CHAPTER-05 QUADRATIC EQUATIONS

LEARNING OBJECTIVES

This chapter will enable our students to learn the concepts and application of:

- Quadratic Equations and its solution system
- Solving quadratic simultaneous equations
- Relationships between roots and coefficient of a quadratic equation
- Formation of quadratic equations whose roots are given
- Nature of the roots of a quadratic equations
- Business application of quadratic equations

5.1 INTRODUCTION

An equation in one unknown quantity (let it be x) in the form $ax^2+bx+c=0$ is known as a quadratic equation, where *a, b, c* are constants and $a \neq 0$ while *b* and *c* may be zero. Here *a* is called the coefficient of x^2, *b* is the coefficient of *x* and *c* is a constant term. The values of *x* which satisfy the equation are named as the roots of the quadratic equation. The roots of the quadratic equation $ax^2+bx+c=0, (a \neq 0)$ are $x=\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

5.2 QUADRATIC EQUATIONS

An equation, which contains the square of the unknown variable and no higher power, is called a **quadratic equation** or an equation of the second degree. An equation of the form $ax^2+bx+c=0, (a \neq 0)$ is called a quadratic equation or, an equation of the second degree, where a,b,c are arbitrary constants, and $a \neq 0$ while x is unknown. Here $ax^2+bx+c=0, (a \neq 0)$ is the **general form** of the quadratic equation. If $b=0$, then given equation reduces to $ax^2+c=0$. Such an equation is called a pure quadratic equation. That is, an equation, which contains only the square of the unknown and not the first power, is called a **pure quadratic equation**. Example: $3x^2=15$. But if $b \neq 0$, then such an equation is called complete or mixed quadratic equation, i.e. an equation which contains the square as well as the first power of the unknown is called an **adfactor or complete or mixed quadratic equation**. For examples: (i) $ax^2+bx+c=0$, (ii) $3x^2+5x+4=0$.

Sometimes, it is difficult to judge whether the given equation is quadratic as in the following case:

$$\sqrt{5x+1}+\sqrt{x}=3 \quad \text{or, } \sqrt{5x+1}=3-\sqrt{x}$$

Squaring on both sides, we get

$$5x+1=9+x-6\sqrt{x} \quad \text{or, } 4x-8=-6\sqrt{x}$$

Squaring again, we have

$$16x^2+64-64x=36x \quad \text{or, } 16x^2-100x+64=0$$

Thus this equation reduces to quadratic form and contains the square of x and no higher power

5.3 SOLUTIONS OF A GENERAL QUADRATIC EQUATION

General methods of solving a quadratic equation are;

(a) Factorization Method

(b) Method of completing perfect square.

(a) Factorization Method: In this method the quadratic equation: $ax^2+bx+c=0$ is first decomposed into two factors, say $(\text{Ax}+\text{b})\times(Cx+D)$ each of the first degree, and then equating them separately to zero to find the values of the unknown variable x, i.e.

$$ax^2+bx+c=0 \Rightarrow (Ax+b)(Cx+D)=0$$

Then either $Ax+B=0$, i.e. $x=-B/A$ or, $\text{Cx}+\text{D}=0$ i.e. $\text{x}=\text{-D/C}$

For example: Suppose we are interested to solve the equation: $x^2+16x+60=0$

We have, $x^2+16x+60=0 \Rightarrow x^2+(10+6)x+60=0$

$$\Rightarrow x^2+10x+6x+60=0 \quad \text{or, } x(x+10)+6(x+10)=0$$

$$\Rightarrow (x+10)(x+6)=0$$

Thus, either $x+10=0$, i.e. $x=-10$ or, $x+6=0$, i.e. $x=-6$.

Hence, the required roots are: $x=-10$ and -6

(b) Method of Completing Perfect Square

Consider the general quadratic equation $ax^2+bx+c=0 \Rightarrow ax^2+bx=-c$

Dividing both sides by the coefficient $a(a\neq 0)$ of x^2, we get

$$x^2+\frac{b}{a}x=-\frac{c}{a} \qquad \Rightarrow x^2+2\times x\times\frac{b}{2a}+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}=-\frac{c}{a}$$

$$\Rightarrow\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}=-\frac{c}{a} \qquad \Rightarrow\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$ (By taking square root on both sides)

$$\Rightarrow x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Thus, the two roots, say α and β of the general equation are:

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Remark: To solve an equation by this method, reduce the given equation in the standard form $ax^2+bx+c=0$ and find the values of a, b and c. Put these values in these values in the given formula and get the required roots.

For example: Solve the equation: $5x^2+9x+4=0$.

Solution: Given $5x^2+9x+4=0$ Here $a=5, b=9$ and $c=4$.

By substituting these values in the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, we get

$$x=\frac{-9\pm\sqrt{9^2-4\times5\times4}}{2\times5}=\frac{-9\pm\sqrt{81-80}}{10}=\frac{-9\pm1}{10}$$

$$\therefore x=\frac{-9+1}{10}=-\frac{4}{5} \text{ or, } \frac{-9-1}{10}=-1$$

5.4 RELATIONSHIP BETWEEN ROOTS AND COEFFICIENT OF A QUADRATIC EQUATION:

Consider general form of quadratic equation

$$ax^2+bx+c=0$$

We have two roots of this equation

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

here a, b, c are said to be co-efficients.

Thus, the relationship between roots and co-efficients are : $\alpha+\beta=\frac{-b}{a}$ and $\alpha\beta=\frac{c}{a}$

that is, sum of the roots = $-\dfrac{coefficient\ of\ x}{coefficient\ of\ x^2}$ and product of the roots = $\dfrac{constant\ term}{coefficient\ of\ x^2}$

Example-28: Suppose $2x^2 + 6x + 16 = 0$

$\therefore$ Sum of the roots = $\dfrac{-6}{2} = -3$

and product of the roots = $\dfrac{16}{2} = 8$.

5.5 NATURE OF THE ROOTS OF A QUADRATIC EQUATION

We know the roots of the quadratic equation

$$ax^2 + bx + c = 0,\ (a \neq 0) \text{ are: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, $D = b^2 - 4ac$ is called discriminant of the equation $ax^2 + bx + c = 0$, the nature of the roots depend on the value of $\sqrt{b^2 - 4ac}$. Therefore

if

(i) $b^2 - 4ac > 0$, then roots are real and unequal,

(ii) $b^2 - 4ac < 0$, then roots are imaginary and unequal, and

(iii) $b^2 - 4ac = 0$, then both the roots are real and equal, i. e. $\dfrac{-b}{2a}$

(iv) If discriminant is positive and a perfect square then the roots of equation (1) are real, rational and unequal. And if the discriminant is positive but not a perfect square then the roots of equation (1) are real, irrational and unequal.

(v) If $b^2 - 4ac$ is a perfect square but any one of a or b is irrational then the roots of equation (1) are irrational.

Note that:

Complex and irrational roots are conjugate, i.e.

(i) if one roots is $2 + \sqrt{5}$, then other one will be $2 - \sqrt{5}$

(i) if one root is $3 + 2i$, than other one will be $3 - 2i$.

5.6 FORMATION OF QUADRATIC EQUATIONS WITH GIVEN ROOTS

So far we were calculate the roots of a given quadratic equation. But here roots are given and we have to calculate the quadratic equation.

We know, if α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$ ... (1)

then $\alpha+\beta=-\frac{b}{a}$, and $\alpha\beta=\frac{c}{a}$.

Now equation (1) can be written as

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \quad => x^2-\left(\frac{-b}{a}\right)x+\left(\frac{c}{a}\right)=0 \quad => x^2-(\alpha+\beta)x+\alpha\beta=0$$

which is the required equation whose roots are α and β.

Therefore, if α and β be the roots of a quadratic equation, then the equation will be

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

i.e. x^2- (sum of the roots) $x+$ (product of the roots) $=0$

For Example : If 2 and 3 are the roots of a quadratic equation, the equation will be

$$x^2-(2+3)x+2.3=0 \quad i.e.x^2-5x+6=0$$

5.7 CURVE SKETCHING OF DIFFERENT LINEAR AND NON-LINEAR EQUATIONS

The graph of a linear equation in two variables is a straight line. There is one and only one straight line through any two given points and all ordered pairs generated by a linear equation will fall on the straight line connecting any two points on that line. Thus we can plot the graph of a linear equation by finding any two points.

Example 36: Sketch the following linear equation:

i) $3x-y=-2$; ii) $3x+2y=12$

Solution: i) Given equation is

$3x-y=-2$

$\therefore$ $y=3x+2$

when $x=0$, $y=2$

when $x=1$, $y=5$

So, we have two points (0,2) and (1, 5).

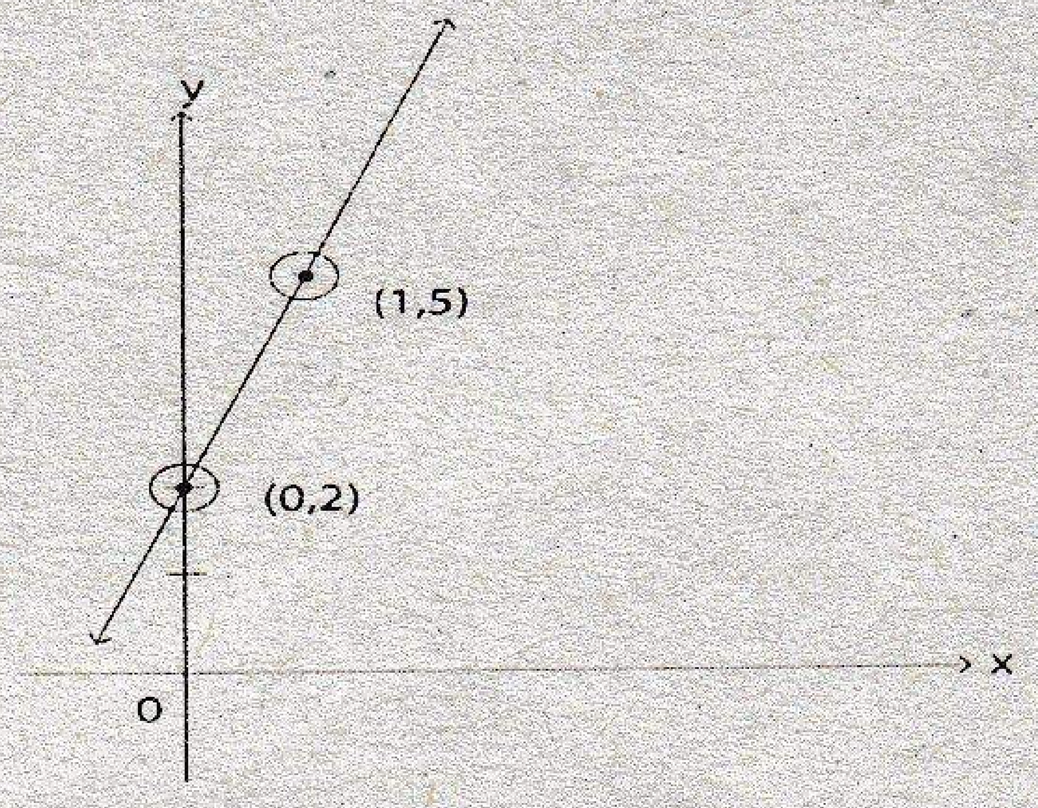


fig : Graphing the equation 3x-y=-2

ii) The given equation is $3x+2y=12$

When $x=o$, $y=6$

When $y=0$, $x=4$

So, we have two points (0, 6) and (4, 0).

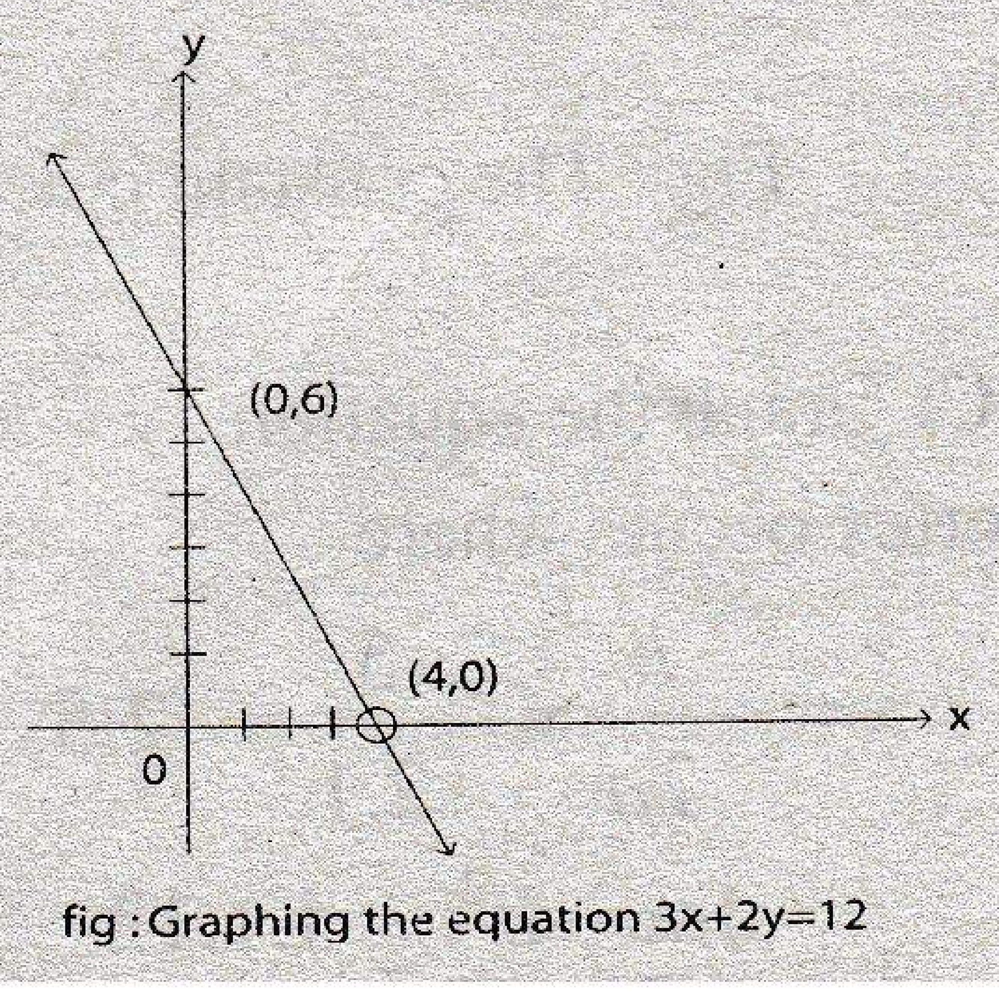
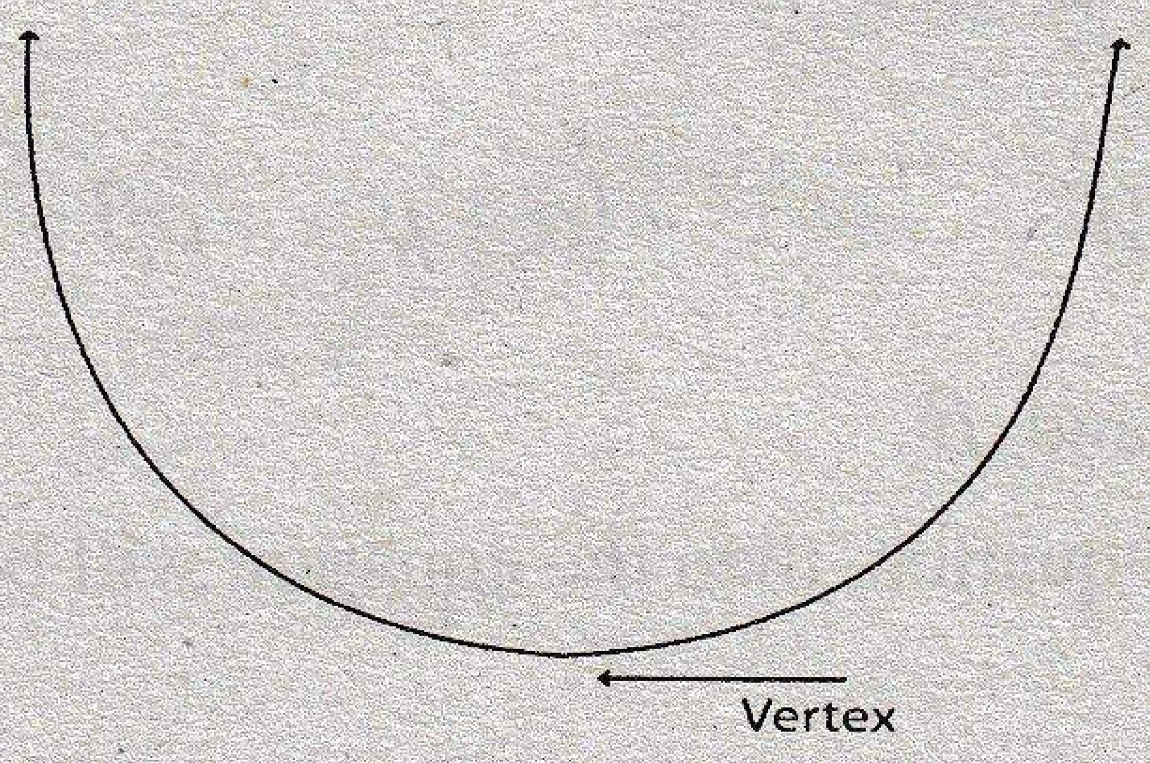


fig : Graphing the equation 3x+2y=12

GRAPHING NON-LINEAR EQUATION OR FUNCTION: Case A :



