CHAPTER-06 THE STRAIGHT LINES

LEARNING OBJECTIVES

After studying this chapter, the students will be able to understand:

- Cartesian Coordinate system
- Distance formulae in R^2
- Different types of formulae in Cartesian coordinate system
- Relationships between Cartesian and polar coordinate system
- Angle between two intersecting lines
- Concept of concurrent of three straight lines
- Different types of equation of a straight line
- Business application of straight lines

6.1 INTRODUCTION

French mathematician and philosopher Rene Descartes (1596-1650) is credited with the invention of this new branch of geometry, which is after his name also, called as Cartesian geometry. The fundamental idea of the analytical or coordinate geometry is the representation of points, called coordinates in the plane, by ordered pair of real numbers and the representation of lines are curves by algebraic equations. Coordinate geometry has enabled the integration of algebra and geometry since algebraic methods are used to represent and prove the fundamental properties of the functions corresponds to particular types of lines and analysis of various geometrical properties of these curves. Due to these features, coordinate geometry is considered as a technique for analysis of geometric figures based on certain axioms suggested by physical consideration such as straight line, parabola, circle, hyperbola, etc.

6.2 ABSCISSA AND ORDINATE OF POINT

The distance of a point from y-axis measured along the x-axis is called abscissa or x-coordinate of the point. In figure-1, PM is the ordinate (y-coordinate of point P) while OM is the abscissa (x-coordinate of point P).

The abscissa and ordinate taken together are called coordinates and these coordinates are always written between brackets, the abscissa being written first and then the ordinate separated by a comma. Thus, if x and y are abscissa and ordinate of a point P respectively, then the position of the point P in the plane with respect to the coordinate axis is denoted by the ordered pair (x, y) meaning that P is a point whose abscissa and ordinate are x and y units of length, respectively.

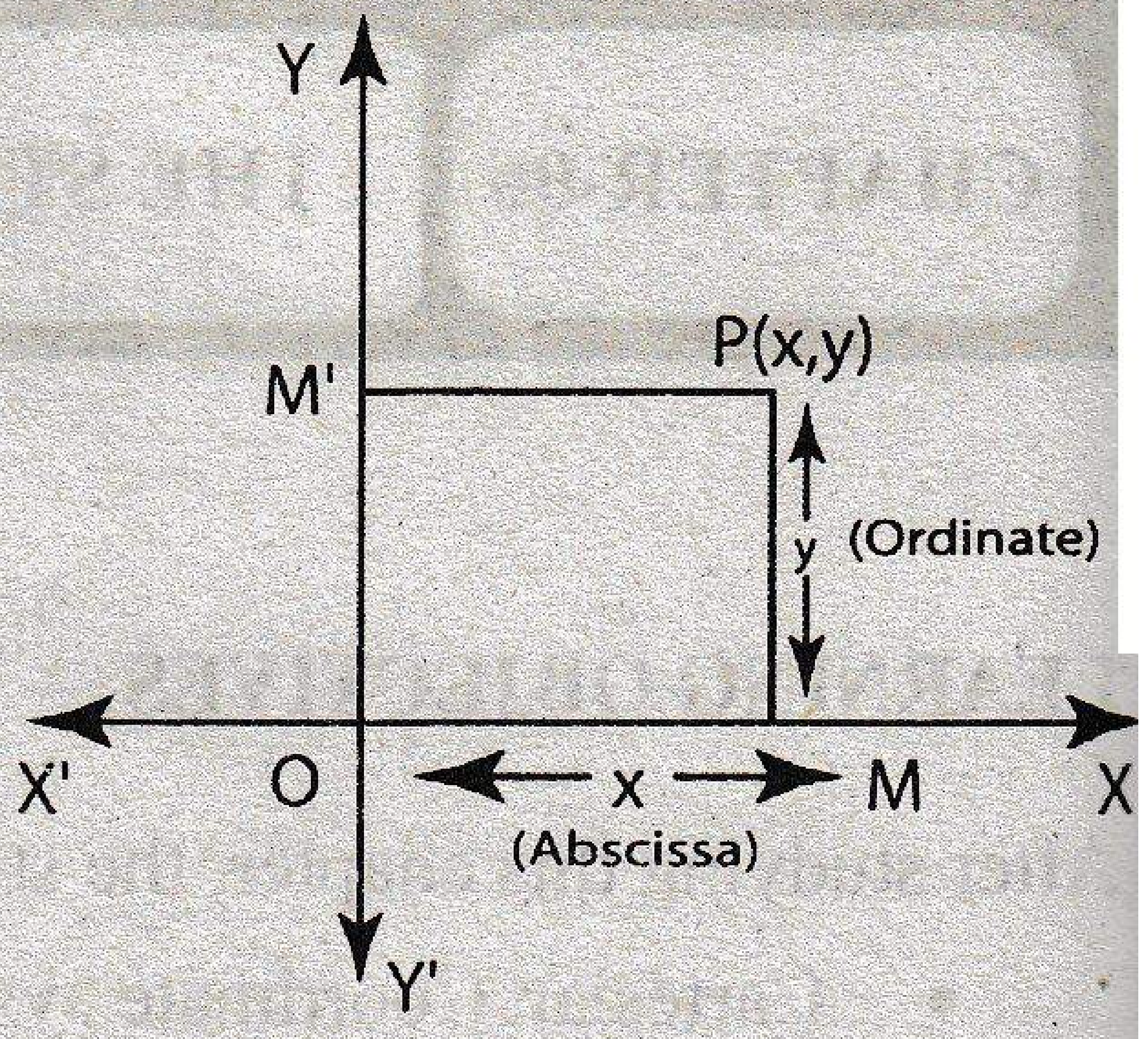


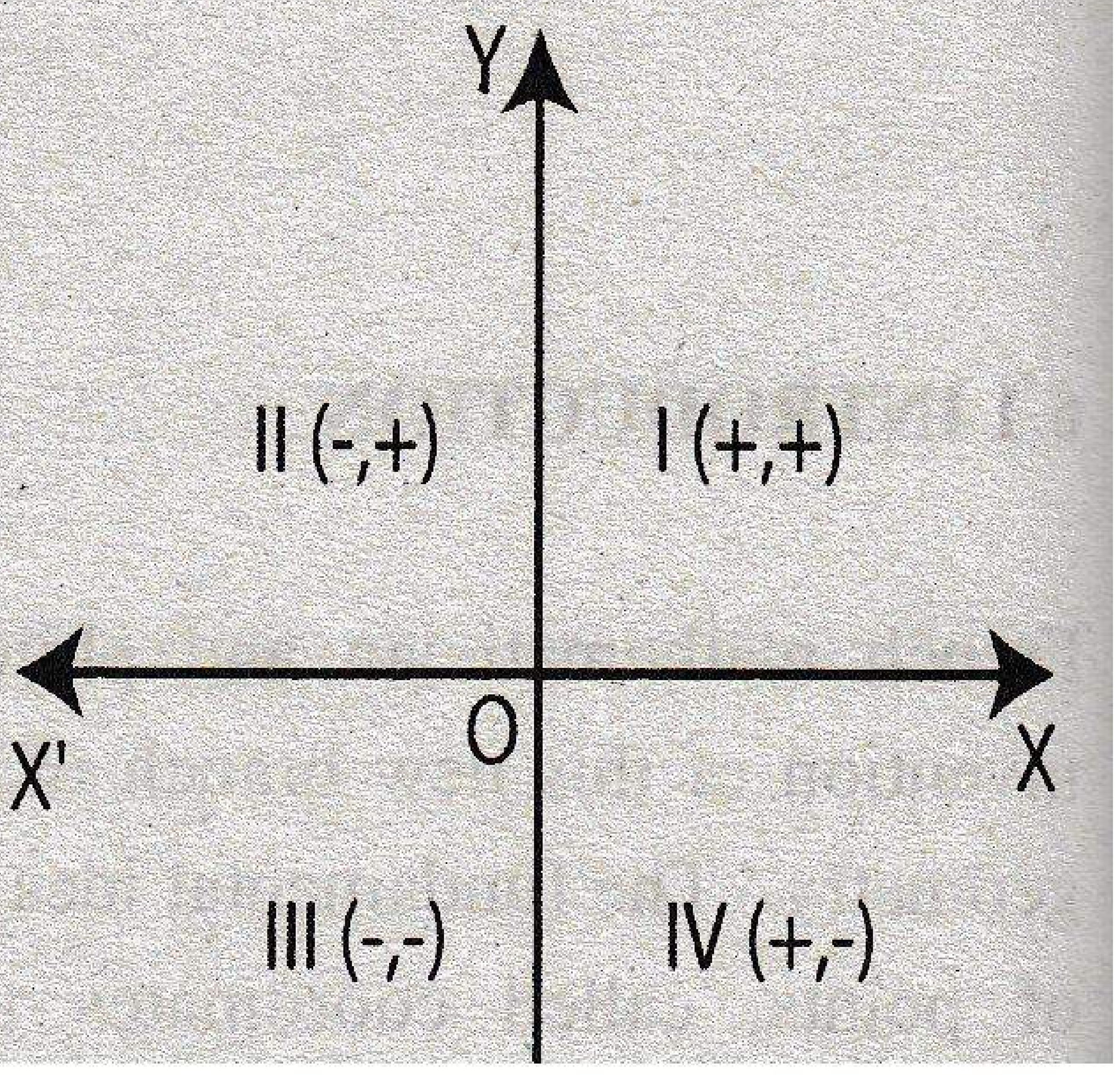
Figure -1: Coordinates of a Point

Various methods of expressing the coordinates of a point in a two-dimensional plane are:

i) Varying alphabets: (x, y), (a,b), (h, k)
ii) Varying subscripts: (x_1, y_1), (x_2, y_2), (x_3, y_3)
iii) Varying dashes: (x, y), (x´ y´), (x´´, y´´)

6.3 RECTANGULAR COORDINATES

When the plane is divided into four parts or regions (also called quadrants) by two mutually perpendicular lines intersecting at right angles at the point O=(0,0) called origin, then such lines are known as rectangular axis of coordinates. Quadrants XOY, X´OY, X´OY´ and XOY´ are respectively leveled as I, II, III and IV quadrant.



The convention of positive and negative signs follows from the definitions of abscissa and ordinates in terms of directed line segments. Positive directions are measured rightwards and upwards from the origin.

These quadrants are also represented as follows:

1^{st} quadrant	:	{(x,y);x>o and y>0}
2^{nd} quadrant	:	{(x,y); x<0 and y>0}
3^{rd} quadrant	:	{(x,y); x<0 and y<0}
4^{th} quadrant	:	{(x,y); x>0 and y<0}

6.4 DISTANCE BETWEEN TWO POINTS

The distance or length between two points P (x_1, y_1) and Q (x_2, y_2) is given by the formula,

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(\text{difference of abscissa})^2+(\text{difference of ordinate})^2}$$

where the square root is taken to be positive.

To prove this formula, let P and Q be any two points in the plane with coordinates (x_1, y_1) and (x_2, y_2) respectively as shown in figure-3. It is now required to express the distance PQ in-terms of coordinates of the two points.

From P and Q draw perpendicular PM and QN on the x-axis. Form P draw perpendicular PR on QN.

Now $OM = x_1$, $MP = y_1$, $ON = x_2$, $NQ = y_2$.

Then $MN = PR = ON - OM = |x_2 - x_1|$,

$QR = QN - RN = |y_2 - y_1|$

Hence, from the right angle triangle QPR, we have

$$PQ^2 = PR^2 + RQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

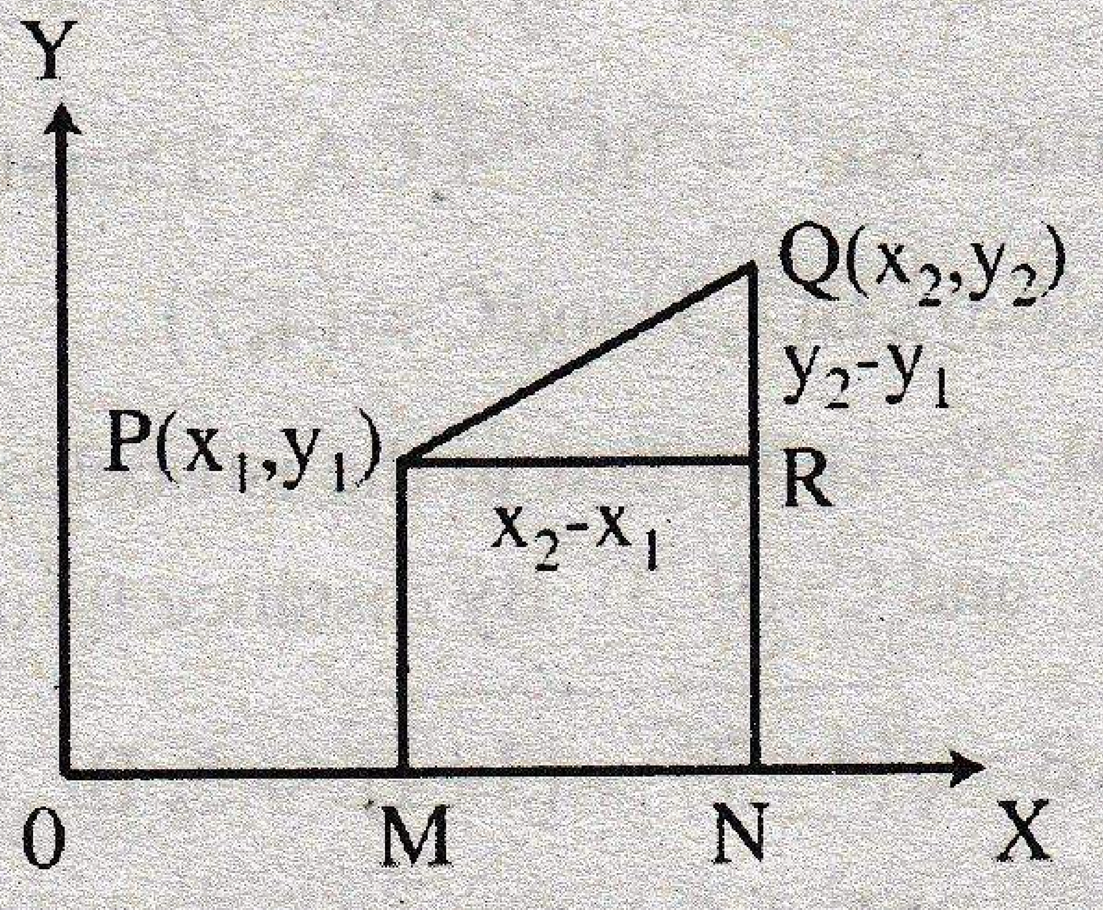


Figure-3: Distance between two point

REMARKS

1. No sign is given with the radical because only the numerical value of the distance and not its direction is needed.
2. The distance between the origin O(0,0) and the point $Q(x, y)$ is given by $\mathbf{OQ} = \sqrt{x^2 + y^2}$. The directed segment OQ is called the radius vector to Q.

6.5 COORDINATES OF MID-POINT

The coordinate of a mid-point from the coordinates of the any two points is given by the formula: $x_m = \dfrac{x_1 + x_2}{2}$, $y_m = \dfrac{y_1 + y_2}{2}$

For example, the coordinate of the mid-point of the join of points (6, 5) and (-2, 3) is:

$$\left(\frac{6+(-2)}{2}, \frac{5+3}{2}\right), \text{ i.e. } (2, 4).$$

Example-1:

(a) Show that the points (6, 6), (2, 3) and (4, 7) are vertices of a right angle.

(b) Prove that the points (4, 3), (7, -1) and (9, 3) are the vertices of an isosceles triangle.

Solution:

(a) Given that, A=(6, 6), B=(2, 3) and C=(4, 7).

