CHAPTER-08	BINOMIAL THEOREM

LEARNING OBJECTIVES

After studying this chapter, the student will be able to understand:

- Binomial theorem
- Binomial expansion
- General term of binomial expansion
- Middle term of a binomial expansion
- Expansion of exponential expression

8.1 INTRODUCTION

A binomial expression in mathematics is one which has two terms, e,g (x + y), (5x + 3y), (p + q) etc. In Business mathematics and statistics, there are various problems based on Binomial theorem. The general type of binomial expression is $(x + y)^n$.

From fundamental algebra, we know

$$(x+y)^2=(x+y)(x+y)=x^2+2xy+y^2$$

$$(x+y)^3=(x+y)(x^2+2xy+y^2)=x^3+3x^2y+3xy^2+y^3$$

These above type of example are quite simple but if the expansion is to a higher order the problem becomes quite complicated. It is here that the rule of expansion stated as the binomial theorem is very useful. The inventor of this theorem was Issac Newton in 1976.

8.2 BINOMIAL THEOREM

Statement: If $(x + a)$ is a binomial expression, for 'n' a positive integer the expansion of $(x + a)^n$ is given by

$$(x+a)^n=x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+{}^nC_3x^{n-3}a^3+.....+{}^nC_rx^{n-r}a^r+....+a^n.$$

Proof: The theorem can be proved by the method of induction.

By actual multiplication, we have

$$(x+a)^2=x^2+2xa+a^2$$

$$={}^2C_0x^2+{}^2C_1xa+{}^2C_2a^2$$

Thus the theorem is true when n has the value 2.

We will prove this theorem by the principle of Mathematical induction.

First we assume that the theorem is true for $n = m$

So $(x+a)^m={}^mC_0x^m+{}^mC_1x^{m-1}a+{}^mC_2x^{m-2}a^2+.....+{}^mC_rx^{m-r}a^r+.....+{}^mC_ma^m$

Multiplying both rides by $(x+a)$ we have

$$(x+a)^m(x+a)=\left[x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + ... + {}^mC_3x^{m-r}a^r + ... + {}^mC_ma^m\right](x+a)$$

$$= x\left[x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + ... + {}^mC_rx^{m-r}a^r + ... + {}^mC_ma^m\right] + a$$

$$\left[x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + ... + {}^mC_rx^{m-r}a^r + ... + {}^mC_ma^m\right]$$

$$= x^{m+1} + {}^mC_1x^ma + {}^mC_2x^{m-1}a^2 + ... + {}^mC_rx^{m-r+1}a^r + ... + {}^mC_mxa^m + x^ma + {}^mc_1x^{m-1}a^2$$

$${}^mC_2x^{m-2}a^3 + ... + {}^mC_rx^{m-r}a^{r+1} + ... + {}^mC_ma^{m+1}$$

$$\therefore (x+a)^{m+1} = x^{m+1} + \left(1 + {}^mC_1\right)x^ma + \left({}^mC_1 + {}^mC_2\right)x^{m-1}a^2 + ... + \left({}^mC_{r-1} + {}^mC_r\right)x^{m-r+1}a^r + ... + {}^mC_ma^m$$

Now

$${}^mC_{r-1} + {}^mC_r = \frac{m!}{(r-1)!(m-r+1)!} + \frac{m!}{r!(m-r)!}$$

$$= \frac{m!(r+m-r+1)}{r!(m-r+1)!} = \frac{m!(m+1)}{r!(m-r+1)!} = \frac{(m+1)!}{r!(m+1-r)!}$$

$$= {}^{m+1}C_r$$

And ${}^mC_m = 1 = {}^{m+1}C_{m+1}$

$$\therefore (x+a)^{m+1} = {}^{m+1}C_0x^{m+1} + {}^{m+1}C_1x^ma + {}^{m+1}C_2a^{m-1}a^2 + ... + {}^{m+1}C_{m+1}a^{m+1}$$

Therefore the theorem is true next higher value $(m+1)$ of n.

Then by principal of mathematical induction the theorem is true for all positive integral values of n.

8.3 CHARACTERISTICS OF BINOMIAL EXPANSION

We observe the following features in the binomial expansion.

1. In all there are $(n+1)$ terms in the expansion.
2. The coefficients of the terms are: ${}^nC_0, {}^nC_1, {}^nC_2, \cdots {}^nC_n$ respectively.
3. In each term of the expansion the sum of the powers of x and a is n.
4. The first term is ${}^nC_0x^n = x^n$ and the last term is ${}^nC_na^n = a^n$.
5. The power of x in various terms decreases by 1 and the power of a increases by 1.
6. The coefficients of the first and the last terms are 1 each and the coefficients of terms at equal distance from the middle term are equal. The coefficient of binomial expansion can also be obtained from the following Pascal's triangle.

Pascal's Triangle for Binominal Coefficients

Power	Coefficients
1	1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1

8.4 DETERMINATION OF GENERAL TERM

The (r+1) th term is called general term in the expansion of $(a+x)^n$. It is denoted by the symbol t_{r+1} or T_{r+1}

1^{st} term $= t_1 = {}^nC_0 a^{n-0} x^0$

2^{nd} term $= t_2 = {}^nC_1 a^{n-1} x^1$

3^{rd} term $= t_3 = {}^nC_2 a^{n-2} x^2$

4^{th} term $= t_4 = {}^nC_3 a^{n-3} x^3$

....

r-th term $= t_r = {}^nC_{r-1} a^{n-(r-1)} x^{r-1}$

$(r+1)-th\ \ term = t_{r+1} = {}^nC_r a^{n-r} x^r$

Therefore, the general term, $T_{r+1} = {}^nC_r a^{n-r} x^r$

Remarks: (i) In the expansion of $(a+x)^n$ the (r+1)–th term = ${}^nC_r a^{n-r} x^r$

(ii) In the expansion of $(a-x)^n$ the (r+1)–th term = ${}^nC_r a^{n-r} (-1)^r x^r$

8.5 MIDDLE TERM OF A BINOMIAL EXPANSION

We will determine the middle term of the Binomial expansion of $(a+x)^n$ when n∈ N.

