2 The Balance Equations and Newtonian Fluid Dynamics

2.1 INTRODUCTION

The engineering science of ''transport phenomena'' as formulated by Bird, Stewart, and Lightfoot (1) deals with the transfer of momentum, energy, and mass, and provides the tools for solving problems involving fluid flow, heat transfer, and diffusion. It is founded on the great principles of conservation of *mass*, *momentum* (Newton's second law), and *energy* (the first law of thermodynamics).[1] These conservation principles can be expressed in mathematical equations in either *macroscopic* form or *microscopic* form.

In this chapter, we derive these equations in some detail using the generalized, coordinate-free formulation of the Reynolds Transport Theorem (2). We believe that it isimportant for every student or reader to work through these derivations at least once. We then discuss the nature of the stress and rate of deformation tensors, demonstrate the use of the balance equations for problem solving with Newtonian fluids using analytical and numerical techniques, discuss the *lubrication approximation*, which is very useful in modeling of polymer processing operations, and discuss the broad principles of *mathematical modeling* of complex processes.

1. See R. Feynman, *The Character of Physical Law*, MIT Press, Cambridge, MA, 1967, where the profound nature of the conservation laws is discussed.

Principles of Polymer Processing, Second Edition, by Zehev Tadmor and Costas G. Gogos.

2.2 THE BALANCE EQUATIONS

Since in "transport processes" mass, momentum, and energy are *transported* from one part of the medium to another, it is essential that proper "bookkeeping" be applied to keep track of these quantities. This can be done using *balance equations*, which are the mathematical statements of the physical laws of conservation. These are very general laws that *always* hold, and they apply to all media: solids or fluids, stationary or flowing. These equations can be formulated over a specified *macroscopic* volume, such as an extruder, or a *microscopic* volume taking the form of a differential (field) equation that holds at every point of the medium. In the former case, the balance holds over the *extensive* quantities of mass, momentum, and energy, whereas in the latter case, it holds over their *intensive* counterparts of density, specific momentum, and specific energy, respectively.

In the formulation of the microscopic balance equations, the molecular nature of matter is ignored and the medium is viewed as a *continuum.* Specifically, the assumption is made that the mathematical points over which the balance field-equations hold are big enough to be characterized by property values that have been averaged over a large number of molecules, so that from point to point there are no discontinuities. Furthermore, *local equilibrium* is assumed. That is, although transport processes may be fast and irreversible (dissipative), from the thermodynamics point of view, the assumption is made that, locally, the molecules establish equilibrium very quickly.

2.3 REYNOLDS TRANSPORT THEOREM

The physical laws of conservation of mass, momentum, and energy are commonly formulated for closed *thermodynamic systems*,[2] and for our purposes, we need to transfer these to open *control volume*[3] formulations. This can be done using the Reynolds Transport Theorem.[4]

Let P represent some extensive property of the system (e.g., mass, momentum, energy, entropy) and let p represent its intensive counterpart (i.e., per unit mass), such that:

$$P = \int_{\mathcal{V}} \rho p \, d\mathcal{V} \tag{2.3-1}$$

where $\mathcal{V}$ is the volume of the system, which can be a function of time, t. The Reynolds Transport Theorem states that the substantial derivative (see Footnote 6) of P is

$$\frac{DP}{Dt} = \int_{V} \frac{\partial}{\partial t}(\rho p) dV + \int_{S} \rho p \mathbf{v} \cdot \mathbf{n} \, dS \tag{2.3-2}$$

2. A *thermodynamic system* is an arbitrary volume of matter without any transportation of matter across its surface.
3. The *control volume* is an arbitrary, fixed volume in space.
4. We assume the reader is familiar with vector notation, which is covered in many texts (e.g., Ref. 1), and except for brief explanatory comments, no summary of vector operation is presented. However, the tabulated components of the balance equations in various coordinate systems presented in this chapter should enable the reader to apply them without any detailed knowledge of vector operations.

where V is the *control volume* fixed in space, S is the surface area of the control volume, $\mathbf{v}$ is the velocity field, and $\mathbf{n}$ is the unit outward normal vector to the control surface. In physical terms, Eq. 2.3-2 states that the rate of change of P of the *system*, at the instant it coincides with the *control volume*, is the sum of two terms: the rate of change of P within the control volume, and the net rate of flow of P out of the control volume.

Proof of Eq. 2.3-2 First we take the substantial derivative of Eq. 2.3-1

$$\left.\frac{DP}{Dt}\right|_{system} = \frac{d}{dt}\int_{V} \rho p \, dV \tag{a}$$

Then by defining $P_s = P_{\text{system}}$ we can express the left-hand side of Eq. (a)

$$\frac{DP_s}{Dt} = \lim_{\Delta t \to 0} \frac{P_{s,t+\Delta t} - P_{s,t}}{\Delta t} \tag{b}$$

Next we let the arbitrary volume of P_s coincide at time t with the control volume. Since the volume is arbitrary, we can do so without losing generality. But because there is the flow of matter in the space, at time $t + \Delta t$ the volume of P_s will be different, as shown in Fig. 2.1.

Looking at the figure we see that there are three distinct volume regions, A, B, and C. The control volume equals the sum $A + B$, and the system equals $B + C$. Therefore P_s at time $t + \Delta t$ can be expressed as

$$P_{S,t+\Delta t} = P_{B,t+\Delta t} + P_{C,t+\Delta t} = P_{CV,t+\Delta t} - P_{A,t+\Delta t} + P_{C,t+\Delta t} \tag{c}$$

and

$$P_{S,t} = P_{CV,t} \tag{d}$$

where the subscript CV stands for control volume. Substituting Eqs. (c) and (d) into Eq. (b) gives

$$\frac{DP_S}{Dt} = \lim_{\Delta t \to 0} \frac{P_{CV,t+\Delta t} - P_{CV,t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{P_{C,t+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{P_{A,t+\Delta t}}{\Delta t} \tag{e}$$

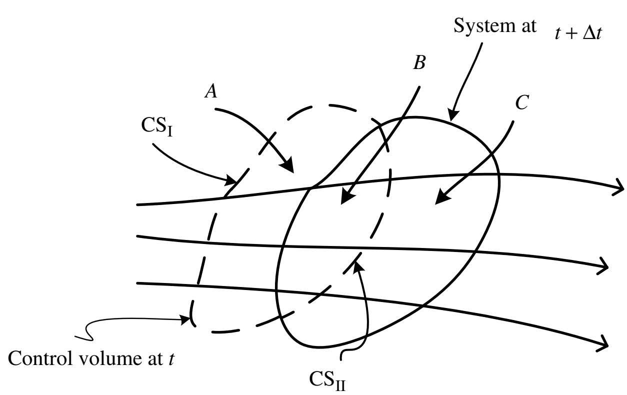


Fig. 2.1 The *control volume* (broken curve) and the *thermodynamic system* (solid curve) at time $t + \Delta t$ in a flowing medium. The *control volume* and the *thermodynamic system* coincide at time, t.

The first expression on the right hand side is the partial differential in time of $P_{CV}, \partial P_{CV}/\partial t$. Now, $P_{C,t+\Delta t}$ is due to flow through surface CS_{II} of the control volume, separating volumes B and C, which can be calculated as follows:

- The local volumetric rate of flow is $\mathbf{v} \cdot \mathbf{n}\, dS$
- The local rate of flow of the property is $\rho p\ \mathbf{v} \cdot \mathbf{n}\, dS$
- The differential quantity of P_C transported over time Δt is $dP_C = \Delta t p \mathbf{v} \cdot \mathbf{n}\, dS$

Thus, the total amount of P_C transported over surface CS_{II} and time Δt is given by:

$$P_{C,t+\Delta t} = \Delta t \int_{CS_{II}} \rho p \mathbf{v} \cdot \mathbf{n}\, dS \tag{f}$$

and, therefore,

$$\lim_{\Delta t \to 0} \frac{P_{C,t+\Delta t}}{\Delta t} = \int_{CS_{II}} \rho p \mathbf{v} \cdot \mathbf{n}\, dS \tag{g}$$

Similarly we can show that

$$\lim_{\Delta t \to 0} \frac{P_{A,t+\Delta t}}{\Delta t} = -\int_{CS_{I}} \rho p \mathbf{v} \cdot \mathbf{n}\, dS \tag{h}$$

The reason for the negative sign is that $\mathbf{v} \cdot \mathbf{n}$ for flow into the system is negative. Substituting Eqs. g and h into Eq. e yields the following equation:

$$\frac{DP_S}{Dt} = \frac{\partial}{\partial t} \int_V \rho p\, dV + \int_S \rho p \mathbf{v} \cdot \mathbf{n}\, dS \tag{i}$$

where S is the total surface $(CS_I + CS_{II})$ of the control volume and V is its volume, which is identical to Eq. 2.3-2. This concludes the proof.

2.4 THE MACROSCOPIC MASS BALANCE AND THE EQUATION OF CONTINUITY

In deriving the balance equations, we use vector notation and the sign convention adopted by R. B. Bird, W. E. Stewart, and E. N. Lightfoot in their classic book *Transport Phenomena* (1).

We begin the derivation of the conservation of mass by simply inserting into Eq. 2.3-2, $P = M$ and $p = 1$, yielding directly the *macroscopic mass balance equation*:

$$\frac{\partial}{\partial t} \int_V \rho\, dV + \int_S \rho \mathbf{v} \cdot \mathbf{n}\, dS = 0 \tag{2.4-1}$$

We can convert the surface integral in Eq. 2.4-1 to a volume integral using the *Gauss Divergence Theorem*[5,6] to yield:

$$\frac{\partial}{\partial t}\int_V \rho \, dV + \int_V (\nabla \cdot \rho \mathbf{v}) \, dV = 0 \tag{2.4-2}$$

But by definition, we have selected a fixed control volume, therefore the order of differentiation by time and integration can be reversed to get:

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v}\right) dV = 0 \tag{2.4-3}$$

For this equation to hold for any arbitrary volume V, the kernel of the integral must vanish, resulting in the *equation of continuity*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{2.4-4}$$

Equation 2.4-4 can be rewritten in terms of the substantial derivative as:

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v}) \tag{2.4-5}$$

Equation 2.4-4 states the mass conservation principle as measured by a *stationary* observer. The derivative $(\partial/\partial t)$ is evaluated at a *fixed position* in space (this is referred to as the Eulerian point of view); whereas, Eq 2.4-5 states the conservation principle, as

5. The Gauss Divergence Theorem states that if V is a volume bounded by a closed surface S, and $\mathbf{A}$ is a continuous vector field, then $\int_{\mathbf{v}}(\nabla \cdot \mathbf{A}) \, dV = \int_{\mathbf{s}}(\mathbf{n} \cdot \mathbf{A}) \, dS$.

6. The recurring vectorial operator ∇, known as del or nabla, is a differential operator that, in rectangular coordination is defined as:

$$\nabla = \boldsymbol{\delta}_1 + \frac{\partial_1}{\partial x_1} + \boldsymbol{\delta}_2 + \frac{\partial_2}{\partial x_2} + \boldsymbol{\delta}_3 + \frac{\partial_3}{\partial x_3}$$

where $\boldsymbol{\delta}_i$ are unit vectors in directions x_1, x_2 and x_3. For the derivation of ∇ in curvilinear coordinates, see Problem 2.1. The "substantial derivative," namely, the change in time of some property in a fluid element while being convected (or riding with) the fluid in terms of ∇, is given by:

$$\frac{D}{DT} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

Recall that the operation of ∇ on a scalar quantity is the gradient, which is a vector. For example, if ∇ is operated on a scalar pressure field P, then ∇P is the pressure gradient vector field, which can have different values in the three spatial directions. The operation of ∇ on a vector field can either be the divergence or the curl of the vector field. The former is obtained by the dot product (also called the scalar product) as $\nabla \cdot \mathbf{v}$ or div $\vec{v}$, where the result is a scalar; whereas, the latter is obtained by the cross product (also called the vector product) $\nabla \otimes \mathbf{v}$, or curl $\mathbf{v}$, and the result is a vector field.

TABLE 2.1 The Equation of Continuity in Several Coordinate Systems

Rectangular Coordinates (x, y, z)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical Coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical Coordinates (r, θ, ϕ)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

Source: Reprinted with permission from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960.

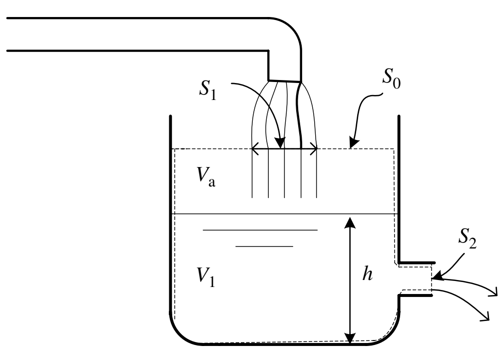
measured (reported) by an observer who is moving with the fluid (this is referred to as the Lagrangian point of view). Table 2.1 gives Eq. 2.4-4 in rectangular, cylindrical, and spherical coordinate systems.

For an incompressible fluid, the density is constant, that is, it does not change in time or spatial position, and therefore Eq. 2.4-5 simplifies to:

$$\nabla \cdot \mathbf{v} = 0 \tag{2.4-6}$$

In fluid dynamics we frequently invoke the incompressibility assumption, even though fluid densities change with pressure and temperature, and these may vary in time and space. If the density change cannot be neglected, then an appropriate *equation of state* of the form $\rho = \rho(T, P)$ must be used in conjunction with the balance equations.

Example 2.1 The Use of the Macroscopic Mass Balance for a Vessel with Salt Solution A liquid-filled vessel shown in the accompanying figure contains a 1000 kg of 10% by weight salt solution. At time $t = 0$ we begin feeding a 2% by weight salt solution at 20 kg/h and extracting 10 kg/h solution. Find the amount of solution M and salt S in the vessel as a function of time.



Solution Using the macroscopic mass balance Eq. 2.4-1, we first define the control volume as shown by the dotted line in the figure. Note that we defined the control volume above the liquid surface. Next, we apply Eq. 2.4-1 to the control volume which, taking into account all streams entering and leaving the tank, yields:

$$\frac{\partial}{\partial t}\int_{V_l} \rho_{\text{sol}}\, dV + \frac{\partial}{\partial t}\int_{V_a} \rho_{\text{air}}\, dV - \rho_{\text{sol},2\%} v_1 S_1 + \rho_{\text{sol}} v_2 S_2 + \rho_{\text{air}} \frac{dh}{dt} S_0 = 0$$

(change in M in CV) (mass flow in) (air mass flow out)
(change in air mass in CV) (mass flow out)

Note that **n** is normal to the surface and points outward from the control volume, hence, streams entering are negative and leaving are positive. Now the second and last terms cancel each other so they can be dropped to yield:

$$\frac{dM}{dt} - 20 + 10 = 0$$

and integration with the given initial condition gives:

$$M(\text{kg}) = 1000 + 10t(\text{min})$$

We next apply Eq. 2.4-1 to the salt:

$$\frac{\partial}{\partial t}\int_V \left(\frac{S}{M}\right) \rho_{\text{sol}}\, dV - 20 \times 0.02 + 10\left(\frac{S}{M}\right) = 0$$

where the first term is $\partial S/\partial t$ and S/M is the instantaneous salt concentration, leading to:

$$\frac{dS}{dt} + \frac{10S}{1000 + 10t} = 0.4$$

which results after integration with initial conditions $S = 100$ at $t = 0$, in

$$S(\text{kg}) = \frac{10000 + 40t + 0.2t^2}{100 + t}$$

where t is in minutes. The ratio of S/M gives the concentration of salt as a function of time. This will show that, as time goes toward infinity, the concentration approaches inlet concentration.

Example 2.2 The Radial Velocity in a Steady, Fully Developed Flow Show that in a steady, fully developed flow of an incompressible liquid in a pipe, the radial velocity component vanishes.

Solution In a pipe flow we have, in principle, three velocity components v_z, v_θ, and v_r. The equation of continuity in cylindrical coordinates is given in Table 2.1. For an incompressible fluid, this equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

However, because of symmetry, v_θ must vanish and, in a fully developed flow $\partial v_z/\partial z = 0$, we therefore obtain

$$\frac{\partial}{\partial r}(rv_r) = 0$$

which, upon integration, yields $rv_r = C$, where C is a constant. However, since there is no flow across the wall, $C = 0$, and hence the radial velocity must vanish everywhere.

2.5 THE MACROSCOPIC LINEAR MOMENTUM BALANCE AND THE EQUATION OF MOTION

Newton's Second Law is a statement of conservation of linear momentum for a system:

$$\frac{D\boldsymbol{P}_s}{Dt} = \sum_i \boldsymbol{F}_i \tag{2.5-1}$$

where $\boldsymbol{P}_s$ is the linear momentum $m\mathbf{v}$ of a body of mass m; $D\boldsymbol{P}_s/Dt$ is the substantial derivative of the linear momentum; and $\sum_i \boldsymbol{F}_i$ are the forces acting on the body. Substituting Eq. 2.5-1 into Eq. 2.3-2 with $p \stackrel{i}{=} \mathbf{v}$, we get:

$$\frac{\partial}{\partial t}\int_V \rho\mathbf{v}\, dV + \int_S \rho\mathbf{v}\mathbf{v}\cdot\mathbf{n}\, dS = \boldsymbol{F}_b + \boldsymbol{F}_s \tag{2.5-2}$$

where $\boldsymbol{F}_b$ are the body forces (e.g., gravitation), and $\boldsymbol{F}_s$ the surface forces (e.g., viscous forces) that are acting on the control volume. If there are other forces, such as electric or magnetic forces, acting on the control volume, they should be added to Eq. 2.5-2 and appropriately accounted for. Within this text, however, the only forces that we will consider are gravitational and viscous forces.

