CHAPTER-02	INDICES AND LOGARITHIM

LEARNING OBJECTIVES

This chapter will enable you to learn the concepts and application of:

- Indices
- Logarithms
- Inter-relationship between indices and logarithm
- Application of indices and logarithms

2.1 INTRODUCTION

We are informed of certain rules of addition and multiplication and now we take up certain higher order operations with powers and roots. These rules are essential for any serious mathematical problem. In this chapter we will discuss about various operation of indices and Logarithms.

2.2 INDEX AND BASE OF A NUMBER

If a positive integer a is multiplied by itself n times, we get a^n,

i.e. $a.a.a..........n\ times = a^n$

Then the constant a is called the base and the positive integer n is called the **index or exponent or power**. For example: $9 = 3.3 = 3^2$, here 3 is base and 2 is index .

2.3 LAWS OF INDICES

For all $a,b \in R$ and $m,n \in N$, we have the following operation on indices.

1. $a^0 = 1\ and\ a^1 = a$
2. $a^n = a.a.a.......n\ \ times$
3. $a^{n+1} = a^n.a$
4. $a^m.a^n = a^{m+n}$
5. $a^m \div a^n = a^{m-n}$
6. $(a^m)^n = a^{mn} = (a^n)^m$
7. $(ab)^n = a^n.b^n$

and $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

8. $\frac{1}{a^m} = a^{-m} \quad ; a \neq 0$

9. $a^x = a^y$ iff $x = y \quad (a \neq 1)$

10. $a^x = b^x$ iff $a = b \quad (x \neq 0)$

2.4 MEANING OF a^m

$a^m = a.a.a...(\ m\ times\ )$; means that a is multiplied m times by itself. We say that m is the power of a; where a is called base of a^m and m is called the index of a^m.

For example: Find $(7^3)^2$ and $(4^2)^3$

Solution: $(7^3)^2 = 7^{3\times2} = 7^6 = 7\times7\times7\times7\times7\times7 = 117649$

$(4^2)^3 = 4^{2\times3} = 4^6 = 4\times4\times4\times4\times4\times4 = 4096$

For example: Find a. $(2.3)^4$ b. $\left(\frac{2}{3}\right)^4$

Solution:

a. $(2.3)^4 = 2^4.3^4 = (2\times2\times2\times2).(3\times3\times3\times3) = 16\times81 = 1296$

b. $\left(\frac{2}{3}\right)^4 = \frac{2\times2\times2\times2}{3\times3\times3\times3} = \frac{16}{81}$

For example: Find : $2^3\times3^3\times4^3$

Solution: $2^3\times3^3\times4^3 = (2\times3)^3\times4^3 = (2\times3\times4)^3 = 24^3 = 13824$

2.5 POSITIVE INDICES

If n is a positive integer and 'a' positive real number but $a \neq 1$. Then, a^n is used to denote the continued product of n factors each equal to 'a' shown below

a^n = a × a × a × .... to n factor

where a is called base and n the positive index or exponent.

For example: $11^6 = 11\times11\times11\times11\times11\times11$

$(3x)^2 = 3x\times3x$

2.6 FRACTIONAL INDICES

In a positive fractional index the numerator represents the power and the denominator, called the root.

For example: $\sqrt[2]{a} = \sqrt{a} = a^{1/2}$; square root

$\sqrt[3]{a} = a^{1/3}$,cubic root

$\sqrt[n]{a} = a^{1/n}$; n^{th} root or n^{th} root radical.

and $a^{m/n} = \sqrt[n]{a^m}$, where m is power and $\sqrt[n]{\ }$ is n^{th} root radical.

In Particular **i)** $16^{\frac{1}{2}} = \sqrt[2]{16} = 4$ **ii)** $(27)^{\frac{1}{3}} = \sqrt[3]{27} = 3$ **iii)** $(16)^{\frac{3}{4}} = (16^3)^{\frac{1}{4}} = \sqrt[4]{16^3} = 8$

2.7 OPERATION WITH POWER FUNCTIONS

The two operations involved in power functions are multiplication and division. As indicated earlier power functions cannot be added or subtracted so as to derive a new resultant function.

MULTIPLICATION WITH COMMON BASE

In the case of multiplication of two or more power functions with a common base, the powers are added.

The formulae is $x^m \times x^n \times x^r = x^{m+n+r}$

This can be shown as follows:

Also $\{a \times a \times a \times a \times a\}\{a \times a \times a\} = a^8$

But remember $a^5 + a^3 \neq a^8$

DIVISION WITH COMMON BASE

In this case, the base will be raised by the difference of the indices.

The formulae is

$$x^m \div x^n = \frac{x^m}{x^n} = x^{m-n} \text{ (where } m > n\text{)}$$

and $x^m \div x^n = \dfrac{x^m}{x^n} = \dfrac{1}{x^{n-m}}$ (where $m < n$)

For example

$$\frac{x^8}{x^4} = \frac{x \times x \times x \times x \times x \times x \times x \times x}{x \times x \times x \times x} = x^{8-4} = x^4$$

and $$\frac{x^4}{x^8} = \frac{x \times x \times x \times x}{x \times x \times x \times x \times x \times x \times x \times x} = x^{4-8} = x^{-4} = \frac{1}{x^4}$$

MULTIPLICATION OF FACTORS WITH DIFFERENT BASES

The rules for this can be stated as follows :

i) $x^m \times y^m = (xy)^m$ ii) $x^m . y^m . z^m = (xyz)^m$

For example: $5^3 \times 3^3 = (5 \times 3)^3 = (15)^3$

and $5^4 \times 4^4 \times 3^4 = (5 \times 4 \times 3)^4 = (60)^4$

DIVISION OF FACTORS WITH DIFFERENT BASES

The formulae for the Purpose can be stated as follows: $\dfrac{x^m}{y^m} = \left(\dfrac{x}{y}\right)^m$

For example: $\dfrac{7^4}{5^4} = \left(\dfrac{7}{5}\right)^4$

2.8 DEFINITION OF LOGARITHM

For any number N, if $N = a^x \,(a > 0, a \neq 1)$, then the index x is called the logarithm of the number N to the base a, and we write $x = \log_a N$.

If $x = \log_a N$ is given, we can write $N = a^x$.

Thus, $N = a^x$ and $x = \log_a N (a > 0, a \neq 1)$ are equivalent relations.

i.e. $N = a^x \Leftrightarrow x = \log_a N \quad (a > 0, a \neq 1)$

Examples:

1. $3^4 = 81 \Rightarrow 4 = \log_3 81$, *i.e.* logarithm of 81 to the base 3 is equal to 4. We can write $4 = \log_3 81$ as: anti $\log_3 4 = 81$.
2. $2^3 = 8 \Rightarrow 3 = \log_2, 8$, *i.e.* logarithm of 8 to the base 2 is equal to 3. We can write $3 = \log_2 8$ as: anti $\log_2 3 = 8$.
3. $(25)^{1/2} = 5 \Rightarrow \frac{1}{2} = \log_{25} 5$, *i.e.* logarithm of 5 to the base 25 is equal to $\frac{1}{2}$. Again $\frac{1}{2} = \log_{25} 5 \Rightarrow anti \log_{25} \frac{1}{2} = 5$.
4. Since $2^6 = 64$, $4^3 = 64$, $8^2 = 64$; we can write $\log_2 64 = 6$, $\log_4 64 = 3$ $\log_8 64 = 2$.

Thus, we see that the logarithms of the same number with different bases will be different.

Remarks:

a. Two equations $a^x = n$ and $x = \log_a n$ are only transformations of each other and should be remembered to change one form of the relation into the other.

b. The logarithm of 1 to any base is zero. This is because any number raised to the power zero is one. Since $a^0 = 1 \Rightarrow \log_a 1 = 0$.

c. The logarithm of any quantity to the same base is unity. This is because any quantity raised to the power 1 is that quantity only. Since $a^1 = a \Rightarrow \log_a a = 1$.

d. Base should not be taken as 0 or 1 because a zero raised to any power is meaningless and 1 raised to any power is one only. The base of logarithm cannot be a negative number because certain values will become imaginary.

2.9 ANTI-LOGARITHM

If $\log_a N = x$, then the number N is called the anti-logarithm of x to base a and we can write $N = \text{anti} \log_a x$. For example: If $\log x = 2$, then $x = anti \log 2 = 100$.

2.10 LAWS OF LOGARITHMS

Law-1: The logarithm of the product of two quantities is equal to the sum of their logarithms taken separately, i.e. $\log_a (mn) = \log_a m + \log_a n$

Proof: Let $\log_a m = x, \log_a n = y$ and $\log_a(mn) = z$

Then, by definition, $a^x = m, a^y = n$ and $a^z = mn$.

Therefore, $a^z = mn = a^x \times a^y = a^{x+y}$;

$\therefore z = x + y, i.e.$ $\log_a(m \times n) = \log_a m + \log_a n$. (Proved)

Corollary: $\log_a(m \times n \times p \times \dots\dots\times y) = \log_a m + \log_a n + \log_a p + \dots\dots\dots + \log_a y$.

Remark: $\log_a(m+n) \neq \log_a m + \log_a n$. It is possible if $m + n = mn$.

For example: Show that $\log(1+2+3) = \log 1 + \log 2 + \log 3$.

We have, $\log(1+2+3) = \log 6 = \log(1 \times 2 \times 3) = \log 1 + \log 2 + \log 3$.

Law-2: The logarithm of the quotients of two numbers is equal to the difference of their logarithms, i.e. $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$,

Proof: Let $\log_a m = x, \log_a n = y$ and $\log_a\left(\frac{m}{n}\right) = z$.

Then by definition, $a^x = m, a^y = n$ and $a^z = \frac{m}{n}$.

Thus, $a^z = \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$ $\quad \therefore z = x - y$

$i.e., \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$. (Proved)

Law-3: The logarithm of a power of a number is the product of the power and the logarithm of the number, i.e. $\log_a(m^n) = n \log_a m$.

Proof: Let $\log_a m = x$ and $\log_a(m^n) = y$.

Then, by definition $a^x = m$ and $a^y = m^n$.

Thus, $a^y = m^n = (a^x)^n = a^{nx}$ $\quad \therefore y = nx$.

$i.e. \log_a(m^n) = n \log_a m$. (Proved)

Law-4: (Change of Base): If the logarithm of a number to any base is given, then the logarithm of the same number to any other base can be determined from the following relation:

$\log_a b = \log_c b \times \log_a c$, where $a, b, c > 0 \Rightarrow \log_c b = \frac{\log_a b}{\log_a c}$.

Proof: Let $\log_a b = x, \quad \log_c b = y$.

Then, by definition, $a^x = b$,

$\therefore a^x = c^y$ or, $c = a^{x/y}$ $\quad \therefore$ From definition, $\frac{x}{y} = \log_a c$

$\Rightarrow x = y \log_a c$

i.e. $\log_a b = \log_a b \times \log_a c$. **(Proved)**

Corollary: If we substitute $b = a$ in the above result, we get

$$\log_c a = \frac{\log_a a}{\log_a c} = \frac{1}{\log_a c}.$$

Therefore, $\log_c a \times \log_a c = 1$.

2.11 COMMON LOGARITHM

The logarithm of a number with '10' as base is called the **common logarithm** or, Briggsian logarithm of that number. When no base is mentioned, it is understood to be base 10, i.e. by the word logarithm, we generally mean common logarithm. Common logarithm $\log_{10} x$ is generally written as: $\log x$.

For example: We have $10^0 = 1 \quad \therefore \log_{10} 1 = 0$ or, $\log 1 = 0$

$10^2 = 100 \quad \therefore \log_{10} 100 = 2$ or, $\log 100 = 2$.

$10^{-1} = \frac{1}{10} = 0.1 \quad \therefore \log 0.1 = -1$.

$10^{-2} = \frac{1}{100} = 0.01 \quad \therefore \log 0.01 = -2$.

2.12 NATURAL LOGARITHM

The logarithm of a number to the base 'e' [$e = 2.718281....... \cong 2.72$] is called **natural logarithm** or, Napierian logarithm of that number. In theoretical calculations, the base 'e' is used where as for numerical calculations; the base '10' is used most conveniently. Natural logarithm $\log_e x$ is generally written as: $\ln x$

For example: $\log_e e = \ln e = 1$

We can write e in series form: $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots\cdots = 2.718281\cdots\cdots \cong 2.72$

Here e is called **Napier's constant**.

ANTILOGARITHM

If $\log M = x$, the base being 10, then M is called the antilogarithm or antilog of x, If x is given, to find M means to find the antilog of x from log-table and we write.

M=antilog of x.

2.13 CHARACTERISTICS AND MANTISSA

We may ask $\log_{10} 68 = x$ i.e. if $10^x = 68$, then $x = ?$ the answer is : $\log_{10} 68 = 1.8325$. thus log 68 consists of two parts, an integral part (which is 1) and a decimal part (0.8325). The integral part is called the **characteristic** and the decimal part (always positive is called the **mantissa** of the logarithm. The mantissa of the logarithm of a number is always found from the log-table, but the characteristic can be found by inspection.

We know, $\log 1 = 0$, $\log 10 = 1$ and all numbers consisting of one digit in the integral portion lie between 1 and 10.

$\therefore$ $\log 2, \log 3, \log 3.5, \log 7, \log 8, \log 9.8$ etc. all lie between 0 and 1.

$\therefore$ we can write $\log 3 = 0....$,

$\log 3.5 = 0......$, *i.e.* o point something;

$\log 9.8 = 0,.....$, 0 point something.