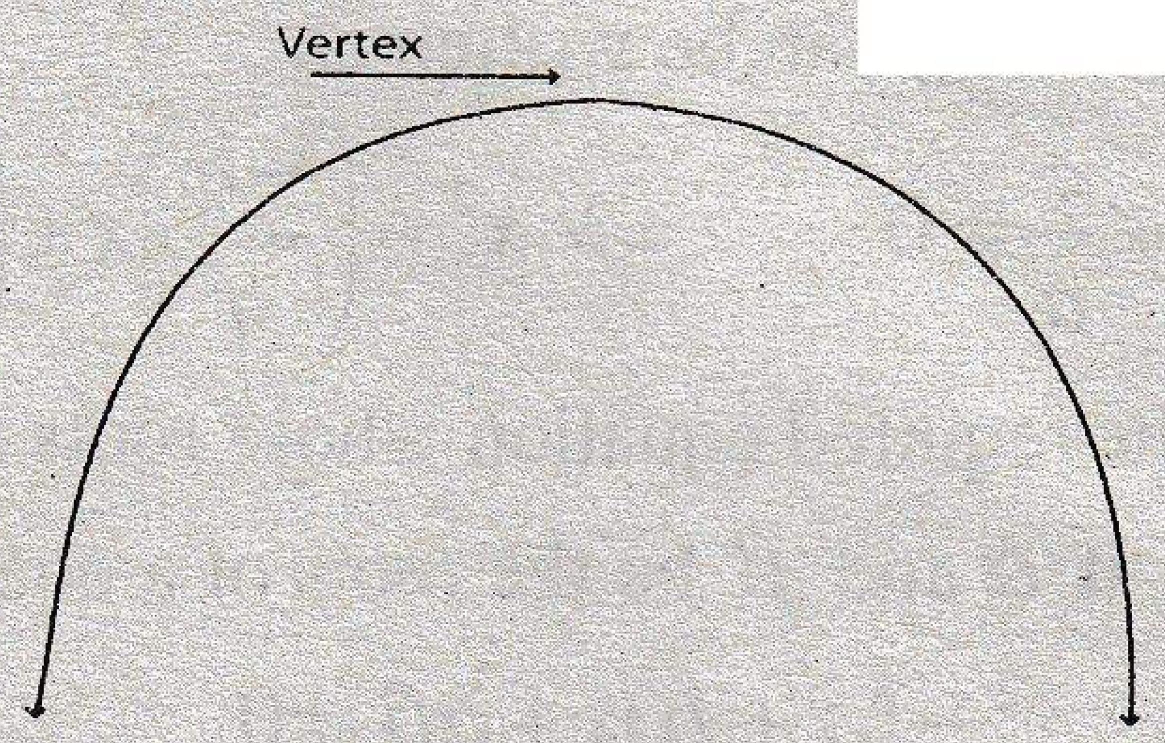
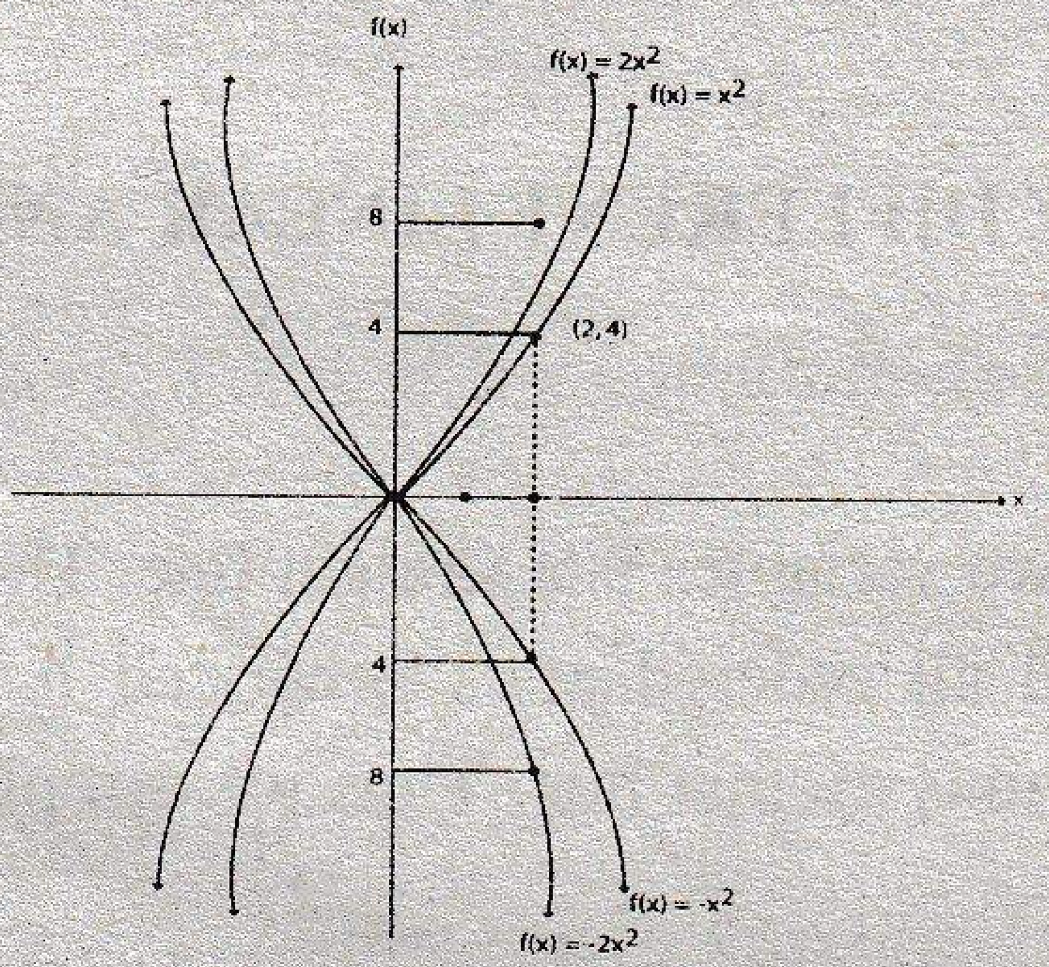


Fig : $f(x)=ax^2+bx+c \quad for \ a>0$ fig : $f(x)=ax^2+bx+c \quad for \ a<o.$



Case B

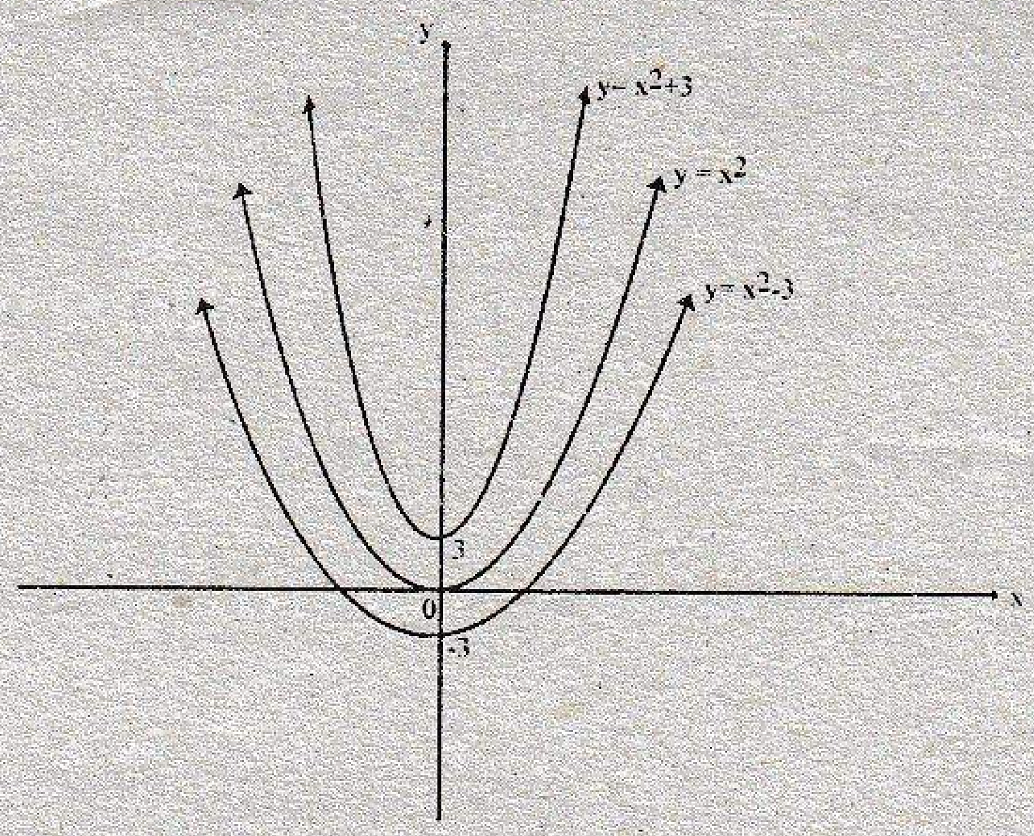


Fig : The constant c determines the vertical displacement of $y= ax^2+c$.

Case C: Graph the equation : $y = x^2+5x+12$

The vertex occurs where

$$x=-\frac{b}{2a}=-\frac{5}{2.1}=-2.5$$

$$y=\frac{4ac-b^2}{4a}=\frac{4.1.12-5^2}{4.1}=\frac{48-25}{4}$$

$=5.75$ When $x = 0, y = 12.$

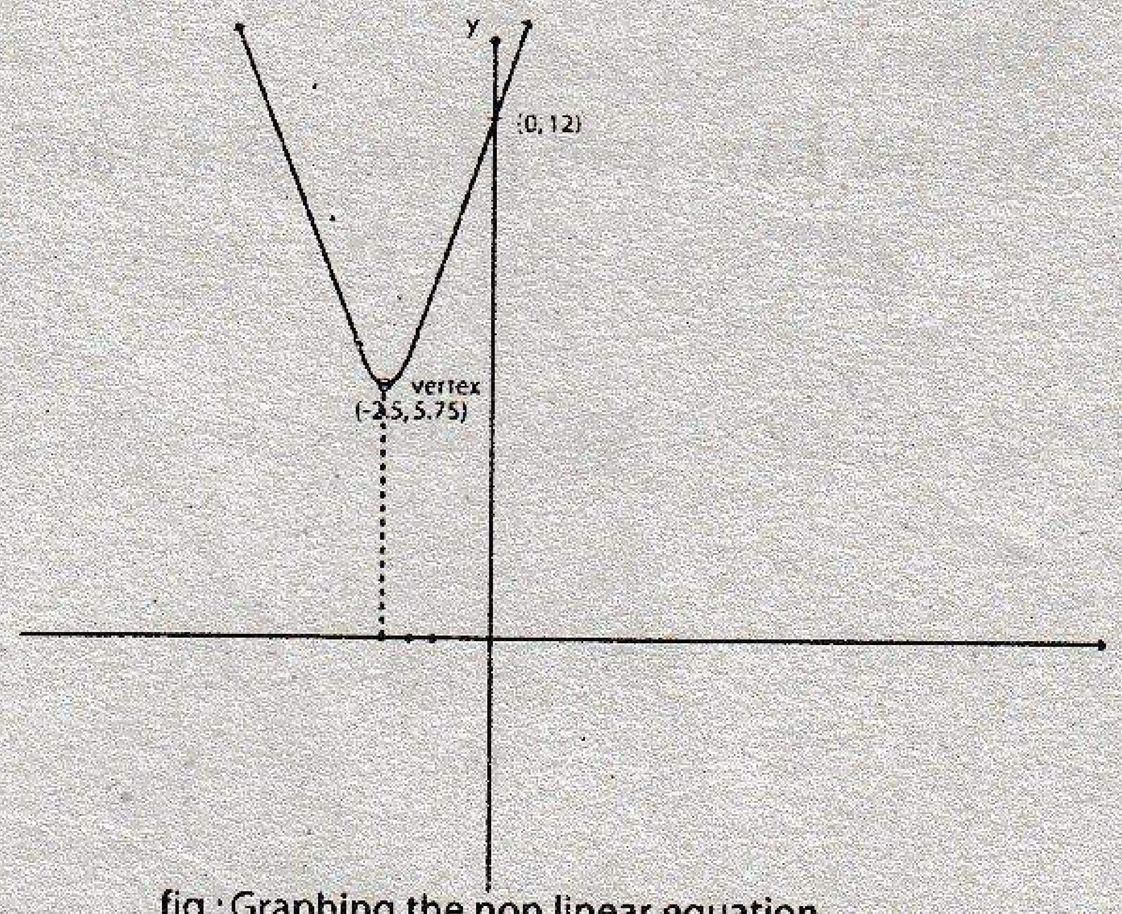


fig : Graphing the non linear equation $y=x^2+5x+12$

ILLUSTRATIONS

Illustration-01: Solve the equation: $\sqrt{2x+1}+\sqrt{3x+2}=\sqrt{5x+3}$

Solution: We have $\sqrt{2x+1}+\sqrt{3x+2}=\sqrt{5x+3}$

Squaring on both sides, we get

$$(2x+1)+(3x+2)+2\sqrt{(2x+1)(3x+2)}=5x+3$$

$$or,(5x+3)+2\sqrt{(2x+1)(3x+2)}=5x+3 \; or,\; 2\sqrt{(2x+1)(3x+2)}=0$$

or, $(2x+1)(3x+2)=0$

Thus, either $2x+1=0$ $i.e. x=-1/2$ $or, 3x+2=0$, $i.e. x=-2/3$

Illustration-02: Solve: $ax^2+bx+c=0;\ a\neq 0$

Solution: $ax^2+bx+c=0$

=> $4a^2x^2+4abx+4ac=0$ (Multiplying both sides by '4a')

=> $(2ax)^2+2.2ax.b+b^2-b^2+4ac=0$ => $(2ax+b)^2=b^2-4ac$

=> $2ax+b=\pm\sqrt{b^2-4ac}$ (Taking square root on both sides)

=> $2ax=-b\pm\sqrt{b^2-4ac}$

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, the general form of a quadratic equation $ax^2+bx+c=0$ has two roots and these are : $\alpha=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Illustration-03: Solve the equation: $\dfrac{2}{x+1}-\dfrac{3}{x-1}=\dfrac{2}{x}$

Solution: We have $\dfrac{2}{x+1}-\dfrac{3}{x-1}=\dfrac{2}{x}\; or, \dfrac{2(x-1)-3(x+1)}{(x+1)(x-1)}=\dfrac{2}{x}$

$$\Rightarrow \frac{-x-5}{x^2-1}=\frac{2}{x}\Rightarrow 2x^2-2==-x^2-5x\Rightarrow 3x^2+5x-2=0 \quad \Rightarrow x=-2,\ \frac{1}{3} \text{ (Ans.)}$$

Illustration-04: Solve the equation: $\frac{x}{b}+\frac{b}{x}=\frac{a}{b}+\frac{b}{a}$

Solution: We have $\frac{x}{b}+\frac{b}{x}=\frac{a}{b}+\frac{b}{a}$ $or, \frac{x}{b}-\frac{a}{b}+\frac{b}{x}-\frac{b}{a}=o$

or, $\frac{1}{b}(x-a)-b\left(\frac{1}{a}-\frac{1}{x}\right)=0$ or, $\frac{1}{b}(x-a)-b\left(\frac{x-a}{ax}\right)=0$ or, $(x-a)\left(\frac{1}{b}-\frac{b}{ax}\right)=0$

Thus, either $x-a=0, i.e. x=a$ or, $\frac{1}{b}-\frac{b}{ax}=0$, $i.e. x=\frac{b^2}{a}$

Hence, the required roots are : a and $\frac{b^2}{a}$.

Illustration-05: Solve the equations: $\frac{x}{x+2}=\frac{x+3}{5(x+11)}$

Solution: Given $\frac{x}{x+2}=\frac{x+3}{5(x+11)}$

$\Rightarrow \frac{x}{x+2}=\frac{x+3}{5x+55}$ $\Rightarrow x(5x+55)=(x+3)(x+2)$

$\Rightarrow 5x^2+55x=x^2+2x+3x+6$ $\Rightarrow 4x^2+50x-6=0$

$\Rightarrow 2(2x^2+25x-3)=0$ $\Rightarrow 2x^2+25x-3=0$

$\Rightarrow x=\frac{-25\pm\sqrt{(25)^2-4(2)(-3)}}{2(2)}$ $\Rightarrow x=\frac{-25\pm\sqrt{649}}{4}$

$\therefore$ Required solutions are : $x=\frac{-25\pm\sqrt{649}}{4}$. [Ans.]

Illustration-06: Solve the equations: $\frac{x}{3}+\frac{3}{x}=\frac{10}{3}$

Solution: Given $\frac{x}{3}+\frac{3}{x}=\frac{10}{3}$

$\Rightarrow \frac{x^2+9}{3x}=\frac{10}{3}$ $\Rightarrow 3x^2+27=30x$

$\Rightarrow 3x^2-30x+27=0$ $\Rightarrow x^2-10x+9=0$

$\Rightarrow x^2-9x-x+9=0$ $\Rightarrow (x-9)(x-1)=0$

$\Rightarrow x=9,1$ $\therefore$ Required solutions are: $x=9,1$. [Ans.]

SOME SPECIAL METHODS OF SOLVING QUADRATIC EQUATION

(a) Equations Reducible to Quadratic Form

Equations of higher degree than second degree can be solved by reducing them to quadratic form by using suitable substitution. Various cases are discussed below:

Illustration-07: Solve the equation: $x^4 - 10x^2 + 9 = 0$

Solution: Putting $x^2 = y$, we get $y^2 - 10y + 9 = 0$.