Now Eq. 2.5-2 is a vectorial equation that has three components, reflecting the fact that linear momentum is *independently* conserved in the three spatial directions. For a rectangular coordinate system, Eq. 2.5-2 becomes:

$$\frac{\partial}{\partial t}\int_V \rho v_x\, dV + \int_S v_x\rho\mathbf{v}\cdot\mathbf{n}\, dS = \boldsymbol{F}_{bx} + \boldsymbol{F}_{sx} \tag{2.5-3}$$

$$\frac{\partial}{\partial t}\int_V \rho v_y\, dV + \int_S v_y\rho\mathbf{v}\cdot\mathbf{n}\, dS = \boldsymbol{F}_{by} + \boldsymbol{F}_{sy} \tag{2.5-4}$$

$$\frac{\partial}{\partial t}\int_V \rho v_z\, dV + \int_S v_z\rho\mathbf{v}\cdot\mathbf{n}\, dS = \boldsymbol{F}_{bz} + \boldsymbol{F}_{sz} \tag{2.5-5}$$

For deriving the *equation of motion*, which is the microscopic counterpart of the macroscopic momentum balance, we proceed as in the case of the mass balance and first rewrite Eq. 2.5-2 using the Gauss Divergence Theorem to get:

$$\int_V \left[\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot \mathbf{v}(\rho \mathbf{v}) \right] dV = \boldsymbol{F}_b + \boldsymbol{F}_s \tag{2.5-6}$$

Next, we consider the forces that act on the control volume. The body forces are due to gravitation and act on all the mass in the control volume:

$$\boldsymbol{F}_b = \int_V \rho \mathbf{g}\, dV \tag{2.5-7}$$

The surface forces that act on the control volume are due to the stress field in the deforming fluid defined by the stress tensor $\boldsymbol{\pi}$. We discuss the nature of the stress tensor further in the next section; at this point, it will suffice to state that $\boldsymbol{\pi}$ is a symmetric second-order tensor, which has nine components. It is convenient to divide the stress tensor into two parts:

$$\boldsymbol{\pi} = P\boldsymbol{\delta} + \boldsymbol{\tau} \tag{2.5-8}$$

where P is a scalar quantity, which is the ‘‘pressure,’’ $\boldsymbol{\delta}$ is the identity tensor defined as:

$$\boldsymbol{\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.5-9}$$

and $\boldsymbol{\tau}$ is the *dynamic* or *deviatoric* component of the stress tensor, which accounts for the viscous stresses created in the fluid as a result of flow.

Thus Eq. 2.5-8 can be written as

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{pmatrix} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \tag{2.5-10}$$

which expresses nine separate scalar equations relating the respective components of the tensors: $\pi_{ij} = P\delta_{ij} + \tau_{ij}$, where $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. For convenience, the tensor $\boldsymbol{\pi}$ is called the *total* stress tensor and $\boldsymbol{\tau}$ is simply the stress tensor. Clearly, $\pi_{ij} = \tau_{ij}$ for $i \neq j$ and $\pi_{ii} = P + \tau_{ii}$ for $i = j$. Thus, the total normal stress incorporates the contribution of the ‘‘pressure,’’ P, which is isotropic. In the absence of flow, at equilibrium, the pressure P becomes identical to the thermodynamic pressure, which for pure fluids is related to density and temperature via a state equation.

Two difficulties are associated with P. First, flow implies nonequilibrium conditions, and it is not obvious that P appearing during flow is the same pressure as the one defined in thermodynamics. Second, when the incompressibility assumption is invoked (generally used in solving polymer processing problems) the meaning of P is not clear, and P is regarded as an arbitrary variable. No difficulty, however, arises in solving practical problems, because we only need to know the pressure *gradient*.

Turning back to Eq. 2.5-6, the surface forces $\boldsymbol{F}_s$ can now be expressed in terms of the total stress tensor $\boldsymbol{\pi}$ as follows:

$$\boldsymbol{F}_s = -\int_S \boldsymbol{\pi} \cdot \mathbf{n}\, dS \tag{2.5-11}$$

where the minus sign is introduced to account for the forces the surrounding fluid applies *on* the control volume.

Substituting Eqs. 2.5-11 and 2.5-7 into Eq. 2.5-6, using the Gauss Divergence Theorem, we obtain:

$$\int_V \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}\cdot\mathbf{v}(\rho\mathbf{v})\right] dV = \int_V \rho\mathbf{g}\, dV - \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\pi}\, dV \tag{2.5-12}$$

or

$$\int_V \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}\cdot\mathbf{v}(\rho\mathbf{v}) + \nabla\cdot\boldsymbol{\pi} - \rho\mathbf{g}\right] dV = 0 \tag{2.5-13}$$

Equation 2.5-13 is valid for any arbitrary control volume. The only way this can hold true is if the kernel of the integral vanishes, that is,

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}\cdot\mathbf{v}\rho\mathbf{v} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} - \rho\mathbf{g} = 0 \tag{2.5-14}$$

which is the *equation of motion*. But, $\boldsymbol{\nabla}\cdot\mathbf{v}\rho\mathbf{v} = \mathbf{v}(\boldsymbol{\nabla}\cdot\rho\mathbf{v}) + \rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$. Thus Eq. 2.5-14 can be written as

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + \mathbf{v}\nabla\cdot\rho\mathbf{v} + \rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} - \rho\mathbf{g} = 0 \tag{2.5-15}$$

Furthermore, the second and third terms express the product of $\mathbf{v}$ with the *equation of continuity*. Thus they equal zero, and Eq. 2.5-15 reduces to

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\boldsymbol{\nabla}\cdot\boldsymbol{\pi} + \rho\mathbf{g} \tag{2.5-16}$$

or, in terms of the substantial derivative, we get:

$$\rho\frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{\pi} + \rho\mathbf{g} \tag{2.5-17}$$

which we recognize as Newton's Second Law, which states that the mass (per unit volume) times acceleration[7] equals the sum of the forces acting on the fluid element.

Next, we substitute Eq. 2.5-8 into Eq. 2.5-16 to yield the common form of the *equation of motion*:

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\boldsymbol{\nabla}P - \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\mathbf{g} \tag{2.5-18}$$

7. Recall that the substantial derivative implies that we "ride" with the fluid element.

where the terms on the left-hand side express accumulation of momentum and the convection of momentum, respectively, and those on the right side express the forces acting on the fluid element by the pressure gradient, the stresses in the flowing fluid, and the gravitational forces. The three components of the equation of motion, in rectangular, cylindrical, and spherical coordinates are given in Table 2.2.

TABLE 2.2 The Equation of Motion in Terms of τ in Several Coordinate Systems

Rectangular Coordinates (x, y, z)

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}-\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial P}{\partial y}-\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}-\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

Cylindrical Coordinates (r, θ, z)

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

Spherical Coordinates (r, θ, ϕ)

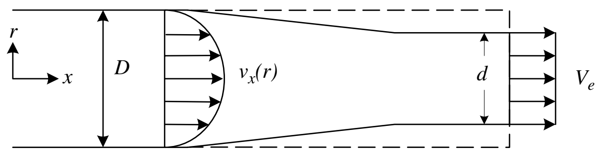
$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)$$
$$=-\frac{\partial P}{\partial r}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{rr}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{r\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi}-\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi}+\frac{\tau_{r\theta}}{r}-\frac{\cot\theta}{r}\tau_{\phi\phi}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)$$
$$=-\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\phi}\right)+\frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi}+\frac{\tau_{r\phi}}{r}+\frac{2\cot\theta}{r}\tau_{\theta\phi}\right)+\rho g_\phi$$

Source: Reprinted with permission from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960.

Example 2.3 The Use of the Macroscopic Momentum Balance to Calculate the Diameter of a Free Jet A free jet of diameter d leaves a horizontal tube of diameter D, as shown in the accompanying figure. Assuming a laminar, fully developed velocity profile at the exit of the tube, and neglecting gravitational forces and the drag of the air on the free jet, prove that $d/D = (0.75)^{0.5}$.



Solution We first select the control volume as shown by the dotted line in the figure, assuming that, at the downstream end of the control volume, the velocity profile in the free jet is flat. Next, we apply the macroscopic momentum balance, Eq. 2.5-3, to the control volume. We need be concerned only with the x component, because this is the only momentum that crosses the control volume boundaries. The flow is steady, and therefore the time-dependent term vanishes, as do the forces, since there are none acting on the control volume. Thus the equation reduces to:

$$\int_S v_x \rho \mathbf{v} \cdot \mathbf{n} \, dS = 0 \tag{E2.3-1}$$

The velocity profile in a laminar flow is given by $v_x = V_0\left[1 - (r/R)^2\right]$, where V_0 is the maximum velocity. At the exit, the velocity is uniform and given by V_e. Substituting these terms into Eq. E2.3-1 gives:

$$-\rho \int_S V_0^2\left[1 - (r/R)^2\right]^2 dS + \rho V_e^2\left(\pi d^2/4\right) = 0 \tag{E2.3-2}$$

or

$$-2\pi V_0^2 (D/2)^2 \int_0^1 \xi\left(1 - \xi^2\right)^2 d\xi + V_e^2\left(\pi d^2/4\right) = 0 \tag{E2.3-3}$$

where $\xi = r/R$, which then yields:

$$\left(\frac{d}{D}\right)^2 = \frac{1}{3}\left(\frac{V_0}{V_e}\right)^2 \tag{E2.3-4}$$

Next we apply the macroscopic mass balance, Eq. 2.4-1, which gives a second relationship between the variables:

$$-\rho \int_S V_0\left[1 - (r/R)^2\right] dS + \rho V_e\left(\pi d^2/4\right) = 0 \tag{E2.3-5}$$

which results in

$$\frac{V_0}{V_e} = 2\left(\frac{d}{D}\right)^2 \tag{E2.3-6}$$

Combining Eqs. E2.3-4 and E2.3-6 gives the desired result of $d/D = 0.75^{0.5}$.

2.6 THE STRESS TENSOR

Consider a point P in a continuum on an arbitrary surface element ΔS (defined by the normal **n**), as in Fig. 2.2. Let $\Delta \mathbf{f}_i$ be the resultant force exerted by the material on the positive side of the surface on that of the negative side across ΔS.

The average force per unit area is $\Delta \mathbf{f}_i/\Delta S$. This quantity attains a limiting nonzero value as ΔS approaches zero at point P (Cauchy's stress principle). This limiting quantity is called the *stress vector*, or traction vector $\mathbf{T}'$. But $\mathbf{T}'$ depends on the orientation of the area element, that is, the direction of the surface defined by normal **n**. Thus it would appear that there are an infinite number of unrelated ways of expressing the state of stress at point P.

It turns out, however, that the state of stress at P can be completely specified by giving the stress vector components in *any three mutually perpendicular planes* passing through the point. That is, only nine components, three for each vector, are needed to define the stress at point P. Each component can be described by two indices ij, the first denoting the orientation of the surface and the second, the direction of the force. Figure 2.3 gives these components for three Cartesian planes. The nine stress vector components form a second-order Cartesian tensor, the stress tensor[8] $\boldsymbol{\pi}'$.

Furthermore, some argumentation based on the principles of mechanics and experimental observations, as well as molecular theories, leads to the conclusion that the stress tensor is symmetric, that is,

$$\pi'_{ij} = \pi'_{ji} \tag{2.6-1}$$

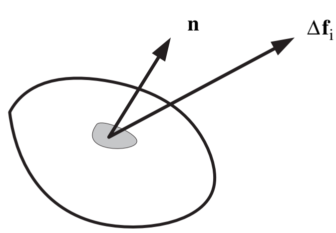


Fig. 2.2 An arbitrary surface element with direction defined by normal **n** with resultant force $\Delta \mathbf{f}_i$ acting at point P.

8. Note that we differentiate the stress tensor $\boldsymbol{\pi}'$ discussed in this section from the previously discussed stress tensor $\boldsymbol{\pi}$ because they are defined on the basis of different sign conventions, as discussed later in the chapter.

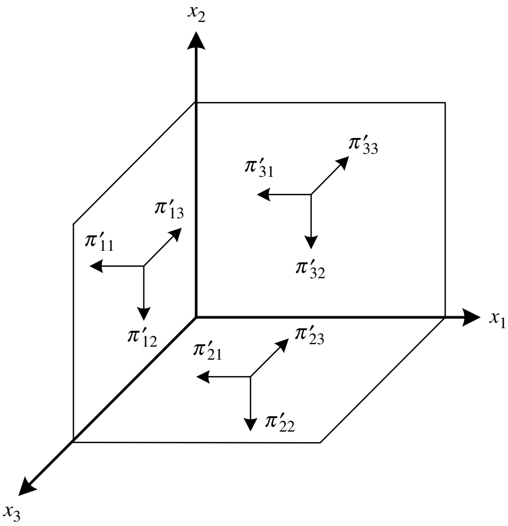


Fig. 2.3 The nine Cartesian components of the stress tensor. In the limit, the cube shrinks to point *P*.

Hence, only six independent components of the stress tensor are needed to fully define the state of the stress at point *P*, where π'_{ii} are the normal stress components, and $\pi'_{ij}(i \neq j)$ are the shear-stress components.

By considering the forces that the material on the positive side of the surface (i.e., material on the side of the surface at which the outward normal vector points) exerts on the material on the negative side, a stress component is *positive* when it acts in the positive direction of the coordinate axes and on a plane whose outer normal points in one of the positive coordinate directions (or if both of the previously mentioned directions are negative).

A stress component is negative if any *one* direction is negative. Hence, by this sign convention, generally used in mechanics and mechanical engineering, tensile stresses are positive and compressive stresses are negative. Moreover, according to this sign convention, all the stresses shown in Fig. 2.3 are positive. Unfortunately, this sign convention is *opposite* to that resulting from momentum transport considerations, thus $\boldsymbol{\pi}'^{\dagger} = -\boldsymbol{\pi}$ (where † stands for 'transpose'). In the latter sign convention, as pointed out by Bird et al. (1), if we consider the stress vector $\boldsymbol{\pi}_n = \mathbf{n} \cdot \boldsymbol{\pi}$ acting on surface $d\mathbf{S}$ of orientation $\mathbf{n}$, the force $\boldsymbol{\pi}_n dS$ is that exerted by the material on the negative side onto that on the positive side. (According to Newton's Third Law, this force is equal and opposite to that exerted by the material of the positive side to the material of the negative side.) It follows, then, that, in this latter convention, tensile stresses are negative. In this book we follow this latter sign convention.

As we pointed out in the introductory remarks, polymer processing is the simultaneous occurrence of momentum, heat, and occasionally, mass transfer in multicomponent systems. This sign convention, as shown in the following paragraphs, leads to consistency among the three transport processes. Nevertheless, it is worth emphasizing that the sign convention used in no way affects the solution of flow problems. Once constitutive equations are inserted into the equation of motion, and stress components are replaced by velocity gradients, the two sign conventions lead to identical expressions.

Generally, from the tensor $\boldsymbol{\pi}$ or tensor $\boldsymbol{\tau}$, which is related to the former via Eq. 2.5-8, three independent scalar invariant entities can be formed by taking the trace of $\boldsymbol{\tau}$. The three invariants are:

$$I_\tau = \text{tr } \boldsymbol{\tau} = \sum_1^3 \tau_{ii} \tag{2.6-2}$$

$$II_\tau = \text{tr } \boldsymbol{\tau}^2 = \sum_1^3 \sum_1^3 \tau_{ij}\tau_{ji} \tag{2.6-3}$$

$$III_\tau = \text{tr } \boldsymbol{\tau}^3 = \sum_1^3 \sum_1^3 \sum_1^3 \tau_{ij}\tau_{jk}\tau_{ki} \tag{2.6-4}$$

and the magnitude of the stress tensor $\boldsymbol{\tau}$ denoted as τ is given by:

$$\tau = |\boldsymbol{\tau}| = \sqrt{\frac{1}{2}(\boldsymbol{\tau}:\boldsymbol{\tau})} = \sqrt{\frac{1}{2} II_\tau} \tag{2.6-5}$$

Example 2.4 The Similarity Between the Three Transport Phenomena Consider an infinite slab of solid shown in Fig. E2.4(a) with a constant temperature difference over its surfaces.

The temperature gradient for $T(0) > T(b)$ is negative and given by

$$\frac{dT}{dy} = -\frac{T(0) - T(b)}{b}$$

and using Fourier's law, the heat is given by:

$$q_y = -k\frac{dT}{dy} = \frac{k}{b}[T(0) - T(b)]$$

where k is the thermal conductivity. Clearly, for the ase shown, the heat flux is in the positive y direction, and flows from high temperature to low temperature.