Here 0 is the characteristics.

Again, $\log 10 = 1$, $\log 100 = 2$ and all numbers consisting of two digits in the integral portion lie between 10 and 100.

$\therefore \log 10.1, \log 11, \log 12.56, ...\log 99.9$ etc. all lie between 1 and 2.

$\therefore \log 10.1 = 1:........, \log 11 = 1:......,$ *i.e.* 1 point something.

$\log 12.56 = 1:......., \log 99.9 = 1:........,$ etc.

Here 1 is the characteristic.

Proceeding as above we can show that logarithm of all numbers consisting of 3 digits in the integral portion can be expressed as 2:.....(i.e.,2 point something) in which 2 is the characteristic and so on for other numbers consisting of 4 and more than 4 digits.

Thus we arrive at the following rule:

"The characteristic of the logarithm of a number greater than 1 is positive and is one less than the number of digits."

Let us now consider the logarithm of the numbers less than 1 but greater than 0.

We have, $\log 1 = 0, \log \frac{1}{10} = \log 1 - \log 10 = -1$ and a number having no zero immediately after the decimal point lies between $\frac{1}{10} = 0.1$ and 1;0.246, 0.9087 lie between 0.1 and 1.

$\therefore$ $\log 0.246$ and $\log 0.9087$ lie between $\log 0.1$ and $\log 1$, i.e. between –1 and 0.

Hence $\log 0.246 = -a$ decimal$= -1 + a$ decimal and $\log 0.9087 = -a$ decimal $= -1 + a$ decimal. To express a negative decimal, say-0.903 as $-1 + a$ decimal, we write

$$-0.903 = -1 + 1 - 0.903 = -1 + (1 - 0.903)$$

$$= -1 + 0.097 \text{ which is written as } \bar{1}.097$$

and $\quad -0.457 = -1 + 1 - 0.457 = -1 + 0.543$ which is written as $\bar{1}.543$.

Hence the characteristic of $\log 0.246$ or, $\log 0.9087$ is -1, *i.e.* the characteristic of the logarithm of all numbers lying between 1 and 1 is –1 or $\bar{1}$.

All numbers having only one zero immediately after the decimal point lie between 0.01 and 0.1.

Also $\log 0.01 = \log \frac{1}{100} = -2$ and $\log 0.1 = -1$.

∴ Logarithms of all those numbers lying between 0.01 and 0.1 lie between –2 and –1. For instance, 0.0546 lies between 0.01 and 0.1

∴ $\log 0.0546$ lies between –2 and –1; i.e. $\log 0.0546 = -1 : .....$

Hence $\log 0.0546$ can be expressed as: $-2 + \text{a}$ decimal.

Here –2 or $\bar{2}$ is the characteristic.

Thus the characteristic of the logarithms of all numbers having one zero immediately after the decimal point is –2 or $\bar{2}$. Similarly, the characteristic of the logarithm of all numbers having two zeros immediately after the decimal point (these numbers lie between 0.001 and 0.01) is –3 or $\bar{3}$ and so on.

Remark: 1. Students must note that $\bar{2}.785$ is not the same as -2.785.

$$\bar{2}.785 = -2 + 0.785 \text{ and } -2.785 = -2 - .785.$$

2. The characteristic of the logarithm of a number may be positive or negative but its mantissa (which is found from log-table) is always positive.

FINDING MANTISSA FROM LOG TABLE

Generally four-figure log tables are used, a copy of which is given at the end of the book. From this table we can find the mantissa of the common logarithm of all numbers of 4 digits or less. In the log-table, figures from 10 to 99 (bold type figure) are arranged vertically in the first column. In the next ten columns, under each of the digits 0, 1, 2, …9, four figures are written, then after a vertical line, in the next nine columns, each under each of the digits 1, 2, 3, …..9, one or two figures are written and this part is called the **mean difference table.**

Suppose we are to find the value of log5687. As the number consists of 4 digits, its characteristics must be 3. For the mantissa, we take the first two digits 56 and in the 1^{st} column finding out 56, we place a scale horizontally under it and then note the number on this scale headed vertically above by the 3^{rd} digit 8, which is found to be 7543. For the 4^{th} digit, 7 the increment in the mantissa is found under the heading 7 in the mean difference table in the row beginning with 56. This increment is found to be 5. Adding 5 to 7543 we get 7548. Thus the mantissa of log5687 is 0.7548. Hence log5648=3.7548.

Instead of log5648, if we take log56.87, we see that the number of digits in the integral portion is two (i.e., 56) and hence the characteristic is 1. To find mantissa, we shall consider the number 5687 (omitting the decimal point). ∴ Mantissa of log56.87 is 0.7548, which is found above. ∴log56.87=1.7548 Similarly, log5.687 = 0.7548, log0.5687 = 1.7548, Log0.05648=2.7548 and so on.

Thus we see that the mantissa of the logarithm of all numbers having the same digits in the same order is the same, i.e. it is unaffected by the position of the decimal point. To find the logarithm of numbers consisting of more than 4 digits, characteristic is first determined by counting the number of digits in the integral portion. Then to find mantissa, we retain the first four digits; for the fifths digit, if it is 5 or greater than 5, we add 1 to the 4^{th} digits; if however it is less than 5, we strike it off without adding 1. Other digits from the 6^{th} place are to be ignored. Take log76.509. Here we strike off the digit 9 and add 1 to the 4^{th} digit 0.

∴log76.509=log76.51=1.8838.

FINDING ANTILOGARITHM FROM ANTILOG TABLE

When log76.51=1.8838, antilog of 1.8838=76.51, by definition of antilogarithm. In the antilog table figures from 0.00 to 0.99 (bold types figures) are arranged vertically in the first column and the arrangements in the other columns are the same as in the log table (logarithm of numbers)

The method of finding antilog from the antilog table is similar to that of finding logarithms from log table. First we find the number corresponding to a given mantissa and then place the decimal point with reference to the characteristic.

Suppose we are to find antilog of 2.5687.

We find the number corresponding to the mantissa 0.5687 from the antilog table; it is 3698+6=3704. Now since the given characteristic is 2, the required antilog is 370.4 Thus antilog of 2.5687=370.4 [check: log 370.4=2.5687 (neatly)]

Similarly, antilog of 1.5687=37.04, antilog of 0.5687=3.704, antilog of 1.5687=0.3704, antilog of 2.5687 =0.03704, etc.

ILLUSTRATIONS (INDICES)

Illustration - 01: Find the values of a. $3^5 \times 3^3$ b. $3^5 + 3^{10}$ c. $\frac{3^5}{3^3}$ d. $3^5 - 3^3$

Solution:

a. $3^5 \times 3^3 = 3^{5+3} = 3^8 = 3 \times 3 \times 3 \times ....8 \ times = 6561$

b. $3^5 + 3^3 = (3 \times 3 \times 3 \times 3 \times 3) + (3 \times 3 \times 3) = 243 + 27 = 270$

c. $\frac{3^5}{3^3} = 3^{5-3} = 3^2 = 3 \times 3 = 9$

d. $3^5 - 3^3 = (3 \times 3 \times 3 \times 3 \times 3) - (3 \times 3 \times 3) = 243 - 27 = 216$

Illustration - 02: Simplify $\frac{(3x^3)^2}{(2x^2)^3} + \frac{(5x^2)^3}{(4x^3)^2}$

Solution: $\frac{(3x^3)^2}{(2x^2)^3} + \frac{(5x^2)^3}{(4x^3)^2} = \frac{3^2(x^3)^2}{2^3(x^2)^3} + \frac{5^3(x^2)^3}{4^2(x^3)^2} = \frac{9x^6}{8x^6} + \frac{125x^6}{16x^6}$

$= \frac{9}{8}x^{6-6} + \frac{125}{16}x^{6-6} = \frac{9}{8} + \frac{125}{16} (\because x^0 = 1) = \frac{18+125}{16} = \frac{143}{16}$

Illustration - 03: Simplify: $\frac{(3x^3)^2}{(2x^2)^3} + \frac{(5x^2)^3}{(4x^3)^2}$

Solution : $\frac{3^2 x^6}{2^3 x^6} + \frac{5^3 x^6}{4^2 x^6}$

$= \frac{9x^6}{8x^6} + \frac{125x^6}{16x^6} = \frac{9}{8}x^{6-6} + \frac{125}{16}x^{6-6} = \frac{9}{8} + \frac{125}{16} = \frac{18+125}{16} = \frac{143}{16}$

Illustration - 04: Find the value (i) $16^{3/4}$ (ii) $25^{-3/2}$

Solution : (i) $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8$ **(ii)** $25^{-\frac{3}{2}} = \frac{1}{25^{\frac{3}{2}}} = \left(\frac{1}{\sqrt{25}}\right)^3 = \frac{1}{5^3} = \frac{1}{125}$

Illustration - 05: Find $\sqrt[2]{16}$ *and* $\sqrt[3]{27}$

Solution: $\sqrt[2]{16} = 16^{1/2} = (4^2)^{1/2} = 4^{2 \times \frac{1}{2}} = 4^1 = 4$

and $\sqrt[3]{27} = (27)^{1/3} = (3^3)^{1/3} = 3^{3 \times \frac{1}{3}} = 3^1 = 3$

Illustration - 06: Find the value i. $16^{3/4}$ ii. $25^{-3/2}$

Solution: i. $16^{3/4} = \left(2^4\right)^{3/4} = 2^{4\times\frac{3}{4}} = 2^3 = 8$

ii. $25^{-3/2} = \dfrac{1}{25^{3/2}} = \dfrac{1}{\left(5^2\right)^{3/2}} = \dfrac{1}{5^{2\cdot\frac{3}{2}}} = \dfrac{1}{5^3} = \dfrac{1}{125}$

Illustration - 07: If $a^x = b,\ b^y = c,\ c^z = a$ then show that $xyz = 1$

Solution: Given, $a^x = b \Rightarrow \left(c^z\right)^x = b \Rightarrow c^{zx} = b \Rightarrow \left(b^y\right)^{zx} = b \Rightarrow b^{yzx} = b = b^1 \Rightarrow xyz = 1$

Illustration - 08: If $a = 3^{1/3} + 3^{-\frac{1}{3}}$ Prove that $3a^3 - 9a = 10$.

Solution: Here $a = 3^{\frac{1}{3}} + 3^{-\frac{1}{3}}$ Cubing both sides, we get

$$a^3 = \left(3^{\frac{1}{3}} + 3^{-\frac{1}{3}}\right)^3 \Rightarrow a^3 = \left(3^{\frac{1}{3}}\right)^3 + \left(3^{-\frac{1}{3}}\right)^3 + 3.3^{\frac{1}{3}}.3^{-\frac{1}{3}}\left(3^{\frac{1}{3}} + 3^{-\frac{1}{3}}\right)$$

$$\Rightarrow a^3 = 3 + 3^{-1} + 3.3^{\frac{1}{3}-\frac{1}{3}}.a \Rightarrow a^3 = 3 + 3^{-1} + 3a$$

$$\Rightarrow a^3 = 3 + \frac{1}{3} + 3a \Rightarrow a^3 = \frac{9 + 1 + 9a}{3}$$

$$\Rightarrow 3a^3 = 9a + 10 \Rightarrow 3a^3 - 9a = 10 \text{ (Showed)}$$

Illustration - 09: Simplify $\dfrac{2^n + 2^{n-1}}{2^{n+1} - 2^n}$

Solution : $\dfrac{2^n + 2^{n-1}}{2^{n+1} - 2^n} = \dfrac{2^n + 2^n.2^{-1}}{2^n.2^1 - 2^n} = \dfrac{2^n + 2^n.\frac{1}{2}}{2^n.2 - 2^n} = \dfrac{2^n(1+\frac{1}{2})}{2^n(2-1)} = \dfrac{2^n.\frac{3}{2}}{2^n.1} = \dfrac{3}{2}$

Illustration - 10: Show that $\left(\dfrac{x^b}{x^c}\right)^a \times \left(\dfrac{x^c}{x^a}\right)^b \times \left(\dfrac{x^a}{x^b}\right)^c = 1$

Solution . L.H.S. $\left(\dfrac{x^b}{x^c}\right)^a \times \left(\dfrac{x^c}{x^a}\right)^b \times \left(\dfrac{x^a}{x^b}\right)^c$

$$= \left(x^{b-c}\right)^a \times \left(x^{c-a}\right)^b \times \left(x^{a-b}\right)^c = x^{ab-ac} \times x^{bc-ab} \times x^{ac-bc}$$

$$= x^{ab-ac+bc-ab+ac-bc} = x^0$$

$= 1$ $\therefore$ R.H.S = 1 (Showed)

Illustration - 11: Simplify : $\left(\frac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \times \left(\frac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \times \left(\frac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$

Solution : $\left(\frac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \times \left(\frac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \times \left(\frac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$

$= \left(x^{a+b}\right)^{a^2-ab+b^2} \times \left(x^{b+c}\right)^{b^2-bc+c^2} \times \left(x^{c+a}\right)^{c^2-ca+a^2}$

$= x^{(a+b)(a^2-ab+b^2)} \times x^{(b+c)(b^2-bc+c^2)} \times x^{(c+a)(c^2-ca+a^2)}$

$= x^{a^3+b^3} \times x^{b^3+c^3} \times x^{c^3+a^3} \quad = x^{a^3+b^3+b^3+c^3+c^3+a^3} \quad = x^{2a^3+2b^3+2c^3} \quad = x^{2(a^3+b^3+c^3)}$

Illustration - 12: Simplify : $\frac{4^y.20^{x-1}.12^{x-y}.15^{x+y-2}}{16^x.5^{2x+y}.9^{x-1}}$

Solution : $\frac{2^{2y}.2^{2(x-1)}.5^{x-1}.3^{x-y}.2^{2(x-y)}.3^{x+y-2}.5^{x+y-2}}{2^{4x}.5^{2x+y}.3^{2(x-1)}}$

$= \frac{2^{2y+2x-2+2x-2y}.3^{x-y+x+y-2}.5^{x-1+x+y-2}}{2^{4x}.5^{2x+y}.3^{2x-2}} = \frac{2^{4x-2}.3^{2x-2}.5^{2x+y-3}}{2^{4x}.3^{2x-2}.5^{2x+y}}$

$= 2^{4x-2-4x}.3^{2x-2-2x+2}.5^{2x+y-3-2x-y} \quad = 2^{-2}.3^0.5^{-3} \quad = \frac{1}{2^2}.1.\frac{1}{5^3} \quad = \frac{1}{4}.1.\frac{1}{125} = \frac{1}{500}$ (Ans.)

