– 6 –

Wave Optics and Quantum–Optical Phenomena

6.1 PHYSICS OF ELECTROMAGNETIC OPTICAL WAVES

Wave optics deals with the propagation of light waves and their interaction with matter.

A schematic drawing of electromagnetic waves was given at the end of Chapter 5 (Figure 5.43b) and their main characteristics were summarized (Table 5.3). The frequencies (the wavelength) corresponding to the range of waves visible to the naked eye are presented in Figure 6.1: relative eye sensitivity S is depicted which reflects the property of the human eye. Two areas adjoin the optical range: infrared (IR) and ultraviolet (UV) radiation (see Table 5.3). Remember also the ratio $\upsilon = c/n$ between light velocity in a vacuum c and in some isotropic media υ. n being the refraction index, $n = \sqrt{\mu\varepsilon}$. A useful scheme of the wavelengths of visible light is also presented.

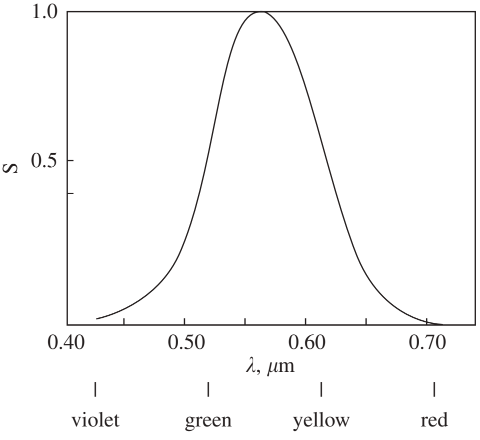


Figure 6.1 The relative eye sensitivity S of the standard observer to different wavelengths, $S(\lambda)$, for a normal level of illumination.

All sets of optical phenomena comprise three general parts: geometrical, wave and physical optics. In this chapter, substantial attention will be given to wave optics and its transition to quantum optics, in which light exhibits as a flow of particles.

To make an atom emit an electromagnetic wave it is necessary to excite the atom, i.e., to transfer it to a higher energy state with a short lifetime. In the ground state, an atom can exist for an indefinitely long time without changing its state. However, an atom can be transferred from the ground state to some other state with higher energy, i.e., to the so-called *excited* state with a limited lifetime τ (in many cases approximately 10^{-8} sec; see Section 7.2 and Section 8). For this time, an atom emits a continuous wave referred to as a *wavetrain or zug*. The wavetrain length l can be estimated as follows: time τ (~10^{-8} sec) multiplied by the light velocity (3.10^8 m/sec) gives value of the order of the wavetrain length $l \sim 1$ m. We can describe a simplified diagram of a wavetrain as a single wave with definite properties. Groups of such wavetrains form a bunch of waves or *a light beam*.

Electromagnetic waves do not require a medium to propagate. Different kinds of electromagnetic waves are produced by different emitters. As a rule, an emitter does not emit a single wave, but a bundle of waves (we don't touch coherent laser radiation here). An emitter produces waves by changing its own state (for instance, at the transition from one energy level to another, the energy difference contributes to the wave's energy). Each specified wave is characterized by some properties. So, a beam consists of an enormous amount of particular waves having a definite length (Section 7.2).

Electric interactions are stronger than magnetic ones. In other words, all physiological, photochemical, photoelectric, etc., actions of light are caused by the operation of an electric field. Therefore, the vector **E** is referred to as the light vector and the plane of its oscillation is called the plane of oscillation. Each light beam consists of many wavetrains with particular orientation of the plane of oscillation, i.e., polarizations.

Figure 6.2 shows the polarization for a number of cases, viewed along the wave propagation direction. In Figure 6.2a, a natural beam of transverse electromagnetic waves is depicted; a cylindrical symmetry can be seen in the beam; the direction of the symmetry axis coincides with the beam propagation direction (axis x in the picture). The beam as a whole is completely *unpolarized*. In Figure 6.2c, the electric field vector **E** (i.e., the single orientation of the oscillation plane) is shown; a beam is totally polarized. Such a wave is called a *plane-polarized wave*. However, in many cases the beam consists of waves with

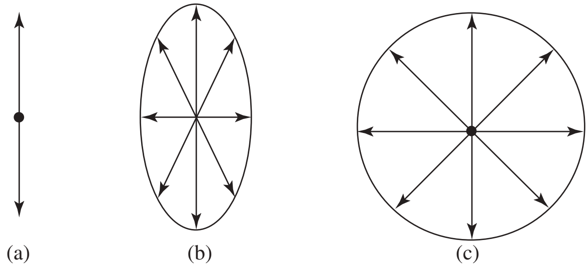


Figure 6.2 Polarization of electromagnetic wave beams: (a) completely polarized beam, (b) partly polarized beam, (c) natural unpolarized beam.

partly ordered planes of oscillations; a partly polarized beam is formed (Figure 6.2b). In solids, in the given propagation direction two perpendicular oscillation planes can exist; they are denoted by the letter s, which can be equal to 1 and 2; in addition, a third, longitudinal, polarization can exist too ($s = 3$). The waves in solids can thus have three polarizations ($s \to 1, 2, 3$), but in liquids and gases only one ($s = 3$, longitudinal) polarization exists.

Different orientations of the planes of oscillation are possible in a beam. However, many planes of vibrations take place in a beam, they can all always be reduced to two mutually perpendicular planes.

The equations of running waves were derived in Chapter 2 (eqs. (2.8.5) and (2.8.7)). In these equations, $\xi(x, t)$ symbolizes the displacement ξ of any point from its equilibrium position x in time instance t. In this chapter we have passed from mechanical to electromagnetic waves; by displacement ξ we shall now mean oscillations of the electric field strength **E** and those of the magnetic field **H** in mutually perpendicular planes (Figure 5.45). The cross line of these planes coincides with the axis of wave propagation, which coincides with a wave vector **k** (refer to Section 2.8.2). A plane that is determined by the vector **k** and the plane of **E** vector oscillations is a plane of oscillation.

Let us present some fundamental laws of optics; later we shall use them from the point of view of the wave nature of light. Consider a beam of light falling on the border of two media 1 and 2 with absolute refraction indexes n_1 and n_2 (Figure 6.3). The medium with the higher refraction index is referred to as the optically denser one (Figure 6.3, $n_1 < n_2$) and vice versa. At a point O, every wavetrain splits: part of it reflects (reflected beam OB) and the other part refracts (refracted beam OO′). The law of reflection–refraction asserts that all beams, namely, initial AO, reflected OB and refracted OO′ together with a normal KL to the border plane in a point O, *lie in one plane*. The angle of incidence α is equal to the reflection angle α. The ratio of the sine of the angle of incidence α to the sine of the refraction angle β is equal to the ratio of the absolute refraction indexes of two media

$$\frac{\sin\alpha}{\sin\beta} = \frac{n_2}{n_1} = n_{2,1}, \tag{6.1.1}$$

where $n_{2,1}$ is the relative refraction index. This equation is referred to as Snell's law. Since the air refraction index is near to 1, the relative refraction index on the border with air is practically equal to the absolute index.

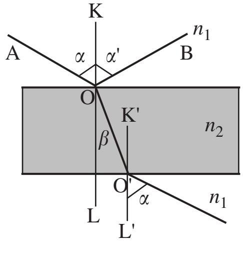


Figure 6.3 Reflection and refraction at an air–glass interface ($n_2 > n_1$).

If, having undergone a number of reflections and refractions, a beam has passed along a path; another beam directed along the same path, but in the opposite direction, will pass along the path; in exactly the same way, having repeated all features. This statement is referred to as *the law of reversibility*. Taking advantage of this law, if we follow the path of the return beam at its propagation from optically more dense medium 2 and on the border of the media contact, then $\sin\alpha = n_{21}\sin\beta$. Since $n_{21} > 1$, at some angle α, the refracted beam will propagate along the border without leaving medium 2 (the refraction angle becomes $\pi/2$; Figure 6.4). The angle β_C being in this very case an incident angle is referred to as the critical angle of total internal reflection. At greater values of β ($\beta > \beta_C$) the total reflection is preserved.

The phenomenon of total internal reflection plays a basic role in the construction of fiber light guides, i.e., thin fibers made up of transparent materials (Figure 6.4): if an optic ray enters the guide, it will be unable to abandon it, propagating inside to its end because any incident angle for it will be larger β_C because of the small fiber diameter.

Consider now a path of beams in a prism in air with a vertex angle θ of a material with a refraction index n (Figure 6.5). At small angles θ a deviation ρ is determined by the refraction index n and angle θ as:

$$\rho = (n-1)\theta, \tag{6.1.2}$$

i.e., the angle of deviation ρ linearly depends on the refraction index n of the material that the prism was made of. Since the refraction index depends on the wavelength λ (see Section 6.5), the prism is the simplest device for decomposition of light in a spectrum on the wavelength. A rainbow in the sky after rain is a widely known example of decomposition of white light into a spectrum by water drops in air.

In Chapter 2 (Section 2.9.2 and Figure 2.27), the effects arising from the reflection of waves from the borders were considered. There, a very simple model of a traveling wave

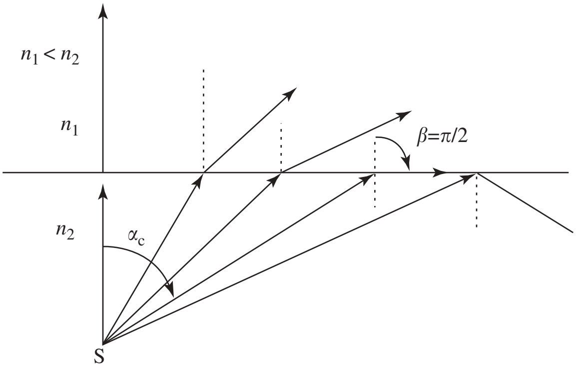


Figure 6.4 A total internal reflection of light from a source S at different incident angles: larger and smaller than the critical angle β_C.

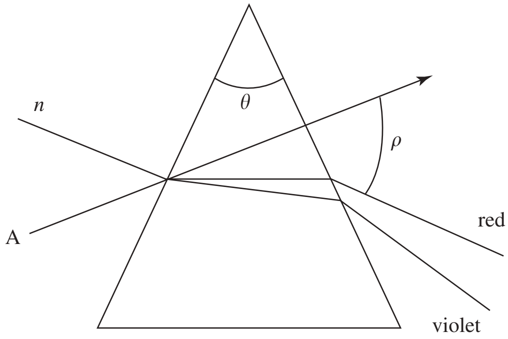


Figure 6.5 Light ray paths in a prism.

Table 6.1

Refraction indexes of some materials

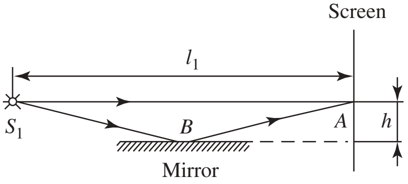
Media	Refraction index (n)	Light speed in the medium (in the light speed unit)
Absolute vacuum	1	1
Air	1.0003	0.9997
Water	1.33	0.75
Glass	1.4 ÷ 1.8	0.56 ÷ 0.71
Diamond	2.4	0.42
Silicon	3.5	0.29

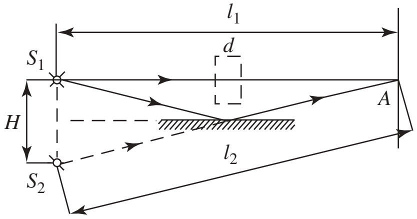
propagated in some media with a reflection index n_1 was used. The reflection was examined for two cases: when the reflecting wave on an interface has a node or antinode. This is of importance since it influences whether or not the wave at reflection loses a half-wavelength. It was shown that if the reflection takes place from the denser media with a refraction index $n_2 > n_1$, the wave loses a half-wavelength and, in contrast, if $n_2 < n_1$ it does not.

In Table 6.1, the refraction indexes of some substances are presented.

EXAMPLE E6.1

From a source of monochromatic light with wavelength λ into a screen point A, there come two beams (Figure E6.1): one beam S_1A comes directly from a source horizontally perpendicular to the vertical screen, and the second beam S_1BA is reflected in a point B from a horizontally located mirror. The distance from the source to the screen l_1 is 1 m, the distance from the horizontal beam to the mirror surface is $h = 2$ mm. Determine (1) what will be observed in point A of the screen—strengthening or lowering of intensity? (2) how will the intensity in point A be changed if a plane-parallel glass plate ($n = 1.55$) of thickness $d = 6$ μm is placed perpendicular to the path of the horizontal beam?





Solution: Let us find the position of an imaginary image S_2 of the source S_1 in the mirror. Sources are coherent; therefore, at superposition of the waves coming from these sources on the screen point A there will be an interference. Strengthening or weakening of intensity depends on the difference Δ of the length of the optical path of two beams; in other words, from the number of half-wavelengths stacked on an optical path difference Δ: $m = \Delta/(\lambda/2)$*; if Δ is a whole even number, intensity will be maximal, if a whole odd number then intensity is minimal.

(1) The optical path difference Δ_1 will consist of both the geometrical difference $l_2 - l_1$ (both beams go in air) and the additional difference in $\lambda/2$ appearing on reflection from the mirror. Therefore,

$$\Delta_1 = l_2 - l_1 + \left(\frac{\lambda}{2}\right)^{**}.$$

Since , $l_2 = \sqrt{\ell^2 + H^2}$,

$$\ell_2 - \ell_1 = \ell_1\sqrt{1+\left(\frac{H}{\ell_1}\right)^2} - \ell_1 = \ell_1\left[\sqrt{1+\left(\frac{H}{\ell_1}\right)^2} - 1\right].$$

Since $(H/l_1) \ll 1$, therefore, for the root calculation we can use the approximation

$$\sqrt{1+a} \approx 1 + \frac{a}{2} + \cdots.$$

Thus,

$$\ell_2 - \ell_1 \approx \ell_1\left[1 + \frac{1}{2}\left(\frac{H}{\ell_1}\right)^2 - 1\right] = \frac{H_2}{2\ell_1}.$$

After substituting this expression into **, we obtain

$$\Delta_1 = \frac{H^2}{2\ell_1} + \frac{\lambda}{2}.$$

Then we have to decide commensurability of the last expression with ($\lambda/2$):

$$m_1 = \frac{\dfrac{H^2}{2\ell_1} + \dfrac{\lambda}{2}}{\dfrac{\lambda}{2}} = \frac{H^2}{\ell_1 \lambda} + 1.$$

Since $H = 2h$, then

$$m_1 = 4\frac{h^2}{\ell_1 \lambda} + 1.$$

Executing calculations, we have $m_1 = 33$. Therefore, in point A there will be a minimum of intensity.

(2) The glass plate of thickness d introduced into the path of the horizontal beam will change the optical path length. The optical path length will be sum of the geometrical path length and optical path length of the beam in the plate itself:

$$L = (\ell_1 - d) + nd = \ell_1 + (n-1)d.$$

The optical path length difference is now

$$\Delta_2 = l_2 - L + \frac{\lambda}{2} = l_2 - [l_1 + (n-1)d] + \frac{\lambda}{2},$$

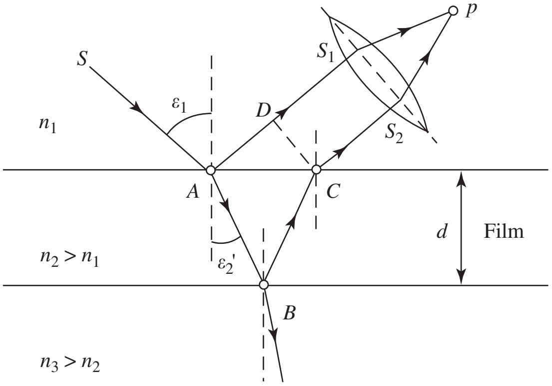
or using *

$$m_2 = \frac{\Delta_2}{\lambda/2} = \frac{\Delta_1 - (n-1)d}{\lambda/2} = m_1 - 2\frac{d(n-1)}{\lambda}.$$

Executing calculations, we obtain $m_2 = 19.8$. Since this number is nearer to 20 (even) than to 19 (odd) in point A there will be a partial increase of intensity.

EXAMPLE E6.2

On a thick glass plate with refraction index $n_3 = 1.5$ covered with a very thin film, the refraction index of which is equal $n_2 = 1.4$, a parallel beam of monochromatic light ($\lambda = 0.6$ μm) falls nearly along a normal. The reflected light, owing to interference, is maximally weakened. Determine the thickness of the film.

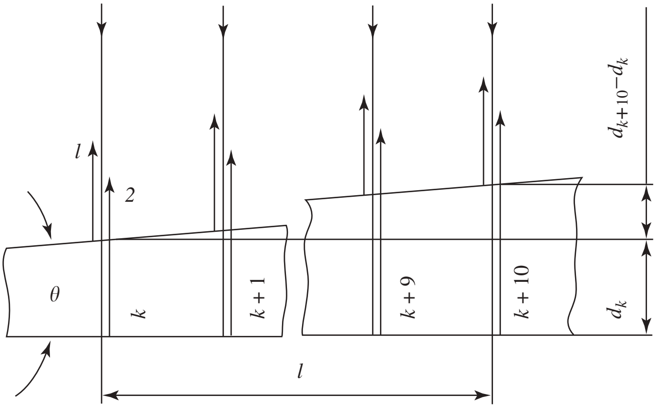


Solution: Let us allocate a narrow light beam from the light falling on the film. The path of this beam in the case when the angle of incidence is ε is shown in Figure E6.2. At point A, the beam is in part reflected and refracted. The reflected beams AS_1 and $ABCS_2$ fall on a convex lens S1−S2 and interfere at point F. As the parameter of refraction of air $n_1 = 1$ is less than the refraction index of the film, which in turn is less than the glass refraction index, in both rays reflection occurs in total without phase change. Since the light is maximally weakening the optical path length $\Delta = l_2 n_2 - l_1 n_1 = (|AB| + |BC|n_2 - |AD|n_1$ should be equal to an odd number of half-wavelengths $(|AB| + |BC|n_2 - |AD|n_1 = (2k + 1)(\lambda/2)$. If the incidence angle tends to zero, $AD \to 0$ and $|AB| + |BC| \to 2d$ and we obtain $\Delta = 2dn_2 = (2k + 1)(\lambda/2)$. Then the film thickness is $d = [(2k+1)\lambda]/4n$. Taking $k = 0, 1, 2, 3$, we arrive at a number of possible values for d:

$$d_0 = \frac{\lambda}{4n_2} = 0.11\ \mu\text{m}, \quad d_1 = \frac{3\lambda}{4n_2} = 3, \quad d_0 = 0.33\ \mu\text{m}, \text{etc.}$$

EXAMPLE E6.3

A monochromatic light of wavelength λ falls onto a nearly parallel glass wedge (two plate) with a very small wedge angle θ , normal to its sides. An interference picture appears. It consists of a sequence of light and dark strings (see Figure 6.9). On a length of wedge $l = 1$ cm, 10 strips are observed. Define the refracting angle of the wedge (Figure E6.3).



