Electric Potential

CHAPTER 29

29–1 Electric Potential

The electric field around a charged rod can be described not only by a (vector) electric field strength **E** but also by a scalar quantity, the *electric potential* V. These quantities are intimately related, and often it is only a matter of convenience which is used in a given problem.

To find the *electric potential difference* between two points A and B in an electric field, we move a test charge q_0 from A to B, always keeping it in equilibrium, and we measure the work W_{AB} that must be done by the agent moving the charge. The electric potential difference * is defined from

$$V_B - V_A = \frac{W_{AB}}{q_0}. \qquad (29\text{–}1)$$

The work W_{AB} may be (*a*) positive, (*b*) negative, or (*c*) zero. In these cases the electric potential at B will be (*a*) higher, (*b*) lower, or (*c*) the same as the electric potential at A.

The mks unit of potential difference that follows from Eq. 29–1 is the joule/coul. This combination occurs so often that a special unit, the *volt*, is used to represent it; that is,

$$1 \text{ volt} = 1 \text{ joule/coul}.$$

* This definition of potential difference, though conceptually sound and suitable for our present purpose, is rarely carried out in practice because of technical difficulties. Equivalent and more technically feasible methods are usually adopted.

Usually point A is chosen to be at a large (strictly an infinite) distance from all charges, and the electric potential V_A at this infinite distance is arbitrarily taken as zero. This allows us to define the *electric potential at a point.* Putting $V_A = 0$ in Eq. 29-1 and dropping the subscripts leads to

$$V = \frac{W}{q_0}, \tag{29-2}$$

where W is the work that an external agent must do to move the test charge q_0 from infinity to the point in question. The student should keep in mind that *potential differences* are of fundamental concern and that Eq. 29-2 depends on the arbitrary assignment of the value zero to the potential V_A at the reference position (infinity); this reference potential could equally well have been chosen as any other value, say -100 volts. Similarly any other agreed-upon point could be chosen as a reference position. In many circuit problems the *earth* is taken as a reference of potential and assigned the value zero.

Bearing in mind the assumptions made about the reference position, we see from Eq. 29-2 that V near an isolated positive charge is positive because positive work must be done by an outside agent to push a (positive) test charge in from infinity. Similarly, the potential near an isolated negative charge is negative because an outside agent must exert a restraining force on (that is, must do negative work on) a (positive) test charge as it comes in from infinity. Electric potential as defined in Eq. 29-2 is a scalar because W and q_0 in that equation are scalars.

Both W_{AB} and $V_B - V_A$ in Eq. 29-1 are independent of the path followed in moving the test charge from point A to point B. If this were not so, point B would not have a unique electric potential (with respect to point A as a defined reference position) and the concept of potential would have limited usefulness.

We can easily prove that potential differences are path-independent for the special case shown in Fig. 29-1. This figure illustrates the case in which the two points A and B are in a field set up by a spherical charge q; the two points are further chosen, for simplicity, to lie along a radial line. Although our

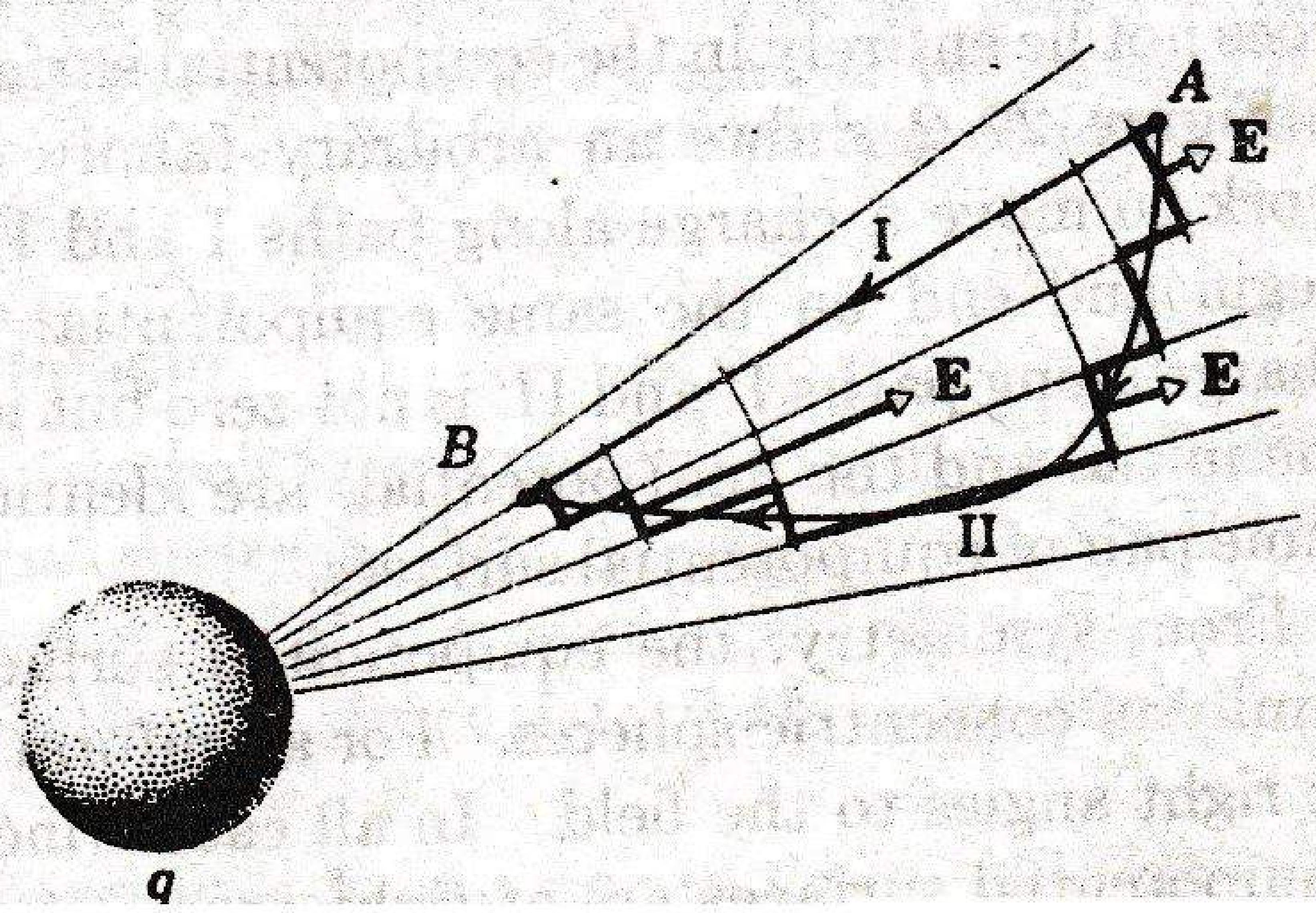

Fig. 29-1 A test charge q_0 is moved from A to B in the field of charge q along either of two paths. The open arrows show **E** at three points on path II.

path-independence proof applies only to this special case, it illustrates the general principles involved.

Point A in Fig. 29–1 may be taken as a defined reference point, and we imagine a positive test charge q_0 moved by an external agent from A to B. We consider two paths, path I being a radial line between A and B and path II being a completely arbitrary path between these two points. The open arrows on path II show the electric force per unit charge that would act at various points on a test charge q_0.

Path II may be approximated by a broken path made up of alternating elements of arc and of radius. Since these elements can be arbitrarily small, the broken path can be made arbitrarily close to the actual path. On path II the external agent does work *only along the radial segments* because along the arcs the force $\mathbf{F}$ and the displacement $d\mathbf{l}$ are at right angles, $\mathbf{F}\cdot d\mathbf{l}$ being zero in such cases. The sum of the work done on the radial segments that make up path II is the same as the work done on path I because each path has the same array of radial segments. Since path II is arbitrary, we have proved that the work done is the same for *all* paths connecting A and B. Although this proof holds only for the special case of Fig. 29–1, the potential difference is path-independent for *any* two points in *any* electrostatic field. We discussed path independence in Section 8–2 for the general class of *conservative forces;* electrostatic forces, like gravitational forces, are conservative.

The locus of points, all of which have the same electric potential, is called an *equipotential surface.* A family of equipotential surfaces, each surface corresponding to a different value of the potential, can be used to give a general description of the electric field in a certain region of space. We have seen earlier (Section 27–3) that electric lines of force can also be used for this purpose; in later sections (see, for example, Fig. 29–15) we explore the intimate connection between these two ways of describing the electric field.

No work is required to move a test charge between any two points on an equipotential surface. This follows from Eq. 29–1,

$$V_B - V_A = \frac{W_{AB}}{q_0},$$

because W_{AB} must be zero if $V_A = V_B$. This is true, because of the path independence of potential difference, even if the path connecting A and B does not lie entirely in the equipotential surface.

Figure 29–2 shows an arbitrary family of equipotential surfaces. The work to move a charge along paths I and II is zero because all these paths begin and end on the same equipotential surface. The work to move a charge along paths I′ and II′ is not zero but is the same for each path because the initial and the final potentials are identical; paths I′ and II′ connect the same pair of equipotential surfaces.

From symmetry, the equipotential surfaces for a spherical charge are a family of concentric spheres. For a uniform field they are a family of planes at right angles to the field. In all cases (including these two examples) the equipotential surfaces are at right angles to the lines of force and thus to $\mathbf{E}$

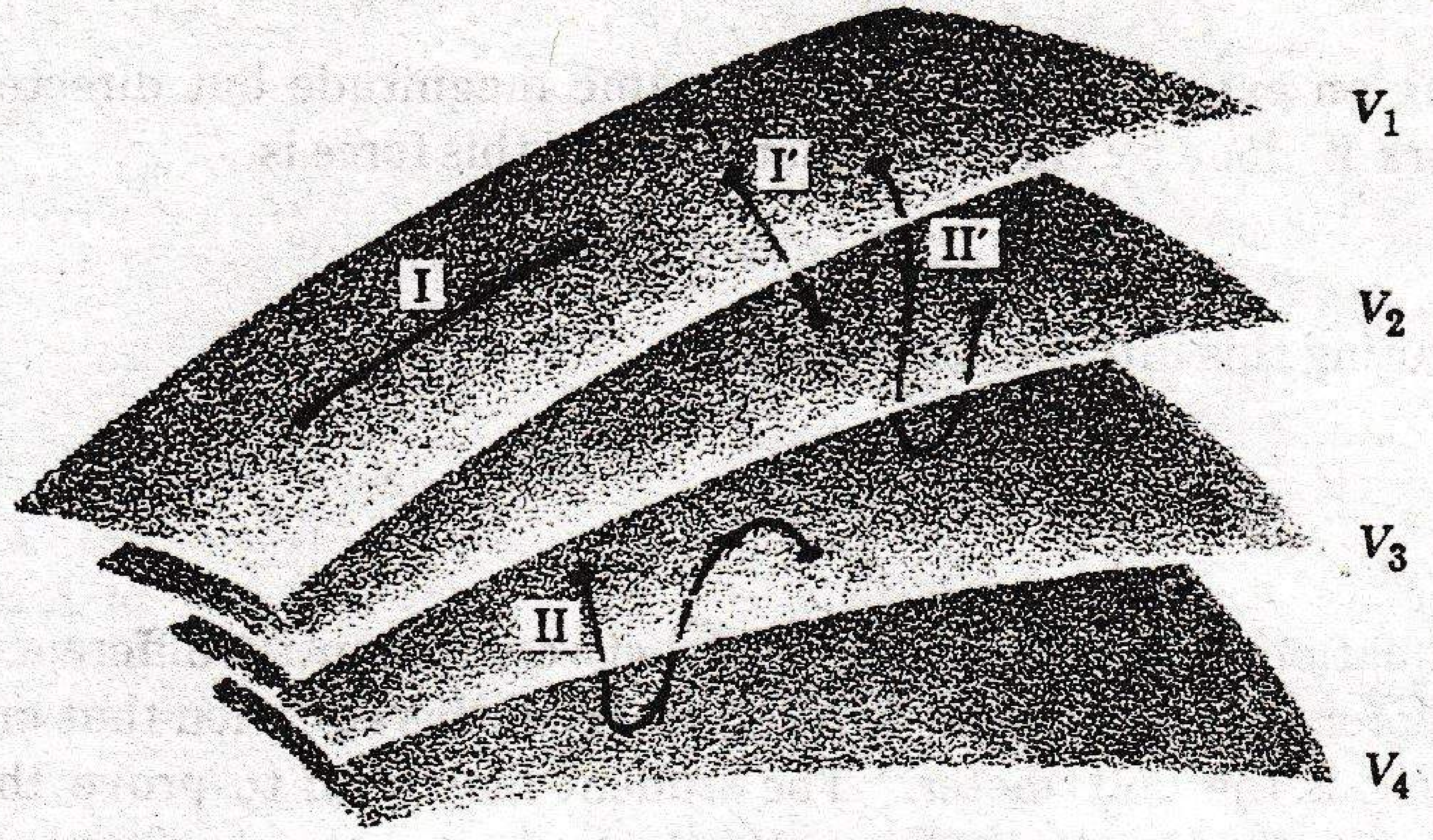

Fig. 29-2 Portions of four equipotential surfaces. The heavy lines show four paths along which a test charge is moved.