Illustration - 13: Simplify : $\left[1-\{1-(1-x^3)^{-1}\}^{-1}\right]^{-1/3}$

Solution : $\left[1-\left\{1-\frac{1}{1-x^3}\right\}^{-1}\right]^{-1/3} = \left[1-\left\{\frac{1-x^3-1}{1-x^3}\right\}^{-1}\right]^{-1/3}$

$= \left[1-\left\{\frac{-x^3}{1-x^3}\right\}^{-1}\right]^{-1/3}$

$= \left[1-\left\{\frac{x^3}{x^3-1}\right\}^{-1}\right]^{-1/3}$

$= \left[1-\frac{x^3-1}{x^3}\right]^{-1/3}$

$= \left[\frac{x^3-x^3+1}{x^3}\right]^{-1/3} = \left[\frac{1}{x^3}\right]^{-1/3} = (x^3)^{\frac{1}{3}} = x$ Ans.

ILLUSTRATIONS (LOGARITHMS)

Illustration - 14: If $\log_a \sqrt{2} = \frac{1}{8}$, find the value of a.

Solution: Given $\log_a \sqrt{2} = \frac{1}{8}$

From definition, we get $a^{\frac{1}{8}} = \sqrt{2} \Rightarrow a = \left(\sqrt{2}\right)^8 = 2^4 = 16.$

Illustration - 15: If $\log_x \frac{1}{2} = \frac{1}{3}$, find the value of x.

Solution: Given $\log_x \frac{1}{2} = \frac{1}{3}$ $\therefore$ By definition, $x^{1/3} = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$

Illustration - 16: Find the value of (i) $\log_8 128$, (ii) $\log_{5\sqrt{5}}(125)$, (iii) $\log_3 \log_3 (27)$.

Solution:

(i) Let $\log_8 128 = x$. Then by definition of logarithm, we have

$$8^x = 128 \Rightarrow \left(2^3\right)^x = 2^7 \Rightarrow 2^{3x} = 2^7$$

$$\therefore 3x = 7 \Rightarrow x = \frac{7}{3} \quad \therefore \log_8 128 = \frac{7}{3}.$$

(ii) Let $\log_{5\sqrt{5}}(125) = x$; then by definition, $\left(5\sqrt{5}\right)^x = 125$

$\Rightarrow \left(5\sqrt{5}\right)^x = 5^2 \times 5 = 5^2\left(\sqrt{5}\right)^2 = \left(5\sqrt{5}\right)^2 \quad \therefore x = 2$ i.e. $\log_{5\sqrt{5}}(125) = 2$.

(iii) $\log_3 \log_3 (27) = \log_3 \log_3 \left(3^3\right) = \log_3 3 \log_3 3 = \log_3 3.1 = \log_3 3 = 1 \quad [\because \log_a a = 1]$

Illustration-17: Find the value of x in the following cases:

(i) $\log_5 x = 3$, **(ii)** $\log_{\sqrt{8}} x = -\frac{2}{3}$, **(iii)** $\log_x \left(\frac{1}{9}\right) = 4$, **(iv)** $\log_x 125 = 3$.

Solution: **(i)** Given $\log_5 x = 3 \Rightarrow \frac{\log x}{\log 5} = 3 \Rightarrow \log x = 3\log 5 \Rightarrow \log x = \log 5^3$

$$\therefore x = 5^3 \Rightarrow x = 125 \text{ [Ans.]}$$

(ii) Given $\log_{\sqrt{8}} x = -\frac{2}{3} \Rightarrow \frac{\log x}{\log\sqrt{8}} = -\frac{2}{3} \Rightarrow \log x = -\frac{2}{3}\log\sqrt{8} \Rightarrow \log x = \log\left(\sqrt{8}\right)^{-\frac{2}{3}}$

$\therefore x = \left(8^{\frac{1}{2}}\right)^{-\frac{2}{3}} \Rightarrow x = (8)^{-\frac{1}{3}} \Rightarrow x = \left(2^3\right)^{-\frac{1}{3}} \Rightarrow x = 2^{-1} \Rightarrow x = \frac{1}{2}$ [Ans.]

(iii) Given $\log_x\left(\frac{1}{9}\right) = 4 \Rightarrow \frac{\log\frac{1}{9}}{\log x} = 4 \Rightarrow \log\frac{1}{9} = 4\log x \Rightarrow \log\frac{1}{9} = \log x^4$

$\therefore \frac{1}{9} = x^4 \Rightarrow \sqrt{\frac{1}{9}} = \sqrt{x^4} \Rightarrow x^2 = \frac{1}{3} \Rightarrow \sqrt{x^2} = \sqrt{\frac{1}{3}} \Rightarrow x = \frac{1}{\sqrt{3}}$ [Ans.]

(iv) Given $\log_x 125 = 3 \Rightarrow \frac{\log 125}{\log x} = 3 \Rightarrow \log 125 = 3\log x \Rightarrow \log 125 = \log x^3$

$\therefore 125 = x^3 \Rightarrow 5^3 = x^3 \Rightarrow x = 5$ [Ans.]

Illustration-18: Prove that:

(a) $\log\frac{41}{35} + \log 70 - \log\frac{41}{2} + 2\log 5 = 2$

(b) $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} + \log\frac{1}{2} = 0$

(c) $\log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243} = \log 2$

Solution: (a) LHS= $\log\frac{41}{35} + \log 70 - \log\frac{41}{2} + 2\log 5$

$= \log\frac{41}{35} + \log 70 - \log\frac{41}{2} + \log 5^2 = \log\left(\frac{41}{35}\times 70\times\frac{2}{41}\times 25\right)$

$= \log(2\times 2\times 25) = \log 100 = 2 = \text{RHS}$ [Proved]

(b) LHS= $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} + \log\frac{1}{2}$

$= \log\left(\frac{2^4}{3\times 5}\right)^7 + \log\left(\frac{5^2}{3\times 2^3}\right)^5 + \log\left(\frac{3^4}{5\times 2^4}\right)^3 + \log\frac{1}{2}$

$= \log\left(\frac{2^{28}}{3^7\times 5^7}\right) + \log\left(\frac{5^{10}}{3^5\times 2^{15}}\right) + \log\left(\frac{3^{12}}{5^3\times 2^{12}}\right) + \log\frac{1}{2}$

$$= \log\left(\frac{2^{28}}{3^7 \times 5^7} \times \frac{5^{10}}{3^5 \times 2^{15}} \times \frac{3^{12}}{5^3 \times 2^{12}} \times \frac{1}{2}\right)$$

$= \log 1 = 0 = \text{RHS}$ [Proved]

(c) LHS= $\log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243}$

$$= \log\frac{75}{16} - \log\left(\frac{5}{9}\right)^2 + \log\frac{32}{243} = \log\left(\frac{75}{16}\right) - \log\left(\frac{25}{81}\right) + \log\left(\frac{32}{243}\right)$$

$= \log\left(\frac{75}{16} \times \frac{81}{25} \times \frac{32}{243}\right) = \log 2 = \text{RHS}$ [Proved]

Illustration-19: If $y = \log_{10}(\log_{10} x)$, find (a) y when $x = 10$ and (b) x when $y = 1$

Solution: We have, $y = \log_{10}(\log_{10} x)$.................(1)

a. When $x = 10$, $y = \log_{10}(\log_{10} 10) = \log_{10} 1 = 0$ $[\log_{10} 1 = 0]$

b. When $y = 1$, from (1), we get $1 = \log_{10}(\log_{10} x)$;

$\therefore \log_{10} x = 10^1 = 10; \quad \therefore x = (10)^{10}$

Illustration-20: If a, b, c are any three consecutive positive integers, prove that $\log(1+ac) = 2\log b$.

Solution: Since a, b, c are any three consecutive positive integers,

$\therefore \quad b - a = 1 \Rightarrow a = b - 1$ and $c - b = 1 \Rightarrow c = b + 1$

$\therefore \quad a = b - 1$ and $c = b + 1$

$\therefore \quad$ LHS= $\log(1+ac) = \log\{1 + (b-1)(b+1)\}$

$= \log\{1 + b^2 - 1^2\} = \log b^2 = 2\log b = \text{RHS}$ [Proved].

Illustration-21: Given that $3\log_{10}(x^2 y) = 4 + 2\log_{10} x - \log_{10} y$ where x and y are both positive. Express y in terms of x. If $x - y = 2\sqrt{6}$, find the value of x and y.

Solution: We have, $3\log_{10}(x^2 y) = 4 + 2\log_{10} x - \log_{10} y$

Or, $\quad 3(\log_{10} x^2 + \log_{10} y) = 4 + 2\log_{10} x - \log_{10} y$

Or, $\quad 3(2\log_{10} x + \log_{10} y) = 4 + 2\log_{10} x - \log_{10} y$

Or, $\quad 6\log_{10} x + 3\log_{10} y = 4 + 2\log_{10} x - \log_{10} y$

Or, $\quad 3\log_{10} y + \log_{10} y = 4 + 2\log_{10} x - 6\log_{10} x$

Or, $\quad 4\log_{10} y = 4 - 4\log_{10} x$

Or, $\quad \log_{10} y = 1 - \log_{10} x$

Or, $\quad \log_{10} y = \log_{10} 10 - \log_{10} x \quad [\because \log_{10} 10 = 1]$

Or, $\quad \log_{10} y = \log_{10}\left(\frac{10}{x}\right) \Rightarrow y = \frac{10}{x}$, which is the required expression.

Again, $x - y = 2\sqrt{6} \Rightarrow x - \frac{10}{x} = 2\sqrt{6} \Rightarrow \frac{x^2 - 10}{x} = 2\sqrt{6}$

$$\Rightarrow x^2 - 10 = 2\sqrt{6}x \Rightarrow x^2 - 2\sqrt{6}x - 10 = 0$$

$$\therefore x = \frac{-(-2\sqrt{6}) \pm \sqrt{(-2\sqrt{6})^2 - 4.1(-10)}}{2.1} = \frac{2\sqrt{6} \pm \sqrt{24+40}}{2} = \frac{2\sqrt{6} \pm \sqrt{64}}{2}$$

$$= \frac{2\sqrt{6} \pm 8}{2} = 4 + \sqrt{6}\text{, taking positive sign } [\because x > 0]$$

$$\therefore x = 4 + \sqrt{6}$$

$$\therefore y = x - 2\sqrt{6} = 4 + \sqrt{6} - 2\sqrt{6} = 4 - \sqrt{6}.$$

Illustration-22: If x, y, z are in G. P., then prove that $\log_a x + \log_a z = \frac{2}{\log_y a}$

Solution: Since x, y, z are in G.P., $\quad \therefore \frac{y}{x} = \frac{z}{y} \Rightarrow xz = y^2$..............(1)

Taking logarithm of both sides of (1) with the base a, we get

$$\log_a x + \log_a z = 2\log_a y = 2 \times \frac{1}{\log_y a} \qquad \left[\log_a b = \frac{1}{\log_b a}\right]$$

Hence $\log_a x + \log_a z = \frac{2}{\log_y a}$

Illustration-23: (a) If $\log_a x = m$, $\log_a y = n$, what are the values of

(i) $\log_a\left(\frac{x}{y}\right)$, **(ii)** $\log_a\left(\frac{x^2}{y}\right)$, **(iii)** $\log_a\left(\frac{x}{y^2}\right)$, **(iv)** $\log_a\left(\frac{x^3}{y}\right)$.

(b) Given that $\log_{10} y = 2 - \log_{10} x$, express y in the form ax^n.

(c) If $3 + \log_{10} x = 2\log_{10} y$, express x in terms of y.

Solution: (a) (i) Given $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y = m - n$ [Ans.]

(ii) Given $\log_a\left(\frac{x^2}{y}\right) = \log_a x^2 - \log_a y = 2\log_a x - \log_a y = 2m - n$ [Ans.]

(iii) Given $\log_a\left(\frac{x}{y^2}\right) = \log_a x - \log_a y^2 = \log_a x - 2\log_a y = m - 2n$ [Ans.]

(iv) Given $\log_a\left(\frac{x^3}{y}\right) = \log_a x^3 - \log_a y = 3\log_a x - \log_a y = 3m - n$ [Ans.]