Solution: A parallel beam of light falling normally onto the wedge reflects from both sides of the wedge, upper and lower. Both beams are coherent and practically parallel; though an optical path length difference Δ is created, therefore an interference picture is observed. Dark strips are visible on those sites of the wedge for which Δ is multiple to an odd number of half-wavelengths: $\Delta = (2k + 1)(\lambda/2)$ with $k = 0, 1, 2\ldots.$ The geometrical difference of paths length is $2dn\cos\theta$; losses of $\lambda/2$ appeared at the reflection from the upper side of the wedge and should be added. Therefore, for dark strips we have $\Delta = (2k + 1)(\lambda/2) = 2d_k n\cos\theta + (\lambda/2)$, where n ($n = 1.5$) is refractive index of glass, d_k is the thickness of the glass wedge in a point of dark strip. The incident angle is assumed to be zero. Therefore after simplification, we obtain $2d_k n = k\lambda$*. Let the thickness for any dark strip be d_k and the thickness of glass in the point $k + 10$ is d_{k+10}, the distance between them being l. Then (θ expressed in radians)

$$\theta = \frac{d_{k+10} - d_k}{\ell}.$$

Calculating values of d's from expression * and substituting them into the last expression we arrive at $\theta = 5\lambda/(nl)$. The angle sought is then $\theta = 2 \times 10^{-4}$ rad. This angle in degrees is $= 2 \times 10^{-4} \times 2.06'' \times 10^5 = 41.2''$.

6.2 AN INTERFERENCE

6.2.1 Superposition of two colinear light waves of the same frequencies

In Section 2.3.1 a summation of two oscillations of the same frequencies and propagating in the same direction was considered analytically and using a vector diagram. It was also shown in Section 2.8.3 that the oscillation intensity is proportional to the square of amplitude. Apply now the conclusions mentioned to the calculation of the light intensity at waves imposed in any fixed point of space; let it be x_0. At the fixed coordinate the equation of a running wave

$E(x_0, t) = E_0 \cos(\omega t - kx_0)$ transforms into the equation of oscillation $E(t) = E_0 \cos(\omega t - \varphi)$, where product kx_0 is included as a certain number into the phase φ. A splintered wavetrain traveled in two parts, 1 and 2; then they superpose (Figure 6.6) enhancement or weakening of the light intensity. The expression for the square of total amplitude (2.3.1) is

$$E_0^2 = E_{0,1}^2 + E_{0,2}^2 + 2\,E_{0,1}E_{0,2}\cos(\varphi_1 - \varphi_2). \tag{6.2.1}$$

Consider a number of cases of the addition of two light waves with equal amplitude (i.e., at $E_{0,1} = E_{0,2}$) at various phase differences $(\varphi_1 - \varphi_2) = \Delta\varphi$. If light waves from two different independent sources are summed, the average value of $\cos\Delta\varphi$ is equal to 0. This means that if in expression (6.2.1), $E_0^2 = 2E_{1,2}^2$, i.e., the intensity of light will increase twice, there will be *an enhancement of intensities.*

If $\Delta\varphi$ remains constant the two waves are coherent (refer to Section 2.9.1). At $\Delta\varphi = (2m + 1)\pi$ (m is an integer), $\cos\Delta\varphi = -1$ and the resulting light amplitude and intensity in the given point are equal to zero. At $\Delta\varphi = 2m\pi$, $\cos\Delta\varphi = 1$ and the resulting light amplitude is twice as high as the amplitudes of each of the initial waves, i.e., *an enhancement of amplitudes* takes place; intensity thus grows four times. *The phenomenon of redistribution of intensity of light in space on imposing two or several coherent waves is referred to as the interference*.

Interference phenomena (including diffraction phenomena) are the direct consequence, and proof of, the wave nature of light.

6.2.2 Interference in thin films

The practical realization of two coherent light sources is very difficult (it can be achieved, for instance, with the use of lasers). However, there is a relatively simple way to carry out an interference. It consists of splitting a single light beam into two components by reflection from a pair of mirrors and then superposing them in a single point; they will interfere, thus

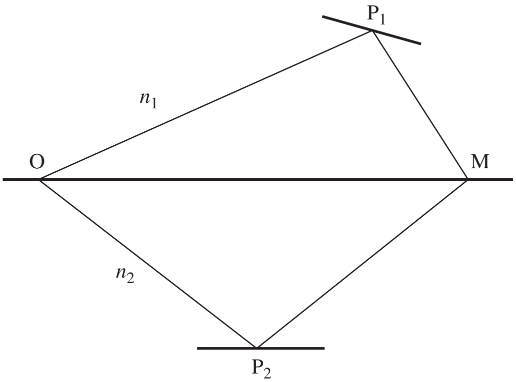


Figure 6.6 Reflection and refraction ray paths of two split parts of a single wavetrain in two media: 1 and 2; P$_1$ and P$_2$ are mirrors.

a splintered wave "interferes with itself." The basic scheme of such an experiment is submitted in Figure 6.6. At point O, on the border of the two media with the refraction indexes n_1 and n_2, a wave is splintered in two parts. With two mirrors P_1 and P_2 both parts go to point M at which they interfere. The speed of propagation of the two beams, due to different media properties, is c/n_1 and c/n_2. At point M, the two parts of the wave will superpose with each other with constant shift in time equal to $S_1/\upsilon_1 - S_2/\upsilon_2$, where S_1 and S_2 are geometrical lengths of the path traveled by the two parts of the wave. The oscillations of electric field strength at point M will be $E_{0,1} \cos \omega(t - S_1/\upsilon_1)$ and $E_{0,2} \cos \omega(t - S_2/\upsilon_2)$. The square of the resulting oscillation amplitude at point M is

$$E_0 = E_{0,1}^2 + E_{0,2}^2 + 2E_{0,1}E_{0,2} \cos\left[\left(\omega\frac{S_1}{v_1}\right) - \left(\omega\frac{S_2}{v_2}\right)\right]. \tag{6.2.2}$$

Since $\omega = 2\pi/T$ and $\upsilon = c/n$, the expression in square brackets is equal to $\Delta\varphi = (2\pi/cT)(S_2n_2 - S_1n_1) = (2\pi/\lambda_0)(S_2n_2 - S_1n_1)$. The product of the path traveled S and refraction index n is referred to as optical path length denoted by Δ. Keeping in mind that $cT = \lambda_0$ (λ_0 being the wavelength in vacuum),

$$\Delta\varphi = \frac{2\pi}{\lambda_0}\Delta. \tag{6.2.3}$$

This expression joins the phase differences and the optical length traveled in the splintered wave. $\Delta\varphi$ defines the interference effects. Indeed, $\cos \Delta\varphi = 1$ corresponds to the maximum intensity since $\Delta\varphi = (2\pi/\lambda_0)\,\Delta = 2\pi m$. From this, the condition of the intensity maximum can be derived:

$$\Delta = m\lambda \tag{6.2.4}$$

The largest diminishing of the light intensity corresponds to $\cos \Delta\varphi = -1$, i.e., $\Delta\varphi = (2m + 1)\pi$. Then $(2m + 1)\pi = (2\pi/\lambda_0)\,\Delta$ or

$$\Delta = (2m+1)\left(\frac{\lambda_0}{2}\right). \tag{6.2.5}$$

It is easy to see that the summation of waves described above with fourfold enhancement of intensity corresponds to the displacement of the two "parts" of the splintered wave from each other by the difference in lengths equal to the integer wavelengths (or, accordingly, to the phase difference $\Delta\varphi = 2\pi m$). The complete extinction of the wave's intensity is observed at the displacement of the two wave parts on the wavelength half (on an odd number of the wavelength half, i.e., $\Delta\varphi = (2m + 1)\pi$).

Consider as an example the interference of light at the reflection from thin films (or from a thin plane-parallel plate; Figure 6.7). The direction of a beam falling on the film is shown in the figure by an arrow. Splitting of the wavetrains occurs in this case at partial reflection of each part of it on the upper (point A) and the lower surfaces (point B) of the film. We shall consider that the light beam goes from air and leaves after a point B into air (with air refraction index equal to unity) whereas the parameter of this film material is equal to n. Every wave of the beam falling at an angle α at point A is split into two parts: one of them is reflected (beam AD) and the other refracts (beam AB). At point B every wave of the refracted beam is split again: part is reflected from the lower film surface and part refracts leaving the film. At point C the wave is again split, but we will follow that wave part which leaves the film at the same angle as beam AD. The two reflected beams are gathered by a lens (not shown in the picture) at one point. Being parts of the same primary wave the beams are coherent and can participate in the interference, the intensity being dependent on the difference of their optical traveled lengths (or differences in phases).

The phase difference in waves 1 and 2 is accumulated in traveling along path lengths AD and ABC. The optical path length is $\Delta = (AB + BC)n - AD$, where $AB + BC = 2d/\cos\beta$ and $AD = 2d \sin\beta \sin\alpha/\cos\beta$. Remembering that $\sin\alpha = n\sin\beta$, then $\Delta = (2dn/\cos\beta)(1 - \sin^2\beta)$ or $\Delta = 2dn\cos\beta$. Since angle α is usually given in problems but not β, it is more convenient to present the Δ value in the form

$$\Delta = 2d\sqrt{n^2 - \sin 2\alpha}. \tag{6.2.6}$$

When defining the conditions of light intensity (maximum and/or minimum), it was necessary to equate the Δ value to the integer or half integer to the number of wavelengths (eqs. (6.2.4) and (6.2.5)). However, as well as estimating the optical path length difference Δ, it is also necessary to analyze the opportunity of the loss of half a wavelength during reflection. This depends on a specific condition, namely, whether the media from which the reflection occurs is more or less dense. So, if the film with $n > 1$ is surrounded with air with $n = 1$, the loss of half-wavelength occurs at point A (Figure 6.7). If the film is on a surface of a medium whose reflection index is higher than for the film material, the loss

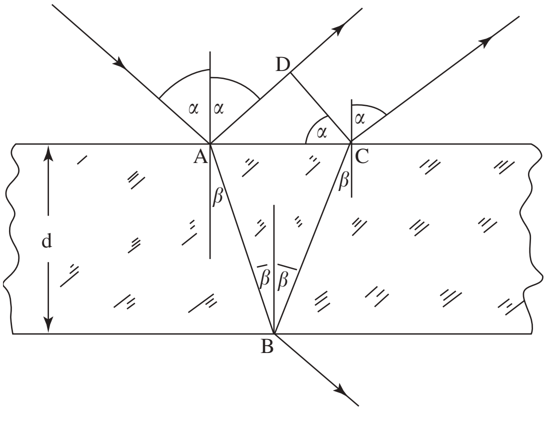


Figure 6.7 Ray paths in a thin film.

of half a wavelength occurs at two points—A and B; as in this case the whole wavelength is accumulated, this effect should not be taken into account at all. It follows, that specific tasks demand individual consideration. The main principle consists in finding the whole difference of optical lengths Δ, to consider the possible loss of half a wavelength at reflection, if necessary, to add (or subtract, it does not matter) it to (or from) Δ and to bring it into correlation with the conditions presented. In the case of a film in air represented in Figure 6.7, the condition of maximum interference looks like

$$\Delta = 2d(n^2 - \sin^2\alpha)^{1/2} + \frac{\lambda_0}{2} = m\lambda_0. \tag{6.2.7}$$

Since the refraction index depends on the wavelength (see Section 6.5), the interference conditions are qualitatively different. Therefore, the film will decompose falling light in a spectrum, i.e., in falling white light the thin film always looks as if it has been painted. We all have met examples of this: observing multicolored soap bubbles or an oil stain on the surface of water.

Consider now the example of a thin air wedge. This wedge is opposite to the thin film picture (Figure 6.8). A plate with well-polished surfaces lies on another, similarly perfect plate. At a definite place between the two plates a thin subject (e.g., a thin wire) is introduced, so an air wedge is formed. Consider a beam of light falling normally onto the upper plate. We shall accept that there is no divergence at surface points on reflection and refraction, keeping in mind that the wedge angle δ is very small. Admit that A is a point on the lower plate where the optical path length between plates Δ is equal to integer m of wavelengths λ plus $\lambda/2$ (due to reflection from the optically more dense lower plate); two reflected waves are nearly parallel to each other. Suppose that there is the condition of maximum interference intensity at this point. An equation describes this condition (factor 2 appears because the beam runs the distance twice):

$$2\Delta = m\lambda + \frac{\lambda}{2} = \lambda\left(m + \frac{1}{2}\right). \tag{6.2.8}$$

If we look at this picture from above (for this purpose a simple optical system is required), it is possible to see geometrical strips in which, at certain m, light (or dark)

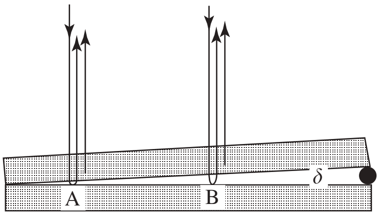


Figure 6.8 An air wedge.

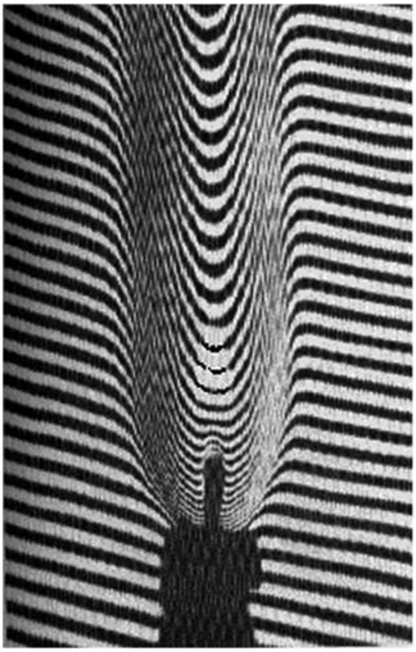


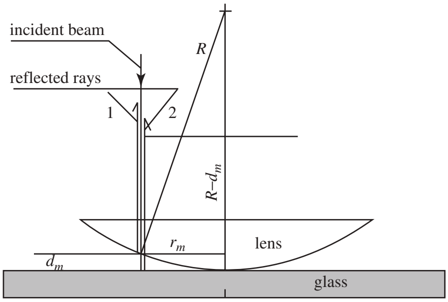
Figure 6.9 Lines of constant thickness.

strips are formed. Along this strip condition (6.2.8) is fulfilled, i.e., along it the backlash of air has the same thickness. Such strips referred to as *strips of equal thickness*. Provided that the plates are made carefully, the strips of equal thickness are represented by parallel straight lines. If, however, there are defects in the plates, the appearance of the strips changes appreciably, and the position and form of the defects develop clearly. Fringes of equal thickness are shown in Figure 6.9: in an air wedge a narrow stream of warm air is produced, the density of which and, accordingly, the refraction index, differ from the values for cold air. The curvature of the lines of constant thickness is visible.

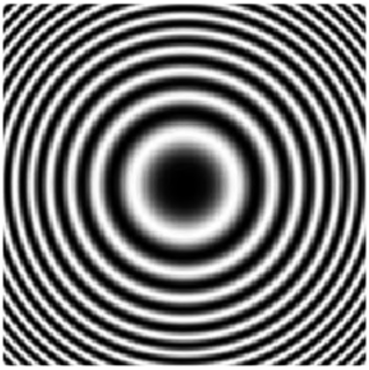
If a convex lens touches a perfect flat plate, at a favorable ratio of the lens curvature radius, light wavelength and the presence of an optical magnifying system, so-called *Newton's rings* can be observed. They represent the fringes of equal thickness in the form of concentric circles.

EXAMPLE E6.4

A vivid example of strips of equal thickness is Newton rings. They appear when a lens of a radius of curvature R lying on a carefully processed glass plate is irradiated with monochromatic light in the wavelength λ. Determine the radius of the mth ring.



(a)



(b)

Solution: A geometrical diagram and the interference picture are presented in Figure E6.4a.

The condition of interference maxima is

$$d_m = \left(m + \frac{1}{2}\right)\frac{\lambda}{2}.$$

The air refraction index is assumed to be unity. In Figure 4a, it can be seen that

$$d_m = R - \sqrt{R^2 - r_m^2} = R - R\sqrt{1 - \frac{r_m^2}{R^2}},$$

where r is the radius of the mth ring. Taking into account the fact that the radius of curvature R is much larger than the size of the interference picture, we can expand the last equation into binominal series limiting ourselves by two terms

$$d = R - R\left[1 - \frac{1}{2}\frac{r^2}{R^2} + \cdots\right] \approx \frac{r^2}{2R}.$$

The ring's radii r can be deduced as

$$r = \sqrt{\left(m + \frac{1}{2}\right)\lambda R *}.$$

The character of rings—light or dark—depends on the loss of $\lambda/2$ at the reflection from the more dense matter: in each particular case this needs to be derived separately. In our problem, we use a light falling from above, therefore, there is only one reflection from a denser media (from a glass plate). Therefore, for the light ring

$$r_m = \sqrt{\left(m + \frac{1}{2}\right)\lambda R + \frac{\lambda}{2}}.$$

An overall picture of Newton's rings is depicted in Figure E6.4b.

If the light is directed from below (and the results are observed from above), there will be no loss of a half-wavelength because the reflection from the denser media takes place twice: from the lens and from the glass plate).

EXAMPLE E6.5

Find the radii of the second r_2 dark, and fifth r_5 light Newton rings in a monochromatic light $\lambda_0 = 0.56$ μm provided the lens radius is $R = 1.2$ m.

