have cancelled each other out.

Determine now an initial phase φ. From Figure 2.7, $\mathrm{tg}\, \varphi = (\mathrm{BD}/\mathrm{OD})$ can be derived; however, $\mathrm{BD} = \mathrm{A}_1 \sin \varphi_1 + \mathrm{A}_2 \sin \varphi_2$ and $\mathrm{OD} = \mathrm{A}_1 \cos \varphi_1 + \mathrm{A}_2 \cos \varphi_2$. Therefore

$$\mathrm{tg}\, \varphi = \frac{A_1 \sin \varphi_1 + A_2 \sin \varphi_2}{A_1 \cos \varphi_1 + A_2 \cos \varphi_2},$$

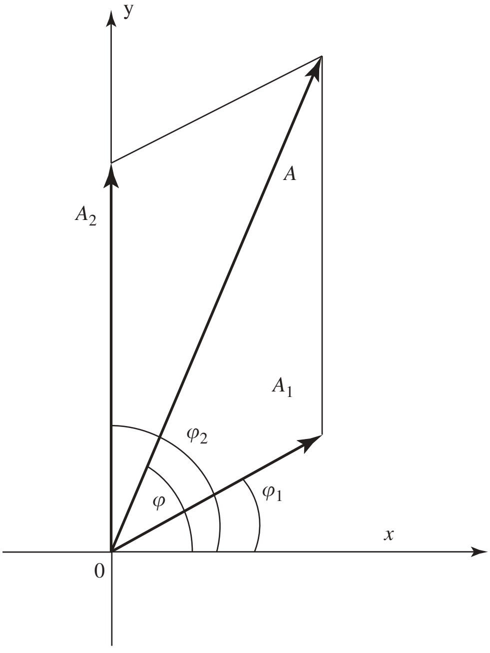
and the phase is

$$\varphi = \arctan\left(\frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A_1 \cos\varphi_1 + A_2 \cos\varphi_2}\right) \tag{2.3.6}$$

Hence, summing up two harmonic oscillations, we also obtain the harmonic oscillation with the same cyclic frequency and amplitudes given by expression (2.3.1) and the initial phase given above (2.3.6).

EXAMPLE E2.3

Two oscillations take place along one and the same direction. They are expressed by equations $x_1 = A_1 \cos\omega(t + \tau_1)$ and $x_2 = A_2 \cos\omega(t + \tau_2)$, where $A_1 = 1$ cm, $A_2 = 2$ cm, $\tau_1 = 1/6$ sec, $\tau_2 = 1/2$ sec, $\omega = \pi\ \text{sec}^{-1}$. Determine (1) initial phases φ_1 and φ_2 of the component oscillation; and (2) the amplitude and initial phase of the resulting oscillation. Write down the equation of the resulting oscillation and draw the corresponding vector diagram.



Solution: (1) In general form the equation of oscillations is $x = A\cos(\omega t + \varphi_1)$. Rewriting the given equation for both oscillations: $x_1 = A_1\cos(\omega t + \omega\tau_1)$ and $x_2 = A_2\cos(\omega t + \omega\tau_2)$. From the conditions given above we can find $\varphi_1 = w\tau_1 = (\pi/6)$ rad and $\varphi_2 = \omega\tau_2 = (\pi/2)$ rad. (2) Keeping in mind the vector diagram (Figures 2.4 and 2.7), we can find $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\Delta\varphi}$. We can find $\Delta\varphi$ from the data given above; $\Delta\varphi = (\pi/3)$ rad, and further, $\tan\varphi = (A_1\sin\varphi_1 + A_2\sin\varphi_2)/(A_1\cos\varphi_1 + A_2\cos\varphi_2)$. Executing the calculations we arrive at $\varphi = \arctan(5/\sqrt{3}) = 70.9° = 0.394\pi$ rad.

2.3.2 Summing up two codirectional oscillations with slightly different frequencies: beatings

For simplicity, consider that $\omega_1 \leq \omega_2$, so $\Delta\omega << (\omega_1 - \omega_2)$ and $A_1 = A_2 = A$. This summation can be made analytically without difficulty. However, we will use here the summation method of oscillations based on the vector diagram as we did earlier in this chapter to define the qualitative nature of the result. Turn first to Figure 2.7. In this case vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ rotate with slightly different angular velocities. This signifies that at some point in time both vectors can be in antiphase and the resulting oscillation amplitude is zero. Hereon, vector $\mathbf{A}_2$ runs after $\mathbf{A}_1$ and at another time instant "catches" it up, the phases of both vectors coinciding. The resulting oscillation amplitude will become $2A$, whereupon vector $\mathbf{A}_2$ will overrun vector $\mathbf{A}_1$ and the total oscillation amplitude will gradually decrease. At least vector $\mathbf{A}_2$ overruns $\mathbf{A}_1$ by π: once again the oscillations will disappear in a moment. Then the whole process will be repeated again.

It is easy to see that oscillations like those depicted in Figure 2.8 are composed oscillations. These types of oscillations are called "beatings" and are easily observable. When two sources of oscillations (for instance, two engines in an aircraft) rotate at close frequencies,

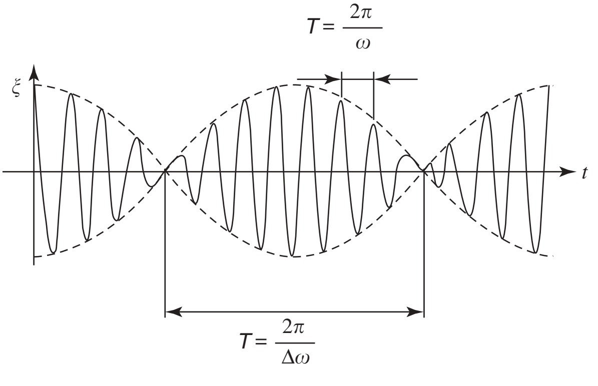


Figure 2.8 Beatings.

a resulting sound can be heard: low-frequency fluctuations $\Delta\omega$ on a background of the high-frequency roar ω of the engines.

2.4 DYNAMICS OF THE HARMONIC OSCILLATION

2.4.1 Differential equations of harmonic oscillations

It was shown earlier (refer to eq. (2.2.6)) that there exists a simple correlation between displacement and acceleration of an oscillating MP:

$$\ddot{\xi}(t) = -\omega^2 \xi(t)$$

or

$$\ddot{\xi}(t) + \omega^2 \xi(t) = 0. \tag{2.4.1}$$

This equation is referred to as the differential equation of harmonic oscillations. By analyzing this equation, we can arrive at an important conclusion: when solving a problem and arriving at an equation like that presented above, it means that the problem can be reduced to harmonic oscillations and the coefficient before the displacement function is the square of its cyclic frequency.

A general equation is a sum

$$\xi = a\cos\omega t + b\sin\omega t \tag{2.4.2}$$

or, in exponential form

$$\xi = A\exp(\mathrm{i}\omega t).$$

This is proved by substitution of any of the proposed solutions into eq. (2.4.1).

2.4.2 Spring pendulum

A system consisting of a body with a mass m, which, moving without friction (!), can oscillate under the action of elastic force (weightless springs) with the rigidity coefficient β, is called a spring pendulum. One end of the spring is attached to the weight and the other end is fastened to a rigid wall (Figure 1.22). (This system has already been considered in Sections 1.4.1 (Figure 1.22) and 1.5.4 (Figure 1.31).) The starting position is that of a nondeformed spring. According to the second law of dynamics, we can write

$$m\ddot{\xi} = -\beta\xi, \tag{2.4.3}$$

where ξ is the value of the shift from the origin. In this case, the elastic force is a single force acting on the body (since the projection of the gravitational force to the abscissa x is zero and friction is neglected). A negative sign is stipulated by the fact that the force acting on the weight is always directed toward the origin. Transferring both terms of eq. (2.4.3) to the left-hand side and subdividing them by m, we arrive at

$$\ddot{\xi} + \frac{\beta}{m}\xi = 0. \tag{2.4.4}$$

Compare the expression obtained with the general type of differential equation of harmonic oscillation (eq. (2.4.1)). From the fact that both equations have a similar form, it can be stated that the weight makes a harmonic oscillation. Thereof, the other definition of a harmonic oscillation is an oscillation that occurs under the action of an elastic force. By equating the multipliers in the similar terms of the equation, we can derive an expression for the cyclic frequency of the spring pendulum:

$$\omega = \sqrt{\frac{\beta}{m}}. \tag{2.4.5}$$

The same expression can be obtained for the cyclic frequency of free-oscillating bodies appearing by the action of any quasi-elastic force, which is linearly proportional to displacement ξ and directed opposite to it.

According to eq. (2.2.2), the period T of harmonic oscillations produced by the action of elastic and quasi-elastic forces is given by the formula

$$T = 2\pi\sqrt{\frac{m}{\beta}}. \tag{2.4.6}$$

2.4.3 The mathematical pendulum

Another example of harmonic oscillations is that of a mathematical pendulum. An MP suspended on a weightless, nonstretched and ideally flexible thread, is referred to as a mathematical pendulum. Consider small displacements of a pendulum from the equilibrium position, i.e., $\xi \ll l$, where l is the length of the mathematical pendulum. At a certain instant of time, let the pendulum occupy the position depicted in Figure 2.9. Using the second Newtonian law, equation of motion can be written as

$$m\ddot{\xi} = -mg\sin\alpha \tag{2.4.7}$$

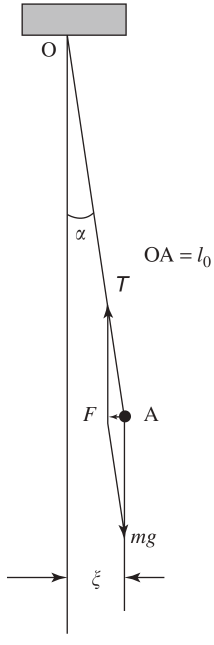


Figure 2.9 A mathematical pendulum.

At small pendulum's angle deflection ($\xi/l \ll 1$), $\sin\alpha \approx \alpha$ and the returning force $F = -(mg/l)\xi$ can be considered as a quasi-elastic one. The coefficient characterizing the "rigidity" of the quasi-elastic force for the mathematical pendulum is $\beta = mg/l$. Introducing this expression for the rigidity of a quasi-elastic force into eqs. (2.4.5) and (2.4.6), we can obtain an expression for the cyclic frequency and period of small oscillations of a mathematical pendulum:

$$\omega = \sqrt{\frac{g}{\ell}} \tag{2.4.8}$$

and

$$T = 2\pi\sqrt{\frac{\ell}{g}}. \tag{2.4.9}$$

These formulas are valid only for small displacements ($\xi \ll l$), under which approximation $\sin\alpha \approx \alpha$ is also valid. This approximate equality will be executed if angle $\alpha \ll 1$. So, for instance, at $\alpha = 5°$ ($\alpha \approx 0.1$ rad) replacing $\sin\alpha$ by α brings about an inaccuracy of the order 0.2%. On reducing angle α this inaccuracy quickly decreases: at $\alpha = 1°$, it reaches an insignificantly small value 0.005%. By contrast, at greater amplitudes it is impossible to consider oscillations to be harmonic and their period will depend on amplitude.

2.4.4 A physical pendulum

Any body having the possibility to oscillate freely under a gravitational force around a horizontal axis, not passing through the body's CM, is referred to as a physical pendulum. In this case, all points of a rigid body move along an arc of concentric circles. Consequently, for the description of a physical pendulum's oscillations, the rotational laws of dynamics should be applied.

Let an axis of rotation z pass horizontally through point O (Figure 2.10) perpendicular to the plane of drawing. Also, let the physical pendulum be deflected from the position of equilibrium by angle α, which, as previously, is considered to be small. Then, the main law of dynamics of rotational motion can be written as

$$I_z\ddot{\alpha} = -M_z, \tag{2.4.10}$$

where I_z is the moment of inertia of the physical pendulum regarding axis Oz, $\ddot{\alpha}$ is a time second derivative of α and M_z is the moment of external force with respect to axis z returning the pendulum to the position of equilibrium. In a given case this moment is stipulated by gravitational force mg, attached to the CM of the physical pendulum. In Figure 2.10, the

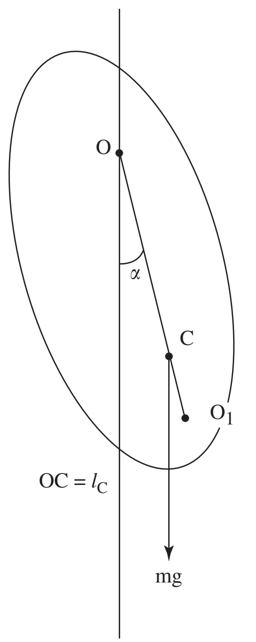


Figure 2.10 A physical pendulum.

physical pendulum CM is marked by the letter C whose distance from the oscillation axis O is marked by the letter l_c. From Figure 2.10 it can be seen that for small angular displacements

$$M_z = -mg\,\ell_c \sin\alpha \approx -mg\,\ell_c\,\alpha. \qquad (2.4.11)$$

Sign "−" corresponds to the accepted sign rule for the returning force moment of the Oz axis. Thereby, the differential equation for small physical pendulum oscillations according to eqs. (2.4.10) and (2.4.11) can be written as

$$\ddot{\alpha} + \frac{mg\ell_c}{I_z}\alpha = 0. \qquad (2.4.12)$$

Comparing this expression with eq. (2.4.1) we can conclude that the physical pendulum makes harmonic oscillations with cyclic frequency

$$\omega = \sqrt{\frac{mg\ell_c}{I_z}} \qquad (2.4.13)$$

and the period of small oscillations is

$$T = 2\pi\sqrt{\frac{I_z}{mg\ell_c}}. \qquad (2.4.14)$$

The length of such a mathematical pendulum, which is equal to the physical pendulum's oscillation period, is called the reduced length of a physical pendulum. An expression for the reduced length of a physical pendulum can be found by comparing eqs. (2.4.9) and (2.4.14):

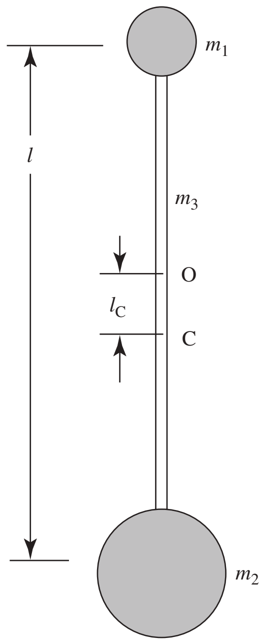
$$OO_1 = L = \frac{I_z}{m\ell_c}. \qquad (2.4.15)$$

Point O_1 on the line OC (Figure 2.10) at a distance L from the axis of rotation z is called the center of swing of the physical pendulum. It is noteworthy that if a pendulum is turned over and hung up on the horizontal axis passing through the point O_1 the period of its oscillation does not change, point O being the new center of oscillation. We will leave the proof of this property as an exercise for the reader.