We have

$$(a+x)^2 = a^2 + {}^2C_1 ax + x^2$$

$$(a+x)^3 = a^3 + {}^3C_1 a^2 x + {}^3C_2 ax^2 + x^3$$

$$(a+x)^4 = a^4 + {}^4C_1 a^3 x + {}^4C_2 a^2 x^2 + {}^4C_3 ax^3 + x^4$$

$$(a+x)^5 = a^5 + {}^5C_1 a^4 x + {}^5C_2 a^3 x^2 + {}^5C_3 a^2 x^3 + {}^5C_3 ax^4 + x^5$$

We observe that when n = 2 and n = 4 then the expansion have only one middle term and these middle terms are $\left(\frac{2}{2}+1\right)$ or, 2nd and $\left(\frac{4}{2}+1\right)$ or 3rd respectively

Also when n = 3 the and n = 5 then the expansion have two middle terms.

For n = 3 the middle terms are $\left(\frac{3-1}{2}\right)+1$ or, 2nd and $\left(\frac{3+1}{2}+1\right)$ or, 3rd term

And also for n = 5 the middle terms are $\left(\frac{5-1}{2}+1\right)$ or, 3rd and $\left(\frac{5+1}{2}+1\right)$ or, 4th term.

From the above discussion we can conclude that (i) if n is even then in the expansion of $(a+x)^n$ will have only one middle term and the middle term is $\left(\frac{n}{2}+1\right)th$ term.

(ii) Again if n is odd then in the expansion of $(a+x)^n$ will have two middle term and these middle terms are

$\left(\frac{n-1}{2}+1\right)$-th term and $\left(\frac{n+1}{2}+1\right)$- th term respectively

ILLUSTRATIONS

Illustration-01: (a) $(a + b)^7$=?

$$(a+b)^7 = a^7 + {}^7C_1a^{7-1}b + {}^7C_2a^{7-2}b^2 + {}^7C_3a^{7-3}b^3 + {}^7C_4a^{7-4}b^4 + {}^7C_5a^{7-5}b^5 + {}^7C_6a^{7-6}b^6 + b^7$$

$$= a^7 + \frac{7!}{1!(7-1)!}a^6b + \frac{7!}{2!(7-2)!}a^5b^2 + \frac{7!}{3!(7-3)!}a^4b^3 + \frac{7!}{4!(7-4)!}a^3b^4 + \frac{7!}{5!(7-5)!}a^2b^5 + \frac{7!}{6!(7-6)!}ab^6 + b^7$$

$$= a^7 + \frac{7.6!}{6!}a^6b + \frac{7.6.5!}{2!\ 5!}a^5b^2 + \frac{7.6.5.4!}{3!.4!}a^4b^3 + \frac{7.6.5.4.3!}{.4!.3!}a^3b^4 + \frac{7.6.5!}{5!.2!}a^2b^5 + \frac{7.6!}{6!}ab^6 + b^7$$

$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

(b) $(a-b)^7 = ?$

$$(a-b)^7 = a^7 - {}^7C_1 a^{7-1}b + {}^7C_2 a^{7-2}b^2 - {}^7C_3 a^{7-3}b^3 + {}^7C_4 a^{7-4}b^4 - {}^7C_5 a^{7-5}b^5 + {}^7C_6 a^{7-6}b^6 - b^7$$

$$= a^7 - \frac{7!}{1(7-1)!}a^6 b + \frac{7!}{2!(7-2)!}a^5 b^2 - \frac{7!}{3!(7-3)!}a^4 b^3 + \frac{7!}{4!(7-4)!}a^3 b^4 - \frac{7!}{5!(7-5)!}a^2 b^5 + \frac{7!}{6!(7-6)!}ab^6 - b^7$$

$$= a^7 - \frac{7.6!}{1!6!}a^6 b + \frac{7.6.5!}{2!.5!}a^5 b^2 - \frac{7.6.5.4!}{3!.4!}a^4 b^3 + \frac{7.6.5.4!}{4!.3!}a^3 b^4 - \frac{7.6.5!}{5!.2!}a^2 b^5 + \frac{7.6!}{6!.1!}ab^6 - b^7$$

$$= a^7 - 7a^6 b + 21a^5 b^2 - 35a^4 b^3 + 35a^3 b^4 - 21a^2 b^5 + 7ab^6 + b^7$$

$$\therefore (a-b)^7 = a^7 - 7a^6 b + 21a^5 b^2 - 35a^4 b^3 + 35a^3 b^4 - 21a^2 b^5 + 7ab^6 - b^7.$$

(c) Write down the expansion of $(x+2y)^5$

$$(x+2y)^5 = x^5 + {}^5C_1 x^4(2y) + {}^5C_2 x^3(2y)^2 + {}^5C_3 x^2(2y)^3 + {}^5C_4 x(2y)^4 + (2y)^5$$

$$= x^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + (2y)^5$$

$$= x^5 + 10x^4 y + 40x^3 y^2 + 80x^2 y^3 + 80xy^4 + 32y^5$$

(d) Hence find the expansions of $(x-2y)^5$

$$(x-2y)^5 = [x + (-2y)]^5$$

$$= x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + (-2y)^5$$

$$= x^5 - 10x^4 y + 40x^3 y^2 - 80x^2 y^3 + 80xy^4 - 32y^5$$

(e) Using bionomical formula expand $(2x+y)^6$

$$(2x+y)^6 = (2x)^6 + {}^6C_1(2x)^{6-1}y + {}^6C_2(2x)^{6-2}y^2 + {}^6C_3(2x)^{6-3}y^3 + {}^6C_4(2x)^{6-4}y^4 + {}^6C_5(2x)^{6-5}y^5 + y^6$$

$$= 64x^6 + \frac{6!}{1!(6-1)!}(2x)^5 y + \frac{6!}{2!(6-2)!}(2x)^4 y^2 + \frac{6!}{3!(6-3)!}(2x)^3 y^3 + \frac{6!}{4!(6-4)!}(2x)^2 y^4 + \frac{6!}{5!(6-5)!}(2x)y^5 + y^6$$

$$= 64x^6 + \frac{6.5!}{5!}32x^5 y + \frac{6.5.4!}{2!.4!}16x^4 y^2 + \frac{6.5.4.3!}{3!.3!}8x^3 y^3 + \frac{6.5.4!}{4!.2!}4x^2 y^4 + \frac{6.5!}{5!.1!}2xy^5 + y^6$$

$$= 64x^6 + 192x^5 y + 15.16x^4 y^2 + 20.8x^3 y^3 + 15.4x^2 y^4 + 6.2xy^5 + y^6$$

$$= 64x^6 + 192x^5 y + 240x^4 y^2 + 160x^3 y^3 + 60x^2 y^4 + 12xy^5 + y^6$$

$$\therefore (2x+y)^6 = 64x^6 + 192x^5 y + 240x^4 y^2 + 160x^3 y^3 + 60x^2 y^4 + 12xy^5 + y^6 \quad (Ans).$$