Solution: Using the star equations from the previous example, keeping in mind that the air refraction index is 1 and taking into account that we are first searching for the radius of the dark ring, we can write $\Delta = (2m+1)(\lambda/2) = (r_m^2/R)+(\lambda_0/2)$, therefore, $r_m = \sqrt{mR\lambda}$. Therefore, at $m = 2$, $r_2 = \sqrt{2 \times 1.2 \times 5.6 \times 10^{-7}}$ $m = 1.16$ mm.

For the light ring, $\Delta = m\lambda_0 = (r_m^2/R) + (\lambda_0/2)$ and $r_m = \sqrt{(2m-1)R(\lambda/2)}$. Executing calculations, we obtain $r_5 = \sqrt{(2.5 - 1)1.2(5.6/2)10^{-7}} = 1.74$ mm.

The phenomena of interference find wide application in chemistry and the chemical industry. In particular, they are used in interferometry in defining the refraction indexes of substances in their three states: solid, liquid and gaseous. There is a large number of various interferometers which differ by their assignment.

Let us illustrate the determination of the refraction index of substances by a simple interferometer intended for the measurement of the refraction indexes of liquid and gaseous substances (Figure 6.10). Two completely identical thick plane-parallel glass plates A and B are fixed in parallel to each other. The light from source S falls onto the surface of plate A at an angle close to 45°. As a result of its reflection from both sides

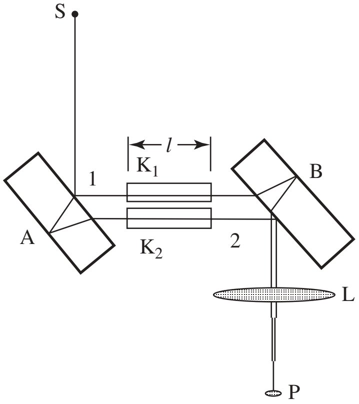


Figure 6.10 The diagram of a simple interferometer.

of plate A, two parallel beams 1 and 2 are produced. Running through two identical glass cells K_1 and K_2, these beams fall onto plate B and are again reflected from both its sides and are gathered at a point P by a lens L. At this point, they interfere, and the interference strips are examined with an ocular, which is not shown in the figure. If one of the cells (e.g., K_1) is filled by gas with a known absolute refraction index n_1 and the second with a substance with a measured refraction index n_2, the optical path length difference between plates will be equal to $\delta = (n_1 - n_2)\, l$, where l is the cell length. With the help of a special device, the displacement of the interference strips concerning their position with empty cells can be observed. Displacement δ is proportional to the difference $(n_1 - n_2)$, which allows one to determine one parameter knowing another. We note that while there are rather low requirements as to the accuracy of the measurement of the strips' position, the relative accuracy in defining the refraction indexes can achieve values of 10^{-6}–10^{-7}. This accuracy enables the study of small impurities in gases and liquids, measurement of the different dependences of the refraction indexes on temperature, pressure, humidity, etc.

There are still many other designs of interferometer construction, intended for various physical measurements. In particular, using a specially designed interferometer, Michelson and Morley in 1881 established the independence of light speed from the speed of its source (refer to Section 1.6). Einstein took this fact as a principle of his Special Theory of Relativity.

6.3 DIFFRACTION

Diffraction is a set of phenomena arising from the propagation of light in a media with pronounced heterogeneity and consisting of light deviations from the laws of geometrical optics. Diffraction leads to light deviating from rectilinear distribution, bending around opaque obstacles and penetrating into an area of geometrical shadow.

6.3.1 Huygens–Fresnel principle: Fresnel zones

Taking into account experiments in which light exhibited its wave nature C. Huygens assumed that each point of a primary light wavefront serves in its turn as a source of secondary spherical wavelets. The new position of the wavefront will be the enveloping surface of these secondary waves (Figure 6.11); in turn, each point of the secondary wavefront is again the source of the next generation of waves and so on. In Figure 6.12, this principle is illustrated with an example where a light wave is passed through an aperture; it can be seen that, due to secondary waves, light can penetrate into the area of geometrical shadow. These phenomena are only exhibited in an appreciable measure

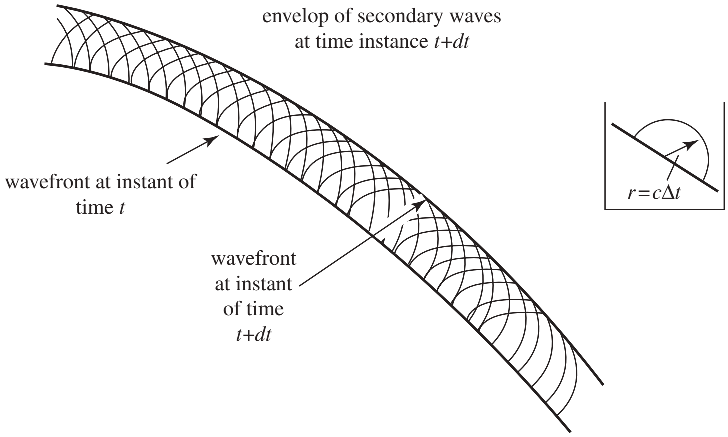


Figure 6.11 Huygens principle (the wavefront at time instants t and $t + \Delta t$ is shown, each point is the source of the second waves).

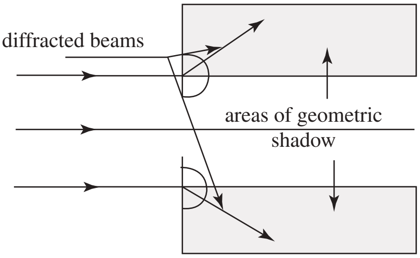


Figure 6.12 Diagram of penetration of the diffraction radiation into the area of geometrical shadow.

when the diameter of the aperture is commensurable with the light wavelength: only in this case do the angles of diffraction appear to be measurable (see below).

A.J. Fresnel enlarged this principle by assuming the same laws of interference which had already been developed for primary waves were also applicable to secondary waves.

It is rather difficult to calculate the distribution of intensity in a diffraction picture. However, a method allowing an essentially simplified calculation of the diffraction effects, at least at a qualitative level, has been offered: Fresnel has suggested mentally breaking a wave surface into zones, the distance from respective points of which up to the sighting point differs from the previous one by $\lambda/2$. In this case light waves from the adjacent zones are in an antiphase (because of the shift by $\lambda/2$); this leads to the mutual cancellation of such waves; in other words, *the adjacent zones extinguish each other*. The method has been successfully used to solve different problems of wave optics, in particular, in the explanation of rectilinear distribution of light. We shall take advantage of this principle by considering the diffraction on a slit.

Two kinds of diffraction can be distinguished: diffraction in parallel light rays from a plane front wave (referred to as Fraunhofer diffraction), and diffraction in converging beams (Fresnel diffraction). Here we will consider only the Fraunhofer diffractions. The scheme of this diffraction is presented in Figure 6.13: point S marks the light source, a condenser lens K provides a parallel light beam, and lens L with a focal length f concentrates the result of the diffraction at an angle φ at a screen point P. A central ray O and axis sin φ along which the figure is expanded are shown.

6.3.2 Diffraction on one rectangular slit

We use the Fresnel zone principle for qualitative consideration of the Fraunhofer diffraction on a single rectangular slit. By definition, each following zone extinguishes the previous one. This means that if, at the slit width d and wavelength λ at an angle φ, an even number

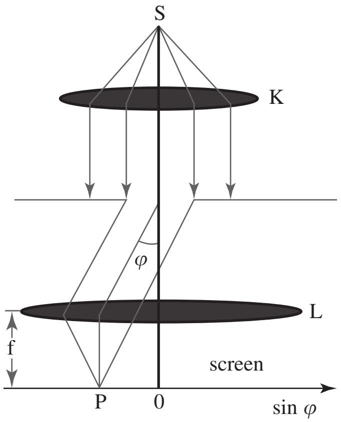


Figure 6.13 Diffraction in parallel rays (Fraunhofer diffraction).

of Fresnel zones opens, all zones extinguish each other; in this direction the intensity of diffraction becomes equal to zero. On the contrary, if at another angle φ, an odd number of zones open, there should be a maximum in the spectrum. It can be seen in Figure 6.14 that if the condition is even, the zone number corresponds to the length of a segment MN = $d\sin\varphi$ on which the integer of wavelengths is stacked (the even half-wavelengths). This is an indication of the minimum intensity. Mathematically, it looks like

$$d\sin\varphi = \pm\kappa\lambda, \tag{6.3.1}$$

where k is an integer which shows the diffraction order. The maximum intensity appears when an odd number of half-wavelengths stack up in the segment MN

$$d\sin\varphi = \pm(2\kappa+1)\frac{\lambda}{2}. \tag{6.3.2}$$

In Figure 6.15, an experimental diffraction spectrum on a single slit is schematically depicted: at $\varphi = 0$ in a direct beam the maximum is seen because only one zone is opened

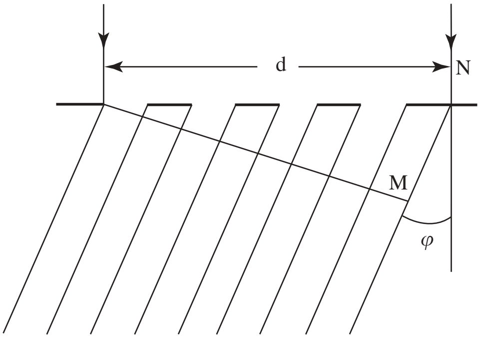


Figure 6.14 Fraunhofer diffraction from a single slit (the φ angle corresponds to eight Fresnel zones half of them faintly marked in the figure).

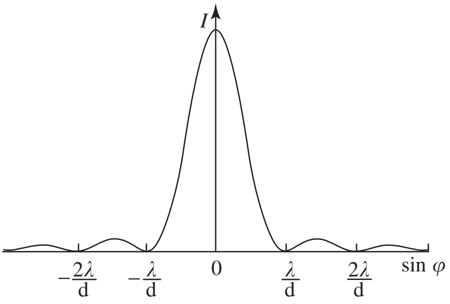


Figure 6.15 Light intensity distribution after single slit diffraction.

(the whole slit width). On changing φ in both directions the maximums and minimums alternate in an orderly way. The quantitative solution of this plot shows that

$$A_{\varphi} = \frac{A_0 \sin\left[\frac{\pi}{\lambda} d \sin\varphi\right]}{\left[\frac{\pi}{\lambda} d \sin\varphi\right]},$$

the square of which gives the so-called interference function

$$I(\sin\varphi) = A_{\varphi}^2. \tag{6.3.3}$$

This function describes the intensity distribution at the diffraction on a single slit (Figure 6.15). At the diffraction on a single slit the intensity of diffraction rapidly decreases with angle.

6.3.3 Diffraction grating

A diffraction grating is composed of a large number of identical, regularly distributed alternating transparent strips on an opaque flat carrier. A constant of the diffraction grating b is the distance between corresponding points of two adjacent strips (Figure 6.16).

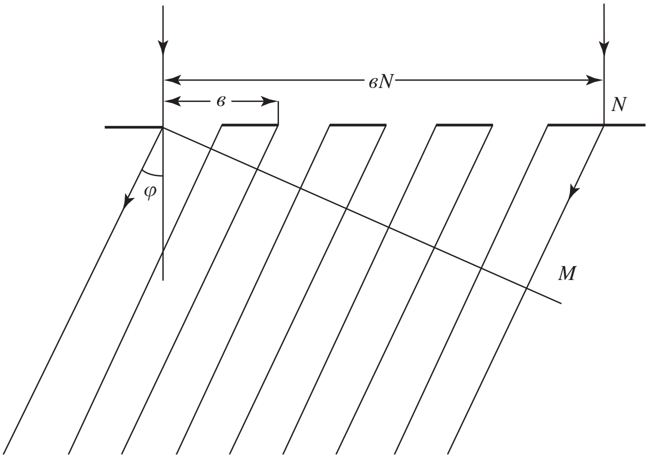


Figure 6.16 Diffraction from a diffraction grating: $\mathit{в}$ is a grating constant, N is the number of slits, $\mathit{в}N$ is the total width of the grating, φ is the diffraction angle; the faintly marked zones in Figure 6.14 are now completely nontransparent.

For simplicity we shall consider that the grating consists of transparent and opaque strips of identical width. We shall designate N as the number of transparent and opaque pairs. Then the general width of the grating will be bN. Compare the condition of minimum intensity for one slit (eq. 6.3.1) with that for diffraction grating. Imagine that we could close every second Fresnel zone at the diffraction on a single slit. Therefore all open zones, having no "antagonists," make a full contribution to the diffraction spectrum.

Figure 6.16 presents a scheme to illustrate this idea. What was the slit width d, is now bN, the minimum condition (6.3.1) transforms into the maximum condition $бN \sin\varphi = \pm \kappa \lambda$ or $б\sin\varphi = \pm k/N \lambda$, where k and N are integers.

There can be two cases, the most important is when κ is divisible by N, i.e., when $(\kappa/N) = m$, where m is a simple integer. In this case, a so-called main maximum of order m is obtained; it corresponds to diffraction maximum when all transparent slits are "in a phase." The main maximum condition can be written as:

$$b \sin\varphi = \pm m\lambda. \tag{6.3.4}$$

The second case is when in expression (κ/N) both numbers are integers, but are not divisible by each other. This gives the so-called subsidiary maxima of small intensity, which are obtained due to diffraction only on a single-grating slit. As a result, the spectrum consists of a rear strong main and many weak subsidiary maxima, as shown in Figure 6.17*a*.

The intensity of the diffraction maxima (eq. (6.3.2) and (6.3.3)) increases N^2 times in comparison with one slit, and the maxima width decreases by $1/N$.

The condition of the main maximum (6.3.4) is of primary importance. It shows that for a given diffraction grating (at fixed b), a different wavelength gives maxima at different points of the spectrum. This is the basis of the use of diffraction gratings in optical spectroscopy.

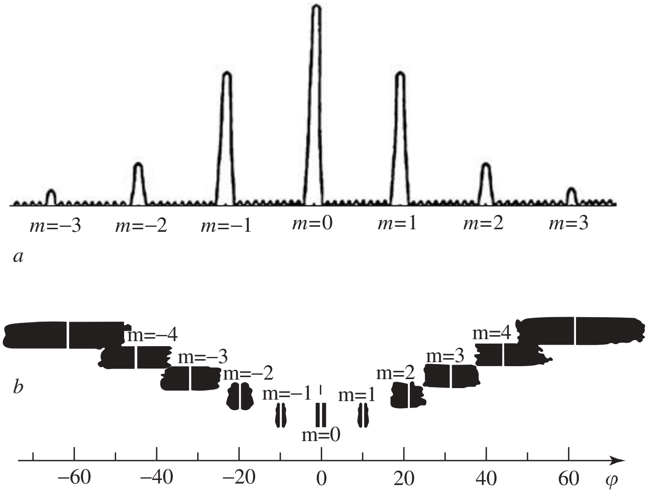


Figure 6.17 Spectrum of white light as viewed in a grating instrument. The different orders of spectra identified by the order number m are shown separated vertically. The central line in each order corresponds to $\lambda = 0.55$ μm.

If white light falls on a diffraction grating each of the main maxima is broadened. The width of a white spectrum is defined by boundary wavelengths 0.4–0.7 μm of visible light. Spectrum of adjacent orders can sometimes overlap. This overlapping is shown in Figure 6.17*b* where, for clarity, spectra of different orders are given on a different vertical level. It can be seen that spectra of zero-, first- and second-orders exist separately, whereas spectra of the third- and fourth-orders are partially overlapped.

Notice that a diffraction grating can also be used in a reflecting position.

6.3.4 Diffraction grating as a spectral instrument

Spectroscopy is the method of studying the composition and structure of a substance or the control of technological processes (refer to Chapter 7, Section 7.8). The main stage in spectroscopy is the decomposition of electromagnetic radiation in a spectrum on the wavelength or frequency. Optical spectroscopy deals with the optical range of electromagnetic radiation, including UV and IR. The basic units of optical spectrometers are either a prism or a diffraction grating.

The most important characteristics of the quality of a spectral device are dispersion D and resolution R. Distinguish an angular and linear dispersion. The value numerically equal to the ratio of the angular distance between spectral lines $\delta\varphi$ to the difference of wavelengths $\delta\lambda$ of these spectral lines is referred to as angular dispersion. The angular dispersion is equal

$$D_\varphi = \frac{\delta\varphi}{\delta\lambda}, \tag{6.3.5}$$

In order to obtain an expression for angular dispersion, we should find a derivative $d\varphi/d\lambda$ from eq. (6.3.4) and change further the differentials into finite increments (neglecting the minus sign). At small angles, eq. (6.3.4) can be rewritten as $b\varphi \approx m\lambda$ and then

$$D_\varphi \approx \frac{m}{b}. \tag{6.3.6}$$

Linear dispersion D_l is the value numerically equal to the ratio of the linear distance in the spectrum between the spectral lines δl to the difference $\delta\lambda$ corresponding to those lines

$$D_\mathrm{l} = \frac{\delta l}{\delta\lambda}; \tag{6.3.7}$$

and at small φ

$$D_\mathrm{l} \approx f\frac{m}{\text{в}}, \tag{6.3.8}$$

where f is the focal length of lens L (Figure 6.13). From the above formulas, it can be seen that the dispersion (both angular and linear) is larger for higher order of the spectrum m.

Another device that permits the decomposition of incident radiation in the spectrum is a prism. Decomposition in the spectrum by a prism is due to the dependence of the angle of refraction on the wavelength. The corresponding formula for the prism can be obtained using expression (6.1.2). Notice that the sign of the derivative $\delta\varphi/\delta\lambda$ for the diffraction grating and for the prism is different.