EXAMPLE E2.4

On the ends of a thin rod of weight m_3 and length l, small-sized balls of weights m_1 and m_2 are fixed. The rod makes small oscillations about a horizontal axis perpendicular to the rod and passing through its middle point. Define the period T and frequency ω of the oscillation of the pendulum. Solve the problem for l =1 m, m_1 = 200 g, m_2 =300 g and m_3 = 400 g.

Solution: The frequency of the physical pendulum's oscillation and the period of its oscillations can be found according to eqs. (2.4.14) and (2.4.13). There are three values to define: total weight m, the MI of the pendulum relative to the axis of oscillation I_z and the distance from the axis of oscillations up to the center of weights l_c. Oscillation axis z passes through the center of the rod perpendicular to the plane of drawing: let us direct an axis x vertically downward (parallel to the rod) and superpose its origin with an oscillation axis. The coordinates of all bodies included in the system can be determined from Figure E2.4.



We can find the distance l_c from eq. (1.3.32):

$$l_c = \frac{\sum_1^2 m_i x_i}{m} = \frac{m_1(-(l/2)) + m_2(l/2)}{m} = \frac{(m_2 - m_1)}{2m} = 5.55 \text{ cm}.$$

Remember that the oscillation axis coincides with the position of the oscillation axis, i.e., the coordinate of the CM numerically coincides with l_c. The weight of a physical pendulum is equal to $m = m_1 + m_2 + m_3$ (0.9 kg).

We shall find the MI of a physical pendulum relative to oscillation axis as the sum of the moments of inertia of three bodies:

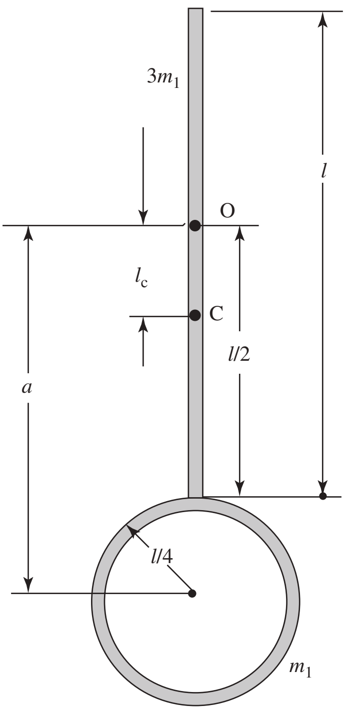
$$I = I_1 + I_2 + I_3 = m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2 + \frac{m_3\ell^2}{12} = \frac{\ell^2(3m_1 + 3m_2 + m_3)}{12} = 0.158\,\text{kg}\,\text{m}^2.$$

Substituting the data into the general eqs. (2.4.13) and (2.4.14) we arrive at $T = 2.17$ sec and $\omega = 2.89$ rad/sec.

EXAMPLE E2.5

A physical pendulum consists of a rod and a hoop of masses $m_{\text{rod}} = 3m_1$ and $m_{\text{hoop}} = m_1$; the length of the rod is $l = 1$ m. The horizontal axis of oscillation Oz is perpendicular to the rod and passes it at its center O. Determine the oscillation period of such a pendulum. The rest of the definitions are given in Figure E2.5.

Solution: The period of oscillation is expressed by eq. (2.4.14), $T = 2\pi\sqrt{\frac{I}{mgl_c}}$.



To find the period, we must first choose a reference frame (axis x), mark a zero position on it (see Figure E2.5) and find the MI of parts of the pendulum, $I_{z,1}$ is the MI of the rod and $I_{z,2}$ is the MI of the hoop and the total MI is I. It is also necessary to find the distance l_c between the oscillation axis (point O) and the CM. The MI of the physical pendulum relative to the oscillation axis is the sum of I_1 (the MI of a rod) and I_2 (the MI of the hoop both relative to the same axis $I = I_1 + I_2$).

The rod MI is $I_1 = m_1 l^2/4$ (because the rod mass is $3m_1$ and the axis passes through the center of the rod). The MI of the hoop is the sum of the MI of the hoop itself (first item) and the addition from the parallel axis theorem (second item):

$$I_2 = m_1\left(\frac{\ell}{4}\right)^2 + m_1\left(\frac{3\ell}{4}\right)^2 = \frac{5m_1\ell^2}{8}.$$

The total MI of the pendulum is the sum $I=\frac{m_1 l^2}{4}+\frac{5m_1 l^2}{8}=\frac{7m_1 l^2}{8}$.

The distance

$$\ell_c = \frac{\sum m_i x_i}{\sum m_i} = \frac{3m_1 \times 0 + m_1(3\ell/4)}{3m_1 + m_1} = \frac{\frac{3}{4}m_1\ell}{4m_1} = \frac{3}{16}\ell.$$

(In order to simplify calculation of l_c it is useful to mark zero on axis x at the same level as point O; in this case the CM coordinate is simultaneously l_c). Obtaining these preliminary results we can place all the values under the square root:

$$T = 2\pi\sqrt{\frac{\frac{7}{8}m_1\ell_c^2}{4m_1 g\frac{3}{16}\ell_c}} = 2\pi\sqrt{\frac{7\ell_c}{6g}},$$

therefore, $T = 2.17$ sec.

EXAMPLE E2.6

A small weight of mass $m = 5$ g performs harmonic oscillations under the action of a gravitational force with frequency v = 0.5 Hz. Amplitude is $A = 3$ cm. Find (1) the velocity of the weight at a time instant when $x = 1.5$ cm; (2) the maximum force F acting on the weight; and (3) the total energy of the oscillator.

Solution: (1) The oscillation equation is $x(t) = A\cos(\omega t + \varphi)$, whereas the expression for velocity can be obtained by time derivation of $\xi(t)$: $(d\xi/dt) = \boldsymbol{\upsilon} = -A\omega\sin(\omega t + \varphi)$*. In order to find the relation between the velocity $\boldsymbol{\upsilon}$ and ξ, we should exclude the time from the last two equations. For this it is necessary to square both equations, divide the first by A^2, the second by $A^2\omega^2$ and sum the results: $(x^2/A^2)+(\boldsymbol{\upsilon}^2/A^2\omega^2) = 1$.

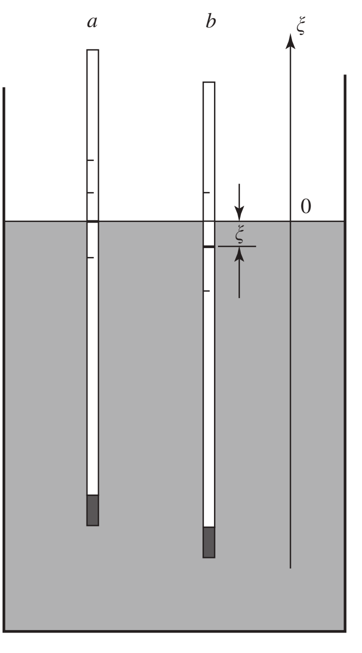
Solving the equation relative to υ we can arrive at $\boldsymbol{\upsilon} = 2\pi v\sqrt{A^2 - x^2}$ (keeping in mind that $\omega = 2\pi v$). Executing all calculations, we obtain $\upsilon = \pm 8.2$ cm/sec.

The sign "+" is valid when the velocity is directed along the positive direction of x-axis and vice versa. The same result can be obtained if the sin function is used instead of cos. (2) The force value can be found according to the Newton's second law. The time first and second derivative can be seen in eqs. (2.2.4) and (2.2.5): $a = \ddot{x} = (d\upsilon/dt) = -A\omega^2\cos(\omega t + \varphi)$ or $a = -4\pi^2\nu^2 A\cos(\omega t + \varphi)$. Using acceleration amplitude in any of these equations can yield $F_{max} = \omega^2 mA$ or 1.49 mN. (3) The total energy is the sum of the kinetic and potential energies. Therefore, the total energy can be calculated at any position; for instance, in the lower position the kinetic energy is maximum at this point. Therefore using the equation marked as * (see above) and taking $\sin(\omega t + \varphi) = 1$ we obtain $E = K_{max} = (m\upsilon^2_{max}/2)$. Finally, $E = 2\pi^2\nu^2 A^2$; executing the calculation we arrive at 22.1×10^{-6} J or 22.1 μJ.

EXAMPLE E2.7

An areometer (densitometer) consisting of a long tube of diameter $d = 1$ cm weighing $m = 50$ g freely floats vertically in still water. It is submerged a little and then released; it begins to oscillate up and down. Neglecting the water viscosity find the period of its oscillations.

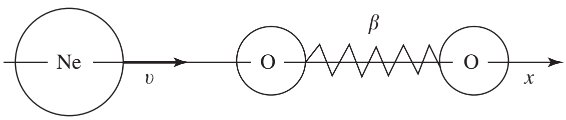
Solution. Choose an axis ξ vertically and denote an origin $\xi = 0$ at the areometer tube prior to its oscillation (Figure E2.7,a). In this state its gravity and Archimedes force are equalized. Oscillations will be accomplished by the periodically changing Archimedes force because the gravity remains constant. The value of areometer's



immersion can be arranged as ξ (Figure E2.7,*b*) which will determine the Archimedes buoyancy force $(\pi d^2/4)\rho g\ \xi$; this is the restoring force in our case. According the second Newtonian law, $m\ddot{\xi} = (\pi d^2/4)\ \rho g\xi$ and then $\ddot{\xi} + (\pi d^2/4m)\rho g\xi = 0$. Firstly, we obtained the confirmation that our system accomplishes harmonic oscillation (compare this equation with eq. (2.4.1)), and secondly can easily extract the ω^2 value: $\omega = (2\pi/T) = (\mathrm{d}/4\pi)\sqrt{(\pi\rho g/m)}$ whence $T = (4/\mathrm{d})\sqrt{(\pi m/\rho g)} = 1.6$ sec.

EXAMPLE E2.8

A neon atom (Ne) collides with an oxygen atom of a molecule (O_2) along its bond direction (Figure E2.8). The kinetic energy of the Ne atom is $K_1 = 6 \cdot \times 10^{-21}$ J. The oxygen bond rigidity coefficient β is $1.18 \cdot \times 10^3$ N/m. Relative masses of the Ne and O atoms equal $A_{r,Ne} = 20$; $A_{r,O} = 16$. Consider the collision to be elastic and the oscillations of the O_2 molecule after impact to be harmonic. Determine: (1) the translational kinetic energy $K_{z,tr}$ of the oxygen molecule after impact; (2) the oscillation energy $E_{2,osc}$ of the oxygen molecule acquired by the impact (suppose that the O_2 molecule did not oscillate before the collision); (3) the average values of kinetic $<K_{2,osc'}>$ and potential $<U_{2,osc'}>$ energies; (4) amplitude A of the harmonic oscillation; and (5) the angular oscillation frequency ω.



Solution: (1) Assume that impact is elastic and head-on impact takes place. Using eq. (1.5.16) we can obtain

$$u_1 = \frac{m_1 - m_2}{m_1 + m_2}\upsilon_1,$$

where υ_1 is the velocity of the Ne atom prior to impact. After collision its kinetic energy is

$$K_1' = \frac{m_1 u_1^2}{2} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 K_1.$$

The CM velocity can be found using only the momentum conservation law because part of the kinetic energy is taken for the inner (oscillation) energy of the O_2 molecule

$$V_c = \frac{m}{2m}\left(\upsilon_1 - \frac{m_1 - m_2}{m_1 + m_2}\upsilon_1\right) = \frac{m_1}{m_1 + m_2}\upsilon_1.$$

Then the kinetic energy of the translational movement of the O_2 molecule $K_{2,\mathrm{trs}'}$ is

$$K_{2,\mathrm{trs}}^{'} = \frac{2m_2V_c^2}{(m_1+m_2)^2} = \frac{2m_1m_2}{(m_1+m_2)^2}K_1.$$

Executing calculations we arrive at

$$K_{2,\mathrm{trs}}^{'} = \frac{2\times 20\times 16}{20+16}\ 6\times 10^{-21} = 2.96\times 10^{-21}\ \mathrm{J}$$

(2) The energy of oscillation can be found using the energy conservation law:

$$K_1 = K_1^{'} + K_{1,\mathrm{trs}}^{'} + E_{2,\mathrm{osc}}^{'} \ \text{Therefore}$$

$$E_2^{'} = K_1 - K_2^{'} - E_{2,\mathrm{trs}},$$

or

$$E_{2,\mathrm{osc}}^{'} = K_1 - \left(\frac{m_1-m_2}{m_1+m_2}\right)^2 K_1 - \frac{2m_1m_2}{(m_1+m_2)}K_1$$
$$= \frac{2m_1m_2}{(m_1+m_2)^2}K_1 \quad \text{Since} \quad E_{2,\mathrm{osc}} = K_{2,\mathrm{trs}}^{'},$$

therefore, $E_{2,\mathrm{osc}} = 2.96 \times 10^{-21}$ J.

(3) The average values of oscillation kinetic and potential energies can be found from eqs. (2.5.7) and (2.5.8). It follows that

$$\langle K_{2,\mathrm{osc}}\rangle = \langle U_{2,\mathrm{osc}}\rangle = \frac{m_1m_2}{m_1+m_2}K_1 = \mu\times K = 1.48\times 10^{-21}\ \mathrm{J}.$$

(4) The amplitude of the harmonic oscillations can be calculated from the expression

$$E_{2,\mathrm{osc}} = (\tfrac{1}{2})\ \mu\omega^2 \mathrm{A}^2$$

The rigidity coefficient $\beta = \mu\omega^2$, therefore $E_{2,\mathrm{osc}} = (1/2)\beta A^2$ and then

$$A = \sqrt{\frac{2E_{2,\mathrm{osc}}}{\beta}}.$$

Using the oscillation energy we can execute all operations

$$A = \sqrt{\frac{2\times 2.96\times 10^{-21}}{1.18\times 10^{3}}} = 2.25\times 10^{-12}\ \text{m}.$$

(5) The angular frequency can be found from equation $\beta = \mu\omega^2$. Since $\mu = (m_2/2)$,

$$\omega = \sqrt{\frac{\beta}{\mu}} = \sqrt{\frac{2\beta}{m_2}}.$$

And finally we arrive at

$$\omega = \sqrt{\frac{2\times 1.18\times 10^{3}}{16\times 1.66\times 10^{-27}}} = 2.98\times 10^{14}\ \text{sec}^{-1}.$$

2.4.5 Diatomic molecule as a linear harmonic oscillator

The diatomic molecule is an example of a linear harmonic oscillator provided that the interatomic force is an elastic one. Consider a molecule to be close to an isolated system. This signifies that two atoms of a molecule make oscillations relative to their CM, so that such oscillation can be reduced to an oscillation of a single body (with the mass equal to the reduced mass system) regarding the motionless fixed point under the action of the same interatomic force.

Superpose the origin into point C, which is the CM of two points with masses m_1 and m_2 (Figure 2.11). Then their coordinates x_1 and x_2 determine the equilibrium positions of both.