By distance formula, we have

$$AB^2 = (6-2)^2 + (6-3)^2 = 25$$

$$BC^2 = (2-4)^2 + (3-7)^2 = 29$$

$$CA^2 = (4-6)^2 + (7-6)^2 = 5$$

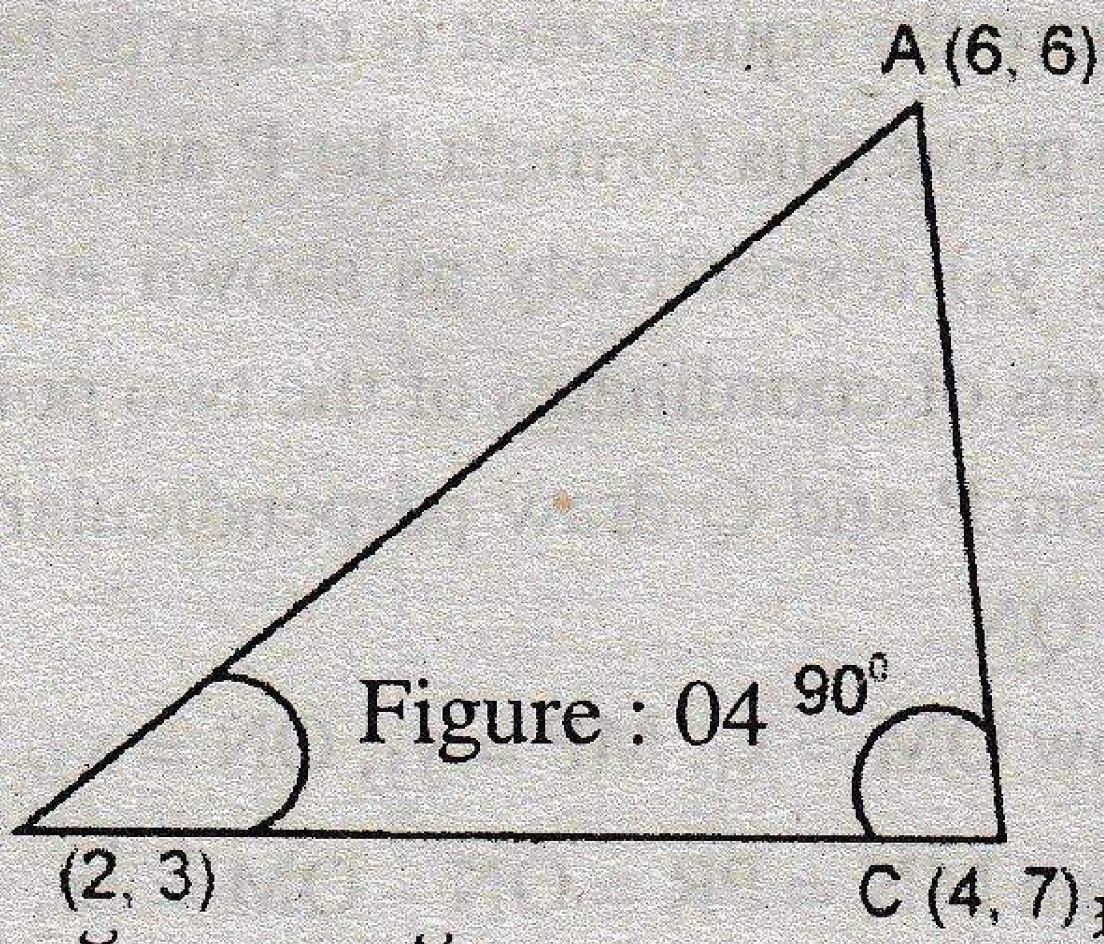


Figure : 04

Since $AB^2 = BC^2 + CA^2$ Therefore, ΔABC is the righ are $A(6,6), B(2,3)$ and $C = (4,7)$

(b) We know that in the isosceles triangle, two sides are equal. Given that A=(4,3), B=(7, -1) and C=(9, 3). By distance formula

$$AB = \sqrt{(4-7)^2 + (3+1)^2} = 5,\ BC = \sqrt{(7-9)^2 + (-1,-3)^2} = 2\sqrt{5},\ CA = \sqrt{(9-4)^2 + (3-3)^2} = 5$$

Since $AB = CA$, therefore the triangle is an isosceles triangle.

Example-2: Show that the points (3, 2), (5, 4), (3, 6) and (1, 4) are the vertices of a square.

Solution: Given that $A = (3,2), B = (5,4), C = (3,6)$ and $D = (1,4)$. By distance formula,

$$AB = \sqrt{(3-5)^2 + (2-4)^2} = 2\sqrt{2} \quad BC = \sqrt{(5-3)^2 + (4-6)^2} = 2\sqrt{2}$$

$$CD = \sqrt{(3-1)^2 + (6-4)^2} = 2\sqrt{2} \quad DA = \sqrt{(1-3)^2 + (4-2)^2} = 2\sqrt{2}$$

Since, $AB = BC = CD = DA$, therefore, ABCD is a rhombus.

Now, $(AC)^2 = (AB)^2 (BC)^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2 = 16$ or, AC=4

Since $(AC)^2 = (AB)^2 + (BC)^2, \angle B$ is right angle.

Therefore ABCD is a square.

6.6 DIVISION OR SECTION FORMULA

The coordinates of the point, R (x, y) which divides the line joining two point (x_1,y_1) and Q $(x_2\ y_2)$ in the ratio $m_1:m_2$ (a) internally, and (b) externally are given below:

(a) $x=\dfrac{m_1x_2+m_2x_1}{m_1+m_2}$: $y=\dfrac{m_1y_2+m_2y_1}{m_1+m_2}$ (internal division)

(b) $x=\dfrac{m_1x_2-m_2x_1}{m_1-m_2}$; $y=\dfrac{m_1y_2-m_2y_1}{m_1-m_2}$ (External division)

REMARK: If $m_1 : m_2$, then the coordinates of he middle point R (x, y) of the line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is: $\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$

6.7 CENTROID OF A TRIANGLE

The centroid of a triangle is the point of intersection of its three medians. Each median bisects the side opposite to the vertex into two equal parts. The coordinates of the centroid of the triangle are: $x=\dfrac{x_1+x_2+x_3}{3}$, $y=\dfrac{y_1+y_2+y_3}{3}$. To obtain these coordinates, suppose $A(x_1,y_1)$, $B(x_2,y_2)$and $C(x_3,y_3)$ are the vertices of the triangle

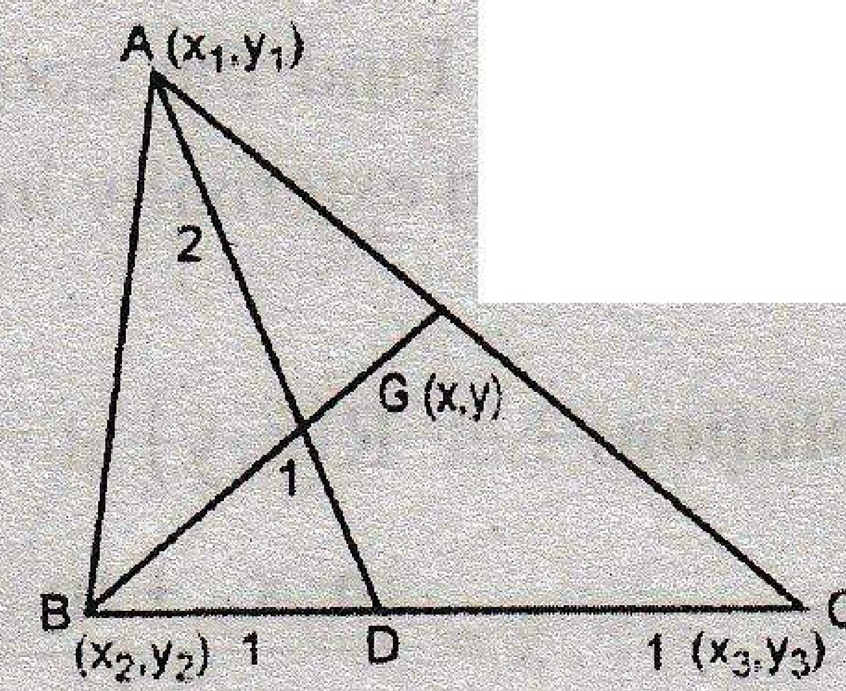


Fig-5: Centroid of a Triangle

ABC and AD be the median bisecting its bases BC at D. Then the coordinates of the middle point D will be $\left(\dfrac{x_2+x_3}{2},\dfrac{y_2+y_3}{2}\right)$.

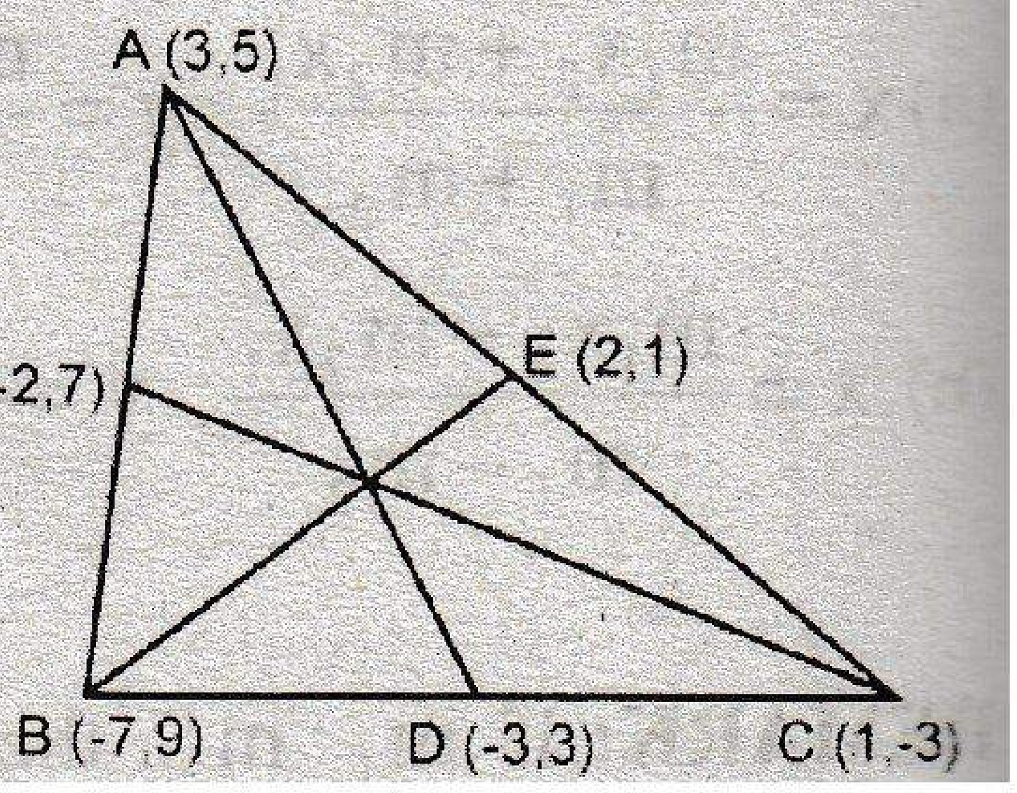
Since point G divides AD internally in the ratio 2:1, therefore the coordinates of point G are:

$$x=\frac{2\{(x_2+x_3)/2\}+1\times x_1}{2+1}=\frac{x_1+x_2+x_3}{3}$$

$$y=\frac{2\{(y_2+y_3)/2\}+1\times y_1}{2+1}=\frac{y_1+y_2+y_3}{3}$$

Example-3: The vertices of a triangle are A=(3,5), B=(-7,9) and C=(1,-3), Find the lengths of the three medians of the triangle.

Solution: Let D, E, and F be the mid-points of the three sides BC, AC and AB respectively 0 the triangle ABC.



By definition, the coordinates of these mid-point are:

$$\text{Coordinates of } D = \left(\frac{-7+1}{2}, \frac{9-3}{2}\right) = (-3,3)$$

$$\text{Coordinates of } E = \left(\frac{3+1}{2}. \frac{5-3}{2}\right) = (2,1)$$

$$\text{Coordinates of } F = \left(\frac{3-7}{2}, \frac{5+9}{2}\right) = (-2,7)$$

Hence, by the distance formula, the length of the three medians is given by

$$AD = \sqrt{(3+3)^2 + (5-3)^2} = \sqrt{40},\ BE = \sqrt{(-7,-2)^2 + (9-1)^2} = \sqrt{145}$$

$$CE = \sqrt{(1+2)^2 + (-3-7)^2} = \sqrt{109}$$

Example-4: Find the coordinates of the points which divide the join of (4, 7) and (2, 4) internally and externally in the ratio 3:5.

Solution: Let $R(x,y)$ divide the join of points $P(4,7))$ and $Q(2,4))$ internally in the ratio 3:5. $\therefore x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} = \frac{3\times2+5\times4}{3+5} = \frac{13}{4}$

$$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} = \frac{3\times4+5\times7}{3+5} = \frac{47}{8}$$

$$\therefore R(x,y) = (13/4,\ 47/8)$$

When $R(x,y)$ divide externally in the ratio 3:5, then

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} = \frac{3\times2-5\times4}{3-5} = 7$$

$$y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2} = \frac{3\times4-5\times7}{3-5} = \frac{23}{2}$$

$$\therefore R(x,y) = (7, 23/2)$$

Example-5: Find the ratio in which the join of points (2,-3) is divide by x-axis.

Solution: Let $P(2,-3)$ and $Q(5,6)$ be the given points. Let x-axis divide the line PQ in the ratio 1:K, then the coordinates of the point of division are :

$$\frac{2\times k+5\times 1}{k+1}=\frac{2k+5}{k+1} \text{ and } \frac{-3k+6\times 1}{k+1}=\frac{-3k+6}{k+1}$$

Since the point lines on the x-axis, therefore its y-coordinates is zero, Thus $\frac{-3k+6}{k+1}=0\Rightarrow k=2$. Hence, the ratio is 1:2

Example-6: Find the ratio in which the point (2, 1) divides the join of the points (1, -2) and (4, 7).

Solution: Let the point R(2, 1) divide the join of P(1,-2) and Q(4, 7) in the ratio $\lambda:1$.

By definition, the coordinates of the point of division are: $2=\frac{4\lambda+1}{1+\lambda}\Rightarrow \lambda=\frac{1}{2}$ Hence, R divides PQ internally in the ratio, $\frac{1}{2}:1\Rightarrow 1:2$.

REMARK: If the value of λ happens to be negative, then it implies that the division is external.

Example-7: Prove that the points (-2,-1), (1,0), (4,3) and (1,2) be the vertices of a parallelogram.

Solution: Let $A(-2,-1)$, $B(1,0)$, $C(4,3)$ and $D(1,2)$ are the vertices of a quadrilateral,

Then midpoint of, $AC=\left(\frac{-2+4}{2},\frac{-1+3}{2}\right)=(1,1)$

Similarly, midpoint of, $BD=\left(\frac{1+1}{2},\frac{2+0}{2}\right)=(1,1)$

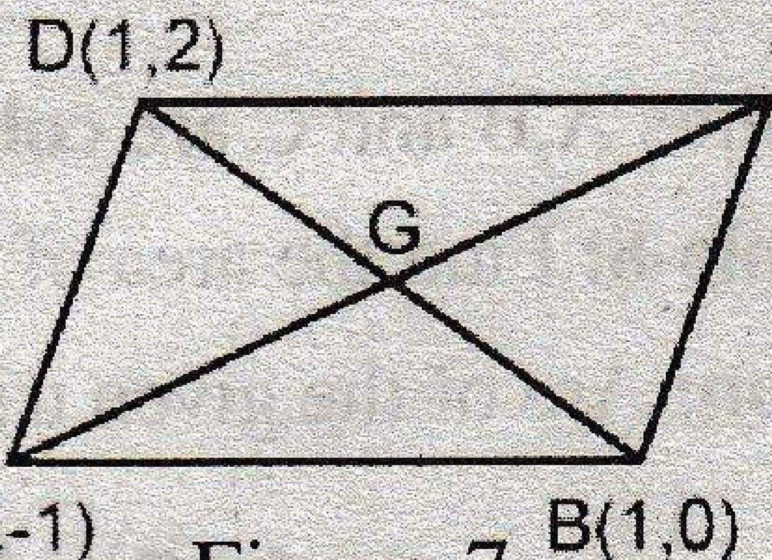


Figure-7

6.8 AREA OF A TRIANGLE

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of the triangle ABC as shown in fig-8. draw perpendiculars AA′, BB′ and CC′ from A,B and C respectively on X-axis. Here OA′=x_1, OB′=x_2, OC′=x_3 and AA′=y_1, BB′=y_2, CC′=y_3. area of the triangle ABC, Δ=Area of trapezium AA′CC′+Area of trapezium CC′BB′-Area of trapezium AA′BB′=$\frac{1}{2}$(Sum of parallel sides)$\times$(Perpendicular distance between the parallel sides).

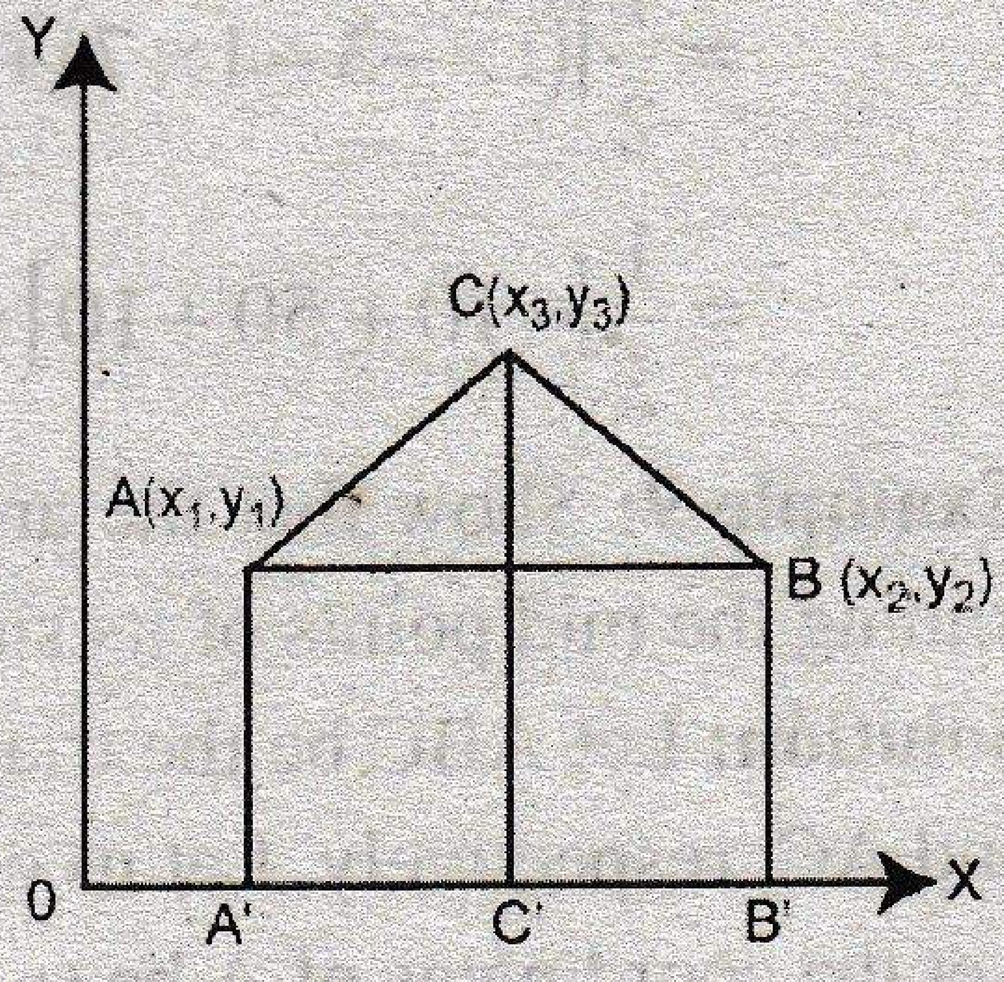


Figure-8: Area of Triangle

Hence, Area of the triangle, ABC is:

$$\Delta=\frac{1}{2}(AA'+CC')\times A'C'+\frac{1}{2}(CC'+BB')\times C'B'-\frac{1}{2}(AA'+BB')\times A'B'$$

$$=\frac{1}{2}(y_1+y_3)(x_3-x_1)+\frac{1}{2}(y_3+y_2)(x_2-x_3)-\frac{1}{2}(y_1+y_2)(x_2-x_1)$$

$$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_2-y_1)]$$

$$=\frac{1}{2}[(x_1y_2+x_2y_3+x_3y_1)-(y_1x_2+y_2x_3+y_3x_1)]$$

WORKING METHODS

First Method: Write down the coordinates of the vertices in a row and repeat the first set at the end. Multiply across as indicated by the arrows and attach positive signs to those obtained by thick lines and negative signs to those obtained by dotted lines. Finally, and up and take half of the sum i.e.