Now consider the case of one-dimensional diffusion of component A shown in Fig. E2.4(b). Similarly, the concentration gradient for $C_A(0) > C_A(b)$ is negative and given by:

$$\frac{dC_A}{dy} = -\frac{C_A(0) - C_A(b)}{b}$$

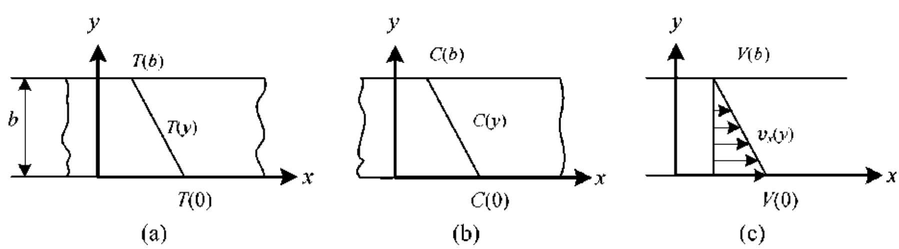


Fig. E2.4 (a) Temperature, (b) concentration, and (c) velocity profile over infinite slabs of material. In (c) the fluid is confined between two parallel plates in relative motion.

Using Fick's law (assuming constant density and low concentration of the diffusing component), the mass flux is positive and is given by

$$J_{Ay} = -\mathscr{D}_{AB}\frac{dC_A}{dy} = \mathscr{D}_{AB}[C_A(0) - C_A(b)]$$

where $\mathscr{D}_{AB}$ is the binary diffusion coefficient. As in the case of heat flux, the flux of the A component is positive, and it flows from high concentration to low concentration.

Finally, let us examine the flow of viscous fluid between two parallel plates in relative motion [Fig. E2.4(c)]. Because of intermolecular forces, the fluid layer next to the bottom plate will start moving. This layer will then transmit, by viscous drag, *momentum* to the layer above it, and so on. The velocity gradient for $v_x(0) > v_x(b)$ is positive and given by

$$\frac{dv_x}{dy} = -\frac{V(0)}{b}$$

Using Newton's law, which holds for an important class of fluids, we get:

$$\tau_{yx} = -\mu\frac{dv_x}{dy} = \frac{\mu}{b}V(0)$$

where μ is the viscosity. Clearly, the flux of x momentum is the shear stress, and it is in the y direction from the lower plate to the upper one; that is, it flows downstream the velocity profile, from high velocity to low velocity, and there is a *positive* momentum flux according to the coordinate system used. This example demonstrates the similarity of the three transport processes, and the reason for defining the fluxes of heat, mass, and momentum in Fourier's, Fick's, and Newton's laws with a negative sign.

2.7 THE RATE OF STRAIN TENSOR

We know from everyday experience that applying a given tensile or shear stress to a solid material results in a given deformation. In the elastic range, Hooke's law predicts a linear deformation with the applied stress. The elastic modulus in Hooke's law specifies the nature of the particular solid. Yet in viscous fluids, the applied stress is not related to the *deformation* of the fluid, but to the *rate* at which the fluid is being deformed, or to the *rate of strain*. As we shall see in this section, in order to define the rate of strain of a fluid at a given point, we need nine (six independent) numbers. Therefore, just like *stress*, the *rate of strain* is a second-order symmetric tensor. It is the nature of the relationship between the *stress* and the *rate of strain* tensors that the *constitution* of the particular fluid being deformed is manifested. The generally empirical equations relating the two, therefore, are called *constitutive equations.*

In an important class of materials, called *Newtonian*, this relationship is linear and one parameter—the viscosity—specifies the constitution of the material. Water, low-viscosity fluids, and gases are Newtonian fluids. However, most polymeric melts are non-Newtonian and require more complex constitutive equations to describe the relationship between the stress and the rate of strain. These are discussed in Chapter 3.

Geometric Considerations of the Rate of Strain Tensor

We first consider a small rectangular element at time t in shear flow, as shown in Fig. 2. 4. This element is a vanishingly small differential element, and therefore without loss of generality we can assume that the local velocity field is linear, as shown in the figure.

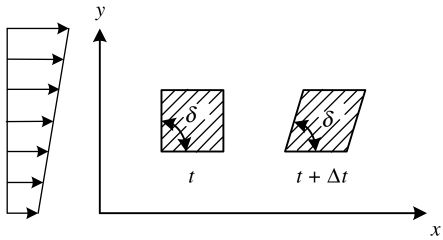


Fig. 2.4 The deformation of a fluid element in unidirectional shear flow.

At time $t + \Delta t$ the rectangular fluid element is translated in the x direction and deformed into a parallelogram. We define the *rate of shear* as $-d\delta/dt$, where δ is the angle shown in the figure.

Out of simple geometrical considerations, we express the rate of shear in terms of velocity gradients as follows:

$$\begin{aligned} -\frac{d\delta}{dt} &= \lim_{\Delta t \to 0} \frac{\delta_{t+\Delta t} - \delta_t}{\Delta t} \\ &= -\lim_{\Delta t \to 0} \left\{ \frac{\pi/2 - \arctan\left[\left(v_{x,y+\Delta y} - v_{x,y}\right)\Delta t/\Delta y\right] - \pi/2}{\Delta t} \right\} \\ &= \lim_{\Delta y \to 0} \frac{v_{x,y+\Delta y} - v_{x,y}}{\Delta y} = \frac{dv_x}{dy} \end{aligned} \tag{2.7-1}$$

Thus, we find that the rate of shear (or *shear rate*, as it is commonly referred to), or the rate of change of the angle δ, simply equals the velocity gradient.

We can extend this analysis to general flow fields $v(x, y, z, t)$ by considering the deformation of the fluid element in the *x,y, z,y*, and *x,z* planes. In such a case, for the *x,y* plane we get (see Problem 2.6):

$$-\frac{d\delta_{x,y}}{dt} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} = \dot{\gamma}_{xy} \tag{2.7-2}$$

and for the other two planes we get:

$$-\frac{d\delta_{y,z}}{dt} = \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} = \dot{\gamma}_{yz} \tag{2.7-3}$$

$$-\frac{d\delta_{x,z}}{dt} = \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} = \dot{\gamma}_{xz} \tag{2.7-4}$$

where we defined the shear components of the rate of deformation tensor $\dot{\boldsymbol{\gamma}}$ in Cartesian (rectangular) coordinates.

Now that we have discussed the geometric interpretation of the rate of strain tensor, we can proceed with a somewhat more formal mathematical presentation. We noted earlier that the (deviatoric) stress tensor $\boldsymbol{\tau}$ related to the flow and deformation of the fluid. The kinematic quantity that expresses fluid flow is the velocity gradient. Velocity is a vector and in a general flow field each of its three components can change in any of the three spatial directions, giving rise to nine velocity gradient components. We can therefore define a *velocity gradient tensor* $\nabla\mathbf{v}$ (i.e., the dyadic product of ∇ with $\mathbf{v}$), which in Cartesian coordinates can be written as:

$$\nabla\mathbf{v} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{2.7-5}$$

A fluid in motion may simultaneously deform and rotate. Decomposing the velocity gradient tensor into two parts can separate these motions:

$$\nabla\mathbf{v} = \frac{1}{2}(\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}) \tag{2.7-6}$$

where $\dot{\boldsymbol{\gamma}}$ and $\boldsymbol{\omega}$ are the *rate of strain* and the *vorticity* tensors, respectively, defined as:

$$\dot{\boldsymbol{\gamma}} = \nabla\mathbf{v} + (\nabla\mathbf{v})^{\dagger} \tag{2.7-7}$$

and

$$\boldsymbol{\omega} = \nabla\mathbf{v} - (\nabla\mathbf{v})^{\dagger} \tag{2.7-8}$$

where $(\nabla\mathbf{v})^{\dagger}$ is the *transpose*[9] $\nabla\mathbf{v}$. Thus by inserting Eq. 2.7- 5 and its transpose into Eq. 2.7- 7, we get the following expression for the rate of deformation tensor in Cartesian coordinates:

$$\dot{\boldsymbol{\gamma}} = \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{2.7-9}$$

For simple shear flow (as between parallel plates in relative motion) Eq. 2.7-9 reduces to:

$$\dot{\boldsymbol{\gamma}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma} \tag{2.7-10}$$

where $\dot{\gamma}$ is the *shear rate*, which is a scalar quantity related to the second invariant of $\dot{\boldsymbol{\gamma}}$ (see Eqs. 2.6-5 and 2.6-3) as follows:

$$\dot{\gamma} = \sqrt{\frac{1}{2}(\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}})} \tag{2.7-11}$$

9. The indices are "transposed"—that is, the rows and columns are interchanged (180° flip on the diagonal).

For simple shear flow we get:

$$\dot{\gamma} = \frac{dv_x}{dy} = \frac{d}{dy}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{dx}{dy}\right) = \frac{d\gamma}{dt} \tag{2.7-12}$$

where γ is the shear strain.

2.8 NEWTONIAN FLUIDS

In the previous section we discussed the nature and some properties of the stress tensor $\boldsymbol{\tau}$ and the rate of strain tensor $\dot{\boldsymbol{\gamma}}$. They are related to each other via a *constitutive equation*, namely, a generally empirical relationship between the two entities, which depends on the nature and *constitution* of the fluid being deformed. Clearly, imposing a given stress field on a body of water, on the one hand, and a body of molasses, on the other hand, will yield different rates of strain. The simplest form that these equations assume, as pointed out earlier, is a linear relationship representing a very important class of fluids called *Newtonian fluids.*

In 1687 Isaac Newton proposed a simple equation relating the shear stress to the velocity gradient in fluids, and defined *viscosity* as the ratio between the two:

$$\mu = -\frac{\tau_{yx}}{\left(\dfrac{dv_x}{dy}\right)} \tag{2.8-1}$$

This equation is known as 'Newton's law'. Of course, it is not really a 'physical law', but only an empirical relationship describing a limited, yet very important class of fluids. Newton's law is generally valid for ordinary fluids with molecular weights below 1000. Gases, water, low molecular weight oils, and so on, behave under most normal conditions according to Newton's law, namely, they exhibit a linear relationship between the shear stress and the consequent shear rate.

Equation 2.8-1 holds only for simple shearing flow, namely, when there is one velocity component changing in one (normal) spatial direction. The most general Newtonian constitutive equation that we can write for an arbitrary flow field takes the form:

$$\boldsymbol{\tau} = -\mu\dot{\boldsymbol{\gamma}} + (2\mu/3 - \kappa)(\boldsymbol{\nabla} \cdot \mathbf{v})\boldsymbol{\delta} \tag{2.8-2}$$

where κ is the *dilatational* viscosity. For an incompressible fluid (and polymers are generally treated as such), $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and Eq. 2.8-2 reduces to:

$$\boldsymbol{\tau} = -\mu\dot{\boldsymbol{\gamma}} \tag{2.8-3}$$

Equations 2.8-2 and 2.8-3 are coordinate-independent compact tensorial forms of the Newtonian constitutive equation. In any particular coordinate system these equations break up into nine (six independent) scalar equations. Table 2.3 lists these equations in rectangular, cylindrical and spherical coordinate systems.

TABLE 2.3 The Components of $\boldsymbol{\tau} = -\mu\dot{\boldsymbol{\gamma}} + (2\mu/3 - \kappa)(\nabla \cdot \mathbf{v})\boldsymbol{\delta}$ in Several Coordinate Systems

Rectangular Coordinates (x, y, z)

$$\tau_{xx} = -\mu\left[2\frac{\partial v_x}{\partial x} - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{yy} = -\mu\left[2\frac{\partial v_y}{\partial y} - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{xy} = \tau_{yx} = -\mu\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right]$$

$$\tau_{yz} = \tau_{zy} = -\mu\left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right]$$

$$\tau_{zx} = \tau_{xz} = -\mu\left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right]$$

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Cylindrical Coordinates (r, θ, z)

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Spherical Coordinates (r, θ, ϕ)

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{\phi\phi} = -\mu\left[2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) - \frac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Source: Reprinted with permission from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960.

Inserting Eq. 2.8-3 into the equation of motion, 2.5-18, we get[10] the celebrated Navier–Stokes[11] equation:

$$\rho \frac{\partial \mathbf{v}}{dt} + \rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\boldsymbol{\nabla} P + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \tag{2.8-4}$$

The symbol defined as $\boldsymbol{\nabla}^2$ is called the *Laplacian*. Table 2.4 lists the components of the Navier–Stokes equation in the various coordinate systems.

We should note that the Navier–Stokes equation holds *only* for Newtonian fluids and incompressible flows. Yet this equation, together with the equation of continuity and with proper initial and boundary conditions, provides *all* the equations needed to solve (analytically or numerically) *any* laminar, isothermal flow problem. Solution of these equations yields the pressure and velocity fields that, in turn, give the stress and rate of strain fields and the flow rate. If the flow is nonisothermal, then simultaneously with the foregoing equations, we must solve the thermal energy equation, which is discussed later in this chapter. In this case, if the temperature differences are significant, we must also account for the temperature dependence of the viscosity, density, and thermal conductivity.

Polymer processing flows are always laminar and generally *creeping* type flows. A creeping flow is one in which viscous forces predominate over forces of inertia and acceleration. Classic examples of such flows include those treated by the hydrodynamic theory of lubrication. For these types of flows, the second term on the left-hand side of Eq. 2.5-18 vanishes, and the Equation of motion reduces to:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} P + \mu \boldsymbol{\nabla}^2 \mathbf{v} + \rho \mathbf{g} \tag{2.8-5}$$

and the Navier–Stokes equation for creeping flows reduces to:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} P + \mu \boldsymbol{\nabla}^2 \mathbf{v} + \rho \mathbf{g} \tag{2.8-6}$$

On the other extreme of negligible viscosity, which is of little interest to the subject matter of this book, but is added for the sake of comprehensiveness, the equation of motion reduces to

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\boldsymbol{\nabla} P + \rho \mathbf{g} \tag{2.8-7}$$

which is the well-known Euler equation, after the Swiss mathematician Leonard Euler, who derived it in 1775.

Finally, for the no-flow situation ($\mathbf{v} = 0$), the equation of motion reduces to

$$\boldsymbol{\nabla} P = \rho \mathbf{g} \tag{2.8-8}$$

which is the basic equation of hydrostatics.

10. Note that: $-\boldsymbol{\nabla} \cdot \boldsymbol{\tau} = \mu \boldsymbol{\nabla} \cdot \dot{\boldsymbol{\gamma}} = \mu \boldsymbol{\nabla} \left[\boldsymbol{\nabla} \mathbf{v} + (\boldsymbol{\nabla} \mathbf{v})^{\dagger} \right] = \mu \left[\boldsymbol{\nabla}^2 \mathbf{v} + \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{v}) \right] = \mu \boldsymbol{\nabla}^2 \mathbf{v}$

11. Claude Louis Navier (1785–1836) was a French scientist who, using molecular arguments, derived the equation in 1882; George Gabriel Stokes (1819–1903) was a British physicist who made many contributions to the theory of viscous flow in the period 1845–1850.

TABLE 2.4 The Navier–Stokes Equation in Several Coordinate Systems

Rectangular Coordinates (x, y, z)

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

Cylindrical Coordinates (r, θ, z)

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z$$

Spherical Coordinates (r, θ, ϕ)

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)$$

$$=-\frac{\partial P}{\partial r}+\mu\left(\nabla^2 v_r-\frac{2}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2}{r^2}v_\theta\cot\theta-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)$$

$$=-\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi$$

Source: Reprinted with permission from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960. In these equations

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial \phi^2}\right)$$

Although polymeric melts are generally non-Newtonian, many problems in polymer processing are initially solved using the Newtonian assumption, because these solutions (a) provide simple results that help gain insight into the nature of the process; (b) provide quick, rough, quantitative estimates; and (c) the rigorous non- Newtonian solution may be too time-consuming for the problem at hand. Yet, for a true appreciation of polymer processing, the non-Newtonian character of the material must be considered. The study of non-Newtonian behavior forms an active branch of the science of rheology, and is discussed in Chapter 3.

In the meantime, we will solve a number of flow problems that are highly relevant to polymer processing problems, which demonstrate the rather straightforward use of the equation of motion and continuity.

Example 2.5 Parallel Plate Flow The methodology for formulating and solving flow problems involves the following well-defined and straightforward steps:

Step 1. Draw a schematic figure of the flow configuration, visualize the flow on physical grounds, pick the most appropriate coordinate system, and make some sensible assumptions about the velocity components.

Step 2. Reduce the equation of continuity to the form appropriate for the problem at hand.

Step 3. Reduce the equation of motion or the Navier–Stokes equation to the form appropriate for the problem at hand. Take advantage of the results of the equation of continuity.

Step 4. State the boundary and initial conditions, if any.

Step 5. Solve the differential equations for the velocity profiles, which then lead to the volumetric flow rate expression, shear stress, and rate distribution, power consumption, and so forth.

Step 6. Sketch out the velocity profiles and velocity gradient profiles and see if they are reasonable for the problem at hand.

In this example, we consider the viscous, isothermal, incompressible flow of a Newtonian fluid between two infinite parallel plates in relative motion, as shown in Fig. E2.5a. As is evident from the figure, we have already chosen the most appropriate coordinate system for the problem at hand, namely, the rectangular coordinate system with spatial variables x, y, z. We placed the coordinate system at the stationary lower plate, with the coordinate y pointing across the flow field, and z pointing in the direction of the flow. The upper plate is moving at constant velocity V_0 and the lower plate is stationary. Derive (a) the velocity profile; (b) the flow rate; (c) the shear stress and shear rate distributions, and (d) the power consumption.