(see Fig. 29-15). If **E** were *not* at right angles to the equipotential surface, it would have a component lying in that surface. Then work would have to be done in moving a test charge about on the surface. Work cannot be done if the surface is an equipotential, so **E** must be at right angles to the surface.

There is a strong analogy between electrostatic forces and gravitational forces, based on the fact that their fundamental laws are inverse square laws (see Eqs. 26-3 and 16-1):

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2} \quad \text{and} \quad F_g = G\frac{m_1m_2}{r^2}.$$

Thus we can define the *gravitational potential* V_g (compare Eq. 29-2) from

$$V_g = \frac{W}{m},$$

where W is the work required to move a test body of mass m from infinity to the point in question. Gravitational equipotential surfaces can also be constructed; they prove to be everywhere at right angles to the gravitational field strength vector **g**. For a uniform gravitational field, such as that near the surface of the earth, these surfaces are horizontal planes. This correlates with the facts that (*a*) no net work is required to move a stone of mass m between two points with the same elevation and (*b*) the same net work is required to move a stone along any path starting on a given horizontal surface and ending on another.

29-2 Potential and Field Strength

Let A and B in Fig. 29-3 be two points in a uniform electric field **E**, set up by an arrangement of charges not shown, and let A be a distance d in the field direction from B. Assume that a positive test charge q_0 is moved, by an external agent and without acceleration, from A to B along the straight line connecting them.

The *electric* force on the charge is $q_0\mathbf{E}$ and points down. To move the charge in the way we have described we must counteract this force by applying an external force **F** of the same magnitude but directed upward. The work W done by the agent that supplies this force is

$$W_{AB} = Fd = q_0Ed. \tag{29-3}$$

Substituting this into Eq. 29-1 yields

$$V_B - V_A = \frac{W_{AB}}{q_0} = Ed. \tag{29-4}$$

This equation shows the connection between potential difference and field strength for a simple special case. Note from this equation that another mks unit for **E** is the volt/meter. The student may wish to prove that a volt/meter is identical with a nt/coul; this latter unit is the one first presented for **E** in Section 27-2.

In Fig. 29-3 B has a higher potential than A. This is reasonable because an external agent would have to do positive work to push a positive test charge from A to B. Figure 29-3 could be used as it stands to illustrate the act of lifting a stone from A to B in the uniform gravitational field near the earth's surface.

What is the connection between V and **E** in the more general case in which the field is *not* uniform and in which the test body is moved along a path that is *not* straight, as in Fig. 29-4? The electric field exerts a force $q_0\mathbf{E}$ on the test charge, as shown. To keep the test charge from accelerating, an external agent must apply a force **F** chosen to be exactly equal to $-q_0\mathbf{E}$ for all positions of the test body.

If the external agent causes the test body to move through a displacement $d\mathbf{l}$ along the path from A to B, the element of work done by the external agent

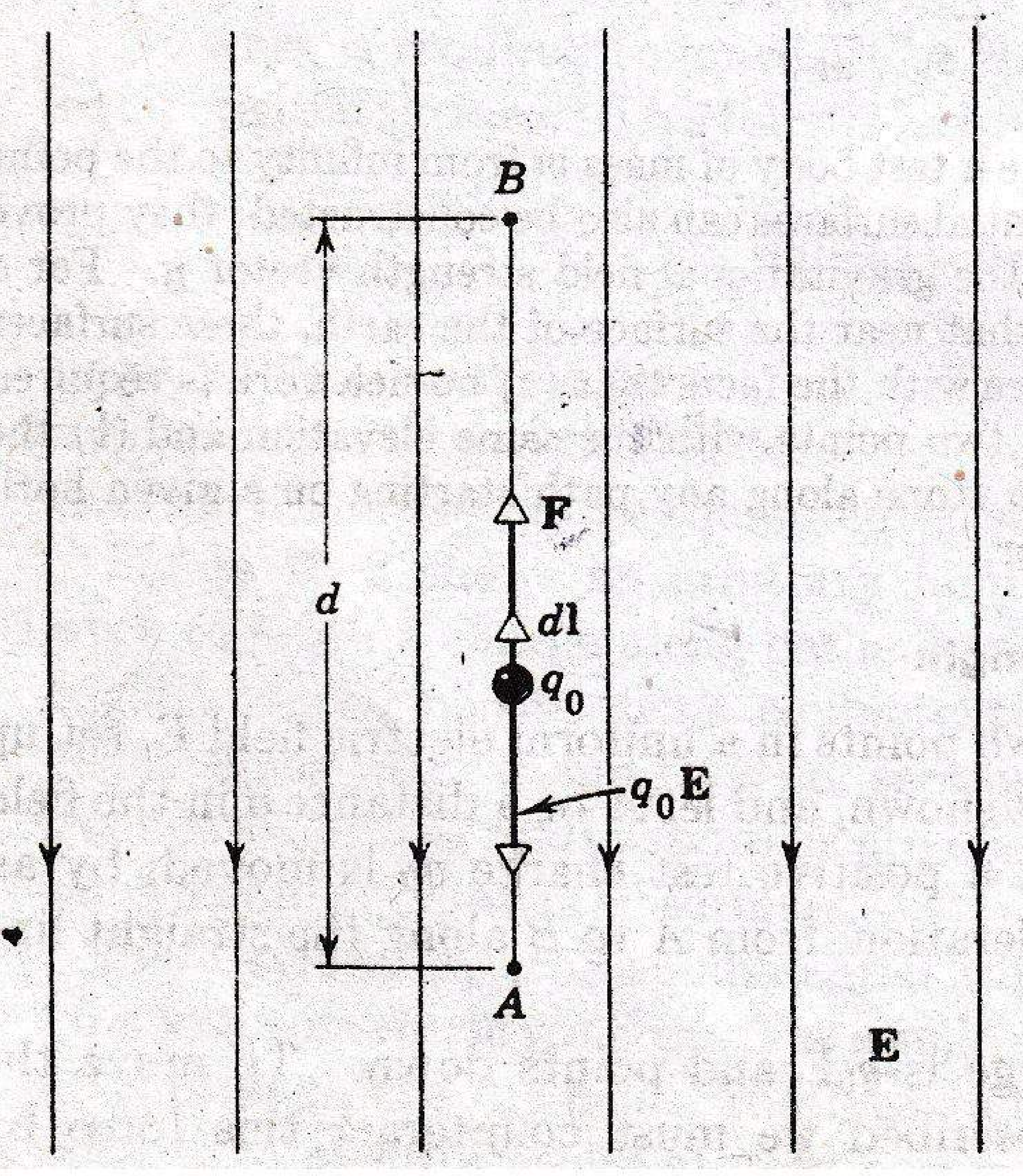

Fig. 29-3 A test charge q_0 is moved from A to B in a uniform electric field **E** by an external agent that exerts a force **F** on it.

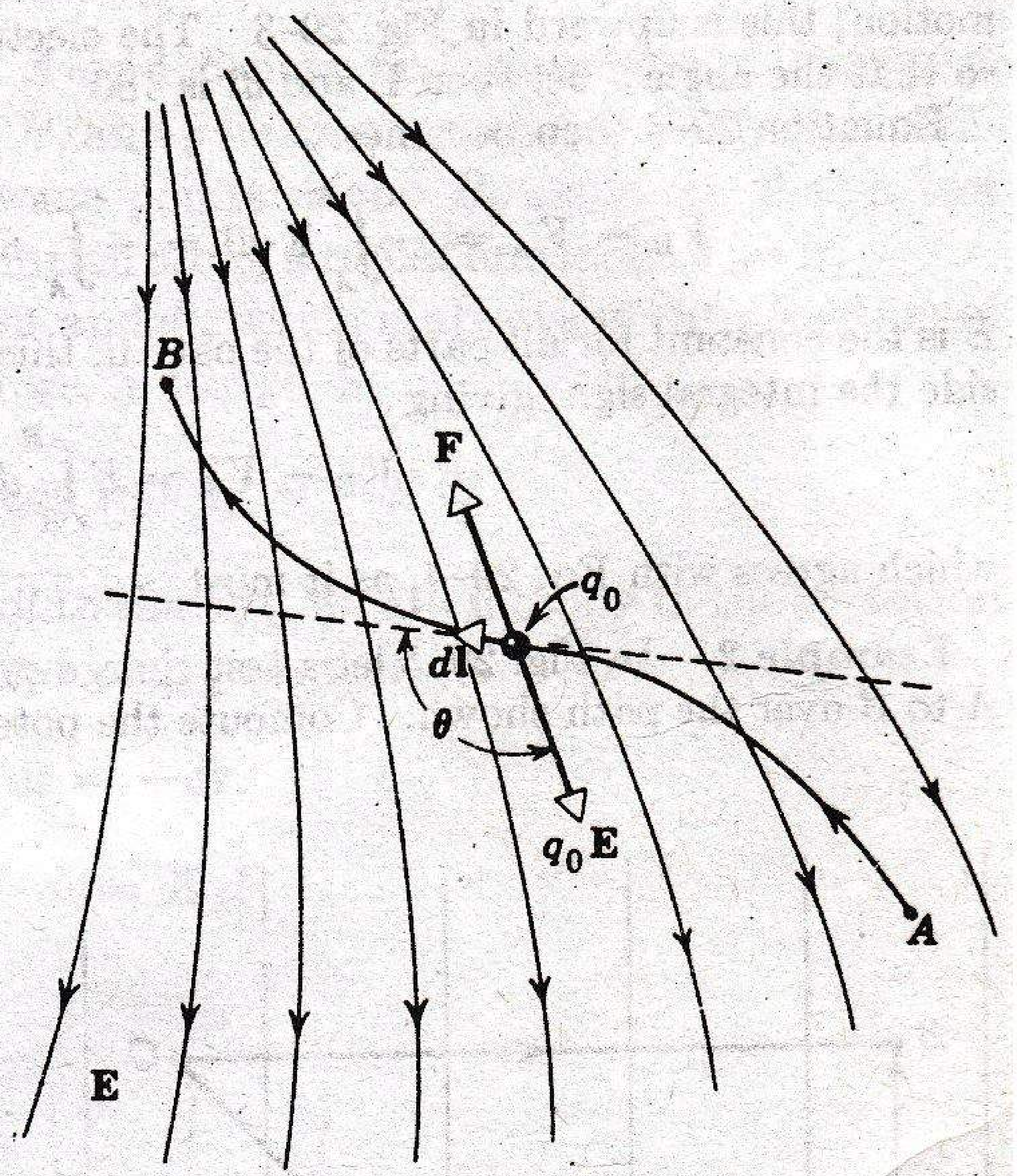

Fig. 29-4 A test charge q_0 is moved from A to B in a nonuniform electric field by an external agent that exerts a force **F** on it.

is $\mathbf{F}\cdot d\mathbf{l}$. To find the total work W_{AB} done by the external agent in moving the test charge from A to B, we add up (that is, integrate) the work contributions for all the infinitesimal segments into which the path is divided. This leads to

$$W_{AB} = \int_A^B \mathbf{F}\cdot d\mathbf{l} = -q_0 \int_A^B \mathbf{E}\cdot d\mathbf{l}.$$

Such an integral is called a *line integral.* Note that we have substituted $-q_0\mathbf{E}$ for its equal, **F**.

Substituting this expression for W_{AB} into Eq. 29-1 leads to

$$V_B - V_A = \frac{W_{AB}}{q_0} = -\int_A^B \mathbf{E}\cdot d\mathbf{l}. \tag{29-5}$$

If point A is taken to be infinitely distant and the potential V_A at infinity is taken to be zero, this equation gives the potential V at point B, or, dropping the subscript B,

$$V = -\int_\infty^B \mathbf{E}\cdot d\mathbf{l}. \tag{29-6}$$

These two equations allow us to calculate the potential difference between any two points (or the potential at any point) if **E** is known at various points in the field.