Area of the triangle, $\Delta=\frac{1}{2}\begin{pmatrix} x_1 & \searrow\swarrow & x_2 & \searrow\swarrow & x_3 & \searrow\swarrow & x_1 \\ y_1 & & y_2 & & y_3 & & y_1 \end{pmatrix}$

Second Method: Area of the triangle = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

The points A,B and C are **collinear** if the area of the triangle formed by them is zero.

Example-8: Find the area of the triangle whose vertices are $(1,2),(7,-3),(12,2)$

Solution: Are of the given triangle is

$$\frac{1}{2}[(1\times-3-1\times7)+(7\times2-12\times-3)+(12\times1-2\times1)]$$

$$=\frac{1}{2}[(1\times-3-1\times7)+(7\times2-12\times-3)+(12\times1-2\times1)]$$

$$=\frac{1}{2}[-10+50+10]=25 \text{ Square units.}$$

Example-9: Show that the area of a triangle is four times the area of the triangle formed by joining the mid points of its sides.

Solution: Let ABC be the triangle with base BC=2a, D, E and F are the mid points of BC, AB and AC, respectively. Let us consider D as origin and BC as x-axis.

Let the coordinates of A be (x, y).

Then area of $\Delta ABC = \frac{1}{2}\begin{bmatrix} x & -a & a & x \\ y & 0 & 0 & y \end{bmatrix}$

$$= \frac{1}{2}\left[(x\times 0 + y\times a) + (-a\times 0 - 0\times a)(a\times y - 0\times x)\right] = ay$$

coordinates of E= $\left(\frac{x-a}{2}, \frac{y}{2}\right)$ and

coordinates of F= $\left(\frac{x+a}{2}, \frac{y}{2}\right)$

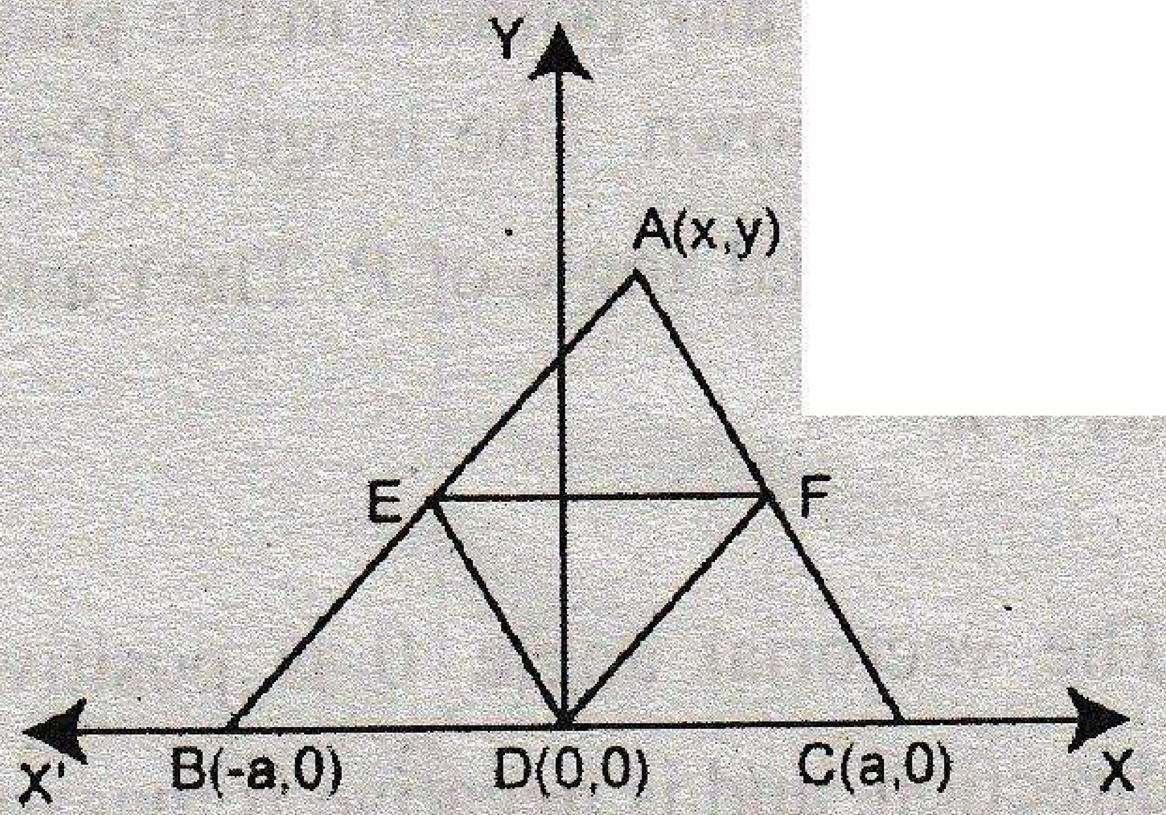


Figure-9

Area of $\Delta DEF = \frac{1}{2}\begin{bmatrix} 0 & \frac{x-a}{2} & \frac{x+a}{2} & 0 \\ 0 & y/2 & y/2 & 0 \end{bmatrix}$

$$= \frac{1}{2}\left[\left(\frac{0\times y}{2} - 0\times\frac{x+9}{2}\right) + \left(\frac{x-a}{2}\times\frac{y}{2} - \frac{x+a}{2}\times\frac{y}{2}\right) + \left(\frac{x+a}{2}\times 0 - 0\frac{y}{2}\right)\right] = \frac{1}{4}ay$$

Hence, Δ ABC= 4 Δ DEF

Example-10: Show that the points $P(3,-2)$, $Q(-1, 1)$ and $R(-5, 4)$ are collinear:

Solution: The points P, Q and R are collinear if the area of the triangle formed by them is zero. Therefore, area of

$$\Delta PQR = \frac{1}{2}\begin{bmatrix} 3 & -1 & -5 & 3 \\ -2 & 1 & 4 & -2 \end{bmatrix}$$

$$= \frac{1}{2}\left[\{(3\times 1) - (-2\times -1)\} + \{(-1\times 4) - (1\times 5)\} + (-5\times 2) - (4\times 3)\right]$$

$$= \frac{1}{2}[3 - 2 - 4 + 5 + 10 - 12] = 0$$ Hence, the points are collinear.

Example-11: if the points (2, 3/2), (-3, -7/2) and (k, 9/2) are collinear, then find the value of k.

Solution: The area of the triangle formed by the given collinear points is given by

$$\frac{1}{2}\begin{bmatrix} 2 & -3 & k & 2 \\ 3/2 & -7/2 & 9/2 & 3/2 \end{bmatrix} = 0 \Rightarrow \frac{1}{2}\left[2\left(-\frac{7}{2}\right) - \left(-3\times\frac{3}{2}\right) + \left(-3\times\frac{9}{2}\right) - \left(k\times\frac{-7}{2}\right) + \left(k\times\frac{3}{2}\right) - \left(2\times\frac{9}{2}\right)\right] = 0$$

$$\Rightarrow -\frac{14}{2} + \frac{9}{2} - \frac{27}{2} + \frac{7k}{2} + \frac{3k}{2} - \frac{18}{2} = 0 \Rightarrow, k = 5 \text{ [Ans.]}$$

6.9 POLAR COORDINATES

Polar coordinate system is used to determine the position of any point in the plane. Let OX be a given straight line called the initial line through a fixed point O, which is called the origin or pole. Take any point P in the plane, then its position is fixed if the distance OP and the angle Pox are known. The length OP=r is called the radius vector and the angle XOP $=\theta$ in called the Vectorial angle of P. The r and θ are called the polar co-ordinates of P and is denoted by P (r, θ).

The vectorial angle θ is positive if it is measured from the initial line in the clockwise direction, and is negative if measured in the opposite direction.

6.10 RELATION BETWEEN POLAR AND CARTESIAN COORDINATES OF A POINT

Let P be any point whose Cartesian coordinates with –respect to the axes OX and OY be (x, y) and the polar coordinates with respect to the origin O and initial line OX be (r, θ). Draw PM perpendicular on OX, we have

OM=x, MP=y, $\angle$ MOP= θ and OP=r, Now, in Δ OPM,

$$\frac{x}{r} = \cos\theta \Rightarrow x = r\cos\theta \text{..............................(i)}$$

$$\frac{y}{r} = \sin\theta \Rightarrow y = r\sin\theta \text{..............................(ii)}$$

Squaring and adding (i) and (ii), we get

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

Dividing (ii) by (i) we get, $\tan\theta = \frac{y}{x}$. Or $\theta = \tan^{-1}\frac{y}{x}$.

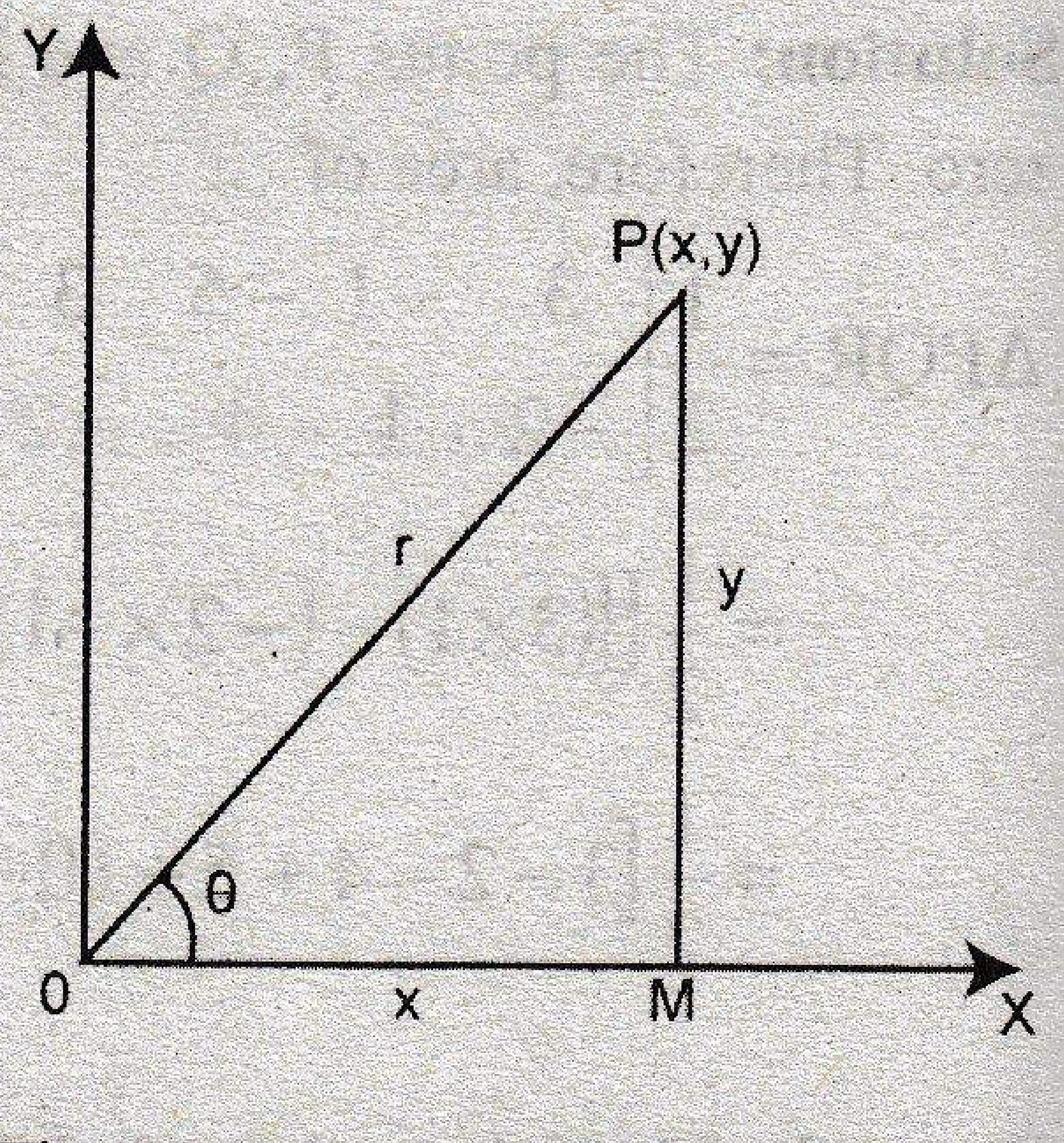


Fig: 10: Relation between Polar and Cartesian Coordinates

6.11 THE STRAIGHT LINE AND STANDARD FORM OF ITS EQUATIONS

Straight line plays an important role in the development and study of curves. Mathematically, a straight line may be defined as the shortest distance between two az distinct points.

(i) Slope (or Gradient) of a straight line

The slope of a line is defined as a tangent of the angle which the line makes with the positives direction of x-axis and is generally denoted by m. Thus, if a line makes an angle θ with the positive direction of the x-axis, then its slope is given by $m = \tan\theta$

If θ is acute, the slope is positive and if θ is obtuse, the slope is negative.

The slope of a line passing through two points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ as shown in Fig. 12 is given by

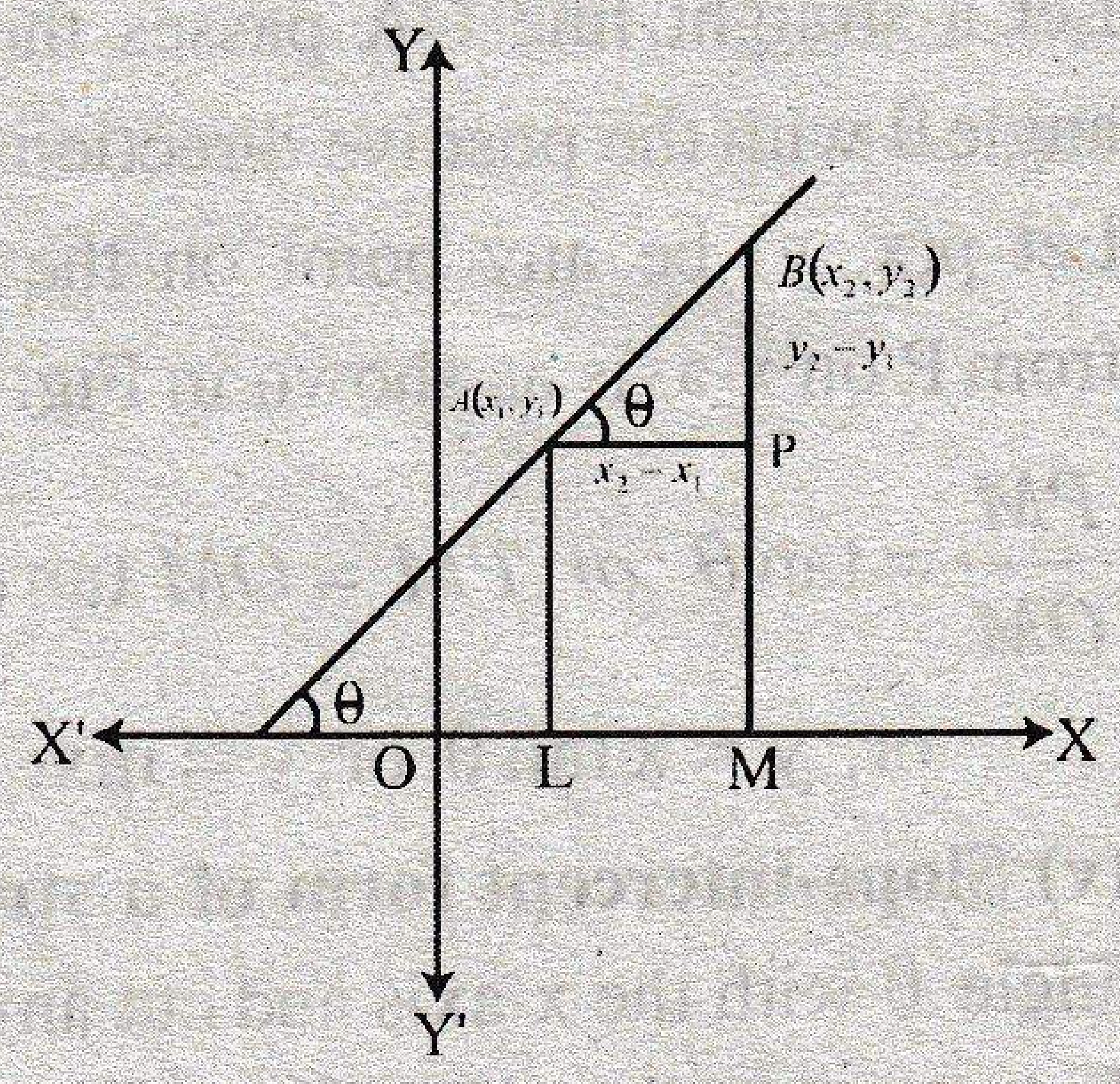


Fig: 12 Slope of Straight line

$$m = \tan\theta = \frac{BP}{AP} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Difference\ of\ ordinates}{Difference\ of\ abscissase}$$

(ii) Equations of the Coordinates Axes

(i) If any point P (x, y) lies on the x-axis, then its ordinate y is always zero (Fig. 13). Therefore, $y = o$ is the equation of x-axis.