(b) Given $\log_{10} y = 2 - \log_{10} x \Rightarrow \log_{10} x + \log_{10} y = 2 \Rightarrow \log_{10}(xy) = 2$

$$\Rightarrow \frac{\log(xy)}{\log 10} = 2 \Rightarrow \log(xy) = 2\log 10 \Rightarrow \log(xy) = \log(10)^2$$

$$\therefore (xy) = (10)^2 \Rightarrow xy = 100 \Rightarrow y = \frac{100}{x} \Rightarrow y = 100x^{-1} \quad \text{[Ans.]}$$

(c) Given $3 + \log_{10} x = 2\log_{10} y \Rightarrow 3 = \log_{10} y^2 - \log_{10} x \Rightarrow 3 = \log_{10}\frac{y^2}{x} \Rightarrow 3 = \frac{\log\frac{y^2}{x}}{\log 10}$

$$\Rightarrow 3\log 10 = \log\frac{y^2}{x} \Rightarrow \log(10)^3 = \log\frac{y^2}{x} \Rightarrow 1000 = \frac{y^2}{x} \Rightarrow 1000x = y^2$$

$$\therefore x = \frac{y^2}{1000} \quad \textbf{[Ans.]}$$

Illustration-24: Prove that $\log_3 5 \times \log_{25} 27 = \frac{3}{2}$.

Solution: LHS$=\log_3 5 \times \log_{25} 27 = \frac{\log_a 5}{\log_a 3} \times \frac{\log_a 27}{\log_a 25}$ $\left[\because \log_b m = \frac{\log_a m}{\log_a b}\right]$

$$= \frac{\log_a 5}{\log_a 3} \times \frac{\log_a (3^3)}{\log_a (5^2)} = \frac{\log_a 5}{\log_a 3} \times \frac{3\log_a 3}{2\log_a 5} = \frac{3}{2} = \text{RHS} \quad \text{[Proved]}$$

Illustration-25: Prove that: $\frac{\log\sqrt{27}+\log 8+\log\sqrt{1000}}{\log 120}=\frac{3}{2}$

Solution: LHS$=\frac{\log(27)^{1/2}+\log(2^3)+\log(1000)^{1/2}}{\log(3\times 2^2\times 10)}=\frac{\frac{1}{2}\log(3^3)+3\log 2+\frac{1}{2}\log(10^3)}{\log 3+2\log 2+\log 10}$

$$=\frac{\frac{3}{2}\log 3+\frac{3}{2}.2\log 2+\frac{3}{2}\log 10}{\log 3+2\log 2+\log 10}=\frac{\frac{3}{2}(\log 3+2\log 2+\log 10)}{\log 3+2\log 2+\log 10}=\frac{3}{2}=\text{RHS [Proved]}$$

Illustration-26: Show that **(a)** $\frac{\log 343}{1+\frac{1}{2}\log\left(\frac{49}{4}\right)+\frac{1}{3}\log\left(\frac{1}{125}\right)}=3$

(b) $\frac{1}{6}\sqrt{\frac{3\log 1728}{\frac{1}{2}\log 36+\frac{1}{3}\log 8}}=\frac{1}{2}$.

Solution:

LHS$=\frac{\log 343}{1+\frac{1}{2}\log\left(\frac{49}{4}\right)+\frac{1}{3}\log\left(\frac{1}{125}\right)}=\frac{\log 7^3}{1+\frac{1}{2}\log\left(\frac{7}{2}\right)^2+\frac{1}{3}\log\left(\frac{1}{5}\right)^3}=\frac{3\log 7}{1+\frac{1}{2}.2\log\left(\frac{7}{2}\right)+\frac{1}{3}.3\log\left(\frac{1}{5}\right)}$

$$=\frac{3\log 7}{1+\log\left(\frac{7}{2}\right)+\log\left(\frac{1}{5}\right)}=\frac{3\log 7}{1+\log\left(\frac{7}{2}\times\frac{1}{5}\right)}=\frac{3\log 7}{1+\log\left(\frac{7}{10}\right)}=\frac{3\log 7}{1+\log 7-\log 10}$$

$$=\frac{3\log 7}{1+\log 7-1}=\frac{3\log 7}{\log 7}=3=\text{RHS [Proved]}$$

(b) LHS$=\frac{1}{6}\sqrt{\frac{3\log 1728}{\frac{1}{2}\log 36+\frac{1}{3}\log 8}}=\frac{1}{6}\sqrt{\frac{3\log(12)^3}{\frac{1}{2}\log(6)^2+\frac{1}{3}\log(2)^3}}=\frac{1}{6}\sqrt{\frac{9\log 12}{\log 6+\log 2}}$

$$=\frac{1}{6}\sqrt{\frac{9\log 12}{\log 12}}=\frac{1}{6}\times\sqrt{9}=\frac{1}{6}\times 3=\frac{1}{2}=\text{RHS [Proved]}$$

Illustration-27: Prove that **(a)** $\log_y x.\log_z y.\log_x z=1$

(b) $\log_y(x^3)\log_z(y^3)\log_x(z^3)=27$

(c) $\log_y(\sqrt{x})\log_z(y^3)\log_x(\sqrt[3]{z^2})=1$

Solution: **(a)** LHS= $\log_y x.\log_z y.\log_x z = \frac{\log x}{\log y}\times\frac{\log y}{\log z}\times\frac{\log z}{\log x} = 1 =$ RHS (Proved)

(b) LHS= $\log_y(x^3).\log_z(y^3).\log_x(z^3)$

$$= \frac{\log x^3}{\log y}\times\frac{\log y^3}{\log z}\times\frac{\log z^3}{\log x} = \frac{3\log x}{\log y}\times\frac{3\log y}{\log z}\times\frac{3\log z}{\log x}$$

$= 3\times3\times3 = 27 =$RHS [Proved]

(c) LHS= $\log_y(\sqrt{x}).\log_z(y^3).\log_x(\sqrt[3]{z^2})$

$$= \frac{\log x^{\frac{1}{2}}}{\log y}\times\frac{\log y^3}{\log z}\times\frac{\log z^{\frac{2}{3}}}{\log x} = \frac{\frac{1}{2}\log x}{\log y}\times\frac{3\log y}{\log z}\times\frac{\frac{2}{3}\log z}{\log x}$$

$= \frac{1}{2}\times3\times\frac{2}{3} = 1 =$ RHS [Proved]

Illustration-28: Prove that, (a) $x = a^{1/\log_x a}$, (b) $\frac{1}{\log_x xy}+\frac{1}{\log_y xy} = 1$,

(c) $\frac{1}{\log_6 24}+\frac{1}{\log_8 24}+\frac{1}{\log_{12} 24} = 2$,

Solution: (a) Let $m = a^{1/\log_x a}$

$\Rightarrow \log m = \log a^{\frac{1}{\log_x a}}$ [Taking 'log' on both sides]

$\Rightarrow \log m = \frac{1}{\log_x a}.\log a \Rightarrow \log m = \frac{1}{\frac{\log a}{\log x}}.\log a$

$\Rightarrow \log m = \frac{\log x}{\log a}.\log a \Rightarrow \log m = \log x \Rightarrow m = x$

$\therefore x = a^{1/\log_x a}$ [Proved]

(b) LHS= $\frac{1}{\log_x xy}+\frac{1}{\log_y xy}$

$= \log_{xy} x + \log_{xy} y$ $\quad (\because \frac{1}{\log_y x} = \log_x y)$

$= \log_{xy} xy = 1 =$RHS [Proved]

(c) $\text{LHS}=\frac{1}{\log_6 24}+\frac{1}{\log_8 24}+\frac{1}{\log_{12} 24}$

$=\log_{24}6+\log_{24}8+\log_{24}12=\log_{24}(6\times8\times12)=\log(576)$

$=\log_{24}(24)^2=2\log_{24}24=2=$ RHS [Proved]

Illustration-29: **(a)** If $x=\frac{e^y-e^{-y}}{e^y+e^{-y}}$, show that $y=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$.

(b) If $(3.7)^x=(.37)^y=1000$, show that $\frac{1}{x}-\frac{1}{y}=\frac{1}{3.}$

(c) If $a^{2x-3}.b^{2x}=a^{6-x}.b^{5x}$, prove that $3\log a=x\log\frac{a}{b}$

Solution: **(a)** Given $x=\frac{e^y-e^{-y}}{e^y+e^{-y}}\Rightarrow\frac{1}{x}=\frac{e^y+e^{-y}}{e^y-e^{-y}}$

$\Rightarrow\frac{1+x}{1-x}=\frac{(e^y+e^{-y})+(e^y-e^{-y})}{(e^y+e^{-y})\ (e^y-e^{-y})}$ [By componendo-dividendo]

$\Rightarrow\frac{1+x}{1-x}=\frac{e^y+e^{-y}+e^y-e^{-y}}{e^y+e^{-y}-e^y+e^{-y}}$ $\Rightarrow\frac{1+x}{1-x}=\frac{2e^y}{2e^{-y}}\Rightarrow\frac{1+x}{1-x}=e^{y+y}$

$\Rightarrow\log_e\left(\frac{1+x}{1-x}\right)=\log e^{2y}\Rightarrow\log_e\left(\frac{1+x}{1-x}\right)=2y\log_e e$

$\Rightarrow 2y\log_e e=\log_e\left(\frac{1+x}{1-x}\right)\Rightarrow 2y=\frac{\log_e\frac{1+x}{1-x}}{\log_e e}$ $\Rightarrow y=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$ [Proved]

(b) Given $(3.7)^x=(.37)^y=1000$

Now $(3.7)^x=1000\Rightarrow\log(3.7)^x=\log1000$

$\Rightarrow x\log3.7=3\Rightarrow x=\frac{3}{\log3.7}$

Also $(0.37)^x=1000\Rightarrow\log(3.7)^x=\log1000$

$\Rightarrow x\log0.37=3\Rightarrow x=\frac{3}{\log0.37}$

$\therefore \text{LHS}=x^{-1}-y^{-1}=\frac{1}{x}-\frac{1}{y}$

$$= \frac{1}{\frac{3}{\log 3.7}} - \frac{1}{\frac{3}{\log 0.37}} = \frac{\log 3.7}{3} - \frac{\log 0.37}{3}$$

$$= \frac{\log 3.7 - \log 0.37}{3} = \frac{\log\left(\frac{3.7}{0.37}\right)}{3} = \frac{\log 10}{3} = \frac{1}{3} = \text{RHS} \quad \text{[Proved]}$$

(c) Here $a^{2x-3}.b^{2x} = a^{6-x}.b^{5x}$

$\Rightarrow \log\left(a^{2x-3}.b^{2x}\right) = \log\left(a^{6-x}.b^{5x}\right)$ [Taking 'log' on both sides]

$\Rightarrow \log a^{2x-3} + \log b^{2x} = \log a^{6-x} + \log b^{5x}$

$\Rightarrow (2x-3)\log a + 2x\log b = (6-x)\log a + 5x\log b$

$\Rightarrow 2x\log a - 3\log a + 2x\log b = 6\log a - x\log a + 5x\log b$

$\Rightarrow 3x\log a - 3x\log b = 9\log a \qquad \Rightarrow 3x(\log a - \log b) = 9\log a$

$\Rightarrow x(\log a - \log b) = 3\log a \qquad \Rightarrow x\log\frac{a}{b} = 3\log a \Rightarrow 3\log a = x\log\frac{a}{b}$ [proved]

Illustration-30: (a) If $\log\frac{x+y}{3} = \frac{1}{2}(\log x + \log y)$, show that $\frac{x}{y} + \frac{y}{x} = 7$

(b) If $x^2 + y^2 = 7xy$, prove that $\log\left\{\frac{1}{3}(x+y)\right\} = \frac{1}{2}(\log x + \log y)$

(c) If $x^3 + y^3 = 0$ and $x + y = 0$, prove that $\log(x+y) = \frac{1}{2}(\log x + \log y + \log 3)$

Solution: (a) Given $\log\frac{x+y}{3} = \frac{1}{2}(\log x + \log y)$

$$\Rightarrow \log\frac{x+y}{3} = \frac{1}{2}\log(xy) \Rightarrow \log\frac{x+y}{3} = \log(xy)^{\frac{1}{2}}$$

$$\Rightarrow \frac{x+y}{3} = (xy)^{\frac{1}{2}} \Rightarrow \left(\frac{x+y}{3}\right)^2 = \left\{(xy)^{\frac{1}{2}}\right\}^2$$

$$\Rightarrow \frac{x^2 + 2xy + y^2}{9} = xy \Rightarrow x^2 + 2xy + y^2 = 9xy$$

$$\Rightarrow x^2 + y^2 = 7xy \Rightarrow \frac{x^2 + y^2}{xy} = 7 \Rightarrow \frac{x^2}{xy} + \frac{y^2}{xy} = 7$$

$$\Rightarrow \frac{x}{y} + \frac{y}{x} = 7 \quad \text{[Proved]}$$

(b) Given $x^2 + y^2 = 7xy$

$\Rightarrow (x+y)^2 - 2xy = 7xy \qquad \Rightarrow (x+y)^2 = 9xy \Rightarrow \sqrt{(x+y)^2} = \sqrt{9xy}$

$\Rightarrow (x+y) = 3\sqrt{xy} = \frac{(x+y)}{3} = (xy)^{\frac{1}{2}} \quad \Rightarrow \log\left\{\frac{(x+y)}{3}\right\} = \log(xy)^{\frac{1}{2}}$

$\Rightarrow \log\left\{\frac{1}{3}(x+y)\right\} = \frac{1}{2}\log(xy) \Rightarrow \log\left\{\frac{1}{3}(x+y)\right\} = \frac{1}{2}(\log x + \log y)$ [Proved]

(c) Given $x^3 + y^3 = 0$

$\Rightarrow (x+y)^3 - 3xy(x+y) = 0 \Rightarrow (x+y)^3 = 3xy(x+y)$

$\Rightarrow \frac{(x+y)^3}{(x+y)} = 3xy \Rightarrow (x+y)^2 = 3xy \Rightarrow \sqrt{(x+y)^2} = \sqrt{3xy}$

$\Rightarrow (x+y) = (3xy)^{\frac{1}{2}} \Rightarrow \log(x+y) = \frac{1}{2}\log(3xy)$

$\Rightarrow \log(x+y) = \frac{1}{2}(\log x + \log y + \log 3)$ [Proved]

Illustration-31: (a) If $\frac{\log a}{q-r} = \frac{\log b}{r-p} = \frac{\log c}{p-q}$, prove that $a^{q+r}b^{r+p}c^{p+q} = 1$.