For this case, we can write $m_1x_1 = m_2x_2$ (refer to Section 1.3.7). Considering the oscillations to be symmetrical, for any instant of time it is fair to say that

$$m_1(x_1+\xi_1) = m_2(x_2+\xi_2).$$

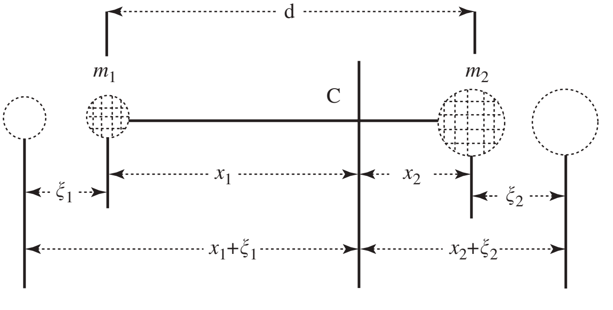


Figure 2.11 A diatomic molecule as a linear rigid harmonic oscillator.

Simplifying,

$$m_1\xi_1 = m_2\xi_2.$$

Assume for the linear oscillator the interatomic force is an elastic one. This corresponds to the problem in harmonic approximation. Then, a force acts on any atom that is out of its equilibrium position:

$$\mathbf{F} = -\beta(\xi_1 + \xi_2).$$

Use Newton's second law for each atom

$$m_1\ddot{\xi}_1 = -\beta(\xi_1 + \xi_2) \quad \text{and} \quad m_2\ddot{\xi}_2 = -\beta(\xi_1 + \xi_2).$$

Express in the first equation ξ_2 through ξ_1 and in the second equation ξ_1 through ξ_2:

$$m_1\ddot{\xi}_1 = -\beta\left(\xi_1 + \frac{m_1}{m_2}\xi_1\right), \quad m_2\ddot{\xi}_2 = -\beta\left(\frac{m_2}{m_1}\xi_2 + \xi_2\right).$$

Rewriting these expression as

$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\xi} = -\beta\xi_1 \quad \text{and} \quad \frac{m_1 m_2}{m_1 + m_2}\ddot{\xi} = -\beta\xi_2$$

and summing both expressions, we arrive at

$$\frac{m_1 m_2}{m_1 + m_2}(\ddot{\xi}_1 + \ddot{\xi}_2) = -\beta(\xi_1 + \xi_2).$$

Because $\xi_1 + \xi_2 = \xi$, where ξ is the displacement of one atom relative to the other, we can write the expression for relative acceleration as $\ddot{\xi}_1 + \ddot{\xi}_2 = \ddot{\xi}$. Value $m_1m_2/(m_1 + m_2)$ is the reduced mass of the molecule, which is denoted by μ (Section 1.3.9). Then the above equation corresponds to the harmonic oscillation of a single material point μ under the action of an elastic force $-\beta\xi$:

$$\ddot{\xi} + \frac{\beta}{\mu}\xi = 0, \tag{2.4.16}$$

where

$$\omega = \sqrt{\frac{\beta}{\mu}}. \tag{2.4.17}$$

The equation derived is similar to that describing the oscillations of a spring pendulum. Here the interatomic force plays the role of a spring. This shows that the problem of the atom vibrating in the molecule is reduced to the problem of harmonic material point oscillations with the mass, equal to the reduced molecular mass.

A physical system, making oscillations, is referred to as an oscillator. If it complies with eq. (2.4.6), it is referred to as a *one-dimensional harmonic oscillator*. In the first approximation any molecule can be considered as a classical one-dimensional harmonic oscillator; this is the simplest physical model explaining some (but by no means all) particularities of atom vibrations in the molecule. In Chapter 7 it will be shown that in quantum mechanics a much better approximation is given by a model of a quantum linear harmonic oscillator. However, the next best approximation is a nonharmonic (nonlinear) model (this model is more complicated; it describes atomic vibrations in more detail and introduces new phenomena).

2.5 ENERGY OF HARMONIC OSCILLATIONS

An oscillating body possesses both potential energy U and kinetic energy K. Its total energy E is the sum: $E = U + K$.

First an expression for the potential energy of the oscillating body is found. When displacing from its equilibrium position, an elastic force acts on the body. The potential energy of the body in this case was determined in eq. (1.5.4):

$$U = \tfrac{1}{2}\beta\xi^2. \tag{2.5.1}$$

The time dependence of displacement is expressed by formula (2.2.1). Then the potential energy is equal to

$$U = \tfrac{1}{2}\beta A^2 \cos^2(\omega t + \varphi). \tag{2.5.2}$$

The kinetic energy is equal to $K = \dfrac{1}{2}\,\mathrm{m}\upsilon^2$. Since $\upsilon = \dot{\xi} = -A\omega\sin(\omega t + \varphi)$,

$$K = \tfrac{1}{2}A^2\omega^2 m \sin^2(\omega t + \varphi). \tag{2.5.3}$$

Substituting $\mathrm{m}\omega^2 = \beta$,

$$K = \tfrac{1}{2}\beta A^2 \sin^2(\omega t + \varphi). \tag{2.5.4}$$

Now the total mechanical energy combining both U (2.5.2) and K (2.5.3) is

$$E = \tfrac{1}{2}\beta A^2[\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)]$$

or

$$E = \tfrac{1}{2}\beta A^2. \tag{2.5.5}$$

It can now be seen that total mechanical energy is defined by the coefficient β characterizing the rigidity of the system, and by the oscillation amplitude squared. Therefore, the total mechanical energy is equal to

$$E = \tfrac{1}{2}m\omega^2 A^2. \tag{2.5.6}$$

It is important to note that the *total energy of harmonic oscillations is proportional to the square of amplitude: $E \sim A^2$*. Obviously, the system considered is conservative and its total energy is conserved, i.e., only the transfer of kinetic energy into potential energy and back again is taking place (refer to eq. (1.5.4)). The potential energy reaches a maximum at the largest (amplitude) displacement, whereas kinetic energy is at its highest possible when the system crosses the origin.

Note that expressions (2.5.2) and (2.5.4) can be presented in the form

$$U = \tfrac{1}{4}\beta A^2[1+\cos^2(\omega t+\varphi)], \tag{2.5.7}$$

and

$$K = \tfrac{1}{4}\beta A^2[1-\cos^2(\omega t+\varphi)]. \tag{2.5.8}$$

It can be seen that the oscillation of kinetic and potential energies accomplishes with the doubled frequency 2ω in comparison with the initial one. Graphs of functions $\xi(t)$, $K(t)$ and $U(t)$ are presented in the Figure 2.12.

The averaged values of kinetic and potential energies of harmonic oscillations for the period are equal to half the total energy, i.e.,

$$\langle K\rangle = \langle U\rangle = \tfrac{1}{2}E \tag{2.5.9}$$

This equation is the consequence of the fact that the average values of sin α and cosα functions for the period are equal to

$$\langle \sin^2\omega t\rangle = \frac{1}{T}\int_0^T \sin^2\omega t\,dt \text{ and } \langle \cos^2\omega t\rangle = \frac{1}{T}\int_0^T \cos^2\omega t\,dt.$$

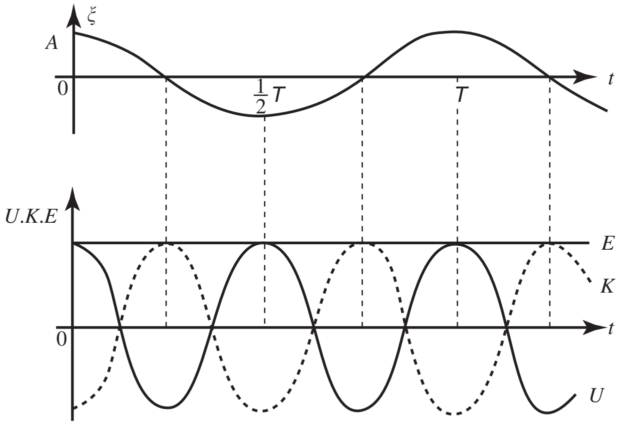


Figure 2.12 Energy graphs of harmonic oscillations.

2.6 DAMPED OSCILLATIONS

Hitherto, we have considered harmonic oscillations appearing under the action of a single (restoring) force in a system. Such oscillations are called free or natural ones, their frequencies being designated by ω_0. Strictly speaking, there are no such systems in surrounding macroscopic nature. In real systems, there are forces other than elastic forces; they are distinguished in nature from quasi-elastic forces and appear when an oscillation system interacts with its surroundings. The final result of these interactions is the transformation of the mechanical energy of the moving bodies into heat. In other words, a dissipation of the mechanical energy occurs. The process of energy dissipation is not purely mechanical and for its description another section of physics is required. Within the framework of mechanics we can describe this process by introducing forces of friction or resistance. As a result of mechanical energy dissipation, the oscillation amplitude decreases. The damping oscillations are no longer harmonic ones, since oscillation amplitude changes. *Oscillations that, in consequence of energy dissipation, have a continuously decreasing amplitude are referred to as damped oscillations.*

Consider nearly free damped oscillation with a small resistance. At small oscillation amplitudes, the velocity of the body will also be small; under small velocities the force of resistance is often proportional to the velocity value (refer to eq. (1.3.5))

$$F = -rv = -r\dot{\xi}, \tag{2.6.1}$$

where $\upsilon = \dot{\xi}$ is the velocity of a body's motion and r is the proportionality factor, called the resistance coefficient. The minus sign in the expression of resistance force is stipulated by the fact that its direction is opposite to the velocity of the moving body.

Taking the quasi-elastic ($-\beta\xi$) and resistance forces ($-r\dot{\xi}$) into account, we can write an equation of motion of damping oscillations as

$$m\ddot{\xi} = -\beta\xi - r\dot{\xi}. \tag{2.6.2}$$

Substituting the coefficient β by $m\omega_0{}^2$ in this expression and dividing both sides by m, we have

$$\ddot{\xi} + \frac{r}{m}\dot{\xi} + \omega_0^2\xi = 0. \tag{2.6.3}$$

Suppose that for damping oscillations the expression sought has the same form as previously discussed:

$$\xi = A_0 \exp(\mathrm{i}\gamma t), \tag{2.6.4}$$

Here γ is as yet an unknown value. The initial phase is taken as zero, i.e., we begin to measure time when the phase crosses the zero position. To find this quality we can substitute the form (2.6.4) into equation (2.6.3) together with their first and second derivatives:

$$\dot{\xi}(t) = \frac{d\xi}{dt} = A_0 \mathrm{i}\gamma \mathrm{e}^{\mathrm{i}\gamma t},$$

$$\ddot{\xi}(t) = \frac{d^2\xi}{dt^2} = A_0 \mathrm{i}^2\gamma^2 \mathrm{e}^{\mathrm{i}\gamma t} = -A_0\gamma^2 \mathrm{e}^{\mathrm{i}\gamma t}.$$

Substitute these equations into (2.6.3):

$$-A_0\gamma^2 \mathrm{e}^{\mathrm{i}\gamma t} + \frac{r}{m} A_0 \mathrm{i}\gamma \mathrm{e}^{\mathrm{i}\gamma t} + \omega_0^2 A_0 \mathrm{e}^{\mathrm{i}\gamma t} = 0.$$

After reducing on $A_0\mathrm{e}^{\mathrm{i}\gamma t}$ and changing signs we obtain

$$\gamma^2 - \mathrm{i}\frac{r}{m}\gamma - \omega_0^2 = 0$$

This quadratic equation relative γ has two roots:

$$\gamma = \frac{\mathrm{i}r}{2m} \pm \sqrt{\omega_0^2 - \frac{r^2}{4m^2}}. \tag{2.6.5}$$

We can find the time dependence of displacement by introducing (2.6.5) into (2.6.4). For the sake of convenience introduce two more values:

$$\delta = \frac{r}{2m} \tag{2.6.6}$$

and

$$\omega = \sqrt{\omega_0^2 - \delta^2}. \tag{2.6.7}$$

Since $\gamma = \mathrm{i}\delta \pm \sqrt{\omega_0^2 - \delta^2} \approx \mathrm{i}\delta \pm \omega$ the displacement in damping oscillation can be presented in the form:

$$\xi(t) = A_0 \mathrm{e}^{\mathrm{i}(\mathrm{i}\delta \pm \omega)t} = A_0 \mathrm{e}^{-\delta t} \times \mathrm{e}^{\pm \mathrm{i}\omega t}. \tag{2.6.8}$$

The choice of the sign corresponds to the phase shift on π. We shall write the solution with a "+" sign. Then eq. (2.6.8) can be rewritten as

$$\xi(t) = \mathrm{A}_0 \mathrm{e}^{-\delta t} \times \mathrm{e}^{\mathrm{i}\omega t}. \tag{2.6.9}$$

This is the expression sought for damped oscillation. In trigonometric form it can be given as

$$\xi = \mathrm{A}_0 \mathrm{e}^{-\delta t} \cos \omega t. \tag{2.6.10}$$

In all the sequent expressions ω is the frequency of damped oscillations. It is always lower than the frequency ω_0.

The time dependence of the amplitude can be given as

$$A(t) = A_0 \exp(-\delta t), \tag{2.6.11}$$

where A_0 is the amplitude in the initial instant of time (at $t = 0$). The constant δ (refer to (2.6.6)), equal to the ratio of the resistance coefficient r to the doubled system mass, is usually called the coefficient of the damped oscillations.

Find the time during which the amplitude of the damped oscillations reduces e times, i.e., $A(t)/A(t+\tau) = e$.Using eq. (2.6.11) we can obtain

$$\frac{A_0 \exp(-\delta t)}{A_0 \exp[-\delta(t+\tau)]} = e$$

or $\exp(\delta\tau) = e$,wherefrom

$$\delta\tau = 1. \tag{2.6.12}$$

Hence, the damping coefficient δ is the value reciprocal to the time of amplitude reduction into e times.

In order to characterize the process of attenuation, the so-called logarithmic decrement of damping (attenuation) λ is also used. It is accepted to be equal to the natural logarithm of a ratio of two oscillation amplitudes, separated from each other by an oscillation period:

$$\lambda = \ln\frac{A(t)}{A(t+T)}. \tag{2.6.13}$$

Using expression (2.6.11), we obtain

$$\lambda = \ln\frac{A_0 \exp(-\delta t)}{A_0 \exp[-\delta(t+T)]} = \ln \exp[\delta T] = \delta T. \tag{2.6.14}$$

Find a physical sense of logarithmic decrement of fading. Let the oscillation amplitude decrease e times after N oscillations. The time τ, for which a system makes N oscillations, can be expressed in periods: $\tau = NT$. Having substituted this value τ in (2.6.12), we obtain $\delta NT = 1$. As far as $\delta T = \lambda$, we can get $\lambda N = 1$ or

$$\lambda = \frac{1}{N}. \tag{2.6.15}$$

Consequently, the logarithmic decrement of fading is a value inverse to the number of oscillations after which the amplitude decreases e times.