(ii) If any point P (x, y) lies on the y-axis, then its abscissa x is always zero (Fig. 13). Therefore, $x = 0$ is the equation of y-axis.

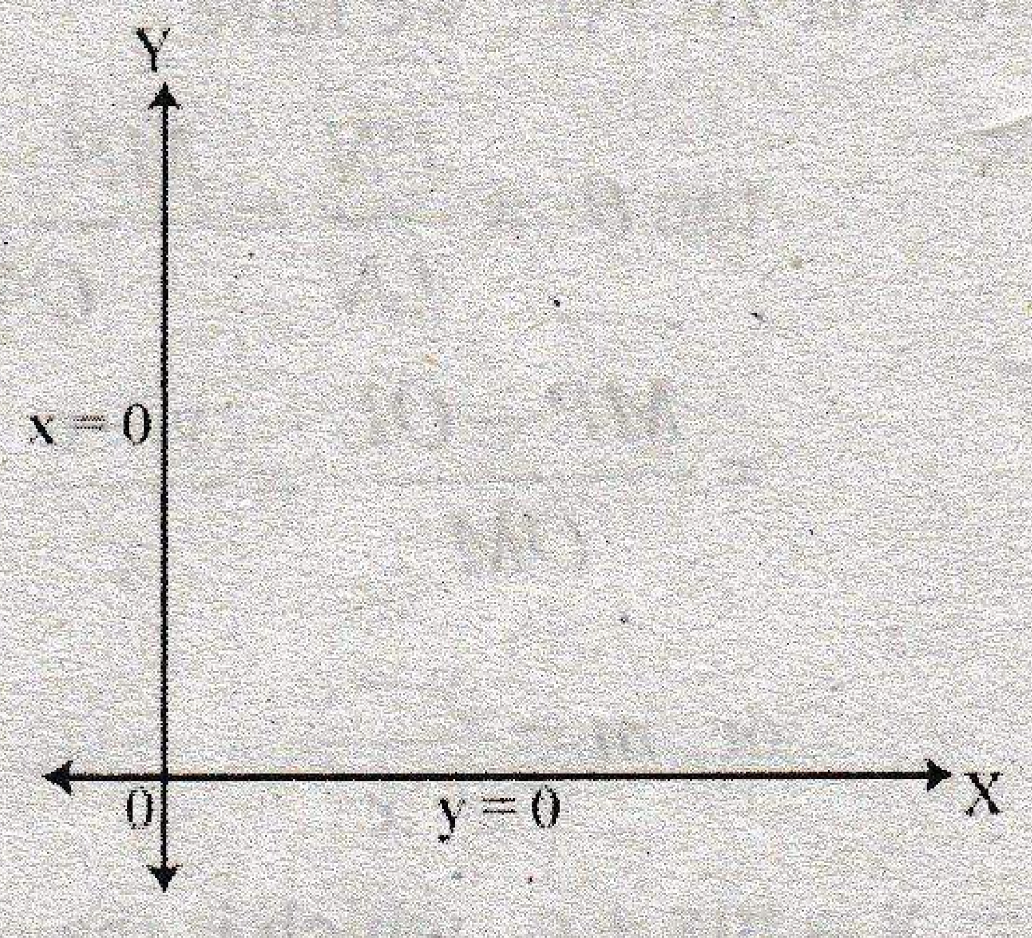


Fig: 13: Equation of the Coordinate Axes

(iii) Equation of lines Parallel to Co-ordinate Axes

Let P(x, y) be any point on the line PR parallel to y-axis and at a distance 'a' from it. Wherever the point P lies on the line PR, its abscissa x is always constant and is equal to 'a'. Hence, the equation to the straight line parallel to the y-axis and at a distance 'a' from it is x = a.

(i) Similarly, the equation to the straight line parallel to x-axis and at a distance 'b' from it is y = b.

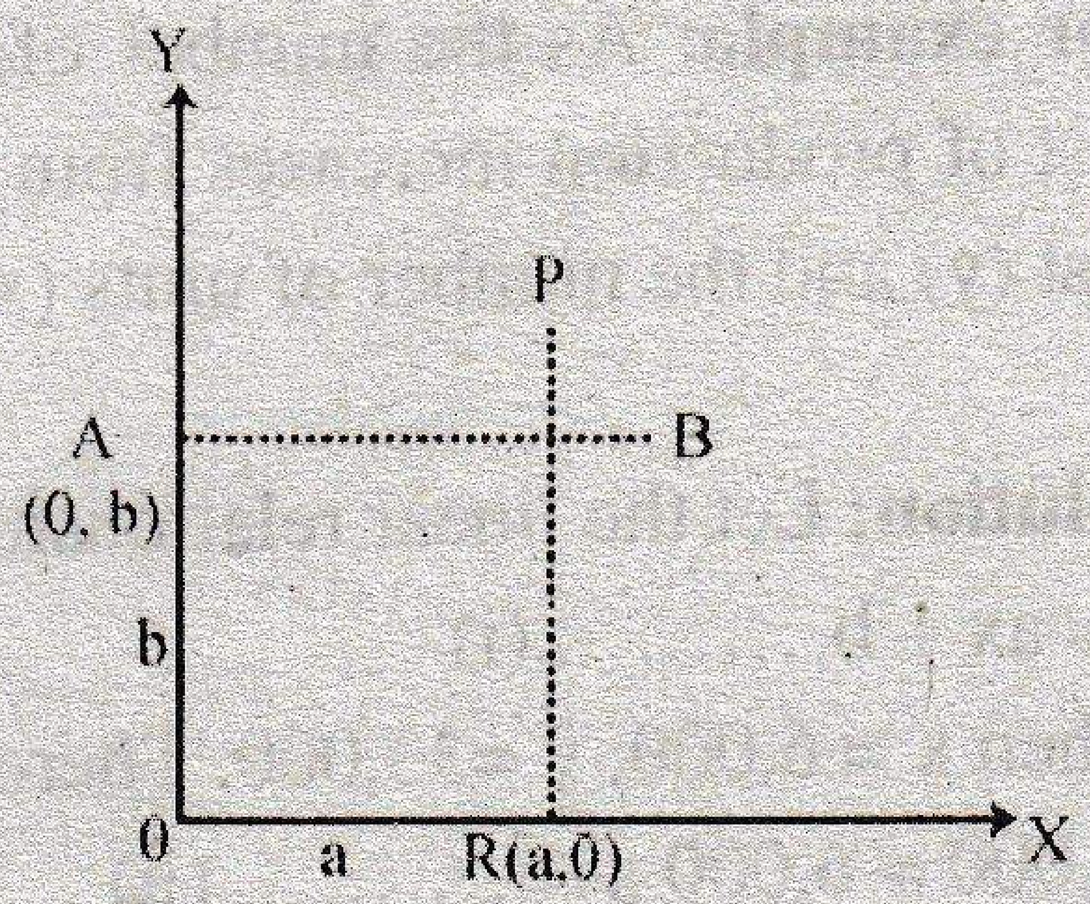


Fig: 14: Line Parallel to X-axis and Y-axis

(iv) Origin-slope form of a straight line: The equation of a straight line through the origin and making an angle θ with x-axis.

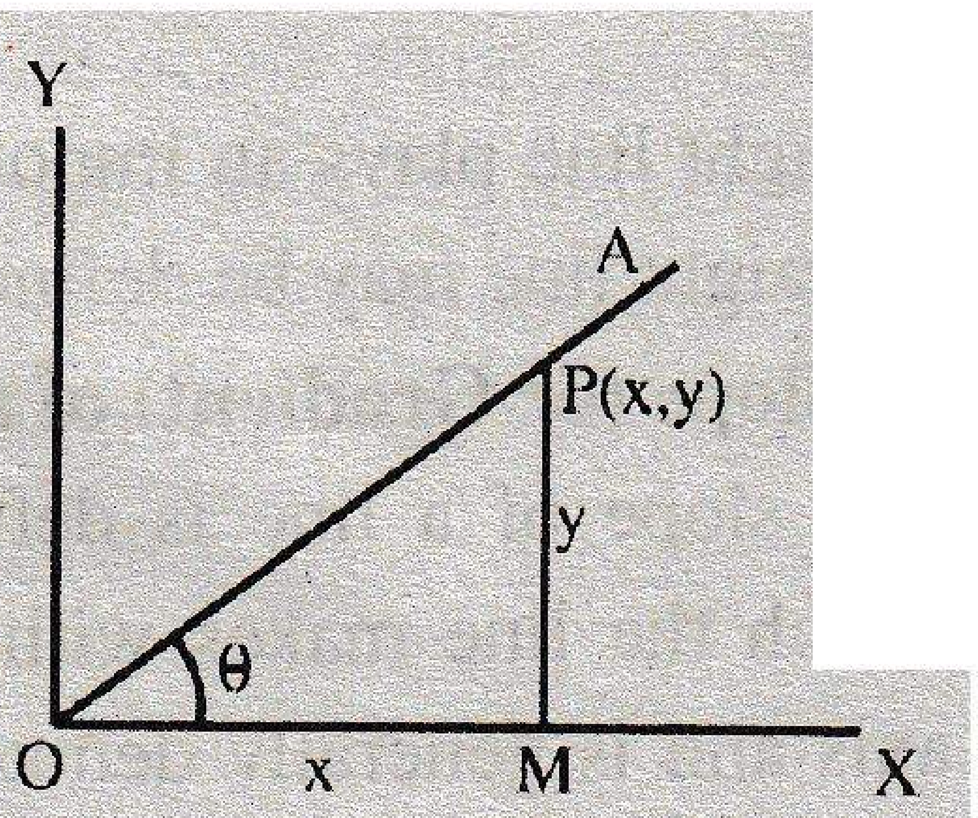


Fig: 15: Equation of Straight Line: Origin-slope Form

Let a straight line OA passes through the origin O and makes angle θ with the positive direction of x-axis, i.e., $\angle XOA = \theta$.
Let p(x, y) be any point on the line. Draw perpendicular PM from P on x-axis as shown in Fig. 15, then in $\Delta\ POM$, we have

$$\frac{PM}{OM} = \tan\theta \quad or, PM = OM \tan\theta$$

or, y = x tan θ or, y = mx, which is the required equation.

(v) Slope-intercept form of a straight line: The equation of a straight line which makes an angle θ with the x-axis and cut any intercept c on y-axis.
Let a straight line AB make an angle θ with the x-axis and cut off an intercept OL=c on the y-axis. Take any point $P(x, y)$ on the line and draw perpendicular PM on $x-axis$ and LN on PM

Now in ΔLNP, we have

$$\tan\theta = \frac{PN}{LN} = \frac{MP - MN}{OM}$$

$$= \frac{MP - OL}{OM} = \frac{y-c}{x}$$

$$\text{or, } m = \frac{y-c}{x}$$

i.e. y = mx + c, which is the required equation.

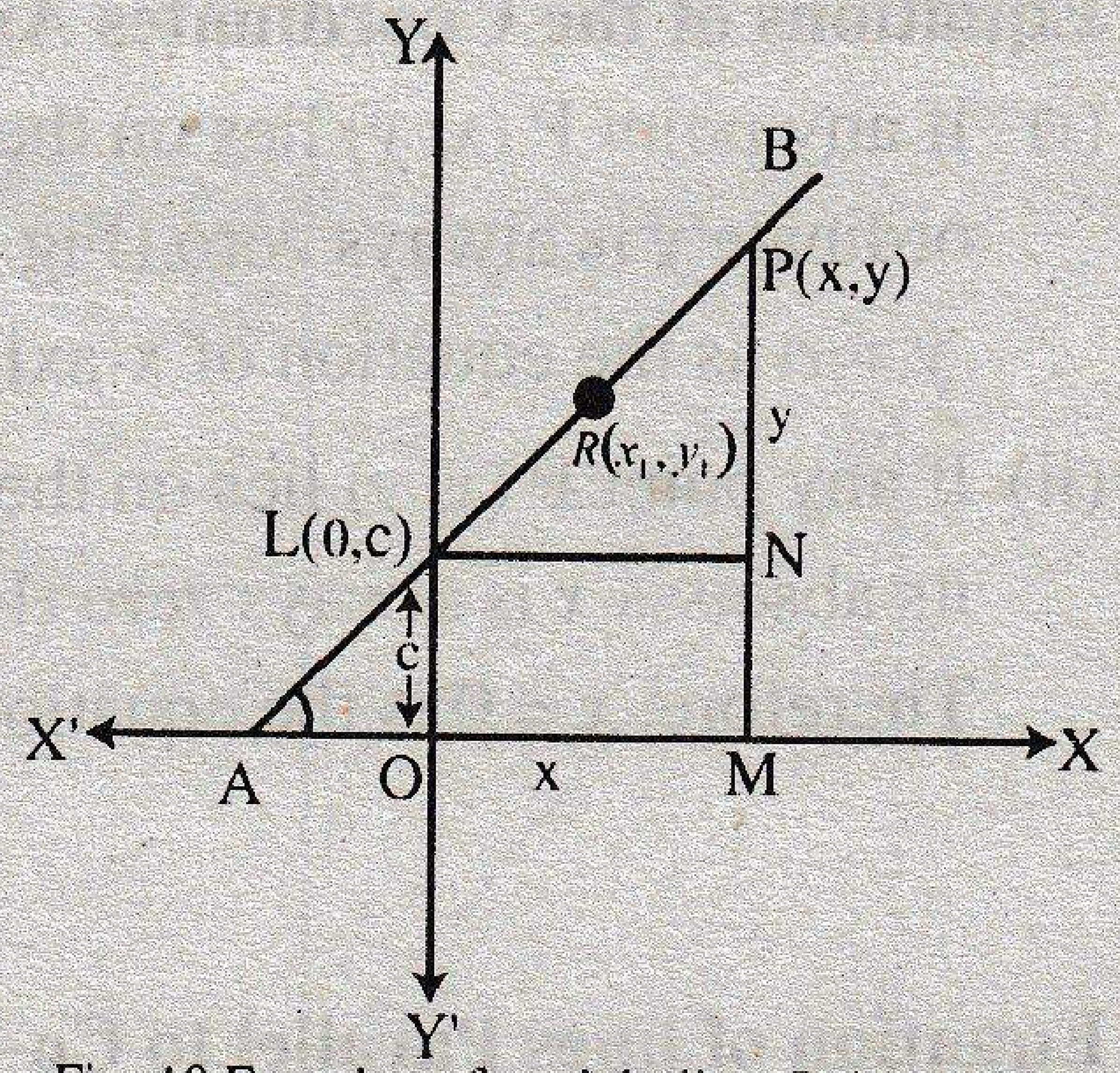


Fig: 18 Equation of straight line: Point-Slpe Form

For example: As the number of units manufactured increases from 6000 to 8000, the total cost of production increases from Tk. 33,000 to Tk. 40,000. Find the relationship between the cost (y) and the number of units (x), if the relationship is linear.

Solution: Let the linear relationship between x and y be given by
y = ax + b...........(i)
When x = 6,000; y =33,000. Therefore from (i), we have,
33,000 = 6,000 a+b...........(ii)
Similarly, when x = 8,000, y = 40,000, therefore from (i), we have,
40,000 = 8,000 a + b..(iii)

Subtracting (ii) from (iii), we get 7,000 = 2,000 a or, a=3.5
Substituting a=3.5 in (ii), we get, 33,000 =6,000×3.5 + b or, b = 12,000
Hence, the relationship is: y=3.5x+12,000

(vi) Two intercept form of a straight line: Let us consider a straight line which cut-off intercepts 'a' and 'b' on the coordinate axis x and y respectively. Suppose this straight line cut off the axis of x and y at the points A and B respectively, such that

OA = b and OB = a.

Let P(x, y) be any point on the line AB and draw perpendicular PM on x-axis.