On wavelength resolution (resolving power) of the spectral device, there is a minimal distance at which two close spectral lines are accepted as being two instead of seeing them as one single widened line (Figure 6.18). Rayleigh has offered a criterion by which the spectral lines are considered as resolved if the middle of the maximum position of one line coincides with the edge of the adjacent line (Figure 6.18, center). The resolving power R of a spectral device is a dimensionless reversed value of the wavelength difference of the resolved neighboring lines to the wavelength of one of them: the value is

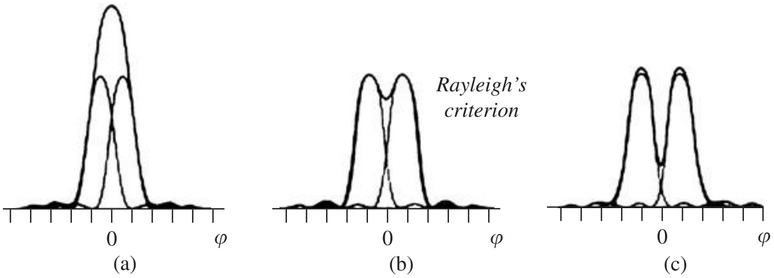
$$R = \frac{\lambda}{\delta\lambda}. \tag{6.3.9}$$

Using the Rayleigh criterion, we arrive at the expression

$$R = mN. \tag{6.3.10}$$

It can be seen that the resolving power is larger when a longer grating length and higher order reflections are used.

In Figure 6.19, two spectral lines obtained with three different diffraction gratings are presented. Gratings I and II are characterized by identical resolution (lines have identical half-widths) but provide a different dispersion, whereas gratings II and III have different resolution (maxima have different half-widths at identical dispersion).



(a) (b) (c)

Figure 6.18 Image of two distant point objects formed by a converging lens; (a) the angular separation of the objects is so small that the images are not resolved, (b) the objects are farther apart and the images obey Rayleigh's criterion of resolution, (c) the objects are still farther apart and the images are well resolved.

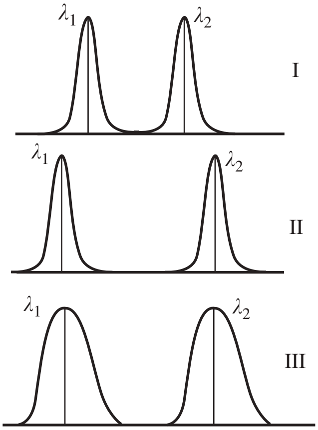


Figure 6.19 The intensity patterns of light with wavelength λ_1 and λ_2 incident on the different gratings; grating II has the same resolution as I however higher dispersion, grating III – the same dispersion but lower resolution.

6.3.5 X-ray diffraction

X-rays, discovered by W.K. Röntgen in 1895, as well as visible light are both electromagnetic waves; however the X-ray wavelength is 10^3–10^4 times shorter (about 10^{-10} m, i.e., 0.1 nm). This circumstance defines their high penetrating ability, which the great majority of mankind has experienced during medical inspections. Our interest here is in X-ray diffraction (XRD) in crystals.

For experimental observation of diffraction, the radiation wavelength should be of the same order of magnitude as the diffraction grating period. This follows from eq. (6.3.4): in order to measure diffraction angle φ the ratio mb/λ should have the order of unity, i.e., b should be commensurable to λ. Therefore, to observe XRD using diffraction gratings is extremely difficult in practice.

At the same time, a diffraction grating with a period of about 1 Å has the nature that the interatomic distances in crystals are about this size. As the interatomic distances are approximately 10^{-10} m and the size of even the smallest crystal is 10^{-7} m (repetition is 10^3 in the majority of cases), the crystal can be considered infinite. If a beam of X-rays falls on a crystal, under the action of an electromagnetic wave the atoms' electrons begin to oscillate and scatter secondary radiation of the same wavelength in all directions (compare with Compton-effect, Section 6.6). As the atoms in a crystal are ordered, these secondary waves are coherent and interfere; this defines the diffraction effect.

The diffraction problem of X-rays in crystals "on transmission" has been solved by M. Laue. However, a more evident picture has been given by W.L. Bragg and also independently by G.V. Wulf. Formalizing the picture described above, they reduced the scattering of secondary waves to the X-ray reflection from so-called crystallographic planes (see Section 9.1). (These are planes drawn through the nodes of a crystal lattice.)

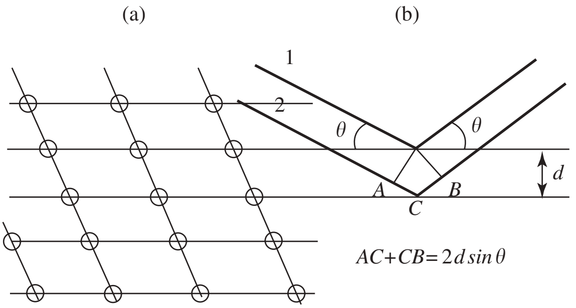


Figure 6.20 The Bragg's law: (a) the ordered atom's array and two arbitrary crystallographic planes, (b) an incident X-ray beam scattered by the entire family of crystallographic planes, the X-ray's paths difference ACB is marked.

The atoms of a crystal and two most rational crystallographic planes are shown in Figure 6.20a. An X-ray beam falls on a crystallographic plane at an incident angle θ. Because of its high penetrating ability, the X-ray radiation passes into the crystal without refraction (the refraction index $n = 1$). Therefore, the difference in the lengths traveled by waves 1 and 2 can be easily counted, making $\Delta = 2d \sin \theta$, where d is the distance between the nearest parallel planes (interplanar distance). The maximum of intensity will be observed if this difference is equal to an integer of wavelengths (refer to Section 6.2):

$$2d \sin \theta = m\lambda, \tag{6.3.11}$$

where m is the reflection order. This formula is referred to as the Bragg formula.

Knowing the arrangement of atoms in a crystal, it is easy to calculate the intensity of X-ray reflection. More difficult, however, is the problem of calculating the arrangement of atoms in a crystal from an experimentally measured diffraction picture. This problem is the essence of modern X-ray crystal structure analysis for which M. von Laue (1914) and W.L. and W.H. Bragg (1915) were awarded Nobel Prizes.

6.4 POLARIZATION

6.4.1 Polarized light: definitions

An important feature of a wave beam is its polarization. A wavetrain has two mutually perpendicular planes in which oscillations of the vectors of **E** and **H** takes place (Figure 5.45). It has already been mentioned that the action of an electromagnetic wave is defined mainly by the vector **E**. Therefore, vector **H** in many cases is neglected in drawings, whereas the plane of **E** vector oscillation is referred to as the plane of oscillations. The wavetrain is, therefore, linearly polarized, i.e., it possesses a single plane of oscillation (Section 2.8.1; Figures 6.2a and 6.21b and c).

Because all atoms of a source emit electromagnetic waves independently, the beam consists of large numbers of independent wavetrains; their planes of oscillations are not correlated, such light being referred to as nonpolarized or natural. In this case, axial symmetry of the oscillation planes disposition takes place (Figure 6.2b and 6.21b and c). The direction of the axis of symmetry coincides with the direction of the wave propagation. If there is a partial infringement of the axial symmetry, the light beam is partly polarized (Figure 6.2b and 6.21b and c).

The following designations are accepted in the schematic representation of light polarization in physics literature (Figure 6.21). The plane of oscillations of the vector **E** is set by arrows. The polarized beam is represented accordingly by a number of parallel arrows. If the plane of oscillations is perpendicular to the drawing plane, arrows are projected in points. A nonpolarized beam is represented by alternate points and arrows.

6.4.2 Malus law

There are devices called *polarizers*, sensitive to polarization of a light beam. These devices freely transmit the incident electromagnetic waves with a plane of oscillation parallel to the plane of the polarizer, and completely absorb oscillations perpendicular to this plane. Hence, behind the polarizer the natural light becomes polarized with the plane of oscillation parallel to the plane of the polarizer. If, in the way of this secondary beam, a second polarizer is installed with a plane perpendicular to the first, it will detain the first polarized beam completely. This second polarizer is in the position of an analyzer; it is sensitive to the degree of polarization of the light beam.

What will happen if the plane of oscillations in the beam makes an angle with the plane of the polarizer? Let the plane polarized beam falls on the analyzer with the oscillation plane oriented at an angle φ relative to this plane and the plane of the polarizer. Separate the $\mathbf{E}_0$ vector into two components: parallel and perpendicular to the polarizer planes (Figure 6.22). The perpendicular component will be completely absorbed by the polarizer, whereas the component of the electric field in the parallel position will be equal to $\mathbf{E}_0 \cos\varphi$, and the corresponding intensity will be

$$I(\varphi) = \mathbf{E}_0^2 \cos^2\varphi = I_0 \cos^2\varphi. \tag{6.4.1}$$

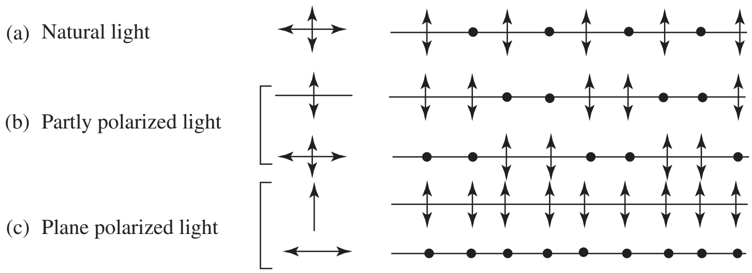


Figure 6.21 Schematic representation of light polarization: (a) natural light, (b) partly polarized light and (c) completely polarized light.

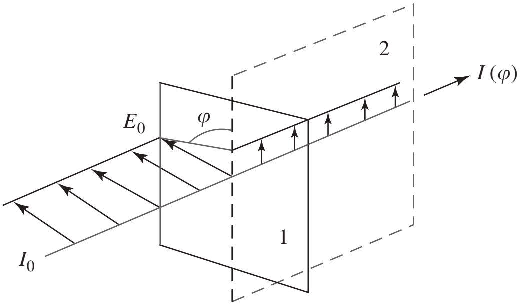


Figure 6.22 Illustration of Malus' law: (1) the polarization device and (2) the plane of polarization.

This equation is called Malus' law. It can be seen that if the wave's plane of oscillation is parallel to the polarizer axis, the beam will pass through with no intensity loss ($\varphi = 0$, $\cos \varphi = 1$). In contrast, at $\varphi = \pi/2$ $\cos \varphi = 0$ and light will be absorbed completely.

If natural light falls on a polarizer the intensity of the passed light is proportional to average value of $\langle \cos^2\varphi \rangle$; since in an interval $0 < \varphi < \pi/2$ the value $\langle \cos^2\varphi \rangle$ is equal to 1/2, $I_{\text{pol.}} = I_{\text{nat}}/2$: intensity of light passed through the polarizer is a half of that of the incident natural light.

6.4.3 Polarization at reflection: Brewster's law

If natural light falls on the border surface of two media the reflected and refracted beams are partly polarized. This occurs because of the fact that from a dielectric surface only the component of the **E** vector which is parallel to the border surface (perpendicular to the incidence plane) is reflected. Then, in the reflected light the oscillations perpendicular to the plane of incidence will predominate, whereas in the refracted beam the oscillations parallel to the plane of incidence will prevail. It has been experimentally established that when reflected from a dielectric surface light is completely polarized if there is a certain relationship between the incidence angle α and the refraction index:

$$\tan \alpha_{\text{B}} = n. \tag{6.4.2}$$

Here the angle α_{B} is referred to as the Brewster angle and the given reflection is known as *Brewster's law.* However, the refracted beam is polarized only in part.

When the beam falls on the two-phase border at the Brewster angle, the angle between the reflecting and refracting beams is equal to $\pi/2$ (Figure 6.23). Indeed, as $\tan \alpha = (\sin \alpha/\cos \alpha)$ and $(\sin \alpha/\sin \beta) = n$ to satisfy Brewster's law ($\tan \alpha_{\text{B}} = n$) it turns out that $\cos \alpha_{\text{B}} = \sin \beta$, which is possible only when $\alpha_{\text{B}} + \gamma = \beta + \gamma = \pi/2$.

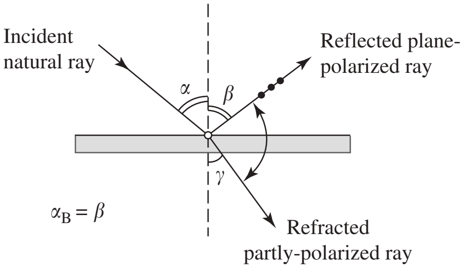
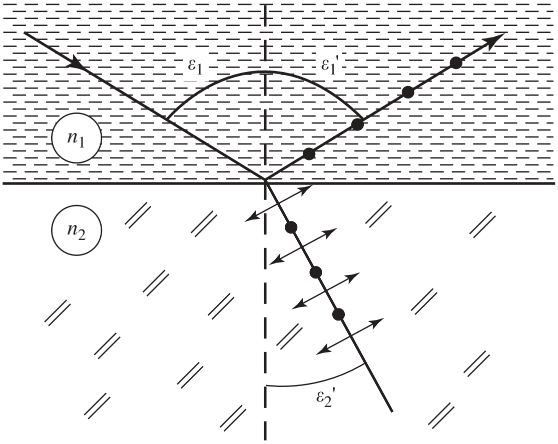


Figure 6.23 Illustration of Brewster's law.

EXAMPLE E6.6

A beam of natural light falls on a polished glass surface plate submerged in a liquid. The beam of light reflected from the plate is at an angle of $\varphi = 97°$ to the incident beam. Define the refraction index of the liquid if the reflected light is completely polarized (Figure E6.6).



Solution: According to Brewster's law, when reflected from a dielectric light is completely polarized if the tangent of the incidence angle is equal to $\tan \varepsilon_1 = (n_2/n_1) = n_{21}$ where n_{21} is the relative index of the second body (glass) relative the first (liquid). The relative refractive index n_{21} is the ratio of absolute indexes, i.e., $\tan \varepsilon_{12} = (n_2/n_1)$. The reflected beam makes an angle of $2\varepsilon = \varphi$ and, consequently, $\tan(\varphi/2) = (n_2/n_1)$. Therefore, $n_1 = n_2/\tan(\varphi/2)$ and the refraction index n_1 is 1.33.

6.4.4 Rotation of the polarization plane

When passing plane-polarized light through some substances, the plane of polarization can change its position in space, namely, it rotates around the light wave vector $\boldsymbol{k}$. Substances possessing such properties are referred to as being *optically active*. Among optically active substances there are many anisotropic crystal (i.e., whose structure does not relate to cubic and hexagonal systems, see Section 9.1) and liquids (e.g., turpentine, nicotine, solutions of many organic and inorganic substances in inactive solvents, etc.).

Experience shows that the angle of rotation of the polarization plane φ around the wave vector $\boldsymbol{k}$ in optically active media is proportional to the length l, traveled by a beam in a sample

$$\varphi = \alpha l \tag{6.4.3}$$

The coefficient α, generally dependent on the wavelength, is referred to *a rotation constant* and is expressed in angular degrees on millimeters of distance run. In solutions of optically active substances the angle of polarization plane rotation is proportional to the length traveled l and concentration of the active substance c:

$$\varphi = [\alpha] c\, l, \tag{6.4.4}$$

where $[\alpha]$ is the *specific rotation constant*. So, knowing $[\alpha]$ and having measured l, it is possible to define the concentration of an active substance in a solution.

The direction of rotation of the polarization plane depends on the substance: if the plane of polarization turns clockwise in relation to $\boldsymbol{k}$, the substance is referred to as a right-hand (or dextrorotatory); if it turns anticlockwise, the substance is a left-hand (or laevorotatory) substance. Thus the direction (the wave vector $\boldsymbol{k}$) and the beam direction of rotation in a dextrorotatory substance forms a left-hand system, and in a laevorotatory substance forms a right-hand system.

For an explanation of the rotation of the polarization plane it is supposed that plane polarization in inactive substances is the superposition of two oppositely directed circular polarizations with identical amplitude and angular velocity. In Figure 6.24, a scheme explaining this supposition is given. On the left, vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ rotate around the $\boldsymbol{k}$ vector in opposite directions with equal angular velocities, therefore the total vector $\mathbf{E}$ oscillates in the vertical plane. If the angular velocities differ, the plane of oscillations turns around $\boldsymbol{k}$ (Figure 6.24, on the right).

The angular velocity's characteristic in optically active substances is caused by an asymmetric arrangement of atoms in molecules and crystals. In Figure 6.25, an example of a hypothetical tetrahedron in two various enantiomorphous forms is depicted. In the center of the tetrahedron is an atom (e.g., carbon as a complexation atom, not shown in the picture), and in the vertexes various atoms are arranged A, X, Y and Z. If the tetrahedron is looked at from above at a detour alternation XYZ (Figure 6.25a), a clockwise motion takes place. The tetrahedron in Figure 6.25b is a mirror image of the one Figure 6.25a. Such molecules are referred to as enantiomorphic. Therefore, if a substance with tetrahedrons of *a*-type in the structure is, for instance, dextrorotatory, an isomer with tetrahedrons of *b*-type is a left-handed isomer.

Research into the effects of rotation of the polarization plane is one of the methods of structural chemistry.

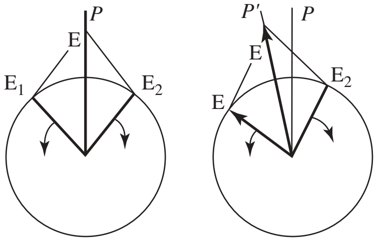


Figure 6.24 The polarization plane rotation.

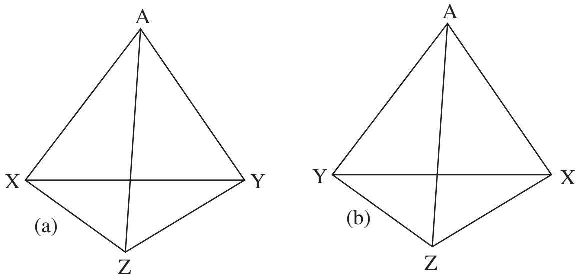


Figure 6.25 Enantiomorphic atomic arrangement.

6.4.5 Birefringence: a Nichol prism

Let us consider now a phenomenon known as *double refraction* in anisotropic crystals. In the XVIIth century, Huygens discovered that light passing through some crystals is split into two beams (Figure 6.26). One passes through the crystal in strict conformity with the laws of geometrical optics and is referred to *as an ordinary* beam (marked on the figure by the letter “o”). The other beam is called *an extraordinary* beam (marked on figure by the letter “e”); it passes the crystal’s surfaces with an infringement of the law of refraction: i.e., it cannot lie in one plane with an incident beam and a normal to an interface. The important thing is that both beams are completely polarized in mutually perpendicular planes. This is the basic, and most practically important, property of birefringent crystals.