(b) If $\frac{\log x}{1} = \frac{\log y}{2} = \frac{\log z}{5}$. Show that $x^4y^3z^{-2} = 1$.

(c) If $p = \log_a bc$, $q = \log_b ca$, $r = \log_c ab$, show that $pqr = p+q+r+2$

Solution: (a) Let $\frac{\log a}{q-r} = \frac{\log b}{r-p} = \frac{\log c}{p-q} = k$

$\therefore \frac{\log a}{q-r} = k \Rightarrow \log a = k(q-r) \Rightarrow (q+r)\log a = k(q-r)(q+r)$

$\Rightarrow \log a^{q+r} = k(q^2 - r^2)$........................(1)

Similarly. $\log b^{r+p} = k(r^2 - p^2)$............................(2)

And $\log c^{p+q} = k(p^2 - q^2)$......................(3)

Adding (1), (2) and (3), We get

$\log a^{q+r} + \log b^{r+p} + \text{loc}^{p+q} = k(q^2 - r^2) + k(r^2 - p^2) + k(p^2 - q^2)$

$\Rightarrow \log(a^{q+r}.b^{r+p}.c^{p+q}) = k(q^2 - r^2 + r^2 - p^2 + p^2 - q^2)$

$\Rightarrow \log(a^{q+r}.b^{r+p}.c^{p+q}) = k(0) \qquad \Rightarrow \log(a^{q+r}.b^{r+p}.c^{p+q}) = 0$

$\Rightarrow \log(a^{q+r}.b^{r+p}.c^{p+q}) = \log 1 \Rightarrow a^{q+r}.b^{r+p}.c^{p+q} = 1$ [Proved]

(b) Let $\frac{\log x}{1} = \frac{\log y}{2} = \frac{\log z}{5} = k$

$\therefore \frac{\log x}{1} = k \Rightarrow \log x = k$

$\Rightarrow 4\log x = 4k \Rightarrow \log x^4 = 4k$..........................(1)

$\therefore \frac{\log y}{2} = k \Rightarrow \log y = 2k \Rightarrow 3\log y = 3\times 2k \Rightarrow \log y^3 = 6k$...................(2)

$\therefore \frac{\log z}{5} = k \Rightarrow \log z = 5k \Rightarrow -2\log z = -2\times 5k \Rightarrow \log z^{-2} = -10k$.........(3)

Adding (1), (2) and (3), We get

$\log x^4 + \log y^3 + \log z^{-2} = 4k + 6k - 10k$

$\Rightarrow \log(x^4y^3z^{-2}) = 0 \Rightarrow \log(x^4y^3z^{-2}) = \log 1$

$\therefore x^4y^3z^{-2} = 1$ [Proved]

(c) Given $p = \log_a bc$

$\Rightarrow 1+p = 1+\log_a bc \Rightarrow 1+p = \log_a a + \log_a bc \Rightarrow 1+p = \log_a abc$

$\Rightarrow \frac{1}{1+p} = \frac{1}{\log_a abc} \Rightarrow \frac{1}{1+p} = \log_{abc} a$.....................(1)

Similarly, $\frac{1}{1+q} = \log_{abc} b$................................(2)

And $\frac{1}{1+r} = \log_{abc} c$............................(3)

Adding (1), (2) and (3), we get

$$\frac{1}{1+p} + \frac{1}{1+q} + \frac{1}{1+r} = \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$\Rightarrow \frac{(1+q)(1+r) + (1+p)(1+r) + (1+p)(1+q)}{(1+p)(1+q)(1+r)} = \log_{abc} abc$$

$$\Rightarrow \frac{1+r+q+qr+1+r+p+pr+1+q+pq}{(1+q+p+pq)(1+r)} = 1$$

$$\Rightarrow \frac{3+2r+2q+2p+pq+qr+pr}{1+r+q+qr+p+pr+pq+pqr} = 1$$

$$\Rightarrow 3+2r+2q+2p+pq+qr+pr = 1+r+q+qr+p+pr+pq+pqr$$

$$\Rightarrow p+q+r+2 = pqr$$

$$\Rightarrow pqr = p+q+r+2$$ [Proved]

Illustration-32: (i) If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $1 + xyz = 2yz$.

(ii) If $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$, prove that $1 + abc = 2bc$.

Solution: (i) L.H.S $= xyz + 1$

$$= \log_{2a} a.\log_{3a} 2a.\log_{4a} 3a + 1$$

$$= \frac{\log a}{\log 2a}.\frac{\log 2a}{\log 3a}.\frac{\log 3a}{\log 4a} + 1 = \frac{\log a}{\log 4a} + 1 = \frac{\log a + \log 4a}{\log 4a}$$

$$= \frac{\log(a \times 4a)}{\log 4a} = \frac{\log 4a^2}{\log 4a} = \frac{\log(2a)^2}{\log 4a} = \frac{2\log 2a}{\log 4a}$$

$$= 2.\frac{\log 2a}{\log 4a} = \frac{\log(2a)^2}{\log 4a} = \frac{2\log 2a}{\log 4a} = 2.\frac{\log 2a}{\log 3a}.\frac{\log 3a}{\log 4a} = 2yz = R.H.S$$

(ii) L.H.S $= 1 + abc$

$$= 1 + \frac{\log 12}{\log 24}.\frac{\log 24}{\log 36}.\frac{\log 36}{\log 48}$$

$$= 1 + \frac{\log 12}{\log 48} = \frac{\log 48 + \log 12}{\log 48} = \frac{\log(48 \times 12)}{\log 48} \quad [\because \log a + \log b = \log ab]$$

$$= \frac{\log 576}{\log 48} = \frac{\log(24)^2}{\log 48} = \frac{2\log 24}{\log 48}$$

$$= 2.\frac{\log 24}{\log 36}.\frac{\log 36}{\log 48} = 2bc = R.H.S$$

Illustration-33: Solve: $6^{3-4x}4^{x+5} = 8$. Given $\log 2 = 0.3010; \log 3 = 0.4771$.

Solution: We have $6^{3-4x}4^{x+5} = 8$.

Taking logarithm of both sides, we get

$(3-4x)\log 6 + (x+5)\log 4 = \log 8, \Rightarrow 3\log 6 - 4x\log 6 + x\log 4 + 5\log 4 = \log 8$

$\Rightarrow 3\log 6 + 5\log 4 - \log 8 = 4x\log 6 - x\log 4 \Rightarrow x(4\log 6 - \log 4) = 3\log 6 + 5\log 4 - \log 8$

$$\therefore x = \frac{3\log 6 + 5\log 4 - \log 8}{4\log 6 - \log 4} = \frac{3\log(2\times 3) + 5\log 2^2 - \log 2^3}{4\log(2\times 3) - \log 2^2}$$

$$= \frac{3(\log 2 + \log 3) + 10\log 2 - 3\log 2}{4(\log 2 + \log 3) - 2\log 2} = \frac{10\log 2 + 3\log 3}{2\log 2 + 4\log 3}$$

$$= \frac{10 \times 0.3010 + 3 \times 0.4771}{2 \times 0.3010 + 4 \times .04771} = \frac{3.0100 + 1.4313}{0.6020 + 1.9084} = \frac{4.4413}{2.5104} = 1.7692$$

Illustration-34: If $a = \log_{12} m$, and $b = \log_{18} m$, prove that $\dfrac{a-2b}{b-2a} = \log_3 2$.

Solution: $a - 2b = \log_{12} m - 2\log_{18} m$

$$= \frac{1}{\log_m 12} - \frac{2}{\log_m 18} = \frac{\log_m 18 - 2\log_m 12}{\log_m 12.\log_m 18}$$

and $\quad b - 2a = \log_{18} m - 2\log_{12} m = \dfrac{1}{\log_m 18} - \dfrac{2}{\log_m 12} = \dfrac{\log_m 12 - 2\log_m 18}{\log_m 18.\log_m 12}$

$$\therefore \frac{a-2b}{b-2a} = \frac{\log_m 18 - 2\log_m 12}{\log_m 12 - 2\log_m 18} = \frac{\log_m(2\times 3^2) - 2\log_m(3\times 2^2)}{\log_m(3\times 2^2) - 2\log_m(2\times 3^2)}$$

$$= \frac{\log_m 2 + 2\log_m 3 - 2\log_m 3 - 4\log_m 2}{\log_m 3 + 2\log_m 2 - 2\log_m 2 - 4\log_m 3}$$

$$= \frac{-3\log_m 2}{-3\log_m 3} = \frac{\log_m 2}{\log_m 3} = \log_3 2. \qquad \because \log_a b = \frac{\log_x b}{\log_x a}$$

Illustration-35: If $2\log_g N = p$, $\log_2 2N = q$ and $q - p = 4$, prove that $N = 512$.

Solution: Given $\quad 2\log_g N = p$..................(1)

$\log_2 2N = q$....................(2)

and $q - p = 4$....................(3)

From (1), $\log_g(N^2) = p \quad \therefore 8^p = N^2 \Rightarrow 2^{3p} = N^2$.............(4)

From (2), $2^q = 2N$ or $N = \dfrac{2^q}{2} = 2^{q-1}$

Substituting this value of N in (4), we get

$$2^{3p} = (2^{q-1})^2 = 2^{2q-2}$$

$\therefore 3p = 2q - 2 \Rightarrow 2q - 3p = 2$................(5)

Multiplying (3) by 3, we get

$3q - 3p = 12$..........................(6)

Subtracting (5) from (6), $q = 10$

Hence $N = 2^{q-1} = 2^{10-1} = 2^9 = 512$ [Proved]

Illustration-36: Solve for $x : \log_x(8x-3) - \log_x 4 = 2$.

Solution: We have $\log_x(8x-3) - \log_x 4 = 2 \Rightarrow \log_x \dfrac{8x-2}{4} = 2$

By definition, $x^2 = \dfrac{8x-3}{4} \Rightarrow 4x^2 = 8x-3$

Or, $4x^2 - 8x + 3 = 0$ or, $4x^2 - 6x - 2x + 3 = 0$

Or, $2x(2x-3) - 1(2x-3) = 0 \Rightarrow (2x-3)(2x-1) = 0$

$\therefore$ Either $2x - 3 = 0$ or, $2x - 1 = 0$ $\quad \therefore x = \dfrac{3}{2}$ or, $\dfrac{1}{2}$.

Illustration-37: Solve the following equations:

(a) $\log_2 x + \log_4 x + \log_{16} x = \dfrac{21}{4}$. **(b)** $\log_{10} x + \log_{10}(x-3) = 1$.

(c) $\log(x+3)^2 - 2 = \log\dfrac{1}{x^2}$

Solution: (a) Given $\log_2 x + \log_4 x + \log_{16} x = \dfrac{21}{4}$

$\Rightarrow \dfrac{\log x}{\log 2} + \dfrac{\log x}{\log 4} + \dfrac{\log x}{\log 16} = \dfrac{21}{4}$ $\qquad \Rightarrow \dfrac{\log x}{\log 2} + \dfrac{\log x}{2\log 2} + \dfrac{\log x}{4\log 2} = \dfrac{21}{4}$

$\Rightarrow \dfrac{4\log x + 2\log x + \log x}{4\log 2} = \dfrac{21}{4}$ $\qquad \Rightarrow \dfrac{7\log x}{\log 2} = \dfrac{21}{1} \Rightarrow \dfrac{\log x}{\log 2} = \dfrac{3}{1}$

$\Rightarrow \log x = 3\log 2 \Rightarrow \log x = \log 2^3 \Rightarrow x = 2^3 \quad \therefore x = 8$ **[Ans.]**

(b) Given $\log_{10} x + \log_{10}(x-3) = 1$

$\log_{10}[x(x-3)] = 1 \Rightarrow \dfrac{\log(x^2-3x)}{\log 10} = 1 \Rightarrow \log(x^2-3x) = \log 10 \Rightarrow (x^2-3x) = 10$

$\Rightarrow x^2 - 3x - 10 = 0 \Rightarrow x^2 - 5x + 2x - 10 = 0 \Rightarrow (x-5)(x+2) = 0$

$\therefore x = 5$, **[Ans.]** [$x \neq -2$, because $\log(-2)$ is undefined]

(c) Given $\log_{10}(x+3)^2 - 2 = \log_{10}\dfrac{1}{x^2}$

$\Rightarrow 2\log_{10}(x+3) - 2 = \log_{10} x^{-2}$ $\qquad \Rightarrow 2[\log_{10}(x+3) - 1] = -2\log_{10} x$

$\Rightarrow \log_{10}(x+3) - 1 = -\log_{10} x$. $\qquad \Rightarrow \log_{10}(x+3) + \log_{10} x = 1 \Rightarrow \log_{10}[[x+3]x] = 1$

$$\Rightarrow \frac{\log(x^2+3x)}{\log 10} = 1 \Rightarrow \log(x^2+3x) = \log 10$$

$$\Rightarrow (x^2+3x) = 10 \Rightarrow x^2+3x-10 = 0$$

$$\Rightarrow x^2+5x-2x-10 = 0 \Rightarrow (x+5)(x-2) = 0$$

$\therefore x = -5,\ 2$ **[Ans.]**

Illustration-38: Solve the equations:

(i) $\log_{1/2}[\log_x(\log_4 32)] = 2$.   **(ii)** $\log_3[\log_2(\log_3 x)] = 1$.