From ΔAOB and ΔPMB, we have

$$\frac{PM}{AO}=\frac{MB}{OB}=\frac{OB-OM}{OB}=1-\frac{OM}{OB}$$

or, $\frac{y}{b}=1-\frac{x}{a}$.

Thus $\frac{x}{a}+\frac{y}{b}=1$ is the required equation.

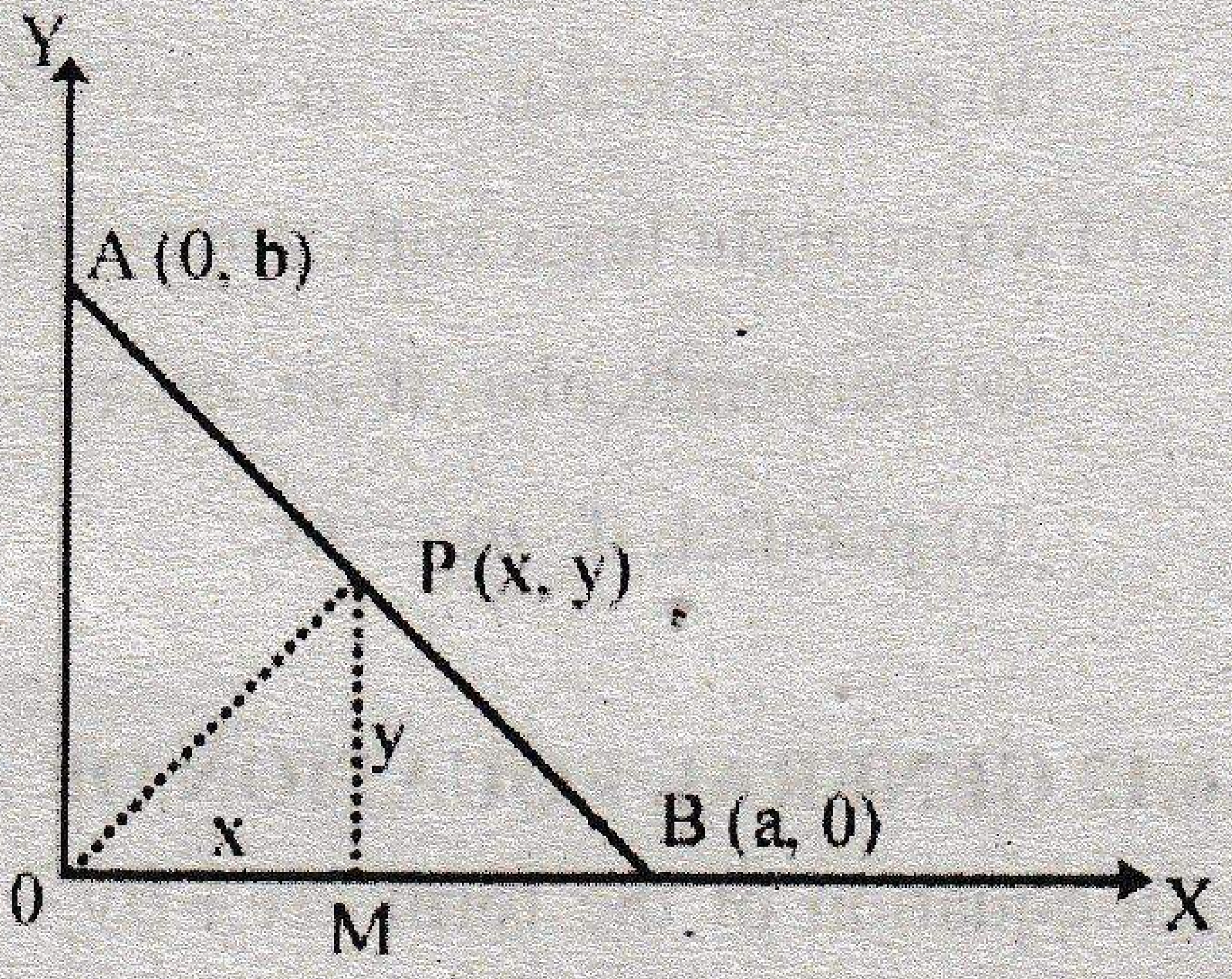


Fig:17: Equation of Straight Line: Two Intercept Form

(vii) Point-slope form of a straight line: The equation of the straight line having slope m and which passes through the point $R(x_1, y_1)$.

The equation of a straight line AB which makes an angle θ with x-axis and makes an intercept OL=c on y-axis is given by $y = mx + c$. ....(i)

If the point $R(x_1, y_1)$ lies on (i) then we have

$y_1 = mx_1 + c$ or $c = y_1 - mx_1$

Putting this value of c in (i), we have,

$y = mx + y_1 - mx_1$

or, $y - y_1 = m(x - x_1)$, which is the required equation.

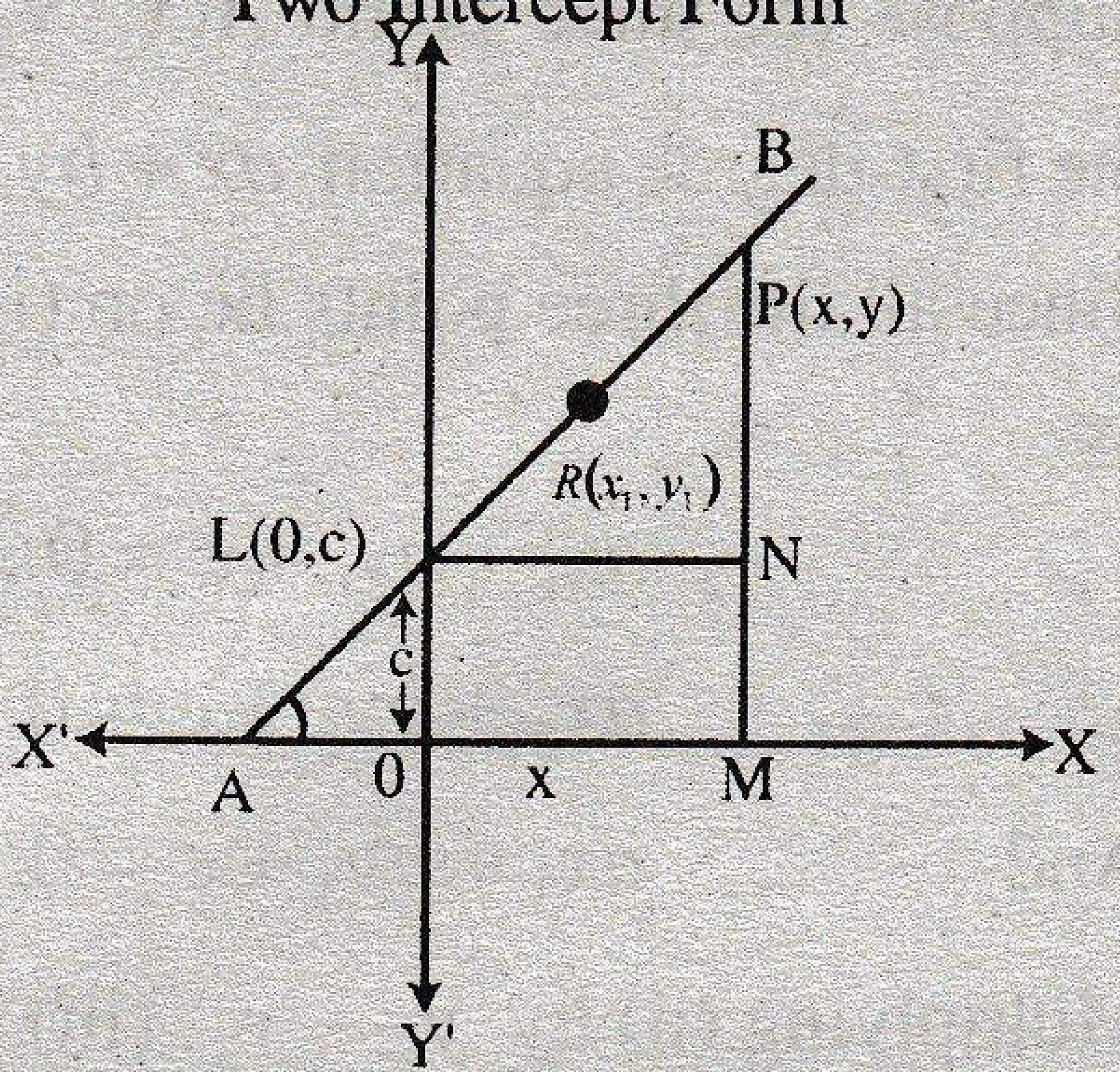


Fig: 18 Equation of straight line: Point-Slpe Form

Remark: The slope 'm' of a line is a real number. Therefore m = 0,m > 0 or, m < 0.

(viii) Two-points form of a straight line: Consider a straight line passing through two points (x_1, y_1) and (x_2, y_2). We have, the equation of a straight line passing through the single point (x_1, y_1) with slope 'm' is given by

$y - y_1 = m(x - x_1)$..........(i).

But, if the point (x_2, y_2) also lies on the same line, then we have

$$y_2 - y_1 = m(x_2 - x_1) \text{ or, } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Putting this value of m in (i), we get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$, Which is the required equation.

(ix) The equation of a straight line

(a) parallel to $ax + by + c = 0$ is $ax + by + k = 0$

(b) perpendicular to $ax + by + c = 0$ is $bx - ay + k = 0$

(x) Two straight lines with slopes m_1 and m_2 are:

(a) perpendicular, if $m_1 m_2 = -1$

(b) parallel, if $m_1 = m_2$

6.12 GENERAL EQUATION OF A STRAIGHT LINE

An equation of the form $ax + by + c = 0$ is called the general equation of the straight where a, b, c are constants and x, y are variables.

Suppose the three point $A(x_1, y_1)$, $B(x_2, y_2)$ *and* $C(x_3, y_3)$ lie on the locus represented by the first-degree equation $ax + by + c = 0$ in x and y.

Since all the three points lie on (i), then coordinates of each of them must satisfy the equation, i.e.

$$ax_1 + by_1 + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$ax_2 + by_2 + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

And $$ax_3 + by_3 + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

Multiplying (ii) by $y_2 - y_3$, (iii) by $y_3 - y_1$ and (iv) by $y_1 - y_2$ and adding, we have

$$a[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_3)] + b[y_1(y_2 -_3) + y_2(y_3 - y_1) + y_3(y_1 - y_2)]$$
$$+ c[y_2 - y_3 + y_3 - y_1 + y_1 - y_2] = 0$$

or, $$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_3) = 0$$

or, $$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_3)] = 0$$

It represents x_1 the area of the triangle formed by joining three points. Since area is zero, therefore three points A, B, C are collinear.

Remark: For determining the slope of the line $ax + by + c = 0$ express it as:

$$y = -\frac{a}{b}x - \frac{c}{b} = mx + d \text{ where } m = -\frac{a}{b} \text{ and } d = -\frac{c}{d}$$

Thus, the slope of the line, $m = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$

6.13 INTERSECTION OF TWO STRAIGHT LINES

Since the point of intersection of two lines lies on both the lines and, therefore, its coordinates satisfy both the equations. Thus, coordinates of the point of intersection are obtained by solving the equation of both the lines.

Let the equation of the two given straight lines be

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0$$

Let $P(x_1, y_1)$ be the point of intersection, then

$$a_1x_1 + b_1y_1 + c_1 = 0 \ldots\ldots\ldots\ldots(i)$$

$$a_2x_1 + b_2y_1 + c_2 = 0 \ldots\ldots\ldots\ldots(ii)$$

Solving (i) and (ii) by the cross multiplication rule, we have

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\text{or, } x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}; \; y_1 = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

6.14 CONDITION FOR CONCURRENT OF THE THREE STRAIGHT LINES

The three straight lines are said to be concurrent if the point of intersection of any two straight lines lies on the third line i.e. all of them have a point common to all of them.

Let the three lines be

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

$$\text{and } a_3x + b_3y + c_3 = 0$$

The coordinates of the point of intersection of (i) and (ii) are (see earlier section)

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

The three lines are concurrent if this point lies on line (iii). That is, its coordinates must satisfy the third line. Therefore

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0$$

$$\text{or, } a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0$$

This may be written in the determinant form as:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

Hence, the condition that three lines be concurrent is that the determinant of the coefficients should be zero.

For Example: Prove that the lines: $3x-4y+5=0,\ 7x-8y+5=0$ *and* $4x+5y=45$ are concurrent.

Solution: It is given that

$$3x-4y+5=0 \ldots\ldots\ldots\ldots(i)$$

$$7x-8y+5=0 \ldots\ldots\ldots\ldots\ldots(ii)$$

$$4x+5y-45=0 \ldots\ldots\ldots\ldots(iii)$$

Solving (i) and (ii) using cross multiplication rule for x and y, we get

$$\frac{x}{-4\times5-5\times(-8)}=\frac{-y}{3\times5-7\times5}=\frac{1}{-8\times3-7\times(-4)}$$

$$\text{or, } x=\frac{-4\times5-5\times(-8)}{-8\times3-7\times(-4)}=\frac{20}{4}=5 \text{ and } y=\frac{7\times5-3\times5}{-8\times3-7\times(-4)}=\frac{20}{4}=5$$

Thus, the coordinates of the point of intersections of (i) and (ii) is: $(x,y)=(5,5)$ Putting this point in (iii) i.e. $4x+5y=45$, we have

$4\times5+5\times5=45$, Which is true. Hence, the three lines are concurrent.

6.15 ANGLE BETWEEN TWO STRAIGHT LINES

Let AB and CD be the straight lines $y = m_1x + c_1$ *and* $y = m_2x + c_2$ inclined to OX and at an angle θ_1 and θ_2 respectively so that $m_1 = \tan\theta_1$ *and* $m_2 = \tan\theta_2$

For Interior Angle: In figure 20, we have

$$\theta_1 = \theta + \theta_2 \quad or, \theta = \theta_1 - \theta_2$$

$$\therefore \tan\theta = \tan(\theta_1 - \theta_2)$$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 . \tan\theta_2} = \frac{m_1 - m_2}{1 + m_1 . m_2}$$

$$= \frac{\text{Difference of slope}}{1 + \text{Product of slopes}}$$

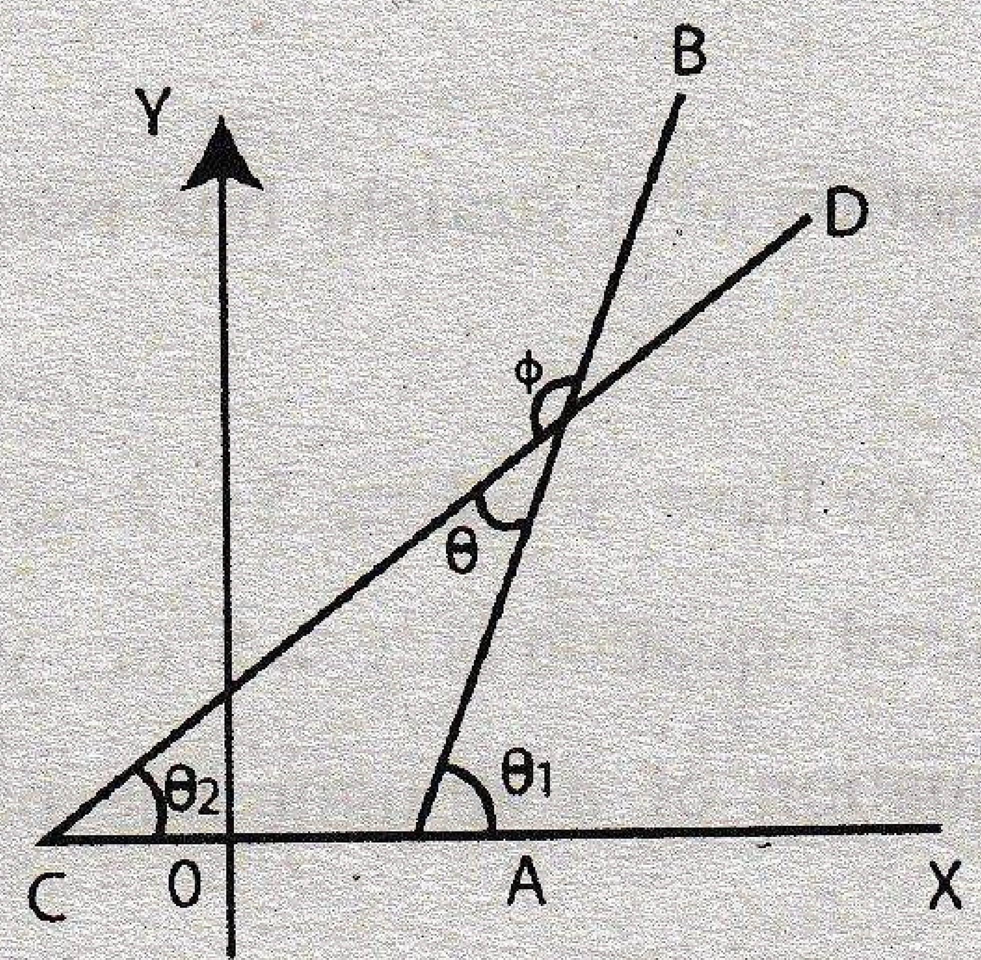


Fig: 20: Angle between two

For the Exterior Angle :

Given that $\pi = \theta + \phi \quad or, \ \phi = \pi - \theta$

Therefore, $\tan\phi = \tan(\pi - \theta)$

$$= -\tan\theta = -\frac{m_1 - m_2}{1 + m_1 . m_2}$$

Condition of parallelism and perpendicularity condition of parallelism: If the two lines are parallel, them the angle between them is zero, i.e.

$$\theta = 0 \ or, \tan\theta = 0, i.e. \frac{m_1 - m_2}{1 + m_1 m_2} = 0 \ or, \ m_1 - m_2 = 0, \text{ i.e. } m_1 = m_2$$

Hence, two lines are parallel, if their slopes are equal.