(f) Using bionomical formula expand (2x-y)^6

$$(2x-y)^6=(2x)^6-{}^6C_1(2x)^{6-1}y+{}^6C_2(2x)^{6-2}y^2-{}^6C_3(2x)^{6-3}y^3+{}^6C_4(2x)^{6-4}y^4-{}^6C_5(2x)^{6-5}y^5+y^6$$

$$=64x^6-\frac{6!}{1!(6-1)!}(2x)^5y+\frac{6!}{2!(6-2)!}(2x)^4y^2-\frac{6!}{3!(6-3)!}(2x)^3y^3+\frac{6!}{4!(6-4)!}(2x)^2y^4-\frac{6!}{5!(6-5)!}(2x)y^5+y^6$$

$$=64x^6-\frac{6.5!}{5!}32x^5y+\frac{6.5.4!}{2!.4!}16x^4y^2-\frac{6.5.4.3!}{3!.3!}8x^3y^3+\frac{6.5.4!}{4!.2!}4x^2y^4-\frac{6.5!}{5!.1!}2xy^5+y^6$$

$$=64x^6-6.32x^5y+15.16x^4y^2-20.8x^3y^3+15.4x^2y^4-6.2xy^5+y^6$$

$$=64x^6-192x^5y+240x^4y^2-160x^3y^3+60x^2y^4-12xy^5+y^6$$

$$\therefore(2x-y)^6=64x^6-192x^5y+240x^4y^2-160x^3y^3+60x^2y^4-12xy^5+y^6.\ (Ans)$$

Note: The expansion of the binomial $(x-a)^n$ is given below:

$$(x-a)^n=x^n+{}^nC_1(-a)x^{n-1}+{}^nC_2(-a)^2x^{n-2}+{}^nC_3(-a)^3x^{n-3}+..........+{}^nC_n(-a)^n$$

$$=x^n-{}^nC_1ax^{n-1}+{}^nC_2a^2x^{n-2}-{}^nC_3a^3x^{n-3}+..........+(-1)^n\ {}^nC_na^n$$

Illustration-02: If the first three terms are b, $\frac{21}{2}bx$ and $\frac{189}{4}bx^2$ respectively in the expansion of $(a+3x)^n$, then find the value of a. b and n.

Solution: By the Binomial Theorem

$$(a+3x)^n=a^n+{}^nC_1a^{n-1}(3x)+{}^nC_2a^{n-2}(3x)^2+{}^nC_3a^{n-3}(3x)^3+.........$$

$$=a^n+3nxa^{n-1}+\frac{9}{2}n(n-1)x^2a^{n-2}+\frac{27n(n-1)(n-2)}{6}x^3a^{n-3}+............$$

$$\therefore a^n=b................(i)$$

$$\therefore 3nxa^{n-1}=\frac{21}{2}bx$$

$$or,\ 3na^{n-1}=\frac{21}{2}b..................(ii)$$

$$\therefore\frac{9}{2}n(n-1)x^2a^{n-2}=\frac{189}{4}bx^2$$

$$or,\frac{9}{2}n(n-1)a^{n-2}=\frac{189}{4}b.........................(iii)$$

Now Dividing (ii) by (i) we get

$$\frac{3na^{n-1}}{a^n}=\frac{21b}{2b}$$

$$\Rightarrow \frac{3n}{a}=\frac{21}{2}$$

$$\Rightarrow \frac{n}{a}=\frac{7}{2} \text{ ....}(iv)$$

Again dividing (iii) by (ii) we get

$$\frac{9(n-1)n\ a^{n-2}}{2.3n\ a^{n-1}}=\frac{189b}{4}\times\frac{2}{21b}$$

$$\Rightarrow \frac{3(n-1)}{2a}=\frac{9}{2}$$

$$or,\quad n-1=\frac{9\times2\times a}{3\times2}$$

$$or,\quad n-1=3a$$

$$or,\quad \frac{7a}{2}-1=3a \quad \text{[by using equation (iv)]}$$

$$or,\quad \frac{7a}{2}-3a=1 \quad or,\quad \frac{a}{2}=1$$

$$\therefore a=2$$

We have $n-1=3a$

$$or,\ n-1=3.2\ \ or,\ n=7\ \therefore\ n=7$$

From equation (i) , We get

$$b=a^n\ or,\ b=2^7\ \therefore b=2^7$$

$\therefore$ Required values are : $a = 1, b = 2^7, n=7$

Illustration-03: Expand $\left(\frac{x}{3}+\frac{2}{a}\right)^3$ by the binomial theorem.

Solution: $\left(\frac{x}{3}+\frac{2}{a}\right)^3=\left(\frac{x}{3}\right)^3+{}^3C_1\left(\frac{x}{3}\right)^{3-1}\left(\frac{2}{a}\right)^1+{}^3C_2\left(\frac{x}{3}\right)^{3-2}\left(\frac{2}{a}\right)^2+{}^3C_3\left(\frac{x}{3}\right)^{3-3}\left(\frac{2}{a}\right)^3$

$$=\frac{x^3}{27}+3.\left(\frac{x}{3}\right)^2.\frac{2}{a}+3\left(\frac{x}{3}\right)^1.\left(\frac{2}{a}\right)^2+1.\left(\frac{x}{3}\right)^0\left(\frac{2}{a}\right)^3$$

$$=\frac{x^3}{27}+\frac{2x^2}{3a}+\frac{4x}{a^2}+\frac{8}{a^3}$$

Illustration-04: Find the third term of $(2a+b)^7$

Solution: We know, (r+1)th term, $T_{r+1} = {}^nC_r\ a^{n-r}\ x^r$

Hence, the 3rd term of $(2a+b)^7$ *is*, $T_{2+1} = {}^7C_2(2a)^{7-2}b^2$

$$=\frac{7(7-1)}{2!}.2^5.a^5b^2=\frac{7.6.2^5}{2}a^5b^2=7.6.16a^5b^2=672a^5b^2.\ \textit{Ans.}$$

Illustration-05: Find the coefficient of x^7 in the expansion of $\left(2x^2-\frac{1}{4x}\right)^{11}$.