In a number of cases, it is suitable to express the dependency of oscillation amplitude from time through the logarithmic decrement of fading. The degree δt in expression (2.6.11) can be written according to (2.6.14) as follows:

$$\delta t = \lambda\frac{\tau}{T}.$$

Then expression (2.6.11) takes the form

$$A = A_0 \exp\left(-\lambda \frac{\tau}{T}\right), \tag{2.6.16}$$

where τ/T is the magnitude, which shows the number of oscillation a system accomplishes for time τ.

The approximate values (orders of magnitude) of the logarithmic damping decrements of some oscillation systems are plotted in Table 2.1.

Let us now analyze the influence of resistance force on oscillation frequency. When displacing a body from its position of equilibrium and allowing it to return to its initial position, the resisting force will act all the time. This signifies that a body will cover the same distance in a longer time. It means that the period of damped oscillation will be larger than that of free oscillations. From expression (2.6.7) it can be seen that the difference in oscillation frequency becomes larger the larger the damping factor β. Under greater resistance force, oscillations degenerate into the aperiodic process.

Figure 2.13 shows a graph of the time dependencies $\xi(t)$ and $A(t)$ (at the initial phase $\varphi = 0$). The dashed line expresses a change of oscillation amplitude (2.6.11) in the course of

Table 2.1

Magnitudes of the logarithmic decrements of damping of some systems

System	Decimal logarithm of the oscillating system
Acoustic waves in gases	−1
Electric oscillating systems	−2
Tuning fork	−3
Quartz plate	−5

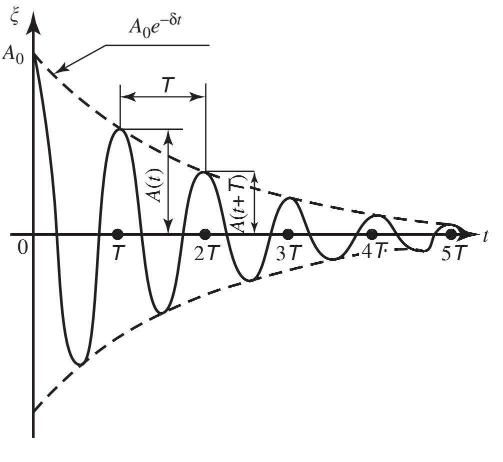


Figure 2.13 Damping oscillations.

time. If friction in the system increases sufficiently to be commensurate with the magnitude of the free oscillation frequency ω_0, the oscillations become less and less. Finally, when $\delta > \omega_0$, oscillations become completely impossible (this corresponds to an imaginary magnitude of frequency, refer to (2.6.7)). Then the system becomes a damping one.

2.7 FORCED OSCILLATIONS

If a periodical force is acting on a system in addition to restoring and damping forces, the so-called forced oscillations will take place. Consider the simplest case of forced oscillations when the driven force changes according to the periodical law

$$F = F_0 \cos \omega t, \tag{2.7.1}$$

where ω is the frequency of the driven force and F_0 is its amplitude. In addition to the driven force, a quasi-elastic $(-\beta\xi)$ and a resistance force $(-r\xi)$ are acting on the system as discussed before. The equation of motion can be written in the following way:

$$m\ddot{\xi} = -\beta\xi - r\dot{\xi} + F_0 \cos \omega t. \tag{2.7.2}$$

Let us divide the above equation by mass m:

$$\ddot{\xi} = -\frac{\beta}{m}\xi - \frac{r}{m}\dot{\xi} + \frac{F_0}{m}\cos \omega t. \tag{2.7.3}$$

We will introduce the definitions used in the previous section: $\omega_0^2 = \beta / m$ and $\delta = \mathrm{r} / 2m$; let us denote F_0 / m by f_0. The equation will then take the form $\ddot{\xi} = -\omega_0^2\xi - 2\delta\dot{\xi} + f_0 \cos \omega t$. For convenience we present the driven force in the complex form $F = F_0 \mathrm{e}^{\mathrm{i}\omega t}$. The equation will then be as follows:

$$\ddot{\xi} + 2\delta\dot{\xi} + \omega_0^2\xi = f_0 \mathrm{e}^{\mathrm{i}\omega t}.$$

The steady forced oscillations, i.e., oscillations occurring after a time long enough to ensure their stability, will be accomplished with a frequency equal to the frequency of the driven force ω. Because of the system's inertia, the displacement will be detained upon the phase φ. Hence, the solution of the above equation can be rewritten as

$$\xi(t) = A\mathrm{e}^{\mathrm{i}\omega t}, \tag{2.7.4}$$

where A is the complex amplitude containing the phase multiplier $\mathrm{e}^{\mathrm{i}\varphi}$. Therefore, the solution of eq. (2.7.2) is now known with accuracy to the phase multiplier and to the magnitude of the oscillation amplitude. Our task is just to find these values.

Introduce into eq. (2.7.2) the expressions $\dot{\xi} = \mathrm{i}A\omega \mathrm{e}^{\mathrm{i}\omega t}$ and $\ddot{\xi} = -A\omega^2\mathrm{e}^{\mathrm{i}\omega t}$. Therefore, $-A\omega^2\mathrm{e}^{\mathrm{i}\omega t} + 2\delta \mathrm{i}A\omega\mathrm{e}^{\mathrm{i}\omega t} + \omega_0^2 A\mathrm{e}^{\mathrm{i}\omega t} = f_0\mathrm{e}^{\mathrm{i}\omega t}$, Divide the equation by a common multiplier and find from it the amplitude as a function of the frequency of the driven force:

$$A(\omega) = \frac{f_0}{\omega_0^2 - \omega^2 + 2\mathrm{i}\delta\omega}$$

The complex quantity $\mathrm{Z}=(\omega_0{}^2-\omega^2) + \mathrm{i}2\delta\omega$ is presented in the denominator whose modulus is $\left|\mathrm{Z}\right|= \sqrt{(\omega^2-\omega^2)^2 + 4\delta^2\omega^2}$. This complex number can be presented in the form $\mathrm{Z}=\left|\mathrm{Z}\right|\mathrm{e}^{\mathrm{i}\phi}$, where $\varphi=\arctan(2\delta\omega/\omega_0{}^2-\omega^2)$. The expression for the amplitude receives the form

$$A(\omega,\varphi) = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\delta^2\omega^2}}\mathrm{e}^{-i\varphi}.$$

As can be seen from this expression, amplitude depends both upon ω and φ. We are not interested in the dependence on φ. Determine only the $A(\omega)$ dependence. It takes the form

$$A(\omega) = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\delta^2\omega^2}}. \tag{2.7.5}$$

Consider first the case of the absence of a resisting force ($\delta = 0$). Then eq. (2.7.5) simplifies to

$$A(\omega) = \frac{f_0}{\left|\omega_0^2-\omega^2\right|}. \tag{2.7.6}$$

Let us analyze this expression. If $\omega \ll \omega_0$, then $A = f_0/\omega_0 = F_0/m\omega^2$ or $A = f_0/\omega^2$. The damped oscillation amplitude in this case turns out to be practically equal to the displacement, caused by constant force F_0 in the system, characterized by the quasi-elastic coefficient β.

If the frequency is increased ($0 < \omega < \omega_0$), the oscillation amplitude increases, as can be seen from expression (2.7.6), and under $\omega = \omega_0$ will go to infinity. With a further increase in frequency ($\omega > \omega_0$) the amplitude decreases and at $\omega \gg \omega_0$ will practically not depend on the elastic characteristics since in this case it is possible to neglect the natural oscillation frequency ω_0 in comparison with ω. Then $A = f_0/\omega^2$ and at $\omega \to \infty$ the forced oscillation amplitude will go to zero.

Figure 2.14 presents the dependence of forced oscillation amplitude on the force frequency. In the absence of damping at $\omega = \omega_0$ the amplitude reaches infinity. This result is practically unrealistic, since it means an infinitely great energy of oscillatory motion. In fact, due to the resistance force, the oscillation amplitude remains finite. Its magnitude is

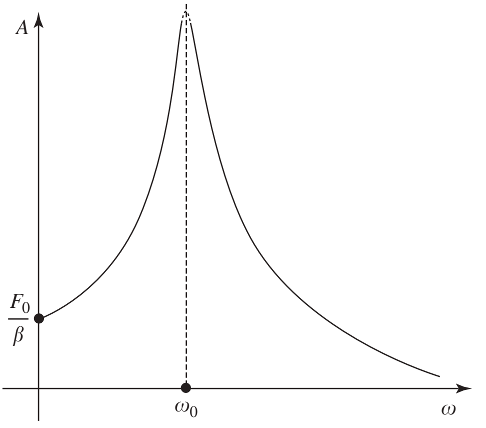


Figure 2.14 Amplitude of forced oscillations versus a frequency of external driving force in the absence of damping.

determined by the formula (2.7.5). (This expression at $\delta = 0$ turns over to (2.7.6), considered above.) Thereby, the amplitude of the given oscillating system depends on damping coefficient value δ and correlations of natural and damped oscillation frequencies.

Consider now how the amplitude of forced oscillation will behave when damping takes place. With a driven force frequency equal to zero (oscillation will not take place, but will become steady-state displacement under the action of force F_0):

$$A_{CT} = \frac{f_0}{\omega_0^2} = \frac{f_0}{m\omega^2} = \frac{F_0}{\beta}.$$

This coincides with the case of the absence of resistance. The amplitude increases with the force frequency increase because the denominator in eq. (2.7.5) decreases. When w approaches ω_0, the amplitude of forced oscillation increases, reaches a maximum and then decreases at $\omega >> \omega_0$. The increase in amplitude of forced oscillation at the approach of the frequency of the driven force to that of natural frequency is called resonance. In order to determine the resonance frequency in the presence of resistance, we must use the well-known principle for finding extrema $\delta A/\delta\omega = 0$. Since the denominator in expression (2.7.5) is constant, it is easier to find the extremum of the radicand in the denominator. We obtain

$$2(\omega_0^2 - \omega_{res}^2)(-2\omega_{res}) + 8\delta^2\omega_{res} = 0,$$

from which we can arrive at the resonance frequency ω_{res}

$$\omega_{res} = \sqrt{\omega_0^2 - 2\delta^2}. \tag{2.7.7}$$

Then the resonance amplitude of forced oscillations will be

$$A_{\text{res}} = \frac{f_0}{2\delta\sqrt{\omega_0^2 - \delta^2}}. \tag{2.7.8}$$

At small damping ($\delta << \omega_0$)

$$A_{\text{res}} \approx \frac{f_0}{2\delta\omega_0}. \tag{2.7.9}$$

The dependence of the forced oscillation amplitude A on the driving force frequency ω under different damping factors δ is depicted in Figure 2.15. The following conclusion can be made from consideration of this graph: the less the resistance force (i.e., δ is small), the sharper the resonance peak and the closer the resonance frequency ω_{res} to the natural frequency ω_0. On the contrary, at significant resistance the resonance peak is smoothed and shifted to the low frequency region.

At small resistance the resonance amplitude is given by eq. (2.7.7), whereas at static shift it is $A_{\text{st}} = f_0/\omega_0$. Now the ratio of the amplitudes is

$$\frac{A_{\text{res}}}{A_{\text{st}}} = \frac{f_0/2\delta\omega_0}{f_0/\omega_0^2} = \frac{\omega_0}{2\delta}. \tag{2.7.10}$$

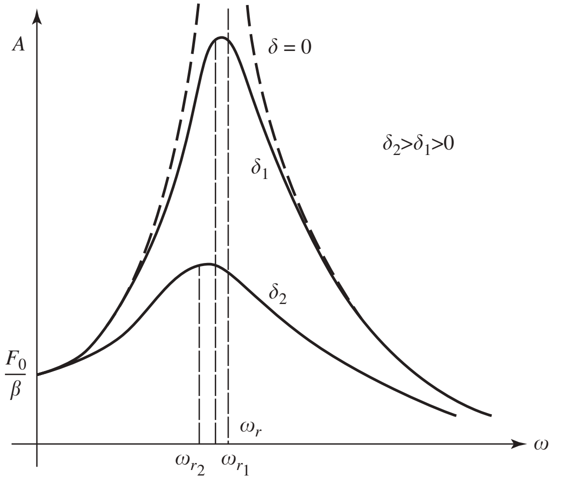


Figure 2.15 Amplitude of forced oscillations versus a frequency of external driving force in the presence of damping: the smaller the damping the higher the oscillation amplitude; ω_r are the resonance cyclic frequencies.

Obviously, at $\delta << \omega_0$ the ratio will be very large. This explains the great significance of resonance phenomena in physics and technology. It is widely used when a weak effect is to be measured. The best example of this is radio broadcasting. On the other hand, if the resonance phenomena can lead to some destruction it must be avoided. The resonance technique is widely used nowadays in methods of chemical structure investigations (refer to Chapter 8). Such a property of an instrument as a *figure of merit* is its critical property. It characterizes the sharpness of the spectral lines in which the main information of the subject studied is included. Let us begin with eq. (2.7.5). Dividing both nominator and denominator by ω_0 we arrive at

$$A(\omega) = \frac{f_0/\omega_0^2}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)+\left(2\delta\frac{\omega}{\omega_0^2}\right)^2}}. \tag{2.7.11}$$

Value $\omega_0\,/2\delta = G$ is called the figure of merit. The above equation can be rewritten using this definition:

$$A(\omega) = \frac{\frac{f_0}{\omega_0^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)+\left(\frac{1}{G}\frac{\omega}{\omega_0}\right)^2}}. \tag{2.7.12}$$

The time of merit characterizes the rate of the energy loss by an oscillating system (oscillator). At small damping $\Delta E = E(t) - E(t - T)$ (T is the period of damping oscillations, which in this case is equal to the period of natural oscillations). Then $E(t) = E_0\exp(-2t/\tau)$ $= E_0 e\ -(\omega_0/G)t$ (because $2/\tau = 2\delta(\omega_0/G)$). Therefore,

$$\Delta E = E(t)\left[1-\exp\left(\frac{2T}{\tau}\right)\right] \approx E(t)\frac{2T}{\tau} = E(t)\frac{2\pi}{G}.$$

The relative energy loss for the period is

$$\frac{\Delta E}{E} = \frac{2\pi}{G}. \tag{2.7.13}$$

Regarding electrical oscillations, the value $E/\Delta\mathrm{E}$ is called the figure of merit of a vibration contour. The width of the spectral line at half of its height at $\omega = \omega_0$ (its *half-width*) is called the *figure of merit.* The values of resonance characteristics in some systems are presented in Table 2.2.

Table 2.2

Values (up to the order of magnitude) of the figures of merits of some systems

System	Decimal logarithm of the natural oscillation frequencies (Hz)	Decimal logarithm of the figures of merit	Decimal logarithm of the ratio $\frac{\Delta E}{E}$
Oscillation counter	4	2	−2
Resonator in a quartz watch	5	4	−4
Optical spectral line	14÷15	5÷7	−5÷−7
CO_2 laser	13	9	−9
γ-Radiation of atomic nuclei in Mössbauer effect (refer to Chapter 8).	19	9÷15	−9÷−15

EXAMPLE E2.9

A body of weight $m = 10$ g makes a damping oscillation in a viscous media. The resistance coefficient is $\delta = 2\,4 \times 10^{-4}$ kg/sec. Determine the ratio $|\Delta E|/E_0$ of energy loss by the body in time $\tau = 1$ min.