▶ **Example 1.** In Fig. 29-3 calculate $V_B - V_A$ using Eq. 29-5. Compare the result with that obtained by direct analysis of this special case (Eq. 29-4).

In moving the test charge the element of path $d\mathbf{l}$ always points in the direction of motion; this is upward in Fig. 29–3. The electric field **E** in this figure points down so that the angle θ between **E** and $d\mathbf{l}$ is 180°.

Equation 29–5 then becomes

$$V_B - V_A = -\int_A^B \mathbf{E}\cdot d\mathbf{l} = -\int_A^B E\cos 180°\, dl = \int_A^B E\, dl.$$

E is the constant for all parts of the path in this problem and can thus be taken outside the integral sign, giving

$$V_B - V_A = E\int_A^B dl = Ed,$$

which agrees with Eq. 29–4, as it must.

Example 2. In Fig. 29–5 let a test charge q_0 be moved without acceleration from A to B over the path shown. Compute the potential difference between A and B.

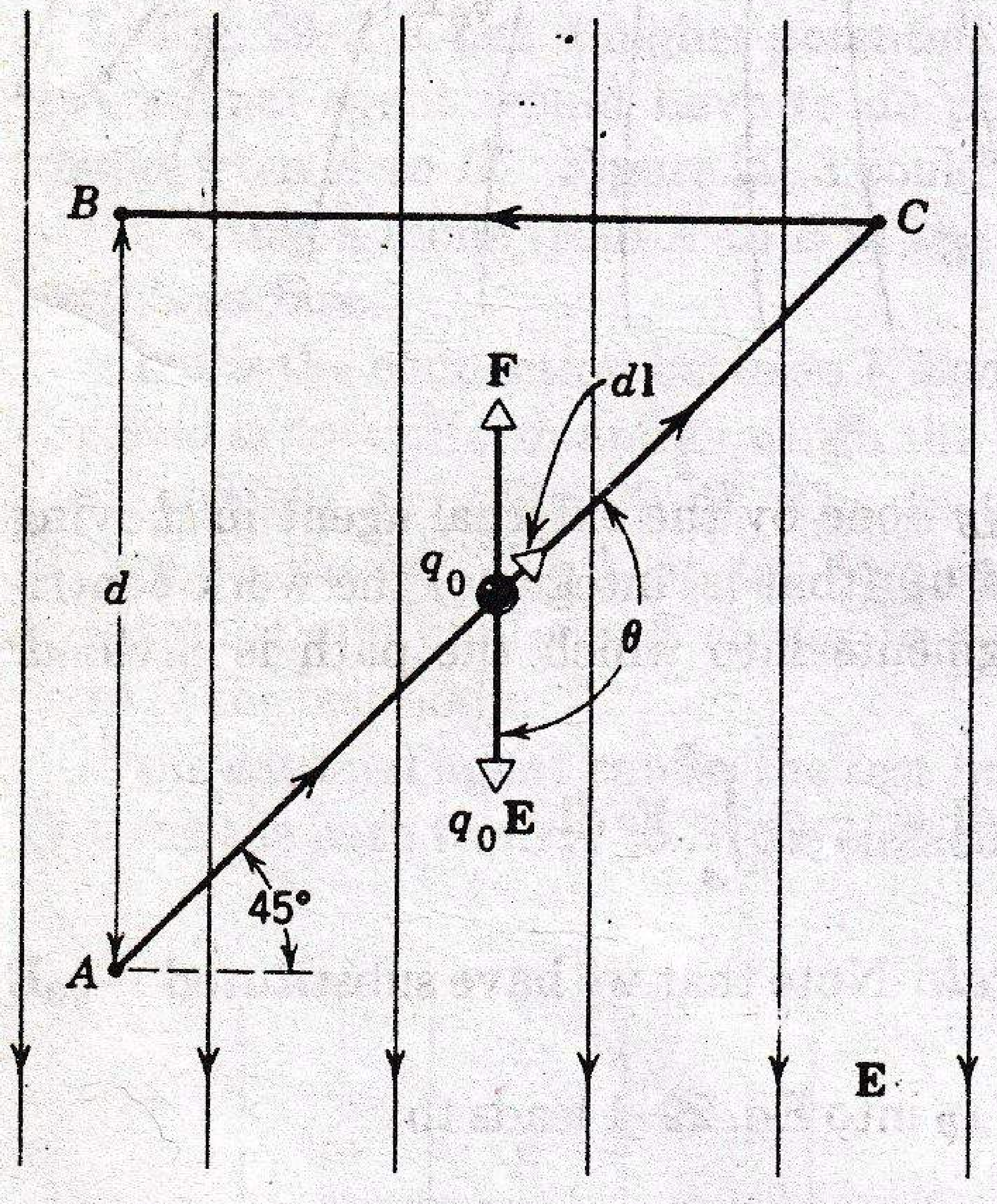

Fig. 29–5 Example 2. A test charge q_0 is moved along path ACB in a uniform electric field by an external agent.

For the path AC we have $\theta = 135°$ and, from Eq. 29–5,

$$V_C - V_A = -\int_A^C \mathbf{E}\cdot d\mathbf{l} = -\int_A^C E\cos 135°\, dl = \frac{E}{\sqrt{2}}\int_A^C dl.$$

The integral is the length of the line AC which is $\sqrt{2}d$. Thus

$$V_C - V_A = \frac{E}{\sqrt{2}}(\sqrt{2}d) = Ed.$$

Points B and C have the same potential because no work is done in moving a charge between them, **E** and $d\mathbf{l}$ being at right angles for all points on the line CB. In other words, B and C lie on the same equipotential surface at right angles to the lines of force. Thus

$$V_B - V_A = V_C - V_A = Ed.$$

This is the same value derived for a direct path connecting A and B, a result to be expected because the potential difference between two points is path independent. ◀

29–3 Potential Due to a Point Charge

Figure 29–6 shows two points A and B near an isolated point charge q. For simplicity we assume that A, B, and q lie on a straight line. Let us compute the potential difference between points A and B, assuming that a test charge q_0 is moved without acceleration along a radial line from A to B.

In Fig. 29–6 **E** points to the right and $d\mathbf{l}$, which is always in the direction of motion, points to the left. Therefore, in Eq. 29–5,

$$\mathbf{E} \cdot d\mathbf{l} = E \cos 180° \, dl = -E \, dl.$$

However, as we move a distance dl to the left, we are moving in the direction of decreasing r because r is measured from q as an origin. Thus

$$dl = -dr.$$

Combining yields

$$\mathbf{E} \cdot d\mathbf{l} = E \, dr.$$

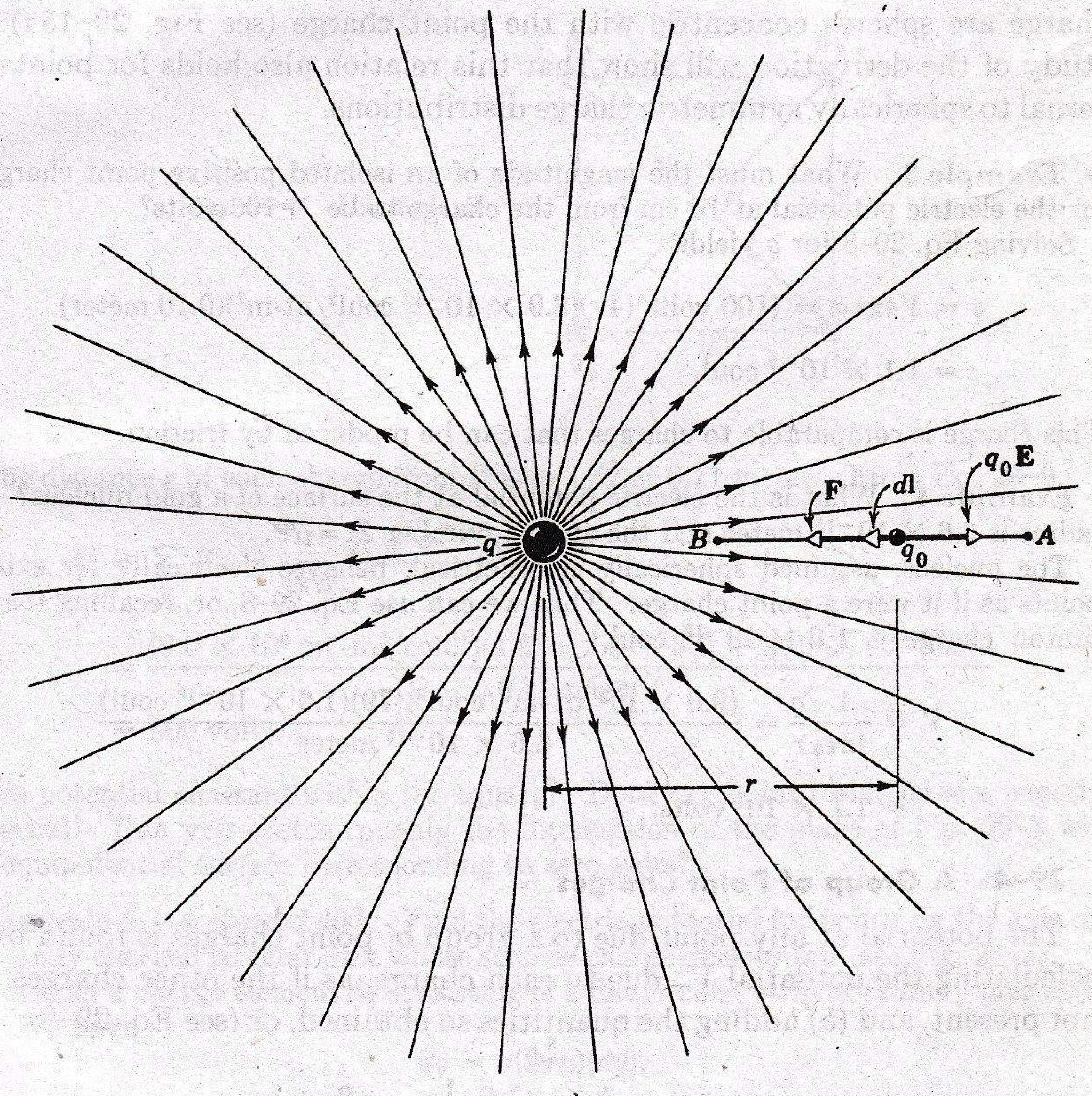

Fig. 29–6 A test charge q_0 is moved by an external agent from A to B in the field set up by a point charge q.

Substituting this into Eq. 29–5 gives

$$V_B - V_A = -\int_A^B \mathbf{E}\cdot d\mathbf{l} = -\int_{r_A}^{r_B} E\,dr.$$

Combining with Eq. 27–4,

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

leads to
$$V_B - V_A = -\frac{q}{4\pi\epsilon_0}\int_{r_A}^{r_B}\frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right). \tag{29–7}$$

Choosing reference position A to be at infinity (that is, letting $r_A \to \infty$). choosing $V_A = 0$ at this position, and dropping the subscript B leads to

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}. \tag{29–8}$$

This equation shows clearly that equipotential surfaces for an isolated point charge are spheres concentric with the point charge (see Fig. 29-15*a*). A study of the derivation will show that this relation also holds for points external to spherically symmetric charge distributions.

▶ **Example 3.** What must the magnitude of an isolated positive point charge be for the electric potential at 10 cm from the charge to be +100 volts?

Solving Eq. 29–8 for q yields

$$q = V4\pi\epsilon_0 r = (100 \text{ volts})(4\pi)(8.9 \times 10^{-12} \text{ coul}^2/\text{nt-m}^2)(0.10 \text{ meter})$$
$$= 1.1 \times 10^{-9} \text{ coul}.$$

This charge is comparable to charges that can be produced by friction.

Example 4. What is the electric potential at the surface of a gold nucleus? The radius is 6.6×10^{-15} meter and the atomic number $Z = 79$.

The nucleus, assumed spherically symmetrical, behaves electrically for external points as if it were a point charge. Thus we can use Eq. 29–8, or, recalling that the proton charge is 1.6×10^{-19} coul,

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} = \frac{(9.0 \times 10^9 \text{ nt-m}^2/\text{coul}^2)(79)(1.6 \times 10^{-19} \text{ coul})}{6.6 \times 10^{-15} \text{ meter}}$$
$$= 1.7 \times 10^7 \text{ volts}.$$ ◀

29–4 A Group of Point Charges

The potential at any point due to a group of point charges is found by (*a*) calculating the potential V_n due to each charge, as if the other charges were not present, and (*b*) adding the quantities so obtained, or (see Eq. 29–8)

$$V = \sum_n V_n = \frac{1}{4\pi\epsilon_0}\sum_n \frac{q_n}{r_n}, \tag{29–9}$$

where q_n is the value of the nth charge and r_n is the distance of this charge from the point in question. The sum used to calculate V is an *algebraic sum* and not a vector sum like the one used to calculate **E** for a group of point charges (see Eq. 27-5). Herein lies an important computational advantage of potential over electric field strength.