Solution: Let the term x^7 exists in (r+1)-th term in the expansion of $\left(2x^2-\frac{1}{4x}\right)^{11}$.

$$\therefore (r+1)-th\ term = {}^{11}C_r\left(2x^2\right)^{11-r}\left(-\frac{1}{4x}\right)^r$$

$$={}^{11}C_r\ 2^{11-r}\left(-\frac{1}{4}\right)^r\left(x^2\right)^{11-r}\left(x^{-1}\right)^r={}^{11}C_r\ 2^{11-r}\left(-\frac{1}{4}\right)^r x^{22-2r-r}$$

$$={}^{11}C_r\ 2^{11-r}\left(-\frac{1}{4}\right)^r x^{22-3r}$$

Since the term x^7 exists here

So 22-3r = 7

Or, -3r = 7-22

Or, 3r = 15

$\therefore$ r = 5

Therefore in (r+1)-th term or 6-th term x^7 exists.

And required coefficient $={}^{11}C_5 2^{11-5}\left(-\frac{1}{4}\right)^5$

$$=-\frac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1}\times\frac{2^6}{4^5}=-\frac{231}{8}.$$

Illustration-06: Find the term in x^4 and the term independent of x in the expansion of

$$\left(x+\frac{1}{x}\right)^{20}.$$

Solution: The general term of the expansion is the (r+1)-th term $= {}^{20}C_r a^{n-r} x^r$

$$= {}^{20}C_r (x)^{20-r}\left(\frac{1}{x}\right)^r = {}^{20}C_r x^{20-r}(x)^{-r} = {}^{20}C_r x^{20-2r}$$

To find the term in x^4:

$$x^{20-2r} = x^4$$

$$or,\ 20-2r=4$$

$$ro,\ r=8$$

$$\therefore\ r=8$$

$\therefore$ the term in x^4 in the $(8+1)^{th}$ term = 9^{th} term

The 9^{th} term $= {}^{20}C_8 x^4$

$$=125970x^4$$

Let the term in x^0 be the $(r+1)^{th}$ term.

The $(r+1)^{th}$ term is ${}^{20}C_r x^{20-2r}$

x^0 occurs when 20-2r = 0

$\therefore$ r = 10

Hence the term independent of x is the $(10+1)^{th}$ term = 11^{th} term

$$11^{th}\text{ term} = {}^{20}C_{10} x^{10}\left(\frac{1}{x}\right)^{10}$$

$$= {}^{20}C_{10} x^{10}.x^{-10} = {}^{20}C_{10} = \frac{20!}{10!(20-10)!} = \frac{20!}{10!\,10!} = 184756$$

Illustration-07: Find the 7^{th} term in the expansion of $\left(1-\frac{1}{x}\right)^{10}$

Solution: Here $\left(1-\frac{1}{x}\right)^{10} = \left\{1+\left(-\frac{1}{x}\right)\right\}^{10}$

Therefore 7 th term = ${}^{10}C_6\left(-\frac{1}{x}\right)^6 = \frac{10!}{6!\,(10-6)!}\left(\frac{1}{x}\right)^6 = \frac{10.9.8.7.6!}{6!\,4!}.\frac{1}{x^6} = \frac{10.9.8.7}{4.3.2.1}.\frac{1}{x^6} = \frac{210}{x^6}.$

Illustration-08: If the coefficient of x^5 and x^{15} in the expansion of $\left(2x^2+\frac{k}{x^3}\right)^{10}$ are equal, find the value of k.

Solution: The (r+1)th term in the expansion of $\left(2x^2+\frac{k}{x^3}\right)^{10}$

$$= {}^{10}C_r\left(2x^2\right)^{10-r}\left(\frac{k}{x^3}\right)^r = {}^{10}C_r 2^{10-r}k^r x^{20-5r}$$

If x^5 exists in this term,

then 20-5r = 5

$\therefore r = 3$

Again if x^{15} exists in (r+1)th term,

then 20-5r = 15

$\therefore r = 1$

Since the coefficient of x^5 *and* x^{15} are equal

$$\therefore {}^{10}C_3 2^{10-3}k^3 = {}^{10}C_1 2^{10-1}k$$

$$or,\ 120\times2^7\times k^2 = 10\times2^9$$

$$or,\ k^2 = \frac{10\times2^9}{120\times2^7}$$

$$or,\ k^2 = \frac{2^2}{12}$$

$$or,\ k^2 = \frac{1}{3}$$

$$\therefore k = \frac{1}{\sqrt{3}}$$

The required value of k is $\frac{1}{\sqrt{3}}$.

Illustration-09: Find the middle term in the expansion of $\left(\frac{a}{x}-bx\right)^{12}$

Solution: The total number of terms in the expansion is 12+1 = 13

Since the number is odd, there is only one middle term

i,e the 7th term, $t_7 = {}^{12}C_6\left(\frac{a}{x}\right)^{12-6}(-bx)^6$

$$= {}^{12}C_6\left(\frac{a}{x}\right)^6(bx)^6 = {}^{12}C_6\left(\frac{a^6}{x^6}\right).b^6x^6 = {}^{12}C_6\ a^6b^6$$

Illustration-10: Find the middle terms in the expansion of $\left(3x-\frac{2x^2}{3}\right)^7$