Solution This flow configuration is of great significance in polymer processing and it is important to understand in depth. We therefore discuss it in some detail.

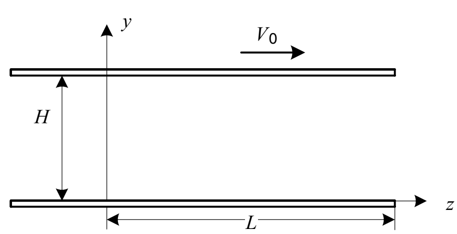


Fig. E2.5a Two parallel plates in relative motion. The upper plate moves at a constant velocity V_0 and the lower plate is stationary.

The *infinite* parallel plates construct may sound theoretical and impractical, but it is not. The flow in screw extruder channels, between the rotor and the wall of an internal mixer or between the rolls of calenders and roll-mills, to mention a few, can be considered to first approximation as taking place locally between parallel plates in relative motion.

We assume a *creeping laminar flow* because, in all practical cases, the very high viscosities of polymeric melts preclude turbulent flow. With increasing Reynolds number prior to reaching turbulence, viscous dissipation heating and degradation will take place. The following table, which illustrates characteristic values of some typical fluids, gives a sense of the magnitude of viscosities of polymeric melts:

Characteristic Viscosities of Some Typical Fluids $(\mathrm{Ns/m^2})$

Fluid	Viscosity	Character	Fluid	Viscosity	Character
Air	10^{-5}	Gas	Polymeric melts	10^2–10^6	toffee-like
Water	10^{-3}	Liquid	Pitch	10^9	stiff
Olive oil	10^{-1}	Liquid	Glass	10^{21}	rigid
Glycerin	1	Thick liquid			

We also assume *isothermal flow*. Of course, no viscous flow can be truly isothermal, because the friction between the sliding layers of fluid generates heat, called *viscous dissipation*. But slow viscous flows in narrow channels can be assumed, at first approximation, to be isothermal. This assumption greatly simplifies the solution and provides simple, useful working equations.

We further assume that the flow is *steady in time*. We make this assumption because most machines operate continuously, and even in reciprocating machines such as, for example, injection-molding machines, the flow can be viewed instantaneously as steady state. Finally, we assume that the fluid is *incompressible* and *Newtonian*, that the *flow is fully developed*, that is, $\partial v_z/\partial z = 0$, and that the *gravitational forces are negligible* compared to viscous forces.

(a) Now we begin the actual solution of the problem. We start with the equation of continuity and, turning to Table 2.1, we find that, for an incompressible fluid (constant density), it reduces to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{E2.5-1}$$

The third term on the left-hand side vanishes because we assumed fully developed flow, as does the first term because we do not expect any flow in the neutral x direction. Thus we are left with $dv_y/dy = 0$, which upon integration, yields $v_y = C$, where C is a constant. But v_y must equal zero on the plate surfaces and therefore, $v_y = 0$ everywhere.

Now we turn to the Navier–Stokes equation in Table 2.4. We take each component and analyze it term by term, dropping those that equal zero. This simple process leads to the following equations:

$$\frac{\partial P}{\partial x} = 0 \tag{E2.5-2}$$

$$\frac{\partial P}{\partial y} = 0 \tag{E2.5-3}$$

$$\mu \frac{\partial^2 v_z}{\partial y^2} = \frac{\partial P}{\partial z} \tag{E2.5-4}$$

Equations E2.5-2 and E2.5-3 tell us that the pressure P is *not* a function of x or y. Thus P can only be a function of z. Considering Eq. E2.5-4, with partial derivatives replaced with ordinary derivatives, we find that the left-hand side of the equation is a function only of y, the right-hand side of the equation is a function only of z, and the only way this can happen is if they both equal a constant:

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dP}{dz} = C \qquad \text{(E2.5-5)}$$

This situation has a number of interesting implications. First, it implies that the *pressure gradient* for such flows must be a constant, that is, the pressure changes (drops or rises—we do not yet know which) linearly with distance. We can further conclude that, in principle, a moving plate that drags liquid with it, as in this case, may, in principle, *generate* pressure in the direction of flow and that this pressure will increase linearly with distance, just as pressure drops linearly with distance, in pipe flow, for example.

Equation E2.5-5 can be integrated, but first we define the following dimensionless variables: $u_z = v_z/V$ and $\xi = y/H$. Since the pressure gradient is constant, we can replace it by the pressure drop:

$$\frac{dP}{dz} = \frac{P_L - P_0}{L} \qquad \text{(E2.5-6)}$$

where P_L and P_0 are the pressure at $z = 0$ and $z = L$, respectively. Clearly, if the pressure at the exit is higher than at the entrance, we know that pressure rises in the direction of flow and the pressure gradient is positive, and vice versa. Rewriting Eq. E2.5.6 in dimensionless form gives:

$$\frac{d^2 u_z}{d\xi^2} = \frac{H^2}{\mu V L_0}(P_L - P_0) \qquad \text{(E2.5-7)}$$

which can be integrated with the boundary condition $u_z(0) = 0$ and $u_z(1) = 1$, to give:

$$u_z = \xi - \xi(1 - \xi)\frac{H^2}{2\mu V_0}\left(\frac{P_L - P_0}{L}\right) \qquad \text{(E2.5-8)}$$

Clearly, the first term on the right-hand side expresses a linear velocity profile due to the drag of the moving plate, and the second term is a parabolic profile due to the pressure gradient. We will explore the velocity profile after we derive the flow rate.

(b) We obtain the flow rate by integrating the velocity over the cross section:

$$\begin{aligned} q &= \int_0^H v_z\,dy = V_0 H \int_0^1 u_z\,d\xi \\ &= \frac{V_0 H}{2} - \frac{H^3}{12\mu L}(P_L - P_0) \end{aligned} \qquad \text{(E2.5-9)}$$

where q is the net flow rate per unit width, the first term on the right-hand side is the *drag flow* q_d:

$$q_d = \frac{V_0 H}{2} \qquad \text{(E2.5-10)}$$

and the second term is the *pressure flow* q_p:

$$q_p = \frac{H^3}{12\mu L}(P_0 - P_L) \tag{E2.5-11}$$

For further insight into the flow rate equation, we can now rewrite Eq. E2.5-9 as follows:

$$\frac{dP}{dz} = \frac{P_L - P_0}{L} = \frac{12\mu}{H^3}(q_d - q) \tag{E2.5-12}$$

This equation now clearly demonstrates that the parallel plate geometry will generate pressure if $q_d > q$, that is, provided that the *moving plate drags more fluid than is actually delivered.* Under these conditions the parallel-plate geometry becomes a *pump.* This requires a *restriction* or die at the discharge end. We can further see that the pressure generation is proportional to the viscosity. Therefore, the high viscosities encountered with polymeric melts increase the device pressurization capability (of course, high viscosities also imply large pressure drops over dies and restrictions). We can further see that, at constant discharge rate q, increasing plate velocity will increase the pressure generation (by increasing q_d).

Plate velocity in actual machines becomes tantamount to speed of rotation and becomes an *operating* variable. Furthermore, we find that pressurization is inversely proportional with the gap size to the cube, which becomes a sensitive *design* variable. The maximum pressure that can be generated is obtained by setting $q = 0$, to get

$$\left(\frac{dP}{dz}\right)_{\max} = \frac{6\mu V_0}{H^2} \tag{E2.5-13}$$

Finally, it can easily be shown (see Problem 2.12) from Eq. E2.5.9 that, for a given *net flow rate* q there is an optimum $H = 3q/V_0$ for a maximum pressure rise of

$$\left(\frac{dP}{dz}\right)_{\max,q} = \frac{6\mu V_0^3}{27q^2} \tag{E2.5-14}$$

Equation E2.5-9 further indicates that, in the absence of a pressure drop, the net flow rate equals the drag flow rate. Note that q_p is positive if $P_0 > P_L$ and pressure flow is in the positive z direction and negative when $P_L > P_0$. The net flow rate is the sum or *linear superposition* of the flow induced by the drag exerted by the moving plate and that caused by the pressure gradient. This is the direct result of the linear Newtonian nature of the fluid, which yields a linear ordinary differential equation. For a non- Newtonian fluid, as we will see in Chapter 3, this will not be the case, because viscosity depends on shear rate and varies from point to point in the flow field.

By dividing Eq. E2.5-11 by Eq. E2.5-10 we get a useful expression for the pressure-to-drag flow ratio:

$$\frac{q_p}{q_d} = \frac{q - q_d}{q_d} = \frac{H^2}{6\mu L V_0}(P_0 - P_L) \tag{E2.5-15}$$

Next we substitute Eq. E2.5-15 into Eq. E2.5-8 to yield:

$$u_z = \xi + 3\frac{q_p}{q_d}\xi(1 - \xi) \tag{E2.5-16}$$

We can plot the dimensionless velocity profile with pressure-to-drag flow ratio as a single parameter. When this ratio is zero we get pure drag flow, when it assumes a value of -1, the

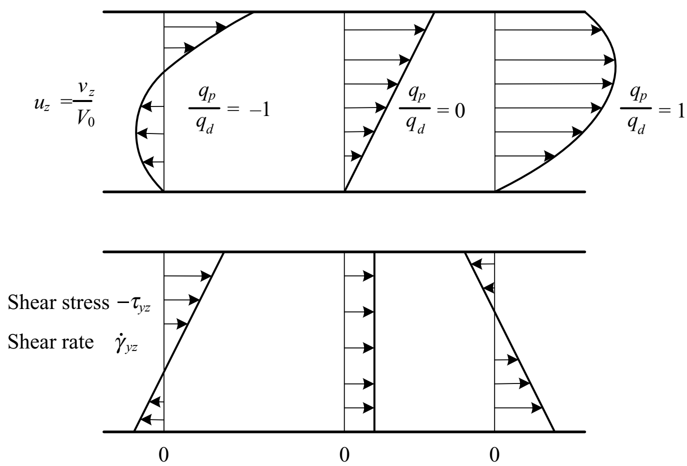


Fig. E2.5b Schematic representation velocity shear rate and shear stress profiles of a Newtonian fluid between parallel plates.

net flow rate is zero, and when the value is $+1$, the net flow rate is twice the drag flow rate. As the value of this ratio increases, the velocity profile approaches a parabolic profile of pure pressure flow between two stationary parallel plates. Figure E2.5b shows the characteristic velocity profiles.

(c) By taking the derivative of the velocity we obtain the shear rate:

$$\dot{\gamma}_{yz} = \frac{dv_z}{dy} = \frac{V_0}{H}\frac{du_z}{d\xi} = \frac{V_0}{H}\left[1 + 3(1 - 2\xi)\frac{q_p}{q_d}\right] \tag{E2.5-17}$$

This equation shows that, when the pressure to drag flow ratio equals $-1/3$, the shear rate at the stationary plate is zero, when it equals $+1/3$, the shear rate at the moving plate is zero, and when it equals zero, the shear is constant and equals V_0/H. In this range the velocity profile exhibits no extremum. In terms of the net flow rate, the condition of no extremum in velocity is:

$$\frac{2q_d}{3} < q < \frac{4q_d}{3} \tag{E2.5-18}$$

With the shear rate at hand, we can calculate the local viscous dissipation per unit volume. From Table 2.3 we note that the only nonvanishing shear-stress component is $\tau_{yz} = \tau_{zy}$ which is given by

$$\tau_{yz} = -\mu\dot{\gamma}_{yz} = -\mu\frac{V_0}{H}\left[1 + 3(1 - 2\xi)\frac{q_p}{q_d}\right] \tag{E2.5-19}$$

and the stress at the moving plate τ_{yz} (1) becomes

$$\tau_{yz}(1) = -\mu\frac{V_0}{H}\left[1 - 3\frac{q_p}{q_d}\right] \tag{E2.5-20}$$

Figure E2.5b depicts the shear rate and shear stress profiles normalized by the pure drag flow values for a number of pressure-to-drag flow ratios.

(d) The *power input per unit area* needed to drag the moving plate is given by

$$p_w = -V_0 \tau_{yz}(1) = \mu \frac{V_0^2}{H}\left[1 - 3\frac{q_p}{q_d}\right] \tag{E2.5-21}$$

where the minus sign is introduced because, according to the sign convention adopted in this book, the shear stress τ_{yz} (1) is the stress exerted by the fluid on the plate. The *total power input* into a system of length L and width W is

$$P_w = \mu \frac{V_0^2 LW}{H}\left[1 - 3\frac{q_p}{q_d}\right] \tag{E2.5-22}$$

For pressure-to-drag flow ratios above 1/3, the P_w becomes negative, implying that power is flowing *out* of the system via the moving plate. In this case, the pressure drop is negative, implying that an outside power source pressurized the liquid and some of that is extracted by the moving (now restraining rather than forward dragging) plate, with the rest of the power dissipated into heat. The *specific power input*, defined as the power input into a unit volume of material leaving the system, is given by

$$\frac{P_w}{qW} = \mu \frac{V_0^2 L}{Hq}\left[1 - 3\frac{q_p}{q_d}\right] = 2\mu \frac{V_0 L}{H^2} \frac{\left(1 - 3\dfrac{q_p}{q_d}\right)}{\left(1 + \dfrac{q_p}{q_d}\right)} \tag{E2.5-23}$$

Clearly, the power input and the specific power input both vanish at a pressure-to-drag flow ratio of 1/3, when the shear stress at the wall is zero. It is also worth noting that the specific power input is proportional to viscosity and plate velocity, and inversely proportional to the distance between the plates squared.

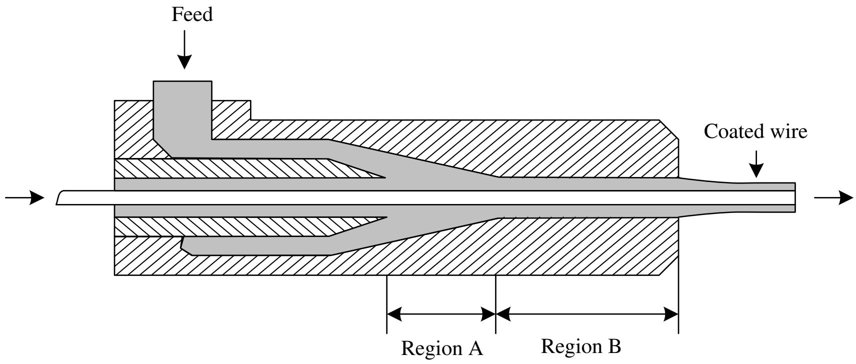
From Eq. E2.5-17 we can calculate the total viscous dissipation between the parallel plates. The second invariant of the rate of strain tensor multiplied by the viscosity gives the viscous dissipation per unit volume. From Table 2.3 we find that, for the case at hand, the second invariant reduces to $\dot{\gamma}_{yz,}^2$; therefore, the total viscous energy dissipation (VED) between the plates will be given by

$$\boldsymbol{\tau} : \dot{\boldsymbol{\gamma}} = \mu \dot{\gamma}^2 = LHW \int_0^1 \mu \dot{\gamma}_{yz}^2 \, d\xi = \frac{\mu V_0^2 LW}{H}\left[1 + 3\left(\frac{q_p}{q_d}\right)^2\right] \tag{E2.5-24}$$

Now the difference between the total power input (Eq. E2.5-22) and the total viscous dissipation (Eq. E2.5-24) is the power converted into pressure. Indeed, if we subtract the latter from the former, we get exactly $q(P_L - P_0)$, which is the power input required for raising the pressure. This pressure also will be converted into heat through a die, and therefore the expression given in Eq. E2.5-23 correctly gives the total power input into the exiting fluid.

Example 2.6 Axial Drag and Pressure Flow between Concentric Cylinders The accompanying figure provides a schematic representation of a wire-coating die. We wish to analyze the flow of polymeric melt in the tip region of the die where the flow is confined in an annular space created by an axially moving wire in a constant-diameter die. This section determines the thickness of the coating. Polymer melt is forced into the die by an extruder at high pressure, bringing it in touch with the moving wire. The wire moves at relatively high speeds of up to 1000–2000 m/min. The wire drags with it the melt and the flow is a combined pressure and drag flow. Derive expressions for (a) the velocity profile in the tip region, (b) the shear rate and stress profile, (c) an expression for the flow rate and (d) an expression for the coating thickness.

In Region A, flow cross-section converges to a constant value in the tip Region A. The wire moves at a constant speed.



Solution

(a) The flow boundaries are best described by a cylindrical coordinate system. We assume an incompressible, Newtonian fluid flowing at steady state in a fully developed isothermal flow. We visualize the flow with one nonvanishing velocity component, $v_z(r)$, which is a function of only r. The θ direction is neutral and we do not expect flow in this direction. Moreover, it is easy to show, along the lines of the previous example, that $v_r = 0$, and, therefore, the components of the Navier–Stokes equation in cylindrical coordinates listed in Table 2.4 reduce to:

$$\frac{\partial P}{\partial r} = 0 \tag{E2.6-1}$$

$$\frac{\partial P}{\partial \theta} = 0 \tag{E2.6-2}$$

$$\frac{\partial P}{\partial z} = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \tag{E2.6-3}$$

Thus we find that the pressure is a function of only z and, since the right-hand side of Eq. E2.6-3 is a function of only r, $\partial P/\partial z$ = constant. We can therefore rewrite this equation as an ordinary differential equation and integrate it with boundary conditions $v_z(R_i) = V_0$ and $v_z(R_o) = 0$, where R_i and R_o are the inner and outer radii, respectively, to give:

$$v_z = V_0\left(\frac{\ln(r/R_o)}{\ln\alpha}\right) + \frac{R_o^2}{4\mu}\left(-\frac{dP}{dz}\right)\left[1 - \left(\frac{r}{R_o}\right)^2 - (1-\alpha^2)\frac{\ln(r/R_o)}{\ln\alpha}\right] \tag{E2.6-4}$$

where $\alpha = R_i/R_o$. Note that the pressure gradient $-(dP/dz) = (P_0 - P_L)/L$, where P_0 and P_L are the pressures at the beginning of the tip region and at the exit, respectively, and L is the length of the tip region, is positive because pressure drops in the direction of motion.