If the charge distribution is continuous, rather than being a collection of points, the sum in Eq. 29-9 must be replaced by an integral, or

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}, \tag{29-10}$$

where dq is a differential element of the charge distribution, r is its distance from the point at which V is to be calculated, and dV is the potential it establishes at that point.

▶ **Example 5.** What is the potential at the center of the square of Fig. 29-7? Assume that $q_1 = +1.0 \times 10^{-8}$ coul, $q_2 = -2.0 \times 10^{-8}$ coul, $q_3 = +3.0 \times 10^{-8}$ coul, $q_4 = +2.0 \times 10^{-8}$ coul, and $a = 1.0$ meter.

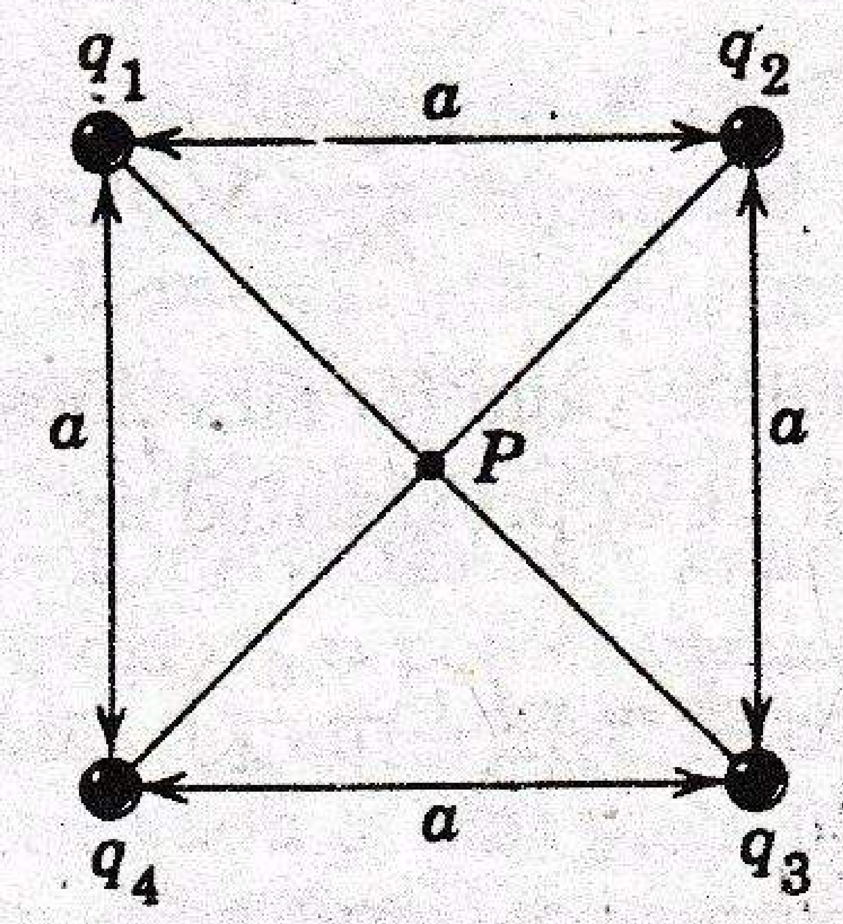

Fig. 29-7 Example 5

The distance r of each charge from P is $a/\sqrt{2}$ or 0.71 meter. From Eq. 29-9

$$V = \sum_n V_n = \frac{1}{4\pi\epsilon_0}\frac{q_1 + q_2 + q_3 + q_4}{r}$$

$$= \frac{(9.0 \times 10^9 \text{ nt-m}^2/\text{coul}^2)(1.0 - 2.0 + 3.0 + 2.0) \times 10^{-8} \text{ coul}}{0.71 \text{ meter}}$$

$$= 500 \text{ volts.}$$

Is the potential constant within the square? Does any point inside have a negative potential? Can you sketch roughly the intersection of the plane of Fig. 29-7 with the equipotential surface corresponding to zero volts?

Example 6. *A charged disk.* Find the electric potential for points on the axis of a uniformly charged circular disk whose surface charge density is σ (see Fig. 29-8).

Consider a charge element dq consisting of a flat circular strip of radius y and width dy. We have

$$dq = \sigma(2\pi y)(dy),$$

where $(2\pi y)(dy)$ is the area of the strip. All parts of this charge element are the same distance r' $(= \sqrt{y^2 + r^2})$ from axial point P so that their contribution dV to the

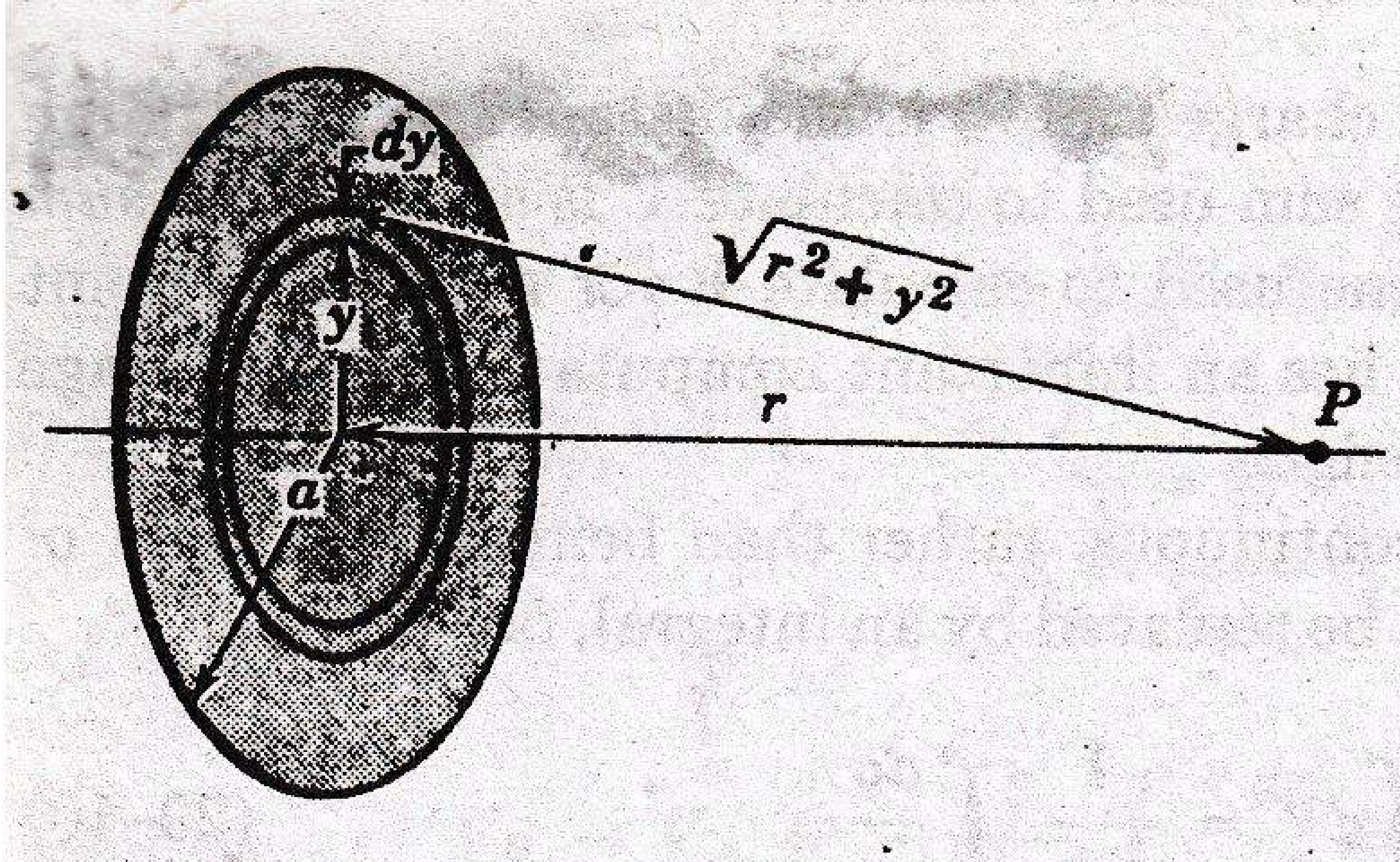

Fig. 29–8 Example 6. A point P on the axis of a uniformly charged circular disk of radius a.

electric potential at P is given by Eq. 29–8, or

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r'} = \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi y\,dy}{\sqrt{y^2+r^2}}.$$

The potential V is found by integrating over all the strips into which the disk can be divided (Eq. 29–10) or

$$V = \int dV = \frac{\sigma}{2\epsilon_0}\int_0^a (y^2+r^2)^{-1/2} y\,dy$$

$$= \frac{\sigma}{2\epsilon_0}(\sqrt{a^2+r^2} - r).$$

This general result is valid for all values of r. In the special case of $r \gg a$ the quantity $\sqrt{a^2+r^2}$ can be approximated as

$$\sqrt{a^2+r^2} = r\left(1+\frac{a^2}{r^2}\right)^{1/2} = r\left(1+\frac{1}{2}\frac{a^2}{r^2}+\cdots\right) \cong r + \frac{a^2}{2r},$$

in which the quantity in parentheses in the second member of this equation has been expanded by the binomial theorem (see Appendix *I*). This equation means that V becomes

$$V \cong \frac{\sigma}{2\epsilon_0}\left(r+\frac{a^2}{2r}-r\right) = \frac{\sigma\pi a^2}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r},$$

where q $(= \sigma\pi a^2)$ is the total charge on the disk. This limiting result is expected because the disk behaves like a point charge for $r \gg a$. ◀

29–5 Potential Due to a Dipole

Two equal charges, q, of opposite sign, separated by a distance $2a$, constitute an electric dipole; see Example 3, Chapter 27. The electric dipole moment **p** has the magnitude $2aq$ and points from the negative charge to the positive charge. Here we derive an expression for the electric potential V at any point of space due to a dipole, provided only that the point is not too close to the dipole.

A point P is specified by giving the quantities r and θ in Fig. 29–9. From symmetry, it is clear that the potential will not change as point P rotates about the z axis, r and θ being fixed. Thus we need only find $V(r,\theta)$ for any plane containing this axis; the plane of Fig. 29–9 is such a plane. Applying

Eq. 29–9 gives

$$V = \sum_n V_n = V_1 + V_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right) = \frac{q}{4\pi\epsilon_0}\frac{r_2 - r_1}{r_1 r_2},$$

which is an exact relationship.

We now limit consideration to points such that $r \gg 2a$. These approximate relations then follow from Fig. 29–9:

$$r_2 - r_1 \cong 2a\cos\theta, \quad \text{and} \quad r_1 r_2 \cong r^2,$$

and the potential reduces to

$$V = \frac{q}{4\pi\epsilon_0}\frac{2a\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}, \tag{29–11}$$

in which p $(= 2aq)$ is the dipole moment. Note that V vanishes everywhere in the equatorial plane ($\theta = 90°$). This reflects the fact that it takes no work to bring a test charge in from infinity along the perpendicular bisector of the dipole. For a given radius, V has its greatest positive value for $\theta = 0$ and its greatest negative value for $\theta = 180°$. Note that the potential does not depend separately on q and $2a$ but only on their product p.

It is convenient to call *any* assembly of charges, for which V at distant points is given by Eq. 29–11, an *electric dipole*. Two point charges separated by a small distance behave this way, as we have just proved. However, other charge configurations also obey Eq. 29–11. Suppose that by measurement at points outside an imaginary box (Fig. 29–10) we find a pattern of lines of force that can be described quantitatively by Eq. 29–11. We then declare that the object inside the box is an *electric dipole*, that its axis is the line zz', and that its dipole moment $\mathbf{p}$ points vertically upward.