In the crystals described there are one or two directions along which the double refraction does not occur. These directions are referred to as the *optical axes* of a crystal (in Figure 6.26 and further defined by line MN). Certainly, they are determined by the atomic structure of a crystal. If the crystal has one such direction it is referred to as a single-axis crystal; there are also biaxial crystals with two such directions. Any plane which runs through the crystal’s optical axis is referred to as *the main section* or *the main plane*. Most interesting is the main section containing the light beam. The plane of the vector **E** oscillations in an ordinary beam is perpendicular to the main section and in extraordinary beam lies in the main plane.

The properties of isotropic media (including optical properties) are identical in all directions. Nearly all gases and liquids and highly symmetric crystals with cubic and, in part, hexagonal structure (see Chapter 9.1) are referred to as isotropic ones. With anisotropic crystals light interacts differently than with isotropic media.

Remember that the refraction index n defines the light speed in a medium $\upsilon = c/n$. For weakly magnetic substances ($M \approx 1$), it is connected to the dielectric permeability of the medium ε by an equation $n = \sqrt{\varepsilon}$. In anisotropic crystals the dielectric permeability depends on direction. In particular, in the optic single-axis crystals the dielectric permeability ε in the direction of the optical axis and perpendicular to it have different values, $\varepsilon_{\|}$ and $\varepsilon_{\perp}$ respectively. In other directions ε has intermediate values. If we draw a sketch of ε values in a single-axial crystal for different directions by segments from an origin, the ends of these segments form a rotation ellipsoid. Its axis of symmetry will coincide with the crystal optical axis. In Figure 6.27, an ellipsoid of the dielectric permeability of a single-axis crystal is presented.

Because the light speed υ in a substance depends on the dielectric permeability ε, the given scheme also represents a diagram of the dependence of υ on the crystal direction. In this case this figure is called an *indicatrix of speeds*. Since the light speed does not depend on direction in isotropic media, the indicatrix is represented by a spherical surface. In anisotropic crystals, the properties of which depend on direction, the indicatrix differs from a spherical one. Moreover, they can differ for ordinary and extraordinary

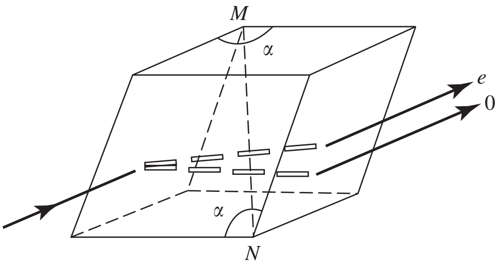


Figure 6.26 The birefraction of natural light by a single crystal of spar $CaCO_3$: MN is the optical axis, o is the ordinary ray (o-ray), e is the extraordinary ray (e-ray).

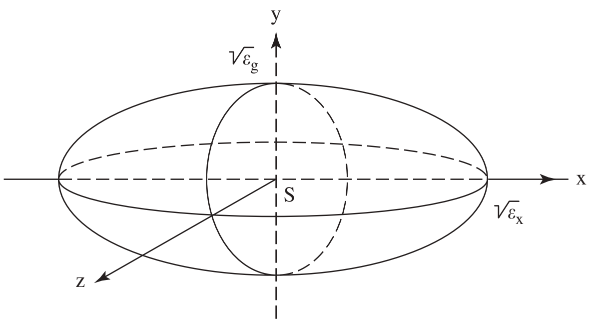
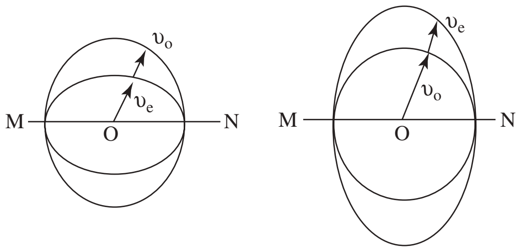


Figure 6.27 Huygens wave surfaces generated by a point source S embedded in calcite.

beams. Therefore, two indicatrixes exist: spherical for ordinary beams and as rotational ellipsoids for extraordinary beams. They appear to be "inserted" in each other. Both indicatrixes touch in the direction of an optical axis because in this direction they have an identical light velocity. In a perpendicular direction, both indicatrixes differ maximally. In single-axis crystals there are two opportunities: in optically positive crystals the velocity of extraordinary beam υ_e is less than that of ordinary υ_0, in optically negative crystals $\upsilon_e > \upsilon_0$ (Figure 6.28). By taking into account the difference in the optical properties of the crystal, it is possible to find the refraction of all rays in all directions graphically.

Nichol suggested a relatively simple method of making a completely polarized beam; the method is based on arranging two split pieces of a crystal of Iceland spar ($CaCO_3$) in such a way that the beam transmitted through it is polarized. Such a device is now referred to as a Nichol (Figure 6.29). In order to obtain such a polarizer the single crystal of Iceland spar should be cut first into two pieces of proper orientation and then be stuck together by a special glue substance. This substance should have the refraction index n, lying in an interval between indexes n_0 and n_e of an initial crystal ($n_0 > n > n_e$); it is the Canadian balm. The angle of fall onto the plane of pasting is selected in order to make the ordinary beam undergo a total internal reflection (refer to Section 6.1) and then be absorbed by the frame of the prism. The extraordinary beam freely passes through the thin layer of balm and leaves the Nichol. Accordingly, it is completely polarized and can be used in optical measurements.



(a) (b)

Figure 6.28 Ellipsoid index of the light wave velocity in a mono-axis crystal: (a) single axis positive and (b) negative. MN is the optical axis.

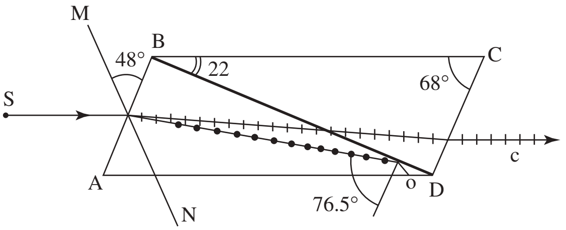


Figure 6.29 Schematic presentation of a Nichol. BD—the Canadian balm layer.

A similar Nichol can be used as an analyzer. If two Nichols are one after another, one as polarizer and the other as analyzer, their rotation around the beam axis essentially influences the transmission of light through this double system. If both Nichols are installed identically the light transmission has maximal intensity, the rotation of the second Nichol (the analyzer) around the beam at a right angle (crossed Nichol) completely extinguishes light. If, however, an optically active substance is placed between the Nichol, the sample becomes visible in crossed Nichols. To determine the angle of polarization plane rotation it is necessary to turn the Nichol-analyzer at a certain angle to achieve extinction again. The angle of rotation will be equal to angle φ (Section 6.4.2). This fact relates to optical methods of substances research.

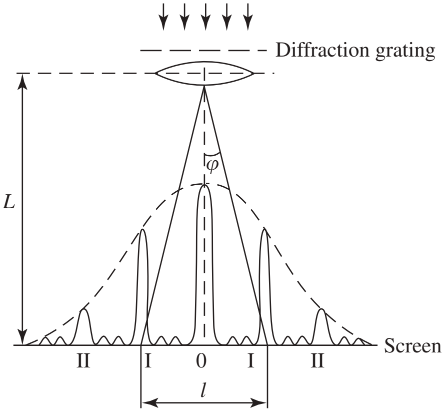
EXAMPLE E6.7

A plate of quartz of thickness $d_1 = 1$ mm cut out perpendicularly to the optical axis of a crystal, turns a polarization plane of monochromatic light of a certain wavelength at an angle of 20°. Define: (1) what the thickness of the quartz plate placed between two parallel Nichols should be in order to extinguish the light completely; (2) what lengths a tube l with a solution of sugar of mass concentration $C = 0.4$ kg/l should be placed between Nichols in order to obtain the same effect? The specific rotation of the sugar solution is $0.665°/(\text{m.kg.m}^{-3})$ (refer to Section 6.6.4).

Solution: (1) An optically active medium rotates a polarization plane at an angle $\varphi = \alpha d$*. Therefore we can present the thickness of the quartz plate as $d_2 = (\varphi_2/\alpha)$**, where φ_2 is an angle totally extinguishing the light ($\varphi_2 = 90°$). The rotation constants α can be found from * formula $\alpha = (\varphi_1/d_1)$. Substituting this expression into **, we obtain $d_2 = (\varphi_2/\varphi_1)\, d_1$. Executing the calculations, we obtain $d_2 = 4.5$ mm. (2) The length of the tube with the sugar solution can be found from the expression $\varphi_2 = [\alpha]Cd$ which defines the sugar solution turning angle of the polarization plane $\ell = \varphi_2[\alpha]C$. Substituting all known data we obtain $l = 0.38$ m.

EXAMPLE E6.8

A parallel beam of light with a wavelength $\lambda = 0.5$ μm falls normal to a diffraction grating. A lens with a focal length $l = 1$ m is behind the diffraction grating to project a diffraction picture on a screen (Figure E6.8). The distance between two first-order maxima is $l = 20.2$ cm. Determine (1) the diffraction grating constant, (2) the specific number of grating grooves (on 1 cm), (3) the limiting diffraction maxima and their total amount N and (4) the angle $\varphi_{\max}$ corresponding to this maximum.



Solution: (1) The diffraction grating formula is $d \sin \varphi = k\lambda$. In our case $k = 1$. $\sin \varphi \approx \tan \varphi$ (because $L \gg l$) and $2 \tan \varphi = l/\mathrm{L}$. Therefore, $d \cdot \ell/2L = \lambda$. From this equation it follows that $d = 2L\lambda/\ell$. Executing calculations, $d = 4.95\ \mu\mathrm{m}$.

(2) The specific number (on 1cm) of grating grooves is $n = (1/d) = 2.02 \times 10^3\ \mathrm{cm}^{-1}$.

(3) In order to find the number of diffraction maxima we need to calculate the $k_{\max}$ corresponding to $k_{\max} = d/\lambda(\sin \varphi_{\max})^*$. $k_{\max}$ corresponds to max $\sin \varphi$. Therefore, $k_{\max} = 9.9$; however for this number $\sin \varphi$ is larger than 1, therefore, $k_{\max}$ is 9. This value allows us to find the general number of diffraction maximums N. The obvious relation exists $N = 2k_{\max} + 1$. Therefore, $N = 19$.

(4) $\varphi_{\max}$ can be found from star relation: $\varphi_{\max} = \arcsin(k_{\max}\lambda/d) = 65.4°$.

6.5 DISPERSION OF LIGHT

All phenomena caused by the dependence of the refraction index on frequency (or on wavelength) are united under the name *light dispersion*. Dispersion is referred to as *normal* if the refraction index steadily falls with an increase of wavelength ($dn/d\lambda < 0$) or, grows with an increase of frequency ($(dn/d\omega) > 0$); otherwise the dispersion is referred to as being anomalous ($(dn/d\omega) < 0$). A typical picture of dependence $n(\lambda)$ for a normal dispersion is given in Figure 6.30.

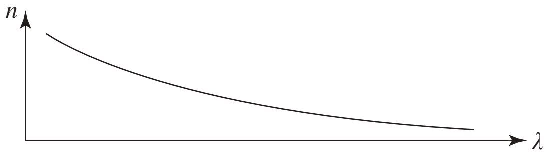


Figure 6.30 Normal dispersion: the relation between refraction index and λ.

For a better understanding of the presence of abnormalities in the dependence $n(\lambda)$, it is useful to consider the process of light interaction with atoms in a substance. In Section 4.2.4 in the analysis of electronic types of dielectric polarization, it was shown that the internal electric field in an atom linearly depends on displacement and creates a force that returns the charge to its initial position. This means that an electron, being forced out of its equilibrium position, begins to make harmonic oscillations with a frequency of its natural frequency ω_0. The strength of an electric field in a light wave acts with its own frequency ω. Under the action of an electric field, an electron starts to oscillate with frequency ω causing secondary radiation. The process of forced oscillation was considered in Chapter 2.7. There it was shown that the amplitude of the forced oscillations of a system $A(\omega)$ depends inversely on the difference of squares of natural frequency and the frequency of the driving force $(\omega_0{}^2 - \omega^2)$. When ω^2 approaches $\omega_0{}^2$, a sharp increase in oscillation amplitude can be observed (eq. (2.7.4) and Figure 2.15). Accordingly, the resonance is accompanied by an additional absorption of the incident wave; which is expressed in more or less sharp lines in spectra of absorption and emission.

The refraction index n is expressed through dielectric permeability ε and magnetic susceptibility μ by an equation $n = (\varepsilon\mu)^{1/2}$. In Section 4.2.5, it was mentioned that only electron polarization will be exhibited in light with a frequency $\approx 10^{15}$ sec^{-1}. Accordingly for substances with $\mu \approx 1$ (which is characteristic of the overwhelming number of chemical compounds) $n^2 = \varepsilon$. Taking into account that $\varepsilon = 1 + \kappa$ and $\kappa = \Re/\varepsilon_0 E$ (refer to Section 4.2.2, eq. (4.2.8)), we obtain $n^2 = 1+(\Re/\varepsilon_0 E)$. For dielectrics in this frequency range the polarization $\Re$ can be presented as: $\Re = np$ (eq. (4.2.2)). In its turn an induced electric moment p, according to definition, is the product of the charge $|e|$ and shoulder x of the induced dipole; then

$$n^2 = 1 + |e| \frac{nx}{\varepsilon_0 E} \cdot \tag{6.5.1}$$

The displacement value x can be found by solving a differential equation of forced oscillations. In this case the electric field strength is acting as the force: $F_{\mathrm{frc}} = |e| E(t) = |e| E_0 \cos\omega t$, a restoring force can be considered as an elastic force $F_{\mathrm{rst}} = -\beta x = -m\omega_0^2 x$. Therefore, the differential equation for x determination has the form:

$$m\left(\frac{d^2x}{dt^2}\right) = -m\omega_0^2 x + |e| E_0 \cos\omega t. \tag{6.5.2}$$

The solution of such an equation is given in Chapter 2.7 where it was shown that:

$$x = \frac{|e| E_0 \cos\omega t}{m(\omega_0^2 - \omega^2)} = \frac{|e| E(t)}{m(\omega_0^2 - \omega^2)}. \tag{6.5.3}$$

Substituting the result into eq. (6.5.1) we obtain

$$n^2(\omega) = 1 + \frac{ne^2}{\varepsilon_o m(\omega_0^2 - \omega^2)}. \tag{6.5.4}$$

At $\omega \ll \omega_0$ or $\omega \gg \omega_0$, the value n^2 does not depend on ω and is near to 1. If the light frequency approaches ω_0, resonance phenomena appear and the refraction index rises significantly (Figure 6.31). In reality the picture, $n^2(\omega)$ is like that presented in Figure 6.32. It is important that the position of abnormal dispersion ($(dn/d\omega) < 0$) coincides with the light absorption line in a substance.

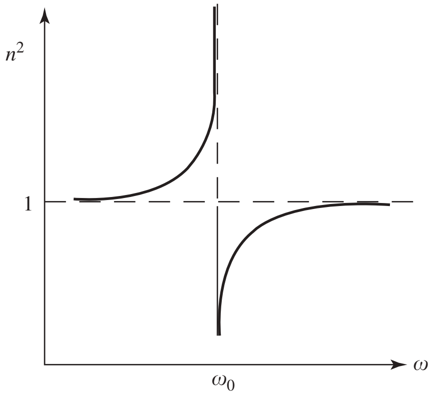


Figure 6.31 Graphic representation of the relationship of the square refraction index and the light frequency (as follows from eq. (6.5.4)).

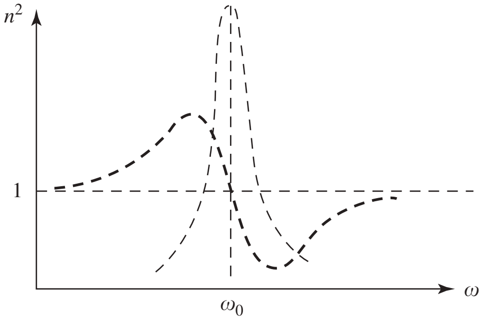


Figure 6.32 The experimentally measured dependence of n^2 versus the incident light angular frequency ω. The position of the anomaly dispersion maximum coincides with the absorption line position.

The figure at which $n < 1$, the light speed in a substance becomes larger than the speed of light c, should not confuse the reader. In this case the phase speed is considered, i.e., the speed of distribution of the given, constant phase. The wave energy and momentum are transferred by group speed which cannot be more than the speed of light (see Section 2.9.4).

6.6 THE QUANTUM-OPTICAL PHENOMENA

6.6.1 Experimental laws of an ideal black body radiation

The radiation of electromagnetic waves by a heated body refers to as *thermal radiation*. Any body radiates at any temperature, however at medium temperature range its intensity cannot be measured by an ordinary device at sure.

For the quantitative characteristic of thermal radiation the concept *of an emittance R* is used: the emittance is referred as to the energy that is emitted by a unit surface of a heated body in all directions within a solid angle 2π (a half of full solid angle, i.e., one side of a plane) in a unit time in a whole interval of frequencies (wavelengths).

Thermal radiation basically contains waves of all frequencies. When allocate an interval of frequencies $d\omega$ at temperature T, part of emittance dR corresponds to it: the wider $d\omega$ the higher dR. However the ratio between them is not linear, it depends on frequency ω (wavelength λ). The value r, connecting dR with $d\omega$ depends also on radiation frequency and referred to *as spectral density of emittance*, i.e.

$$dR(T) = r(\omega,T)d\omega,$$

or

$$\frac{dR(\omega,T)}{d\omega} = r(\omega,T). \tag{6.6.1}$$

The total emittance R can be obtained by integration of function $r(\omega,T)$ over the whole interval of frequencies, therefore R does not depend on frequency and is entirely defined by temperature:

$$R(T) = \int_0^\infty dR(\omega,T) = \int_0^\infty r(\omega,T)d\omega. \tag{6.6.2}$$

Function $r(\omega,T)$ also describes of a body's ability to emit thermal radiation.