(b) Taking the derivative of Eq. E2.6-4 with respect to r gives:

$$\dot{\gamma}_{rz} = \frac{\partial v_z}{\partial r} = \frac{V_0}{\ln\alpha}\cdot\frac{1}{r} - \frac{(P_0 - P_L)}{4\mu L}\left[2r - \frac{(1-\alpha^2)}{\ln\alpha}\cdot\frac{R_0^2}{r}\right] \tag{E2.6-5}$$

The shear stress can be obtained with Eq: E2.6-5 as follows:

$$\tau_{rz} = -\mu\dot{\gamma}_{rz} = \frac{(P_0 - P_L)}{4L}\left[2r - \frac{(1-\alpha^2)}{\ln\alpha}\cdot\frac{R_0^2}{r}\right] - \frac{\mu V_0}{\ln\alpha}\cdot\frac{1}{r} \tag{E2.6-6}$$

(c) The flow rate can be obtained by integrating Eq. E2.6-4 as follows:

$$Q = \frac{\pi R_0^4(P_0 - P_L)(1-\alpha^2)}{8\mu L}\left(1 + \alpha^2 + \frac{1-\alpha^2}{\ln\alpha}\right) - \pi R_0^2 V_0\left(\alpha^2 + \frac{1-\alpha^2}{2\ln\alpha}\right) \tag{E2.6-7}$$

Note that the flow rate increases with the pressure drop and decreases with increasing wire speed at constant die geometry.

(d) We define the polymer coating thickness as δ. The circular cross-section area of the coating lay is given as

$$S = \pi(R_i + \delta)^2 - \pi R_i^2 = \pi\delta(2R_i + \delta) \tag{E2.6-8}$$

In terms of the mass balance in an incompressible fluid, we have

$$Q = V_0 S = \pi V_0 \delta(2R_i + \delta) \tag{E2.6-9}$$

Equation E2.6-9 can be rewritten as

$$\delta^2 + 2R_i\delta - K = 0 \tag{E2.6-10}$$

where

$$K = \frac{Q}{\pi V_0} \tag{E2.6-11}$$

Solving Eq. E2.6-10 according to the limit of $\delta > 0$ gives

$$\delta = R_i\left[\sqrt{1 + \frac{K}{R_i^2}} - 1\right] \tag{E2.6-12}$$

If an assumption of $K/R_i^2 \gg 1$ is made, the preceding equation can be rewritten by first taking two terms of a binomial expansion for it:

$$\delta = \frac{K}{2R_i} = \frac{Q}{2\pi R_i V_0} \tag{E2.6-13}$$

Inserting Eq. E2.6.7 into the preceding equation results in

$$\delta = \frac{R_0^4(P_0 - P_L)(1-\alpha^2)}{16\mu L R_i V_0}\left(1 + \alpha^2 + \frac{1-\alpha^2}{\ln\alpha}\right) - \frac{R_0^2}{2R_i}\left(\alpha^2 + \frac{1-\alpha^2}{2\ln\alpha}\right) \tag{E2.6-14}$$

Note that the thickness of the coating layer is proportional to the pressure drop and inversely proportional to wire speed.

2.9 THE MACROSCOPIC ENERGY BALANCE AND THE BERNOULLI AND THERMAL ENERGY EQUATIONS

Polymer processing operations, by and large, are nonisothermal. Plastics pellets are compacted and heated to the melting point by interparticle friction, solid deformation beyond the yield point, and conduction. The molten polymer is heated or cooled by temperature-controlled processing machine walls, and the deforming viscous polymer melt constantly undergoes heating by internal viscous dissipation. Therefore, we need to account for nonisothermal effects via appropriate equations.

The starting point is the first law of thermodynamics, which states mathematically the great principle of conservation of energy:

$$dE = \delta Q + \delta W \tag{2.9-1}$$

where E is the total energy of a system, δQ is the heat added *to* the system, and δW is the work done *on* the system. The differential δ signifies that the changes on the right-hand side of the equation are *path* dependent. The rate of change of the energy in the systems is given by:

$$\left.\frac{dE}{dt}\right|_{\text{system}} = \dot{Q} + \dot{W} \tag{2.9-2}$$

where

$$E_{\text{system}} = \int_{V} e\rho \, dV \tag{2.9-3}$$

and where e is the specific energy or energy per unit mass. Substituting the energy E and specific energy e for P and p, respectively, *in* the Reynolds Transport Theorem, Eq. 2.3-2 we get the macroscopic total energy balance equation:

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\int_{V} \rho e \, dV + \int_{S} \rho e \mathbf{v} \cdot \mathbf{n} \, dS = \dot{Q} + \dot{W} \tag{2.9-4}$$

The total rate of heat added to the control volume through the control surfaces can be expressed in terms of the local heat flux $\mathbf{q}$ as follows:

$$\dot{Q} = -\int_{S} \mathbf{q} \cdot \mathbf{n} \, dS \tag{2.9-5}$$

where the negative sign was introduced to be consistent with $\dot{Q}$, which defined heat added to the system as positive (recall that $\mathbf{n}$ is the *outward* unit normal vector). The rate of work done on the control volume through the control surfaces and by gravitation is given by

$$\dot{W} = -\int_{S} \boldsymbol{\pi} \cdot \mathbf{n} \cdot \mathbf{v} \, dS + \int_{V} \rho \mathbf{g} \cdot \mathbf{v} \, dV \tag{2.9-6}$$

Substituting Eqs. 2.9-5 and 2.9-6 into Eq. 2.9-4 gives

$$\frac{\partial}{\partial t}\int_{V} \rho e \; dV + \int_{S} \rho e \; \mathbf{v} \cdot \mathbf{n} \; dS + \int_{S} \mathbf{q} \cdot \mathbf{n} \; dS + \int_{S} \boldsymbol{\pi} \cdot \mathbf{n} \cdot \mathbf{v} \; dS - \int_{V} \rho \mathbf{g} \cdot \mathbf{v} \; dV = 0 \tag{2.9-7}$$

and using Gauss' Divergence Theorem, we can rewrite it as

$$\frac{\partial}{\partial t}(\rho e) + \boldsymbol{\nabla} \cdot \mathbf{v}\rho e + \boldsymbol{\nabla} \cdot \mathbf{q} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi} \cdot \mathbf{v} - \rho \mathbf{g} \cdot \mathbf{v} = 0 \tag{2.9-8}$$

Next we break the total specific energy into specific kinetic and internal energies:

$$e = \frac{1}{2}v^2 + u \tag{2.9-9}$$

to give

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho u\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho v^2 + \rho u\right)\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi}\cdot\mathbf{v} - \rho\mathbf{g}\cdot\mathbf{v} = 0 \tag{2.9-10}$$

Equation 2.9-10 is the total differential energy balance, and it contains both thermal and mechanical energies. It is useful to separate the two. We can do this by taking the dot product of the equation of motion with the velocity vector **v** to get the mechanical energy balance equation:

$$\frac{\partial}{\partial t}(\rho v^2) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho v^2\right)\mathbf{v} + \mathbf{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\pi}) - \rho\mathbf{v}\cdot\mathbf{g} = 0 \tag{2.9-11}$$

Integration of Eq. 2.9-11 leads to the *macroscopic mechanical energy balance* equation, the steady-state version of which is the famous *Bernoulli equation*. Next we subtract Eq. 2.9-11 from Eq. 2.9-10 to obtain the differential thermal energy-balance equation:

$$\frac{\partial}{\partial t}(\rho u) + \boldsymbol{\nabla}\cdot(\rho u)\mathbf{v} + \boldsymbol{\nabla}\cdot\mathbf{q} + \boldsymbol{\pi} : \boldsymbol{\nabla}\mathbf{v} = 0 \tag{2.9-12}$$

Substituting $\boldsymbol{\pi} = P\boldsymbol{\delta} + \boldsymbol{\tau}$, we get:

$$\underbrace{\frac{\partial}{\partial t}(\rho u)}_{\substack{\text{rate of increase}\\ \text{in internal energy}\\ \text{per unit volume}}} = \underbrace{-\boldsymbol{\nabla}\cdot(\rho u)\mathbf{v}}_{\substack{\text{net rate of}\\ \text{addition of}\\ \text{internal energy}\\ \text{by convective}\\ \text{transport per}\\ \text{unit volume}}} \quad \underbrace{-\boldsymbol{\nabla}\cdot\mathbf{q}}_{\substack{\text{rate of internal}\\ \text{energy addition}\\ \text{by conduction}\\ \text{per unit volume}}} \quad \underbrace{-P(\boldsymbol{\nabla}\cdot\mathbf{v})}_{\substack{\text{reversible rate}\\ \text{of internal}\\ \text{energy increase}\\ \text{per unit volume}}} \quad \underbrace{-\boldsymbol{\tau}:\boldsymbol{\nabla}\mathbf{v}}_{\substack{\text{irreversible}\\ \text{rate of internal}\\ \text{energy increase}\\ \text{per unit volume}\\ \text{by viscous dissipation}}} \tag{2.9-13}$$

or

$$\rho\frac{Du}{Dt} = -\boldsymbol{\nabla}\cdot\mathbf{q} - P(\boldsymbol{\nabla}\cdot\mathbf{v}) - \boldsymbol{\tau}:\boldsymbol{\nabla}\mathbf{v} \tag{2.9-14}$$

This equation simply states that the increase in internal energy of a fluid element riding with the stream is due to the heat flux, the reversible increase of internal energy per unit volume by compression, and viscous dissipation or the irreversible conversion of internal friction to heat. Should there be another type of heat source (e.g., chemical reaction), it can be added to the equation.

The heat flux can be expressed in terms of temperature gradient by the Fourier equation:

$$\mathbf{q} = -k\boldsymbol{\nabla}T \tag{2.9-15}$$

and the internal energy in terms of enthalpy $u = h - P/\rho$, which in turn is expressed in terms of specific heat to give[12] the two following expressions for the equation of change of temperature:

$$\rho C_v \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot k\boldsymbol{\nabla} T - T\left(\frac{\partial P}{\partial T}\right)_\rho (\boldsymbol{\nabla} \cdot \mathbf{v}) - \boldsymbol{\tau} : \boldsymbol{\nabla}\mathbf{v}$$
$$\rho C_p \frac{DT}{Dt} = \boldsymbol{\nabla} \cdot k\boldsymbol{\nabla} T - \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P \frac{DP}{Dt} - \boldsymbol{\tau} : \boldsymbol{\nabla}\mathbf{v} \tag{2.9-16}$$

The first equation is listed in rectangular, cylindrical, and spherical coordinates in Table 2.5. For incompressible Newtonian fluids with constant thermal conductivity, Eq. 2.9-16 reduces to:

$$\rho C_v \frac{DT}{Dt} = k\nabla^2 T + \frac{1}{2}\mu(\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}) \tag{2.9-17}$$

which is listed in various coordinate systems in Table 2.6.

Clearly, then, the temperature dependence of viscosity, on the one hand, and the viscous dissipation term that depends on the magnitude of the local rate of deformation, on the other hand, couple the energy equation with the equation of motion, and they must be solved simultaneously.

Example 2.7 Nonisothermal Parallel Plate Drag Flow with Constant Thermophysical Properties Consider an incompressible Newtonian fluid between two infinite parallel plates at temperatures $T(0) = T_1$ and $T(H) = T_2$, in relative motion at a steady state, as shown in Fig. E2.7 The upper plate moves at velocity V_0. (a) Derive the temperature profile between the plates, and (b) determine the heat fluxes at the plates.

Solution

(a) By assuming constant thermophysical properties, the equation of motion and energy are decoupled. The velocity profile between the plates is simple drag flow $v_z = V_0(y/H)$, and all other velocity components equal zero. We now turn to the equation of energy in rectangular coordinates in Table 2.6, which reduces to:

$$k\frac{d^2T}{dy^2} + \mu\left(\frac{dv_z}{dy}\right)^2 = 0 \tag{E2.7-1}$$

Substituting the linear drag velocity profile $(dv_z/dy) = V_0/H$ into Eq. 2.7-1, and defining $\xi = y/H$ subsequent to integration, yields:

$$\frac{T - T_1}{T_2 - T_1} = \xi + \frac{\mathrm{Br}}{2}\xi(1 - \xi) \tag{E2.7-2}$$

where Br is the dimensionless Brinkman number defined as

$$\mathrm{Br} = \frac{\mu V_0^2}{k(T_2 - T_1)} \tag{E2.7-3}$$

12. For details see R. Byron Bird, Waren E. Stewart and Edwin N. Lightfoot, *Transport Phenomena*, 2nd ed., Wiley, New York, 2002, pp. 336–340.

TABLE 2.5 The Equation of Energy in Terms of Energy and Momentum Fluxes in Several Coordinate Systems

Rectangular Coordinates (x, y, z)

$$\rho C_v\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)$$
$$=-\left[\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right]-T\left(\frac{\partial P}{\partial T}\right)_\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$
$$-\left\{\tau_{xx}\frac{\partial v_x}{\partial x}+\tau_{yy}\frac{\partial v_y}{\partial y}+\tau_{zz}\frac{\partial v_z}{\partial z}\right\}$$
$$-\left\{\tau_{xy}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\tau_{xz}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\right\}$$

Cylindrical Coordinates (r, θ, z)

$$\rho C_v\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)$$
$$=-\left[\frac{1}{r}\frac{\partial}{\partial r}(rq_r)+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}\right]-T\left(\frac{\partial P}{\partial T}\right)_\rho\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}\right)$$
$$-\left\{\tau_{rr}\frac{\partial v_r}{\partial r}+\tau_{\theta\theta}\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right)+\tau_{zz}\frac{\partial v_z}{\partial z}\right\}$$
$$-\left\{\tau_{r\theta}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]+\tau_{rz}\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)+\tau_{\theta z}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)\right\}$$

Spherical Coordinates (r, θ, ϕ)

$$\rho C_v\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$
$$=-\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2q_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(q_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi}\right]$$
$$-T\left(\frac{\partial P}{\partial T}\right)_\rho\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)$$
$$-\left\{\tau_{rr}\frac{\partial v_r}{\partial r}+\tau_{\theta\theta}\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta}+v_r\right)+\tau_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right)\right\}$$
$$-\left\{\tau_{r\theta}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]+\tau_{r\phi}\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)+\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}\right]\right.$$
$$\left.+\tau_{\theta\phi}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]\right\}$$

Source: Reprinted with permission from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960.

TABLE 2.6 The Equation of Thermal Energy in Terms of Transport Properties (for Newtonian fluids at constant ρ, μ and k. Note that constant ρ implies that $C_v = C_p$)

Rectangular Coordinates (x, y, z)

$$\rho C_v\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right)$$
$$= k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + 2\mu\left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\}$$
$$+ \mu\left\{\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)^2\right\}$$

Cylindrical Coordinates (r, θ, z)

$$\rho C_v\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right)$$
$$= k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + 2\mu\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\right]^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\}$$
$$+ \mu\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2\right\}$$

Spherical Coordinates (r, θ, ϕ)

$$\rho C_v\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$
$$= k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right]$$
$$+ 2\mu\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)^2\right\}$$
$$+ \mu\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}\right]^2\right.$$
$$\left.+ \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2\right\}$$

Source: Reprinted with permission from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960.

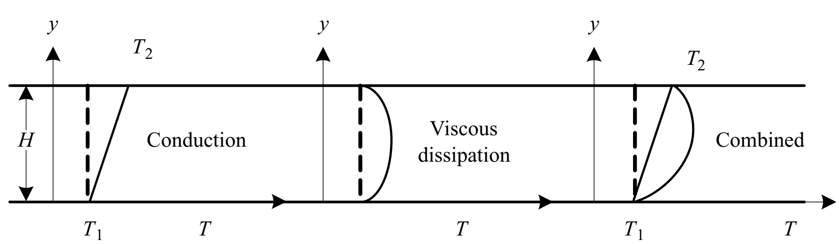


Fig. E2.7 Schematic temperature profiles between the parallel plates in relative motion at different temperatures with temperature-independent physical properties.

which measures the ratio of the rate of thermal heat generation by viscous dissipation to rate of heat conduction. Clearly, in the absence of viscous dissipation, the temperature profile between the plates is linear; whereas, the contribution of viscous dissipation is a parabolic, and the linear combination of the two yields the desired temperature profile as depicted in Fig. E2.7.