Many molecules have electric dipole moments. That for H_2O in its vapor state is 6.1×10^{-30} coul-m. Figure 29–11 is a representation of this molecule, showing the three nuclei and the surrounding electron cloud. The dipole moment $\mathbf{p}$ is represented

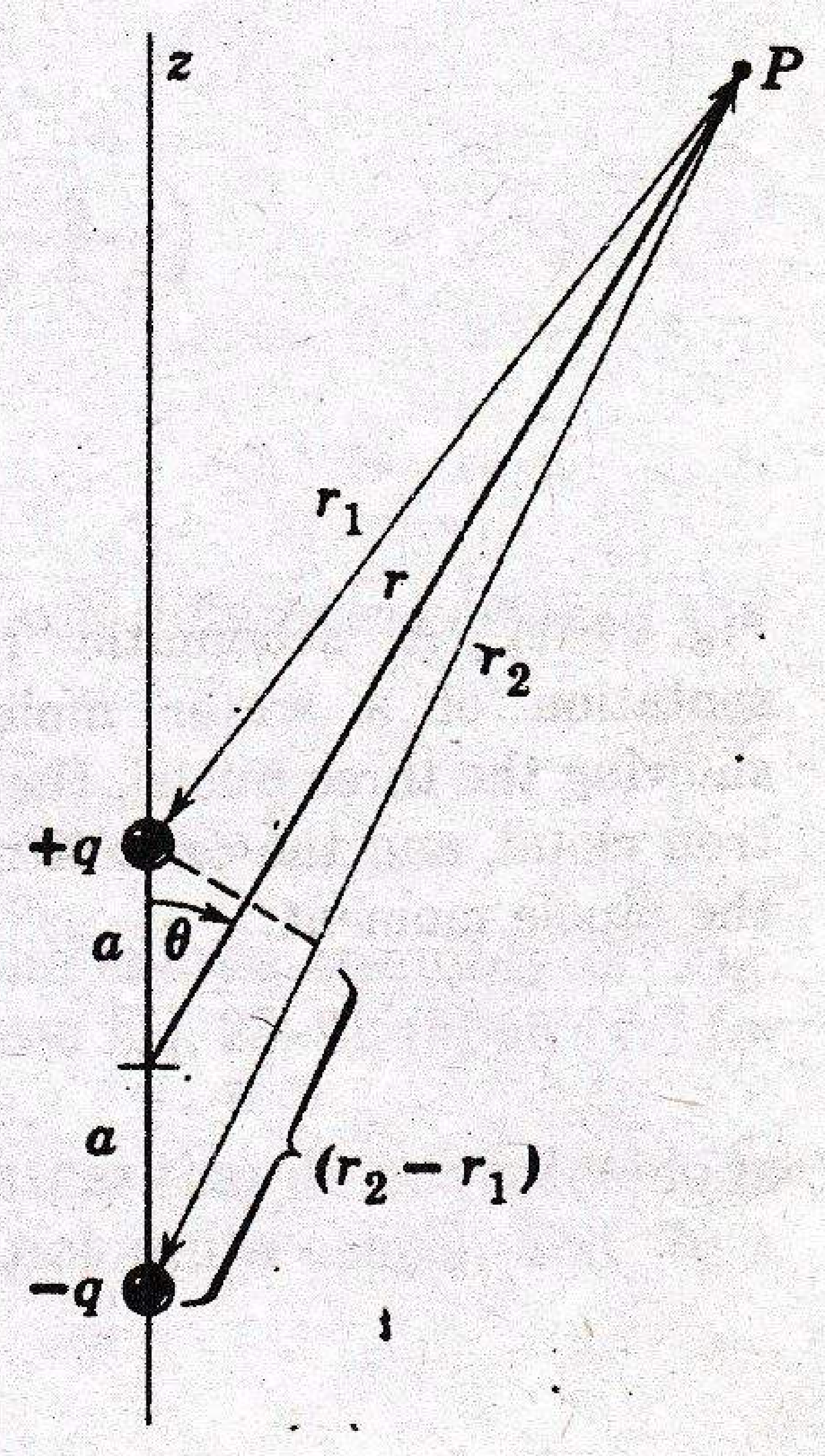

Fig. 29–9 A point P in the field of an electric dipole.

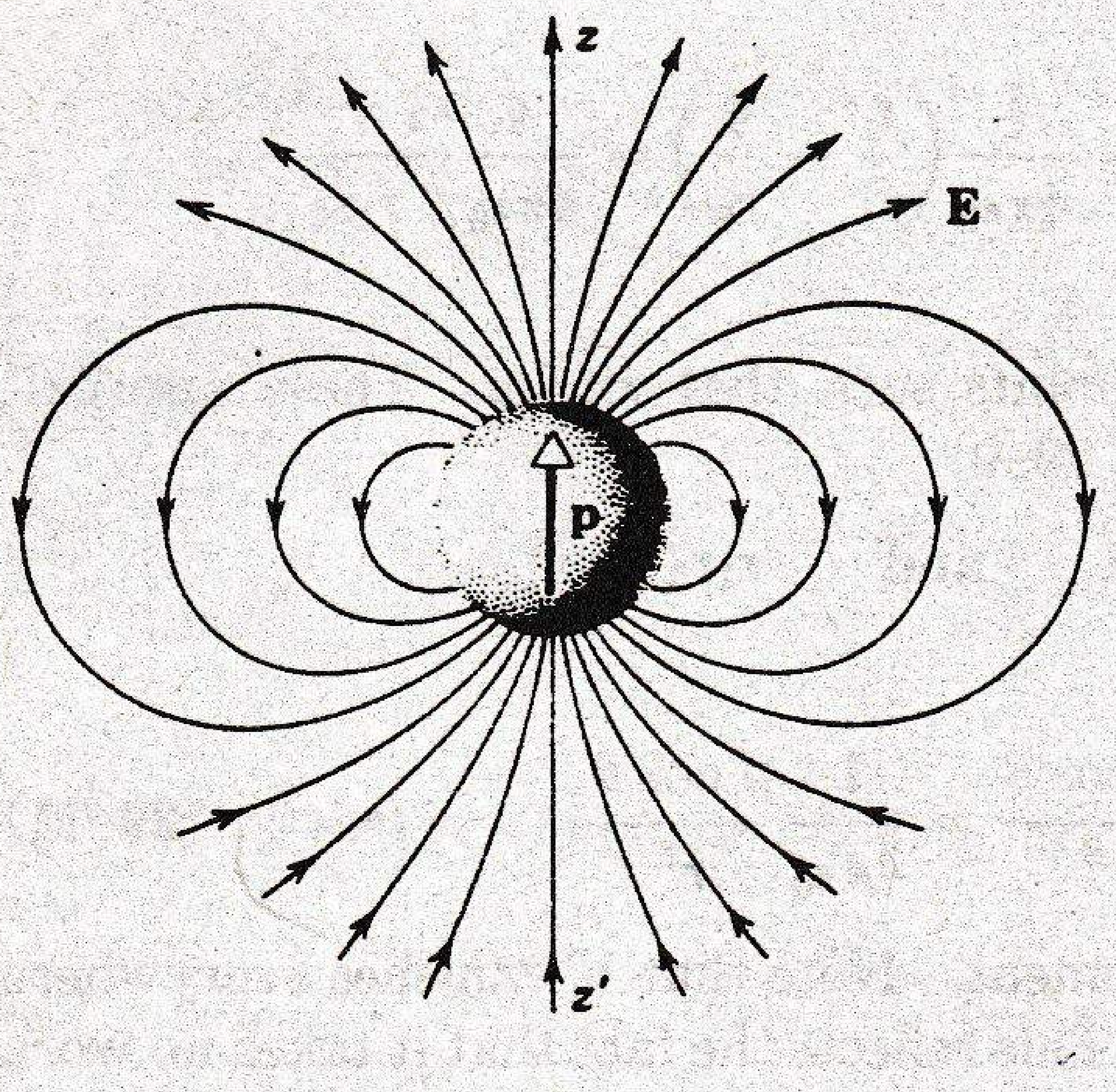

Fig. 29–10 If an object inside the spherical box sets up the electric field shown (described quantitatively by Eq. 29–11), it is an electric dipole.

by the arrow on the axis of symmetry of the molecule. In this molecule the effective center of positive charge does not coincide with the effective center of negative charge. It is precisely because of this separation that the dipole moment exists.

Atoms, and many molecules, do *not* have permanent dipole moments. However, dipole moments may be induced by placing any atom or molecule in an external electric field. The action of the field (Fig. 29–12) is to separate the centers of positive and of negative charge. We say that the atom becomes *polarized* and acquires an *induced electric dipole moment*. Induced dipole moments disappear when the electric field is removed.

Electric dipoles are important in situations other than atomic and molecular ones. Radio and radar antennas are often in the form of a metal wire or rod in which electrons surge back and forth periodically. At a certain time one end of the wire or rod will be negative and the other end positive. Half a cycle later the polarity of the ends is exactly reversed. This is an *oscillating* electric dipole. It is so named because its dipole moment changes in a periodic way with time.

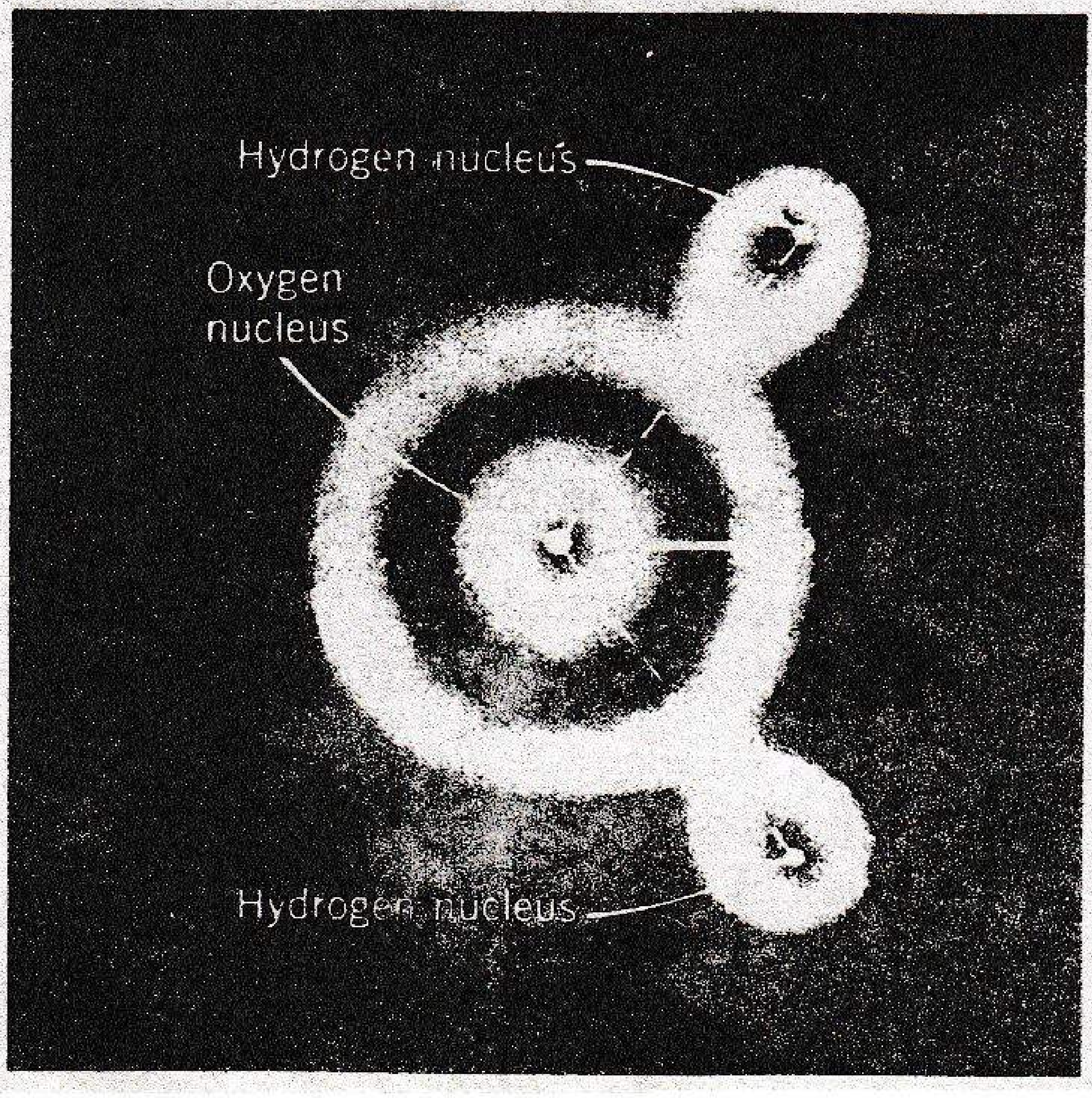

Fig. 29–11 A schematic representation of a water molecule, showing the three nuclei, the electron cloud, and the orientation of the dipole moment.