Then $\quad y^2 - 10y + 9 = 0 \Rightarrow y^2 - 9y - y + 9 = 0$

Making factors, we get $(y-9)(y-1) = 0$.

Thus, either $y - 9 = 0, i.e. y(= x^2) = 9$ or $x = \pm 3$ or $y - 1 = 0, i.e. y(= x^2) = 1$ or $x = \pm 1$

Hence the required roots are ± 3 and ± 1.

Illustration-08: Solve the equation: $\sqrt{\dfrac{x}{1-x}} + \sqrt{\dfrac{1-x}{x}} = \dfrac{13}{6}$

Solution: Let $\sqrt{\dfrac{x}{1-x}} = y$. Then given equation reduce to:

$$y + \frac{1}{y} = \frac{13}{6} \quad \text{or,} \quad \frac{y^2+1}{y} = \frac{13}{6}$$

$$\Rightarrow 6(y^2+1) = 13y \quad \text{or,} \quad 6y^2 - 13y + 6 = 0$$

Making factors, we get $(3y-2)\,(2y-3) = 0$

Thus, either $3y - 2 = 0, i.e. y = 2/3$ or $2y - 3 = 0, i.e. y = 3/2$

Now, if $y = \sqrt{\dfrac{x}{1-x}} = \dfrac{2}{3}$

$\Rightarrow \dfrac{x}{1-x} = \dfrac{4}{9} \Rightarrow 9x = 4(1-x), i.e. x = 4/3$

Again $y = \sqrt{\dfrac{x}{1-x}} = \dfrac{3}{2}$

$\Rightarrow \dfrac{x}{1-x} = \dfrac{9}{4} \Rightarrow 4x = 9(1-x), i.e. x = 9/13$

Hence, required roots are: $4/13$ and $9/13$

Illustration-09: Solve the following equations:

(i) $\sqrt{\dfrac{x}{x+16}} + \sqrt{\dfrac{x+16}{x}} = \dfrac{25}{12}$ (ii) $\dfrac{x-a^2-b^2}{a^2} + \dfrac{c^2}{x-a^2-b^2} = 2.$

Solution: (i) Given $\sqrt{\frac{x}{x+16}} = \sqrt{\frac{x+16}{x}} = \frac{25}{12}$

$\Rightarrow a+\frac{1}{a}=\frac{25}{12}$ [Putting $a=\sqrt{\frac{x}{x+16}}$]

$\Rightarrow \frac{a^2+1}{a}=\frac{25}{12}$ $\Rightarrow 12a^2+12=25a$

$\Rightarrow 12a^2-25a+12=0$ $\Rightarrow 12a^2-16a-9a+12=0$

$\Rightarrow 4a(3a-4)-3(3a-4)=0$ $\Rightarrow (3a-4)(4a-3)=0$

$\Rightarrow (3a-4)=0$ *or*, $(4a-3)=0 \Rightarrow 3a=4$ *or*, $4a=3$

$\Rightarrow a=\frac{4}{3}$ *or*, $a=\frac{3}{4}$

$\Rightarrow \sqrt{\frac{x}{x+16}}=\frac{4}{3}$ or $\sqrt{\frac{x}{x+16}}=\frac{3}{4}$ [Putting $a=\sqrt{\frac{x}{x+16}}$]

$\Rightarrow \frac{x}{x+16}=\frac{16}{9}$ or $\frac{x}{x+16}=\frac{9}{16}$ [Squaring on both sides]

$\Rightarrow 9x=16x+256$ or $16x=9x+144$ $\Rightarrow -7x=256$ or $7x=144$

$\Rightarrow x=-\frac{256}{7}$ or $x=\frac{144}{7}$

$\therefore$ Required solutions: $x=-\frac{256}{7},\frac{144}{7}$. [Ans.]

(ii) Given $\frac{x-a^2-b^2}{c^2}+\frac{c^2}{x-a^2-b^2}=2$

$\Rightarrow k+\frac{1}{k}=2$ [Putting $k=\frac{x-a^2-b^2}{c^2}$]

$\Rightarrow \frac{k^2+1}{k}=2$ $\Rightarrow k^2+1=2k \Rightarrow k^2-2k+1=0$

$\Rightarrow (k-1)^2=0 \Rightarrow (k-1)=0$ $\Rightarrow k=1$

$\Rightarrow \frac{x-a^2-b^2}{c^2}=1$ [Putting $k=\frac{x-a^2-\mathbf{b}^2}{c^2}$]

$\Rightarrow x-a^2-b^2=c^2$ $\Rightarrow x=a^2+b^2+c^2$

$\therefore$ Required solution: $x=a^2+b^2+c^2$. [Ans.]

Illustration-10: Solve the equation: $\sqrt{2x+1}+\sqrt{3x+4}=7$

Solution: Given, $\sqrt{2x+1}+\sqrt{3x+4}=7$ or $\sqrt{2x+1}=7-\sqrt{3x+4}$

Squaring on both sides, we get

$$2x+1=49+3x+4-14\sqrt{3x+4} \quad \text{or, } x+52=14\sqrt{3x+4}$$

Again, squaring on both sides, we get

$$x^2+104x+2704=196(3x+4)$$

$$\text{or, } x^2-484x+1920=0$$

Making factors, we get

$$x(x-4)-480(x-4)=0 \text{ or, } (x-4)(x-480)=0$$

Thus, either $x-4=0, i.e. x=4$, or $x-480=0, i.e. x=480$.

But $x=480$ does not satisfy the equation, hence the only root of this equation is, $x=4$

Illustration-11: Solve the equation: $\sqrt{3x^2+7x-1}+\sqrt{3x^2+7x-10}=9$

Solution: We have, $\sqrt{3x^2+7x-1}+\sqrt{3x^2+7x-10}=9$ ...(i)

Now $(3x^2+7x-1)-(3x^2+7x-10)=9$ ...(ii)

Therefore, we have $\left[\sqrt{(3x^2+7x-1)}\right]^2-\left[\sqrt{(3x^2+7x-10)}\right]^2=9$

Now, applying the formula, $a^2-b^2=(a+b)(a-b)$, where

$a=\sqrt{(3x^2+7x-1)}$ and $b=\sqrt{(3x^2+7x-10)}$, we get

$$\left[\sqrt{(3x^2+7x-1)}+\sqrt{(3x^2+7x-10)}\right]\times\left[\sqrt{(3x^2+7x-1)}-\sqrt{(3x^2+7x-10)}\right]=9$$

$$\Rightarrow 9\times\left[\sqrt{(3x^2+7x-1)}-\sqrt{(3x^2+7x-10)}\right]=9$$

$$\Rightarrow \left[\sqrt{(3x^2+7x-1)}-\sqrt{(3x^2+7x-10)}\right]=1 \quad \text{...(iii)}$$

Adding (i) and (iii), we get

$$2\sqrt{(3x^2+7x-1)}=10 \Rightarrow \sqrt{(3x^2+7x-1)}=5$$

$$\Rightarrow (3x^2+7x-1)=25 \Rightarrow 3x^2+7x-26=0$$

$$\Rightarrow 3x^2-3x+13x-26=0$$

$$\Rightarrow 3x(x-2)+13(x-2)=0 \Rightarrow (3x+13)(x-2)=0$$

Thus, either $3x+13=0, i.e. x=-13/3$ or, $x-2=0, i.e. x=2$.

Hence, the required roots are, $-13/3$ and 2.

Illustration-12: Solve the equation: $(x+1)(x+4)(x+7)(x+10)=360$

Solution: To solve such equation, first reduce it to four factors in which two factors have terms containing x^2 and x are the same. Thus, we get

$$\{(x+1)\ (x+10)\}\{(x+4)\ (x+7)\}=360$$

$$(x^2+11x+10)\ (x^2+11x+28)=360$$

Putting $x^2+11x=y$, we have $(y+10)(y+28)=360$. Thus

$$y^2+38y+280=360 \qquad \text{or } y^2+38y-80=0 \text{ or } (y-2)(y+40)=0$$

Thus, either $y-2=0, i.e. y=2$ or $y+40=0$, $i.e. y=-40$

Taking $y=2$, we have $x^2+11x-2=0$. Therefore

$$x=\frac{-11\pm\sqrt{121+8}}{2}=\frac{-11\pm\sqrt{129}}{2}$$

Similarly, taking $y=-40$, we have $x^2+11x+40=0$. Therefore

$$x=\frac{-11\pm\sqrt{121-160}}{2}=\frac{-11\pm i\sqrt{39}}{2}$$

Illustration-13: Solve the equation: $x+\sqrt{x}=\frac{6}{25}$

Solution: Putting $\sqrt{x}=y$ we have, $y^2+y=\frac{6}{25}$ or, $25y^2+25y-6=0$

Then $y=\frac{-25\pm\sqrt{625+600}}{50}=\frac{-25\pm 35}{50}$

Thus $y=\frac{-25+35}{50}=\frac{1}{5}$ and $y=\frac{-25-35}{50}=-\frac{6}{5}$

Taking $y=-6/5$ gives $\sqrt{x}=-6/5$ or, $x=36/25$.

Similarly taking $y=1/5$, gives $\sqrt{x}=1/5$, or $x=1/25$

EXTRANEOUS ROOTS

When the roots obtained by solving the equation are substituted in the equation, do not satisfy the equation, the such roots are known as extraneous roots.

Illustration-14: Solve the equation: $2x^2 - x + 5\sqrt{2x^2 - x + 4} = 10$.

Solution: Let $\sqrt{2x^2 - x + 4} = y$. Then the given equation is rewritten as:

$$(2x^2 - x + 4) - 4 + 5\sqrt{2x^2 - x + 4} = 10$$

$$\Rightarrow y^2 - 4 + 5y = 10 \Rightarrow y^2 + 5y - 14 = 0$$

On solving this equation, we get $y = \dfrac{-5 \pm \sqrt{25 + 56}}{2} = \dfrac{-5 \pm 9}{2} \Rightarrow y = 2$ or -7

Taking, $y = \sqrt{2x^2 - x + 4} = 2$, we get $2x^2 - x + 4 = 4$ or, $2x^2 - x = 0$, *i.e.* $x = 0$ or $1/2$.

Taking, $y = \sqrt{2x^2 - x + 4} = -7$, we get $2x^2 - x + 4 = 49$ or, $2x^2 - x - 45 = 0$

or, $2x^2 + 9x - 10x - 45 = 0$

or, $x(2x + 9) - 5(2x + 9) = 0$ or, $(x - 5)(2x + 9) = 0$, *i.e.* $x = 5$ or $-9/2$

Since, the roots, $x = 5$ and $-9/2$ do not satisfy the given equation, therefore these are the extraneous roots. So, the required solutions are: $x = 0, 1/2$.

Illustration-15: Solve: $\dfrac{\sqrt{12 - x}}{5} = \dfrac{3}{2 + \sqrt{12 - x}}$

Solution: Given $\dfrac{\sqrt{12 - x}}{5} = \dfrac{3}{2 + \sqrt{12 - x}}$

$\Rightarrow \dfrac{a}{5} = \dfrac{3}{2 + a}$ [Putting $a = \sqrt{12 - x}$]

$\Rightarrow a(2 + a) = 3 \times 5 \Rightarrow 2a + a^2 = 15 \quad \Rightarrow a^2 + 2a - 15 = 0 \Rightarrow a^2 + 5a - 3a - 15 = 0$

$\Rightarrow a(a + 5) - 3(a + 5) = 0 \quad \Rightarrow (a + 5)(a - 3) = 0$

$\Rightarrow (a + 5) = 0$ or, $(a - 3) = 0 \quad \Rightarrow a = -5$ or, $a = 3$

$\Rightarrow \sqrt{12 - x} = -5$ or $\sqrt{12 - x} = 3$ [Putting $a = \sqrt{12 - x}$]

$\Rightarrow 12 - x = 25$ or $12 - x = 9$ [Squaring on both sides]

$\Rightarrow x = -13$ or $x = 3$

But when $x=-13$, then

$$\text{L.H.S.} = \frac{\sqrt{12-x}}{5} \qquad \text{R.H.S.} = \frac{3}{2+\sqrt{12-x}}$$

$$= \frac{\sqrt{12-(-13)}}{5} [\because x=-13] \qquad = \frac{3}{2+\sqrt{12-(-13)}} [\because x=-13]$$

$$= \frac{\sqrt{25}}{5} = \frac{5}{5} = 1 \qquad = \frac{3}{2+5} = \frac{3}{7}$$

$\therefore$ When $x=-13$, then the given equation is not satisfied. Therefore $x=-13$ is not a solution.

Again when $x=3$, then

$$\text{L.H.S.} = \frac{\sqrt{12-x}}{5} \qquad \text{R.H.S.} = \frac{3}{2+\sqrt{12-x}}$$

$$= \frac{\sqrt{12-3}}{5} [\because x=3] \qquad = \frac{3}{2+\sqrt{12-3}} [\because x=3]$$

$$= \frac{\sqrt{9}}{5} = \frac{3}{5} \qquad = \frac{3}{2+3} = \frac{3}{5}$$

$\therefore$ when $x=3$, then the given equation is satisfied.

$\therefore$ the solution is valid for $x=3$

$\therefore$ required solution: $x=3$. [Ans.]

(b) Irrational Equations Reducible to Quadratic Form

Illustration-16: Solve the equation: $\sqrt{(2x-1)(x+3)}+\sqrt{(2x-1)(x-3)}=\sqrt{(2x-1)}$

Solution: Given $\sqrt{(2x-1)(x+3)}+\sqrt{(2x-1)(x-3)}=\sqrt{(2x-1)}$

$$\Rightarrow \sqrt{(2x-1)(x+3)}+\sqrt{(2x-1)(x-3)}-\sqrt{(2x-1)}=0$$

$$\Rightarrow \sqrt{(2x-1)}\left[\sqrt{(x+3)}+\sqrt{(x-3)}-1\right]=0$$

Therefore, either $\sqrt{(2x-1)}=0, i.e. x=1/2$ or, $\sqrt{(x+3)}+\sqrt{(x-3)}=1$

Now $\sqrt{(x+3)}+\sqrt{(x-3)}=1$

Squaring on both sides, we get

$(x+3)+(x-3)+2\sqrt{(x+3)}\sqrt{(x-3)}=1$

$\Rightarrow 2x+2\sqrt{(x+3)}\sqrt{(x-3)}=1 \Rightarrow 2\sqrt{(x+3)}\sqrt{(x-3)}=1-2x$

Squaring on both sides again, we get

$$4(x+3)(x-3)=(1-2x)^2=1+4x^2-4x$$

$$\Rightarrow 4(x^2-9)=4x^2-4x+1 \Rightarrow 4x^2-36=4x^2-4x+1$$

$$\Rightarrow 4x=37.i.e.x=37/4$$

Hence, $x=1/2$ and $37/4$ are the required roots.

Illustration-17: Solve the equation: $\sqrt{2x^2+5x-7}+\sqrt{3x^2-21x+18}=\sqrt{7x^2-6x-1}$

Solution: We get $\sqrt{2x^2+5x-7}+\sqrt{3x^2-21x+18}=\sqrt{7x^2-6x-1}$

$$\Rightarrow \sqrt{(2x+7)(x-1)}+\sqrt{(3x-18)(x-1)}=\sqrt{(x-1(7x+1))}$$

$$\Rightarrow \sqrt{(2x+7)(x-1)}+\sqrt{(3x-18)(x-1)}-\sqrt{(x-1)(7x+1)}=0$$

Taking out the common factor $\sqrt{x-1}$, we get

$$\sqrt{(x-1)}\left[\sqrt{(2x+7)}+\sqrt{(3x-18)}-\sqrt{(7x+1)}\right]=0$$

Thus, either $\sqrt{x-1}=0 \Rightarrow (x-1)=0, i.e.x=1$

or, $\sqrt{2x+7}+\sqrt{3x-18}-\sqrt{7x+1}=0, \Rightarrow \sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$

Squaring both sides, we get

$$2x+7+3x-18+2\sqrt{2x+7}\sqrt{3x-18}=7x+1 \Rightarrow 2\sqrt{2x+7}\sqrt{3x-18}=2x+12$$

Again squaring both sides, we get

$4(2x+7)(3x-18)=4x^2+144+48x \Rightarrow (2x+7)(3x-18)=x^2+36+12x$

$\Rightarrow 6x^2+12x-36x-126=x^2+36+12x \Rightarrow 5x^2-27x-162=0 \Rightarrow (5x+18)(x-9)=0$

Thus either $5x+18=0$, $i.e.x=\frac{18}{5}$ or, $x-9=0$, $i.e.x=9$.

Hence $x=-18/5$ and 9 are the required roots.

Illustration-18: Solve the equation: $5-\sqrt{11x^2-3x+1}=2x$

Solution: We have, $5-\sqrt{11x^2-3x+1}=2x \Rightarrow 5-2x=\sqrt{11x^2-3x+1}$

Squaring on both sides, we get

$(5-2x)^2=11x^2-3x+1. \Rightarrow 25+4x^2-20x=11x^2-3x+1$

$\Rightarrow -7x^2 - 17x + 24 = 0 \Rightarrow 7x^2 + 17x - 24 = 0$

$\Rightarrow 7x^2 - 7x + 24x - 24 = 0 \Rightarrow 7x(x-1) + 24(x-1) = 0 \Rightarrow (x-1)(7x+24) = 0$

Thus, either $x - 1 = 0,\ i.e. x = 1$ or, $7x + 24 = 0,\ i.e. x = -24/7$.

Hence x = 1 and -24/7 are the required roots.