Basically, the curve of dependence $r(\omega)$ at the fixed temperature T can be experimentally measured (Figure 6.33). The figure shows: a muffle furnace with a radiating body inside, aperture in the furnace door and in a screen cutting out the desirable stream of

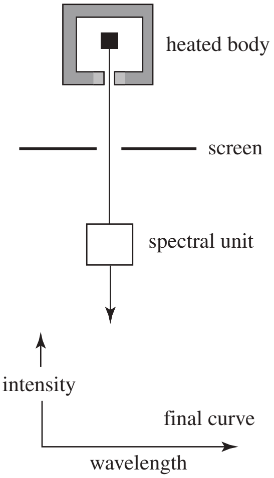


Figure 6.33 The experimental scheme of the measurement of the spectral composition of heat radiation; furnace to heat a body, screen protector, spectral device to measure the radiation intensity; the experimental graph coordinates are shown.

the thermal energy directed further to a spectral device. A purpose of the spectral device is to decompose the whole falling radiation into a spectrum on frequency. Consequently the spectrum of thermal radiation is obtained which should be investigated and explained.

To measure such curve represents significant difficulty as beams of different wavelength demand various techniques. In a result, the various parts of a general curve are “tailored”.

Figure 6.34 schematically depicts the emission spectra of bodies heated to various temperatures, from almost room temperature to temperature of the sun surface. In the same figure the curve of luminosity presents (see also Figure 6.1) which enables one to know radiation at what emitter temperature is radiation perceived by eye as light. Part of a spectrum of thermal radiation can be felt as heat on the skin of the human – this is mainly IR radiation. UV radiation can also be felt by the skin – this we think of as sunburn. All this is an insignificant part of the general thermal radiation. Perhaps, only the sun radiation contains in its spectrum all the wavelengths that humans perceives with almost all their sensory organs.

The curves can be schemed in coordinates $r(\omega)$ and $r(\lambda)$ keeping in mind the relation

$$\omega = 2\pi c/\lambda, \tag{6.6.3}$$

In order to transfer from $r(\lambda)$ to $r(\omega)$ one should to compare equivalent peaces of the graphs areas of both functions

$$r(\omega)d\omega = r(\lambda)d\lambda. \tag{6.6.4}$$

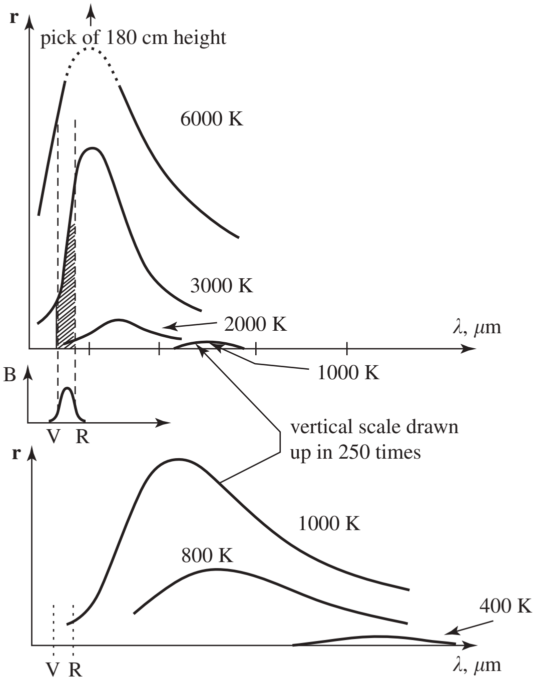


Figure 6.34 Spectral intensity distribution of the heat radiation at different emitter temperature: boiling water (400 K), electric heater (800 K), red incandescence (1000 K), blowtorch (2000 K), voltage arc (3000 K) and the sun radiation (6000 K).

Derivation of the eq. (6.6.3) gives

$$d\lambda = -\frac{2\pi c}{\omega^2}d\omega = -\frac{\lambda^2}{2\pi c}d\omega. \tag{6.6.5}$$

(The minus sign in this expression specifies only that with increase one value another one decreases. Therefore the sign minus further will be omitted). Changing in the equation (6.6.4) $d\lambda$ on $d\omega$ according to (6.6.5) we can obtain $r(\omega)d\omega = r(\lambda)(2\pi c/\omega^2)d\omega = r(\lambda)(\lambda^2/2\pi c)d\omega$, or, finally:

$$r(\omega) = r(\lambda)\frac{2\pi c}{\omega^2} = r(\lambda)\frac{\lambda^2}{2\pi c}. \tag{6.6.6}$$

The radiation power flux $d\,\Phi\,(\omega)$ is the thermal radiation energy emitted in the unit of time from the surface dS of the radiated body

$$d\Phi(\omega) = R(\omega)dS. \tag{6.6.7}$$

Falling on any surface, this flux will be partly absorbed (this part we will call the absorbed flux of energy $d\Phi'(\omega)$), and partly reflected. The dimensionless value

$$a(\omega,T) = \frac{d\Phi'(\omega,T)}{d\Phi(\omega,T)} \tag{6.6.8}$$

is referred to as the absorbing capacity of a body *a*. In general cases, absorbing capacity *a* depends both on frequency and on temperature of the radiator.

For a body that completely absorbs all the thermal radiation falling on it in all ranges of frequencies and at any temperature, the value *a* is constant and equal to 1; such a body is called a *perfect black body* (PBB). If the body's absorbing capacity depends on frequency and/or on temperature, but is constant (less than 1), the body is called *gray.* The majority of bodies, however, are not PBB or gray, their absorbing capacity is less than 1 and depends on frequency and temperature; these are arbitrary bodies.

There is a certain connection between the emitting and absorbing capacity of a body namely the Kirchhoff law: the ratio of the body's emitting and absorbing capacity does not depend on the nature of the body but is identical, i.e., universal, for all bodies' functions of frequency and temperatures (function $f(\omega,T)$

$$\frac{r(\omega,T)}{a(\omega,T)} = f(\omega,T). \tag{6.6.9}$$

This function is referred to as Kirchhoff function.

The Kirchhoff law defines one of the most important properties of thermal radiation, distinguishing it from other types of radiation (fluorescence, luminescent, etc.): thermal radiation is an equilibrium one. From eq. (6.6.9), it follows that the more a body absorbs, the more it radiates. Hence, in an isolated system of bodies their temperature will eventually be equalized, becoming identical. If a body absorbs more, it also radiates more. The values $r(\omega,T)$ and $a(\omega,T)$ can differ, but their ratio is identical.

The analysis of the curves, similar to those presented in Figure 6.34, allows one to understand and formulate some laws of thermal radiation. Thus it has been experimentally established that emittance is proportional to the fourth degree of absolute temperature:

$$R(T) \sim T^4.$$

For a gray body the given ratio can be rewritten:

$$R(T) = a\sigma T^4. \tag{6.6.10}$$

This is the Stefan–Boltzmann law. The value of constant σ in this law is experimentally established: $\sigma = 5.7 \times 10^{-8}$ W/(m^2 K^4). The expression (6.6.2) shows that the area under the curves in Figure. 6.34 is proportional to the fourth degree of absolute temperature.

From the same figure it can be seen that, depending on temperature, the radiation spectrum is shifted to the shorter wavelengths. This law can be written as

$$\lambda_{\max} = \frac{b}{T} \tag{6.6.11}$$

where $\lambda_{\max}$ represents the wavelength corresponding to the maximum of radiation spectral distribution. This dependence is referred to as Wien's law of displacement, constant b is equal to 2.90×10^{-3} mK.

All the above-mentioned experimental facts and laws require a theoretical explanation.

6.6.2 Theory of radiation of an ideal black body from the point of view of wave theory: Rayleigh–Jeans formula

Before examining the theory of thermal radiation, one should suggest the model of an ideal black body (IBB), i.e., a body that absorbs all falling radiation. The most simple, yet successful, is a model representing an almost completely closed cavity with a small aperture (Figure 6.35): all beams that get inside the cavity lose their intensity after consecutive reflections, and do not leave the cavity.

Because the heated-up walls of the cavity are a source of thermal radiation and only an insignificant part of it leaves, a certain equilibrium density of radiation is established in the cavity. Standing waves (such as in a string (see Section 2.9.3) are produced in the cavity, the wavelength of which is defined by eq. (2.9.8). As for standing waves in a string, the maximum wavelength of a standing wave is determined by the size of the cavity. Also, the minimum length of a wave is determined by the discrete character of the material from which the walls are made. Therefore, the density of standing waves in the cavity is finite. After detailed consideration, one can obtain $\omega^2/4\pi^2c^2$.

Within the framework of classical physics, an oscillator with certain frequency ω can be put in conformity to each standing wave. According to the law of uniform distribution of energy on degrees of freedom (see Chapter 3), to every oscillator an average energy κT can be attributed (because two $(1/2)\kappa T$ goes to the oscillation degree of

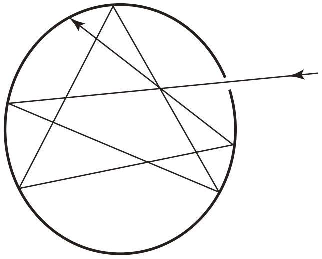


Figure 6.35 Model of an ideal black body (IBB).

freedom connected to kinetic and potential energy). Hence, the total energy of radiation of an IBB is

$$r(\omega,T)=\frac{\omega^2}{4\pi^2c^2}\kappa T. \tag{6.6.12}$$

or in terms of wavelengths

$$r(\lambda)=\frac{2\pi c}{\lambda^4}\kappa T. \tag{6.6.12$'$}$$

This formula was first suggested by Rayleigh and Jeans, and was irreproachable from the point of view of the wave nature of light. It equally concerns experimentally measured dependence of heat radiation of an ideal black body.

At the same time, the theory and experiment are in a glaring contradiction with one another, mainly in the area of small wavelengths (large frequencies). From Figure 6.36b, it can be seen that at small wavelengths (or at increase of frequency, in other figures) the theoretical curve soars sharply upwards, whereas the curve achieved by experiment (Figures 6.34 and 6.36a) goes downwards. This also leads to the incorrect conclusion that the luminosity R (6.6.2) becomes senselessly infinite. Because of the area where the divergence takes place, the discrepancy between theory and experiment has been called an “UV accident,” thus recognizing the inability of the theory to explain the laws of radiation within the framework of wave theory as it existed at that time.

A completely different approach to the theory of radiation was, therefore, necessary. One was proposed by W. Wien (awarded the Nobel Prize in 1911) who, on the basis of the laws of thermodynamics, obtained a “bell-shaped” theoretical curve. A revolutionary approach was suggested by Planck (1900), which resulted in the full agreement between theory and experiment.

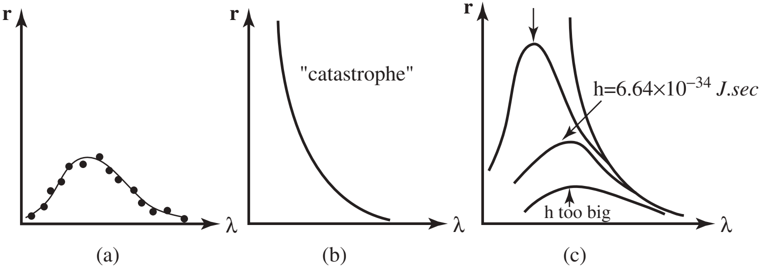


Figure 6.36 The spectral density of heat radiation r versus the wavelength λ: (a) experimental curve, (b) “ultraviolet catastrophe,” (c) the correspondence of the theory and experiment in the framework of Planck theory.

EXAMPLE E6.9

The maximum of spectral density of the sun's radiation emission corresponds to the wavelength $\lambda = 500$ nm. Assuming that the sun radiates like an IBB, determine:

(1) emittance of sun R^* (R^* means the emittance of IBB), (2) energy flux of sun radiation Φ, (3) total mass m of electromagnetic radiation irradiated by the sun in one sec (refer to Section 6.6.1).

Solution: (1) The Stefan–Boltzmann law describes the radiation emittance R^* of an IBB $R^* = \sigma T^4$, where σ equals to 5.67×10^{-8} W/(m^{2}K^4). The sun's surface temperature can be determined using the Wien's law $\lambda_{\max} = b/dT$, where b = 2.90×10^{-3} mK. Combining these two formulas, we obtain

$$R^* = \sigma\left(\frac{b}{\lambda_{\max}}\right)^4 .$$

Executing calculations, we arrive at $R^* = 6.4 \times 10^7$ W/m^2.

(2) The energy flux Φ radiated by the sun is the product of radiation emittance and the sun's surface area $\Phi = \mathrm{RS}$ or $\Phi = 4\pi r^2 R^*$ where r is the sun's radius. Substituting all data into the last formula, we arrive at $\Phi = 3.9 \times 10^{26}$ W.

(3) The total mass of electromagnetic radiation emitted by the sun in 1 sec can be determined using the correspondence mass and energy $E = mc^2$. The energy of the electromagnetic radiation in the time t is equal to the product of the flux and time $E = \Phi t$. Therefore $\Phi t = mc^2$ and further $m = \Phi(t/c^2)$. Executing calculations, we arrive at $m = 4.3 \times 10^9$ kg.

6.6.3 Planck's formula: a hypothesis of quanta—intensity of light from wave and quantum points of view

Analysis of the state of theory and experiment concerning IBB radiation and the mathematical description of the phenomenon led Planck to recall Newton's hypothesis that light is a stream of particles (corpuscles); this had been rejected on the basis of successive works on interference and diffraction. Planck suggested a revolutionary idea: that each particle of radiation is a corpuscle or quantum, i.e., a particle bears a portion of energy $hv = \hbar\omega$ where h is a certain constant, and $\hbar = h/2\pi$. Then the total radiation energy flux should be expressed by the total number of quanta, $\hbar\omega$, i.e., $E = \Sigma_{i=1}^{N} n_i \hbar\omega_i$, where $\hbar\omega$ is the energy of a single quantum, n_i is the amount of such quanta and N is their total number. The distribution of quanta on energy is set by the Boltzmann factor.

According to the statistical method of average values calculation (see eq. (3.2.11)), we can write a similar expression for the average energy of a quantum oscillator $\langle\varepsilon\rangle$ having replaced integrals by sums. Therefore, following Planck, we arrive at:

$$\langle\varepsilon\rangle = \frac{\sum_{n=0}^{N} n\hbar\omega \exp\left[-\frac{n\hbar\omega}{\kappa T}\right]}{\sum_{n=0}^{N} \exp\left[-\frac{n\hbar\omega}{\kappa T}\right]} = \frac{\hbar\omega[\exp(-x)+2\exp(-2x)+\cdots]}{1+\exp(-x)+\exp(-2x)+\cdots},$$

where $x = -(\hbar\omega/\kappa T)$. In the last expression a fraction can be rewritten as

$$\begin{aligned}\langle\varepsilon\rangle &= \hbar\omega\frac{d}{dx}\log[1+\exp x+\exp 2x+\cdots]\\ &= \hbar\omega\frac{d}{dx}\log\frac{1}{1-\exp x}=\frac{\hbar\omega}{\exp(-x)-1}\end{aligned}$$

from which the sought value $\langle\varepsilon\rangle$ is derived:

$$\langle\varepsilon\rangle=\frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{\kappa T}\right)-1}. \tag{6.6.13}$$

Leaving unchanged the part of the calculation in which the oscillation density in the cavity of an IBB was counted, and attributing to every oscillator the above-mentioned average energy $\langle\varepsilon\rangle$, Planck came to the formula of function $r(\omega,T)$ for an IBB

$$r(\omega,T)=\frac{\omega^2}{4\pi^2c^2}\times\frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{\kappa T}\right)-1}. \tag{6.6.14}$$

The formula obtained not only correctly reflected an agreement between the theory and experiment, but also allowed the determination of the h value; in Figure 6.36c the theoretical results are “adjusted” to the experimental, from which the value of the constant h (= 6.626×10^{-34} J sec) has been determined. This value was later named after M. Planck; it is typical that Planck’s constant has the dimension of the momentum or quantum of action (i.e., the product of energy and time of its action).

Proceeding further to corresponding functions from wavelength ((6.6.1) and (6.6.6)), it is possible to obtain the dependences r and f on λ:

$$r(\lambda,T)=f(\lambda,T)=\frac{4\pi^2\hbar c^2}{\lambda^5}\times\frac{1}{\exp\left(\frac{2\pi\hbar c}{\lambda\kappa T}\right)-1}. \tag{6.6.15}$$

Planck’s formula well describes limiting transitions. So, at $\hbar\omega \ll \kappa T$ the exhibitor in the denominator of function (6.6.14) can be decomposed in a series and be limited by two terms. This leads to Rayleigh–Jeans formula which describes very well experiment in this area of frequencies. In contrast, at $\hbar\omega \gg \kappa T$ the unit in the denominator can be neglected and functions r and f fall according to the exponent that is found by experiment.

The Planck formula suggests how to find numerical values of constants in Stefan–Boltzmann and Wien laws. In particular on integration of Kirchhoff's law on the whole frequency range one can arrive at the Stefan–Boltzmann formula. The constant in Wien's law *b* can be found by derivation of the Kirchhoff's function on frequency and equalizing it to zero. We hope that readers can carry out these calculations themselves.

Thus in the phenomena described, radiation is represented by a flow of corpuscles, quanta of energy which have been called *photons*. The energy of each photon is defined by product $\hbar\omega$. Though for a long time it was known that in other experiments the same radiation manifests itself as a flow of waves. There is a problem, which has occupied the minds of many physicists, which is now known as particle-wave dualism and to which we shall pay more attention.

Wave and quantum theory lead to completely different representations of the intensity of light. We should remember that intensity is understood as energy falling normally on a unit of area in a unit of time. Within the framework of wave theory, the intensity of a monochromatic beam of light is defined by the square of the wave amplitude and does not depend on frequency, i.e., $I \sim A^2$. In quantum theory at a fixed wavelength (and, correspondingly, frequency) intensity is defined by the number of quanta, i.e., $I \sim N\hbar\omega$. As will be shown below, a number of experiments have excellently confirmed Planck's quantum hypothesis.

Let us emphasize once again that the theory of thermal radiation became the starting point for quantum mechanics, which has subsequently received confirmation in many areas of physics.

Laws of thermal radiation are widely used in technology to initiate and support chemical processes and to measure the temperature of bodies, contact with which is either impossible or complicated (e.g., measuring the temperature of the stars, the heated up gases during the launch of missiles, etc). These laws form the basis of optical pyrometry. It can be seen in Figure 6.34 that measurement of an integral of luminosity *R* (the area under the curves) can be a measure of body's temperature. There are several kinds of pyrometry: one is based on color and another on brightness. The first is based on the position of the curve maximum and the second on the ordinate at fixed wavelength. Certainly, all the laws used here are only fair for IBB, however there are ways to account for the uncertainty arising in experiments.