Solution. Two forces operate on an oscillating body in the presence of damping. One of them is a quasi-elastic force $F = -\beta\xi$, where β is the coefficient of the quasi-elastic force and ξ is deformation. The other force is the force of resistance dependent on the speed of the moving body $F_c = -rv = -r\dot{\xi}$.

According to the second Newtonian law, the equation of movement in a projection on an x-axis can be written as $m\ddot{\xi} = -\beta\xi - r\dot{\xi}$. Dividing both sides of the equality by mass m, making some replacements $r/m = 2\delta$ and $\beta/m = \omega_0^2$ and rearranging the terms, we obtain $\ddot{\xi} + (r/m)\dot{\xi} + (\beta/m)\xi = 0$ or in compact form $\ddot{\xi} + 2\delta\dot{\xi} + \omega_0^2\xi = 0$ (refer to Section 2.6).

Solving this differential equation we obtain the time dependence of the amplitude of damping oscillations $A(t) = A_0\exp(-\delta t)$*, where A_0 is the initial amplitude of oscillations. The total mechanical energy is dependent on amplitude $E(t) = ½\beta A^2(t)$ (refer to eq. (2.5.5)) where $A(t)$ is substituted by the equation *. We shall obtain $E(t) = ½\beta A_0^2\exp(-2\delta t)$, where $½\beta A_0^2$ is the initial energy E_0. Therefore, the energy of the damping oscillation's time dependence can be expressed as $E(t) = E_0\exp(-2\delta t)$.

The lost energy ratio at damping oscillation in time τ can be found by dividing $|\Delta E| = E_0 - E(\tau)$ by the initial energy E_0:

$$\frac{|\Delta E|}{E_0} = \frac{E_0 - E(\tau)}{E_0} = 1 - \frac{E(\tau)}{E_0}$$

or

$$\frac{\Delta E}{E_0} = 1 - \exp(-2\delta\tau).$$

Returning to the initial definitions, we substitute δ into r/m and arrive at the final expression

$$\frac{|\Delta E|}{E_0} = 1 - \exp\left(-\frac{r}{m}\tau\right).$$

After all calculations, we obtain

$$\frac{|\Delta E|}{E_0} = 1 - \exp\left(-\frac{2\times 10^{-4}}{10^{-2}}\times 60\right) = 1 - \exp(-1.2) = 1 - 0.301 = 0.699.$$

That is, the loss of energy is approximately 70% of the initial value.

EXAMPLE E2.10

A body of weight m = 0.2 kg is attached to one end of a spring with rigidity β = 2 N/cm = 200 N/m. The body can move along a horizontal pivot without friction. The other end of the spring is fixed (refer to Figure 1.22 and Section 2.7). The oscillations occur in viscous media. An external harmonic variable force operates on the body: $F(t) = F_0 \cos \omega t$ where F_0 is the force amplitude value (F_0 = 3 N) and ω is its angular frequency. For this system, define the resonant frequency ω_{res} and resonant amplitude A_{res}. Make calculations for two values of the resistance coefficients r_1 = 0.5 kg/sec and r_2 = 5 kg/sec.

Solution: Let us consider the forces working on the body. There are several forces acting in a vertical direction; however, they all mutually compensate each other (according the third Newtonian law) and are therefore excluded from our consideration. Operating along the horizontal direction are: (1) a periodically changing with frequency ω external force $F(t) = F_0 \cos \omega t$, (2) an elasticity force $F_{el} = -\beta\xi$, (3) the velocity-dependent force of resistance $F_r = -rv$ (see Section 2.6, eq. (2.6.1)). Therefore, the equation of the body's movement can be written as: $m\ddot{\xi} = -\beta\xi - r\dot{\xi} + F_0 \cos \omega t$. This equation was solved in Section 2.7 and the results are $\omega_{res} = \sqrt{\omega_0^2 - 2\delta^2}$ * and amplitude $A_{res} = f_0 / 2\delta\sqrt{\omega_0^2 - \delta^2}$ **. There are two frequencies: ω characterizes the driven force and ω_0 is the natural frequency of the free system.

For visualization we first calculate separately the natural frequency $\omega_0 = \sqrt{\beta / m}$. $= \sqrt{200 / 0.2} = 31.62\ \mathrm{sec}^{-1}$. The fourth significant digit is useful here because it ensures that we do not lose the accuracy at intermediate calculations. $\delta_1 = r_1 / 2m$

$= 0.5 / (2 \times 0.2) = 1.25\ \text{sec}^{-1}$. The same is valid for $\delta_2 = 12.5\ \text{sec}^{-1}$. In the first case $2\delta_1^2 \ll \omega_0^2$ ($3.12 \ll 1000$), and hence $\omega_{res} \approx \omega_0 = 31.62\ \text{sec}^{-1}$. In fact, calculations according to the precise formula* give $31.57\ \text{sec}^{-1}$, so the coincidence is up to the third significant digit.

In the second case $2\delta_2^2 = 312\ \text{sec}^{-2}$ and $\omega_0^2 = 1000\ \text{sec}^{-2}$, and therefore calculations should be executed mainly according the precise formula *: $\omega_{res} = \sqrt{1000 - 312}\ \text{s}^{-1} = 26.2\ \text{sec}^{-1}$.

Let us next calculate the resonance amplitude. In the first case of δ_1 we can use the insignificance of δ_1^2 in comparison with ω_0^2 ($1.56 \ll 1000$). An approximate formula gives

$$A_{res,1} = \frac{f_0}{2\delta_1\omega_0} = \frac{15}{2\times1.25\times31.6} = 0.19\ \text{m} = 19\ \text{cm}$$

In the second case ($\delta_2 = 12.5\ \text{sec}^{-1}$) the approximate formula is invalid ($\delta_2^2 = 156\ \text{sec}^{-2}$ is commensurable with $\omega_0^2 = 1000\ \text{sec}^{-2}$) and calculations have to be made according to precise formula **:

$$\text{A}_{res,2} = \frac{\text{f}_0}{2\delta_2\sqrt{\omega_0^2 - \delta^2}} = \frac{15}{2\times12.5\sqrt{1000-156}} = 0.0207\ \text{m or A}_{res,2} = 2.07\ \text{cm.}$$

In conclusion we can present the static displacement under the action of the force F_0($\xi_{max} = F_0/\beta$) (Hooke's law), i.e., $\xi_{max} = (3/200)$ m $= 1.5$ cm. One can see that in the first case at small damping the resonance phenomena are more pronounced than at high damping: $A_{res} / \xi_{max} = 29 / 1.5 = 12.7$.

2.8 WAVES

2.8.1 Introductory remarks

Oscillations originating from any source propagate further in space. The propagating oscillations are referred to as waves.

It was noted in the preceding sections that the mathematical descriptions of different kinds of oscillations are similar, thus allowing a general mathematical description to be made regardless of the type oscillation. Different waves exist (mechanical, electromagnetic, acoustic, etc.), depending on what physical value is "propagated"; herewith their mathematical description will once again be the same. Thus, we shall mostly consider mechanical waves, bearing in mind the possibility of applying the results to other kinds of waves, e.g., electromagnetic waves.

Mechanical waves can propagate only in an elastic media. If particle vibrations are agitated in a region of an elastic medium (solid, liquid or gaseous), as a consequence of the interaction between particles, this disturbance will be transmitted to surrounding particles, which in turn, will distribute excitation further. In this manner, the wave appears.

The process is not instantaneous; a wave propagates with a speed υ, which depends on the properties of the medium. However, it must be noted that no transportation of the medium's particles take place, particles oscillate around their permanent equilibrium positions.

We can distinguish different kinds of waves by considering how the motion of the constituent particles is related to the direction of propagation. Distinction is made between two kinds of waves. A wave is called *longitudinal* if the direction of particle oscillations coincides with the direction of wave propagation (Figure 2.16). Longitudinal waves can be agitated in a medium that is elastic in terms of compression and stretching. All media—solid, liquid and gaseous—possess these properties.

A wave is called *transverse* if the medium particles oscillate in a direction perpendicular to the direction of propagation (Figure 2.17). It follows from this definition that the transverse

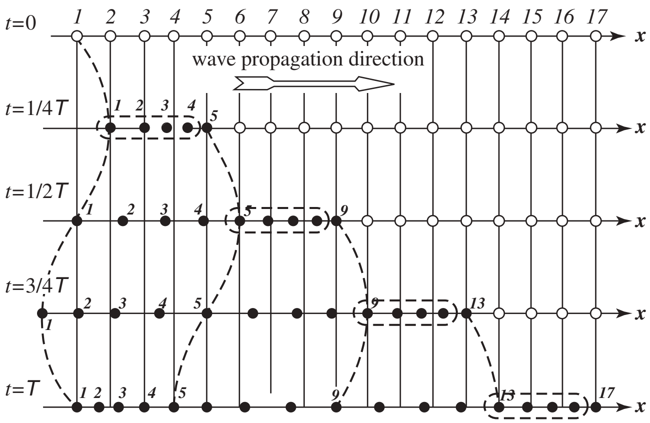


Figure 2.16 Generation of a longitudinal wave.

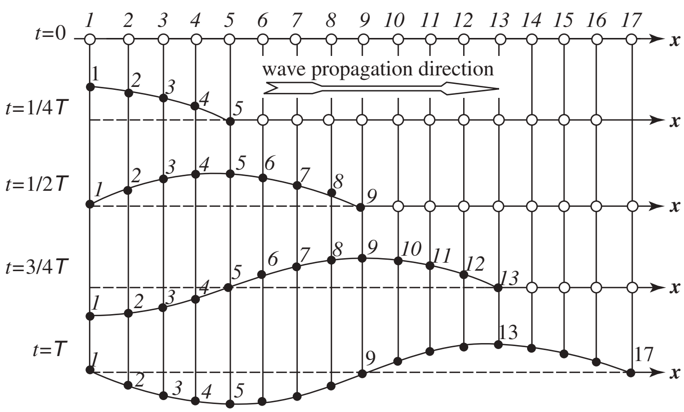


Figure 2.17 Generation of a transverse wave.

wave can propagate in media that possess the property of elastic deformation of slip. Only solid media possess this property; only in solid media can the transverse wave propagate. Hence, only in solid media can both transverse and longitudinal waves be propagated.

The space in which the wave propagation occurs is called the *wavefield*. The geometrical locus of points that, at a given instant, the propagation process reaches, is called the *wavefront*. For a periodic wave one can draw a surface through the points that oscillate in phase. This surface is called a *wave surface*. The extreme wave surface is the wavefront. The direction of propagation in isotropic media is always perpendicular to the wavefront. A line perpendicular to the wavefront is called a *ray*. From the wave's viewpoint a ray is an imaginary line along the direction of travel of the wave. A bundle of parallel rays form a *beam.*

In isotropic elastic media all waves propagate at the same speed. Therefore, if the source of waves is tightened down to a point the wavefront is *spherical* and the wave is also *spherical*. If the wave front is a plane, a *plane wave* is produced. If the initiating oscillation is harmonic, the wave produced in isotropic media is also harmonic.

2.8.2 An equation of a plane traveling wave

For the majority of problems it is important to know the dependence of oscillations of different points of media at a given instant. This dependence can be considered as determined if the amplitudes and phases of oscillation are known. For transverse waves it is also necessary to know the polarization. For a plane one-dimensional polarized wave it is sufficient to have an expression defining the displacement of any wave point of $\xi(x,t)$ with the coordinate x in the instant of time t. Such an expression is called an *equation of wave.*

Consider a so-called *traveling wave*, i.e., a wave propagated in one direction. Direct the x-axis along the wave propagation. In this case the wavefront is perpendicular to axis x. Let particles of media, just verging on the source of plane waves, accomplish harmonic oscillations according to the harmonic law $\xi(0,t) = A \cos \omega t$ (Figure 2.18). In Figure 2.18a the

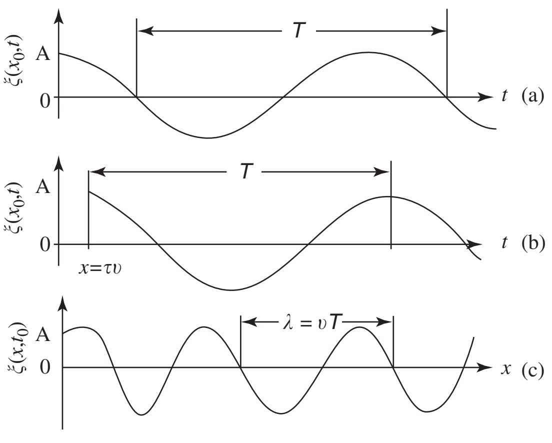


Figure 2.18 Schematic representation of a traveling wave: (*a*) oscillations at the oscillation source origin $\xi(0,t)$, (*b*) oscillations at the distance x from the origin ($x = \tau\upsilon$, τ being the delay relative to the source oscillations), (*c*) deviation of the wave's particles from their equilibrium positions $\xi(x,t_0)$ at a time instant t_0. The periods T and wavelength λ are shown.

displacement of particles at $x = 0$ is presented. Zero time has been chosen to ensure the initial phase is also zero. Particles lying in this plane are all oscillating in the same phase.

Find the expression $\xi(x,t)$ for the displacement of particles that are at distance x from the wave's source (origin). The wavefront covered this distance in time $\tau = x/v$. This means that the vibrations at point (plane) x will be behind by the time τ from that in origin. These points will also accomplish the harmonic oscillations but with propagation delay τ. In the absence of damping the oscillation amplitude is constant. Therefore,

$$\xi(x,t) = A\cos\omega(t-\tau). \tag{2.8.1}$$

This is the plane *traveling wave equation*. As already noted, the equation allows us to define a displacement ξ of the media particles with coordinate x at the instant of time t. The phase of oscillation $\omega\ [t - (x / \upsilon)]$ depends on two variables: particle coordinate x and time t. At a given fixed instant of time the phases of the different particles will, generally speaking, be different. However, it is possible to select particles oscillating in the same phase (in-phase). The phase difference is $2\pi m$ (where $m = 1, 2, 3$). The shortest distance between two particles of a traveling wave, oscillating in phase, is called the *wavelength* λ.

Find the relationship of wavelength λ with other values, characterizing the wave propagation in a definite media. In accordance with the definition of the wavelength we can write

$$\omega\left(t - \frac{x+\lambda}{\upsilon}\right) - \omega\left(t - \frac{x}{\upsilon}\right) = 2\pi$$

or, after cancellations, $\omega\lambda / \upsilon = 2\pi$. Since $\omega = 2\pi / T$, then

$$\lambda = \upsilon T. \tag{2.8.2}$$

This expression allows another definition of the wavelength: wavelength is the distance a wave can propagate for a time equal to the period of oscillation.