Condition of Perpendicularity: If the two lines are perpendicular, then the angle between them is $90^0, i.e. \theta = 90^0 \ or, \ \tan\theta = \tan 90^0 = \infty$. that is,

$$\frac{m_1 - m_2}{1 + m_1 . m_2} = \infty \text{ or, } \frac{1 + m_1 . m_2}{m_1 - m_2} = \frac{1}{\infty} \ or, 1 + m_1 m_2 = 0, i.e. \ m_1 m_2 = -1 \quad \left[\because \frac{1}{\infty} = 0\right]$$

Hence, two lines are perpendicular, if the product of their slopes is –1, i.e. the slope of one is the negative reciprocal of the other.

COST EQUATION

The equation of straight line $y = mx + c$ is known as **cost equation or cost function**, where y = total cost, $mx =$ variable cost, $c =$ fixed cost, $m =$ marginal cost or variable cost per unit, and $x =$ number of units. Therefore we have the relationship

That is: total cost = variable cost + fixed cost

Illustrations

Illustration-01: (a) Find the equation of a straight line parallel to the x-axis and passing through the point (-8, 5)

(b) Find the equation of a straight line parallel to the y-axis and passing through the point $(-3,-2)$.

Solution:

(a) The equation of a straight line parallel to the x-axis is $y = k$. which passes through the point $(-8,5)$.

$\therefore$ we have $y = k \Rightarrow 5 = k \Rightarrow 5$

So the required equation is $y = 5$ or $y - 5 = 0$ [Ans.]

(b) The equation of a straight line parallel to the y-axis is $x = k$, which passes through $(-3,-2)$.

$\therefore$ we have $x = k \Rightarrow -3 = k \Rightarrow k = -3$

So the required equation is $x = -3$ or, $x + 3 = 0$ [Ans.]

Illustration-02: Find the equation of the straight line:

(i) Parallel to $2x - 3y - 5 = 0$ and passing through (4, 5).

(ii) Perpendicular to $2x + 3y + 4 = 0$ and passing through (3,-2).

Solution: (i) The equation of a straight line parallel to $2x - 3y - 5 = 0$ is $2x - 3y + k = 0$. It passes through (4,5).

$\therefore 2x - 3y + k = 0 \Rightarrow 2(4) - 3(5) + k = 0 \Rightarrow 8 - 15 + k = 0 \Rightarrow -7 + k = 0 \Rightarrow k = 7$

Hence the required equation is $2x - 3y + k = 0$

$\Rightarrow 2x - 3y + 7 = 0$ [Ans.]

(ii) The equation of a straight line perpendicular to $2x + 3y + 4 = 0$ *is* $3x - 2y + k = 0$.

It passes through (3,-2).

$\therefore 3x - 2y + k = 0 \Rightarrow 3(3) - 2(-2) + k = 0 \Rightarrow 9 + 4 + k = 0$

$\Rightarrow 13 + k = 0 \Rightarrow k = -13$

Hence the required equation is: $3x - 2y + k = 0 \Rightarrow 3x - 2y - 13 = 0$ [Ans.]

Illustration-03: Find the equation of the straight line through (2, 5) and making equal intercepts of opposite sign on the axis.

Solution: Let the line cut off intercepts a and –a from the axis of x and y, respectively. Hence, the equation of the straight line in the intercept form is

$$\frac{x}{a}+\frac{y}{-a}=1 \Rightarrow x-y=a \ldots\ldots\ldots\ldots\ldots(i)$$

Since the line passes through (2, 5), therefore putting x = 2 and y = 5 in (i), we get

$$2-5=a \text{ or, } a=-3$$

Hence, on substituting $a=-3$, the equation (i) becomes: $x-y=-3$, which is the required equation.

Illustration-04: Find the equation of a straight line, which passes through the point (-5, 4) and is such that the portion of the line intercepted between the axis is divided at this point in the ratio 1:2

Solution: The equation of a straight line in the intercept form is: $\frac{x}{a}+\frac{y}{b}=1$

Since the point (-5, 4) divides straight line in the ratio 1:2, therefore

$$-5=\frac{2\times a+1\times 0}{1+2} \Rightarrow -5=\frac{2a}{3} \text{ or, } a=-\frac{15}{2}$$

and $$4=\frac{2\times 0+1\times b}{1+2} \Rightarrow 4=\frac{b}{3} \text{ or, } b=12$$

Putting values of a and b in the equation, we get

$\frac{2x}{-15}+\frac{y}{12}=1$ or, $8x-5y+60=0$, which is the required equation.

Illustration-05: Express the line $3x+4y=12$ in the intercept form and calculate the area of the triangle formed by this line and the axis of coordinates.

Solution: The given line $3x+4y=12$ can be represented in the intercept form as:

$$\frac{x}{12/3}+\frac{y}{12/4}=1 \textbf{ or, } \frac{x}{4}+\frac{y}{3}=1$$

It follows that this line meets the x-axis at the Pont (4, 0) and y-axis at the point (0, 3).
Hence, the area of the triangle whose vertices are (0, 0) (4, 0) and (0, 3) is:

$$\text{Area}=\frac{1}{2}\left[0\times(0-3)+4\times(3-0)+0\times(0-0)\right]=6 \text{ square units.}$$

Illustration-06: Find the equation of the line through the point (4,3) and (a) parallel to as well as (b) perpendicular to the line $3x+4y+7=0$.

Solution: (a) Suppose the line parallel to $3x+4y+7=0 \, is \, 3x+4y+k=0$ ......(1)

Here , the equation (1) passes through the point (4,3).

$$\therefore 3\times4+\times3+k=0 \Rightarrow k=-24$$

Putting the value of k in (1), we get $\quad 3x+4y-24=0$

(b) Suppose the line perpendicular to $3x+4y+7=0 \, is \, 4x+3y+k=0$ ......(2)

Here, the equation (2) passes through the point (4, 3).

$$\therefore 4\times4-3\times3+k=0 \Rightarrow k=-7$$

Putting the value of k in (1), we get $\quad 4x-3y-7=0$.

Illustration-07: Find the equation of line, which has y-intercept equal to 3 and is perpendicular to the line $2x+3y+5=0$

Solution: Given $2x+3y+5=0 \Rightarrow 3y=-2x-5 \Rightarrow y=-\frac{2}{3}x-\frac{5}{3}$

$\therefore$ Slope of the line $2x+3y+5=0 \, is -\frac{2}{3}$

Let the slope of the required line be m. Since the given line is perpendicular to the required line, therefore, $m_1m_2=-1 \Rightarrow m(-2/3)=-1 \Rightarrow m=(3/2)$

Hence, the equation of the required straight line is:

$$y=mx+c \Rightarrow y=\frac{3}{2}x+3.$$

Illustration-08: Find the equation of two straight lines through the point (4, -2) making an angle of 45^0 with the line $8x+7y-1=0$. Show that these lines are at right angles to one another.

Solution: The equation of line passing through (4, -2) is given by

$y+2=m(x-4)$.......(i) Where m is the slope of the straight line.

Given $8x+7y-1=0 \Rightarrow 7y=-8x+1 \Rightarrow y=-\frac{8}{7}x+\frac{1}{7}$

$\therefore$ Slope of the given line $8x+7y-1=0 \;\; is \; -\frac{8}{7}$.

Since (i) makes an angle of 45^0 with the given line, therefore we have

$$\tan 45^0. = \pm\frac{m-(-8/7)}{1+m(-8/7)} = \pm\frac{7m+8}{7-8m} \quad or, 1=\pm\frac{7m+8}{7-8m}$$

$or,\ m = -\frac{1}{15}$, taking positive sign,

$m = 15$, taking negative sign.

Putting these values of m in (i), we get

$$y+2=-\frac{1}{15}(x-4) \text{ and } y+2=15(x-2)$$

or, $x+15y+26=0$ *and* $15x-y-32=0$

Since the product of the slope-1/15 and 15 is –1, therefore the required lines are perpendicular to each other.

Illustration-09: Find the equation of the straight line passing through the intersection of $2x-3y+4=0$ and $3x+4y-5=0$ and is perpendicular to the line $6x-7y+8=0$

Solution: The equation of any line passing through the intersection of given lines is:

$$2x-3y+4+\lambda(3x+4y-5)=0 \text{ or, } (2+3\lambda)x-(3-4\lambda)y+(4-5\lambda)=0 \quad \text{.......(i)}$$

where λ is any arbitrary constant.

Slope of (i) is $=\frac{2+3\lambda}{3-4\lambda}$

Since (i) is perpendicular to the line $6x-7y+8=0$, whose slope is 6/7. Thus, we have

$$\frac{2+3\lambda}{3-4\lambda}\times\left(\frac{6}{7}\right)=-1 \text{ or, } 12+18\lambda=(21-28\lambda) \text{ i.e. } \lambda=9/46$$

So the equation of the required line is

$$\left(2+\frac{27}{46}\right)x-\left(3-\frac{36}{46}\right)y+\left(4-\frac{45}{46}\right)=0 \text{ or, } 119x+102y=139.$$

Illustration-10: Find the co-ordinates of vertices of the triangle formed by the line x=1, $x-3x-1=0$ and $x+y-5=0$. Show that the line joining the point (2, 1) to any vertex is perpendicular to the opposite side.

Solution: Given lines are $x=1$...............*(i)*

$x-3y-1=0$..................*(ii)*

$x+y-5=0$......................*(iii)*

Let the vertices of the triangle formed by the above lines be A, B, C

Solving (i) and (ii), we get $x=1, y=0$

Solving (ii) and (iii), we get $x=4, y=1$

Solving (i) and (iii), we get $x=1, y=4$

Thus, the co-ordinates of vertices are: $A(1,0)$, $B(4,1)$ and $C(1,4)$

Consider the line joining the point $E(2,1)$ to the vertex $C(1,4)$. Then slope of

$$CE=\frac{1-4}{2-1}=-3 \text{ (say } m_1)$$

The slope of line joining the points AB is 1/3 (say m_2). Since $m_1m_2=-3\times(1/3)=-1$, therefore, AB perpendicular to CE.

Similarly, it can be show that AE perpendicular to BC and BE perpendicular to AC. Hence, The line joining (2, 1) to any vertex is perpendicular to the opposite side.

Illustration-11: Find the coordinates of the foot of the perpendicular from the point $(6,-1)$ on the line $3x-5y+11=0$

Solution: Given $3x-5y+11=0 \Rightarrow 5y=3x+11 \Rightarrow y=\frac{3}{5}x+\frac{11}{5}$

$\therefore$ Slope of he given line is 3/5. Therefore, the slope of the line perpendicular to this is -5/3.

Now the equation of the line passing through the point $(6,-1)$ and perpendicular to the given line is given by $y+1=\frac{-5}{3}(x-6)$ *or*, $5x+3y-27=0$.......(i)

The foot of the perpendicular is the point of intersection of (i) and the given line. Thus, coordinates are obtained by solving simultaneous equations:

$$3x-5y+11=0 \text{ and } 5x+3y-27=0$$

Solving these two equations, we get $x=3$ and $y=4$

Hence, the foot of the perpendicular is (3, 4)

Illustration-12: (a) Find the equation of a straight line passing through the point (3, 4) such that the sum of its intercepts on the axes is 14.

(b) Find the equation to the straight line which passes through (-5,2) and is such that the portion of it between the axes is divided by the point in the ratio 2:3

Solution: (a) Let the equation be $\frac{x}{a}+\frac{y}{b}=1$, which passes through (3, 4).

Given sum of intercepts =14 i.e.

$$a+b=14 \Rightarrow a=14-b\text{..................}(1)$$

Now $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{3}{a}+\frac{4}{b}=1 \Rightarrow \frac{3b+4a}{ab}=1 \Rightarrow 3b+4a=ab$

$$\Rightarrow 3b+4(14-b)=(14-b)b \Rightarrow 3b+56-4b=14b-b^2$$

$$\Rightarrow b^2-15b+56=0 \Rightarrow b^2-8b-7b+56=0$$

$$\Rightarrow b(b-8)-7(b-8)=0 \Rightarrow (b-8)(b-7)=0$$

$$\Rightarrow (b-8)=0 \ or (b-7)=0$$

$$\therefore b=8 \ or, b=7$$

When $b=8$, then $a=14-b=14-8=6$

Required equation is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{6}+\frac{y}{8}=1 \Rightarrow \frac{4x+3y}{24}=1 \Rightarrow 4x+3y=24$ [Ans.]

Again when $b=7$, then $a=14-b=14-7=7$

Required equation is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{7}+\frac{y}{7}=1 \Rightarrow \frac{x+y}{7}=1 \Rightarrow x+y=7$ [Ans.]

(b) Let $P \equiv (-5,2)$, $m_1 : m_2 = 2:3$, $A \equiv (a,0), B \equiv (0,b)$

Point of division $\equiv \left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$

$$\Rightarrow (-5,2) \equiv \left(\frac{2(0)+3(a)}{2+3}, \frac{2(b)+3(0)}{2+3}\right).$$

$$\Rightarrow (-5,2) \equiv \left(\frac{3a}{5}, \frac{2b}{5}\right)$$

$$\Rightarrow -5=\frac{3a}{5}, 2=\frac{2b}{5} \Rightarrow -5=\frac{3a}{5}, \ 2=\frac{2b}{5}$$

$$\Rightarrow 3a=-25, \ 2b=10 \Rightarrow a=-\frac{25}{3}, b=5$$

Hence the required equation is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{-\frac{25}{3}}+\frac{y}{5}=1$

$$\Rightarrow -\frac{3x}{25}+\frac{y}{5}=1 \Rightarrow \frac{-3x+5y}{25}=1$$

$$\Rightarrow -3x+5y=25 \Rightarrow 3x-5y+25=0 \ \text{[Ans.]}$$

Illustration-13: Find the equation of a line, which passes through the point (1, -2) and makes the intercepts on the axes equal in magnitude and opposite in sign.

Solution: Let the equation be $\frac{x}{a}+\frac{y}{b}=1$, which passes through (1, -2)

From the condition, $a=-b$

$$\therefore \frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{1}{-b}+\frac{-2}{b}=1$$

$$\Rightarrow -\frac{1}{b}-\frac{2}{b}=1 \Rightarrow \frac{-1-2}{b}=1$$

$$\Rightarrow -3=b \Rightarrow b=-3$$

$$\therefore a=-b=-(-3)=3$$

Required equation is: $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{3}+\frac{y}{-3}=1 \Rightarrow \frac{x}{3}-\frac{y}{3}=1 \Rightarrow \frac{x-y}{3}=1 \Rightarrow x-y=3$.

Illustration-14: Find the equation of the line passing through the points (a, b) and $(a+b, a-b)$.

Solution: Let $A \equiv (a,b)$, $B \equiv (a+b, a-b)$.

Required equation of the line AB is

$$\frac{x-x_1}{x_1-x_2}=\left(\frac{y-y_1}{y_1-y_2}\right)$$

$$\Rightarrow \frac{x-a}{a-a-b}=\frac{y-b}{b-a+b}$$

$$\Rightarrow \frac{x-a}{-b}=\frac{y-b}{2b-a} \Rightarrow (2b-a)(x-a)=(y-b)(-b)$$

$$\Rightarrow 2bx-2ab-ax+a^2=-by+b^2$$

$$\Rightarrow -ax+2bx+by-b^2-2ab+a^2=0$$

$$\Rightarrow ax-2bx-by+b^2+2ab-a^2=0$$

$$\Rightarrow (a-2b)x-by+b^2+2ab-a^2=0$$ [Ans.]

Illustration-15: Show that the line joining (2, 1) and (3, 4) is perpendicular to the line joining (7, 5) and (4, 6).

Solution: Let $A \equiv (2,1),\ B \equiv (3,4),\ C \equiv (7,5),\ D \equiv (4,6)$

Here slope of the line $AB(m_1) = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{4-1}{3-2} = \dfrac{3}{1} = 3$

Slope of the line $CD(m_2) = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{6-5}{4-7} = \dfrac{1}{-3} = -\dfrac{1}{3}$

Product of two slopes $= m_1 \times m_2 = 3 \times \left(-\dfrac{1}{3} \right) = -1$

Since the product of two slopes is equal to –1, given first line is perpendicular to the second line. [Proved]

Illustration-16: Find the equation of a line which is parallel to $2x - y - 9 = 0$ and which passes through the intersection of $5x + y + 4 = 0$ and $2x + 3y - 1 = 0$.