Illustration-19: Solve the equation: $\sqrt{3x^2 + 5x + 6} + \sqrt{3x^2 + 5x - 9} = 5$

Solution: Putting $3x^2 + 5x = y$, we have

$$\sqrt{y+6} + \sqrt{y-9} = 5 \text{ or, } \sqrt{y-9} = 5 - \sqrt{y+6}$$

Squaring both sides, we get

$$y - 9 = 25 + (y+6) - 10\sqrt{y+6}$$

$$\Rightarrow -40 = -10\sqrt{y+6} \text{ or, } 4 = \sqrt{y+6}$$

Again, squaring both sides, we get, $16 = y + 6, i.e. y = 10$

Thus, for $y = 10$, we have

$$3x^2 + 5x = 10 \text{ or, } 3x^2 + 5x - 10 = 0$$

$$\text{or, } x = \frac{-5 \pm \sqrt{25 + 120}}{6} = \frac{-5 \pm \sqrt{145}}{6}$$

Illustration-20: Solve the equation: $\sqrt{(2x-1)} - \sqrt{(5x-4)} = \sqrt{(4x-3)} - \sqrt{(3x-2)}$

Solution: Squaring on both sides of the given equation, we get

$$(2x-1) + (5x-4) - 2\sqrt{(2x-1)}\sqrt{(5x-4)} = (4x-3) + (3x-2) - 2\sqrt{4x-3} - 3\sqrt{3x-2}$$

$$\text{or, } \sqrt{4x-3}\sqrt{3x-2} = \sqrt{2x-1}\sqrt{5x-4}$$

Squaring again on both sides, we get

$$(4x-3)(3x-2) = (2x-1)(5x-4)$$

$$\Rightarrow 12x^2 - 17x + 6 = 10x^2 - 13x + 4 \Rightarrow x^2 - 2x + 1 = 0 \text{ or, } (x-1)^2 = 0, i.e. x = 1$$

Illustration-21: Solve: $\sqrt{3x^2 - 4x - 5} + \sqrt{2x^2 - 4x - 1} = x + 2$

Solution: The given equation can also written as:

$$\frac{1}{\sqrt{3x^2 - 4x - 5} + \sqrt{2x^2 - 4x - 1}} = \frac{1}{x+2}$$

or, $\frac{1}{\sqrt{3x^2-4x-5}+\sqrt{2x^2-4x-1}} \times \frac{\sqrt{3x^2-4x-5}-\sqrt{2x^2-4x-1}}{\sqrt{3x^2-4x-5}-\sqrt{2x^2-4x-1}} = \frac{1}{x+2}$

or, $\frac{\sqrt{3x^2-4x-5}-\sqrt{2x^2-4x-1}}{(3x^2-4x-5)-(2x^2-4x-1)} = \frac{1}{x+2}$ or, $\frac{\sqrt{3x^2-4x-5}-\sqrt{2x^2-4x-1}}{x^2-4} = \frac{1}{x+2}$

or, $\sqrt{3x^2-4x-5}-\sqrt{2x^2-4x-1} = \frac{x^2-4}{x+2} = \frac{(x+2)(x-2)}{x+2} = x-2 ....(i)$

Subtracting (i) from the given equation, we get

$2\sqrt{2x^2-4x-1} = 4$ or, $\sqrt{2x^2-4x-1} = 2$ or, $2x^2-4x-1=4$ or, $2x^2-4x-5=0$

or, $x = \frac{4 \pm \sqrt{16-4\times2\times(-5)}}{2\times2} = \frac{4\pm\sqrt{56}}{4} = \frac{2\pm\sqrt{14}}{2}$

Illustration-22: Solve the following equations:

(i) $3x^2-18+\sqrt{3x^2-4x-6}=4x$. (ii) $\sqrt{3x+10}+\sqrt{9x+7}=9$.

(iii) $\sqrt{x+5}+\sqrt{x+12}=\sqrt{2x+41}$. (iv) $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$.

Solution: (i) Given $3x^2-18+\sqrt{3x^2-4x-6}=4x$

$\Rightarrow 3x^2-4x-18+\sqrt{3x^2-4x-6}=0 \Rightarrow a-18+\sqrt{a-6}=0$ [Putting $a=3x^2-4x$]

$\Rightarrow \sqrt{a-6}=18-a$ $\Rightarrow a-6=324-36a+a^2$ [Squaring on both sides]

$\Rightarrow a^2-37a+330=0$ $\Rightarrow a^2-22a-15a+330=0$

$\Rightarrow a(a-22)-15(a-22)=0 \Rightarrow (a-22)(a-15)=0$

$\Rightarrow (a-22)=0$ or $(a-15)=0$

$\Rightarrow 3x^2-4x-22=0$ or $3x^2-4x-15=0$ [Putting $a=3x^2-4x$]

$\Rightarrow x = \frac{-(-4)\pm\sqrt{(-4)^2-4(3)(-22)}}{2(3)}$ or, $3x^2-9x+5x-15=0$

$\Rightarrow x = \frac{4\pm\sqrt{280}}{6}$ or $3x(x-3)+5(x-3)=0$

$\Rightarrow x = \frac{4\pm\sqrt{4\times70}}{6}$ or $(x-3)(3x+5)=0$

$\Rightarrow x = \frac{4\pm2\sqrt{70}}{6}$ or $x-3=0$ or $3x+5=0$

$\Rightarrow x = \frac{2(2\pm\sqrt{70})}{6}$ or $x=3$ or $3x=-5$

$\Rightarrow x = \frac{2 \pm \sqrt{70}}{3}$ or $x = 3$ or $x = -\frac{5}{3}$

$\therefore$ Required solutions: $x = \frac{2 \pm \sqrt{70}}{3}, 3, -\frac{5}{3}.$ [Ans.]

(ii) Given $\sqrt{3x+10} + \sqrt{9x+7} = 9$

$\Rightarrow \sqrt{3x+10} = 9 - \sqrt{9x+7}$ $\quad \Rightarrow 3x + 10 = 81 - 18\sqrt{9x+7} + 9x + 7$ [Squaring on both sides]

$\Rightarrow 18\sqrt{9x+7} = 78 + 6x$ $\quad \Rightarrow 324(9x+7) = 6084 + 936x + 36x^2$ [Squaring on both sides]

$\Rightarrow 2916x + 2268 = 6084 + 936x + 36x^2$

$\Rightarrow 36x^2 - 1980x + 3816 = 0$ $\quad \Rightarrow 36(x^2 - 55x + 106) = 0$

$\Rightarrow x^2 - 55x + 106 = 0$ $\quad \Rightarrow x^2 - 53x - 2x + 106 = 0$

$\Rightarrow x(x-53) - 2(x-53) = 0$ $\quad \Rightarrow (x-53)(x-2) = 0$

$\Rightarrow (x-53) = 0$ or $(x-2) = 0$ $\quad \Rightarrow x = 53$ or $x = 2$

But when $x = 53$, then the given equation is not satisfied. Here $x = 53$ is called extraneous root.

When $x = 2$, then the given equation is satisfied.

$\therefore$ The solution is valid for $x = 2$.

$\therefore$ Required solution: $x = 2$. [Ans.]

(iii) Given $\sqrt{x+5} + \sqrt{x+12} = \sqrt{2x+41}$

$\Rightarrow x + 5 + 2\sqrt{x+5}\sqrt{x+12} + x + 12 = 2x + 41$ [Squaring on both sides]

$\Rightarrow 2\sqrt{x+5}\sqrt{x+12} = 24$ $\quad \Rightarrow \sqrt{x+5}\sqrt{x+12} = 12$

$\Rightarrow (x+5)(x+12) = 144$ [Squaring on both sides]

$\Rightarrow x^2 + 12x + 5x + 60 = 144$ $\quad \Rightarrow x^2 + 17x - 84 = 0$

$\Rightarrow x^2 + 21x - 4x - 84 = 0$ $\quad \Rightarrow x(x+21) - 4(x+21) = 0$

$\Rightarrow (x+21)(x-4) = 0$ $\quad \Rightarrow x + 21 = 0$ or $x - 4 = 0$

$\Rightarrow x = -21$ or $x = 4$

But when $x = -21$, then the given equation is not satisfied. Here $x = -21$ is called extraneous root.

Again when $x = 4$, then the given equation is satisfied.

$\therefore$ The solution is valid for $x = 4$.

$\therefore$ Required solution: $x = 4$. [Ans.]

(iv) Given $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$

$\Rightarrow \sqrt{a+36}-\sqrt{a+9}=3$ [Putting $x^2-3x=a$] $\Rightarrow \sqrt{a+36}=3+\sqrt{a+9}$

$\Rightarrow a+36=9+6\sqrt{a+9}+a+9$ [Squaring on both sides] $\Rightarrow 18=6\sqrt{a+9}$

$\Rightarrow 3=\sqrt{a+9}$

$\Rightarrow 9=a+9$ [Squaring] $\Rightarrow a=0 \Rightarrow x^2-3x=0$ [$\because a=x^2-3x$] $\Rightarrow x(x-3)=0$

Therefore, either $x=0$ or $(x-3)=0 \Rightarrow x=0$ or $x=3$

$\therefore$ Required solutions: $x=0,3$. [Ans.]

(c) Reciprocal Equation: If an equation in x remains unchanged by changing x to 1/x, then such type of equations are known as reciprocal equations. Moreover, if α is the root of such equation, then $1/\alpha$ is also a root of this equation. Thus, the roots of a reciprocal equation occur in pair $\alpha, 1/\alpha, \beta, 1/\beta$, etc. If the degree of equation is odd, then there must be a root which is its own reciprocal.

Illustration-23: Solve the equation: $8x^4-54x^3+101x^2-54x+8=0$

Solution: Rearranging the terms, we have

$$8(x^4+1)-54(x^3+x)+101x^2=0$$

Dividing both sides with x^2, we have

$$8\left(x^2+\frac{1}{x^2}\right)-54\left(x+\frac{1}{x}\right)+101=0$$

$$\Rightarrow 8\left\{\left(x+\frac{1}{x}\right)^2-2\right\}-54\left(x+\frac{1}{x}\right)+101=0 \quad \left[\because a^2+b^2=(a+b)^2-2ab\right]$$

Let $x+\frac{1}{x}=y$. Then we have

$8(y^2-2)-54y+101=0$ or, $8y^2-54y+85=0$

or, $8y^2-20y-34y+85=0$ or, $4y(2y-5)-17(2y-5)=0$ or, $(4y-17)(2y-5)=0$

Thus, either $4y-17=0$, *i.e.* $y=17/4$ or, $2y-5=0$, *i.e.* $y=5/2$

Taking $y=\frac{17}{4}$, gives $x+\frac{1}{x}=\frac{17}{4} \Rightarrow \frac{x^2+1}{x}=\frac{17}{4}$

$\Rightarrow 4x^2-17x+4=0 \Rightarrow (4x-1)(x-4)=0$, *i.e.* $x=1/4$ or, 4

Taking $y=\frac{5}{2}$, gives $x+\frac{1}{x}=\frac{5}{2} \Rightarrow \frac{x^2+1}{x}=\frac{5}{2}.$

$\Rightarrow 2x^2-5x+2=0 \Rightarrow (2x-1)(x-2)=0$, *i.e.* $x=1/2$ or, 2

QUADRATIC SIMULTANEOUS EQUATIONS

Cases: (a) when one equation is linear and the other is quadratic

(b) When both equations are quadratic

(a) One Linear and One Quadratic Equation

There is no particular method for solving the simultaneous equations when one equation is linear and the other one is quadratic. However, the method which is generally used to solve such type of problems consists of

(i) Expressing one variable say x, in terms of another, say y from linear equation,

(ii) Substituting the value of x in quadratic equation, and solve for y,

(iii) Obtaining the corresponding values of x.

Illustration-24: Solve the equations: $x^2+y^2=29$ and $x-y=3$

Solution: Given $x-y=3 \Rightarrow x=3+y$

Substituting $x=3+y$ in $x^2+y^2=29$, we get

$$(3+y)^2+y^2=29 \text{ or } y^2+6y+9+y^2=29$$

or, $2y^2+6y-20=0$ or $y^2+3y-10=0$

or, $y^2+5y-2y-10=0$ or, $y(y+5)-2(y+5)=0$

or, $(y+5)(y-2)=0$ or, $y=-5,2$

For $y=-5, x=3-5=-2$ and for $y=2, x=3+2=5$.

Thus, the solutions are $(x=-2, y=-5)$ and $(x=5, y=2)$

Illustration-25: Solve the equations: $\frac{1}{x^2}+\frac{1}{y^2}=\frac{17}{4}$ and $\frac{1}{x}+\frac{1}{y}=\frac{3}{2}$

Solution: Let $1/x=u$ and $1/y=v$, then the given equations reduces to $u^2+v^2=17/4$ and $u+v=3/2$ or $u=3/2-v$

Substituting the value of u in quadratic equation, we get

$$\left(\frac{3}{2}-v\right)^2+v^2=\frac{17}{4} \text{ or } 2v^2-3v+\frac{9}{4}=\frac{17}{4}$$

or $\frac{8v^2-12v+9}{4}=\frac{17}{4}$ or $8v^2-12v+9=17$

or $8v^2-12v-8=0$ or $8v^2-16v+4v-8=0$

or $8v(v-2)+4(v-2)=0$ or $(8v+4)(v-2)=0$

Thus, either $v-2=0, i.e. v=2$ or $8v+4=0, i.e. v=-1/2$

For $v=2, u=(3/2)-2=-1/2$ and for $v=-1/2, u=(3/2)+1/2=2$.

Since $u=1/x$, therefore for $u=-1/2, x=-2$ and for $u=2, x=\frac{1}{2}$. Also, $v=1/y$,

therefore for $v=2, y=\frac{1}{2}$ and for $v=-1/2, y=-2$

Hence, the solutions are $(x=-2, y=1/2)$ and $(x=1/2, y=-2.)$.

Illustration-26: Solve the equations: $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{13}{6}$ and $xy=36$

Solution: Given $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{13}{6}$, $xy=36$,

Now $xy=36$,

$\Rightarrow y=\frac{36}{x}$.............(1)

Also $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{13}{6}$

$\Rightarrow \sqrt{\frac{x}{36/x}}+\sqrt{\frac{36/x}{x}}=\frac{13}{6}$ [From (1), we get]

$\Rightarrow \sqrt{\frac{x}{1}\times\frac{x}{36}}+\sqrt{\frac{36}{x}\times\frac{1}{x}}=\frac{13}{6}$ $\Rightarrow \sqrt{\frac{x^2}{36}}+\sqrt{\frac{36}{x^2}}=\frac{13}{6}\Rightarrow \frac{x}{6}+\frac{6}{x}=\frac{13}{6}$

$\Rightarrow \frac{x^2+36}{6x}=\frac{13}{6}\Rightarrow \frac{x^2+36}{x}=\frac{13}{1}$ $\Rightarrow x^2+36=13x\Rightarrow x^2-13x+36=0$

$\Rightarrow x^2-9x-4x+36=0\Rightarrow x(x-9)-4(x-9)=0$

$\Rightarrow (x-9)(x-4)=0\Rightarrow x=9$ or $x=4$

When $x=9$, then

$y=\frac{36}{x}$

$\Rightarrow y=\frac{36}{9}\Rightarrow y=4$

When $x=4$, then

$y=\frac{36}{x}$

$\Rightarrow y=\frac{36}{4}\Rightarrow y=9$

Required solutions: $\left.\begin{matrix} x=9,4 \\ y=4,9 \end{matrix}\right\}$ [Ans.]

(c) TWO QUADRATIC SIMULTANEOUS EQUATIONS

When the equations are homogeneous and of same degree, i.e.all the terms containing the unknown x and y are of same degree, then the solution may be obtained by reducing them to one linear (first degree) equation and one quadratic equation.

Illustration-27: Solve the equation: $3x^2 - 31xy + 5y^2 = -45$ and $3x^2 + xy + y^2 = 15$

Solution: Putting $y = mx$ in both the equations, we have

$$3x^2 - 31x(mx) + 5(mx)^2 = -45 \text{ or, } x^2(3 - 31m + 5m^2) = -45 \text{......}(i)$$

and $$3x^2 + x(mx) + (mx)^2 = 15 \text{ or, } x^2(3 + m + m^2) = 15 \text{...............}(ii)$$

Dividing (i) by (ii), we get

$$\frac{3 - 31m + 5m^2}{3 + m + m^2} = -\frac{45}{15} = -\frac{3}{1} \text{ or, } 8m^2 - 28m + 12 = 9$$

or, $2m^2 - 7m + 3 = 0$ or $(m-3)(2m-1) = 0$, i.e. $m = 3$ or $1/2$.

If $m = 3$, then from (i) we have

$$-45x^2 = -45 \text{ i.e. } x = \pm 1 \text{ and } y = mx = \pm 3.$$

If $m = \frac{1}{2}$, then from (i) we have

$$x^2 = 4, \ i.e, \ x = \pm 2 \ and \ y = mx = \pm 1$$

Thus the required solution set is $(\pm 1, \pm 3), (\pm 2, \pm 1)$

Illustration-28: Solve the equation:

$4^x.2^y = 128$ ...(i)

$3^{3x+2y} = 9^{xy}$ ...(ii)

for x and y.