Solution: (i) Given $\log_{1/2}[\log_x(\log_4 32)] = 2$

$$\Rightarrow \frac{\log[\log_x(\log_4 32)]}{\log\frac{1}{2}} = 2 \qquad \Rightarrow \log[\log_x(\log_4 32)] = 2\log\frac{1}{2}$$

$$\Rightarrow \log[\log_x(\log_4 32)] = \log\left(\frac{1}{2}\right)^2 \qquad \Rightarrow \log_x(\log_4 32) = \left(\frac{1}{2}\right)^2 \Rightarrow \frac{\log(\log_4 32)}{\log x} = \frac{1}{4}$$

$$\Rightarrow \log(\log_4 32) = \frac{1}{4}\log x \Rightarrow \log(\log_4 32) = \log x^{\frac{1}{4}}$$

$$\Rightarrow \log_4 32 = x^{\frac{1}{4}} \Rightarrow \frac{\log 32}{\log 4} = x^{\frac{1}{4}} \Rightarrow \frac{5\log 2}{2\log 2} = x^{\frac{1}{4}}$$

$$\frac{5}{2} = x^{\frac{1}{4}} \Rightarrow \left(x^{\frac{1}{4}}\right)^4 = \left(\frac{5}{2}\right)^4 \quad \therefore x = \frac{625}{16} \quad \text{[Ans.]}$$

(ii) Given $\log_3[\log_2(\log_3 x)] = 1$

$$\Rightarrow \frac{\log[\log_2(\log_3 x)]}{\log 3} = 1 \qquad \Rightarrow \log[\log_2(\log_3 x)] = \log 3 \Rightarrow \log_2(\log_3 x) = 3$$

$$\Rightarrow \frac{\log(\log_3 x)}{\log 2} = 3 \Rightarrow \log(\log_3 x) = 3\log 2 \qquad \Rightarrow \log(\log_3 x) = \log 2^3 \Rightarrow \log_3 x = 2^3$$

$$\Rightarrow \frac{\log x}{\log 3} = 8 \Rightarrow \log x = 8\log 3 \Rightarrow \log x = \log 3^8$$

$\therefore x = 3^8 \Rightarrow x = 6561$ [Ans.]

Illustration-39: Solve $\log_x 2.\log_{x/16} 2 = \log_{x/64} 2$.

Solution: We have, $\log_x 2.\log_{x/16} 2 = \log_{x/64} 2$

Or, $\dfrac{1}{\log_2 x} \times \dfrac{1}{\log_2 \dfrac{x}{16}} = \dfrac{1}{\log_2 \dfrac{x}{64}}$ $\left[\log_a b = \dfrac{1}{\log_b a}\right]$

Or, $\log_2 x.\log_2 \dfrac{x}{16} = \log_2 \dfrac{x}{64}$

Or, $\log_2 x(\log_2 x - 4\log_2 2) = \log_2 x - \log_2(2^6)$

Or, $\log_2 x(\log_2 x - 4.1) = \log_2 x - 6\log_2 2$

Or, $y(y-4) = y-6$, where $y = \log_2 x$

Or, $y^2 - 5y + 6 = 0 \Rightarrow y^2 - 2y - 3y + 6 = 0 \Rightarrow y(y-2) - 3(y-2) = 0$

Or, $(y-2)(y-3) = 0$ $\therefore y = 2$ or 3

If $y = 2$, then $\log_2 x = 2$ $\therefore x = 2^2 = 4$

If $y = 3$. then $\log_2 x = 3$ $\therefore x = 2^3 = 8$

Hence the required solutions are: $x = 4,8$

Illustration-40: (a) Given $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find the value of $\log 45$.
(b) Given $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find the value of $\log 0.0075$.

Solution: (a) $\log 45 = \log 3^2.5 = \log 3^2 + \log 5 = 2\log 3 + \log 5 = 2\times 0.4771 + \log\dfrac{10}{2}$

$= .9542 + \log 10 - \log 2 = 0.9542 + 1 - 0.3010 = 1.9542 - 0.3010$

$= 1.6532.$

(b) $\log 0.0075 = \log\dfrac{75}{10000} = \log 75 - \log 10000$

$= \log 3\times 5^2 - \log(10)^4 = \log 3 + 2\log 5 - 4\log 10$

$= \log 3 + 2\times\left(\log\dfrac{10}{2}\right) - 4 = \log 3 + 2\log 10 - 2\log 2 - 4$

$= 0.4771 + 2 - 2\times 0.3010 - 4 = 2.4771 - 4.6020 = -2.1249.$

Illustration-41: Using log table, find the value of (a) 0.8176×13.46, (b) $\dfrac{17.51 \times 43.27}{0.0034}$.

Solution: (a) Let $A = 0.8176 \times 13.64$

$$\therefore \log A = \log 0.8176 + \log 13.64 = -0.0875 + 1.1348 = 1.0473$$

$$\therefore A = \text{ antilog } (1.0473) = 11.15$$

(b) Let $A = \dfrac{17.51 \times 43.27}{0.0034}$; then

$$\log A = \log 17.51 + \log 43.27 - \log 0.0034$$

$$= 1.2433 + 1.6362 - (-2.4685) = 5.3480$$

$$\therefore A = \text{ antilog } (5.3480) = 222844$$

Illustration-42: Find the value of (a) $(3.814)^{1/5}$, (b) $\sqrt[11]{598.92}$, (c) $\sqrt[7]{\dfrac{1}{1.235}}$ correct to three decimal places.

Solution: (a) Let $x = (3.814)^{1/5}$. Then, we have

$$\log x = \frac{1}{5}\log 3.814 = \frac{1}{5} \times 0.58137 = 0.116274$$

Hence $x = \text{antilog}(0.116274) \Rightarrow x = 1.307$ i.e. $(3.814)^{1/5} = 1.307$.

(b) Let $A = \sqrt[11]{598.92} = (598.92)^{1/11}$.

$$\therefore \log A = \frac{1}{11}\log 598.92 = \frac{1}{11} \times 2.7774 = 0.2525$$

$\therefore A = \text{ antilog}(0.2525) = 1.788$. Hence $\sqrt[11]{598.92} = 1.778$ (approx)

(c) Let $A = \sqrt[7]{\dfrac{1}{1.235}} = \left(\dfrac{1}{1.235}\right)^{1/7}$

$$\therefore \log A = \frac{1}{7}\log\frac{1}{1.235} = \frac{1}{7}(\log 1 - \log 1.235)$$

$$= \frac{1}{7}(0 - 0.0917) = -\frac{1}{7} \times 0.0917 = -0.0131 = -1 + 1 - 0.0131 = \bar{1}.9869;$$

$$\therefore A = \text{ antilog}\left(\bar{1}.9869\right) = 0.9703.$$

Illustration-43: Find with the help of log tables, the value of

(a) $\dfrac{1}{5.7002\times 6.0818\div 69.732}$ **(b)** $\dfrac{2.389\times 0.004679}{0.00556\times 52.14}$

Solution: (a) Let $x=\dfrac{1}{5.7002\times 6.0818\div 69.732}$

$\Rightarrow \log x=\log\dfrac{1}{5.7002\times 6.0818\div 69.732}$ [Taking 'log' on both sides]

$\Rightarrow \log x=\log 1-\log(5.7002\times 6.0818\div 69.732)$

$\Rightarrow \log x=0-(\log 5.7002+\log 6.0818-\log 69.732)$

$\Rightarrow \log x=-\log 5.7002-\log 6.0818+\log 69.732$

$\Rightarrow \log x=-0.7559-0.7840+1.8434$ [Using 'log' tables]

$\Rightarrow \log x=0.3035$ $\Rightarrow x=\text{Antilog}0.3035$

$\Rightarrow x=\text{Antilog}0.3035$ [using 'antilog' tables] $\Rightarrow x=2.01$ [Ans.]

(b) Let $x=\dfrac{2.389\times 0.004679}{0.00556\times 52.14}$

$\Rightarrow \log x=\log\left(\dfrac{2.389\times 0.004679}{0.00556\times 52.14}\right)$ [Taking 'log' on both sides]

$\Rightarrow \log x=\log 2.389+\log 0.004679-\log 0.00556-\log 52.14$

$\Rightarrow \log x=0.3782+\bar{3}.6702-\bar{3}.7451-1.7172$ [Using 'log' tables]

$\Rightarrow \log x=0.3782-3+.6702-(-3+.7451)-1.7172$

$\Rightarrow \log x=0.3782-3+.6702+3-.7451-1.7172$

$\Rightarrow \log x=-1.4139\Rightarrow \log x=\bar{2}.5861\Rightarrow x=\text{Antilog}\,\bar{2}.5861$

$\Rightarrow x=0.03856$ [Using 'antilog' tables] $\therefore x=0.03856$ [Ans.]

Illustration-44: Simplify by using log tables:

(a) $\dfrac{\sqrt{85.82}-\sqrt[4]{9172}}{\sqrt[5]{125.7}}$ **(b)** $\dfrac{(.0437)^{\frac{2}{3}}\times(1.407)^{2}}{(.0015)^{\frac{1}{3}}\times(1.235)^{\frac{1}{7}}}$

Solution: (a) Let $\dfrac{\sqrt{85.82}-\sqrt[4]{9172}}{\sqrt[5]{125.7}}=\dfrac{x-y}{z}$

$\therefore x=\sqrt{85.82}\Rightarrow \log x=\dfrac{1}{2}\log 85.82$ [Taking 'log' on both sides]

$\Rightarrow \log x = \frac{1}{2} \times 1.9336$ [Using 'log' tables]

$\Rightarrow \log x = 0.9668 \Rightarrow x = \text{Antilog} 0.9668$

$\Rightarrow x = 9.2640$ [Using 'antilog' tables]

$\therefore y = \sqrt[4]{9172} \Rightarrow \log y = \frac{1}{4} \log 9172$ [Taking 'log' on both sides]

$\Rightarrow \log y = \frac{1}{4} \times 3.9625$ [Using 'log' tables]

$\Rightarrow \log y = 0.9906 \Rightarrow y = \text{Antilog} 0.9906$

$\Rightarrow y = 9.7864$ [Using 'antilog' tables]

$\therefore z = \sqrt[5]{125.7} \Rightarrow \log z = \frac{1}{5} \log 125.7$ [Taking 'log' on both sides]

$\Rightarrow \log z = \frac{1}{5} \times 2.0993$ [Using 'log' tables]

$\Rightarrow \log z = 0.4199 \Rightarrow z = \text{antilog } 0.4199$

$\Rightarrow z = 2.6297$ [Using 'antilog' tables]

$\therefore \frac{\sqrt{85.82} - \sqrt[4]{9172}}{\sqrt[5]{125.7}} = \frac{x - y}{z} = \frac{9.2640 - 9.7864}{2.6297} = -0.1986$ [Ans.]

(b) Let $x = \frac{(.0437)^{\frac{2}{3}} \times (1.407)^2}{(.0015)^{\frac{1}{3}} \times (1.235)^{\frac{1}{7}}}$

$\Rightarrow \log x = \log\left[\frac{(.0437)^{\frac{2}{3}} \times (1.407)^2}{(.0015)^{\frac{1}{3}} \times (1.235)^{\frac{1}{7}}}\right]$ [Taking 'log' on both sides]

$\Rightarrow \log x = \frac{2}{3} \log .0437 + 2 \log 1.407 - \frac{1}{3} \log 0.0015 - \frac{1}{7} \log 1.235$

$\Rightarrow \log x = \frac{2}{3}\left(\bar{2}.6405\right) + 2 \times .1483 - \frac{1}{3}\left(\bar{3}.1761\right) - \frac{1}{7}(.0917)$ [Using log tables]

$\Rightarrow \log x = \frac{2}{3}(-2 + .6405) + .2966 - \frac{1}{3}(-3 + .1761) - 0.0131$

$\Rightarrow \log x = .3185$

$\Rightarrow x = \text{Antilog } 0.3185$

$\Rightarrow x = 2.082$ [Using 'antilog' tables]

$\therefore x = 2.082$ [Ans.]

Illustration-45: Evaluation with the help of logarithmic tables: $(125)^{1/10}\times\frac{0.001834}{0.043160}$, correct to the nearest integer.

Solution: Let $x=(125)^{1/10}\times\frac{0.001834}{0.043160}$

$$\therefore \log x=\frac{1}{10}\log 125+\log 0.001834-\log 0.043160$$

$$=\frac{1}{10}\times 2.0969+\bar{3}.2634-\bar{2}.6351=\bar{2}.83799=\bar{2}.8380 \text{ (approx.)}$$

$$\therefore x=\text{antilog}\left(\bar{2}.8380\right)=0.06887=0, \quad \text{correct to the nearest integer.}$$

Illustration-46: Evaluate with the help of logarithm tables $\left\{\frac{(0.32)^9\times(625)^4}{(0.00432)^2\times(0.3125)^3\times 8}\right\}^{1/4}$, correct to the nearest thousands.