The stated theory of thermal radiation also allows an explanation of a phenomenon that has an influence on life on earth. This is the so-called *green-house effect.* The sun's radiation (the spectrum is depicted in Figure 6.34), passes through open space, and reaches the external layers of the earth's atmosphere, naturally with a loss of intensity, but without a special change of spectral composition. In the atmosphere there is selective absorption of the sun's radiation by natural and industrial gases. This selectivity is defined by the structure of molecules, by their concentration and properties. It is natural also, that absorption of radiation depends on humidity, dust content and other properties of the atmospheric layers close to the surface of the earth.

The sun's radiation reaches the surface of the earth and heats it up and, together with the internal heat of the planet, defines the temperature of its parts (depending on geographical place). At the same time, the earth's surface also radiates thermal energy. The temperature of the "radiator" in this case is essentially less than the sun's temperature, accordingly the entire spectrum, under Wien's law, is shifted to the long-wavelength region area (see Figure 6.34).

This radiation is directed from the earth surface and should again penetrate through the atmosphere in the opposite direction. However, being long wavelength the earth's radiation is absorbed by the atmosphere differently than the radiation from the sun. The transmission ability of long-wavelength radiation is less than solar radiation, the earth's radiation is appreciably "absorbed" in the atmosphere, heating it up. This is the green-house effect.

Taking billions of years to establish, the thermal equilibrium in the solar system defines life on earth. Every large-scale action can affect the established balance to some extent, displacing it in one way or another. In particular, the industrial activity of mankind leads to a change in the chemical composition of the atmosphere, increasing the concentration of industrial waste products. This change influences the absorption of the radiation falling to earth and leaving it. However, to a much greater degree, it concerns the radiation of the earth rather than that of the sun. All these events cause "over-warming" of the atmosphere and disturbance of the equilibrium.

The effect is probably not so large: it is estimated at approximately 1–2°. The results, however, can be catastrophic. One example is the appreciable effect on the people living in those European countries that are below sea-level and protected from the sea by dams (e.g., Denmark, The Netherlands). The increase in atmospheric temperature can melt much more ice than would normally maintain the existing balance. The consequences are dangerous for large cities such as Venice, Saint Petersburg and will affect the climate of Florida and many other pearls of human civilization.

In this connection it is also worth mentioning the so-called "ozone holes"—the local destruction of a centuries-old balance in the composition of the atmosphere resulting from the products of industrial activity (e.g., chlorofluorocarbons—freons) which create areas (holes) in the atmosphere that are transparent to short-wave UV radiations. These holes in the ozone layer are making the affected areas dangerous to live in because of the excess of UV radiation, which is harmful to life on earth. It is also probable that short wavelengths can cause undesirable mutations in living organisms.

6.6.4 Another quantum-optical phenomena

Planck's hypothesis was confirmed and developed by Einstein's theory of an external photo-effect.

The photo-effect consists of knocking electrons out of the surface layers of some metals and oxides on their irradiation by quanta of electromagnetic radiation. The scheme of an experiment on the photo-effect is presented in Figure 6.37. The main part of the experimental equipment is a vacuum bulb with two electrodes C and A with a window allowing the irradiation of electrode C. The interaction of electromagnetic radiation of definite frequency ω and amplitude A results in knocking the electrons out of the surface of electrode C. A voltage difference is applied to the electrodes (Figure 6.38). First a negative pole is applied to K accelerating the kicked-out electrons. As the voltage increases, all the electrons reach electrode A, and saturation takes place (the horizontal line in the scheme). However, a device permits the signs of the electrodes to be changed; when a decelerating voltage difference is applied, part of the electrons are not able to reach electrode A. When the difference is zero only those electrons that have their own high enough kinetic energy can reach A. Furthermore, even at a coercive force some of the energetically active electrons can still reach A. Only a locked-out voltage can stop the current through the bulb. If the light intensity (i.e., the amplitude of the incident light, A) is increased and the same ω is kept, the saturation increases but the locked-in voltage remains the same. This means that it is not the light electric field amplitude (intensity) but the frequency which is responsible for kicking electrons out from the electrode and locking them. In fact, if the light frequency ω is increased, the locking in voltage also increases.

The result obtained in the experiment is in agreement with the supposition that, in a given phenomenon in the photo-effect, light behaves as a flux of particles (photons). The results obtained regarding the volt–ampere characteristic shown in Figure 6.38 are deceptive. These results can be explained as follows.

The energy of the falling quantum $\hbar\omega$ is transferred to an electron in photocathode producing the work A. Part of the energy is expended for the work A_1 of moving the electron from the deep layers of the photocathode up to its surface, then in overcoming electron binding to the photocathode body A_2; the remaining energy is left to the

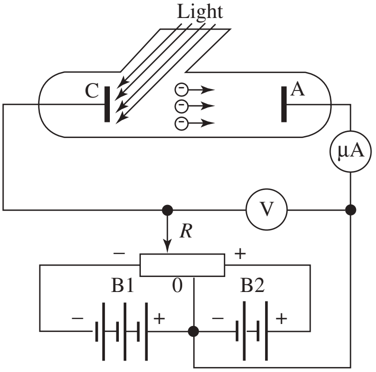


Figure 6.37 Diagram of a device for the photo-effect measurements: an electric device permits to change an electric field polarity.

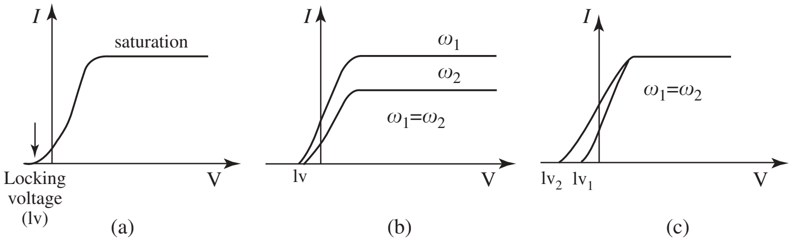


Figure 6.38 The V–A characteristics of the photo-effect at different light frequencies: (a) typical V–A characteristic, (b) the same frequency and different light intensity, (c) the same intensity of light however different frequencies.

photoelectron as its kinetic energy. According to the law of energy conservation we can write:

$$\hbar\omega = A_1 + A_2 + \frac{mv^2}{2}. \tag{6.6.16}$$

In this expression, work A_1 is undetermined since it is not known at precisely what point of the cathode body the collision took place. The work A_2 is a characteristic of every metal and oxide and is referred to as work function. If we exclude the unknown term from this line, i.e., remove the term A_1, the equation become valid only for those electrons which, at the moment of collision, were on the surface of electrode A; they have the highest possible kinetic energy. Therefore:

$$\hbar\omega = A_2 + \frac{mv_{\mathrm{max}}^2}{2}. \tag{6.6.17}$$

It is possible to determine the kinetic energy of the photoelectron using the experimental value of the locking out voltage. Then we can write:

$$eU_{\mathrm{loc}} = \frac{mv_{\mathrm{max}}^2}{2}. \tag{6.6.18}$$

It follows from this equation that there is a limiting frequency ω_{k} below which the photo-effect in a given photocathode disappears completely. In fact, the quantum $\hbar\omega_{\mathrm{k}}$ does not have enough energy to tear an electron out of the surface of electrode C. This happens when $\hbar\omega$ is lower than the work function. The so-called photo-electric threshold takes place at $\hbar\omega_{\mathrm{k}} = A_2$.

In addition to the theory of an IBB, it has been proved in quite another experiment that, in some circumstances, light behaves as a particle flux rather than a wave.

Albert Einstein was awarded a Nobel Prize in 1921 for his outstanding contribution to physical science in general and especially for the photo-effect theory, which belonged mainly to Einstein and provided convincing confirmation of Planck's hypothesis energy quanta not only in the theory of heat radiation but also in some other physical events. Einstein was also responsible for the concept of *a photon* which is widely used in modern physics.

A short-wave border of X-ray radiation is another phenomenon, which supports the quantum idea. Discovered by W.C. Röntgen in 1895 and referred to as X-rays, this is the electromagnetic radiation with a wavelength of the order 10^{-10} m (see Chapter 5, Table 5.3), arising on the electron transition in atoms and also on electron movement with acceleration (linear or centripetal). In the majority of countries this radiation is referred to as X-rays as it was called by Röntgen himself, but in Germany and Russia the term "Röntgen rays" is used. The generator of such radiation is the X-ray tube, the principle scheme of which is given in Figure 6.39. There are two electrodes in vacuum glass cylinder. The cathode represents a heated string and the anode is a massive metal cylinder, compulsorily cooled from inside by flowing water. The cathode's task is to emit electrons. A potential difference from tens to hundreds of keV (and more) is applied between the cathode and the anode. This electric field accelerates all emitted electrons up to an energy of 10–100 keV. The main part of the electron energy, allocated in the anode as heat, is taken away by flowing water. The remaining energy is used in the excitation of X-rays. Two kinds of X-ray radiation are known. *Characteristic* radiation results from the return electron' transitions from excited to ground state levels in atoms. This radiation has a linear spectrum and is widely used in modern science and technology for the analysis of chemical structures (refer to 7.6.4).

Bremschtralung radiation arises at the instant of the electron stopping in the anode substance. According to classical theory, the distribution of a frequency (wavelength), arising due to electron stopping X-ray radiation should cover a wide range of spectrum from zero to infinity (as in thermal radiation spectra). Experimental results contradict this supposition: in Figure 6.40 the X-ray intensity versus their wavelength is plotted, the curves sharply terminating at the shortest wavelengths.

An explanation of this fact can be found in the quantum theory of radiation. The law of energy conservation in this case can be written as:

$$eU = \sum_{i=1}^{N} \hbar\omega_i. \tag{6.6.19}$$

The electron energy before impact with an anode is written on the left-hand side of this expression; on right-hand side is the sum of all the photons' energies, which appeared on collision. Since the process of electron braking is uncontrollable in this process, photons of all energies are produced, and the spectrum contains all wavelengths (so-called "white" spectrum). However, a limiting case exists when an electron gives all its energy to produce only one single unique photon. In this experiment this photon possesses the largest energy. It defines the boundary value of the wavelengths; photons of larger energy (smaller wavelength) in the bremschtralung X-ray spectrum cannot appear. For such a photon, expression (6.6.19) becomes simpler

$$eU = \hbar\omega_{\max} = \frac{2\pi\hbar c}{\lambda_{\min}}, \tag{6.6.20}$$

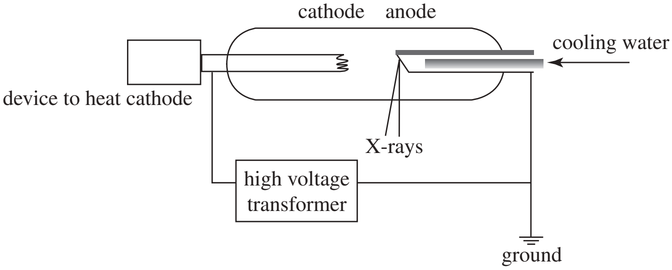


Figure 6.39 Scheme of an X-ray tube source.

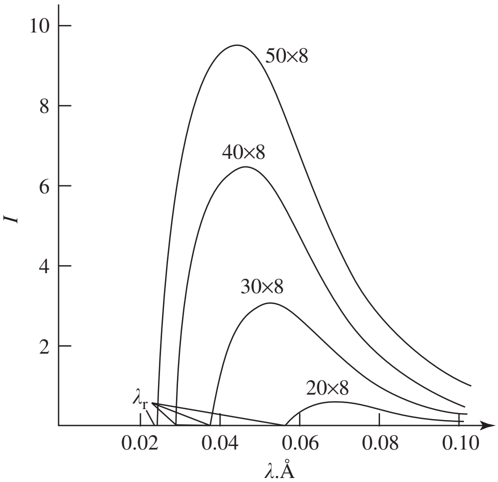


Figure 6.40 The relation of the X-ray bremschtralung short-range limit versus wavelength λ.

whence

$$\lambda_{\min} = \frac{hc}{eU}. \tag{6.6.21}$$

The expression excellently coincides with experiment.

The presence of a short-wave limit of X-ray radiation in the X-ray tube spectrum is a fact that cannot be explained by wave theory; in this experimental arrangement, the photon with larger energy cannot appear under the law of energy conservation: all the electron energy has already been given to the single X-ray quantum, the photon with smaller wavelength (greater energy) simply has no electron energy to appear;

The *Compton effect* is another phenomenon contradicting classical wave theory. This effect arises on X-rays scattering by electrons weakly bonded to atoms. The scheme of the experiment is given in Figure 6.41. A beam of monochromatic X-rays (with wavelength λ_0) falls through a collimator onto a sample and is scattered. A special device investigates the intensities of both incident and secondary radiation scattered at an angle θ.

Proceeding from wave theory, it follows that the scattered radiation should contain only one wavelength: the one that falls on the sample, i.e., λ_0. In fact, the electric field of an electromagnetic wave in the X-ray range should oscillate the electrons, which in turn should radiate secondary waves of the same wavelength. However, in the scattered radiation, experiment reveals that besides one *unshifted* λ_0, there is one more component, referred to as a *shifted* component with wavelength λ greater than λ_0 (Figure 6.42). It is experimentally established that the value of displacement (shift) $\Delta\lambda = \lambda - \lambda_0$ does not depend on the sample material and that this displacement is greater, the larger the scattering angle θ, namely,

$$\Delta\lambda = \lambda_0(1-\cos\theta) = 2\lambda_0 \sin^2\frac{\theta}{2}. \tag{6.6.22}$$

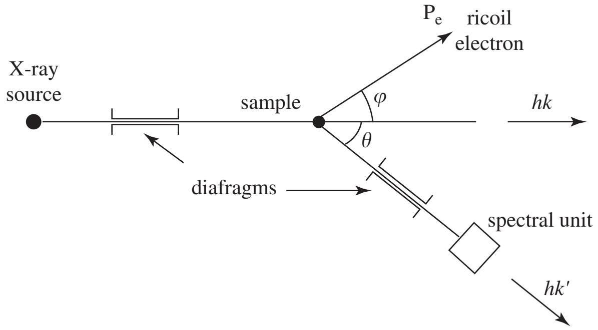


Figure 6.41 The Compton scattering experiment.

It is possible to explain all the features of the Compton effect if we consider them as a process of elastic collisions of X-ray photons with peripheral atomic electrons (in terms of the theory of particle collisions, see Section 1.4.5). We recall that, in elastic collision, both conservation laws (kinetic energy and momentum) are valid. Since a feedback electron can have a speed commensurable with the speed of light, it is more appropriate to use the relativistic theory (refer to Chapter 1.6) for the analysis.

At an initially rested electron, weakly bonded to an atom (with its kinetic energy and momentum practically equal to zero), the photon falls with energy $\hbar\omega$ and a momentum $\hbar\boldsymbol{k}$. In this case, the above-mentioned conservation laws in this case look like:

$$h\nu = h\nu' + (m - m_0)c^2 \tag{6.6.23}$$

the energy conservation law, and

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right). \tag{6.6.24}$$

the momentum conservation law, $\beta = \upsilon/c$.

For the feedback electron, having lost its bonding to the atom, the momentum relativistic expression p_e can be given as:

$$p_e = \frac{m_0\boldsymbol{\upsilon}}{\sqrt{1-\beta^2}}, \tag{6.6.25}$$

where υ is its speed.

Using a vector diagram (Figure 6.43), projecting electron and photon momentums on x-axis, we obtain

$$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\varphi + p_e\cos\theta, \tag{6.6.26}$$

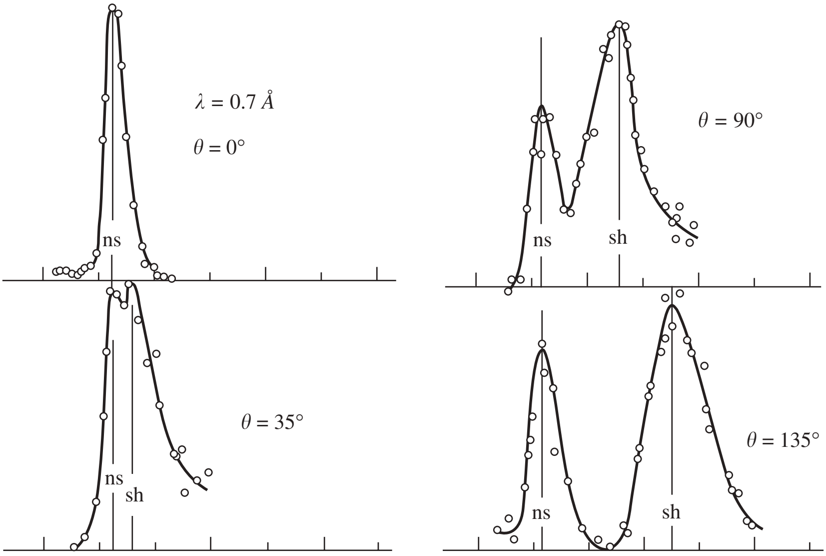


Figure 6.42 Results of the measurement of the Compton effect versus the scattering angle θ: ns, nonshifted component; sh, shifted component.

and on the *y*-axis

$$0 = \frac{h}{\lambda'} \sin\varphi - p_e \sin\theta. \tag{6.6.27}$$

Exclude from the last expressions the electron parameters and we arrive at the final expression:

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\varphi) = \Lambda(1 - \cos\varphi). \tag{6.6.28}$$

The expression h/m_0c is referred to as Compton wavelength and is denoted as Λ.

The same results can be obtained in the framework of nonrelativistic physics though the Compton effect belongs to the relativistic case. Both arrive at the same result, but the nonrelativistic derivation is simpler. However, when examining the feedback electron, it is necessary to use relativistic theory.

For his discovery and explanation of the effect, A. Compton was awarded the Nobel Prize in 1927. At present this effect is used for the study of atomic valence electrons in the structure of the chemical compounds.

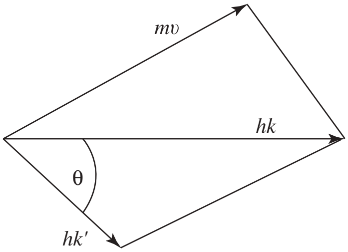


Figure 6.43 Vector diagram of Compton scattering.