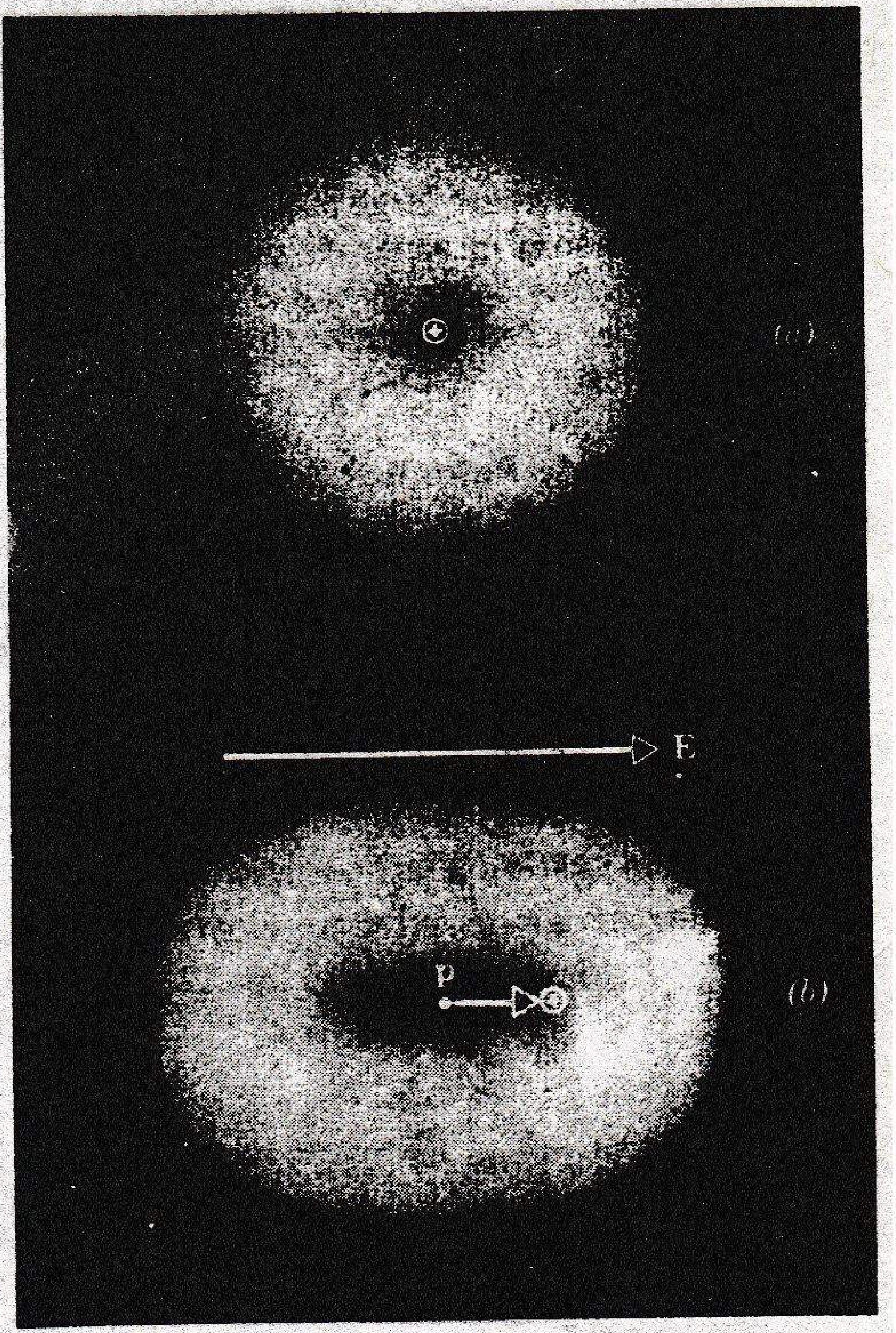

Fig. 29-12 (*a*) An atom, showing the nucleus and the electron cloud. The center of negative charge coincides with the center of positive charge, that is, with the nucleus. (*b*) If an external field E is applied, the electron cloud is distorted so that the center of negative charge, marked by the dot, and the center of positive charge no longer coincide. An electric dipole appears.

▶ **Example 7.** *An electric quadrupole.* An electric quadrupole, of which Fig. 27-21 is an example, consists of two electric dipoles so arranged that they almost, but not quite, cancel each other in their electric effects at distant points.* Calculate $V(r)$ for points on the axis of this quadrupole.

Applying Eq. 29-9 to Fig. 27-21 yields

$$V = \sum_n V_n = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r-a} - \frac{2q}{r} + \frac{q}{r+a}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{2a^2}{(r-a)(r)(r+a)}.$$

Assuming that $r \gg a$ allows us to put $a = 0$ in the sum and difference terms in the denominator, yielding

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^3},$$

where Q $(= 2qa^2)$ is the *electric quadrupole moment* of the charge assembly of Fig. 27-21. Note that V varies (*a*) as $1/r$ for a point charge (see Eq. 29-8), (*b*) as $1/r^2$ for a dipole (see Eq. 29-11), and (*c*) as $1/r^3$ for a quadrupole.

Note too that (*a*) a dipole is two equal and opposite charges that do not coincide in space so that their electric effects at distant points do not quite cancel, and (*b*) a

* See Problem 18, Chapter 27.

quadrupole is two equal and opposite dipoles that do not coincide in space so that their electric effects at distant points again do not quite cancel. This pattern can be extended to define higher orders of charge distribution such as *octupoles*.

The potential at points at distances from an *arbitrary* charge distribution (continuous or discrete) that are large compared with the size of the distribution can always be written as the sum of separate potential distributions due to (*a*) a single charge—sometimes, in this context, called a *monopole*—(*b*) a dipole, (*c*) a quadrupole, etc. This process is called an *expansion in multipoles* and is a very useful technique in problem solving. ◀

29–6 Electric Potential Energy *

If we raise a stone from the earth's surface, the work that we do against the earth's gravitational attraction is stored as *potential energy* in the system earth + stone. If we release the stone, the stored potential energy changes steadily into kinetic energy as the stone drops. After the stone comes to rest on the earth, this kinetic energy, equal in magnitude just before the time of contact to the originally stored potential energy, is transformed into heat energy in the system earth + stone.

A similar situation exists in electrostatics. Consider two charges q_1 and q_2 a distance r apart, as in Fig. 29–13. If we increase the separation between

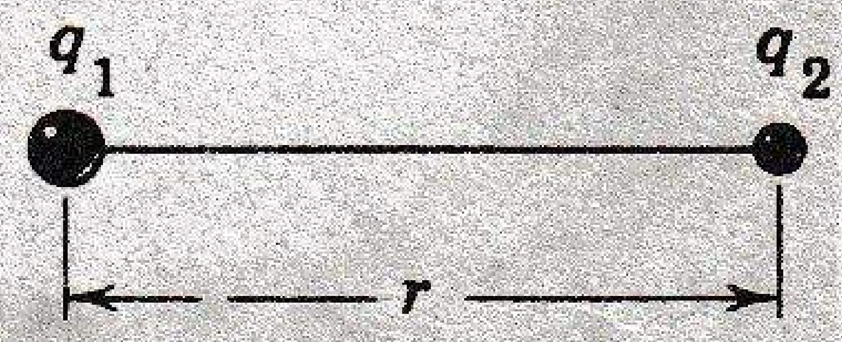

Fig. 29–13

them, an external agent must do work that will be positive if the charges are opposite in sign and negative otherwise. The energy represented by this work can be thought of as stored in the system $q_1 + q_2$ as *electric potential energy*. This energy, like all varieties of potential energy, can be transformed into other forms. If q_1 and q_2, for example, are charges of opposite sign and we release them, they will accelerate toward each other, transforming the stored potential energy into kinetic energy of the accelerating masses. The analogy to the earth + stone system is exact, save for the fact that electric forces may be either attractive or repulsive whereas gravitational forces are always attractive.

We define the electric potential energy of a system of point charges as the work required to assemble this system of charges by bringing them in from an infinite distance. We assume that the charges are all at rest when they are infinitely separated, that is, they have no initial kinetic energy.

In Fig. 29–13 let us imagine q_2 removed to infinity and at rest. The

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* In mechanics the concept of *potential energy* (of compressed springs, falling masses, etc.) is more commonly used than the concept of *potential*. In electrostatics the reverse is true, electric potential being perhaps a more common concept than electric potential energy. In what follows the student must be careful not to confuse these quite different quantities, potential and potential energy.

electric potential at the original site of q_2, caused by q_1, is given by Eq. 29-8, or

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r}.$$

If q_2 is moved in from infinity to the original distance r, the work required is, from the definition of electric potential, that is, from Eq. 29-2,

$$W = Vq_2. \qquad (29\text{-}12)$$

Combining these two equations and recalling that this work W is precisely the *electric potential energy* U of the system $q_1 + q_2$ yields

$$U\,(= W) = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}}. \qquad (29\text{-}13)$$

The subscript of r emphasizes that the distance involved is that between the point charges q_1 and q_2.

For systems containing more than two charges the procedure is to compute the potential energy for every pair of charges separately and to add the results algebraically. This procedure rests on a physical picture in which (*a*) charge q_1 is brought into position, (*b*) q_2 is brought from infinity to its position near q_1, (*c*) q_3 is brought from infinity to its position near q_1 and q_2, etc.

The potential energy of continuous charge distributions (an ellipsoid of charge, for example) can be found by dividing the distribution into infinitesimal elements dq, treating each such element as a point charge, and using the procedures of the preceding paragraph, with the summation process replaced by an integration. We have not considered such problems in this text.

▶ **Example 8.** Two protons in a nucleus of U^{238} are 6.0×10^{-15} meter apart. What is their mutual electric potential energy?

From Eq. 29-13

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r} = \frac{(9.0 \times 10^9 \text{ nt-m}^2/\text{coul}^2)(1.6 \times 10^{-19} \text{ coul})^2}{6.0 \times 10^{-15} \text{ meter}}$$

$$= 3.8 \times 10^{-14} \text{ joule} = 2.4 \times 10^5 \text{ ev.}$$

Example 9. Three charges are arranged as in Fig. 29-14. What is their mutual potential energy? Assume that $q = 1.0 \times 10^{-7}$ coul and $a = 10$ cm.

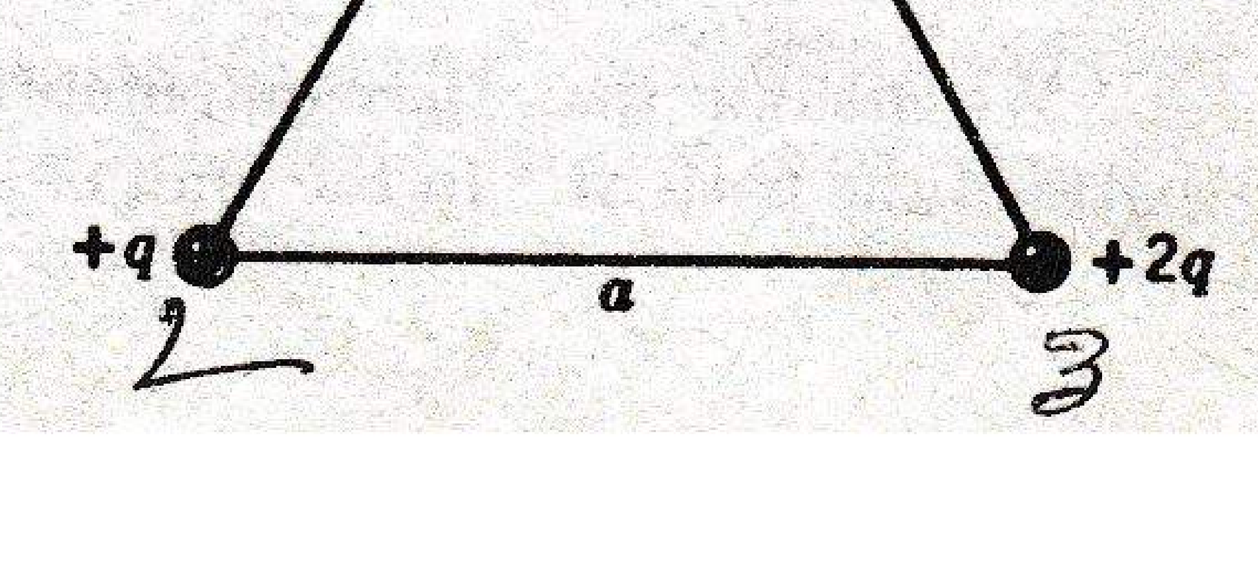


Fig. 29-14 Example 9. Three charges are fixed rigidly, as shown, by external forces.