The traveling wave equation develops thereby a *double periodicity*: on the coordinate x and on time t. One can, for instance, fix a particle coordinate (x = const.) and consider its displacement as a function of time. Alternatively, one can fix a moment of time (t = const.) and consider particle displacement as a function of coordinates. So, standing on a pier one can take a picture of the surface of the sea at time instant t, or having thrown an object into the sea (i.e., having fixed a coordinate x), one can check its oscillation in time. Both these cases are given as graphs in Figure 2.18.

Equation of wave (2.8.1) can be written in another way:

$$\xi(x,t) = A\cos\left(\omega t - \frac{\omega}{\upsilon}x\right).$$

The ratio ω / υ is usually defined by letter k, referred to as the *wavenumber*

$$k = \frac{\omega}{\upsilon}. \tag{2.8.3}$$

Since $\omega = (2\pi/T)$ and $(2\pi/T\upsilon) = (2\pi/\lambda)$,

$$k = \frac{2\pi}{\lambda}. \tag{2.8.4}$$

Hence, the wavenumber shows how many wavelengths can be placed in a length of 2π units. One can now rewrite the equation for the traveling wave in the most popular form

$$\xi(x,t) = A\cos(\omega t - kx). \tag{2.8.5}$$

Find the relation of the oscillation phase difference $\Delta\varphi$ of two particles with a difference of their coordinates $\Delta x = x_2 - x_1$. Using eq. (2.8.5) $\Delta\varphi$ can be written as

$$\Delta\varphi = (\varphi t - kx_1) - (\varphi t - kx_2) = k(x_2 - x_1).$$

Then $\Delta\varphi = k\Delta x$ or, according to (2.8.4),

$$\Delta\varphi = \frac{2\pi}{\lambda}\Delta x. \tag{2.8.6}$$

The plane wave propagating in an arbitrary direction can be expressed as

$$\xi(x,t) = A\exp[\mathrm{i}(\omega t - \mathbf{kr})],$$

where $\mathbf{r}$ is a radius vector, drawn from the origin to a point where a particle occurs, $\mathbf{k}$ is the wave vector equal modulo as eq. (2.8.4) and coinciding with the direction of propagation (or to the normal to the wave surface).

The exponential form is also appropriate for wave description. So, in the case of a plane wave propagating along x-axis

$$\xi(x,t) = A\exp[\mathrm{i}(\omega t - kx)] \tag{2.8.7}$$

and in the general case of an arbitrary direction

$$\xi(x,t) = A\exp[\mathrm{i}(\omega t - \mathbf{kr})]. \tag{2.8.8}$$

The traveling wave equation can be obtained as the solution of a differential wave equation referred to as the *wave equation*. Knowing the solution in the forms (2.8.5) and/or (2.8.7) we can find the wave equation itself. Differentiation of the equation of a plane wave $\xi(x, t)$ twice upon the time and upon the coordinate gives

$$\frac{\partial^2 \xi}{\partial x^2} = -k^2 A \exp[\mathrm{i}(\omega t - kx)] = -k^2 \xi$$

$$\frac{\partial^2 \xi}{\partial t^2} = -\omega^2 \xi$$

and equating the ξ value from both equations, we can reduce it to

$$\frac{\partial^2 \xi}{\partial x^2} \times \frac{1}{k^2} = \frac{\partial^2 \xi}{\partial t^2} \times \frac{1}{\omega^2}.$$

Taking eq. (2.8.3) into account, we can write

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{\upsilon^2} \frac{\partial^2 \xi}{\partial t^2} \tag{2.8.9}$$

This is the one-dimensional wave equation.

In the general case this equation looks as follows:

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = \frac{1}{\upsilon^2} \frac{\partial^2 \xi}{\partial t^2} \tag{2.8.10}$$

or

$$\Delta \xi = \frac{1}{\upsilon^2} \frac{\partial^2 \xi}{\partial t^2}, \tag{2.8.11}$$

where Δ is the Laplace operator

$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right).$$

The speed of propagation of the constant phase point is referred to as the phase speed. In other words, this is the propagation speed of the wave crest or hollow as well as any other wave point preserving the constant phase. The wavefront is the surface of the constant phase. Consequently, the speed of the wavefront is just the phase speed. Therefore, the speed, determined by eq. (2.8.3),

$$\upsilon = \frac{\omega}{k}, \tag{2.8.12}$$

is also the wave phase speed.

The same results can be obtained by finding the speed of propagation of the wave points with constant phase, $\omega t - kx = \text{const}$. Finding the dependence of coordinates upon the time $x = \frac{1}{k}(\omega t - \text{const.})$ we can derive the speed of points with a constant phase $\upsilon = dx/dt = \omega/k$, that coincides with eq. (2.8.12).

The equation of the plane traveling wave in the opposite direction is written as $\xi(x, t) = A\cos(\omega t + kx)$. The phase speed in this case is negative

$$\frac{dx}{dt} = -\frac{\omega}{k} = -\upsilon.$$

The phase velocity in a given media depends both on the properties of the medium and the frequency of the source of oscillations. This relationship is called dispersion, whereas the media relates to dispersive media. (One should not think that expression (2.8.12) is the dependence discussed. The point is that in the absence of dispersion the wavenumber k is directly proportional to the frequency ω, therefore, $(\omega/k) = \text{const}$. Dispersion takes place when $\omega(k)$ is nonlinear (refer to (2.9.4).)

The plane traveling wave is monochromatic if the source of oscillations is harmonic. Eqs. (2.8.5) and (2.8.7) describe monochromatic waves. In linear media this corresponds to the fixed wavelength.

It must be emphasized that the amplitude A and frequency ω of a monochromatic wave is accepted as being independent of time. This means that a monochromatic wave must be infinite in time and space, i.e., is an idealized model. Any real wave is not monochromatic from a formal point of view (refer to Section 7.2). However, the longer the time such a wave is maintained, the nearer the wave is to being monochromatic. In practice, a wave is considered as monochromatic if it lasts sufficiently long.

2.8.3 Wave energy

As mentioned earlier, there is no macroscopic transfer of matter accompanying a wave. This, however, does not mean that there is no energy transfer with wave propagation. On the contrary, forcing every particle to oscillate, a wave carries that energy which is consumed in its creation. This energy can be easily calculated. To make this calculation, the total kinetic energy of all particles participating in oscillation must be counted:

$$W = N\frac{m\upsilon_{\max}^2}{2} = \frac{\rho\upsilon_{\max}^2}{2}\Delta V. \tag{2.8.13}$$

Here, W is the total energy of all particles in the volume ΔV, N and m are their number and mass, respectively, and ρ is the media density. To estimate the $\upsilon_{\max}$ value, we can proceed from expression (2.2.4):

$$\upsilon_{\max} = \omega A,$$

as the trigonometric term is equal to unity. Substituting this equation into eq. (2.8.4) we can find the total energy contained in the volume ΔV of the oscillating media:

$$W = \frac{\rho\omega^2 A^2}{2}\Delta V$$

or

$$\frac{W}{\Delta V} = \frac{\rho\omega^2 A^2}{2} = w, \tag{2.8.14}$$

where $w = (W/\Delta V)$ is the volumetric energy density.

Find the *wave energy flux* Φ, i.e., the energy carried by a wave through the area S perpendicular to wave propagation. So it is defined as a scalar flux averaged upon the area S. It can, however, be different both at different points of the area and in different directions. In order to characterize the energy flux locally the value of the flux density j is introduced:

$$j(\mathbf{r}) = \frac{d\Phi}{dS_\perp}, \tag{2.8.15}$$

which is equal to the energy flux through the unit area perpendicular to the propagation direction. The energy flux density can depend on the direction. Therefore, it has to be defined as a vector numerically equal to $d\Phi/dS_\perp$. Therefore,

$$j(\mathbf{r}) = \frac{d\Phi}{dS_\perp}\cdot\frac{\boldsymbol{\upsilon}}{\upsilon}, \tag{2.8.16}$$

where $\boldsymbol{\upsilon}/\upsilon$ is the unit vector of the wave propagation direction.

Now find connection of the energy flux density with the wave phase velocity. Choose the area $\Delta S_\perp$ perpendicular to the unit vector $\boldsymbol{\upsilon}/\upsilon$ and calculate the energy carried through this area in the unit time interval Δt (Figure 2.19). In the time interval Δt through the area defined above there will be energy ΔE will carried through; it contains in the cylindrical volume with a height $\upsilon\Delta t$ and a base area $\Delta S_\perp$. This energy can be expressed as the product of the volumetric energy density w (2.8.14) and volume ΔV. The energy carried will be $\Delta E = w\Delta S_\perp \boldsymbol{\upsilon}\Delta t$ and the energy flux $\Delta\Phi = \Delta E/\Delta t = w\Delta S_\perp \boldsymbol{\upsilon}$. The scalar energy flux density is $\mathrm{j} = \Delta\Phi/\Delta S_\perp = w\boldsymbol{\upsilon}$. Taking the vector characters $\mathbf{j}$ and $\boldsymbol{\upsilon}$ into account we obtain

$$\mathbf{j} = w\boldsymbol{\upsilon}. \tag{2.8.17}$$

The total energy flux $d\Phi$ (2.8.15) can be determined as

$$d\Phi = jdS_\perp. \tag{2.8.18}$$

In order to obtain a more general significance of the last expression, attach to the elementary area dS the vector character by taking into account the different orientations of the area dS regarding the vector field. Figure 2.20 shows the disposition of the area $dS_\perp$ as part of the more general area dS projected onto a plane perpendicular to the vector $\mathbf{j}$.

Attribute to the scalar value dS a vector character by multiplying dS by the normal unit vector:

$$d\mathbf{S} = dS\mathbf{n}, \text{or } dS = |d\mathbf{S}|\cos\alpha, \tag{2.8.19}$$

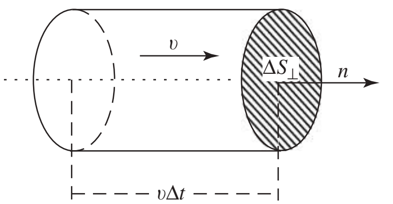


Figure 2.19 An elementary flow $d\Phi$ of a vector $\boldsymbol{\upsilon}$ through the area dS normal to $\boldsymbol{\upsilon}$.

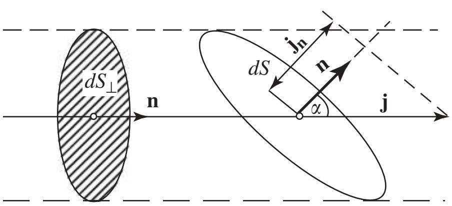


Figure 2.20 An elementary flow $d\Phi$ of a vector $\mathbf{j}$ through the area dS at their arbitrary orientation.

where α is an angle between vectors **n** and **j**. At $\alpha = 0$ the equation $|d\mathbf{S}| = dS$ takes place. Now expression (2.8.18) can be given in another form:

$$d\Phi = (\mathbf{j}\, d\mathbf{S}), \tag{2.8.20}$$

i.e., elementary flux $d\Phi$ of the energy flux density **j** through the elementary area $d\mathbf{S}$ is the scalar product of vectors **j** and $d\mathbf{S}$. The total flux Φ through the surface S can be written more generally as

$$\Phi = \int_S \mathbf{j}\, d\mathbf{S}. \tag{2.8.21}$$

and

$$\Phi = \int_S \mathbf{j}_n\, d\mathbf{S} \text{ or } \Phi = \int_S \mathbf{j}\, d\mathbf{S}_n. \tag{2.8.22}$$

Expression (2.8.21) is the particular case of the vector flux calculations through the surface. It can also be used in evaluation of liquid mass consumption in pipes, electrostatic and magnetic strength vectors, etc. (refer to Chapters 4 and 5).

2.8.4 Acoustic Doppler effect

Hitherto, we have considered the propagation of waves in a media in a coordinate system in which both source and receiver are unmovable. Let us examine further how a wave frequency will change when turning from one inertial coordinate system to another. In other words, what frequency will a wave detector measure if it moves with respect to the source in the same direction? The relationship between frequency of the relative motion of the wave source and detector, which was originally established by P. Doppler, is well known both in acoustics and optics; in spite of the fact that they are related to different principles, the resulting mathematical expressions are mainly the same. Note, however, that we deal with nonrelativistic cases.

We will restrict ourselves only to acoustic waves. In the resting reference system K, let a plane wave be generated propagating in direction x. Denote the frequency by ω. Determine what frequency ω' will the wave detector perceive in system K' moving with velocity V_0 along the same direction x relative to the system K (Figure 2.21).

Apply the wave equation to the system K, i.e., $\xi\,(x,t) = a\exp\{-\mathrm{i}\omega t + \mathrm{i}kx\}$. Turning to another coordinate system K' we should use the Galileo transforms (1.3.1). In the other system, the equation takes the form

$$\xi(x',t') = a\exp\{-\mathrm{i}\omega t' + \mathrm{i}k(x' + V_0 t')\} = a\exp\{-\mathrm{i}(\omega - kV_0)t' + \mathrm{i}kx'\}.$$

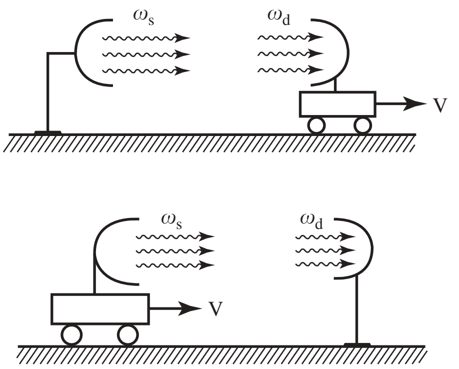


Figure 2.21 An acoustic Doppler effect; V is the mutual speed of source and detector, ω_s and ω_d are emitted and measured frequencies, d and s are detector and source.

Comparing these two equations, it can be seen that the detector in system K' will perceive a wave number process with a frequency equal to $\omega' = \omega - kV_0$ and the same wave $k' = k$. Hence, $k = \omega/\upsilon$, where υ is the propagation wave speed in system K; then

$$\omega' = \omega\left(1 - \frac{V_0}{\upsilon}\right). \tag{2.8.23}$$

Note that there is no contradiction between the change of frequency and stability of wavenumber: In fact, according to eq. (1.3.1) the velocity of wave propagation in system K' is equal to $\boldsymbol{\upsilon}' = \boldsymbol{\upsilon} - V_0$. Therefore,

$$k' = \frac{\omega'}{\upsilon'} = \frac{\omega'}{\upsilon - V_0} = \frac{\omega(1-(V_0/\upsilon))}{\upsilon - V_0} = \frac{\omega}{\upsilon} = k.$$

We use here the relation (2.8.23). This relation allows us to determine how much the frequency measured by the detector ω_{det} differs from that emitted by the wave source ω_{sor} provided they both move relative to each other.