Solution: From equation (i) we have

$$(2^2)^x.2^y = 2^7 \text{ or, } 2^{2x}2^y = 2^7 \text{ or, } 2^{2x+y} = 2^7 \text{ or, } 2x + y = 7 \text{ ....(iii)}$$

Using this, we have from equation (ii)

$$3^{3x}.3^{2y} = (3^2)^{xy} \text{ or, } 3^{3x+2y} = 3^{2xy}, \text{ i.e. } 3x + 2y = 2xy \text{ ....(iv)}$$

From (iii), we get $y = 7 - 2x$. Sub stituting this value of y in (iv), we get

$$3x + 2(7 - 2x) = 2x(7 - 2x) \text{ or } 3x + 14 - 4x = 14x - 4x^2$$

or, $4x^2 - 15x + 14 = 0$ or $(x-2)(4x-7) = 0$, i.e. $x = 2, 7/4$

If $x = 2$, from(iii)we get: $y = 7 - 2(2) = 3$ and for $x = 7/4$, we get $y = 7 - 2(7/4) = 7/2$.

Hence, the required solution set is: $(x = 2, y = 3)$ and $(x = 7/4, y = 7/2)$ [Ans]

Illustration-29: Solve the equations: $x + \frac{1}{y} = 6$ and $y + \frac{1}{x} = \frac{3}{4}$.

Solution: From first equation, we get $xy + 1 = 6y$ and from second equation, we get $xy + 1 = \frac{3x}{4}$. Since left hand side in both the cases is same, therefore

$$6y = \frac{3x}{4} \text{ or } y = \frac{x}{8}$$

Substituting, $y = x/8$ in equation: $xy + 1 = 6y$, we get

$$x \times (x/8) + 1 = 6(x/8) \text{ or } x^2 + 8 = 6x \text{ or } x^2 - 6x + 8 = 0$$

or $(x-2)(x-4) = 0, i.e. x = 2$ or 4.

Now for $x = 2; y = x/8 = \frac{1}{4}$, and for $x = 4; y = x/8 = 4/8 = \frac{1}{2}$

Hence, the required solutions are: $\left(x = 2, y = \frac{1}{4}\right)$ and $\left(x = 4, y = \frac{1}{2}\right)$

Illustration-30: Solve the following equations:

(i) $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 = 0$, (ii) $x^{\frac{2}{13}} + 12 = 7x^{\frac{1}{13}}$, (iii) $6x^{\frac{3}{4}} + 3x^{\frac{1}{4}} = 11x^{\frac{1}{4}}$,

(iv) $\frac{16}{x^{3/2}} + \frac{x^{1/2}}{2} = \frac{6}{x^{1/2}}$

Solution: (i) Given $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 = 0$

$\Rightarrow \left(x^{\frac{1}{3}}\right)^2 + 2x^{\frac{1}{3}} - x^{\frac{1}{3}} - 2 = 0$ $\Rightarrow x^{\frac{1}{3}}\left(x^{\frac{1}{3}} + 2\right) - 1\left(x^{\frac{1}{3}} + 2\right) = 0$

$\Rightarrow \left(x^{\frac{1}{3}} + 2\right)\left(x^{\frac{1}{3}} - 1\right) = 0$

Therefore, either $\left(x^{\frac{1}{3}} + 2\right) = 0$ or, $\left(x^{\frac{1}{3}} - 1\right) = 0$

$\Rightarrow x^{\frac{1}{3}} = -2$ or, $x^{\frac{1}{3}} = 1$ $\Rightarrow \left(x^{\frac{1}{3}}\right)^3 = (-2)^3$ or, $\left(x^{\frac{1}{3}}\right)^3 = (1)^3$

$\Rightarrow x = -8$ or $x = 1$

$\therefore$ Required solutions: $x = -8, 1$. [Ans.]

(ii) Given $x^{\frac{2}{13}} + 12 = 7x^{\frac{1}{13}}$

$\Rightarrow \left(x^{\frac{1}{13}}\right)^2 - 4x^{\frac{1}{13}} - 3x^{\frac{1}{13}} + 12 = 0$ $\Rightarrow x^{\frac{1}{13}}\left(x^{\frac{1}{13}} - 4\right) - 3\left(x^{\frac{1}{13}} - 4\right) = 0$

$\Rightarrow \left(x^{\frac{1}{13}} - 4\right)\left(x^{\frac{1}{13}} - 3\right) = 0$

Therefore, either $\left(x^{\frac{1}{13}} - 4\right) = 0$ or, $\left(x^{\frac{1}{13}} - 3\right) = 0$

$\Rightarrow x^{\frac{1}{13}} = 4$ or, $x^{\frac{1}{13}} = 3$ $\Rightarrow \left(x^{\frac{1}{13}}\right)^{13} = (4)^{13}$ or, $\left(x^{\frac{1}{13}}\right)^{13} = (3)^{13}$

$\Rightarrow x = 4^{13}$ or, $x = 3^{13}$

$\therefore$ Required solution: $x = 4^{13}, 3^{13}$. [Ans.]

(iii) Given $6x^{\frac{3}{4}} + 3x^{\frac{1}{4}} = 11x^{\frac{1}{4}}$

$\Rightarrow 6x^{\frac{3}{4}} - 11x^{\frac{1}{4}} + \frac{3}{x^{\frac{1}{4}}} = 0$ $\Rightarrow \frac{6x^{\frac{3}{4}}.x^{\frac{1}{4}} - 11x^{\frac{1}{4}}.x^{\frac{1}{4}} + 3}{x^{\frac{1}{4}}} = 0$

$\Rightarrow 6x^{\frac{3+1}{4}} - 11x^{\frac{1+1}{4}} + 3 = 0 \times x^{\frac{1}{4}}$ $\Rightarrow 6x - 11x^{\frac{1}{2}} + 3 = 0$

$\Rightarrow 6\left(x^{\frac{1}{2}}\right)^2 - 9x^{\frac{1}{2}} - 2x^{\frac{1}{2}} + 3 = 0$ $\Rightarrow 3x^{\frac{1}{2}}\left(2x^{\frac{1}{2}} - 3\right) - 1\left(2x^{\frac{1}{2}} - 3\right) = 0$

$\Rightarrow \left(2x^{\frac{1}{2}} - 3\right)\left(3x^{\frac{1}{2}} - 1\right) = 0$

Therefore, either $2x^{\frac{1}{2}} - 3 = 0$ or $3x^{\frac{1}{2}} - 1 = 0$

$\Rightarrow 2x^{\frac{1}{2}} = 3$ or $3x^{\frac{1}{2}} = 1$ $\Rightarrow x^{\frac{1}{2}} = \frac{3}{2}$ or $x^{\frac{1}{2}} = \frac{1}{3}$

$\Rightarrow \left(x^{\frac{1}{2}}\right)^2 = \left(\frac{3}{2}\right)^2$ or $\left(x^{\frac{1}{2}}\right)^2 = \left(\frac{1}{3}\right)^2$ [Squaring on both sides]

$\Rightarrow x = \frac{9}{4}$ or $x = \frac{1}{9}$ $\therefore$ Required solutions: $x = \frac{9}{4}, \frac{1}{9}$. [Ans.]

(iv) We have, $\frac{16}{x^{3/2}} + \frac{x^{1/2}}{2} = \frac{6}{x^{1/2}}$ or, $\frac{16}{x^{3/2}} - \frac{6}{x^{1/2}} = -\frac{x^{1/2}}{2}$

$\Rightarrow \frac{1}{x^{1/2}}\left[\frac{16}{x} - \frac{6}{1}\right] = -\frac{x^{1/2}}{2} \Rightarrow 2\left[\frac{16}{x} - 6\right] = -x^{1/2} \times x^{1/2}$.

$\Rightarrow \frac{32}{x} - 12 = -x \Rightarrow \frac{32 - 12x}{x} = -x \Rightarrow 32 - 12x = -x^2$

$\Rightarrow x^2 - 12x + 32 = 0$ or, $(x-8)(x-4) = 0$

Thus, either $x - 8 = 0$, i.e., $x = 8$ or, $x - 4 = 0$, i.e. $x = 4$.

Hence, required roots are, $x = 8$ and 4.

Illustration-31: Solve the following equations:

(i) $\left(x - \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 12 = 0$, **(ii)** $\left(x - \frac{1}{x}\right)^2 - 10\left(x - \frac{1}{x}\right) + 24 = 0$,

(iii) $\left(x - \frac{1}{x}\right)^2 + 9 = \frac{5}{2}\left(x + \frac{1}{x} + 2\right)$.

Solution: (i) Given $\left(x - \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 12 = 0$

$\Rightarrow \left(x + \frac{1}{x}\right)^2 - 4.x.\frac{1}{x} - 6\left(x + \frac{1}{x}\right) + 12 = 0$ $\Rightarrow \left(x + \frac{1}{x}\right)^2 - 4 - 6\left(x + \frac{1}{x}\right) + 12 = 0$

$\Rightarrow \left(x + \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 8 = 0$ $\Rightarrow a^2 - 6a + 8 = 0$ [Putting $a = x + \frac{1}{x}$]

$\Rightarrow a^2 - 4a - 2a + 8 = 0$ $\Rightarrow a(a-4) - 2(a-4) = 0$

$\Rightarrow (a-4)(a-2) = 0$

Therefore, either $a - 4 = 0$ or $a - 2 = 0$

$\Rightarrow x+\frac{1}{x}-4=0$ or $x+\frac{1}{x}-2=0$ [Putting $a=x+\frac{1}{x}$]

$\Rightarrow \frac{x^2+1-4x}{x}=0$ or $\frac{x^2+1-2x}{x}=0$ $\Rightarrow x^2-4x+1=0$ or $x^2-2x+1=0$

$\Rightarrow x=\frac{-(-4)\pm\sqrt{(-4)^2-4.1.1}}{2.1}$ or $(x-1)^2=0$ $\Rightarrow x=\frac{4\pm\sqrt{12}}{2}$ or $(x-1)=0$

$\Rightarrow x=\frac{4\pm2\sqrt{3}}{2}$ or $x=1$ $\Rightarrow x=2\pm\sqrt{3}$ or $x=1$

$\therefore$ Required solutions: $x=2\pm\sqrt{3},1.$ [Ans.]

(ii) Given $\left(x-\frac{1}{x}\right)^2-10\left(x-\frac{1}{x}\right)+24=0$

$\Rightarrow a^2-10a+24=0$ [Putting $a=x-\frac{1}{x}$] $\Rightarrow a^2-6a-4a+24=0$

$\Rightarrow a(a-6)-4(a-6)=0$ $\Rightarrow (a-6)(a-4)=0$

Therefore either $a-6=0$ or $a-4=0$

$\Rightarrow x-\frac{1}{x}-6=0$ or $x-\frac{1}{x}-4=0$ [Putting $a=x-\frac{1}{x}$]

$\Rightarrow \frac{x^2-1-6x}{x}=0$ or $\frac{x^2-1-4x}{x}=0$

$\Rightarrow x^2-6x-1=0$ or $x^2-4x-1=0$

$\Rightarrow x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-1)}}{2(1)}$ or $x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$

$\Rightarrow x=\frac{6\pm\sqrt{40}}{2}$ or $x=\frac{4\pm\sqrt{20}}{2}$ $\Rightarrow x=\frac{6\pm2\sqrt{10}}{2}$ or $x=\frac{4\pm2\sqrt{5}}{2}$

$\Rightarrow x=3\pm\sqrt{10}$ or, $x=2\pm\sqrt{5}$

$\therefore$ Required solutions: $x=3\pm\sqrt{10},2\pm\sqrt{5}.$ [Ans.]

(iii) Given $\left(x-\frac{1}{x}\right)^2+9=\frac{5}{2}\left(x+\frac{1}{x}+2\right)$

$\Rightarrow \left(x+\frac{1}{x}\right)^2-4.x.\frac{1}{x}+9=\frac{5}{2}\left(x+\frac{1}{x}+2\right)$ $\Rightarrow \left(x+\frac{1}{x}\right)^2-4+9=\frac{5}{2}\left(x+\frac{1}{x}+2\right)$

$\Rightarrow a^2+5=\frac{5}{2}(a+2)$ [Putting $a=x+\frac{1}{x}$]

$\Rightarrow 2a^2+10=5a+10$ $\Rightarrow 2a^2-5a=0 \Rightarrow a(2a-5)=0$

Therefore either $a=0$ or $2a-5=0$

$\Rightarrow x+\frac{1}{x}=0$ or $2\left(x+\frac{1}{x}\right)-5=0$ [Putting $a=x+\frac{1}{x}$]

$\Rightarrow \frac{x^2+1}{x}=0$ or $2\left(\frac{x^2+1}{x}\right)-5=0$ $\Rightarrow x^2+1=0$ or $\frac{2x^2+2-5x}{x}=0$

$\Rightarrow x^2=-1$ or $2x^2-5x+2=0$ $\Rightarrow x=\pm\sqrt{-1}$ or $2x^2-4x-x+2=0$

$\Rightarrow x=\pm i$ or $2x(x-2)-1(x-2)=0$ $\Rightarrow x=\pm i$ or $(x-2)(2x-1)=0$

$\Rightarrow x=\pm i$ or $(x-2)=0$ or $(2x-1)=0$ $\Rightarrow x=\pm i$ or $x=2$ or $2x=1$

$\Rightarrow x=\pm i$ or $x=2$ or $x=\frac{1}{2}$

$\therefore$ Required solutions: $x=\pm i, 2, \frac{1}{2}$. [Ans.]

Illustration-32: Solve the following equations;

(i) $10x^4+63x^3+52x^2-63x+10=0$, **(ii)** $x^4+2x^3-13x^2+2x+1=0$,

(iii) $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=3$.

Solution: (i) Given $10x^4+63x^3+52x^2-63x+10=0$

$\Rightarrow 10(x^4+1)+63(x^3-x)+52x^2=0$

Dividing both sides by x^2, we get

$10\left(x^2+\frac{1}{x^2}\right)+63\left(x-\frac{1}{x}\right)+52=0$ $\Rightarrow 10\left\{\left(x-\frac{1}{x}\right)^2+2.x.\frac{1}{x}\right\}+63\left(x-\frac{1}{x}\right)+52=0$

$\Rightarrow 10\left(x-\frac{1}{x}\right)^2+63\left(x-\frac{1}{x}\right)+72=0$ $\Rightarrow 10a^2+63a+72=0$ [Putting $a=x+\frac{1}{x}$]

$\Rightarrow 10a^2+15a+48a+72=0$ $\Rightarrow 5a(2a+3)+24(2a+3)=0$

$\Rightarrow (5a+24)(2a+3)=0$

$\therefore a = -\frac{24}{5}$, or $-\frac{3}{2}$.

When $a = -\frac{24}{5}$, we get

$x - \frac{1}{x} = -\frac{24}{5}$ [Putting $a = x - \frac{1}{x}$] $\Rightarrow \frac{x^2 - 1}{x} = -\frac{24}{5} \Rightarrow 5x^2 - 5 = -24x$

$\Rightarrow 5x^2 + 24x - 5 = 0$

$$\Rightarrow x = \frac{-24 \pm \sqrt{(24)^2 - 4(5)(-5)}}{2(5)} = \frac{-24 \pm \sqrt{576 + 100}}{10} = \frac{-24 \pm 26}{10} \quad \Rightarrow x = -5, \frac{1}{5}.$$

Again when $a = -\frac{3}{2}$, we get

$x - \frac{1}{x} = -\frac{3}{2}$ [Putting $a = x - \frac{1}{x}$] $\Rightarrow \frac{x^2 - 1}{x} = -\frac{3}{2} \Rightarrow 2x^2 - 2 = -3x \quad \Rightarrow 2x^2 + 3x - 2 = 0$

$$\Rightarrow x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm \sqrt{9 + 16}}{4} = \frac{-3 \pm 5}{4} \quad \Rightarrow x = -2, \frac{1}{2}.$$

Therefore, required solutions are: $-5, -2, \frac{1}{5}, \frac{1}{2}$. [Ans].

(ii) Given $x^4 + 2x^3 - 13x^2 + 2x + 1 = 0$

$\Rightarrow (x^4 + 1) + 2x(x^2 + 1) - 13x^2 = 0 \quad \Rightarrow x^2\left(x^2 + \frac{1}{x^2}\right) + 2x^2\left(x + \frac{1}{x}\right) - 13x^2 = 0$

$\Rightarrow x^2\left\{\left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 13\right\} = 0 \quad \Rightarrow \left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 13 = 0$

$\Rightarrow \left(x + \frac{1}{x}\right)^2 - 2.x.\frac{1}{x} + 2\left(x + \frac{1}{x}\right) - 13 = 0 \quad \Rightarrow a^2 - 2 + 2a - 13 = 0$ [Putting $a = x + \frac{1}{x}$]

$\Rightarrow a^2 + 2a - 15 = 0 \Rightarrow a^2 + 5a - 3a - 15 = 0 \Rightarrow a(a + 5) - 3(a + 5) = 0 \Rightarrow (a + 5)(a - 3) = 0$

Therefore, either $a + 5 = 0$ or $a - 3 = 0$

$\Rightarrow x + \frac{1}{x} + 5 = 0$ or $x + \frac{1}{x} - 3 = 0$ [Putting $a = x + \frac{1}{x}$]

$\Rightarrow \frac{x^2 + 1 + 5x}{x} = 0$ or $\frac{x^2 + 1 - 3x}{x} = 0 \quad \Rightarrow x^2 + 5x + 1 = 0$ or $x^2 - 3x + 1 = 0$

$$\Rightarrow x = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(1)}}{2(1)} \text{ or } x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$

$\Rightarrow x = \frac{-5 \pm \sqrt{21}}{2}$ or $x = \frac{3 \pm \sqrt{5}}{2}$

$\therefore$ Required solution: $x = \frac{-5 \pm \sqrt{21}}{2}, \frac{3 \pm \sqrt{5}}{2}$. [Ans.]