Solution: The total number of terms in the expansion are 7+1= 8 (even) and so there will be two middle terms. i.e., $\left(\frac{n-1}{2}+1\right) or, \left(\frac{7-1}{2}+1\right) or,$ 4th term

and $\left(\frac{n+1}{2}+1\right) or, \left(\frac{7+1}{2}+1\right) or,$ 5th term

$\therefore$ The 4th term, $t_4 = {}^7C_3(3x)^{7-3}\left(-\frac{2x^2}{3}\right)^3$

$$= {}^7C_3(3x)^4\left(-\frac{2x^2}{3}\right)^3 = \frac{7!}{3!(7-3)!}.3^4.x^4.\left(-\frac{2}{3}\right)^3.x^6 = \frac{7.6.5.4!}{3.2.1.4!}.81.x^4.\left(-\frac{8}{27}\right).x^6 = -840x^{10}$$

And the 5th term, $t_5 = {}^7C_4(3x)^{7-4}\left(-\frac{2x^2}{3}\right)^4$

$$= \frac{7!}{4!(7-4)!}3^3.x^3\left(\frac{2}{3}\right)^4 x^8 = \frac{7.6.5.4!}{4!.3!}.27.\left(\frac{16}{81}\right)x^{11} = \frac{560}{3}x^{11}.$$

Illustration-11: For $n \in N$, show that the middle term in the expansion of

$$\left(x^2+2+\frac{1}{x^2}\right)^n \text{ is } \frac{(2n)!}{(n!)^2}$$

Solution: Here $\left(x^2+2+\frac{1}{x^2}\right)^n = \left\{(x)^2+2.x.\frac{1}{x}+\left(\frac{1}{x}\right)^2\right\}^n = \left\{\left(x+\frac{1}{x}\right)^2\right\}^n = \left(x+\frac{1}{x}\right)^{2n}$

Since the power of the binomial is 2n, even number so there is only one middle term and that will be $(n+1)$-th term $\therefore$ Middle term or, (n+1)th term $= {}^{2n}C_n x^{2n-n}\left(\frac{1}{x}\right)^n$

$$= \frac{(2n)!}{n!(2n-n)!}x^n.\frac{1}{x^n} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \quad \text{(Showed)}$$

Illustration-12: Evaluate $(19)^4$.

Solution: $(19)^4 = (20-1)^4$

$$= {}^4C_0(20)^4 - {}^4C_1(20)^3(1) + {}^4C_2(20)^2(1)^2 - {}^4C_3(20)^1(1)^3 + {}^4C_4(1)^4$$
$$= (1{,}60{,}000) - 4(8{,}000) + 6(400) - 4(20) - 1$$
$$= 16{,}00{,}000 - 32{,}000 + 2{,}400 - 80 - 1$$
$$= 1{,}30{,}321$$

Illustration-13: Obtain 6th term of the expansion of $(2x+y)^9$.

Solution: We have, $T_{r+1} = {}^nC_r x^{n-r} a^r$

Now, In the expansion of $(2x+y)^9$

$$T_{r+1} = {}^9C_r(2x)^{9-r}(y)^r$$

$$\therefore T_{5+1} = {}^9C_5(2x)^{9-5}(y)^5$$ (For the sixth term put $r=5$)

$$\therefore T_6 = 126 \times 16 \times (x)^4(y)^5$$
$$= 2{,}016x^4y^5$$ (Ans.)

Illustration-14: Obtain the fifth term in the expansion of $\left(\frac{5a}{4}+\frac{4}{5a}\right)^{12}$.

Solution: In the expansion of $\left(\frac{5a}{4}+\frac{4}{5a}\right)^{12}$, $T_{r+1} = {}^{12}C_r\left(\frac{5a}{4}\right)^{12-r}\left(\frac{4}{5a}\right)^r$

$$\therefore T_{4+1} = {}^{12}C_4\left(\frac{5a}{4}\right)^{12-4}\left(\frac{4}{5a}\right)^4 = 495\frac{5^8a^84^4}{4^85^4a^4} = \frac{495\times5^4}{4^4}a^4 = \frac{(495)(625)}{256}a^4$$

$$\therefore T_5 = \frac{3{,}09{,}375}{256}a^4$$ (Ans.)

Illustration-15: Obtain the fourth term in the expansion of: $\left(\frac{2x}{3}-\frac{3}{2x}\right)^9$.

Solution: In the expansion of $\left(\frac{2x}{3}-\frac{3}{2x}\right)^9$ we have

$$T_{r+1} = {}^9C_r\left(\frac{2x}{3}\right)^{9-r}\left(\frac{-3}{2x}\right)^r \therefore T_4 = {}^9C_3\left(\frac{2x}{3}\right)^{9-3}\left(\frac{-3}{2x}\right)^3$$ (For the fourth term put $r = 3$)

$$= 84\frac{2^6 \times x^6}{3^6}\frac{(-1)^3 \times 3^3}{2^3 \times x^3} = -\frac{84 \times 2^{6-3}}{3^{6-3}}x^3$$

$$= -\frac{84(8)}{27}x^3 = -\frac{672}{27}x^3$$ (Ans.)

Illustration-16: Obtain the middle term in the expansion of $\left(\frac{a}{x} - \frac{x}{a}\right)^{10}$.

Solution: In the expansion of $\left(\frac{a}{x} - \frac{x}{a}\right)^{10}$ there will be $10 + 1 = 11$ terms in the expansion.

$\therefore \frac{10}{2} + 1 =$ Sixth term will be the middle term

Now, $T_{r+1} = {}^{10}C_r\left(\frac{a}{x}\right)^{10-r}\left(\frac{-x}{a}\right)^r \therefore T_6 = {}^{10}C_5\left(\frac{a}{x}\right)^{10-5}\left(\frac{-x}{a}\right)^5$ (For the sixth term put $r = 5$)

$$= 252\frac{a^5}{x^5}(-1)^5\frac{x^5}{a^5} = -252$$ (Ans.)

Illustration-17: Obtain the middle term in the expansion of $\left(\frac{2x}{3} - \frac{3}{2y}\right)^9$.

Solution: In the expansion of $\left(\frac{2x}{3} - \frac{3}{2y}\right)^9$ there will be $9 + 1 = 10$ terms.