The total energy of the configuration is the sum of the energies of each pair of particles. From Eq. 29-13,

$$U = U_{12} + U_{13} + U_{23}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{(+q)(-4q)}{a} + \frac{(+q)(+2q)}{a} + \frac{(-4q)(+2q)}{a}\right]$$

$$= -\frac{10}{4\pi\epsilon_0}\frac{q^2}{a}$$

$$= -\frac{(9.0 + 10^9\ \text{nt-m}^2/\text{coul}^2)(10)(1.0 \times 10^{-7}\ \text{coul})^2}{0.10\ \text{meter}} = -9.0 \times 10^{-3}\ \text{joule.}$$

The fact that the total energy is negative means that negative work would have to be done to assemble this structure, starting with the three charges separated and at rest at infinity. Expressed otherwise, 9.0×10^{-3} joule of work must be done to dismantle this structure, removing the charges to an infinite separation from one another.

When, as is common practice, infinity is taken as the zero of electric potential, a positive potential energy (as in Example 8) corresponds to repulsive electric forces and a negative potential energy (as in this example) to attractive electric forces. If the protons in Example 8 were not held in place by attractive (nonelectrical) nuclear forces, they would move away from each other. If the three particles in this example were released from their fixed positions, in which they are held by external forces, they would move toward each other. ◀

29–7 Calculation of E from V

We have stated that V and **E** are equivalent descriptions and have determined (Eq. 29–6) how to calculate V from **E**. Let us now consider how to calculate **E** if V is known throughout a certain region.

This problem has already been solved graphically. If **E** is known at every point in space, the lines of force can be drawn; then a family of equipotentials can be sketched in by drawing surfaces at right angles. These equipotentials describe the behavior of V. Conversely, if V is given as a function of position, a set of equipotential surfaces can be drawn. The lines of force can then be found by drawing lines at right angles, thus describing the behavior of **E**. It is the mathematical equivalent of this second graphical process that we seek here. Figure 29–15 shows some examples of lines of force and of the corresponding equipotential surfaces.

Figure 29–16 shows the intersection with the plane of the figure of a family of equipotential surfaces. The figure shows that **E** at a typical point P is at right angles to the equipotential surface through P, as it must be.

Let us move a test charge q_0 from P along the path marked $\Delta\mathbf{l}$ to the equipotential surface marked $V + \Delta V$. The work that must be done by the agent exerting the force **F** (see Eq. 29–1) is $q_0\Delta V$.

From another point of view we can calculate the work from *

$$\Delta W = \mathbf{F}\cdot\Delta\mathbf{l},$$

* We assume that the equipotentials are so close together that **F** is constant for all parts of the path $\Delta\mathbf{l}$. In the limit of a differential path ($d\mathbf{l}$) there will be no difficulty.

where $\mathbf{F}$ is the force that must be exerted on the charge to overcome exactly the electrical force $q_0\mathbf{E}$. Since $\mathbf{F}$ and $q_0\mathbf{E}$ have opposite signs and are equal in magnitude,

$$\Delta W = -q_0\mathbf{E}\cdot\Delta\mathbf{l} = -q_0E\cos(\pi - \theta)\,\Delta l = q_0E\cos\theta\,\Delta l.$$

These two expressions for the work must be equal, which gives

$$q_0\,\Delta V = q_0E\cos\theta\,\Delta l$$

or

$$E\cos\theta = \frac{\Delta V}{\Delta l}. \tag{29-14}$$

Now $E\cos\theta$ is the component of $\mathbf{E}$ in the direction $-\mathbf{l}$ in Fig. 29-16; the quantity $-E\cos\theta$, which we call E_l, would then be the component of $\mathbf{E}$ in the $+\mathbf{l}$ direction. In the differential limit Eq. 29-14 can then be written as

$$E_l = -\frac{dV}{dl}. \tag{29-15}$$

In words, this equation says: If we travel through an electric field along a straight line and measure V as we go, the rate of change of V with distance that we observe, when changed in sign, is the component of $\mathbf{E}$ in that direction. The minus sign implies that $\mathbf{E}$ points in the direction of decreasing V, as in Fig. 29-16. It is particularly clear from Eq. 29-15 that appropriate units for $\mathbf{E}$ are the volts/meter.

There will be one direction $\mathbf{l}$ for which the quantity dV/dl is a maximum. From Eq. 29-15, E_l will also be a maximum for this direction and will in fact be E itself. Thus

$$E = -\left(\frac{dV}{dl}\right)_{\max} \tag{29-16}$$

The maximum value of dV/dl at a given point is called the *potential gradient* at that point. The direction $\mathbf{l}$ for which dV/dl has its maximum value is always at right angles to the equipotential surface, corresponding to $\theta = 0$ in Fig. 29-16.

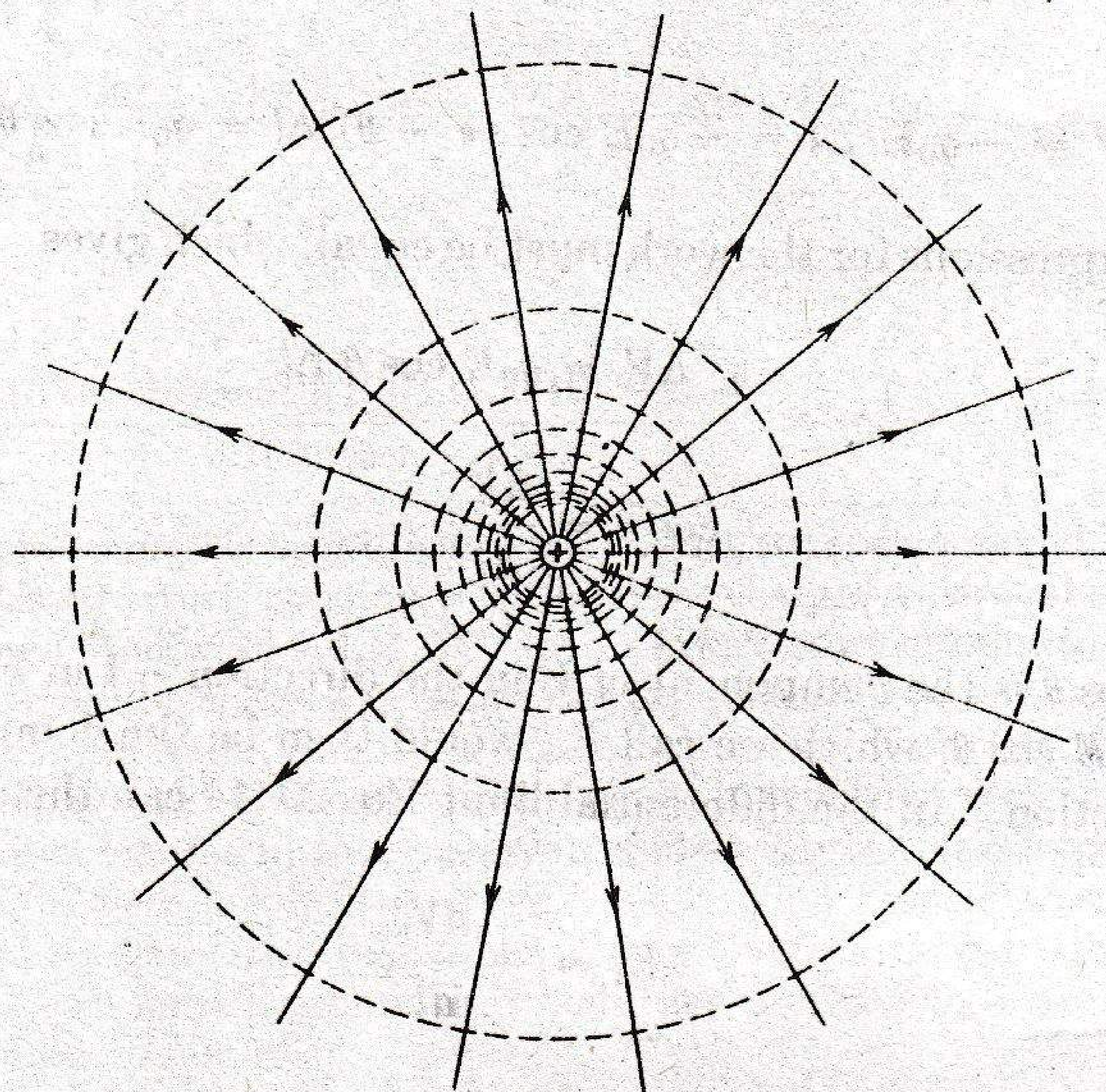
If we take the direction $\mathbf{l}$ to be, in turn, the directions of the x, y, and z axes, we can find the three components of $\mathbf{E}$ at any point, from Eq. 29-15.

$$E_x = -\frac{\partial V}{\partial x}; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z}. \tag{29-17}$$

Thus if V is known for all points of space, that is, if the function $V(x, y, z)$ is known, the components of $\mathbf{E}$, and thus $\mathbf{E}$ itself, can be found by taking derivatives.*

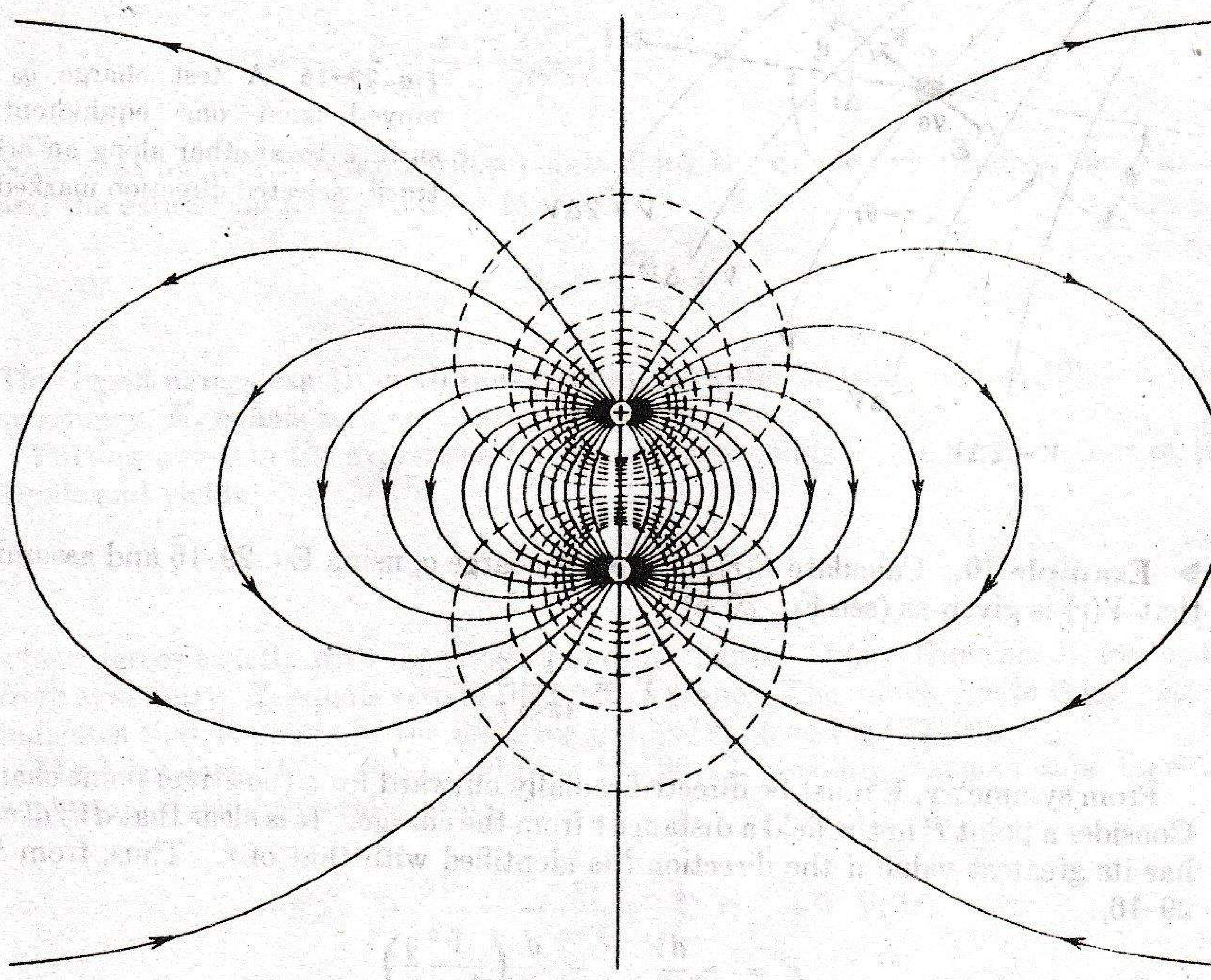
---

* The symbol $\partial V/\partial x$ is a *partial derivative*. It implies that in taking this derivative of the function $V(x, y, z)$ the quantity x is to be viewed as a variable and y and z are to be regarded as constants. Similar considerations hold for $\partial V/\partial y$ and $\partial V/\partial z$.