(iii) Given $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = 3$

$\Rightarrow \sqrt{1+x^2}+\sqrt{1-x^2} = 3\sqrt{1+x^2} - 3\sqrt{1-x^2}$ $\qquad \Rightarrow 4\sqrt{1-x^2} = 2\sqrt{1+x^2}$

$\Rightarrow 4(1-x^2) = 1+x^2$ [Squaring on both sides] $\qquad \Rightarrow 4-4x^2 = 1+x^2$

$\Rightarrow -5x^2 = -3$ $\qquad \Rightarrow x^2 = \frac{3}{5}$ $\qquad \Rightarrow x = \pm\sqrt{\frac{3}{5}}$

$\therefore$ Required solutions: $x = \pm\sqrt{\frac{3}{5}}$. **[Ans.]**

Illustration-33: Solve the following simultaneous equations:

(a) $x^2 + y^2 = 25, x + y = 7,$ $\qquad$ (b) $x^2 + y^2 = 185, x - y = 3$

Solution:

(a) Given $x^2 + y^2 = 25,\ x + y = 7,$

Now $x + y = 7$ $\qquad \Rightarrow y = 7 - x$............(1)

Also $x^2 + y^2 = 25$

$\Rightarrow x^2 + (7-x)^2 = 25$ [From (1) we get]

$\Rightarrow x^2 + 49 - 4x + x^2 = 25$ $\qquad \Rightarrow 2x^2 - 14x + 24 = 0$

$\Rightarrow x^2 - 7x + 12 = 0$ $\qquad \Rightarrow x^2 - 4x - 3x + 12 = 0$

$\Rightarrow x(x-4) - 3(x-4) = 0$ $\qquad \Rightarrow (x-4)(x-3) = 0$

$\Rightarrow x - 4 = 0$ *or* $x - 3 = 0$ $\qquad \Rightarrow x = 4$ *or* $x = 3$

when x=4 then $\qquad$ when x=3, then

$y = 7 - x$ $\qquad$ $y = 7 - x$

$\Rightarrow y = 7 - 4 = 3$ $\qquad \Rightarrow y = 7 - 3 = 4$

Required solutions: $\left.\begin{array}{l} x = 3, 4 \\ y = 4, 3 \end{array}\right\}$ [Ans]

(b) Given $x^2 + y^2 = 185, x - y = 3,$

Now $x - y = 3$

$\Rightarrow x = 3 + y$...............(1)

Also $x^2 + y^2 = 185$

$\Rightarrow (3+y)^2 + y^2 = 185$ [From (1), we get]

$\Rightarrow 9 + 6y + y^2 + y^2 = 185$ $\Rightarrow 2y^2 + 6y - 176 = 0$

$\Rightarrow y^2 + 3y - 88 = 0$ $\Rightarrow y^2 + 11y - 8y - 88 = 0$

$\Rightarrow y(y+11) + 8(y-11) = 0$ $\Rightarrow (y+11)(y-8) = 0$

$\Rightarrow y + 11 = 0$ *or* $y - 8 = 0$ $\Rightarrow y = -11$ *or* $y = 8$

when $y = -11$, then When $y = 3$, then

$x = 3 + y$ $x = 3 + y$

$\Rightarrow x = 3 - 11$ $\Rightarrow x = 3 + 8$

$\Rightarrow x = -8$ $\Rightarrow x = 11$

Required solutions: $\left.\begin{array}{l} x = 11, -8 \\ y = 8, -11 \end{array}\right\}$ [Ans]

Illustration-34: Solve the following simultaneous equations:

(a) $\frac{1}{x} + \frac{1}{y} = \frac{5}{6},\ x + y = 5.$ (b) $\frac{1}{x} + \frac{1}{y} = \frac{7}{12},\ xy = 12$

(c) $\frac{x}{y} + \frac{y}{x} = 5\frac{1}{5},\ x + y = 6$ (d) $\frac{x}{2} + \frac{y}{5} = 5,\ \frac{2}{x} + \frac{5}{y} = \frac{5}{6}$

Solution: (a) Given $\frac{1}{x} + \frac{1}{y} = \frac{5}{6},\ x + y = 5$

Now $x + y = 5$

$\Rightarrow y = 5 - x$.............(1)

Also $\frac{1}{x} + \frac{1}{y} = \frac{5}{6} \Rightarrow \frac{1}{y} = \frac{5}{6} - \frac{1}{x}$

$\Rightarrow \frac{1}{y} = \frac{5x - 6}{6x} \Rightarrow \frac{y}{1} = \frac{6x}{5x - 6}$ [inversing on both sides]

$\Rightarrow \frac{5 - x}{1} = \frac{6x}{5x - 6}$ [From (1), we get]

$\Rightarrow (5 - x)(5x - 6) = 6x$ $\Rightarrow 25x - 30 - 5x^2 + 6x = 6x$

$\Rightarrow -5x^2 + 25x - 30 = 0$ $\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2-3x-2x+6=0$ $\Rightarrow x(x-3)-2(x-3)=0 \Rightarrow (x-3)(x-2)=0$

$\Rightarrow x=3$ *or* $x=2$

when $x=3$ then $\Rightarrow y=5-x$ $\Rightarrow y=5-3$ $\Rightarrow y=2$

when $x=2$, then $y=5-x$ $\Rightarrow y=5-2$ $\Rightarrow y=3$

Required solutions: $\left.\begin{array}{l} x=3,\ 2 \\ y=2,\ 3 \end{array}\right\}$ [Ans]

(b) Given $\frac{1}{x}+\frac{1}{y}=\frac{7}{12}$, $xy=12$.

Now $xy=12$

$\Rightarrow y=\frac{12}{x}$...............(1)

Also $\frac{1}{x}+\frac{1}{y}=\frac{7}{12}$ $\Rightarrow \frac{1}{y}=\frac{7}{12}-\frac{1}{x}$

$\Rightarrow \frac{1}{\frac{12}{x}}=\frac{7x-12}{12x}$ [From (1). we get]

$\Rightarrow \frac{x}{12}=\frac{7x-12}{12x}$ $\Rightarrow \frac{x}{1}=\frac{7x-12}{x}$

$\Rightarrow x^2=7x-12$ $\Rightarrow x^2-7x+12=0$

$\Rightarrow x^2-4x-3x+12=0$ $\Rightarrow x(x-4)-3(x-4)=0$

$\Rightarrow (x-4)(x-3)=0$ $\Rightarrow x=4$ *or* $x=3$

when $x=4$, then $y=\frac{12}{x}$ $\Rightarrow y=\frac{12}{4} \Rightarrow y=3$

when $x=3$, then $y=\frac{12}{x}$ $y=\frac{12}{3} \Rightarrow y=4$

Required Solutions: $\left.\begin{array}{l} x=4,\ 3 \\ y=3,\ 4 \end{array}\right\}$

[Ans].

(c) Given $\frac{x}{y}+\frac{y}{x}=5\frac{1}{5}$, $x+y=6$,

Now $x+y=6$

$\Rightarrow y=6-x$.................(1)

Also $\frac{x}{y}+\frac{y}{x}=5\frac{1}{5}$

$\Rightarrow \frac{x}{6-x}+\frac{6-x}{x}=\frac{26}{5}$ [From (1), we get]

$\Rightarrow \frac{x^2+(6-x)(6-x)}{(6-x)x}=\frac{26}{5}$

$\Rightarrow \frac{x^2+36-6x-6x+x^2}{6x-x^2}=\frac{26}{5}$

$\Rightarrow \frac{2x^2-12x+36}{6x-x^2}=\frac{26}{5}$

$\Rightarrow 10x^2-60x+180=156x-26x^2$

$\Rightarrow 36x^2-216x+180=0$

$\Rightarrow x^2-6x+5=0$

$\Rightarrow x^2-5x-x+5=0$

$\Rightarrow x(x-5)-1(x-5)=0$

$\Rightarrow (x-5)(x-1)=0$

$\Rightarrow x=5\ or\ x=1$

when x=5, then

$y=6-x$

$y=6-5\Rightarrow y=1$

when x=1, then

$y=6-x$

$\Rightarrow y=6-1\Rightarrow y=5$

Required Solutions: $\left.\begin{array}{l} x=5,\ 1 \\ y=1\ 5 \end{array}\right\}$ [Ans]

(d) Given $\frac{x}{2}+\frac{y}{5}=5$, $\frac{2}{x}+\frac{5}{y}=\frac{5}{6}$

Now $\frac{x}{2}+\frac{y}{5}=5$ $\Rightarrow \frac{5x+2y}{10}=5$

$\Rightarrow 5x+2y=50$ $\Rightarrow 2y=50-5x$

$\Rightarrow y=\frac{50-5x}{2}$..................(1)

Also $\frac{2}{x}+\frac{5}{y}=\frac{5}{6}$ $\Rightarrow \frac{2}{x}+\frac{50-5x}{2}=\frac{5}{6}$ [From (1), we get]

$\Rightarrow \frac{2}{x}+\frac{10}{50-5x}=\frac{5}{6}$ $\Rightarrow \frac{2}{x}+\frac{2}{10-x}=\frac{5}{6}$

$\Rightarrow \frac{20-2x+2x}{x(10-x)}=\frac{5}{6}$ $\Rightarrow \frac{20}{10x-x^2}=\frac{5}{6}$

$\Rightarrow 120=50x-5x^2$ $\Rightarrow 5x^2-50x+120=0$

$\Rightarrow x^2-10x+24=0$ $\Rightarrow x^2-6x-4x+24=0$

$\Rightarrow x(x-6)-4(x-6)=0$ $\Rightarrow (x-6)(x-4)=0$

$\Rightarrow x=6 \ or \ x=4$

when $x=6$ then

$y=\frac{50-5x}{2}$

$\Rightarrow y=\frac{50-5(6)}{2} \Rightarrow y=10$

when $x=4$, then

$y=\frac{50-5x}{2}$

$y=\frac{50-5(4)}{2} \Rightarrow y=15$

Required Solutions: $\left.\begin{array}{l} x=6,\ 4 \\ y=10,\ 15 \end{array}\right\}$ [Ans]

Illustration-35: Solve the following simultaneous equations:

(a) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{6}, \frac{1}{x}+\frac{1}{y}=\frac{3}{2}$

(b) $x+y=a+b, \frac{a}{x}+\frac{b}{y}=2$

(c) $\frac{x+y}{1-xy}=3. \frac{x-y}{1+xy}=\frac{1}{3}$

(d) $2x^2+3xy=26,\ 3y^2+2xy=39$

(e) $y^2-5xy+6x^2=0,\ x^2+y^2=45$

Solution: (a) Given $\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{4}, \frac{1}{x}+\frac{1}{y}=\frac{3}{2}$

Now $\frac{1}{x}+\frac{1}{y}=\frac{3}{2}$ $\Rightarrow \frac{1}{y}=\frac{3}{2}-\frac{1}{x}$

$\Rightarrow \frac{1}{y}=\frac{3x-2}{2x}$ $\Rightarrow y(3x-2)=2x$

$\Rightarrow y=\frac{2x}{3x-2}$ ....(1)

Also $\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{4}$ $\Rightarrow \frac{1}{x^2}+\frac{1}{\left(\frac{2x}{3x-2}\right)^2}=\frac{5}{4}$ [From (1), we get]

$\Rightarrow \frac{1}{x^2}+\frac{1}{\frac{4x^2}{9x^2-12x+4}}=\frac{5}{4}$ $\Rightarrow \frac{1}{x^2}+\frac{9x^2-12x+4}{4x^2}=\frac{5}{4}$

$\Rightarrow \frac{4+9x^2-12x+4}{4x^2}=\frac{5}{4}$ $\Rightarrow \frac{9x^2-12x+8}{x^2}=\frac{5}{1}$

$\Rightarrow 9x^2-12x+8=5x^2$ $\Rightarrow 4x^2-12x+8=0$

$\Rightarrow x^2-3x+2=0$ $\Rightarrow x^2-2x-x=2=0$

$\Rightarrow x(x-2)-1(x-2)=0$ $\Rightarrow (x-2)(x-1)=0$

$\Rightarrow x=2 \text{ or } x=1$

when $x=2$, then

$y=\frac{2x}{3x-2}$

$\Rightarrow y=\frac{2(2)}{3(2)-2} \Rightarrow y=1$

when $x=1$, then

$y=\frac{2x}{3x-2}$

$y=\frac{2(1)}{3(1)-2} \Rightarrow y=2$

Required solutions: $\left.\begin{array}{l} x=2,\ 1 \\ y=1,\ 2 \end{array}\right\}$ [Ans]

(b) Given $x+y=a+b,\ \frac{a}{x}+\frac{b}{y}=2.$

Now $x+y=a+b$ $\Rightarrow y=a+b-x$...............(1)

Also $\frac{a}{x}+\frac{b}{y}=2$ $\Rightarrow \frac{a}{x}+\frac{b}{a+b-x}=2$ [From (1), we get]

$\Rightarrow \frac{a(a+b-x)+bx}{x(a+b-x)}=2$ $\Rightarrow \frac{a^2+ab-ax+bx}{ax+bx-x^2}=2$

$\Rightarrow a^2+ab-ax+bx=2ax+2bx-2x^2$ $\Rightarrow 2x^2-2ax-ax+a^2-bx+ab=0$

$\Rightarrow 2x(x-a)-a(x-a)-b(x-a)=0$ $\Rightarrow (x-a)\{2x-(a+b)\}=0$

$\Rightarrow (x-a)=0 \text{ or } \{2x-(a+b)\}=0$ $\Rightarrow x=a \text{ or } 2x=(a+b)$

$\Rightarrow x=a \text{ or } x=\frac{(a+b)}{2}$ $\Rightarrow x=a \text{ or } x=\frac{1}{2}(a+b)$

when $x = a$, then

$y = a + b - x$

$\Rightarrow y = a + b - a$

$\Rightarrow y = b$

when $x = \frac{1}{2}(a+b)$, then

$y = a + b - x$

$\Rightarrow y = a + b - \frac{1}{2}(a+b)$

$\Rightarrow y = \frac{2a + 2b - a - b}{2}$

$\Rightarrow y = \frac{a+b}{2} \Rightarrow y = \frac{1}{2}(a+b)$

Required solutions: $\left.\begin{matrix} x = a, & \frac{1}{2}(a+b) \\ y = b, & \frac{1}{2}(a+b) \end{matrix}\right\}$ [Ans]

(c) Given $\frac{x+y}{1-xy} = 3, \frac{x-y}{1+xy} = \frac{1}{3}$

Now $\frac{x+y}{1-xy} = 3$ $\Rightarrow x + y = 3 - 3xy$

$\Rightarrow y + 3xy = 3 - x$ $\Rightarrow y(1+3x) = 3 - x$

$\Rightarrow y = \frac{3-x}{1+3x}$ ....(1)

Also $\frac{x-y}{1+xy} = \frac{1}{3}$ $\Rightarrow \frac{x - \frac{3-x}{1+3x}}{1 + x\left(\frac{3-x}{1+3x}\right)} = \frac{1}{3}$ [from (1), we get]

$\Rightarrow \frac{\frac{x+3x^2-3+x}{1+3x}}{\frac{1+3x+3x-x^2}{1+3x}} = \frac{1}{3}$ $\Rightarrow \frac{3x^2+2x-3}{x+6x-x^2} = \frac{1}{3}$

$\Rightarrow 9x^2 + 6x - 9 = 6x - x^2 + 1$ $\Rightarrow 10x^2 - 10 = 0$

$\Rightarrow x^2 - 1 = 0$ $\Rightarrow x^2 = 1$

$\Rightarrow x = \pm\sqrt{1}$ $\Rightarrow x = \pm 1$ $\Rightarrow x = +1$ *or* $x = -1$

when $x=+1$, then

$$y=\frac{3-x}{1+3x}$$

$$y=\frac{3-1}{1+3(1)}=\frac{1}{2}$$

when $x=-1$, then

$$y=\frac{3-x}{1+3x}$$

$$y=\frac{3-(-1)}{1+3(-1)}=-2$$

Required solutions: $\left.\begin{array}{l} x=1,\ -1 \\ y=\frac{1}{2},\ -2 \end{array}\right\}$ [Ans]

(d) Given $2x^2+3xy=26,\ 3y^2+2xy=39$

Now $2x^2+3xy=26 \quad \Rightarrow 3xy=26-2x^2$

$$\Rightarrow y=\frac{26-2x^2}{3x}\ldots\ldots\ldots\ldots\ldots(1)$$

Also $3y^2+2xy=39$

$$\Rightarrow 3\left(\frac{26-2x^2}{3x}\right)^2+2x\left(\frac{26-2x^2}{3x}\right)=39 \quad \text{[From (1) we get]}$$

$$\Rightarrow 3\left(\frac{676-104x^2+4x^4}{9x^2}\right)+\frac{52-4x^2}{3}=39 \qquad \Rightarrow \frac{676-104x^2+4x^4}{3x^2}+\frac{52-4x^2}{3}=39$$

$$\Rightarrow \frac{676-104x^2+4x^4+52x^2-4x^4}{3x^2}=39 \qquad \Rightarrow 676-52x^2=117x^2$$

$$\Rightarrow 169x^2=676 \qquad \Rightarrow x^2=4$$

$$\Rightarrow x=\pm\sqrt{4} \qquad \Rightarrow x=\pm 2$$

when $x=\pm 2$, then

$$y=\frac{26-x^2}{3x}=\frac{26-2(\pm 2)^2}{3(\pm 2)}=\frac{26-8}{\pm 6}=\frac{18}{\pm 6}=\pm 3$$

Required solutions: $\left.\begin{array}{l} x=\pm 2 \\ y=\pm 3 \end{array}\right\}$ [Ans]