$\therefore$ 5th and 6th term will be the two middle terms

Now, $T_{r+1} = {}^nC_r x^{n-r} a^r$

Now putting $r = 4$ we will get 5[th] term and which is: $T_5 = {}^9C_4\left(\frac{2x}{3}\right)^{9-4}\left(\frac{-3}{2y}\right)^4$

$$= 126\frac{2^5 x^5}{3^5}\frac{(-1)^4 3^4}{2^4 y^4} = \frac{126 \times 2^{5-4}}{3^{5-4}}\frac{x^5}{y^4} = -\frac{126(2)}{3}\frac{x^5}{y^4} = \frac{84x^5}{y^4}$$

Again putting $r = 5$ we will get 6^{th} term and which is: $T_6 = {}^9C_5\left(\frac{2x}{3}\right)^{9-5}\left(\frac{-3}{2y}\right)^5$

$$=126\frac{2^4x^4}{3^4}\frac{(-1)^5 3^5}{2^5 y^5} = \frac{126\times 3^{5-4}}{2^{5-4}}\frac{x^4}{y^5} = -\frac{126(3)}{2}\frac{x^4}{y^5} = \frac{-189x^4}{y^5}$$

Answer: $\frac{84x^5}{y^4}$ and $\frac{-189x^4}{y^5}$ are the middle terms.

Illustration-18: Obtain the coefficient of x^2 in the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^8$

Solution: In the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^8$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^8C_r\left(\frac{x}{2}\right)^{8-r}\left(\frac{2}{x}\right)^r = {}^8C_r\frac{x^{8-r}2^r}{2^{8-r}x^r} = \frac{{}^8C_r}{2^{8-2r}}x^{8-2r}$$

For the coefficient of x^2

$$x^{8-2r} = x^2 \quad \Rightarrow 8-2r = 2 \quad \therefore 6 = 2r \qquad \therefore r = 3$$

$$\therefore T_{3+1} = {}^8C_3\frac{x^2}{2^2} = \frac{56}{4}x^2 = 14x^2$$

Therefore the coefficient of x^2 in the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^8$ is 14. (Ans.)

Illustration-19: Obtain the coefficient of x in the expansion of $\left(2x-\frac{1}{x}\right)^5$.

Solution: In the expansion of $\left(2x-\frac{1}{x}\right)^5$ we have

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$= {}^5C_r (2x)^{5-r}\left(\frac{-1}{x}\right)^r$$

$$= {}^5C_r (2)^{5-r} x^{5-r} (-1)^r \frac{1}{x^r}$$

$$T_{r+1} = \left[{}^5C_r 2^{5-r}(-1)^r\right] x^{5-2r}$$

Now, For the coefficient of x, we must let, $x^{5-2r} = x \quad \therefore 5-2r=1 \quad \therefore 4=2r \quad \therefore r=2$

$$\therefore T_{2+1} = {}^5C_2 2^{5-2}(-1)^2 x = 10(8)(1)x \quad = 80x$$

$\therefore$ Coefficient of $x = 80$ (Ans.)

Illustration-20: Obtain the coefficient of x^{-8} in the expansion of $\left(2x^2-\frac{1}{3x}\right)^{14}$

Solution: In the expansion of $\left(2x^2-\frac{1}{3x}\right)^{14}$ we have

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^{14}C_r \left(2x^2\right)^{14-r}\left(\frac{-1}{3x}\right)^r = {}^{14}C_r 2^{14-r} x^{28-2r} \frac{(-1)^r}{3^r x^r}$$

$$\Rightarrow T_{r+1} = \frac{\left[{}^{14}C_r 2^{14-r}(-1)^r\right]}{3^r} x^{28-2r-r}$$

$$\therefore T_{r+1} = \left[\frac{{}^{14}C_r 2^{14-r}(-1)^r}{3^r}\right] x^{28-3r}$$

Now, For the coefficient of x^{-8}, we choose $x^{28-3r} = x^{-8} \Rightarrow 28-3r=-8 \Rightarrow 36=3r$

$$\therefore r = 12$$

$$\therefore T_{13} = \left[\frac{^{14}C_{12}\, 2^{14-12}(-1)^{12}}{3^{12}}\right]x^{-8}$$

$$= \left(\frac{^{14}C_{12}\, 2^{2}}{3^{12}}\right)x^{-8} = \frac{91\times 4}{3^{12}}\cdot x^{-8} = \frac{364}{3^{12}}x^{-8}$$

Therefore, the coefficient of x^{-8} is $\frac{364}{3^{12}}$.

Illustration-21: Find the constant term in the expansion of $\left(x+\frac{2}{x}\right)^4$

Solution: In the expansion of $\left(x+\frac{2}{x}\right)^4$ we have

$$T_{r+1} = {}^{n}C_r x^{n-r} a^r$$

$$\therefore T_{r+1} = {}^{4}C_r (x)^{4-r}\left(\frac{2}{x}\right)^r = {}^{4}C_r x^{4-r}\frac{2^r}{x^r} = \left({}^{4}C_r 2^r\right)x^{4-2r} \cdots\cdots(i).$$

For the constant term, we have, $x^{4-2r} = x^0 \therefore 4-2r=0 \quad \therefore 4=2r \quad \therefore r=2$

From (i) we have, $T_3 = \left({}^{4}C_2 2^2\right)x^0$

$$= (6\times 4)1 = 24$$

$\therefore$ The constant term = 24

Illustration-22: Find the value of the independent term of x in the expansion of $\left(\frac{4x^2}{3}-\frac{3}{x}\right)^9$

Solution: In the expansion of $\left(\frac{4x^2}{3}-\frac{3}{x}\right)^9$ we have

$$T_{r+1} = {}^{9}C_r\left(\frac{4x^2}{3}\right)^{9-r}\left(\frac{-3}{x}\right)^r$$

$$= {}^{9}C_r \frac{4^{9-r}x^{18-2r}}{3^{9-r}}\frac{(-1)^r 3^r}{x^r}$$

$$\therefore T_{r+1} = \left[\frac{^{9}C_r 4^{9-r}(-1)^r 3^r}{3^{9-r}}\right]x^{18-3r}$$

Now For the constant term $x^{18-3r} = x^0 \quad \therefore 18-3r=0 \therefore 18=3r \quad \therefore r=6$

$$\therefore T_7 = \left[\frac{{}^9C_6 4^{9-6}(-1)^6 3^6}{3^{9-6}}\right]x^0$$

$$\therefore T_7 = {}^9C_6 \times 4^3 \times 3^{6-3} = {}^9C_6 \times 4^3 \times 3^3 = {}^9C_6(64)(27) = (84)(64)(27) = 1{,}45{,}152$$

Therefore the constant term is $= 1{,}45{,}152$ (Ans.)