Solution: Given $5x + y + 4 = 0$ ....(1)

$2x + 3y - 1 = 0$ ....(ii)

Solving (1) and (2), we get

From (1) $\Rightarrow 5x + y + 4 = 0 \Rightarrow y = -5x - 4$ .... (3)

From (2) $\Rightarrow 2x + 3y - 1 = 0 \Rightarrow 2x + 3(-5x - 4) - 1 = 0$

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

Putting the value of x in (3), we get from (3)

$y = -5x - 4 \Rightarrow y = -5(-1) - 4 \Rightarrow y = 5 - 4 \Rightarrow y = 1$

$\therefore$ Point of intersection is $(-1,1)$

The equation of a line parallel to $2x - y - 9 = 0$ is $2x - y + k = 0$, which passes through $(-1,1)$

$\therefore 2x - y + k = 0$

$\Rightarrow 2(-1) - 1 + k = 0 \Rightarrow -2 - 1 + k = 0 \Rightarrow k = 3$

Required equation of attitude is: $2x - y + k = 0$

$\Rightarrow 2x - y + 3 = 0$

$\Rightarrow y = 2x + 3$ [Ans.]

Illustration-17: Find the equation of that line which passes through the point of intersection of $5x + y + 4 = 0$ and $2x + 3y - 1 = 0$ and is perpendicular to $2x - y = 9$.

Solution: Given $5x + y + 4 = 0$ ....(1)

$2x + 3y - 1 = 0$ ....(2)

Solving (1) and (2), we get

From (1) $\Rightarrow 5x + y + 4 = 0 \Rightarrow y = -5x - 4$ ....(3)

From (2) $\Rightarrow 2x + 3y - 1 = 0 \Rightarrow 2x + 3(-5x - 4) - 1 = 0$

$\Rightarrow 2x - 15x - 12 - 1 = 0 \Rightarrow -13x = 13 \Rightarrow x = -1$

Putting the value of x in (3), we get from (3)

$y = -5x - 4 \Rightarrow y = -5(-1) - 4 \Rightarrow y = 5 - 4 \Rightarrow y = 1$

$\therefore$ Point of intersection is (-1, 1)

The equation of a line perpendicular to $2x - y = 9$ is $x + 2y + k = 0$, which passes through $(-1,1)$.

$\therefore x + 2y + k = 0$

$\Rightarrow -1 + 2(1) + k = 0$

$\Rightarrow -1 + 2 + k = 0 \Rightarrow k = -1$

Required equation of attitude is: $x + 2y + k = 0 \Rightarrow x + 2y - 1 = 0 \Rightarrow x + 2y = 1$ [Ans.]

Illustration-18: Find the length of the perpendicular and the coordinates of the foot of the perpendicular from the point (3, 4) to the line $8x + 15y + 1 = 0$.

Solution: First Part: The equation of a line perpendicular to $8x + 15y + 1 = 0$ is $15x - 8y + k = 0$, which passes through (3, 4).

$\therefore 15x - 8y + k = 0 \quad \Rightarrow 15(3) - 8(4) + k = 0$

$\Rightarrow 45 - 32 + k = 0 \Rightarrow k - 13$

$\therefore$ Equation of perpendicular line is: $15x - 8y + k = 0 \Rightarrow 15x - 8y - 13 = 0$ ......(1)

Given $8x + 15y + 1 = 0$ .... (2)

Solving (1), and (2), we get

(1) – (2) gives: $(15x - 8y - 13) - (8x + 15y + 1) = 0 - 0$

$\Rightarrow 15x - 8y - 13 - 8x - 15y - 1 = 0$

$\Rightarrow 7x - 23y - 14 = 0$

$\Rightarrow 7x = 23y + 14 \qquad \therefore x = \frac{23y + 14}{7}$ .... (3)

Now from (1), we have $15x-8y-13=0$

$$\Rightarrow 15\left(\frac{23y+14}{7}\right)-8y-13=0 \Rightarrow \frac{345y+210-56y-91}{7}=0$$

$$\Rightarrow 289y+119=0 \Rightarrow 289y=-119 \Rightarrow y=-\frac{119}{289} \Rightarrow y=-\frac{7}{17}$$

Putting the value of y in (3), we get

$$x=\frac{23y+14}{7} \Rightarrow x=\frac{23\left(-\frac{7}{17}\right)+14}{7} \Rightarrow x=\frac{\frac{77}{17}}{7}$$

$$\Rightarrow x=\frac{77}{119} \Rightarrow x=\frac{11}{17}$$

Required foot is $\left(\frac{11}{17},-\frac{7}{17}\right)$ [Ans.]

Second Part: Let the foot, $A \equiv \left(\frac{11}{17},-\frac{7}{17}\right)$, Given point, $B \equiv (3,4)$.

Required length of perpendicular line $AB=\sqrt{\left(\frac{11}{17}-3\right)^2+\left(-\frac{7}{17}-4\right)^2}$

$$=\sqrt{\left(\frac{40}{17}\right)^2+\left(-\frac{75}{17}\right)^2}=\sqrt{\frac{1600}{289}+\frac{5625}{289}}$$

$$=\sqrt{\frac{7225}{289}}=\sqrt{25}=5 \text{ units}$$ [Ans.]

Illustration-19: Find the equation of a straight line passing through the point of intersection of the lines $x-2y+3=0$, $2x-3y+4=0$ and parallel to the line joining the points (1, 1) and (0, -1).

Solution: Given $x-2y+3=0$ ....(1)

$2x-3y+4=0$ ....(2)

Solving equations (1) and (2), we get

(1) – (2) gives: $(x-2y+3)-(2x-3y+4)=0-0$

$$\Rightarrow x-2y+3-2x+3y-4=0$$

$$\Rightarrow -x+y-1=0 \Rightarrow x=y-1 \text{ ....(3)}$$

Putting $x=y-1$ in (1), we get

$$x-2y+3=0 \Rightarrow y-1-2y+3=0 \Rightarrow -y+2=0 \Rightarrow y=2$$

Putting the value of y in (3), we get

$$x=y-1 \Rightarrow x=2-1 \Rightarrow x=1$$

$\therefore$ Point of intersection is (1, 2)

The equation of a line joining (1, 1) and (0, -1) is:

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}$$

$$\Rightarrow \frac{x-1}{1-0}=\frac{y-1}{1+1} \Rightarrow \frac{x-1}{1}=\frac{y-1}{2}$$

$$\Rightarrow 2x-2=y-1 \Rightarrow 2x-y-1=0 \Rightarrow 2x-2=y-1.$$

The equation of a line parallel to $2x-y-1=0$ is $2x-y+k=0$, which passes through (1, 2).

$$\therefore 2x-y+k=0$$

$$\Rightarrow 2(1)-2+k=0$$

$$\Rightarrow 0+k=0 \Rightarrow k=0.$$

Required equation is $2x-y+k=0 \Rightarrow 2x-y=0 \Rightarrow y=2x$ (Ans.)

$$\frac{Rise}{Run}=\frac{y_1-y_2}{x_1-x_2}$$

$$=\frac{300-500}{20-200}$$

$$=\frac{-200}{-180}=\frac{10}{9} \quad [Ans.]$$

Illustration-20: A line passes through the point of intersection of the lines $x+2y-1=0$ and $2x+3y-4=0$ and makes equal intercepts on the coordinate axis, show that its equation is $x+y=3$.

Solution: Given $x+2y-1=0$ .... (1)

$2x+3y-4=0$ .... (2)

Solving equations (1) and (2), we get

(1) – (2) gives: $(x+2y-1)-(2x+3y-4)=0-0$

$$\Rightarrow x+2y-1-2x-3y+4=0$$

$$\Rightarrow -x-y+3=0 \Rightarrow x=-y+3 \text{ .... (3)}$$

Putting $x = -y + 3$ in (1), we get $x + 2y - 1 = 0$

$$\Rightarrow -y + 3 + 2y - 1 = 0 \Rightarrow y + 2 = 0 \Rightarrow y = -2$$

Putting the value of y in (3), we get $x = -y + 3$

$$\Rightarrow x = -(-2) + 3 \Rightarrow x = 5$$

$\therefore$ Point of intersection is $(5,-2)$

Let the equation be $\frac{x}{a} + \frac{y}{b} = 1$, which passes through $(5,-2)$

From the condition, two intercepts are equal. $\therefore a = b$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{5}{a} + \frac{-2}{a} = 1 \; [\because a = b]$$

$$\Rightarrow \frac{5-2}{a} = 1 \Rightarrow 5 - 2 = a \Rightarrow a = 3$$

$$\therefore a = b = 3$$

Required equation is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \frac{x}{3} + \frac{y}{3} = 1 \Rightarrow \frac{x+y}{3} = 1 \Rightarrow x + y = 3 \text{ [Proved]}$$

Illustration-21: The line containing points (-8, 3) and (2, 1) is parallel to the line containing the points (11, -1) and (k, 0), show that $k = 6$.

Solution: Slope of the line containing the points $(-8,3)$ and $(2,1)$ is:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{2-(-8)} = \frac{-2}{10} = -\frac{1}{5}.$$

Again the slope of the line containing the points $(11,-1)$ and $(k,0)$ is:

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-1)}{k-11} = \frac{1}{k-11}.$$

Since the above two lines are parallel to each other, we get

$$\therefore -\frac{1}{5} = \frac{1}{k-11} \Rightarrow k - 11 = -5 \Rightarrow k = -5 + 11 = 6.$$

$$\therefore k = 6 \text{ [Proved]}$$

BUSINESS APPLICATION

Problem-01: If total factory cost y of making x units of a product is $y = 3x + 20$ and if 50 units are produced.

a. What is the variable cost?
b. What is the total cost?
c. What is the variable cost per unit?
d. What is the average cost per unit?
e. What is the marginal cost of the 50[th] unit?

Solution: Given cost function: $y = 3x + 20$

We know, total cost = variable cost + fixed cost

Here variable cost = 3x and fixed cost = 20

a. For 50 units, variable cost $= 3 \times 50 = Tk.150$
b. Total cost = 150+20 = Tk 170
c. Variable cost per unit = Tk. 3.
d. Total cost = Tk. 170. Therefore, average cost $= \frac{170}{50} = Tk.3.40$
e. Marginal cost of 50[th] unit = Tk. 3. [Ans.]

Problem-02: A firm invests Tk. 10,000 in a business, which has a net return of Tk. 500 per year. An investment of Tk. 20,000 would yield an income of Tk. 2000 per year. What is the linear relationship between investment and annual income? what would be the annual return on an investment of Tk. 12,000?

Solution: Let $x =$ investment , $y =$ return (profit)

For $x_1 = 10000,\ y_1 = 500;\ 1st\ point\ is\ (10000,500)$

Fox $x_2 = 20000,\ y_2 = 2000;\ 2nd\ point\ is\ (20000,2000)$

First Part:

Required linear relationship between investment (x) and annual income (y) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

$$\Rightarrow \frac{x - 10000}{10000 - 20000} = \frac{y - 500}{500 - 2000} \qquad \Rightarrow \frac{x - 10000}{-10000} = \frac{y - 500}{-1500}$$

$\Rightarrow \frac{x-10000}{20} = \frac{y-500}{3}$ $\Rightarrow 3x - 30000 = 20y - 10000$

$\Rightarrow 3x - 20000 = 20y$

$\Rightarrow 20y = 3x - 20000$ [Ans.]

Second Part:

When investment $(x) = 12000$, then the required annual return will be

$20y = 3x - 20000$ $\Rightarrow 20y = 3(12000) - 20000$

$\Rightarrow 20y = 36000 - 20000$ $\Rightarrow 20y = 16000 \Rightarrow y = 800$ [Ans.]

Problem-03: An investment of Tk. 90000 in a certain business yields an income of Tk. 8000. An investment of Tk. 50000 yields an income of Tk. 5000. If the income is a linear function of investment, determine the equation for this relation. what is the slope? Interpret the slope in terms of the money involved.

Solution: Let $x = investment,\ \ y = income$

For $x_1 = 90000,\ y_1 = 8000$; 1[st] Point is $(90000, 8000)$

For $x_2 = 50000,\ y_2 = 5000$; 2[nd] Point is $(50000, 5000)$

First Part:

Required linear relationship between investment (x) and income (y) is

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$$

$\Rightarrow \frac{x-90000}{90000-50000} = \frac{y-8000}{8000-5000}$ $\Rightarrow \frac{x-90000}{40000} = \frac{y-8000}{3000}$

$\Rightarrow \frac{x-90000}{40} = \frac{y-8000}{3}$ $\Rightarrow 3x - 27000 = 40y - 320000$

$\Rightarrow 3x + 50000 = 40y$ $\Rightarrow 40y = 3x + 50000$ [Ans.]

Second Part:

ii) Slope (m) $= \frac{Rise}{Run} = \frac{y_1-y_2}{x_1-x_2} = \frac{8000-5000}{90000-50000} = \frac{3000}{40000} = \frac{3}{40}$ [Ans.]

Interpretations of Slope:

Here the slope $\left(m = \frac{3}{40}\right)$ indicates that Tk. $\frac{3}{40}$ is added to the total cost (y) for every additional one unit produced

Problem-04: An investment of Tk. 100 in a certain business yields an income of Tk. 20. An investment of Tk. 1000 yields an income of Tk. 90. If the income is a linear function of investment, find the equation for this relation. what is the slope? interpret the slope in terms of the money involve.

Solution: Let $x = investment,\ y = income$

For $x_1 = 100, y_1 = 20$; 1st point is $(100,20)$

For $x_2 = 1000, y_2 = 90$; 2nd Point is $(1000,90)$

First Part:

Required linear relationship between investment (x) and income (y) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \Rightarrow \frac{x - 100}{100 - 1000} = \frac{y - 20}{20 - 90}$$

$$\Rightarrow \frac{x - 100}{-900} = \frac{y - 20}{-70} \Rightarrow \frac{x - 100}{90} = \frac{y - 20}{70}$$

$$\Rightarrow 7x - 700 = 90y - 1800 \Rightarrow 7x - 90y + 1100 = 0 \text{ [Ans.]}$$

Second Part:

ii) Slope $(m) = \frac{Rise}{Run} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{20 - 90}{100 - 1000} = \frac{-70}{-900} = \frac{7}{90}$ [Ans.]

Interpretations of Slope:

Here the slope $\left(m = \frac{7}{90}\right)$ indicates that Tk. $\frac{7}{90}$ is added to the total cost (y) for every additional one unit produced.

Problem-05: M/s. R. K. Industry spends Tk. 4000 to process 100 orders and Tk. 6000 to process 200 orders. Find the linear relation between The processing money and the number of orders. Find the money spent for 300 orders?

Solution: Let x = number of orders, y = expense

For $x_1 = 100,\ y_1 = 4000$; 1st point is (100, 4000)

For $x_2 = 200,\ y_2 = 6000$; 2nd point is (200, 6000)

First part:

Required linear relationship between number of orders (x) and expense (y) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \Rightarrow \frac{x - 100}{100 - 200} = \frac{y - 4000}{4000 - 6000} \Rightarrow \frac{x - 100}{-100} = \frac{y - 4000}{-2000}$$

$$\Rightarrow \frac{x - 100}{1} = \frac{y - 4000}{20} \Rightarrow 20x - 2000 = y - 4000 \Rightarrow 20x - y + 2000 = 0 \text{ [Ans.]}$$

Second Part:

When number orders (x) = 300, then the required expense will be

$20x - y + 2000 = 0 \quad \Rightarrow 20(300) - y + 2000 = 0 \quad \Rightarrow 6000 - y + 2000 = 0$

$\Rightarrow 8000 - y = 0 \Rightarrow y = 8000$ [*Ans.*]

Problem-06: A factory produces 200 bulbs for a total cost of Tk. 800 and 400 bulbs for a total cost of Tk. 1200. Given that the cost curve is a straight line, find the equation of the straight line and use it to find the cost of producing 300 bulbs.

Solution: Let x = number of bulbs, y = cost

For $x_1 = 200$, $y_1 = 800$; 1st point is (200, 800)

For $x_2 = 400$, $y_2 = 1200$, 2nd point is (400, 1200)

First Part:

Required linear relationship between number of bulbs (x) and cost (y) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \Rightarrow \frac{x - 200}{200 - 400} = \frac{y - 800}{800 - 1200} \Rightarrow \frac{x - 200}{-200} = \frac{y - 800}{-400}$$

$$\Rightarrow \frac{x - 200}{1} = \frac{y - 800}{2} \Rightarrow 2x - 400 = y - 800 \Rightarrow 2x - y + 400 = 0 \quad [Ans.]$$

Second Part:

When number of bulbs (x) = 300, then the required expense will be

$2x - y + 400 = 0 \Rightarrow 2(300) - y + 400 = 0 \Rightarrow 600 - y + 400 = 0 \Rightarrow 1000 - y = 0$

$\Rightarrow y = 1000$ [*Ans.*]

Problem-07: For sending non-wagon of wheat, Food Corporation of Bangladesh spends Tk. 300 for a distance of 20 kilometers and Tk. 500 for a distance of 200 kilometers. What is the linear relation between the amount spent and number of kilometers covered? What are the slope and intercepts of the line? Also find the cost of sending through 400 kilometers.