Suppose now that the wave source is resting and the detector moves away from it at a speed V_0 (Figure 2.21). Then in expression (2.8.23) $\omega = \omega_{\text{sor}}$ and $\omega' = \omega_{\text{det}}$. Therefore,

$$\omega_{\text{det}} = \omega_{\text{sor}}\left(1 - \frac{\upsilon}{V_0}\right) < \omega_{\text{sor}}. \tag{2.8.24}$$

Changing the motion direction, the relative sign of the velocity also changes. Hence, in this case

$$\omega_{\text{det}} = \omega_{\text{sor}}\left(1 + \frac{\upsilon}{V}\right) > \omega_{\text{sor}}. \tag{2.8.25}$$

An analogous result is obtained when the detector is unmovable but the source is moving.

The Doppler effect is widely used, since it gives the possibility to change wave frequency, changing, for instance, velocity of source motion. This is exactly the way to obtain the resonance absorption of γ-rays in Mössbauer effect in γ-resonance spectroscopy (refer to Chapter 8).

2.9 SUMMATION OF WAVES

2.9.1 Superposition of waves

Now consider a situation in which, instead of one source, there are several sources of waves (oscillators). In a certain area of space, these waves can interact with each other. Here again we come across the very important physical phenomena – the principle of superposition. It relates equally to waves and to many other types of excitations. Its essence is exceedingly simple. Suppose that instead of a single source of excitation there are a number of sources in space (it can be mechanical vibrations, oscillation of electrical charges, electrical currents, etc.). What result will the measuring instrument register, accepting simultaneously all excitements from all sources? If each component does not influence the others, the total result will simply be the sum of the separate excitations. This precisely implies the superposition principle. This principle is common for many phenomena, but its mathematical expression can sometimes be different depending on the nature of the events considered, e.g., vector or scalar.

The principle of wave superposition is not valid for all cases but only for the so-called harmonic sources and linear media. A medium can be considered linear if its particles are under the action of quasi-elastic forces. Otherwise, a medium is nonlinear. Very unusual and important phenomena can appear in the latter case, e.g., the propagation of ultrasound and/or laser rays in nonlinear media. Extremely interesting and technically important phenomena can appear. Scientific and technical investigations dealing with nonlinear phenomena are referred to as nonlinear acoustics and optics.

Although nonlinear effects are of great importance in certain modern devices, we will only consider linear effects further. When applied to waves, the principle of superposition affirms that each wave is propagated regardless of the presence in the given media of other sources of waves. This can be mathematically expressed as

$$\xi(x,t) = \sum_{i=1}^{N} \xi_i(x,t) = \sum_{i=1}^{N} A_i \mathrm{e}^{-\mathrm{i}(\omega t - k_i x)}, \tag{2.9.1}$$

where N is the number of wave sources, $\xi(x, t)$ is the total wave.

Consider superposition of two waves generated by two sources $\xi_1(x, t) = A_1 \cos(\omega t + k_1 x)$ and $\xi_2(x, t) = \mathrm{A}_2 \cos(\omega t + k_2 x)$. Fix an arbitrary point M and examine the result of the superposition at this point. Fixing the point, we transform a wave into oscillations: $\xi_{1,\mathrm{M}}(t) = A_1 \cos(\omega t + k_1 x_\mathrm{M})$ and $\xi_{2,\mathrm{M}}(t) = A_2 \cos(\omega t + k_2 x_\mathrm{M})$, since a product $k_1 x_\mathrm{M}$ can be considered as a phase. In order to find the resulting oscillation process $\xi(t)$ we should sum ξ_1 and ξ_2 at the point M: $\xi(t) = \xi_1(t) + \xi_2(t)$. Such a problem was solved in Section 2.3.1

(refer to Figure 2.7 and eq. (2.3.1)). Correspondingly, we can write a total amplitude A through partial amplitudes A_1 and A_2:

$$A_M = \sqrt{A_1^2 + A_2^2 \, 1 \, 2A_1A_2 \cos(\varphi_2 \, 2 \, \varphi_1)}. \tag{2.9.2}$$

The value A_M depends on the difference of oscillation phases $\Delta\varphi = \varphi_2 - \varphi_1$. In Section 2.3.1 the situation was analyzed in detail. In particular, the total amplitude A_M can be changed from zero to $2A$ provided $A = A_1 = A_2$ and the phase difference $\Delta\varphi$ remains stable in time.

In order to observe interference, the phase difference should be constant. This can be obtained if the wave's sources are coherent and, vice versa, two sources are referred to as coherent if the phase difference remains constant. If waves are coherent the sources are coherent as well.

The methods of experimental interference phenomena will be considered in more detail in Chapter 6, which is devoted to wave optics.

2.9.2 Standing waves

A wave that appears as the result of the superposition of two similar waves coming from opposite directions is referred to as a standing wave. Find the equation of the standing wave. Let us assume that a flat traveling wave with amplitude A and angular frequency ω, extending along the positive direction of an axis x, meets a counter wave of the same amplitude and frequency. Equations of these primary waves can be written in trigonometric form as follows: $\xi_1 = A\cos(\omega t - kx)$ and $\xi_2 = A\cos(\omega t + kx)$, where ξ_1 and ξ_2 are the displacement of the medium's points caused by two running waves. According to the superposition principle, in an arbitrary point with coordinate x in time instance t the displacement ξ is the sum $\xi_1 + \xi_2$ or $\xi = A\cos(\omega t - kx) + A\cos(\omega t + kx)$. Using a well-known trigonometric equation

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2},$$

we can obtain

$$\xi(t) = 2A\cos kx \cos\omega t. \tag{2.9.3}$$

There are two trigonometric terms in this expression. The first term, $\cos kx$, is a function of a point's coordinates only and can be considered as a variable amplitude of the standing wave changing from point to point, i.e.,

$$A_{st} = |2A\cos kx|. \tag{2.9.4}$$

(Since the oscillation amplitude is essentially positive, the sign on the modulus is written.) The second term, cos ωt, depends only on time and describes harmonic oscillation of the point with the fixed coordinate x. Thus, all wave points make harmonic oscillations with different (dependent on x) amplitudes. It is clear from eq. (2.9.4) that the amplitude of a standing wave depends on x and changes from zero up to $2A$. Points in which the amplitudes of oscillations are maximum are referred to as antinodes of the standing wave. The points with permanently zero displacements are referred to as nodes of a standing wave (Figure 2.22).

Find coordinates of the standing wave nodes. Write the obvious expression $|2A\cos kx| = 0$ whence $\cos kx = 0$. Therefore, $kx = \pm(2n+1)\pi/2$, where $n = 0, 1, 2, \ldots$. Having replaced wavenumber k by its expression $k = 2\pi/\lambda$ we obtain $(2\pi + \lambda)x = \pm(2n+1)(\pi/2)$. So we find the nodes' coordinates

$$x_{\mathrm{knt}} = \pm(2n+1)\frac{\lambda}{2}, n = 0,1,2,... \tag{2.9.5}$$

The antinodes' coordinates can be found from eq. (2.9.4). Indeed, the point's coordinate in which oscillation acquires the maximum displacement satisfies equation $kx = \pm n\pi$ with $n = 0, 1, 2, \ldots$ So we can obtain the antinodes' coordinates:

$$x_{\mathrm{ank}} = \pm n\frac{\lambda}{2}, n = 0,1,2,... \tag{2.9.6}$$

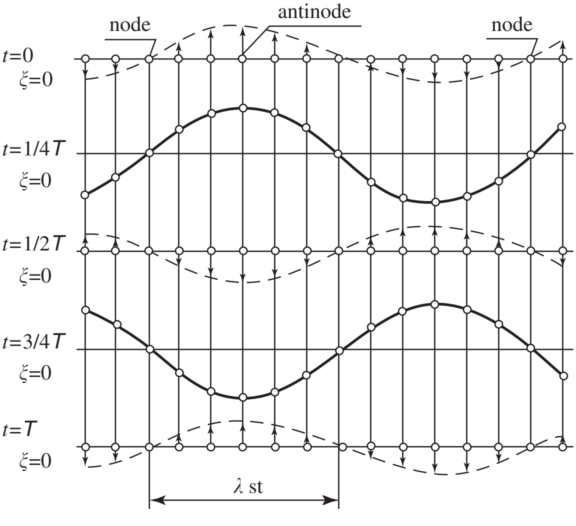


Figure 2.22 Standing transverse waves. Open circles represent a position of oscillating particles, whereas arrows show a direction and the magnitudes of their speed.

The distance between adjacent nodes (or antinodes) is referred to as the standing wave wavelength. It can be seen from eqs. (2.9.5) and (2.9.6) that this length is $\lambda/2$, i.e.,

$$\lambda_{st} = \frac{\lambda}{2}. \tag{2.9.7}$$

Nodes and antinodes are shifted relative to each other on a quarter of the original wavelength. In Figure 2.22 an origin ($x = 0$) is imposed with the node point at $n = 0$ (2.9.6). For $t = 0$ deviations of all the wave's points pass through equilibrium positions and thus the wave is degenerated into a straight line. This instant of time is taken as zero. However, at this very instant each point (except for the nodes) possesses a certain speed specified in the figure by arrows. Displacements achieve a maximum at $t = T/4$, the wave is represented by sine, but the speed of each point of the wave becomes equal to zero. The instant $t = T/2$ again corresponds to the straight line but the speeds of all points are directed to the opposite side, and so on.

Comparison of traveling and standing waves reveals the following difference. In a plane traveling wave all media points oscillate with identical amplitude, but their phases are different and repeat in $\Delta t = T$. In a standing wave all points (from node to node) make oscillations in one phase, but the amplitudes of their oscillations are different. The points of the wave shared by a node oscillate in antiphase. In fact, standing waves do not transfer energy.

An example of a standing wave is the oscillation of a flexible cord fixed rigidly at one end, with the other end in the hands of the experimenter; the latter generates oscillations. When an antinode reaches the fixed end (Figure 2.23a) the wave affects a fastening. Under Newton's third law the fastening produces a reciprocal influence on the cord, equal in magnitude and oppositely directed. It generates a return wave; the displacement of the cord's

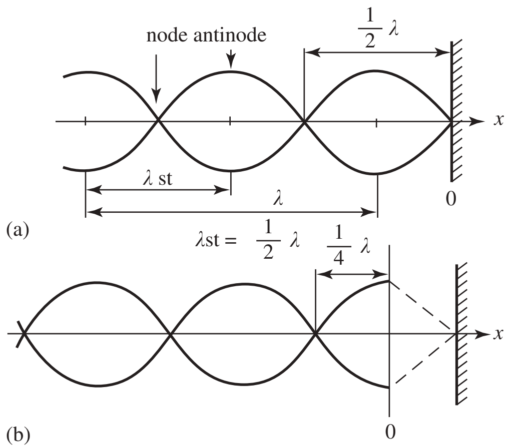


Figure 2.23 A model of a wave reflection in the case of a more dense (a) and less dense (b) medium; the solid curves represent a cord vibrations and a dashed lines show the "less dense" media.

point at a border is in the opposite direction to the displacement of the "incident" wave. As a result, both waves are in an antiphase; therefore, the loss of half a wavelength occurs at the border point. An example of the mobile (less dense) border is a thin weightless braid connecting the cord end with the fastening (Figure 2.23b). The cord end generates free oscillations running along the cord wave in the opposite direction.

Analysis of the wave reflections in these two cases shows that at reflection from the more dense border (Figure 2.23a) loss of half the wavelength occurs, and there is a phase shift of π. Reflection from the less dense border is not accompanied by a phase change; therefore, at the junction point of the cord and the braid there will always be an antinode (Figure 2.23b).

2.9.3 String oscillations

The oscillations of a tightly stretched string can be considered as a special case of standing waves: on both fixed ends of the string there is the reflection of a running wave resulting in the formation of standing waves. This steady picture of standing waves will take place at the string fastening in any of the nodes; they do not seem to participate in oscillations. Hence, the integer of half wavelength m can be confined on the length of string between the fastenings:

$$L = m\frac{\lambda}{2}, m = 1,2,3.$$

The definite wavelength corresponds to these oscillations

$$\lambda_m = \frac{2L}{m}. \tag{2.9.8}$$

We can transform these wavelengths into frequencies:

$$\nu_m = \frac{\upsilon}{\lambda_m} = \frac{\upsilon}{2L}m. \tag{2.9.9}$$

The vibrations of a string with the smallest frequency when only one antinode is confined corresponds to the basic tone in a sounding string (Figure 2.24, above). Other vibrations with multiple frequencies are overtones.

The rules presented operate not only in many musical instruments and the formation of standing waves in them (in the resonator of a guitar, for instance) but are also used in models of an ideal black body, in quantum mechanics, in physics of solid-state properties, etc. All of these are discussed in other sections of the book.

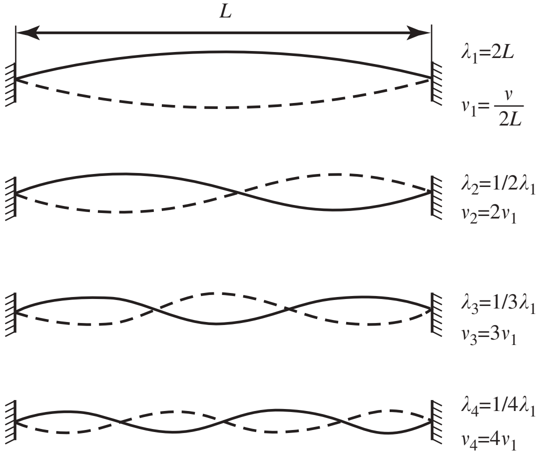


Figure 2.24 String oscillations.

EXAMPLE E2.11

A transverse wave runs along an elastic cord at a speed $\upsilon = 15$ m/sec. The period of oscillations of the cord points is $T = 1.2$ sec, amplitude $A = 2$ cm. Define (1) wavelength, (2) phase of oscillation φ, displacement ξ, velocity $\dot{\xi}$ and acceleration $\ddot{\xi}$ of the point at a distance $x = 45$ m from the source of waves at time instant $t = 4$ sec and (3) phase difference $\Delta\varphi$ of the oscillation of two points lying on the cord at a distance $x_1 = 20$ cm and $x_2 = 30$ cm from the source of waves.

Solution: (1) According to definition, wavelength is $\lambda = \upsilon T$. Substituting the values we obtain $\lambda = 18$ m. (2) The wave equation is $\xi = A \cos \omega\,(t - (x/\upsilon))$*. The phase of the point oscillations with coordinate x in the time instant t stays under cos sign $\varphi = \omega(t - (x/\upsilon))$ or $\varphi = (2\lambda/T)\,(t - (x/\upsilon))$. Calculation gives $\varphi = 5.24$ rad or 300°. We can find the displacement from the equation * substituting the A and φ values: ξ appears equal to $\xi = 1$ cm. The velocity of the point can be found by the time derivation of ξ: substituting all the values derived we arrive at $\dot{\xi} = 9$ cm/sec. The acceleration is the second time derivative on the displacement

$$\ddot{\xi} = -A\omega^2 \cos\omega\left(t - \frac{x}{\upsilon}\right) = \frac{4\pi^2 A}{T^2}\cos\varphi.$$

Therefore, $\ddot{\xi} = 27.4$ cm/sec^2. (3) The phase difference is related to the distance between the points Δx by equation $\Delta\varphi = (2\pi/\lambda)\Delta x$. This gives $\varphi = 3.49$ rad or 200°.