Illustration-23: Prove that there is no constant term in the expansion of $\left(3x^2 - \frac{2}{x}\right)^7$

Solution: In the expansion of $\left(3x^2 - \frac{2}{x}\right)^7$

$$T_{r+1} = {}^7C_r\left(3x^2\right)^{7-r}\left(\frac{-2}{x}\right)^r = {}^7C_r 3^{7-r} x^{14-2r}\frac{(-1)^r 2^r}{x^r}$$

$$= \left[{}^7C_r 3^{7-r}(-1)^r 2^r\right]x^{14-2r-r}$$

$$= \left[{}^7C_r \cdot 3^{7-r} \cdot (-1)^r \cdot 2^r\right]x^{14-3r}$$

$\therefore$ For the constant terms, we have to let, $x^{14-3r} = x^0$ $\therefore 14 - 3r = 0$ $\therefore 3r = 14$

$$\therefore r = \frac{14}{3} \notin Z^+ or\ N \quad (\because \text{ the value of } r \text{ can never be fraction })$$

Therefore , there is no constant term in the expansion of $\left(3x^2 - \frac{2}{x}\right)^7$

Illustration-24: If the middle term in the expansion of $\left(\frac{k}{2} + 2\right)^8$ is 1,120 then find the value of k.

Solution: In the expansion of $\left(\frac{k}{2} + 2\right)^8$ we have $8 + 1 = 9$ terms. Then the 5th term is the middle term. So, $T_5 = {}^8C_4\left(\frac{k}{2}\right)^{8-4} \times (2)^4 = {}^8C_4\frac{k^4}{2^4}2^4 = {}^8C_4 k^4 = 70k^4$

Now $70k^4 = 1{,}120$ (since given that $\therefore T_5 = 1{,}120$)

$\therefore k^4 = \frac{1{,}120}{70}$ $\therefore k^4 = 16$ $\therefore k^4 = (2)^4$ $\therefore k = 2$ (Ans.)

Illustration-25: If the ratio of coefficient of r th term to $(r+1)$ th term in the expansion of $(1+x)^{20}$ is $1:2$, find the value of r.

Solution: In the expansion of $(1+x)^{20}$

$(r+1)$ th term, $T_{r+1} = {}^{20}C_r(1)^{20-r}x^r = {}^{20}C_r \times x^r$

and r-th term, $T_r = T_{(r-1)+1} = {}^{20}C_{r-1}(1)^{20-(r-1)}x^{r-1} = {}^{20}C_{r-1} \times x^{r-1}$

Now given that, $\dfrac{coefficient\ of\ r-th\ term}{coefficient\ of\ (r+1)-th\ term} = \dfrac{1}{2}$

$$\therefore \frac{{}^{20}C_{r-1}}{{}^{20}C_r} = \frac{1}{2}$$

$$\Rightarrow \frac{\dfrac{20!}{(r-1)!(20-r+1)!}}{\dfrac{20!}{r!\ (20-r)!}} = \frac{1}{2}$$

$$\Rightarrow \frac{20!}{(20-r+1)(20-r)!(r-1)!} \times \frac{(20-r)!r(r-1)!}{20!} = \frac{1}{2}$$

$$\Rightarrow \frac{r}{(20-r+1)} = \frac{1}{2} \quad \Rightarrow \frac{r}{(21-r)} = \frac{1}{2} \qquad \therefore 2r = 21-r \quad \therefore 3r = 21$$

$\therefore r = 7$ (Ans.)

Illustration-26: Prove that the sum of coefficient of x^{32} and x^{-17} in the expansion of $\left(x^4 - \dfrac{1}{x^3}\right)^{15}$ is zero.

Solution: In the expansion of $\left(x^4 - \dfrac{1}{x^3}\right)^{15}$ we have

$$\therefore T_{r+1} = {}^{15}C_r\left(x^4\right)^{15-r}\left(\frac{-1}{x^3}\right)^r$$

$$= {}^{15}C_r x^{60-4r}(-1)^r \frac{1}{x^{3r}} \quad = (-1)^r \times {}^{15}C_r x^{60-7r}$$

Now, for the coefficient of x^{32}, we have, $x^{60-7r} = x^{32}$ $\therefore 60-7r = 32$ $\therefore 7r = 28$ $\therefore r = 4$

$$\therefore T_5 = (-1)^4 \times {}^{15}C_4 x^{32}$$

$\therefore$ Coefficient of x^{32} is ${}^{15}C_4$ (1)

Again, for the coefficient of x^{-17}, $x^{60-7r} = x^{-17}$ $\therefore 60-7r = -17$ $\therefore 77 = 7r$ $\therefore r = 11$

$$\therefore T_{12} = (-1)^{11} \times {}^{15}C_{11}(x)^{-17}$$

$\therefore$ Coefficient of $(x)^{-17}$ is $-({}^{15}C_4)$ (2)

$\therefore$ The sum of both coefficients (using (1) and (2)) $= {}^{15}C_4 - {}^{15}C_{11}$

$= {}^{15}C_4 - {}^{15}C_4 \qquad \left(\because {}^nC_r = {}^nC_{n-r} \therefore {}^{15}C_{11} = {}^{15}C_4\right)$

$= 0$ (Proved)

BRIEF REVIEW

Defintion

Binomial expression: A binomial expression in mathematics is one which has two terms, e.g. (x + y), (5x + 3y), (p + q) etc.

Binominal theorem: If $(x + a)$ is a binomial expression, for 'n' a positive integer the expansion of $(x + a)^n$ is given by

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + ... + {}^nC_rx^{n-r}a^r + ... + a^n.$$

This is called Binomial theorem.

Middle term of Binominal Expansion: If n is even, then in the expansion of $(a+x)^n$ will have only one middle term and which is $\left(\frac{n}{2}+1\right)$-th term.

Again if n is odd, then in the expansion of $(a+x)^n$ will have two middle terms and these terms are : $\left(\frac{n-1}{2}+1\right)$-th term and $\left(\frac{n+1}{2}+1\right)$- th term respectively.