Solution: Let x = distance, y = expense

For x1 = 20, y1 = 300; 1st point is (20, 300)

For x2 = 200, y2 = 500; 2nd point is (200, 500)

First Part:

Required linear relationship between distance (x) and expense (y) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \Rightarrow \frac{x - 20}{20 - 200} = \frac{y - 300}{300 - 500} \Rightarrow \frac{x - 20}{-180} = \frac{y - 300}{-200} \Rightarrow \frac{x - 20}{9} = \frac{y - 300}{10}$$

$$\Rightarrow 10x - 200 = 9y - 2700 \Rightarrow 10x - 9y + 2500 = 0 \quad [Ans.]$$

Second Part: $Slope(m) = \frac{Rise}{Run} = \frac{y_1 - y_2}{x_1 - x_2}$

$$= \frac{300-500}{20-200} = \frac{-200}{-180} = \frac{10}{9} \quad [Ans.]$$

Third Part:

We know, the intercept formula is $\frac{x}{a} + \frac{y}{b} = 1$

We have $10x - 9y + 2500 = 0$

$\Rightarrow 10x - 9y = -2500$

$\Rightarrow \frac{10x - 9y}{-2500} = \frac{-2500}{-2500}$ [*Dividing by − 2500 on both sides*]

$\Rightarrow \frac{10x}{-2500} - \frac{9y}{-2500} = 1 \Rightarrow \frac{10x}{-2500} + \frac{9y}{2500} = 1$

$\Rightarrow \frac{x}{\frac{-2500}{10}} + \frac{y}{\frac{2500}{9}} = 1 \Rightarrow \frac{x}{-250} + \frac{y}{\frac{2500}{9}} = 1$

Here intercept on x-axis (a) $= -250$,, intercept on y-axis (b) $= \frac{2500}{9}$

Problem-08: The salary of an employee in 1995 was Tk. 1,200. In 1997, it will be Tk. 1,350. Express salary as a liner function of time and estimate his salary in 1998

Solution: Let (*s*, *t*) represent salary in TK. and time in years respectively. The equation to a straight line passing through (s_1, t_1) and (s_2, t_2) is given by:

$t - t_1 = \frac{t_2 - t_1}{s_2 - s_1}(s - s_1)$........................(i)

If 1995 is considered as the base year, then two points are:

$(s_1, t_1) = (1200, 1)$ and $(s_2, t_2) = (1350, 3)$

Thus equation (i) becomes:

$$t - 1 = \frac{3-1}{1350-1200}(s - 1200) = \frac{2}{150}(s - 1200)$$

or, $150t - 150 = 2s - 2400$ or, $s = 75t + 1125$, the required function.

Hence, the estimated salary in the year 1998 corresponds to $t = 4$ is given by:

$s = 75 \times 4 + 1125 = Tk.1,435$

Problem-09: A firm invested Tk. 10 million in a new factory that has a net return of Tk. 5,00,000 per year. An investment of Tk. 20 million would yield net income of Tk. 2 million per year. What is the linear relationship between investment and annual income? What would be the annual return on an investment of Tk. 15 million?

Solution: Let x and y represent investment and annual income, respectively.
The linear relationship between investment and income would be the equation of the straight line joining the points $[(10{,}000{,}000),(5{,}00{,}000)]\, and\, [(20{,}000{,}000),(2{,}000{,}000)]$:

$$\frac{x-10{,}000{,}000}{10{,}000{,}000-20{,}000{,}000}=\frac{y-5{,}00{,}000}{5{,}00{,}000-2{,}000{,}000}$$

$$\Rightarrow \frac{x-10{,}000{,}000}{-10{,}000{,}000}=\frac{y-5{,}00{,}000}{-1500{,}000} \Rightarrow \frac{x-10{,}000{,}000}{y-5{,}00{,}000}=\frac{10{,}000{,}000}{1500{,}000}$$

$$\Rightarrow \frac{x-10{,}000{,}000}{y-5{,}00{,}000}=\frac{100}{15}=\frac{20}{3}$$

$$\Rightarrow 20y-10{,}000{,}000=3x-30{,}000{,}000$$

$$\Rightarrow 20y=3x-30{,}000{,}000+10{,}000{,}000$$

$$\Rightarrow 20y=3x-20{,}000{,}000 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$\Rightarrow y=\frac{3}{20}x-1{,}000{,}000$$

The annual return y can be found by putting the value of investment x =15,000,000 in (i):

$$y=\frac{3}{20}\times 15{,}000{,}000-1{,}000{,}000=Tk.(2{,}250{,}000-1{,}000{,}000)=Tk.1{,}250{,}000$$

Problem-10: The total cost y, for x units of a certain product consists of fixed cost and the variable cost (proportional to the number of units produced). It is known that the total cost is Tk. 6000 for 500 units and Tk. 9000 for 1000 units.

(i) Find the linear relationship between x and y,
(ii) Find the slope of the line, what does it indicate?
(iii) Find the number of units that must be produced so that
(a) There is neither profit nor loss.
(b) There is a profit of Tk. 1000.
(c) There is a loss of Tk. 300. It being given that the selling price is Tk. 8 per unit.

Solution: Let x = unit, y = cost
For $x_1 = 500$, $y_1 = 6000$; Ist point is (500, 6000)
For $x_2 = 1000$, $y_2 = 9000$; 2nd point is (1000, 9000)

(i) Required linear relationship between x and y is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}$$

$$\Rightarrow \frac{x-500}{500-1000}=\frac{y-6000}{6000-9000} \qquad \Rightarrow \frac{x-500}{-500}=\frac{y-6000}{-300}$$

$$\Rightarrow \frac{x-500}{1}=\frac{y-6000}{6} \qquad \Rightarrow 6x-3000=y-6000 \Rightarrow 6x+3000=y$$

$\therefore \quad y=6x+3000 \quad [Ans.]$

(ii) Slope (m) = $\frac{Rise}{Run}=\frac{y_1-y_2}{x_1-x_2}=\frac{6000-9000}{500-1000}=\frac{-3000}{-500}=6$

Interpretations of Slope: Here the slope (m=6) indicates that Tk. 6 is added to the total cost (y) for every additional one unit produced.

(iii) (a) For neither profit nor loss (breakeven point),

Revenue = Cost

$\Rightarrow$ Price$(p)\times Quantity(x)=Cost(y) \qquad \Rightarrow p\times x=y$

$\Rightarrow 8\times x=6x+3000 \qquad \Rightarrow 8x=6x+3000$

$\Rightarrow 8x-6x=3000$

$\Rightarrow 2x=3000 \Rightarrow x=1500\, units\ (Ans.)$

(b) For a profit of Tk. 1000,

Profit = Revenue - cost

$\Rightarrow 1000=8x-(6x+3000) \Rightarrow 1000=8x-6x-3000 \Rightarrow 1000=2x-3000$

$\Rightarrow 2x=4000 \Rightarrow x=2000\, units\ [Ans.]$

(c) For a loss of Tk. 300,

Loss = Cost- Revenue

$\Rightarrow 300=6x+3000-8x \qquad \Rightarrow 1000=3000-2x$

$\Rightarrow 2x=2000 \Rightarrow x=1000\ units \quad [Ans.]$

Problem-11: The total cost y, for x units of a certain product consists of fixed costs and the variable cost (proportional to the number of units produced). It is known that the total cost is Tk. 1200 for 100 units and Tk. 2700 for 400 units.

i) Find the linear relationship between x and y.

ii) Find the slope of the line and what does it indicate

iii) If the selling Price is Tk. 7 per unit, find the number of units that must be produced so that (a) there is neither profit nor loss, (b) there is a profit of Tk. 300, (c) there is a loss of Tk. 300.

Solution: Let $x =$ no. of units , $y =$ cost

For $x_1 = 100,\ y_1 = 1200;$ 1st point is $(100, 1200)$

For $x_2 = 400,\ y_2 = 2700;$ 2nd point is $(400, 2700)$

i) Required linear relationship between x and y is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

$$\Rightarrow \frac{x - 100}{100 - 400} = \frac{y - 1200}{1200 - 2700} \qquad \Rightarrow \frac{x - 100}{-300} = \frac{y - 1200}{-1500}$$

$$\Rightarrow \frac{x - 100}{1} = \frac{y - 1200}{5} \qquad \Rightarrow 5x - 500 = y - 1200$$

$$\Rightarrow 5x + 700 = y \qquad \Rightarrow y = 5x + 700 \text{ [Ans.]}$$

ii) Slope (m)= $\frac{Rise}{Run} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1200 - 2700}{100 - 400} = \frac{-1500}{-300} = 5$ [Ans.]

Interpretations of Slope: Here the slope $(m = 5)$ indicates that Tk. 5 is added to the total cost (y) for every additional one unit produced.

iii) a. For neither profit nor loss (breakeven point),

Revenue = Cost

$\Rightarrow Price(p) \times Quantity(x) = Cost(y)$

$\Rightarrow p \times x = y \qquad \Rightarrow 7 \times x = 5x + 700$

$\Rightarrow 7x = 5x + 700 \qquad \Rightarrow 7x - 5x = 700$

$\Rightarrow 2x = 700 \qquad \Rightarrow x = 350$ units [Ans.]

b. For a profit of Tk. 300,

Profit =Revenue-cost

$\Rightarrow 300 = 7x - (5x + 700)$

$\Rightarrow 300 = 7x - 5x - 700$

$\Rightarrow 300 = 2x - 700$

$\Rightarrow 2x = 1000 \Rightarrow x = 500$ units [Ans.]

c. For a loss of Tk. 300,

Loss = cost-Revenue

$\Rightarrow 300 = 5x + 700 - 7x$

$\Rightarrow 300 = 700 - 2x$

$\Rightarrow 2x = 400 \Rightarrow x = 200$ units [Ans.]

BRIEF REVIEW

Definition

Straight line: A straight line may be defined as the shortest distance between two distinct points.

In other words, the first degree equation in x and y of the form $ax+by+c=0$ is called the general equation of the straight line, where a, b, c are constants and x, y are variables.

Slope: The Gradient (also called Slope) of a straight line shows how steep a straight line is.

$$\text{i.e. Gradient} = \frac{Change\ in\ Y}{Change\ in\ X} = \frac{y_2 - y_1}{x_2 - x_1}$$

In other words, the slope of a straight line is denoted by m and is defined by

$$m = \tan\theta$$

Where θ is the angle between the positives direction of $x-axis$ and the given line.

y- intercept of a straight line: The y-intercept of a straight line is simply where the line crosses the Y axis.

x- intercept of a straight line: The x-intercept of a straight line is simply where the line crosses the y-axis.

Equation of x-axis: The equation of x-axis is : $y=0$

Equation of y-axis: The equation of y-axis is : $x=0$

Equation of a straight line parallel to x-axis: The equation of a straight line parallel to x-axis is : $y=k$.

Equation of a straight line parallel to y-axis: The equation of a straight line parallel to y-axis is : $x=k$.

Perpendicular lines: Two lines are said to be perpendicular if the multiplication of their slopes is equal to -1. i.e. if $m_1 \times m_2 = -1$.

Parallel lines: Two lines are said to be parallel if they have equal slopes, i.e. if $m_1 = m_2$.

Area of a triangle: Area of the triangle = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$; whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Quiz Questions

Multiple Choice Questions

1. The fundamental idea of the coordinate geometry is-
   (a) point (b) line (c) plane (d) sphere
2. The distance of a point from y-axis measured along x-axis is called-
   (a) Ordinate (b) abscissa (c) y-coordinate (d) Co-ordinate
3. How many quadrants of a two dimensional rectangular co-ordinate system have?
   (a) One (b) Two (c) Three (d) Four
4. The point of intersection of three medians of a triangle is called-------of the triangle?
   (a) Mid-point (b) Solution (c) Centroid (d) Vertex
5. What is the distance of the two points (3,2) and (5,4)?
   (a) 8 (b) 2 (c) $2\sqrt{2}$ (d) $\sqrt{2}$
6. What is the centroid of the triangle whose vertices are $(3,5),(-7,9)\ and\ (1,-3)$?
   (a) $\left(-1,\frac{11}{3}\right)$. (b) $\left(-\frac{10}{3},\frac{11}{3}\right)$ (c) $\left(\frac{2}{3},6\right)$ (d) $\left(-2,\frac{10}{3}\right)$
7. If the area of a triangle become zero, then the vertices of the triangle is called.
   (a) Centroid (b) Median (c) Collinear (d) Line
8. What is the equation of y-axis?
   (a) x = a (b) y = o (c) x = o (d) y = b
9. Which is not the equation of straight line?
   (a) $x+y=2$ (b) $2x-y=-y+2x+3$
   (c) $x-y+3=2x$ (d) $2x-2y=y+3$
10. What is the slope of the straight line $2y=10x+5$?
    (a) 5 (b) -5 (c) 10 (d) -10
11. Which line is parallel to the line $2x-3y+2=0$?
    (a) $3x+2y+k=0$ (b) $2x-3y+k=0$
    (c) $2x+3y=0$ (d) $3x-2y+k=0$
12. When two straight lines with slopes m_1 and m_2 are perpendicular, which one is correct?
    (a) $m_1=m_2$ (b) $m_1=-m_2$ (c) $m_1 m_2=1$ (d) $m_1 m_2=-1$
13. What is the equation of x-axis?
    (a) x = a (b) y = o (c) x = o (d) y = b
14. Which line is perpendicular to the line $2x-3y+2=0$?
    (a) $3x+2y+k=0$ (b) $2x-3y+k=0$
    (c) $2x+3y=0$ (d) $3x-2y+k=0$

Which one of the following statement is true/false?

1. Every equation is an identity
2. Every equation is not identity
3. $5x-2=7$ is an equation of a straight line
4. The highest power of a variable in a linear equation is one
5. The graph of a quadratic equation is always a straight line
6. The general equation of a straight line is $ax+by+c=0$
7. The slope of the equation $ax+by+c=0$ is $m=-b/a$
8. The graph of a linear equation is always a straight line
9. The general form of a linear equation is $ax+b=0$
10. The slope of x- axis is zero (0)
11. The slope of y -axis is zero
12. If the slope of a straight line is zero, then it is horizontal
13. The segment $A(2,4)$ and $B(2,8)$ is horizontal

Brief Questions

1. What is the slope or gradient of a straight line?
2. What is the mathematical formula of slope of a straight line passing through two points (x_1, y_1) and (x_2, y_2)?
3. What is the equation of a straight line which passes through the points (x_1, y_1) and (x_2, y_2)?
4. What is the equation of a straight line parallel to $x-$axis?
5. What is the equation of straight line parallel to y axis?
6. What is the equation of a straight line having intercept "a" on the x axis and intercept "b" on the y axis?
7. What is the equation of a straight line having a slop m and passing through the points (x_1, y_1)?
8. Write down formula of angle between two lines?
9. When two lines are said to be perpendicular considering their slopes?
10. When two lines are said to be parallel considering their slopes?
11. What is the equation of $x-$axis?
12. What is the equation of y axis?
13. What is the slope of the straight line $3y-5x=12$?
14. What is the y-intercept of the straight line $3y-5x=12$?
15. What is the x-intercept of the straight line $3y-5x=12$?

Conceptual, Analytical & Numerical Questions

1. What is slope of a straight line? Give Geometrical interpretation.
2. What do you mean by a straight line?
3. Discuss the relationships between Cartesian ad polar coordinate system.
4. Discuss the condition of the three straight lines.
5. Determine the point of intersection between any two straight lines.
6. Determine the condition for the concurrent of three straight lines.
7. Establish the formula of a straight line passing through two points (x_1, y_1) and (x_2, y_2)
8. Determine the angle between two straight lines.

Numerical Questions

1. (a) Find the equation of a straight line parallel to x-axis and passing through the point (6,4)
   (b) Find the equation of a straight line perpendicular to x-axis and passing through the point (7,4).
2. (a) Find the equation of a straight line making an angle of 30^0 with the x-axis and whose intercepts on the y-axis is –2
   (b) Find the equation to the straight line passing through the point (3, 2) and cutting off intercepts equal but opposite in sign from the two axis
   (c) Find the equation to the straight line whose intercepts on the axis of x and y respectively are (i) 3 and –2 (ii) –2 and –3.
3. Find the equation of the straight line passing through the point (4, 3) such that the sum of its intercepts on the axis is 14.
4. Find the equation of the straight line through the point (4, -5) and is (a) parallel as well as (b) perpendicular to the line 3x +4y=0.
5. As the number of units manufactured increases from 5000 to 7000, the total cost of production increases from Tk. 26,000 to Tk. 34,000. Find the relationship between the cost (y) and the number of units made (x) if the relationship is linear.
6. As the number of units manufactured increase from 4,000 to 6,000 the total cost of production increases from Tk. 22,000 to Tk. 30,000. Assuming a linear relationship between the cost y and the number of units made x, find y as a function of x. what will be the cost when 4,500 units are produced?

7. A firm produces 20 units of an item for Tk. 73 and 50 units of Tk. 97. Assuming the cost function to be linear, find the equation of this line and use it to estimate the cost of producing 40 units.
8. The total expenses of a mess y are partly constant and partly proportional to the number of the inmates of the mess x. The total expenses are Tk. 1040 when there are 12 members in the mess, and Tk. 1600 for 20 members.
   (i) Find the linear relationship between y and x
   (ii) Find the constant expenses and the variable expenses per member.
   (iii) What would be the total expenditure if the mess has 15 members?
9. If total factory cost y of making x units of a product is y=10x+500 and if 1,000 units are produced: i. What is the variable cost? ii. What is the total cost? iii. What is the variable cost per unit? iv. What is the marginal cost of the last unit produced?

ANSWER

1. (a). y = 4, (b). x = 7 2. (a). $y=\frac{x}{\sqrt{3}}-2$, (b). $x-y=1$, c. i) $2x-3y=6$, ii) $3x+2y=6$ 3. $4x+3y=24, x+y=7$ 5. a. $3x+4y+8=0$, b. $4x-3y-31=0$
5. $y=4x+6{,}000$. 6. Tk. 24,000. 7. $y=0.80x+57$, $y=89$
8.i) $y=70x+200$, ii) 200,70 iii) 1250 9. (a). 10,000, (b). 10,500 (c). 10, (d). 10.