Solution: Let $x=\left\{\frac{(0.32)^9\times(625)^4}{(0.00432)^2\times(0.3125)^3\times 8}\right\}^{1/4}$

$$\therefore \log x=\frac{1}{4}\{9\log 0.32+4\log 625-(2\log 0.00432+3\log 0.3125+\log 8)\}$$

$$=\frac{1}{4}\left\{9\times\bar{1}.5051+4\times 2.7959-\left(2\times\bar{3}.6355+3\times\bar{1}.4949+0.9031\right)\right\}$$

$$=\frac{1}{4}\left\{\bar{5}.5459+11.1836-\left(\bar{5}.2710+\bar{2}.4847+0.9031\right)\right\}$$

$$=\frac{1}{4}\left\{6.7295-\bar{6}.6588\right\}=\frac{1}{4}\times 12.077=3.0177;$$

$\therefore x=\text{antilog}\ (3.0177)=1042=1000,$ correct to the nearest thousands.

Illustration-47: Find the square root of $\frac{\sqrt[3]{0.0125}\times\sqrt{31.15}}{0.00081}$ [Using log tables]

Solution: Let $x=\frac{\sqrt[3]{0.0125}\times\sqrt{31.15}}{0.00081}$

$$\Rightarrow \sqrt{x}=\sqrt{\frac{\sqrt[3]{0.0125}\times\sqrt{31.15}}{0.00081}}$$

$$\Rightarrow \log\sqrt{x} = \log\left(\frac{\sqrt[3]{0.0125}\times\sqrt{31.15}}{0.00081}\right)^{\frac{1}{2}} \quad \text{[Taking 'log' on both sides]}$$

$$\Rightarrow \log\sqrt{x} = \frac{1}{2}\log\left(\frac{\sqrt[3]{0.0125}\times\sqrt{31.15}}{0.00081}\right)$$

$$\Rightarrow \log\sqrt{x} = \frac{1}{2}\left[\log\sqrt[3]{0.0125} + \log\sqrt{31.15} - \log 0.00081\right]$$

$$\Rightarrow \log\sqrt{x} = \frac{1}{2}\left[\frac{1}{3}\log 0.0125 + \frac{1}{2}\log 31.15 - \log 0.00081\right]$$

$$\Rightarrow \log\sqrt{x} = \frac{1}{2}\left[\frac{1}{3}\times\bar{2}.0969 + \frac{1}{2}\times 1.4935 - \bar{4}.9085\right] \quad \text{[Using 'log' tables]}$$

$$\Rightarrow \log\sqrt{x} = \frac{1}{2}\left[\frac{1}{3}\times(-2+.0969) + \frac{1}{2}\times 1.4935 - (-4+.9085)\right]$$

$$\Rightarrow \log\sqrt{x} = \frac{1}{2}[3.2039] \qquad \Rightarrow \log\sqrt{x} = 1.6020$$

$$\Rightarrow \sqrt{x} = \text{Antilog}\,1.6020 \qquad \Rightarrow \sqrt{x} = 39.99 \quad \text{[Using 'antilog' tables]} \quad \therefore \sqrt{x} = 39.99$$

Illustration-48: Pareto law of income for a certain place is $N = \frac{5\times 10^{10}}{x^{12}}$, where x is income level, and N is the number of persons earning incomes Tk. X and over. Find the number of persons earning Tk. 327500 and over. You can use logarithm tables.

Solution: For $x = 327500$,

$$N = \frac{5\times 10^{10}}{x^{1.2}} \qquad \Rightarrow N = \frac{5\times 10^{10}}{(327500)^{1.2}} \quad \text{[Putting the value of x]}$$

$$\Rightarrow \log N = \log 5 + 10\log 10 - 1.2\log 327500$$

$$\Rightarrow \log N = .6990 + 10\times 1 - (12\times 5.5152)$$

$$\Rightarrow \log N = 4.0807$$

$$\Rightarrow N = \text{anti}\log 4.0807$$

$$\Rightarrow N = 12043 \quad \text{(approx)} \quad \text{[Ans.]}$$

Illustration-49: Find the value of N, when $x = 3362$ in the following:

$$N(2.5)^3 = \frac{0.00603\times(4.6378)^{-2}}{x^{1.2}}$$

Solution: For $x = 3362$,

$$N(2.5)^3 = \frac{0.00603\times(4.6378)^{-2}}{x^{1.2}}$$

$$\Rightarrow N(2.5)^3 = \frac{0.00603\times(4.6378)^{-2}}{(3362)^{1.2}} \quad \text{[Putting the value of x]}$$

$$\Rightarrow \log N(2.5)^3 = \log\frac{0.00603\times(4.6378)^{-2}}{(3362)^{1.2}}$$

$$\Rightarrow \log N + \log(2.5)^3 = \log 0.00603 + \log(4.6378)^{-2} - \log(3362)^{1.2}$$

$$\Rightarrow \log N + 3\log(2.5) = \log 0.00603 - 2\log(4.6378) - 1.2\log 3362$$

$$\Rightarrow \log \text{N} + 3\times.3989 = \bar{3}.7803 - 2\times.6663 - 1.2\times3.5266$$

$$\Rightarrow \log \text{N} + 1.1937 = -3 + .7803 - 1.3326 - 4.2319$$

$$\Rightarrow \log \text{N} = -8.9779$$

$$\Rightarrow \log \text{N} = \bar{9}.0221 \qquad \Rightarrow \text{N} = \text{anti}\log \bar{9}.0221 \quad \Rightarrow \text{N} = .000000001052 \quad \textbf{[Ans.]}$$

Illustration-50: Use log tables to find the values of x (correct to three places of decimal) if x satisfies the equation:

$$\frac{20}{14.7} = \left[\frac{0.0613}{\text{x}}\right]^{1.32}$$

Solution: Given $\frac{20}{14.7} = \left[\frac{0.0613}{\text{x}}\right]^{1.32}$

$$\Rightarrow \log\frac{20}{14.7} = \log\left[\frac{0.0613}{\text{x}}\right]^{1.32} \qquad \Rightarrow \log 20 - \log 14.7 = 1.32(\log 0.0613 - \log \text{x})$$

$$\Rightarrow 1.3010 - 1.1673 = 1.32\left(\bar{2}.7875 - \log \text{x}\right) \qquad \Rightarrow 0.1337 = 1.32(-2 + .7875 - \log \text{x})$$

$$\Rightarrow 0.1337 = 1.32(-1.2125 - \log \text{x}) \qquad \Rightarrow 0.1337 = -1.6005 - 1.32\log \text{x}$$

$$\Rightarrow 1.32\log \text{x} = 1.7342 \qquad \Rightarrow \log \text{x} = \frac{-1.7342}{1.32}$$

$$\Rightarrow \log \text{x} = -1.3138 = -2 + 2 - 1.3138 = -2 + 0.6862$$

$$\Rightarrow \log x = \bar{2}.6862 \qquad \Rightarrow \text{x} = \text{anti}\log \bar{2}.6862$$

$$\Rightarrow \text{x} = 0.0486 \quad \text{[Ans.]}$$

BRIEF REVIEW

Definition

- **Index/Exponent:** If a positive integer a is multiplied by itself n times, we get a^n,

  i.e. $a.a.a..........n \ times = a^n$

  Then the constant a is called the base and the positive integer n is called the index or exponent or power. For example: $9 = 3.3 = 3^2$, here 3 is base and 2 is index .
- **Fractional Indices:** In a positive fractional index the numerator represents the power and the denominator, called the root. For example: (i) $\sqrt[2]{a} = \sqrt{a} = a^{1/2}$; square root ; (ii) $\sqrt[3]{a} = a^{1/3}$,cubic root (iii) $\sqrt[n]{a} = a^{1/n}$; n^{th} root or n^{th} root radical.
- **Logarithm:** For any number N, if $N = a^x \left(a > 0, a \neq 1\right)$, then the index x is called the logarithm of the number N to the base a, and we write $x = \log_a N$.

  For examples: If $3^4 = 81 \Rightarrow 4 = \log_3 81$, *i.e.* logarithm of 81 to the base 3 is equal to 4.
- **Common Logarithm:** The logarithm of a number with '10' as base is called the commor logarithm Common logarithm $\log_{10} x$ is generally written as: $\log x$.

  For example: We know $10^2 = 100 \quad \therefore \log_{10} 100 = 2$ or, $\log 100 = 2$.
- **Natural Logarithm:** The logarithm of a number to the base 'e' $[e = 2.718281....... \cong 2.72]$ is called natural logarithm of that number. Natural logarithm $\log_e x$ is generally written as: $\ln x$

  For example: (i) $\log_e e = \ln e = 1$ (ii) $y = e^x \Rightarrow x = \log_e y = \ln x$
- **Anti Logarithm:** If $\log_a N = x$, then the number N is called the anti-logarithm of x to base a and we can write $N = \text{anti} \log_a x$. For example: If $\log x = 2$, then $x = anti \log 2 = 100$.
- **Characteristics and Mantissa:** The logarithm of a number consists of two parts: (i) integral part (ii) decimal or fractional part. The whole part or the integral part is called the characteristic and the positive decimal part is called the mantissa. The characteristic of the logarithm of a number may be positive or negative but its mantissa (which is found from log-table) is always positive. For examples: (i) We have $500 = 5 \times 10^2 \Rightarrow \log 500 = 2 + \log 5$, here 2 is called characteristic and $\log 5$ is called mantissa of $\log 500$.(ii) We have $\log_{10} 68 = 1.8325$, here 1 is called characteristic and 0.8325 is called mantissa of $\log_{10} 68$..

Quiz Questions

Multiple Choice Questions (Indices)

1. What is the value of 2^0 ?

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) 2

2. *If* $a \in R, a \neq 0$ and $m, n \in N$ then $\frac{a^m}{a^{-n}} = ?$

(a) $m+n$ (b) a^{m-n} (c) $m-n$ (d) a^{m+n}

3. What is the value of $\sqrt[3]{27}$?

(a) 3 (b) 9 (c) -3 (d) 81

4. What is the index of $2^5 \times 2^2$?

(a) 2^7 (b) 14 (c) 7 (d) 2^3

5. When $x > 0;\ m, p \in z;\ n, q \in N,\ n > 1, q > 1$, which is the correct form of $x^{\frac{m}{n}}.x^{\frac{p}{q}}$?

a) $\frac{mp}{nq}$ b) $x^{\frac{m}{n}+\frac{p}{q}}$ c) $x^{\frac{mp}{nq}}$ d) $x^{\frac{m+p}{n+q}}$

6. Find out the value of x in expression $5^x = 1$?

(a) $\frac{1}{5}$ (b) 1 (c) 0 (d) 5

7. Which is the correct answer of the expression $\sqrt[3]{a} \times \sqrt{a}$?

(a) $a^{5/6}$ (b) $a^{3/2}$ (c) $\frac{5}{6}$ (d) $a^{\frac{1}{6}}$

8. *If* $x^2 = 4$, then what is the value of x^8 ?

(a) 64 (b) 128 (c) 256 (d) 16

9. What is the value of $(2^3)^{-2}$? (a) -6 (b) 2^6 (c) $\frac{1}{64}$ (d) 64

10. What is simplified form of $\sqrt[8]{a^7} \div \sqrt{a}$? (a) $a^{8/7}$ (b) $a^{\frac{3}{8}}$ (c) $\frac{7}{8}$ (d) $a^{7/2}$.

Multiple Choice Questions (logarithms)

1. What is the correct value of $\log_a 1$?

(a) 1 (b) a (c) $\frac{1}{a}$ (d) 0

2. Which is right of $\log_a a$?

(a) a (b) 1 (c) 0 (d) $\frac{1}{a}$

3. If $\log_x \frac{1}{3} = \frac{1}{3}$ then which is the correct value of x?

(a) $\frac{1}{27}$ (b) 27 (c) 0 (d) 1

4. Chouse the correct value of $\log_{10} 100$?

(a) 0 (b) 1 (c) 2 (d) 3

5. Find out the right value of $\log_2 \log_2(4)$?

(a) 1 (b) 2 (c) 4 (d) 8

6. What is the value of x of $\log_3 x = 4$?

(a) 12 (b) 81 (c) 27 (d) 64

7. Choose the value of x of $\log_x 64 = 3$;

(a) 2 (b) 3 (c) 4 (d) 5

8. If $\log_a x = p$ and $\log_a y = q$ then what is the value of $\log_u\left(\frac{x}{y}\right)$?

(a) $\frac{q}{p}$ (b) pq (c) $p+q$ (d) $p-q$

9. What is the correct value of $lag_b a \log_c b \log_a c$?

(a) 1 (b) abc (c) $a^2b^2c^2$ (d) 0

10. Choose the correct value of $\log_4 64$.

(a) 4 (b) 2 (c) 3 (d) 8

Which one of the following statement is true/false?

a. If $a^p = a^q$ then $p = q$ $(a \neq 1)$

b. If $a^m = b^m$ then $a = b$ $(m \neq 0)$

c. $2^{3^2} = (2^3)^2$

d. $3^{3^2} = (3^3)^2$

e. $\log_a(mn) = \log_a m \times \log_a n.$

f. $\log_a\left(\frac{m}{n}\right) = \frac{\log_a m}{\log_a n}$

g. If $\log_a N = x$ then $x^a = N.$

h. $\log a + \log b = \log(a+b)$

i. $\log a - \log b = \log(a-b)$

j. The logarithm of the product of two quantities is equal to the sum of their logarithms.

k. The logarithm of the quotients of two numbers is equal to the difference of their logarithm.

l. The logarithm of a power of a number is the product of the power and the logarithm of the number.

m. e is called Napier's constant.

n. $\bar{2}.785 = -2 + 0.785$ (in case of logarithm)

Brief Questions

1. Write down the basic laws of indices.
2. What do you mean by indices?
3. Find the value of $(.001)^{-\frac{1}{3}}$.
4. Define logarithm and antilogarithm of a number.
5. Write down the any five laws of logarithm.
6. What is the value a^0?
7. What is the value of $\log_{10} 0$?
8. What is the value of x, if $\log_3 x = 4$?
9. What is correct value of $\log_b a.\log_c b.\log_a c$?
10. Write down the value of x in expression $5^x = 1$?
11. What is the value of $\log_a a$?
12. What is the correct value of $\log_4 64$?
13. What is the correct value of $\log_a 1$?
14. What is meant by common logarithm?
15. What is meant by Natural logarithm?