Binominal Coefficient: The Coefficients of various term in the expansion of $(x+a)^n$ are ${}^nC_0, {}^nC_1, {}^nC_2 .... {}^nC_n$. These are called Binomial Coefficients.

General Term of Binominal Expansion: The (r+1) th term is called general term in the expansion of $(a+x)^n$. It is denoted by the symbol t_{r+1} or T_{r+1}

Therefore, the general term, $T_{r+1} = {}^nC_ra^{n-r}x^r$

Quiz Questions

Multiple Choice Questions

1. What is the 3rd term in the expansion of $(2+x^2)^5$?
   (a) ${}^5C_1x^4$ (b) $40x^6$ (c) ${}^5C_3x^3$ (d) 60
2. What is the correct expansion of $(a+2b)^3$?
   (a) $a^3+6a^2b+12ab^2+8b^3$ (b) a^3+8b^3 (c) ${}^3C_1(a^3+8b^3)$ (d) a^3-8b^3
3. What is the middle term in the expansion of $\left(\frac{a}{x}-bx\right)^{11}$?
   (a) ${}^{12}C_5a^4b^4$ (b) ${}^{12}C_6a^6b^6$ (c) $240a^6b^6$ (d) ${}^{12}C_7a^5b^5$
4. What is the middle term of $\left(3x-\frac{2x^2}{3}\right)^7$?
   (a) $-240x^{10}$ (b) $\frac{560}{3}x^3$ (c) $-840x^{10}, \frac{560}{3}x^{11}$ (d) $240x^9$
5. What is the coefficient of x^4 in the expansion of $\left(x^4+\frac{1}{x^3}\right)^{15}$?
   (a) $\frac{15}{8!7!}$ (b) $\frac{15!}{8!}$ (c) $\frac{15!}{8!7!}$ (d) $\frac{15!}{7!}$
6. If the coefficient of x in the expansion of $\left(x^2+\frac{1}{x^3}\right)^5$ is 270 what is the value of K ?
   (a) 7 (b) 4 (c) 5 (d) 3
7. What is the last term of expansion of $\left(3x-\frac{y}{4}\right)^4$?
   (a) $\frac{y^4}{256}$ (b) $\frac{y^2}{128}$ (c) 256 (d) y^4
8. What is the term independent of x in the expansion of $\left(x^2+\frac{1}{x}\right)^6$?
   (a) 160 (b) 360 (c) 240 (d) 84
9. What is the first term in the expansion of $(3-x)^4$?
   (a) ${}^4C_2$ (b) 3^4 (c) ${}^4C_1$ (d) 3^4x^2
10. What is the general term of $(1+x)^n$?
   (a) ${}^nC_rx^r$ (b) ${}^nC_rx^n$ (c) ${}^nC_ra^{n-r}x^r$ (d) ${}^nC_r$

Which one of the following statement is true/false?

a. A binomial expression has three terms.
b. The $(r+1)^{th}$ term is called general term in the expansion of $(a+x)^n$.
c. In the expansion of $(1+x)^n$. The sum of even Coefficient is equal to the sum of the odd Coefficient and which is equal to $2^n - 1$
d. $(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$
e. In the expansion of $(a+x)^n$, total number of terms will be: $n+1$

Brief Questions

1. Who is the inventor of Binomial Theorem?
2. For "n" a positive integer write down the expansion of $(a+x)^n$.
3. Write down the general term in the expansion of $(a+x)^n$.
4. If n is odd , then in the expansion of $(a+x)^n$,write down the middle term(s).
5. What is the last term of expansion of $\left(3x-\frac{y}{4}\right)^4$?
6. What is the middle term of expansion of $\left(3x-\frac{y}{4}\right)^4$?
7. What is the general term of $(1+x)^n$?
8. What is the 1^{st} term in the expansion of $(3-x)^4$?
9. In the expansion of $(a-x)^n$ write down the $(r+1)^{th}$ term.
10. What is the 2^{nd} term in the expansion of $(2+x^2)^7$?
11. If n is even in the expansion of $(a+x)^n$ have only one middle term. Write down the law of middle term in this situation.

Conceptual, Analytical & Numerical Questions

1. Define Binominal Theorem. Discuss the characteristics of Binominal Theorem.
2. How would you determine the middle term of binomial expansion?
3. Define Binominal Co-efficients.
4. How would you determined of general term?
5. Give the statement of Binomial theorem.

Numerical Questions

1. State and proof the Binomial theorem.
2. Using Binomial theorem expand the following :

(i) $\left(x-\frac{1}{x}\right)^6$ (ii) $\left(x^2-y^3\right)^7$ (iii) $\left(1-\frac{x}{2}\right)^6$ (iv) $\left(x^2+x\right)^5$ (v) $\left(x^2+x-1\right)^4$

3. Find the 4th term of $\left(x+\frac{1}{x}\right)^6$
4. Find the 19th term of $\left(2x^{\frac{1}{2}}-y^{\frac{1}{3}}\right)^{20}$
5. Calculate the coefficient of x^{32} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$.
6. From the expansion of $\left(x^2+\frac{3a}{x}\right)^{15}$ find the coefficient of x^{18}.
7. If the coefficient of x^4 and x^5 in the expansion of $(3+px)^{10}$ are equal, find the value of p.
8. Find and simplify the coefficient of x^7 in the expansion of $\left(x^2+\frac{2}{x}\right)^8$.
9. Find and calculate the value term independent of x in the expansion of $\left(x^2-2+\frac{1}{x^2}\right)^6$.
10. Find the value of the independent term of x in the expansion $\left(2x-\frac{1}{4x^2}\right)^{12}$.
11. Find the value of the term independent of y in the expansion of $\left(3y-\frac{2}{y^2}\right)^{15}$
12. Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5......(2n-1)}{n!}2^n x^n$.
13. Find the middle term of the expansion of $\left(x^4-\frac{1}{x^3}\right)^{11}$
14. Show that the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5.......(2x-1)}{n!}(-2)^n$.

ANSWERS

3. ${}^6C_3$ or 20 4. $760xy^6$ 5. 1365 6. $110565a^4$ 7. $\frac{5}{2}$

8. 448 9. 924 10. 495 11. $-1001\times2^5\times3^{11}$ 12. $-462x^9$, $462x^2$