(b) The heat fluxes at the two plates are obtained by differentiating Eq. E2.7-2, and substituting it at $y = 0$ and $y = H$ into the Fourier equation to give:

$$q_y(H) = -k\frac{T_2 - T_1}{H} + \frac{\mu V_0^2}{2H} \tag{E2.7-4}$$

$$q_y(0) = -k\frac{T_2 - T_1}{H} - \frac{\mu V_0^2}{2H} \tag{E2.7-5}$$

If $T_2 > T_1, q_y(0)$ will always be negative. The flux of heat into the lower plate is the sum of conduction and one half of the rate of heat generated by viscous dissipation. At the upper plate, on the other hand, the flux may be either negative (into the fluid) or positive (into the plate) or zero, depending on the relative values of heat flux due to conduction and viscous dissipation.

2.10 MASS TRANSPORT IN BINARY MIXTURES AND THE DIFFUSION EQUATION

Subsequent to polymer manufacture, it is often necessary to remove dissolved volatiles, such as solvents, untreated monomer, moisture, and impurities from the product. Moreover, volatiles, water, and other components often need to be removed prior to the shaping step. For the dissolved volatiles to be removed, they must diffuse to some melt–vapor interface. This mass-transport operation, called *devolatilization,* constitutes an important elementary step in polymer processing, and is discussed in Chapter 8. For a detailed discussion of diffusion, the reader is referred to the many texts available on the subject; here we will only present the equation of continuity for a binary system of constant density, where a low concentration of a minor component A diffuses through the major component:

$$\frac{DC_A}{Dt} = \mathscr{D}_{AB}\nabla^2 c_A + \dot{R}_A \tag{2.10-1}$$

where the diffusivity $\mathscr{D}_{AB}$ was assumed constant, c_A is the molar concentration of the species A, and $\dot{R}_A$ is the molar rate of production of A per unit volume (e.g., by chemical reaction). The equation, containing the flux and source terms, is identical in form to Eq. 2.9-17, hence, the components of the equation in the various coordinate systems can be easily obtained from Table 2.6.

2.11 MATHEMATICAL MODELING, COMMON BOUNDARY CONDITIONS, COMMON SIMPLIFYING ASSUMPTIONS, AND THE LUBRICATION APPROXIMATION

Mathematical Modeling

Engineering design, analysis, control, optimization, trouble shooting, and any other engineering activity related to specific industrial *processes*, *machines,* or *structures* can best be performed using a *quantitative* study of effect of the parameters as well as of the design and process variables on the process, machine, or structure. In any of these contexts, this undertaking calls for *mathematical modeling*[13] of the specific entity. Hence, engineering mathematical modeling, as the name implies, refers to the attempt to mimic (describe) the actual engineering entity through mathematical equations, which will always contain simplifications about the nature of the substances involved, the relative magnitudes of the various physical effects, and the geometry of the space in which the phenomena take place. But "simplification" is not quite the right definition for what is done in modeling. A better description would be *construction of analogs*. These may be physical analogs or mental analogs, which are amenable to mathematical formulation. A successful modeler is someone with a thorough understanding of the physical mechanisms, who is imaginative enough to create the analog in such a way that it captures the essential elements of the process, and is then able to cast it into mathematical equations.

Aris (3) more formally defined a mathematical model thus: "a system of equations, Σ, is said to be a model of prototypical system, S, if it is formulated to express the laws of S and its solution is intended to represent some aspect of the behavior of S." Seinfeld and Lapidus (4) gave a more specific definition: "Mathematical model is taken to mean the formulation of mathematical relationships, which describe the behavior of actual systems such that the dependent and independent variables and parameters of the model are directly related to physical and chemical quantities in the real system."

All the mathematical formulations presented in the following chapters are mathematical models of polymer processing subsystems and systems that generally consist of a series of intricate, mostly transport-based, physical phenomena occurring in complex geometrical configurations.

Clearly, then, a mathematical model is always an *approximation* of the real system. The better the model, the closer it will approximate the real system.

It is worth noting at this point that the various scientific theories that quantitatively and mathematically formulate natural phenomena are in fact *mathematical models of nature*. Such, for example, are the kinetic theory of gases and rubber elasticity, Bohr's atomic model, molecular theories of polymer solutions, and even the equations of transport phenomena cited earlier in this chapter. Not unlike the engineering mathematical models, they contain simplifying assumptions. For example, the transport equations involve the assumption that matter can be viewed as a continuum and that even in fast, irreversible processes, local equilibrium can be achieved. The paramount difference between a mathematical model of a natural process and that of an engineering system is the required level of accuracy and, of course, the generality of the phenomena involved.

An engineering mathematical model may consist of a single algebraic equation, sets of partial differential equations, or any possible combination of various kinds of equations and mathematical operations, often in the form of large computer programs. Indeed, the revolutionary developments in computer technology have immensely increased the modeling possibilities, their visualization and their interpretation, bringing all engineering models much closer to the real process. They have also vastly expanded the practical use of numerical solutions such as finite difference methods and finite elements.

The quantitative study of the *process*, which as we stressed at the outset, is the reason for modeling, is called *simulation.* But modeling and simulation have useful functions

---

13. The word "model" derives from the Latin word *modus* which means a "measure," hinting toward a change in scale.

beyond the quantitative study of the process. An attempt to build a model for a complex process requires first of all a clear definition of objectives, which is often both useful and educational. In addition, by repeated simulations, a better understanding of the process is achieved, greatly improving our insight and developing our engineering intuition. Using a model, we can study extrapolation or scale-up problems and the effect of individual variables, and explore sensitivity and stability problems. All of these are often difficult, costly, or even impossible to carry out in the actual processes.

Model building consists of assembling sets of various mathematical equations, originating from engineering fundamentals, such as the balance equations which, together with the appropriately selected boundary conditions, bear the interrelations between variables and parameters corresponding to those in the actual processes. Modeling a complex process, such as a polymer processing operation, is done by breaking it down into clearly defined *subsystems.* These are then assembled into the complete model. The latter is tested for *experimental verification*. A mathematical model, no matter how sophisticated and complicated, is of little use if it does not reflect reality to a satisfactory degree as proved by experimentation.

There are various ways to classify mathematical models (5). First, according to the nature of the process, they can be classified as *deterministic* or *stochastic*. The former refers to a process in which each variable or parameter acquires a certain specific value or sets of values according to the operating conditions. In the latter, an element of uncertainty enters; we cannot specify a certain value to a variable, but only a most probable one. Transport-based models are deterministic; residence time distribution models in well-stirred tanks are stochastic.

Mathematical models can also be classified according to the mathematical foundation the model is built on. Thus we have *transport phenomena*–based models (including most of the models presented in this text), *empirical* models (based on experimental correlations), and *population–based* models, such as the previously mentioned residence time distribution models. Models can be further classified as *steady* or *unsteady, lumped parameter* or *distributed parameter* (implying no variation or variation with spatial coordinates, respectively), and *linear* or *nonlinear.*

In polymer processing, the mathematical models are by and large deterministic (as are the processes), generally transport based, either steady (continuous process, except when dynamic models for control purposes are needed) or unsteady (cyclic process), linear generally only to a first approximation, and distributed parameter (although when the process is broken into small, finite elements, locally lumped-parameter models are used).

Common Simplifying Assumptions

In the examples discussed so far, as well as those to be discussed throughout this book, several common simplifying assumptions are introduced without proof or discussion. Their validity for polymeric materials is not always obvious and they merit further discussion.

The No-slip Condition The no-slip condition implies that, at a solid–liquid interface, the velocity of the liquid equals that of the solid surface. This assumption, based on extensive experimentation, is widely accepted in fluid mechanics, though its validity is not necessarily obvious.

The slip of viscoelastic polymeric materials (and flow instabilities) was reviewed in detail by Denn (6). Apparent slip at the wall was observed with highly entangled linear polymers, but not with branched polymers or linear polymers with insufficient numbers of entanglements per chain. The slip was observed at stresses below the onset of visible extrudate distortions. Yet more advanced experimental tools need to be developed to examine slip and its length scales.

Three theories were proposed to explain wall-slip: (a) adhesive failure at the wall, (b) cohesive failure within the material as a result of disentanglement of chains in the bulk and chains absorbed on the wall, and (c) the creation of a lubricating surface layer at the wall either by a stress-induced transition, or by a lubricating additive. If the polymer contains low molecular weight components or slip-additives, their diffusion to the wall will create a thin lubricating layer at the wall, generating apparent slip.

Slip at the wall is closely related to extrudate instabilities, but in normal flow situations within machines, in virtually all but exceptional cases, the no-slip condition is assumed for solving flow problems.

Liquid–liquid interface At the interface between two immiscible liquids, the boundary conditions that must be satisfied are (a) a continuity of both the tangential and the normal velocities (this implies a no-slip condition at the interface), (b) a continuity of the shear stress, and (c) the balance of the difference in normal stress across the interface by the interfacial (surface) force. Thus the normal stresses are not continuous at the interface, but differ by an amount given in the following expression:

$$P_1 - P_0 = \Gamma\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2.11-1}$$

where $P_1 - P_0$ is the pressure difference, due to the surface tension Γ, action on a curved surface of radii of curvature of R_1 and R_2.

The Steady State Assumption A physical process has reached a *steady state* when a stationary observer, located at *any* point of the space where the process is taking place, observes no changes in time. Mathematically, this statement reduces to the condition where, in the field equations describing the process, all the $\partial/\partial t$ terms vanish. In reality, processes are very rarely truly steady. Boundary conditions, forcing functions, system resistance, and composition or constitution of the substances involved change periodically, randomly, or monotonically by small amounts. These changes bring about process response fluctuations. In such cases the process can still be treated as if it were steady using the *pseudo–steady state approximation*.

To illustrate this approximation, let us consider a pressure flow in which the driving-force pressure drop varies with time. We set $\partial\rho/\partial t$ and $\partial\mathbf{v}/\partial t$ in the equations of continuity and motion, respectively, equal to zero and proceed to solve the problem as if it were a steady-state one, that is, we assume ΔP to be constant and not a function of time. The solution is of the form $\mathbf{v} = \mathbf{v}(x_i, \Delta P(t), \text{geometry, etc.})$. Because ΔP was taken to be a constant, $\mathbf{v}$ is also a constant with time. The pseudo–steady-state approximation "pretends" that the foregoing solution holds for any level of ΔP and that the functional dependence of $\mathbf{v}$ on time is $\mathbf{v}(x_i, t) = \mathbf{v}(x_i, \Delta P(t), \text{geometry, etc.})$. The pseudo–steady state approximation is not valid if the values of $\Delta(\rho\mathbf{v})/\Delta t$ (Δt being the characteristic time of fluctuation of ΔP) obtained using this approximation contribute to an appreciable fraction of the mean value of the applied ΔP.

The Constant Thermophysical-Properties Assumption The last commonly used set of assumptions in liquid flow (isothermal, as well as nonisothermal) and in conductive heat transfer is to treat k, C_p, and ρ as *constant* quantities, independent of T and P. In polymer processing, where both heat transfer and flow take place, typical temperature variations may reach up to 200 °C and pressure variations, 50 MN/m^2. Under such significant variations, the density of a typical polymer would change by 10 or 20%, depending on whether it is amorphous or crystalline, while k and C_p would undergo variations of 30 to 40%.

Under normal conditions, when solving momentum and energy equations, we can usually assume the polymer melt to be incompressible, but the melt density at the prevailing pressures and temperatures should be carefully evaluated. Assuming constant C_p and k (taken at the average temperature), though it may affect the results of heat transfer or coupled heat transfer and flow in polymer processing, do give very good approximations.

The Lubrication Approximation

In polymer processing, we frequently encounter creeping viscous flow in slowly tapering, relatively narrow, gaps as did the ancient Egyptians so depicted in Fig. 2.5. These flows are usually solved by the well-known *lubrication approximation*, which originates with the famous work by Osborne Reynolds, in which he laid the foundations of hydrodynamic lubrication.[14] The theoretical analysis of lubrication deals with the hydrodynamic behavior of thin films from a fraction of a mil (10^{-3} in) to a few mils thick. High pressures of the

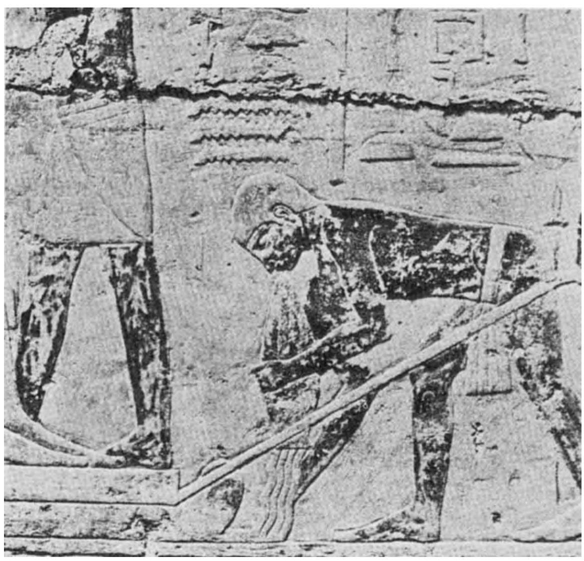


Fig. 2.5 Lubrication of a sledge used to transport the statue of Ti in ancient Egypt, about 2400 B.C. [Reprinted by permission from G. Hähner and N. Spencer, "Rubbing and Scrubbing," *Physics Today*, September, 22 (1998).]

14. Osborne Reynolds published his monumental paper on lubrication in 1886 (*Phil. Trans. R. Soc.*, **177**, 157–234). The paper was entitled "On the Theory of Lubrication and Its Application to Mr. Beauchamps Tower's Experiments." Mr. Tower was an engineer working for the railroads who was trying to understand the mechanism of lubrication of railroad cars. He observed experimentally that a very thin layer of lubricating oil appears to be able to support the huge load of a railroad car. Unable to explain these observations, he turned to Reynolds. Honoring Reynolds contribution to the field of lubrication, the commonly used engineering unit for viscosity, $lb_f s/in^2$, is called a "reyn" (just as the unit "poise" is named after Poiseuille).

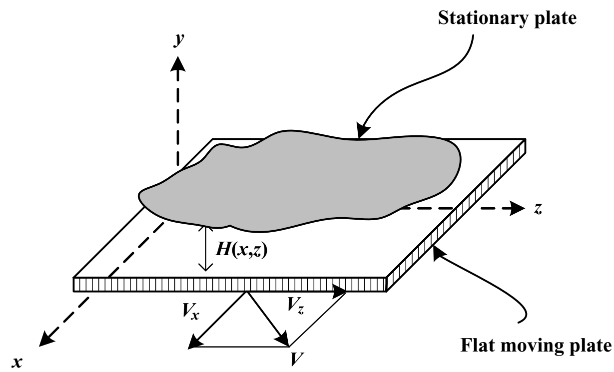


Fig. 2.6 Flow region formed by two closely spaced plates with variable gap $H(x,z)$. The lower plate is flat and moves at velocity V. The upper plate is slowly undulating.

order of thousands of psi (millions of newtons per square meter) may develop in these films as a result of the relative motion of the confining walls. In polymer processing we are generally dealing with films that are several orders of magnitude thicker, but since the viscosity of polymeric melts is also several orders of magnitude higher than the viscosity of lubricating oils, the assumptions leading to the lubrication approximation are valid in polymer processing as well. Next we review the principles of hydrodynamic lubrication.

Consider a narrow two-dimensional gap with slowly varying thickness in the x,z plane with the containing wall in relative motion. Specifically, the characteristic length in the x,z plane is much larger than the characteristic length in the perpendicular direction. Without loss in generality, we can assume that the flow is confined between a flat surface moving in the x,z, plane and a slowly undulating fixed surface at distance $H(x,z)$ from the flat plate, as shown in Fig. 2.6.

According to the lubrication approximation, we can quite accurately assume that *locally* the flow takes place between two parallel plates at $H(x,z)$ apart in relative motion. The assumptions on which the theory of lubrication rests are as follows: (a) the flow is laminar, (b) the flow is steady in time, (c) the flow is isothermal, (d) the fluid is incompressible, (e) the fluid is Newtonian, (f) there is no slip at the wall, (g) the inertial forces due to fluid acceleration are negligible compared to the viscous shear forces, and (h) any motion of fluid in a direction normal to the surfaces can be neglected in comparison with motion parallel to them.