Conceptual, Analytical & Numerical Questions

1. What is meant by index? Discuss the basic laws of indices.
2. Define logarithm and antilogarithm of a number.
3. Mention the laws or properties of logarithm.
4. Distinguish between common logarithm and natural logarithm.
5. Define Characteristics and Mantissa of logarithm of a number with examples.

INDICES

1. Simplify: i) $49^{\frac{3}{2}}$ ii) $(243)^{\frac{3}{5}}$ iii) $\left(\frac{32}{243}\right)^{\frac{4}{5}}$ iv) $\frac{3^5.27^3.9^4}{3.(81)^4}$

2. Show that $\frac{3^5.(3^3)^3.(3^2)^4}{3.(3^4)^4} = 3^5$

3. Find the value of : i) $\frac{x^{\frac{4}{7}}.\sqrt[5]{x^3}.\sqrt[7]{x^3}}{\sqrt[8]{x^{-3}}\sqrt[5]{y^5.x^8}}.\frac{y^2}{\left(x^{1/8}\right)^3}$ ii) $\left(\frac{5^{-1}.7^2}{5^2.7^{-4}}\right)^{\frac{7}{2}} \times \left(\frac{5^{-2}.7^3}{5^3.7^{-5}}\right)^{-\frac{5}{2}}$

4. Express in fractional indices and then simplify: $\sqrt[3]{p^4} \times \sqrt[4]{p^2} \times \sqrt[6]{p^7}$

5. Show that i) $\frac{2.3^{a+1}+7.3^{a-1}}{3^{a+2}-2\left(\frac{1}{3}\right)^{1-a}} = 1$ ii) $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a} = 1$

6. Show that $x^3 - 6x = 6$ when $x = 2^{\frac{1}{3}} + 2^{\frac{2}{3}}$

7. *If* $x = a^{\frac{1}{3}} + a^{-\frac{1}{3}}$ then show that $x^3 - 3x = a + \frac{1}{a}$.

8. *If* $a.\sqrt[3]{x^2} + b.\sqrt[3]{x} + c = 0$, then prove that $a^3x^2 + b^3x + c^3 = 3abcx$

9. *If* $m = a^p, n = a^z$ and $(m^q.n^p)^z = a^2$ than show that $xyz = 1$.

10. *If* $a = 2 + 2^{\frac{2}{3}} + 2^{\frac{1}{3}}$ than show that $a^3 - 6a^2 + 6a - 2 = 0$

11. Simplify: $\sqrt[bc]{\frac{x^{\frac{b}{c}}}{x^{\frac{c}{b}}}} \times \sqrt[ca]{\frac{x^{\frac{c}{a}}}{x^{\frac{a}{c}}}} \times \sqrt[ab]{\frac{x^{\frac{a}{b}}}{x^{\frac{b}{a}}}}$.

12. *If* $\frac{9^b.3^2.(3^{-b})^{-1} - 27^b}{3^{3a}.2^3} = \frac{1}{27}$. Then show that a-b= 1.

ANSWERS

1. (i) 343 (ii) 27 (iii) $\frac{16}{81}$ (iv) 243 3. (i) y (ii) 175 4. p^3 11. 1

OGARITHMS

Find the value of (a) $\log_2 8$; (b) $\log_3 243$; (c) $\log_5\left(\frac{1}{125}\right)$.

Find the value of x in each of the following:

(a) $\log_2 x = 3$; (b) $\log_3 81 = x$; (c) $\log_x 9 = 2$; (d) $\log_{\sqrt{5}} x = 4$; (e) $\log\sqrt{x}8 = 6$.

(a) Find the value of (i) $\log_4 256$. (ii) $\log_2 64$.

(b) Find the logarithm of (i) 784 to the base $2\sqrt{7}$, (ii) 19683 to the base $3\sqrt{3}$.

(c) Find x in the following cases: (i) $\log_{\sqrt{8}} x = -\frac{2}{3}$, (ii) $\log_x\left(\frac{1}{9}\right) = 4$, (iii) $\log_x 125 = 3$

Prove that (a) $\log_2 \log_3 81 = 2$; (b) $\log_2 \sqrt{6} + \log_2 \sqrt{\frac{2}{3}} = 1$

Prove that (i) $\frac{\log_9 11}{\log_5 13} \div \frac{\log_3 11}{\log_5 13} = \frac{1}{2}$, (ii) $(\log a)^2 - (\log b)^2 = \log(ab)\log(a/b)$.

Prove that (a) $\frac{\log_3 8}{\log_9 16 \log_4 10} = 3\log_{10} 2$, (b) $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1$.

(c) $\frac{2\log 6 + 6\log 2}{4\log 2 + \log 27 - \log 9} = 2$, (d) $\frac{\log\sqrt{27} + \log\sqrt{8} - \log\sqrt{125}}{\log 6 - \log 5} = \frac{2}{3}$.

Show that (a) $\frac{\log_3 8}{\log_9 16 \log_4 10} = 3\log_{10} 2$, (b) $\log_y x^2 . \log_z y^3 . \log_x z^4 = 24$.

(a) If $\log(x+y) = \log x + \log y$ show that $\frac{1}{x} + \frac{1}{y} = 1$

(b) If $\log\frac{a+b}{2} = \frac{1}{2}(\log a + \log b)$, show that $a = b$.

(c) If $\log\frac{x+y}{5} = \frac{1}{2}(\log x + \log y)$, show that $\frac{x}{y} + \frac{y}{x} = 23$.

(d) If $x^2 + y^2 = 11xy$, then show that $2\log(x-y) = 2\log 3 + \log x + \log y$.

(e) If $x^2 + y^2 = 7xy$, show that $2\log(x+y) = \log x + \log y + 2\log 3$.

(a) Find $\log_8 25$ given that $\log_{10} 2 = 0.3010$.

(b) If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, then find the value of $\log \dfrac{(16)^{1/5}(5)^2}{(108)^3}$.

10. Find the simplest value of $\log_3 \sqrt[4]{729}.\sqrt[3]{9^{-1}} \times 27^{-4/3}$.

11. Prove that (i) $(yz)^{\log y/z}.(zx)^{\log z/x}.(xy)^{\log x/y} = 1$.

(ii) $\log\dfrac{x^n}{y^n} + \log\dfrac{y^n}{z^n} + \log\dfrac{z^n}{x^n} = 0$, (iii) $\log\dfrac{x^2}{yz} + \log\dfrac{y^2}{zx} + \log\dfrac{z^2}{xy} = 0$.

(iv) $\log\dfrac{81}{8} - 2\log\dfrac{3}{2} + 3\log\dfrac{2}{3} + \log\dfrac{3}{4} = 0$, (vi) $\log\dfrac{384}{2} + \log\dfrac{81}{32} + 3\log\dfrac{5}{3} - \log 9 = 2$

12. Prove that (a) $\dfrac{1}{\log_{xy}(xyz)} + \dfrac{1}{\log_{yz}(xyz)} + \dfrac{1}{\log_{zx}(xyz)} = 2$

(b) $\dfrac{1}{(\log_a bc)+1} + \dfrac{1}{(\log_b ca)+1} + \dfrac{1}{(\log_c ab)+1} = 1$.

13. (a) If $\log_2[\log_3(\log_2 x)] = 1$, find the value of x.

(b) Find the value of $\log_2[\log_2\{\log_3(\log_3 27^3)\}]$.

14. If $a^{3-x}.b^{5x} = a^{x+5}.b^{3x}$, then show that $x\log\left(\dfrac{b}{a}\right) = \log a$.

15. (a) Prove that $\dfrac{1}{6}\sqrt{\dfrac{3\log 1728}{\frac{1}{2}\log 36 + \frac{1}{3}\log 8}} = \dfrac{1}{2}$.

(b) If $a = \log_{24} 12, b = \log_{36} 24, c = \log_{48} 36$ then prove that $1 + abc = 2bc$.

(c) If $\dfrac{\log a}{q-r} = \dfrac{\log b}{r-p} = \dfrac{\log c}{p-q}$, prove that abc=1.

(d) If $\dfrac{\log x}{2} = \dfrac{\log y}{3} = \dfrac{\log z}{5}$, show that $x^4 = yz$.

16. (a) Given that $\log_q(xy) = 3$ and $\log_q(x^2y^3) = 4$. Calculate the value of $\log_q x$ and $\log_q y$. Hence calculate the values of x and y when q=2.

(b) Given that $2\log(x^2y) = 3 + \log x - \log y$ where x and y are both positive, express y in terms of x. If $x - y = 3$, find the values of x and y.

(c) Given that $\log_a(x^2y^2)=6$ and $\log_a\left(\frac{x}{y}\right)=2$. Calculate the values of $\log_a x$ and $\log_a y$.

(d) Given that $3\log(x^2y)=5+\log x-2\log y$, where x and y are positive; express y in terms of x.

17. If $x=\log_{2a} a, y=\log_{3a} 2a, z=\log_{4a} 3a$, prove that $1+xyz=2yz$.

18. Evaluate using log tables:

(a) $\frac{24.395\times(3.16)^3}{8.79}$, (b) $\frac{0.0357\times\sqrt{0.235}}{\sqrt[3]{0.0637}}$, (c) $\sqrt[7]{\frac{1}{0.8176\times36.21}}$,

(d) $\frac{(63)^{\frac{1}{2}}\times(0.0781)^{\frac{1}{4}}\times(46)^{\frac{1}{6}}}{0.0032\times(24.8)^{\frac{1}{3}}}$, (e) $\frac{(17.5)^{\frac{1}{2}}+(15.2)^{\frac{1}{3}}}{(56.3)^{\frac{3}{5}}-(12.4)^{\frac{1}{4}}}$.

19. Find the value of (a) $\frac{(435)^3-(0.56)^{\frac{1}{2}}}{(380)^4}$, (b) $\frac{937.6\times(11.059)^3\times(0.02097)}{\sqrt{6004}\times10^3\times8.06}$.

20. Simplify by using log tables (a) $\sqrt[6]{\frac{9268\times4.573\times0.0864}{87.65\times0.5432}}$, (b) $\frac{(6.45)^3\times(0.00034)^{\frac{1}{3}}\times(981.4)}{(9.37)^2\times(8.93)^{\frac{1}{4}}\times(0.0167)}$

21. Using log tables, find the numerical value of x from the relation: $2x=\log_{10}26.54+\log_{10}0.004321-\log_{10}0.00001357$ and find the value of $\sqrt{\frac{26.54\times0.004321}{0.00001357}}$, correct to the nearest integer.

22. Find the value of (a) $30\{(1+0.035)^{15}-1\}$, (b) $\frac{400}{0.06}\left[1-\frac{1}{(1.06)^4}\right]$, (c) $\frac{45}{0.04}\left[(1.05)^{10}-1\right]$.

23. Solve the following equations:(i) $\log_x 4+\log_x 16+\log_x 64=12$

24. Solve for x the equation: (i) $2^x.3^{2x+1}=7^{4x+3}$, (ii) $4^x.20^{2x-2}=40^x.2^{3x-1}$.

25. Solve the following equation:

(a) $\log_x 3+\log_x 9+\log_x 729=9$, (b) $\log_{10}(x-9)+\log_{10}x=1$.

ANSWERS

1. (a) 3, (b) 5, (c)-3. ,
2. (a) 8, (b) 4, (c) 3, (d) 25, (e) 2.
3. (a) (i) 4, (ii) 6,; (b) (i) 4, (ii) 6; (c) (i) $\frac{1}{2}$, (ii) $\frac{1}{\sqrt{3}}$, (iii) 5.
9. (a) 1.55, (b) –4.4611.
10. 1.
13. (a) 512, (b) 0.
16. (a) 5, -2, 32, $\frac{1}{4}$, (b) $y = \frac{10}{x}$,5,2 , (c) 2.5, 05, (d) $y = \frac{10}{x}$.
18. (a) 87.60, (b) 0.04334, (c) 0.6103, (d) 7166, (e) 0.489.
19. (a) 0.0009342, (b) 0.0426
20. (a) 2.063, (b)1963
21. 1.9635, 92
22. (a) 20.292, (b) 1386, (c) 707.6
23. (i) 2, (ii) 2.
24. (i) –0.9672, (ii) 2.301.
25. (a) 3, (ii) 10.