(e) Given $y^2-5xy+6x^2=0\ldots\ldots\ldots\ldots\ldots(1)$

$x^2+y^2=45\ldots\ldots\ldots\ldots\ldots(2)$

Subtracting (2) form (1) we get,

$$-5xy+5x^2=-45$$
$$\Rightarrow xy-x^2=9$$
$$\Rightarrow xy=9+x^2$$
$$\Rightarrow y=\frac{9+x^2}{x} \text{......................}(3)$$

Putting the value of y in (2), we get

$$x^2+y^2=45$$
$$\Rightarrow x^2+\left(\frac{9+x^2}{x}\right)=45 \quad \text{[From (3), we get]}$$
$$\Rightarrow \frac{x^4+81+18x^2+x^4}{x^2}=45$$
$$\Rightarrow 2x^4-27x^2+81=0$$
$$\Rightarrow 2x^2(x^2-9)-9(x^2-9)=0$$
$$\Rightarrow (x^2-9)=0 \; or \; (2x^2-9)=0$$
$$\Rightarrow x^2=9 \; or \; x^2=\frac{9}{2}$$
$$\Rightarrow x=\pm\sqrt{3} \; or \; x=\pm\frac{3}{\sqrt{2}}$$

when $x=\pm3$, then

$$y=\frac{9+x^2}{x}$$
$$\Rightarrow y=\frac{9+(\pm3)^2}{\pm3}$$
$$\Rightarrow y=\frac{9+9}{\pm3}=\frac{18}{\pm3}=\pm6$$

$$\Rightarrow x^2+\frac{81+18x^2+x^4}{x^2}=45$$
$$\Rightarrow 2x^4+18x^2+81=45x^2$$
$$\Rightarrow 2x^4-18x^2-9x^2+81=0$$
$$\Rightarrow (x^2-9)(2x^2-9)=0$$
$$\Rightarrow x^2=9 \; or \; 2x^2=9$$
$$\Rightarrow x=\pm\sqrt{9} \; or \; x=\pm\sqrt{\frac{9}{2}}$$

when $x=\pm\frac{3}{\sqrt{2}}$, then

$$y=\frac{9+x^2}{x}$$
$$\Rightarrow y=\frac{9+\left(\pm\frac{3}{\sqrt{2}}\right)^2}{\pm\frac{3}{\sqrt{2}}}$$
$$\Rightarrow y=\frac{9+\frac{9}{2}}{\pm\frac{3}{\sqrt{2}}}=\pm\left(\frac{27}{2}\times\frac{\sqrt{2}}{3}\right)=\pm\frac{9}{\sqrt{2}}$$

Required solutions: $\left.\begin{aligned} x &= \pm 3,\ \pm\frac{3}{\sqrt{2}} \\ y &= \pm 6,\ \pm\frac{9}{\sqrt{2}} \end{aligned}\right\}$ [Ans]

Illustration-36: Solve the following simultaneous equations:

$$\text{(a) } \frac{9^x}{3^{x+y}} = 27, \frac{4^x}{32^y} = 1, \quad \text{(b) } 4^x.2^y = 128,\ 3^{3x+2y} = 9^{xy}$$

Solution: (a) Given $\frac{9^x}{3^{x+y}} = 27$

$\Rightarrow 9^x = 27 \times 3^{x+y}$ $\Rightarrow 3^{2x} = 3^3 \times 3^{x+y}$

$\Rightarrow 3^{2x} = 3^{3+x+y}$ $\Rightarrow 2x = 3 + x + y$

$\Rightarrow x = 3 + y$.................(1)

Also $\frac{4^x}{32^y} = 1$

$\Rightarrow 4^x = 32^y$ $\Rightarrow 2^{2x} = 2^{5y}$

$\Rightarrow 2x = 5y$ $\Rightarrow 2(3+y) = 5y$ [from (1), we get]

$\Rightarrow 6 + 2y = 5y$ $\Rightarrow 3y = 6$ $\Rightarrow y = 2$

putting the value of y in (1, we get

$x = 3 + y \Rightarrow x = 3 + 2 \Rightarrow x = 5$

required solutions: x = 5 and y = 2 [Ans]

(b) Given , $4^x \cdot 2^y = 128$

$\Rightarrow 2^{2x} \times 2^y = 2^7$ $\Rightarrow 2^{2x+y} = 2^7$ $\Rightarrow 2x + y = 7$ $\Rightarrow y = 7 - 2x$............(1)

Also $3^{3x+2y} = 9^{xy}$ $\Rightarrow 3^{3x+2y} = 3^{2xy} \Rightarrow 3x + 2y = 2xy$

$\Rightarrow 3x + 2(7 - 2x) = 2x(7 - 2x)$ [from (1), we get]

$\Rightarrow 3x + 14 - 4x = 14x - 4x^2$ $\Rightarrow 4x^2 - 15x + 14 = 0$

$\Rightarrow 4x^2 - 8x - 7x + 14 = 0$ $\Rightarrow 4x(x-2) - 7(x-2) = 0$

$\Rightarrow (x-2)(4x-7) = 0$ $\Rightarrow (x-2) = 0$ or $(4x-7) = 0$

$\Rightarrow x = 2$ or $4x = 7$ $\Rightarrow x = 2$ or $x = \frac{7}{4}$

when $x=2$,

$y=7-2x$

$\Rightarrow y=7-2(2)$

$\Rightarrow y=7-4=3$

when $x=\frac{7}{4}$

$y=7-2x$

$\Rightarrow y=7-2\left(\frac{7}{4}\right)=7-\frac{7}{2}=\frac{7}{2}$

Required solutions: $\left.\begin{matrix} x=2, & \frac{7}{4} \\ y=3, & \frac{7}{2} \end{matrix}\right\}$ [Ans]

Illustration-37: Discuss the nature of the roots of the following equations :

(i) $x^2+2x+5=0$; ii) $2x^2+6x+3=0$; iii) $x^2-6x+9=0$

Solution: i) Here $a=1, b=2, c=5$

$\therefore b^2-4ac=(2)^2-4.1.5$

$=4-20=-16$ which is negative.

So, the roots are complex and unequal.

ii) Here $a=1, b=-6, c=9$

$\therefore b^2-4ac=(6)^2-4.2.3$ $=36-24=12$, which is positive

So, the roots are real and unequal.

iii) Here $a=2, b=-6, c=3$

$b^2-4ac=(-6)^2-4.1.9$ $=36-36$ $=0$

So, the roots are real and equal.

Illustration-38: Find the value of k, if

i) the roots of $2x^2+3x+k=0$ are equal; ii) one root of $x^2-6x+k=0$ is $3+i\sqrt{2}$.

iii) one root of $x^2-4x-k=0$ is $2+2\sqrt{3}$. iv) One root of $x^2-6x+k=0$ is double the other.

Solution: i) Since the roots of $2x^2+3x+k=0$ are equal

$\therefore b^2-4ac=0$

=> $(3)^2-4.2k=0$ => $-8k=-9$

$\therefore k = \frac{9}{8}$ (Answer).

ii) Since one root of $x^2 - 6x + k = 0$ is $3 + i\sqrt{2}$

So another root is $3 - i\sqrt{2}$ ($\because$ complex roots are conjugate)

Now, product of the roots $= \frac{c}{a}$

i.e. $(3 + i\sqrt{2})(3 - i\sqrt{2}) = \frac{k}{1}$

=> $(3)^2 - (i\sqrt{2})^2 = k$ => $9 - i^2.2 = k$

=> $9 + 2 = k \, (\because i^2 = -1)$

$\therefore$ $k = 11$ *Ans.*

iii) Since one root of $x^2 - 4x - k = 0$ is $2 + 2\sqrt{3}$.

So, another one is $2 - 2\sqrt{3}$.

Therefore $(2 + 2\sqrt{3})(2 - 2\sqrt{3}) = \frac{-k}{1}$

=> $(2)^2 - (2\sqrt{3})^2 = -k$ => $4 - 12 = -k$ => $-8 = -k$

$\therefore$ $k = 8$ (Answer)

iv) Let one root is α of the given equation $x^2 - 6x + k = 0$

So, another one is 2α.

Therefore $(\alpha)\ (2\alpha) = k$

=> $2\alpha^2 = k$ ...... (1)

and sum of the roots $\alpha + 2\alpha = -\left(\frac{-6}{1}\right)$ => $3\alpha = 6$

$\therefore$ $\alpha = 2$

Putting the value of α in (1) we get. $k = 8$ Ans.

Illustration-39: Form the quadratic equation, whose roots are :

i) 8, 12; ii) $4 + i\sqrt{3},\ 4 - i\sqrt{3}$; iii) $p + \sqrt{q},\ p - \sqrt{q}$

Solution : (i) The quadratic equation whose roots are 8, 12 is

$x^2 - (8 + 12)x + (8)(12) = 0$ => $x^2 - 20x + 96 = 0$ Ans.

ii) The quadratic equation whose roots are

$4+i\sqrt{3}$ and $4-i\sqrt{3}$ is

$x^2-(4+i\sqrt{3}+4-i\sqrt{3})x+(4+i\sqrt{3})(4-i\sqrt{3})=0$

=> $x^2-8x+(4)^2-(i\sqrt{3})^2=0$ => $x^2-8x+16-i^2.3=0$

=> $x^2-8x+19=0$ (Ans). ($\because i^2=-1$).

iii) The quadratic equation whose roots are $p+\sqrt{q}$ and $p-\sqrt{q}$ is :

$x^2-(\text{sum})x+(\text{product})=0$

=> $x^2-(p+\sqrt{q}+p-\sqrt{q})x+(p+\sqrt{q})(p-\sqrt{q})=0$

=> $x^2-2px+p^2-(\sqrt{q})^2=0$

=> $x^2-2px+p^2-q=0$ (Ans.)

Illustration-40: If α and β are the roots of $x^2+px+q=0$, then find a quadratic equation whose roots are

$\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

Solution : Given α and β be the roots of $x^2+px+q=0$

So we have, $\alpha+\beta=-\frac{p}{1}=-p$ ....(1)

$\alpha\beta=\frac{q}{1}=q$ ....(2)

Now, the required equation whose roots are

$\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ is

$x^2-(\frac{1}{\alpha^2}+\frac{1}{\beta^2})x+\frac{1}{\alpha^2}.\frac{1}{\beta^2}=0$

=> $x^2-\left(\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right)x+\left(\frac{1}{\alpha\beta}\right)^2=0$ => $x^2-\left\{\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}\right\}x+\left(\frac{1}{\alpha\beta}\right)^2=0$

=> $x^2-\left\{\frac{(-p)^2-2q}{q^2}\right\}x+\frac{1}{q^2}=0$ [using (1) and (2) => $x^2-\left\{\frac{p^2-2q}{q^2}\right\}x+\frac{1}{q^2}=0$

=> $q^2x^2-(p^2-2q)x+1=0$ (Answer).

Illustration-41: If α, β be the roots of $x^2 - 2x + 3 = 0$
Then find an equation whose roots

i) $\alpha + 3,\ \beta + 3$; ii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$; iii) $2\alpha - 3\beta,\ 3\alpha - 2\beta$.

Solution: i) Since α, β be the roots of $x^2 - 2x + 3 = 0$

So we have,

$\alpha + \beta = 2$ .... (1)

$\alpha\beta = 3$ .... (2)

Hence the equation whose roots are
$\alpha + 3$ *and* $\beta + 3$ *is:*

x^2- (sum of the roots) x + (Product of the roots) = 0

$\Rightarrow x^2 - (\alpha + 3 + \beta + 3)x + (\alpha + 3)(\beta + 3) = 0 \Rightarrow x^2 - (\alpha + \beta + 6)x + \alpha\beta + 3(\alpha + \beta) + 9 = 0$

$\Rightarrow x^2 - (2 + 6)x + 3 + 3(2) + 9 = 0$ [using (1) and (2)]

$\Rightarrow x^2 - 8x + 18 = 0$ (Answer).

ii) The equation whose roots are $\frac{\alpha}{\beta}$ *and* $\frac{\beta}{\alpha}$ is

$$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta}.\frac{\beta}{\alpha} = 0$$

$\Rightarrow x^2 - \left(\frac{\alpha^2 + \beta^2}{\alpha\beta}\right)x + 1 = 0 \qquad \Rightarrow x^2 - \left\{\frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}\right\}x + 1 = 0$

$\Rightarrow x^2 - \left(\frac{2^2 - 2.3}{3}\right)x + 1 = 0 \qquad \Rightarrow x^2 - \left(\frac{4 - 6}{3}\right)x + 1 = 0$

$\Rightarrow x^2 + \frac{2x}{3} + 1 = 0 \qquad \Rightarrow 3x^2 + 2x + 3 = 0$ Answer.

iii) Here

Sum of the roots = $2\alpha - 3\beta + 3\alpha - 2\beta$

$= 5\alpha - 5\beta$

$= -5(\alpha + \beta)$

$= -5(2) \quad (\because \alpha + \beta = 2)$

$= -10$

Product of the roots = $(2\alpha - 3\beta)(3\alpha - 2\beta)$

$= 6\alpha^2 - 4\alpha\beta - 9\alpha\beta + 6\beta^2$

$= 6\{(\alpha+\beta)^2 - 2\alpha\beta\} - 13\alpha\beta$

$= 6(2^2 - 2.3) - 13.3$ [using (1) and (2)]

$= 6(4-6) - 39$

$= -12 - 39 = -51$

So, the required quadratic equation whose roots are

$2\alpha - 3\beta$ and $3\alpha - 2\beta$ is :

x^2 (sum of the roots) x+ (product of the roots)= 0

$\Rightarrow x^2 - (-10x) + (-51) = 0$

$\Rightarrow x^2 + 10x - 51 = 0$ (Ans)

Illustration-42: If k be the ratio of the roots of the equation $ax^2 + bx + c = 0$, prove that

$$\frac{(k+1)^2}{k} = \frac{b^2}{ac}$$

Solution: Let the two roots of the equation : $ax^2 + bx + c = 0$, are α and $k\alpha$

So we have $\alpha + k\alpha = -\frac{b}{a} \quad \therefore \alpha = \frac{-b}{a(1+k)}$ .... (i)

And $(\alpha)(k\alpha) = \frac{c}{a}$

=> $k\alpha^2 = \frac{c}{a}$ => $k\left\{\frac{-b}{a(1+k)}\right\}^2 = \frac{c}{a}$ [using (i)] => $\frac{kb^2}{(k+1)^2 a^2} = \frac{c}{a}$

=> $\frac{kb^2}{(k+1)^2 a} = \frac{c}{1}$ => $\frac{k}{(k+1)^2} = \frac{ac}{b^2}$ $\quad \therefore \frac{(k+1)^2}{k} = \frac{b^2}{ac}$ (Proved).

Illustration-43: If the sum of the roots of a quadratic equation is 3 and the sum of their cubes is 7, find the equation.

Solution: Let the roots of the equation are α and β

So according to the question we have

$\alpha + \beta = 3$ ..... (1)

and $\alpha^3 + \beta^3 = 7$

=> $(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = 7$ => $(3)^3 - 3\alpha\beta(3) = 7$

=> $27 - 7 = 9\alpha\beta$ => $20 = 9\alpha\beta \quad \therefore \alpha\beta = \frac{20}{9}$

So, the required equation is:

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$ i.e. $x^2 - 3x + \frac{20}{9} = 0$ => $9x^2 - 27x + 20 = 0$ (Ans.)

5.8 BUSINESS APPLICATION

Problem-1: By selling a table for Tk. 56, gain is as much percent as its cost in Taka. What is the cost price?

Solution: Let the cost price of the table =x
according to the question,
cost price +gain = selling price

$$\Rightarrow x+(x\times x\%)=56 \quad \Rightarrow x+\left(x\times\frac{x}{100}\right)=56 \Rightarrow x+\frac{x^2}{100}=56$$

$$\Rightarrow \frac{100x+x^2}{100}=56 \quad \Rightarrow 100x+x^2=5600 \quad \Rightarrow x^2+100x-5600=0$$

$$\Rightarrow x^2+140x-40x-5600=0 \quad \Rightarrow x(x+140)-40(x+140)=0 \quad \Rightarrow (x+140)(x-40)=0$$

$$\Rightarrow (x+140)=0 \text{ or } (x-40)=0 \Rightarrow x=-140 \text{ or } x=40$$

But cost price cannot be negative.

$\therefore x=40$

So the required cost price of the table $(x)=$ Tk. 40 [Ans]

Problem-2: In a perfect competition, the demand curve of a commodity is $D=20-3p-p^2$ and the supply curve is $S=5p-1$, where p is price, D is demand and S is supply. Find the equilibrium price and the quantity exchanged.

Solution: Given $D=20-3p-p^2$

$S=5p-1$

Under perfect competition, equilibrium condition is:

$$D=S$$

$$\Rightarrow 20-3p-p^2=5p-1 \quad \Rightarrow 21-8p-p^2=0 \quad \Rightarrow p^2+8p-21=0$$

$$\Rightarrow p=\frac{-8\pm\sqrt{(8)^2-4(1)(-21)}}{2(1)} \Rightarrow p=\frac{-8\pm\sqrt{148}}{2} \quad \Rightarrow p=\frac{-8\pm\sqrt{4\times 37}}{2}$$

$$\Rightarrow p=\frac{-8\pm 2\sqrt{37}}{2} \Rightarrow p=\frac{2(-4\pm\sqrt{37})}{2} \quad \Rightarrow p=-4\pm\sqrt{37}$$

, But price can never be negative.

Therefore, the equilibrium price, $p=Tk.(-4+\sqrt{37})=Tk.(-4+6.08)=Tk.2.08$

Putting the value of p in the demand or supply function, we get the equilibrium quantity as follows: Equilibrium quantity, $S = 5p - 1 = 5(-4+\sqrt{37}) - 1$

$= -20 + 5\sqrt{37} - 1 \qquad = -21 + 5\sqrt{37} = -21 + 5(6.08) = 9.08 \cong 9$ units.