According to these assumptions, the only nonvanishing velocity components are v_x and v_z, and the equations of continuity and motion in the Cartesian coordinate system in Tables 2.1 and 2.4 reduce, respectively, to:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \tag{2.11-2}$$

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2} \tag{2.11-3}$$

$$\frac{\partial P}{\partial y} = 0 \tag{2.11-4}$$

$$\frac{\partial P}{\partial z} = \mu \frac{\partial^2 v_z}{\partial y^2} \tag{2.11-5}$$

Equation 2.11-4 implies that there is no transverse pressure gradient. The boundary conditions for solving the equations are $v_x(H) = v_z(H) = 0$ and $v_x(0) = V_x, v_z(0) = V_z$. Equations 2.11-3 and 2.11-5 can be directly integrated to give the velocity profiles, recalling that P is not a function of y:

$$v_x(y) = V_x\left(1 - \frac{y}{H}\right) + \frac{yH}{2\mu}\left(\frac{\partial P}{\partial x}\right)\left(\frac{y}{H} - 1\right) \tag{2.11-6}$$

$$v_z(y) = V_z\left(1 - \frac{y}{H}\right) + \frac{yH}{2\mu}\left(\frac{\partial P}{\partial z}\right)\left(\frac{y}{H} - 1\right) \tag{2.11-7}$$

which upon integration gives the volumetric flow rates per unit width, q_x *and* q_z:

$$q_x = \frac{V_x H}{2} + \frac{H^3}{12\mu}\left(-\frac{\partial P}{\partial x}\right) \tag{2.11-8}$$

$$q_z = \frac{V_z H}{2} + \frac{H^3}{12\mu}\left(-\frac{\partial P}{\partial z}\right) \tag{2.11-9}$$

The equation of continuity is next integrated over y:

$$\int_0^H \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right) dy = 0 \tag{2.11-10}$$

and substituting Eqs. 2.11-6 and 2.11-7 into Eq. 2.11-10 gives

$$\frac{\partial}{\partial x}\left(H^3 \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(H^3 \frac{\partial P}{\partial z}\right) = 6\mu \frac{\partial H}{\partial x} V_x + 6\mu \frac{\partial H}{\partial z} V_z \tag{2.11-11}$$

which is known as the *Reynolds equation* for incompressible fluids. By solving it for any $H(x,z)$ the two-dimensional pressure distribution $P(x,z)$ is obtained, from which the local pressure gradients can be computed and, via Eqs. 2.11-6 to 2.11-9, the local velocity profiles and flow rates obtained.

The lubrication approximation facilitates solutions to flow problems in complex geometries, where analytical solutions either cannot be obtained or are lengthy and difficult. The utility of this approximation can well be appreciated by comparing the almost exact solution of pressure flow in slightly tapered channels to that obtained by the lubrication approximation.

The lubrication approximation as previously derived is valid for purely viscous Newtonian fluids. But polymer melts are viscoelastic and also exhibit normal stresses in shearing flows, as is discussed in Chapter 3; nevertheless, for many engineering calculations in processing machines, the approximation does provide useful models.

Example 2.8 Flow between Two Infinite Nonparallel Plates in Relative Motion Consider an incompressible Newtonian fluid in isothermal flow between two non-parallel plates in relative motion, as shown in Fig. E2.8, where the upper plate is moving at constant velocity V_0 in the z direction. The gap varies linearly from an initial value of H_0 to H_1 over length L, and the pressure at the entrance is P_0 and at the exit P_1. Using the lubrication approximation, derive the pressure profile.

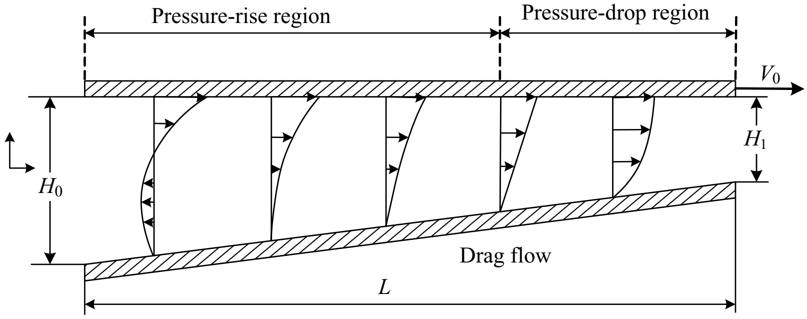


Fig. E2.8 Two non-parallel plates in relative motion, with schematic velocity profiles corresponding to a pressure-rise zone followed by a pressure-drop zone when entrance and exit pressures are equal.

Solution We can gain insight into the nature of the flow if we first consider the special case where the pressure at the entrance P_0 equals the pressure at the exit P_1. Figure E2.8 shows the schematic velocity profiles at different locations. At a steady state, the net volumetric flow rate of an incompressible fluid must be constant. Since the gap between the plates is wide at the entrance and narrow at the exit, the drag flow decreases linearly from entrance to exit. Hence, in order to maintain a uniform *net* flow rate, pressure must initially rise (with opposing pressure flow reducing drag flow), and drop toward the exit (with the pressure flow augmenting the drag flow). Clearly, the pressure profile must reach a maximum, at which point the pressure gradient is zero and the flow is pure drag flow. Of course, if $P_0 \neq P_1$ the pressure may rise continuously, drop continuously, or go through a maximum, depending on the conditions.

The Reynolds equation (Eq. 2.11-11) for one-dimensional flow, as in the case at hand, reduces to:

$$\frac{d}{dz}\left(H^3\frac{dP}{dz}\right) = 6\mu V_0 \frac{dH}{dz} \tag{E2.8-1}$$

where z is the flow direction. Equation E2.8-1 can be integrated with respect to z to give

$$H^3\frac{dP}{dz} = 6\mu V_0 H + C_1 \tag{E2.8-2}$$

where C_1 is an integration constant, which can be conveniently expressed in terms of H^* defined as the separation between the plates where $dP/dz = 0$. If the pressure exhibits a maximum within $0 \le z \le L$, then H^* is the separation between the plates at that location; if the pressure profile exhibits no maximum in this range, the mathematical function describing the pressure as a function of z will still have a maximum at $z < 0$ or $z > L$, and H^* will be the "separation" between the virtual plates extended to that point. Thus, Eq. E2.8-2 can be written as

$$\frac{dP}{dz} = 6\mu V_0 \frac{H - H^*}{H^3} \tag{E2.8-3}$$

and integrated to give the pressure profile:

$$P = P_0 + 6\mu V_0 \int_0^z \frac{H - H^*}{H^3}\, dz \tag{E2.8-4}$$

where $P(0) = P_0$. For a constant taper, the dimensionless gap size as a function of distance is given by:

$$\zeta = \zeta_0 - (\zeta_0 - 1)\frac{z}{L} \tag{E2.8-5}$$

where $\zeta = H/H_1$ and $\zeta_0 = H_0/H_1$. Substituting Eq. E2.8-5 into Eq. E2.8-4 and integrating, the latter gives the desired pressure profile:

$$P = P_0 + \frac{6\mu V_0 L}{H_0 H_1}\left[\frac{\zeta_0 - \zeta}{\zeta(\zeta_0 - 1)} - \frac{q}{V_0 H_0}\frac{\zeta_0^2 - \zeta^2}{\zeta^2(\zeta_0 - 1)}\right] \tag{E2.8-6}$$

where q is the net flow rate per unit width:

$$q = \frac{1}{2}V_0 H^* \tag{E2.8-7}$$

The pressure distribution therefore depends on a number of variables: geometrical (H_0, H_1, and L), operational (V_0 and q), and physical properties (μ). The maximum pressure that can be attained is at $\zeta = 1$ $(z = L)$, at closed discharge conditions ($q = 0$):

$$P_{\max} = P_0 + \frac{6\mu L V_0}{H_0 H_1} \tag{E2.8-8}$$

If the entrance and discharge pressures are equal, the pressure profile will exhibit a maximum value at $H^* = 2H_0/(1 + \zeta_0)$. This conclusion therefore focuses attention on an important difference between parallel-plate and non–parallel-plate geometries. In the former, equal entrance and discharge pressure implies no pressurization and pure drag flow, whereas, in the latter, it implies the existence of a maximum in the pressure profile. Indeed, this pressurization mechanism forms the foundation of the lubrication, as is shown in the next example, and explains the experimental observation of pressure profiles along SSEs as we discuss in Chapter 6.

Example 2.9 The Journal–Bearing Problem[15] A journal of radius r_1 is rotating in a bearing of radius r_2 at an angular velocity Ω. The length of the journal and bearing in the z direction is L. Viscous Newtonian oil fills the narrow gap between the journal and bearing. The oil lubricates the bearing, that is, it prevents solid-solid frictional contact between the journal and the bearing. This is accomplished, of course, as a result of the pressure field generated within the film. We wish to derive a mathematical model that explains this mechanism and enables us to compute the forces acting on the journal and torque needed to turn the journal.

Solution We assume that the bearing is eccentric to the rotating journal by a displacement of magnitude, a, as shown in Fig. E2.9a.

The concentric gap is $c = r_2 - r_1$, and clearly $a \leq c$. The gap is very small, and locally we can assume flow between parallel plates. Thus we define a rectangular coordinate system X, Y, Z located on the surface of the journal such that X is tangential to the journal, as indicated in Fig. E2.9a. The gap between the journal and bearing is denoted as $B(\theta)$ and is well approximated as a function of angle θ by the following expression:

$$r_1 + B(\theta) \cong r_2 + a\cos\theta \tag{E2.9-1}$$

15. We follow the solution presented in R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Second Edition *Vol. 1*, *Fluid Mechanics*, Wiley, New York, 1987, p. 48.

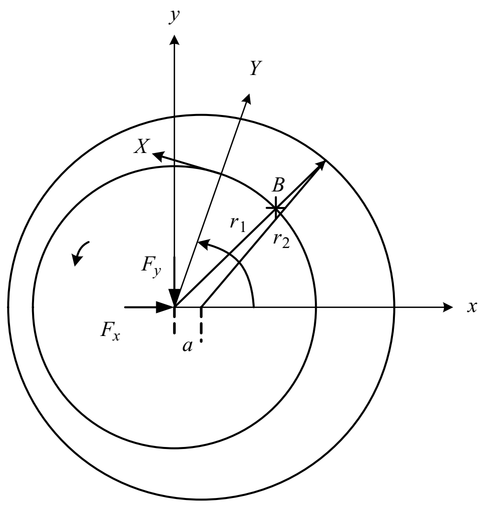


Fig. E2.9a Journal–bearing configuration with centers separated by displacement a. The force acting by the pressure field on the journal is given by component F_y and F_x.

or

$$B(\theta) = c + a\cos\theta \tag{E2.9-2}$$

Invoking the lubrication approximation, the local velocity profile (at a given angle θ) in rectangular coordinates X, Y, with boundary conditions $v_X(0) = \Omega r_1$ and $v_X(B) = 0$ (see Example 2.5) is given by

$$v_X(Y) = \Omega r_1\left(1 - \frac{Y}{B}\right) - \frac{B^2}{2\mu}\left(\frac{Y}{B}\right)\left(1 - \frac{Y}{B}\right)\frac{dP}{dX} \tag{E2.9-3}$$

Integrating Eq. E2.9-3 gives the flow rate:

$$Q = \frac{1}{2}\Omega r_1 BL - \frac{B^3 L}{12\mu}\frac{dP}{dX} \tag{E2.9-4}$$

In this case, we are not interested in the flow rate, but rather the pressure profile around the journal. Therefore, we express the flow rate, which (at steady state and neglecting leaks on the sides) is constant, in terms of the gap size B_0 at locations where the pressure profile attains maximum or minimum, and where the flow rate equals the local drag flow:

$$Q = \frac{1}{2}\Omega r_1 B_0 L \tag{E2.9-5}$$

There will be two such locations, as schematically indicated in Fig. E2.9b.
Substituting Eq. E2.9-5 into Eq. E2.9-4, gives

$$\frac{dP}{dX} = 6\mu\Omega r_1\left(\frac{1}{B^2} - \frac{B_0}{B^3}\right) \tag{E2.9-6}$$

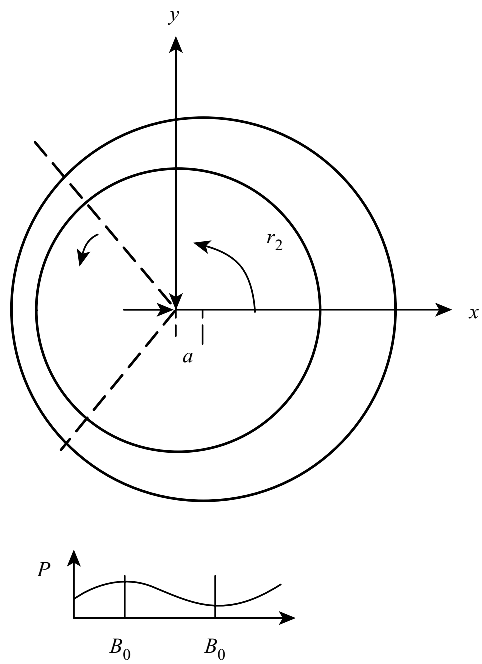


Fig. E2.9b The two broken lines show schematically the locations where the gap size is B_0 and where the pressure profile exhibits a maximum and a minimum.

Next, we obtain an expression for the shear stress by substituting Eq. E2.9-6 into Eq. E2.9-3 subsequent to taking its derivative and multiplying it by viscosity:

$$\tau_{YX}|_{Y=0} = -\mu \frac{dv_X}{dY} = \mu \Omega r_1 \left(\frac{4}{B} - \frac{3B_0}{B^2} \right) \tag{E2.9-7}$$

Next, we substitute Eq. E2.9-2 into Eqs. E2.9-6 and E2.9-7, recalling that $dX = r_1 d\theta$, to get

$$\frac{1}{r_1} \frac{dP}{d\theta} = 6\mu \Omega r_1 \left(\frac{1}{B^2} - \frac{B_0}{B^3} \right) \tag{E2.9-8}$$

$$\tau_{r\theta}|_{r=r_1} = \mu \Omega r_1 \left(\frac{4}{B} - \frac{3B_0}{B^2} \right) \tag{E2.9-9}$$

By integrating Eq. E2.9-8 between $\theta = 0$ and $\theta = 2\pi$, we get an equation that we can solve for B_0:

$$\int_{P_0}^{P_0} dP = 6\mu \Omega r_1^2 \int_0^{2\pi} \left(\frac{1}{B^2} - \frac{B_0}{B^3} \right) d\theta = \int_0^{2\pi} \left(\frac{1}{B^2} - \frac{B_0}{B^3} \right) d\theta = 0 \tag{E2.9-10}$$

which yields:

$$B_0 = \frac{J_2}{J_3} = c \left(\frac{c^2 - a^2}{c^2 + \frac{1}{2} a^2} \right) \tag{E2.9-11}$$

where J_n is defined as

$$J_n = \int_0^{2\pi} \frac{d\theta}{(c + a\cos\theta)^n}$$

and

$$\begin{aligned} J_1 &= 2\pi\left(c^2 - a^2\right)^{-1/2} \\ J_2 &= \frac{dJ_1}{dc} = 2\pi\left(c^2 - a^2\right)^{-3/2} \\ J_3 &= -\frac{1}{2}\frac{dJ_2}{dc} = 2\pi\left(c^2 + \frac{1}{2}a^2\right)\left(c^2 - a^2\right)^{-5/2} \end{aligned}$$

Now we can compute the torque given by

$$\mathscr{T} = L\int_0^{2\pi} [-\tau_{r\theta}]_{r=r_1} \Omega r_1^2 \, d\theta \tag{E2.9-12}$$

by substituting Eq. E2.9-9 into Eq. E.2.9-12 to give

$$\begin{aligned} \mathscr{T} &= -\mu L\Omega r_1^3 (4J_1 - 3B_0 J_2) \\ &= -\frac{2\pi\mu L\Omega r_1^3}{\sqrt{c^2 - a^2}} \frac{c^2 + 2a^2}{c^2 + a^2/2} \end{aligned} \tag{E2.9-13}$$

Next, we calculate the net force the fluid exerts on the journal. The components F_x and F_y of this force, as shown in Fig. E2.9c, are obtained by integrating around the circumference the respective contributions of the pressure and shear stress.

The force in the positive y direction is

$$\begin{aligned} F_y &= L\int_0^{2\pi} (-P\sin\theta - \tau_{r\theta}\cos\theta)_{r=r_1} r_1 \, d\theta \\ &= L\left\{ [P\cos\theta]_0^{2\pi} + \int_0^{2\pi} -\left[\left(\frac{dP}{d\theta}\right) - \tau_{r\theta}\right] \cos\theta r_1 \, d\theta \right\} \\ &= L\int_0^{2\pi} -\left[\left(\frac{dP}{d\theta}\right) - \tau_{r\theta}\right] \cos\theta r_1 \, d\theta \end{aligned} \tag{E2.9-14}$$

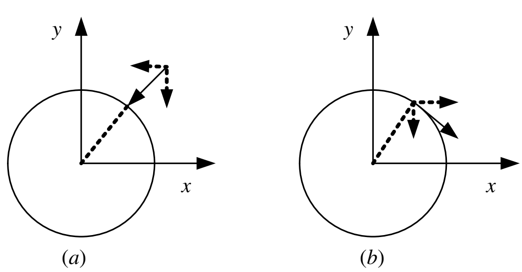


Fig. E2.9c (a) The normal force generated by the pressure and its x and y components. (b) The tangential shear force and its x and y components.