EXAMPLE E6.10

An electron runs accelerating voltage 10^4 V in an X-ray tube. Determine the wavelength corresponding to the short-wave limit of the bremschtralung spectrum of the X-ray radiation.

Solution: A simplified construction of an X-ray generating device is given in Chapter 6.6. The accelerating electrons in the X-ray tube knock on an anode, X-rays being emitted in this process. In this example, a continuous spectrum is of interest. The shortest wavelength limit in this bremschtralung spectrum appears. It corresponds to the case when the whole electron energy transfers to a single X-ray quantum. Therefore, for this point the energy conservation law is valid $\varepsilon = eU = \hbar\omega = hc/\lambda$.

Therefore,

$$\lambda_{\min} = \frac{hc}{eU}.$$

Substituting all the values we arrive at

$$\lambda_{\min} = \frac{6.63\times10^{-34}\times3\times10^{8}}{1.6\times10^{-19}\times10^{4}} = 1.24\ \mathring{A}.$$

EXAMPLE E6.11

Determine the cesium photoelectric threshold λ_0 if at its surface irradiation by violet light $\lambda = 400$ nm a maximal speed of photoelectrons $\upsilon_{\max}$ is equal to 0.65×10^6 m/sec (refer to Section 6.6.4).

Solution: The threshold corresponds to the situation where both the speed and energy of photoelectrons are equal to zero. Therefore, Einstein's equation is

$\varepsilon = A$ where A is the photoelectric work function and ε is the electron kinetic energy. We can obtain the expression

$$\frac{2\pi\hbar c}{\lambda_0} = A \quad \text{or} \quad \lambda_0 = \frac{2\pi\hbar c}{A}*.$$

The photoelectric work function can be determined using Einstein's equation

$$A = \varepsilon - K_{\text{max}} = \frac{2\pi\hbar c}{\lambda} - \frac{mv_{\text{max}}^2}{2}.$$

To execute calculations we should express all the values in the SI system: $\hbar = 1.05 \times 10^{-34}$ J sec, $c = 3 \times 10^8$ m/sec, $\lambda = 400$ nm $= 4 \times 10^{-7}$ m, $m = 9.11 \times 10^{-31}$ kg, $\upsilon_{\text{max}} = 6.5 \times 10^5$ m/sec. Calculations give us $A = 3.05 \times 10^{-19}$ J. To define photoelectric threshold λ_0, we should substitute the already known data and obtain $\lambda_0 = 651$ nm.

EXAMPLE E6.12

A photon of energy $\varepsilon = 0.75$ MeV is scattered by a nearly free electron at an angle $\theta = 60°$. Assuming that the electron's kinetic energy and momentum before the collision were negligible small, define: (1) the energy ε' of the scattered phonon, (1) the kinetic energy K of the recoil electron and (3) the direction of its movement.

Solution: (1) According to the Compton formula (refer to Section 6.6.4):

$$\lambda' - \lambda = \frac{2\pi\hbar}{m_o c}(1 - \cos\theta),$$

we can express λ and λ' using energy of photons ε and ε':

$$\frac{2\pi\hbar c}{\varepsilon'} - \frac{2\pi\hbar c}{\varepsilon} = \frac{2\pi\hbar}{m_0 c}(1 - \cos\theta) \quad \text{or}$$
$$\frac{1}{\varepsilon'} - \frac{1}{\varepsilon} = \frac{(1 - \cos\theta)}{m_0 c^2}.$$

Solving this equation regarding ε' we obtain:

$$\varepsilon' = \frac{\varepsilon}{\frac{\varepsilon}{m_0 c^2}(1 - \cos\theta) + 1}$$

Executing calculations we arrive at $\varepsilon' = 0.43$ MeV.

(2) The kinetic energy can be found from the energy conservation law $K = \varepsilon - \varepsilon' = 0.32$ MeV.

(3) The direction of the electron recoil motion can be found by applying the momentum conservation law (see Figure 6.43) $\mathbf{p} = \mathbf{p}' + \mathrm{m}_0\upsilon$. From a triangle OCD we can find

$$\tan\varphi = \frac{|\mathrm{CD}|}{|\mathrm{OD}|} = \frac{|\mathrm{CA}|\sin\theta}{|\mathrm{OA}| - |\mathrm{CA}|\cos\theta} \quad \text{or}$$

$$\tan\varphi = \frac{p'\sin\theta}{p - p'\cos\theta} = \frac{\sin\theta}{\dfrac{p}{p'} - \cos\theta} = \frac{\sin\theta}{\dfrac{\varepsilon}{\varepsilon'} - \cos\theta}.$$

$\varepsilon' = \dfrac{\varepsilon}{\dfrac{\varepsilon}{m_0c^2}(1-\cos\theta)+1}$.0 (We use here the general equation $p = \varepsilon/c$). Let us express $\tan\varphi$ through the given data; therefore we find the ratio

$$\frac{\varepsilon}{\varepsilon'} = \frac{\varepsilon}{m_oc^2}(1-\cos\theta)+1.$$

Hence

$$\tan\varphi = \frac{\sin\theta}{\left(1+\dfrac{\varepsilon}{m_oc^2}\right)(1-\cos\theta)}.$$

Taking into account some trigonometric relation we arrive at

$$\tan\varphi = \frac{c\tan\dfrac{\theta}{2}}{1+\dfrac{\varepsilon}{m_0c^2}}.$$

Executing calculations we obtain $\tan\varphi = 0.701$ and correspondingly $\varphi = 35°$.

6.7 THE BOHR MODEL OF A HYDROGEN ATOM

In previous sections, experiments which do not have explanations within the framework of Newtonian physics have been described. It is also known that reliably measured linear spectra of atoms, primarily hydrogen, also required the development of a completely new approach. This approach was suggested by Bohr, who formulated the theory of the hydrogen atom (Nobel Prize, 1922). Today it is considered to be semi-quantitative although it has not lost its significance.

If we accept that atoms radiate electromagnetic waves with an energy $h\nu$, it is necessary to establish where this energy originates from. We can equate energy $h\nu$ to the loss of energy E_2 - E_1 but it is then necessary to explain the nature of these energies. It is tempting to accept Rutherford's planetary atomic model, but this seems impossible since it is known that the movement of a charged particle on a curvilinear trajectory undergoes a continuous loss of energy and the electron will inevitably fall onto a nucleus. However, in order to connect energy E_2 and E_1 with orbital movement, it is necessary to understand the stability of their orbits. The answers to all these questions were given by Niels Bohr in his theory of hydrogen atom.

Bohr's planetary model of the atom states that electrons in a hydrogen atom move in a circular orbit of radius r around a proton. The proton is so heavy in comparison with the electron that the center of mass of this system coincides with the position of the nucleus. Following Bohr, calculate the total electron energy.

According to Newton's second law

$$\frac{e^2}{4\pi\varepsilon_0 r^2} = m\frac{\upsilon^2}{r}, \tag{6.7.1}$$

where a Coulomb force electron–proton interaction equates to ma, a being centripetal acceleration. The kinetic energy K can be derived from this equation:

$$K = \frac{m\upsilon^2}{2} = \frac{e^2}{8\pi\varepsilon_0 r}, \tag{6.7.2}$$

whereas the potential energy of a negative charge in a field of the positive nucleus can also be found

$$U = \varphi(-e) = \frac{e^2}{4\pi\varepsilon_0 r}. \tag{6.7.3}$$

where φ is the nucleus electrostatic potential at distance r (refer to eqs. (4.1.21) and (4.1.22)). The total energy is then

$$E = K + U = -\frac{e^2}{8\pi\varepsilon_0 r}. \tag{6.7.4}$$

Since the orbit radius can apparently take on any value, so can the energy E. The problem of E quantization reduces to quantization of r.

All the values listed above are unequivocally connected with r, other values are expressed with its help as well. In particular, the linear electron speed υ can be expressed as

$$\upsilon = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m r}}. \tag{6.7.5}$$

The frequency v is

$$\upsilon = \frac{\upsilon}{2\pi r} = \sqrt{\frac{e^2}{16\pi^3\varepsilon_0 m r^2}}. \tag{6.7.6}$$

The linear momentum p is

$$p = m\upsilon = \sqrt{\frac{me^2}{4\pi\varepsilon_0 r}}. \tag{6.7.7}$$

And the angular momentum L is

$$L = pr = \sqrt{\frac{me^2 r}{4\pi\varepsilon_0}}. \tag{6.7.8}$$

Thus if r is known the orbit parameters K, U, E, υ, v, p and L are also known. If any one of them are quantized all others must be also.

Up to this point only classical physics has been used. Here, Bohr suggested that the necessary quantization of the orbit's parameters shows up most simply when applied to the angular momentum and that, specifically, L can take on only values given by

$$L = n\hbar = n\frac{h}{2\pi}, \tag{6.7.9}$$

where n can accept integer values 1, 2, 3, etc. The Planck constant appears again in a fundamental way.

Combining eqs. (6.7.8) and (6.7.9) leads to

$$r = \frac{h^2\varepsilon_o}{\pi m e^2} n^2 \tag{6.7.10}$$

and

$$E = -\frac{me^4}{8\varepsilon_0^2 h^2}\,\frac{1}{n^2} \tag{6.7.11}$$

i.e., to the energy quantization.

If we accept $n = 1$, the well-known radius of the first orbit in H-atom can be obtained

$$r_0 = \frac{h^2 \varepsilon_o}{\pi m e^2}. \tag{6.7.12}$$

Further, the atom radiates or absorbs energy only when an electron passes from one stationary orbit to another. This portion of radiation has been referred to as a quantum (of energy).

As the energy is connected to the frequency of the quantum and, accordingly, to wavelength (in vacuum) the frequency of the quantum can be expressed through the quantum numbers corresponding to two orbits (j and k being their quantum numbers) (compare with (7.5.33)):

$$v = \frac{me^4}{8\varepsilon_0^2 h^3}\left(\frac{1}{j^2} - \frac{1}{k^2}\right).$$

This is the famous serial formula which allows calculation of all the spectral lines in the hydrogen atom spectrum.

An expression $me^4/8\varepsilon_0{}^2h^3$ is referred to as a Rydberg constant.

The quantum mechanical theory of the hydrogen atom is given below (see Chapter 7.5)

The Bohr model of the hydrogen atom is a transition from purely classical presentations to quantum mechanical ones: the motion of electrons along the orbits is accepted; however not all orbits are permitted, the angular momentum is accepted, though its values and orientations are subject to strict limitation. One can consider the Bohr model as the transition from classical mechanics to quantum mechanics with the preservation of many its attributes. As a result, many of the ideas of the Bohr model will often be met in order to simplify the students' understanding.

A typical quantum mechanical object such as an atom possesses some classical characteristics unexplainable within the framework of generally accepted presentations (no orbital motion, yet the existence of angular momentum; no rotation of an electron around its own axis, yet intrinsic angular and magnetic moments, i.e., spin, etc.). As a result, these terms are used irrespective of their classical sense.

EXAMPLE E6.13

In the framework of the Bohr model of the hydrogen atom (refer to Section 6.6.7) calculate the radius of the first electron orbit r and the linear electron speed υ.

Solution: From Section 6.6.7, we know that r and υ are united in equation $m\upsilon r = n\hbar$ (in our case $n = 1$). In order to find two values one more relation is needed. For the rotation of an electron around a nucleus we can write the Newton's second law equation

$$m\frac{\upsilon^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}.$$

or

$$mv^2 = \frac{1}{4\pi\varepsilon_o}\frac{e^2}{r}.$$

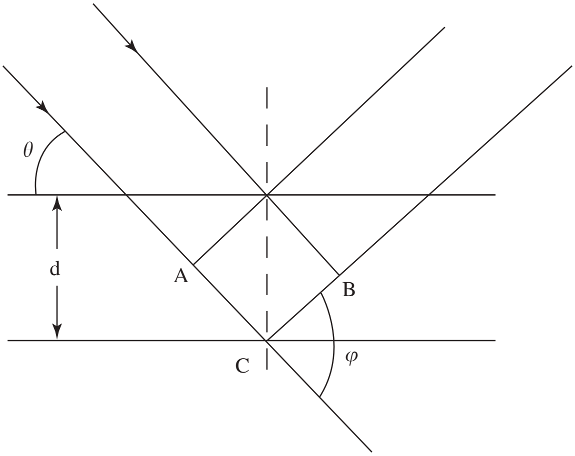
Solving all equation we can find

$$r = \frac{4\pi\varepsilon_o\hbar^2}{me^2}.$$

Substituting all known values, we arrive at $r_1 = a_o = 5.29 \times 10^{-11}$ m and $v = \hbar/mr = 2.18 \times 10^6$ m/sec.

EXAMPLE E6.14

It is measured experimentally that the CuK_α X-ray beam ($\lambda = 1.542$ Å) being diffracted by a corundum single crystal deviates from its initial direction at an angle $\varphi = 41.66°$ (compare to Figure 6.20). The diffraction takes place from the crystallographic plane (600). Find the deviation angle φ from the same crystallographic plane using the MoK_α radiation ($\lambda = 0.710$ Å).



Solution: Bragg's law should be used to solve this problem (refer to Section 6.3.5 and eq. (6.3.11), Figure 6.20). In the equation mentioned, θ is the incident angle and φ is the deflection angle. The picture is repeated here to make the situation clearer (Figure E6.14; an atomic arrangement is not depicted substituted by two reflected planes with interplanar distance d_{600}). It is seen in the picture that $\varphi = 2\theta$. The crystallographic plane's index is (600), in fact there is no such plane in the crystal; this should be understood as a 6th order of reflection from the plane (100) (i.e., at distance

ACB in Figure 6.20 six wavelengths stack). Therefore, $\sin(\varphi/2) = 6\lambda/2d_{600}$; this equation is valid for both wavelengths. Dividing two equations for two wavelengths we obtain $(\sin(\varphi_{Mo}/2) = \sin(\varphi_{Cu}/2)(\lambda_{Mo}/\lambda_{Cu})$. Executing calculations we arrive at $\sin(\varphi_{Mo}/2) = 0.164$, therefore, φ_{Mo} is equal to 4.72°.

PROBLEMS/TASKS

6.1. The surfaces of a glass wedge form an angle $\theta = 0.2'$. On the wedge perpendicular to its surfaces a beam of monochromatic light with wavelength $\lambda = 0.55$ μm falls. Determine the width of interference strips b (the distance between the adjacent maxima).

6.2. The diameters of two light Newton rings are $d_1 = 4.0$ and $d_2 = 4.8$ mm. It is known that three light rings settle between the two measured rings. The rings were observed in reflected light. Find the curvature radius R of the plane-convex lens.

6.3. In the experiment with Newton rings, a liquid oil was poured between a lens and a sample stage table, with its refraction index less than that of glass. The radius of the eighth dark ring is $d_8 = 2$ mm ($\lambda = 700$ nm) whereas the radius R of the plane-convex lens is 1 m. Find the refraction index n of oil.

6.4. On illuminating diffraction gratings by a white light, the spectra of second- and third-orders partly overlap. On what wavelength in the second-order spectrum, does ultraviolet (UV) of the third order ($\lambda = 0.4$ μm) fall.

6.5. A monochromatic light with wavelength $\lambda = 600$ nm falls on diffraction gratings with the period $d = 10$ μm at an angle 30°. Find the diffraction angle φ corresponding to the second main maximum.

6.6. The energy flux Φ radiated through a muffle's sight hole is $\Phi = 34$ W. Assume that the muffle radiates as an IBB and find its temperature T if its area is $S = 6$ cm^2.

6.7. Assume that the sun radiates as an IBB; calculate its emittance R and surface temperature T. The solar constant (the energy radiated by the sun per unit area measured on the outer surface of earth's atmosphere) is $C = 1.4$ kJ/m^2 s. Assume the sun–earth distance to be $d = 1.49 \times 10^{11}$ m.

6.8. Because of the change of temperature of an IBB, the position of the maximum spectral density emittance shifts from $\lambda_1 = 2.4$ μm to $\lambda_2 = 0.8$ μm. How many times are the emittance R and the maximum spectral density of emittance r changed?

6.9. The temperature T of an IBB is $T = 2000$ K. Calculate: (1) the spectral density of emittance $r(\lambda)$ for the wavelength $\lambda = 600$ nm, (2) emittance R in an interval of wavelength from $\lambda_1 = 590$ nm to $\lambda_2 = 610$ nm. Assume that the averaged spectral density of emittance in this interval is equal to that for $\lambda = 600$ nm.

6.10. Compton scattering of X-ray $\lambda = 55.8$ pm occurs by graphite plate. Find the wavelength λ' of light scattered at an angle $\theta = 60°$

6.11. A photon ($\lambda = 1$ pm) is scattered by a free electron at an angle $\theta = 90°$. What part of its energy W does the photon transmit to the electron?

6.12. The spectral density maximum $r_{\max}$ of the emittance (λ) of the bright star Arcturus corresponds to $\lambda = 580$ nm. Assuming that the surface of this star emits as an IBB, determine its temperature T.

6.13. Find the wavelength λ of a 1 MeV photon. Compare it with resting electron mass m_0.

6.14. A photon's wavelength is equal to Compton length λ_C. Determine the photon energy ε and momentum p.

ANSWERS

6.1. $b = \lambda/(2n\theta)$ =3.15 mm.
6.2. $R = 880$ mm.
6.3. $n = 1.4$.
6.4. 0.6 μm.
6.5. $\varphi = \arcsin(\sin\alpha + m\lambda/d) = 38.3°$.
6.6. $T = 1$ kK.
6.7. $R = 64.7$ MW/m^2; $T = 5.8$ kK.
6.8. Increases $R = 81$ and $r = 243$ times.
6.9. $r(\lambda) = 30$ MW/m^2 mm, R =600 W/m^2 .
6.10. $\lambda' = 57$ nm.
6.11. 0.70.
6.12. $T = 4.98$ kK.
6.13. $\lambda = 1.24$ pm, $m_{ph} = 1.8 \times 10^{-30}$ kg, $p_{ph} = 5.3.10^{-22}$ kg m/sec, $m_{ph} \approx 2m_{oe}$.
6.14. $\varepsilon = 0.511$ MeV, $p = 2.7 \times 10^{-22}$ kg m/sec.