So the required equilibrium price, $p =$ Tk. 2.08 and equilibrium quantity $= 9$ units [Ans]

Problem-3: For a certain commodity, the demand equation giving demand, d in kg. for a price, p in rupees per kg. is $d = p^2 - 94p + 1000$; the supply equation giving the supply s in kg. for a price, p in rupees per kg. is $s = 3000 - 14p$.. The market price is such that the demand equals supply. Find the market price and the quantity that will be bought and sold.

Solution: According to question, we have $d = s$, i.e.

$$p^2 - 94p + 1000 = 3000 - 14p \text{ or } p^2 - 80p - 2000 = 0$$

or, $(p-100)(p+20) = 0$

thus, either $p - 100 = 0, i.e \; p = 100$ or $p + 20, i.e. \; p = -20$ (meaning less)

Hence, the market price is, $P =$ Tk. 100

and consequently quantity bought is, $s = 3000 - 14 \times 100 = 1600$ kg.

Problem-4: The demand and supply equations are $2p^2 + q^2 = 11$ and $p + 2q = 7$. Find the equilibrium price and quantity, where p stands for price and q for quantity.

Solution: Given $2p^2 + q^2 = 11$

$$\Rightarrow 2p^2 = 11 - q^2 \qquad \Rightarrow p^2 = \frac{11-q^2}{2} \Rightarrow p = \sqrt{\frac{11-q^2}{2}} \quad \therefore p_d = \sqrt{\frac{11-q^2}{2}}$$

Also $p + 2q = 7 \Rightarrow p = 7 - 2q \; \therefore p_s = 7 - 2q$

Under perfect competition, equilibrium will be obtained as follows:

$D = S$

$\Rightarrow p_d = p_s \quad \Rightarrow \sqrt{\frac{11-q^2}{2}} = 7 - 2q \quad \Rightarrow \frac{11-q^2}{2} = 49 - 28q + 4q^2$ [Squaring on both sides]

$$\Rightarrow 11 - q^2 = 98 - 56q + 8q^2 \Rightarrow 9q^2 - 56q + 87 = 0 \Rightarrow 9q^2 - 27q - 29q + 87 = 0$$

$$\Rightarrow 9q(q-3) - 29(q-3) = 0 \quad \Rightarrow (q-3)(9q-29) = 0$$

$\Rightarrow q = 3, \frac{29}{9}$, which are the equilibrium quantity.

Putting the value of q in the demand of supply function, we get the equilibrium prices as follows:

When $q = 3$, then

Equilibrium price $= 7 - 2q = 7 - 2(3) = 1$

When $q = \frac{29}{9}$, then equilibrium price $= 7 - 2q = 7 - 2\left(\frac{29}{9}\right) = \frac{5}{9}$

So the required equilibrium prices (**p**) $= 1, \frac{5}{9}$

and the required equilibrium quantities $(q) = 3, \frac{29}{9}$. [Ans.]

BRIEF REVIEW

Definition

Quadratic Equation: An equation, which contains the square of the unknown variable and no higher power, is called a quadratic equation or an equation of the second degree

Pure Quadratic Equation: An equation which contains only the square of the unknown and not the first power is called a pure quadratic equation.

Mixed Quadratic Equation: An equation which contains the square as well as the first power of the unknown variable is called mixed quadratic equation.

Degree of an Equation: The degree of an equation is the highest power of the variable occurring in it, after the equation has been expressed in a form free from radicals and fractions.

Discriminant: We know, the roots of the quadratic equation $ax^2+bx+c=0,(a\neq 0)$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. The nature of the roots shall depend on the numerical value of b^2-4ac. The expression b^2-4ac which discriminates the nature of the roots is called discriminant of the equation $ax^2+bx+c=0$. It is denoted by the symbol D or Δ.

Quiz Questions

Multiple Choice Questions

1. The number of roots of a quadratic equation are :

   a) One b) Two c) Three d) Four

2. If $x^2 = 8$, then $x = ?$

   a) 4 b) $-\sqrt{8}$ c) $\pm\sqrt{8}$ d) $\sqrt{8}$

3. If $\frac{x}{2} + 3 = 5$, then $x + 6 = ?$

   a) 11 b) 10 c) 2 d) 12

4. If one root of a quadratic equation is 3+2i, then what will be another one?

   a) $2+3i$ b) $2-3i$ c) $3-2i$ d) 5

5. If two roots of a quadratic equation are; 5 & 7, then what is the equation?

   a) $5x^2 + 7x + 2 = 0$ b) $x^2 - 2x + 5 = 0$ c) $x^2 - 12x + 35 = 0$ d) $x^2 + 12x - 35 = 0$

6. If a cubic equation has one complex root,,then what will be the number of real root(s)?

   a) 1 b) 2 c) 0 d) 3

7. If $\frac{c}{y} + d = 2$, what is the value of y?

   a) $2-d$ b) $\frac{2-d}{c}$ c) $\frac{c}{2-d}$ d) None of these

8. *If* $x^2 = -1$, then $x = ?$

   a) -1 b) 0 c) 1 d) $\pm i$

9. The equation $ax^2 + bx + c = 0$ is a quadratic if

   (i) $a \neq 0$; (ii) $b = 0$; (iii) $c = 0$; (iv) $a = 0$

10. Which is a pure quadratic equation?

    (i) $x^2 + 5x + 6 = 0$; (ii) $x^2 + 2x + 1 = 0$; (iii) $3x^2 - 15 = 0$; (iv) $3x^2 - 9x = 0$

11. The sets of equations containing two or more variables are known as

    (i) Linear equation; (ii) Simultaneous equation; (iii) Quadratic equation; (iv) Cubic equation.

Which one of the following statement is true/false?

a. $x^2 + 5x + 6 = 0$ be a pure Quadratic equation.

b. In a quadratic equation if discriminant, $\Delta = 0$, then roots are real and unequal.

c. The graph of a quadratic equation is straight line.

d. Quadratic equation has two roots.

e. In a quadratic equation if discriminant, $\Delta > 0$ then, the roots are real and equal.

f. The graph of a quadratic equation is a curve.

Brief Questions

1. Write down the standard form or general form of quadratic equation.
2. If $x^2 = -1$ find the value of x.
3. Write down the roots of a general quadratic equation.
4. If one root of a quadratic equation is $3 + i\sqrt{2}$, then what will be another one?
5. Write down a mixed quadratic equation.
6. Write down a pure quadratic equation.
7. When roots, of a quadratic equation, are real and unequal?
8. When roots, of a quadratic equation, are real and equal?
9. When roots, of a quadratic equation, are imaginary and unequal.
10. If $\frac{x}{2} + 3 = 5$ then $x + 6 = ?$
11. If $x^2 = 8$ find the value of x.
12. Mention the types of quadratic equations?

Conceptual, Analytical & Numerical Questions

1. Define Quadratic Equation. Discuss the types of Quadratic Equation.
2. Discuss the characteristics of Quadratic Equation.
3. Define Degree of an Equation with example.
4. Distinguish between linear and quadratic equations.
5. What is Discriminant? Discuss the nature of the roots of a quadratic equation.
6. Discuss the relationship between roots and co-efficient of a quadratic equation.
7. Distinguish between mixed and pure Quadratic Equation.
8. What do you mean by Solution of an equation? Explain by an example.
9. (a) $x^2 - 2\sqrt{3x} + 1 = 0$ (b) $x - \sqrt{x} - 1 = 0$

   (c) $\frac{1}{x+2} = \frac{1}{x+1} - \frac{1}{x+3}$ (d) $3x^2 - 14x + 11 = 0$

Numerical Questions

1. Solve the following equations:

   (a) $\sqrt{\frac{x}{1+x}} + 6\sqrt{\frac{1+x}{x}} = 5$ (b) $\frac{9x-2}{3} + \frac{4x^2-7}{4x^2+3} = \frac{6x-1}{2}$

   (c) $\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}} = \frac{\sqrt{a}+1}{\sqrt{a}-1}$ (d) $\sqrt{3x^2+1} + \frac{4}{\sqrt{3x^2+1}} = 5$

2. Solve the following equations:

   (a) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$ (b) $x^2 - 4x - 12\sqrt{x^2 - 4x + 19} + 51 = 0$

   (c) $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 2(20 - x)$ (d) $x^2 - 6x - \sqrt{x^2 - 6x - 3} = 5$

   (e) $\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x - 5$ (f) $\sqrt{x^2 - 16} - (x - 4) = \sqrt{x^2 - 5x + 4}$

3. Solve the following equations:

   (a) $(2x+3)(2x+5)(x-1)(x-2) = 0$ (b) $(2x-7)(x^2-9)(2x+5) = 91$

   (c) $(x^2 - 3x)^2 - 8(x^2 - 3x) - 20 = 0$

4. Solve the following equations:

   (a) $x^4 + 8x^2 + 1 = 5x(x^2 + 1)$, (b) $4x^4 - 16x^3 + 23x^2 - 16x + 4 = 0$,

(c) $2x^4+9x^3+14x^2+9x+2=0$ (d) $4x^4-7x^3-4x^2+8x+4=0$.

Solve the following simultaneous equations:

5. (a) $8y-2x=3xy,\ \dfrac{10}{x}+\dfrac{1}{y}=2$

(b) $x^2+xy=15,\ xy=1$

(c) $\dfrac{3}{x}+\dfrac{7}{y}=6,\ \dfrac{5}{x}+\dfrac{8}{y}=\dfrac{166}{21}$

(d) $\dfrac{1}{x^2}+\dfrac{1}{y^2}=13,\ \ \dfrac{1}{x}-\dfrac{1}{y}=1$

6. (a) $x+\dfrac{4}{y}=1, y+\dfrac{4}{x}=25$

(b) $x+\dfrac{3}{y}=2,\ y+\dfrac{3}{x}=-2$

(c) $\dfrac{x^2}{y}+\dfrac{y^2}{x}=18,\ x+y=12$

(d) $\dfrac{3}{x+y}+\dfrac{2}{x-y}=3,\ \dfrac{9}{x+y}-\dfrac{5}{x-y}=4$

7. (a) $\dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{9}{2},\ x+y=3$

(b) $\dfrac{2}{x-1}+\dfrac{3}{y-1}=2,\ \dfrac{48}{x-1}+\dfrac{32}{y-1}=13$

(c) $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2},\ x+y=10$

(d) $\dfrac{1}{x^2}+\dfrac{1}{y^2}=13,\ \dfrac{1}{x}+\dfrac{1}{y}=5$

8. (a) $x^2+xy+y^2=19,\quad x^2-xy+y^2=7$

(b) $x^2-7xy+12y^2=0,\quad x^2+5xy-8y^2=64$.

(c) $x^2+xy=12,\ xy-2y^2=1$

(d) $x^2+xy+y^2=19,\ 3xy+2y^2=36$

9. Discuss the nature of the roots the following equations:

(a) $x^2+2x+3=0$ (b) $(x-a)(x-b)=h^2$

(c) $2x^2+2x+3=0$ (d) $x^2-5x-6=0$

(e) $6x^2-5x+2=0$ (f) $9x^2-42x+49=0$

(g) $3x^2-9x+5=0$ (h) $(x-1)(x-7)=2(x-3)(x-4)$

10. If α,β are the roots of $2x^2+3x+7=0,$ find the values of $\alpha^2+\beta^2, \alpha^3+\beta^3, \alpha^3-\beta^3, \alpha\beta^{-1}+\beta\alpha^{-1}$

11. Form the Quadratic equation whose roots are

(i) $4+i\sqrt{2}, 4-i\sqrt{2}$ (ii) $5+\sqrt{3}, 5-\sqrt{3}.$

12. If α,β are the roots of $x^2-2x+3=0,$ form the quadratic equation whose roots are:

(i) $\alpha^2+\beta, \beta^2+\alpha$ (ii) $\alpha+\frac{1}{\beta}, \beta+\frac{1}{\alpha}$

13. If p:q be the ratio of the roots of the equation $ax^2+bx+c=0$, than show that $ac(p+q)^2=b^2pq$

14. If the roots of the equation $ax^2+bx+c=0$ may be in the ratio $4:5,$ Prove that $20b^2=81ac$

15. For what values of m will the equation $(m+1)x^2+2(m+3)x+(2m+3)=0$ have equal roots.

16. If the roots of the equation $(m-n)x^2+(x-l)x+l=m$ are equal, then show that $2m=n+l$

17. If α and β be the roots of $x^2-px+q=0$, find the equation whose roots are α^2 and β^2.

18. If α and β are the roots of $2x^2-4x+1=0$, find a quadratic equation whose roots are $\alpha^2+\beta$ and $\beta^2+\alpha$.

19. If the equation $ax^2+bx+c=0$ and $bx^2+cx+a=0$ have a common root then prove that $a^3+b^3+c^3-3abc=0$

20. If one root of the equation $ax^2+bx+c=0$ be the square of the other. Prove that $b^3+a^2c+ac^2=3abc$.

21. The profit of two companies satisfy the equation $x^2-8x+a=0.$ Where "a" is a

constant and x is the profit in lakes of Taka's. Find the total amount of the sum of the profits of the two companies.

22. Suppose a firm's profit function for a product is given by $p = \frac{5x}{4} - 300$.

    (i) How many units must be produced to make a profit of \$ 150?

    (ii) producing how many units will result in a profit of "o"?

23. In an election there were only two candidates. The winner polled 53% votes and won by 15600 votes. Find the total number of votes polled.

24. The profit x of a company is given by $x^2 - 8x - 20 = 0$. Where x is in lakhs of Taka. Find the probable value of x.

25. A man sells a book at Tk 24 and finds that the percentage of loss is exactly equal to the amount of the cost price of the book. Find the cost price of the book.

26. Sketch the following equations:

    1. $4x+3y-24=0$ 2. $x-y+5=0$ 3. $5y+6x-18=0$

    4. $2y+8x=10$ 5. $4x=10-5y$ 6. $3x+y-12=0$

27. Sketch the following equations:

    1. $y=3x^2$ 2. $y=-3x^2$ 3. $y=3x^2+4$ 4. $y=-3x^2+4$ 5. $y=3x^2-4x$

    6. $y=-3x^2-4x$ 7. $y=3x^2-4x-5$ 8. $y=-3x^2-4x-5$

28. Mr. Selim receives a total return of Tk. 402 from an investment of Tk. 8001 in two debentures issues of a company. The first one carrying an interest of 6% p.a. was bought for Tk. 110 each and the other one carrying an interest rate of 5% p.a. was bought at 105 each. Find the sum invested in each type of debentures.

29. For a certain commodity, the demand equation for a price of Tk. x per kg. is $d = 50(10-x)$. The supply equation for a price of Tk. x per kg. is, $s = 25(x-4)$. The market price is such that at which demand equals supply. Find the market price and the quantity that will be bought and sold.

ANSWERS

1. (a) -9/8, -4/3 (b) $\pm 3/2$ (c) $x = 3a \ or \ -4a$ (d) $0, \pm\sqrt{5}$

2. (a) $1, 5, 3\pm 2\sqrt{3}$ (d) $-1, 7, 3\pm\sqrt{13}$ (f) 4, 5, -13/3.

4. (a) $1, \frac{3\pm\sqrt{5}}{2}$. (b) $2, \frac{1}{2}, \frac{3\pm i\sqrt{7}}{4}$. (c) $-1, -2, -\frac{1}{2}$. (d) $\frac{1\pm\sqrt{5}}{2}$.

5. (a) $x = 4, \ y = -2$ (b) $x = 3, \ -\frac{5}{2}$; $y = 2, \ -\frac{7}{2}$ (c) $x = \pm 1$; $y = -2, \ \frac{1}{2}$ (d) $x = -\frac{1}{2}, \ \frac{1}{3}$; $y = -\frac{1}{3}, \ \frac{1}{2}$

6. (a) $x = \frac{4}{5}, \ \frac{1}{5}$; $y = 20, \ 20$ (b) $x = 3, -1$; $y = -3, \ 1$ (c) $x = 8, \ 41$; $y = 4, \ 8$ (d) $x = \frac{209}{115}, \ y = -\frac{44}{115}$

7. (a) $x = 1, \ 2$; $y = 2, \ 1$ (b) $x = \frac{-11}{5}, \ y = \frac{15}{7}$ (c) $x = 2, \ 8$; $y = 8, \ 2$ (d) $x = \frac{1}{3}, \ \frac{1}{2}$; $y = \frac{1}{2}, \ \frac{1}{3}$

8. (a) $x = \pm 3, \ \pm 2$; $y = \pm 2, \ \pm 3$ (b) $x = \pm 6, \ \pm\frac{16}{\sqrt{7}}$; $y = \pm 2, \ \pm\frac{4}{\sqrt{7}}$ (c) $x = 4\sqrt{\frac{2}{3}}, -4\sqrt{\frac{2}{3}}, 3, -3$; $y = \frac{1}{2}\sqrt{\frac{2}{3}}, -\frac{1}{2}\sqrt{\frac{2}{3}}, 1, -1$

(d) $x = \frac{1}{7}\sqrt{7}, -\frac{1}{7}\sqrt{7}, 2, -2$; $y = -\frac{12}{7}\sqrt{7}, \frac{12}{7}\sqrt{7}, 3, \ -3$

15. $m = 3, -2$

17. $x^2 - (p^2 - 2q)x + q^2 = 0$

18. $4x^2 - 20x + 23 = 0$

28. Tk. 3,032, Tk. 4969.

29. Price = Tk.8, Quantity = 100 kgs.