We can simplify this equation by neglecting the $\tau_{r\theta}$ contribution with respect to the $dP/d\theta$ contribution, because the former is of the order r_1/c; whereas, the latter is of $(r_1/c)^2$. Thus (neglecting higher-order terms from $\tau_{r\theta}$), we obtain for F_y

$$\begin{aligned} F_y &= -Lr_1 \int_0^{2\pi} 6\mu\Omega r_1^2 \left(\frac{1}{B^2} - \frac{B_0}{B^3} \right) d\theta \\ &= -6\mu L r_1^3 \Omega (K_2 - B_0 K_3) \end{aligned} \tag{E2.9-15}$$

where

$$K_n = \int_0^{2\pi} \frac{\cos\theta \; d\theta}{(c + a\cos\theta)^n}$$

and

$$\begin{aligned} K_2 &= \left(\frac{1}{a}\right)(J_1 - cJ_2) \\ K_3 &= \left(\frac{1}{a}\right)(J_2 - cJ_3) \end{aligned}$$

or

$$F_y = -\frac{3\mu(2\pi r_1 L)(\Omega r_1) r_1}{\left(\sqrt{\left(\frac{c}{a}\right)^2 - 1}\right)\left[\left(\frac{c}{a}\right)^2 + \frac{1}{2}\right] a^2} \tag{E2.9-16}$$

The force in the positive x direction, F_x, is

$$F_x = L \int_0^{2\pi} [-P\cos\theta + \tau_{r\theta}\sin\theta]_{r=r_1} \, d\theta = 0 \tag{E2.9-17}$$

Finally, the pressure distribution is obtained by integrating Eq. E2.9-8 to give

$$P = P_0 + \frac{6\mu\Omega r_1^2 a \sin\theta (c + 0.5a\cos\theta)}{(c^2 + 0.5a^2)(c + a\cos\theta)^2} \tag{E2.9-18}$$

where P_0 is an arbitrary constant pressure.

Thus we see that the net force acts in the negative y direction, and is proportional to viscosity, journal surface area, and tangential speed, and inversely proportional and very sensitive to the displacement a. Indeed, as a approaches zero, the force grows and approaches infinity, so clearly, this force prevents the journal from contacting the barrel with the tight clearance circling the bearing.

REFERENCES

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 2002.
2. D. Pnueli and H. Gutfinger, *Fluid Mechanics*, Cambridge University Press, Cambridge 1997.
3. R. Aris, *Mathematical Modeling Techniques*, Pitman, San Francisco, 1978.

4. J. H. Seinfeld and L. Lapidus, *Mathematical Methods in Chemical Engineering, Vol. 3, Process Modeling, Estimation and Identification*, Prentice Hall, Englewood Cliffs, NJ, 1974.
5. D. M. Himelblau and K. B. Bischoff, *Process Analysis and Simulation; Deterministic Systems*, Wiley, New York, 1968.
6. M. M. Denn, "Extrusion Instabilities and Wall Slip," *Ann. Rev. Fluid Mech.*, **33**, 265–287 (2001).

PROBLEMS

2.1 ***Coordinate Transformation*** (a) Verify the following relationships for the conversion of any function in rectangular coordinates $\phi(x, y, z)$, into a function in cylindrical coordinates $\psi(r, \theta, z)$

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x}, \qquad z = z$$

(b) Show that the derivatives of any scalar function (including components of vectors and tensors) in rectangular coordinates can be obtained from the derivatives of the scalar function in cylindrical coordinates

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} + \left(-\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta}$$
$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial}{\partial\theta}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

(Use the Chain Rule of partial differentiation.)

(c) The unit vectors in rectangular coordinates are $\boldsymbol{\delta}_x, \boldsymbol{\delta}_y, \boldsymbol{\delta}_z$, and those in cylindrical coordinates are $\boldsymbol{\delta}_r, \boldsymbol{\delta}_\theta$, and $\boldsymbol{\delta}_z$. Show that the following relationship between the unit vectors exists

$$\boldsymbol{\delta}_r = \cos\theta\boldsymbol{\delta}_x + \sin\theta\boldsymbol{\delta}_y$$
$$\boldsymbol{\delta}_\theta = -\sin\theta\boldsymbol{\delta}_x + \cos\theta\boldsymbol{\delta}_y$$
$$\boldsymbol{\delta}_z = \boldsymbol{\delta}_z$$

and

$$\boldsymbol{\delta}_x = \cos\theta\boldsymbol{\delta}_r - \sin\theta\boldsymbol{\delta}_\theta$$
$$\boldsymbol{\delta}_y = \sin\theta\boldsymbol{\delta}_r + \cos\theta\boldsymbol{\delta}_\theta$$

(d) From the results of (c), prove that

$$\frac{\partial}{\partial r}\boldsymbol{\delta}_r = 0, \qquad \frac{\partial}{\partial r}\boldsymbol{\delta}_\theta = 0, \qquad \frac{\partial}{\partial r}\boldsymbol{\delta}_z = 0$$
$$\frac{\partial}{\partial\theta}\boldsymbol{\delta}_r = \boldsymbol{\delta}_\theta, \qquad \frac{\partial}{\partial\theta}\boldsymbol{\delta}_\theta = -\boldsymbol{\delta}_r, \qquad \frac{\partial}{\partial\theta}\boldsymbol{\delta}_z = 0$$
$$\frac{\partial}{\partial z}\boldsymbol{\delta}_r = 0, \qquad \frac{\partial}{\partial z}\boldsymbol{\delta}_\theta = 0, \qquad \frac{\partial}{\partial z}\boldsymbol{\delta}_z = 0$$

(e) The operator ∇ in rectangular coordinates is

$$\nabla = \boldsymbol{\delta}_x \frac{\partial}{\partial x} + \boldsymbol{\delta}_y \frac{\partial}{\partial y} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}$$

Using the results of (b) and (d), derive the expression for ∇ in cylindrical coordinates

(f) Evaluate $\nabla \cdot \mathbf{v}$ in cylindrical coordinates.

2.2 ***Interpretation of the Equation of Continuity*** Show that the equation of continuity can be written as

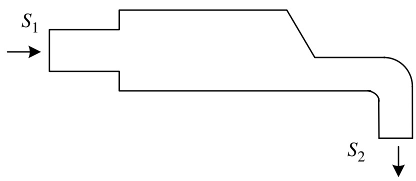
$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \mathbf{v})$$

where D/Dt is the substantial derivative defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

2.3 ***The Equation of Continuity by Differential Mass Balance*** Derive the equation of continuity in cylindrical coordinates by making a mass balance over the differential volume $\Delta r(r\Delta\theta)\Delta z$.

2.4 ***Macroscopic Mass Balance in a Steady Continuous System*** In the flow system shown in the accompanying figure, fluid at velocity V_1 and density ρ_1 enters the system over the inlet surface S_1, and it leaves at density ρ_2 with velocity V_2 over surface S_2. The flow is steady state. Derive a mass balance using Eq. 2.4.1.



2.5 ***The Mean Velocity of Laminar Pipe Flow*** Use the macroscopic mass-balance equation (Eq. 2.4.1) to calculate the mean velocity in laminar pipe flow of a Newtonian fluid. The velocity profile is the celebrated Poisseuille equation:

$$v_z = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

2.6 ***The Rate of Strain Tensor*** Using geometrical considerations, show that in a general flow field

$$\dot{\gamma}_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}$$

$$\dot{\gamma}_{yz} = \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}$$

$$\dot{\gamma}_{xz} = \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}$$

2.7 ***Spatial Variation of Properties*** Let $S(\mathbf{r})$ be a scalar field of a property of the continuum (e.g., pressure, temperature, density) at point P defined by radius vector $\mathbf{r}$.

(a) Show that for any such scalar field an associated vector field ∇S can be defined such that the dot product of which with unit vector $\mathbf{e}$ expresses the change of property S in direction $\mathbf{e}$.

(b) Prove that for a Cartesian coordinate system

$$\nabla S = \boldsymbol{\delta}_x \frac{\partial S}{\partial x} + \boldsymbol{\delta}_y \frac{\partial S}{\partial y} + \boldsymbol{\delta}_z \frac{\partial S}{\partial z}$$

(c) If $S = xy + z$, find the unit vector of maximum gradient at $P(2,1,0)$

(d) Prove that for a cylindrical coordinate system

$$\nabla S = \boldsymbol{\delta}_r \frac{\partial S}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r} \frac{\partial S}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial S}{\partial z}$$

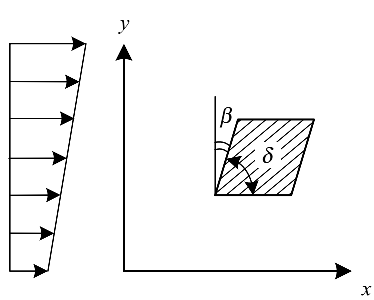
(e) Prove that for a spherical coordinate system

$$\nabla S = \boldsymbol{\delta}_r \frac{\partial S}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r} \frac{\partial S}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{1}{r \sin\theta} \frac{\partial S}{\partial \phi}$$

2.8 ***Viscous Stresses Acting in a Surface Element*** Using the expression $\boldsymbol{\pi} \cdot \mathbf{n}\, ds$, show that the forces acting on a unit surface in plane 2, 3 in a rectangular system is $\boldsymbol{\pi} \cdot \mathbf{n} = \boldsymbol{\delta}_1 \pi_{11} + \boldsymbol{\delta}_2 \pi_{12} + \boldsymbol{\delta}_3 \pi_{13}$.

2.9 ***Sign Convention of the Stress Tensor*** $\boldsymbol{\tau}'$ Consider a linear shear flow and examine the stress components τ'_{ij}

2.10 ***The Relationship between Shear Rate and Strain*** Show that (dv_x/dy) in a simple shear flow is identical to $-(d\gamma/dt)$, where γ is the angle shown in the accompanying figure.



2.11 ***The Invariants of the Rate of Strain Tensor in Simple Shear and Simple Elogational Flows*** Calculate the invariants of a simple shear flow and elongational flow.

2.12 ***Optimum Gap Size in Parallel Plate Flow*** Show that for the flow situation in Example 2.5, for a given net flow rate the optimum gap size for maximum pressure rise is

$$H = 3q/V_0$$

and the maximum pressure gradient is

$$\left.\frac{dP}{dz}\right|_{max} = \frac{6\mu V_0^2}{27q^2}$$

2.13 ***Couette Flow*** Couette flow is the flow in the annular space between two long concentric cylinders of radii R_o and R_i, created by the rotation of one of them. Consider Couette flow with (a) the outer cylinder rotating with angular velocity $\Omega(s^{-1})$; (b) the inner cylinder is rotating with angular velocity $-\Omega(s^{-1})$. (c) Also obtain the result by making a *torque balance* over a thin fluid shell formed by two imaginary fluid cylinders of radii r and $r + \Delta r$ and length $L(R_i < r < R_o)$.

2.14 ***Axial Drag Flow between Concentric Cylinders*** Consider the drag flow created in the space formed by two concentric nonrotating cylinders of radii R_o and R_i, with the inner cylinder moving with an axial velocity V. The system is open to the atmosphere at both ends. (a) Derive the velocity profile. (b) Also obtain the result by making a force balance on a thin fluid shell previously discussed.

2.15 ***Capillary Pressure Flow*** Solve the problem of flow in a capillary of radius R and length L, where $L \gg R$. The fluid is fed from a reservoir under the influence of an applied pressure P_0. The exit end of the capillary is at atmospheric pressure. Consider three physical situations: (a) a horizontal capillary; (b) a downward vertical capillary flow; and (c) an upward vertical capillary flow.

2.16 ***Axial Pressure Flow between Concentric Cylinders*** Solve the problem of flow in the horizontal concentric annular space formed by two long cylinders of length L and radii R_i and R_o, caused by an entrance pressure P_0, which is higher than the exit (atmospheric) pressure. Consider the limit as $(R_0 - R_i)/(R_0 + R_i)$ approaches zero.

2.17 ***Helical Flow between Concentric Cylinders*** Consider the helical flow in an annular space created by a constant pressure drop $(P_0 - P_1)$ and the rotation of the inner cylinder with an angular velocity $\Omega(\mathrm{s}^{-1})$.

2.18 ***Torsional Drag Flow between Parallel Disks*** Solve the torsional drag flow problem between two parallel disks, one of which is stationary while the other is rotating with an angular velocity $\Omega(\mathrm{s}^{-1})$. (*Note*: $v_\theta/r = \text{constant}$.)

2.19 ***Radial Pressure Flow between Parallel Disks*** Solve the problem of radial pressure flow between two parallel disks. The flow is created by a pressure drop $\left(P|_{r=0} - P|_{r=R}\right)$. Disregard the entrance region, where the fluid enters from a small hole at the center of the top disk.

2.20 ***Flow near a Wall Suddenly Set in Motion*** Set up the parallel-plate drag flow problem during its start-up period $t \leq t_{tr}$, when $v_x = f(t)$ in the entire flow region, and show that the resulting velocity profile, after solving the differential equation $v_x/V = 1 - \mathrm{erf}(y/\sqrt{4\,\mu t/\rho})$, if H is very large.

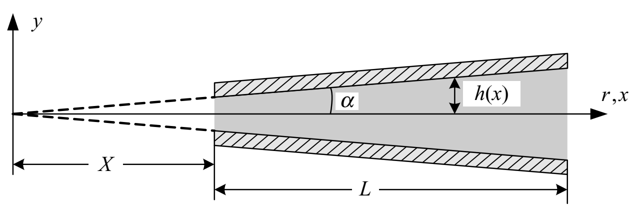
2.21 ***Heat Conduction across a Flat Solid Slab*** Solve the problem of heat transfer across an infinitely large flat plate of thickness H, for the following three physical situations: (a) the two surfaces are kept at T_1 and T_2, respectively; (b) one surface is kept at T_1 while the other is exposed to a fluid of temperature T_b, which causes a heat flux $q_y|_{y=H} = h_2(T_2 - T_b), h_2$ being the heat-transfer coefficient (W/m^2·K); (c) both surfaces are exposed to two different fluids of temperatures T_a and T_b with heat-transfer coefficients h_1 and h_2, respectively.

2.22 ***Heat Transfer in Pipes*** Solve the problem of conductive heat transfer across an infinitely long tube of inside and outside radii of R_i and R_o. Consider the following two physical situations: (a) the surface temperatures at R_i and R_o are maintained at T_i and T_o; (b) both the inside and outside tube surfaces are exposed to heat transfer fluids of constant temperatures T_a and T_b and heat-transfer coefficients h_i and h_o.

2.23 ***Heat Transfer in Insulated Pipes*** Solve case (b) of Problem 2.22 for a composite tube made of material of thermal conductivity k_i for $R_i \leq r \leq R_m$ and of material of thermal conductivity k_o for $R_m \leq r \leq R_o$.

2.24 ***Parallel-Plate Flow with Viscous Dissipation*** Consider the nonisothermal flow of a Newtonian fluid whose ρ, C_p and k are constant, while its viscosity varies with temperature as $\mu = Ae^{\Delta E/RT}$. The flow is between two infinite parallel plates, one of which is stationary while the other is moving with a velocity V. The fluid has a considerably high viscosity, so that the energy dissipated $\left(\frac{1}{2}\mu(\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}\right)$ in Eq. $2.9 - 17)$ cannot be neglected. State the equations of continuity, momentum, and energy for the following two physical situations and suggest a solution scheme: $(a)\,T(0) = T_1,\ T(H) = T_2\ \ (b)\,q_y|_{y=0} = q_y|_{y=H} = 0$.

2.25 ***Flow between Tapered Plates***[16] Consider the steady isothermal pressure flow of a Newtonian and incompressible fluid flowing in a channel formed by two slightly tapered plates of infinite width. Using the cylindrical coordinate system in the accompanying figure and assuming that $v_r(r, \theta), v_\theta = v_z = 0$:



16. W.E. Langlois, *Slow Viscous Flows*, Chapter VIII, Mcmillan, London, 1964.

(a) show that the continuity and momentum equations reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0 \qquad \text{or} \qquad v_r = \frac{F(\theta)}{r} \tag{a}$$

$$\frac{\partial P}{\partial r} = \frac{\mu}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2}\right) \tag{b}$$

$$\frac{\partial P}{\partial \theta} = \frac{2\mu}{r}\left(\frac{\partial v_r}{\partial \theta}\right) \tag{c}$$

Differentiate Eq.(b) with respect to θ and Eq.(c) with respect to r and equate. Solve the resulting equations using the boundary condition

$$v_r(r, \pm\alpha) = 0, \qquad Q = \int_{-\alpha}^{\alpha} v_r r \, d\theta$$

to obtain the velocity and pressure fields:

$$v_r(r,\theta) = \frac{Q}{r}\frac{\sin^2\alpha - \sin^2\theta}{\sin\alpha\cos\alpha - \alpha + 2\alpha\sin^2\alpha} \tag{d}$$

$$P(r,\theta) = P_0 + \frac{\mu Q}{X^2}\frac{\left(\sin^2\alpha - \sin^2\theta\right)\left(X^2/r^2 - 1\right)}{\sin\alpha\cos\alpha - \alpha + 2\alpha\sin^2\alpha} \tag{e}$$

where $P(X, 0) = P_0$.

(b) Show that the two nonvanishing pressure gradients in Cartesian coordinates are

$$\frac{\partial P}{\partial x} = -\frac{2\mu Q(1+D^2)D^3 h}{E}\frac{h^2 - 3D^2y^2}{\left(h^2 + D^2y^2\right)^3} \tag{f}$$

$$\frac{\partial P}{\partial y} = -\frac{2\mu Q(1+D^2)D^4 h}{E}\frac{3h^2 - D^2y^2}{\left(h^2 + D^2y^2\right)^3} \tag{g}$$

where $D = \tan\alpha, h = D(x - X)$ and $E = D - (1 - D^2)\arctan D$.

(c) From the Reynolds equation (Eq. 2.4-11) show that for the tapered channel pressure flow,

$$\frac{\partial P}{\partial x} = -\frac{3Q\mu}{2h^3} \tag{h}$$

Plot the ratio of pressure drops obtained by Eqs. (h) and (f) to show that for $\alpha < 10°$, the error involved using the lubrication approximation is very small.