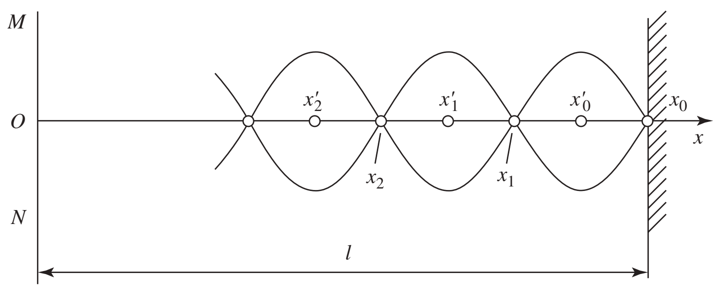
EXAMPLE E2.12

A wall MN is located perpendicular to the wave at a distance $l = 4$ m from the source of a plane wave with frequency $\nu = 440$ Hz. Define the distance from the source of the wave to the points in which the first three nodes and first three antinodes of the wave arising as a result of the superposition of two waves, running to and reflected from the wall, occur. Assume the wave's velocity to be $\upsilon = 440$ m/sec (Figure E2.12).

Solution: Let us choose an axis x directed perpendicularly to the wall and the origin at a distance l from the reflected wave source. Then the equation of the wave will be written as

$$\xi = A\cos(\omega t - kx)^{*}.$$

As in a point with coordinate x the reflected wave will come back covering twice the distance $(l - x)$ and the reflected wave will lose a phase $\pi/2$ at reflection; the reflected wave equation can be written as



$$\xi = A\cos\omega t - k(x + 2(2l - x)) + \pi$$

and further

$$\xi = -A\cos\omega t - k(2l - x))^{**}.$$

Summing two equations * and ** we can find the standing wave:

$$\xi_1 + \xi_2 = \xi = A\cos(\omega t - kx) - A\cos(\omega t - k(2l - x)).$$

According to trigonometry we can obtain

$$\xi = -2A\sin k(l-x)\sin(\omega t - kl).$$

Since A sin $k(l - x)$ does not depend on time it can be considered as the standing wave amplitude (being taken as modulus $|2A \sin k(l - x)|$). From this equation we can obtain the nodes' and antinodes' coordinates. Nodes occur when $|2A \sin k(l - x)| = 0$ and consequently $k(l - x_n) = n\pi$; since , $\lambda = \upsilon/\nu$, $k = 2\pi\nu/\upsilon$. Therefore, $2\pi\nu(l - x_n) = n\pi\upsilon$. The nodes' coordinate can be obtained from the last equation: $x_n = l - (n\upsilon/2\nu)$. The first three nodes' coordinates are $x_0 = 4$ m, $x_1 = 3.61$ m, $x_2 = 3.23$ m. Correspondingly the antinodes appeared when $k(l - x'_n) = (2n + 1)(\pi/2)$; therefore, $4\nu x_n' = 4\nu l - (2n + 1)\upsilon$ and the antinode's coordinates are $x'_n = l - (((2n + 1)\upsilon)/4\nu)$. The first three antinodes will appear at $x_0 = 3.81$ m, $x'_1 = 3.42$ m, $x'_2 = 3.04$ m. The results are depicted in Figure E2.12.

2.9.4 Group velocity of waves: wave package

By definition the monochromatic wave is boundless in space. The real wave is always limited in space and is emitted during a limited interval of time, which is why it cannot be monochromatic in full measure. However, any real wave can be considered as a result of the superposition of a large number of strictly monochromatic flat waves. As a result of interference, in one part of space these waves strengthen each other, and in other parts extinguish. Such waves have some features that can be discovered using a simple model of superposition of two plane monochromatic waves.

Let two plane cross-sectional polarized monochromatic waves with equal amplitudes be distributed along an axis x. Such waves are described by equations: $\xi_1 = A\ \cos(\omega_1 t - k_1 x)$ and $\xi_2 = A\ \cos(\omega_1 t - k_2 x)$. Because of the superposition principle a combined wave can be represented as $\xi = \xi_1 + \xi_2 = A\cos(\omega_1 t - k_1 x) + A\cos(\omega_1 t - k_2 x)$, or

$$\xi = 2A\cos\left(\frac{\omega_2-\omega_1}{2}t - \frac{k_2-k_1}{2}x\right)\cos\left(\frac{\omega_2+\omega_1}{2}t - \frac{k_2+k_1}{2}x\right)$$

Suppose now that the angular frequencies ω_1 and ω_2 and wave vectors k_1 and k_2 differ only slightly, i.e., $\omega_2 - \omega_1 = \Delta\omega$ ($\Delta\omega << \Delta\omega_1$), $k_2 - k_1 = \Delta k$ ($\Delta k << k_1$). Therefore, we can write

$$\frac{\omega_1+\omega_2}{2} \approx \omega_1 \quad \text{and} \quad \frac{k_1+k_2}{2} \approx k_1. \qquad (2.9.10)$$

Then

$$\xi = 2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)\cos(\omega_1 t - k_1 x). \tag{2.9.11}$$

We can see that the superposition of two monochromatic waves with equal amplitudes with slightly different frequencies and wavenumbers produces a new wave with variable in space and time amplitude

$$\left|2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)\right|. \tag{2.9.12}$$

Since $\Delta\omega$ and Δk are small in comparison with ω_1 and k_1 a change of amplitude will take place comparatively slowly. Such a wave is called a wave with modulated amplitudes.

Let us determine the rate of crest displacement of such a composed wave. In order to solve this problem we can repeat the method used in Section 2.8.2 when we evaluate the phase velocity rate. The crest corresponds to the constant phase in eq. (2.9.11), i.e., $\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) =$ const., since the model package velocity dx/dt = g $\approx \Delta x / \Delta t$ appears to be equal to $\Delta\omega/\Delta k$. If now we make the model more realistic and take not just two but a continuous set of waves with k lying in narrow interval Δk, and dependence $\omega(k)$, close to linear, eq. (2.9.11) becomes more complicated. The expression for the crest velocity distribution of the wave package turns out to be obtained from eq. (2.8.13) at $\Delta k \to 0$ (i.e., replacement of final increments by differentials). The corresponding group of waves (or a wave package) is depicted schematically in Figure 2.25.

The group of waves transfers energy and momentum at a velocity that is generally different from the phase velocity, and is characterized by group velocity g. The group velocity is

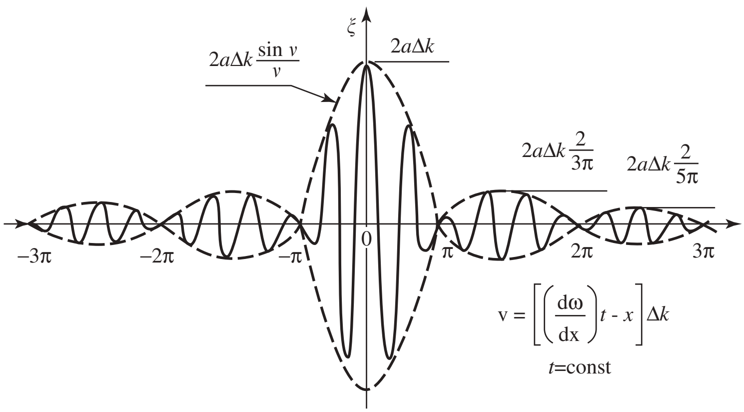


Figure 2.25 A wave packet.

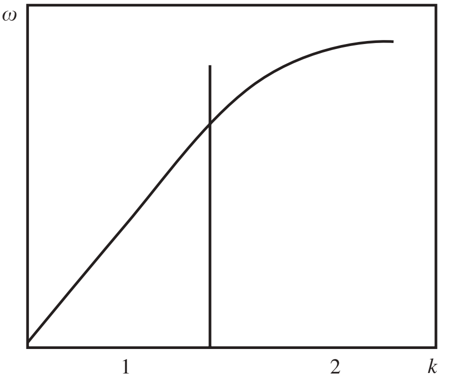


Figure 2.26 An example of a dispersion curve.

the speed of movement of a point corresponding to any fixed wave amplitude. Remember that the phase velocity of a monochromatic wave is $\upsilon = \omega/k$, whereas the group velocity is $g = d\omega / dk$. In the absence of dispersion (frequency is linearly dependent on wavenumber) the group and phase velocities coincide. At the presence of dispersion, group and phase velocities are different. An example of function $\omega(k)$ is given in Figure 2.26. In region 1 the frequency ω linearly depends on k; here the phase and group velocities coincide. In region 2 the function ω (k) is nonlinear; the group and phase velocities differ.

PROBLEMS/TASKS

2.1. A point accomplishes oscillations according to the formula $x = A\cos(\omega t + \varphi)$, where $A = 4$ cm. Determine the initial phase φ if $x(0) = 2$ cm and $\dot{x}(0)<0$.

2.2. Determine the maximum values of speed $\dot{x}_{max}$ and acceleration $\ddot{x}_{max}$ if a point accomplishes harmonic oscillation with amplitude $A = 3$ cm and angular velocity $\omega = \pi/2$ sec^{-1}.

2.3. A physical pendulum consists of a rod of mass m and length $l = 1$ m and of two small balls of masses m and $2m$ fixed to the rod at lengths $l/2$ and l, respectively. The pendulum makes small oscillations relative to a horizontal axis passing perpendicularly to the rod through the middle of the rod. Determine the frequency ν of the harmonic pendulum oscillations.

2.4. A hoop of mass m made of thin metal is suspended on a long nail hammered into a wall. It makes harmonic oscillations in a plane parallel to the wall. The radius of the hoop is $R = 30$ cm. Calculate the oscillation period T.

2.5. A point oscillates according the law: $x = A\cos(\omega t + \varphi)$. In some instant of time the displacement is 5.0 cm, its velocity is $\dot{x} = 20$ cm/sec and acceleration $\ddot{x} = -80$ cm/sec^2. Find the amplitude A, angular velocity ω and phase $(\omega t + \varphi)$ for this time instant.

2.6. Two similar harmonic oscillations with the same periods $T_1 = T_2 = 1.5$ sec and amplitudes $A_1 = A_2$ are summarized. The initial phases are $\varphi_1 = \pi/2$ and $\varphi_2 = \pi/3$.

Determine the amplitude A and initial phase φ of the resulting oscillations. Draw a vector diagram of the superposition in a scale chosen.

2.7. Two tuning forks sound simultaneously. Their frequencies are $\nu_1 = 440$ Hz and $\nu_2 = 440.5$ Hz. Determine the resulting beatings period T.

2.8. An MP oscillates according to equation $x = A\cos\omega t$, where $A = 8$ cm, $\omega = \pi/6$ sec^{-1}. At the instant of time when a force F reaches a value -5 mN for the first time, the potential energy U of the MP becomes equal to 100 μJ. Find this instant of time t and its corresponding phase ωt.

2.9. A small weight is suspended on a spring, which has been expanded by $x = 9$ cm. Find the period T of the weight when it will oscillate freely.

2.10. A mathematical pendulum of $l_1 = 40$ cm in length and a physical pendulum in the form of a thin straight rod of length $l_2 = 60$ cm oscillate around a common horizontal axis. Find the distance a between the rod CM and the oscillation axis.

2.11. An oscillation period of a pendulum $l = 1$ m in length during the time interval $t = 10$ min was lowered two times. Determine the logarithmic decrement of damping λ.

2.12. A body of mass $m = 5$ g accomplishes a damping oscillation. After $t = 50$ sec the system loses 60% of its energy. Determine the friction coefficient.

2.13. Find the number N of total oscillations of a system during which the system energy decreases by $n = 2$ times. The logarithmic decrement of damping $\lambda = 0.01$.

2.14. Under the weight of an electromotor, a bar on which it is fixed deflects on $h = 1$ mm. At what frequency of the motor rotation n can the danger of resonance appear?

2.15. A carriage of $m = 80$ ton has 4 bow springs. The rigidity of each spring is $k = 500$ kN/m. At what speed υ will the carriage begin to intensively swing on the rail's conjunction if the rail is $L = 12.8$ m in length?

2.16. The amplitudes of forced harmonic oscillations at the frequencies $\nu_1 = 400$ Hz and $\nu_2 = 600$ Hz are equal to each other. Determine the resonance frequency ν_{rez}. Neglect damping.

2.17. A swollen log whose section is constant along the whole length, has plunged vertically into the water so that only a small part (in comparison with length) remains above water. The period of oscillation of the log is equal to $T = 5$ sec. Determine the whole length L of the log.

2.18. Mercury with a mass of 200 g is poured quickly into a U-shaped tube with a cross section of $S = 0.4$ cm^2, open at both ends. Determine the period of oscillation of the mercury in the tube. Neglect the viscosity of the mercury. The density of the mercury is $\rho = 13.6$ g/cm^3.

ANSWERS

2.1. $\varphi = \pi/3$ rad.

2.2. $\dot{x}_{max} = 4.71$ cm/sec, $\ddot{x}_{max} = 7.40$ cm/sec^2.

2.3. $\nu = (1/\pi)\sqrt{(3/7)(g/l)} = 0.652$ Hz.

2.4. $T = 2\pi\sqrt{2R/g} = 1.55$ sec.

2.5. $\omega = \sqrt{\ddot{x}/x} = 4\ \text{sec}^{-1}$, $= (2\pi/\omega) = 1.57$ sec, $A = \sqrt{\dot{x}^2 + \omega^2 x^2} = 7.07$ cm, $(\omega t + \varphi) = \arccos(x/A) = (\pi/4)$ rad.

2.6. $A = 3.68$ cm, $\varphi = 0.417\pi$ rad, $x = A\cos(\omega t + \varphi)$, where $\omega = (2\pi/T) = 4.19\ \text{sec}^{-1}$.

2.7. $T = 2$ sec.

2.8. $t = 2$ sec, $\omega t = \pi/3$.

2.9. $T = 0.6$ sec.

2.10. $a_{1,2} = \frac{1}{2}\left(l_1 \pm \sqrt{l_1^2 - (1/3)l_2^2}\right)$; 10 and 30 cm.

2.11 $$\lambda = \frac{2\pi}{\ell}\sqrt{\frac{\ell}{g}}\ln\frac{A_1}{A_2} = 2.31\times 10^{-3}.$$

2.12. $r = 9.16 \times 10^{-5}$ kg/sec.

2.13. $N = 35$.

2.14. $$n = \frac{1}{2\pi}\sqrt{\frac{g}{h}} = 16\ \text{sec}^{-1}.$$

2.15. $$\upsilon = \frac{L}{\pi}\sqrt{\frac{k}{m}} = 10.2\ \text{m/sec}.$$

2.16. $\nu_{\text{rez}} = 510$ Hz.

2.17. $L = 6.21$ m.

2.18. $$T = 2\pi\sqrt{\frac{m}{2\rho \mathbf{g} S}} = 0.86\ \text{sec}.$$

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