(a)

(b)



(c)

Fig. 29-15 Equipotential surfaces (dashed lines) and lines of force (solid lines) for (*a*) a point charge, (*b*) a uniform electric field, and (*c*) an electric dipole. In all figures there is a constant difference of potential ΔV between adjacent equipotential surfaces. Thus from Eq. 29-14, written for the case of $\theta = 180°$ as $\Delta l = -\Delta V/E$, the surfaces will be relatively close together where E is relatively large and relatively far apart where E is small. Similarly (see Section 27-3) the lines of force are relatively close together where E is large and far apart where E is small. See discussion and figures of Section 18-7 for other examples.

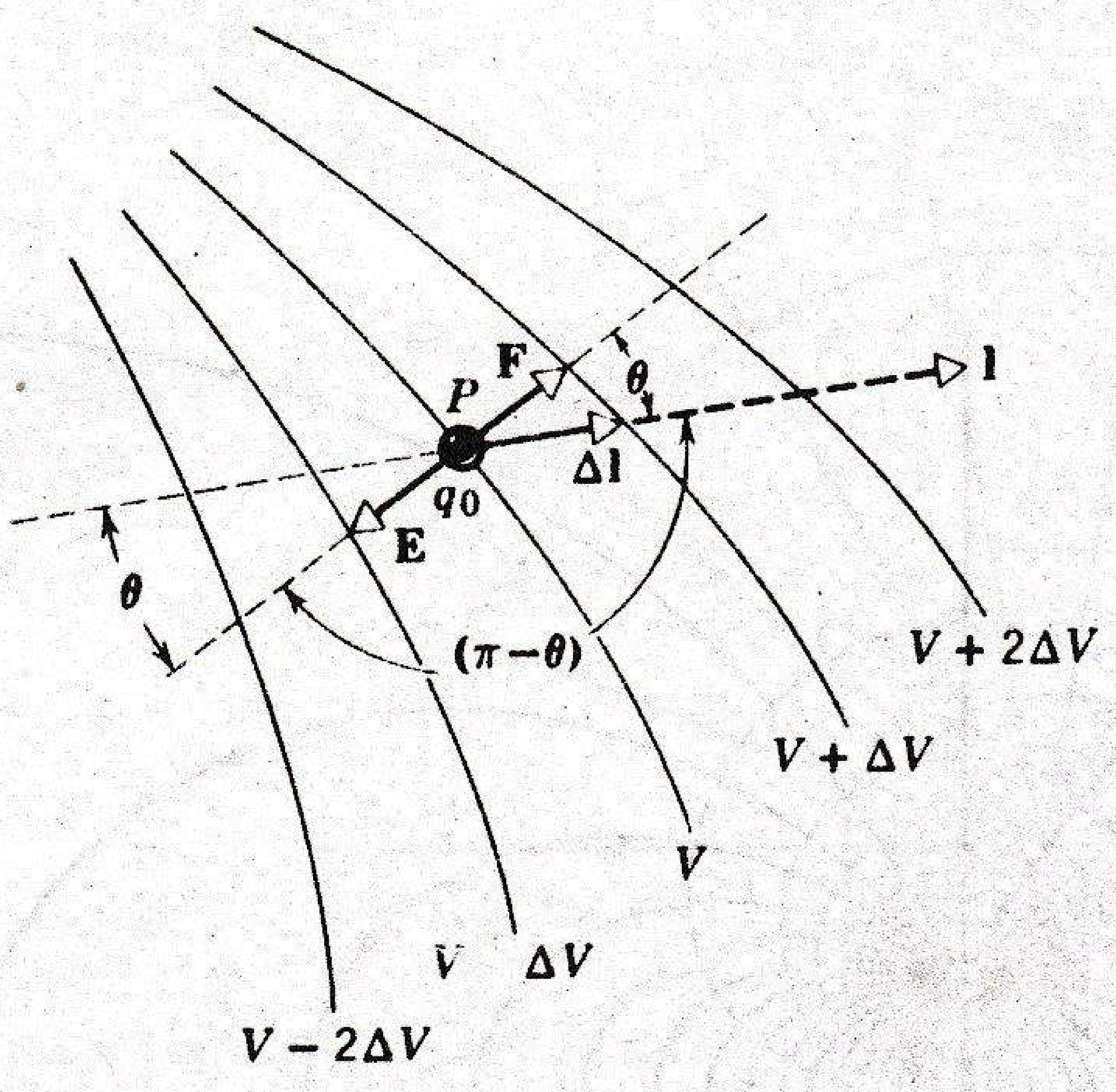


Fig. 29-16 A test charge q_0 is moved from one equipotential surface to another along an arbitrarily selected direction marked **l**.

▶ **Example 10.** Calculate $E(r)$ for a point charge q, using Eq. 29-16 and assuming that $V(r)$ is given as (see Eq. 29-8)

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

From symmetry, **E** must be directed radially outward for a (positive) point charge. Consider a point P in the field a distance r from the charge. It is clear that dV/dl at P has its greatest value if the direction l is identified with that of r. Thus, from Eq. 29-16,

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)$$

$$= -\frac{q}{4\pi\epsilon_0}\frac{d}{dr}\left(\frac{1}{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

This result agrees exactly with Eq. 27-4, as it must.

Example 11. **E** *for a dipole.* Figure 29-17 shows a (distant) point P in the field of a dipole located at the origin of an xy-axis system. V is given by Eq. 29-11, or

$$V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}.$$

Calculate **E** as a function of position.

From symmetry, **E**, for points in the plane of Fig. 29-17, lies in this plane. Thus it can be expressed in terms of its components E_x and E_y. Let us first express the potential function in rectangular coordinates rather than polar coordinates, making use of

$$r = (x^2 + y^2)^{1/2} \quad \text{and} \quad \cos\theta = \frac{y}{(x^2 + y^2)^{1/2}}.$$

The result is

$$V = \frac{p}{4\pi\epsilon_0}\frac{y}{(x^2 + y^2)^{3/2}}.$$

We find E_y from Eq. 29–17, recalling that x is to be treated as a constant in this calculation:

$$E_y = -\frac{\partial V}{\partial y} = -\frac{p}{4\pi\epsilon_0}\frac{(x^2+y^2)^{3/2} - y\frac{3}{2}(x^2+y^2)^{1/2}(2y)}{(x^2+y^2)^3}$$

$$= -\frac{p}{4\pi\epsilon_0}\frac{x^2-2y^2}{(x^2+y^2)^{5/2}}.$$

Note that putting $x = 0$ describes points along the dipole axis (that is, the y axis), and the expression for E_y reduces to

$$E_y = \frac{2p}{4\pi\epsilon_0}\frac{1}{y^3}.$$

This result agrees exactly with that found in Chapter 27 (see Problem 10), for, from symmetry, E_x equals zero on the dipole axis.

Putting $y = 0$ in the expression for E_y describes points in the median plane of the dipole and yields

$$E_y = -\frac{p}{4\pi\epsilon_0}\frac{1}{x^3},$$

which agrees exactly with the result found in Chapter 27 (see Example 3), for, again from symmetry, E_x equals zero in the median plane. The minus sign in this equation indicates that **E** points in the negative y direction (see Fig. 29–10).

The component E_x is also found from Eq. 29–17, recalling that y is to be taken as a constant during this calculation:

$$E_x = -\frac{\partial V}{\partial x} = -\frac{py}{4\pi\epsilon_0}(-\tfrac{3}{2})(x^2+y^2)^{-5/2}(2x)$$

$$= \frac{3p}{4\pi\epsilon_0}\frac{xy}{(x^2+y^2)^{5/2}}.$$

As expected, E_x vanishes both on the dipole axis ($x = 0$) and in the median plane ($y = 0$); see Fig. 29–10. ◀

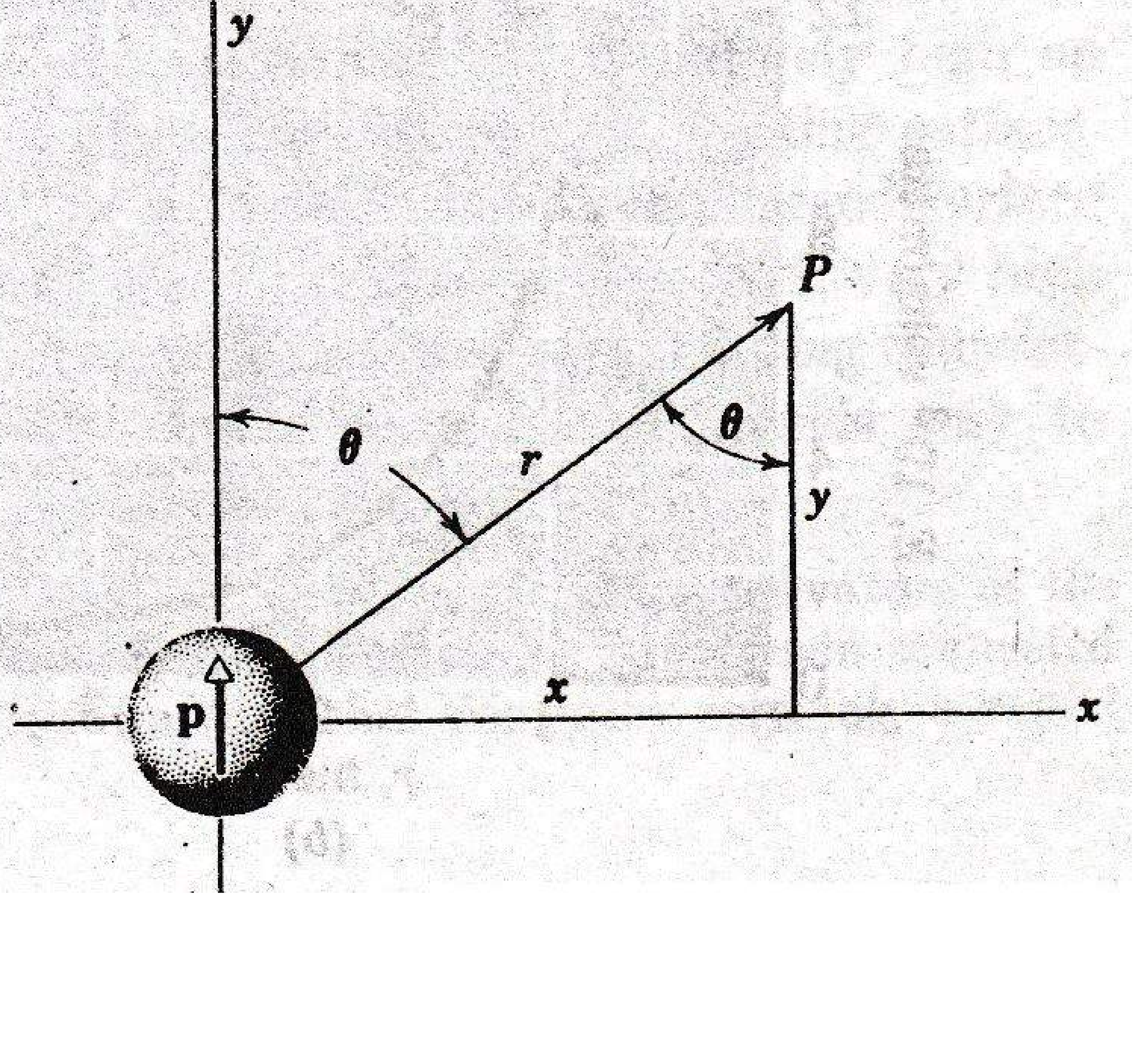


Fig. 29–17 Showing a point P in the field of an electric dipole **p**.

29-8 An Insulated Conductor

We proved in Section 28-4, using Gauss's law, that after a steady state is reached an excess charge q placed on an insulated conductor will move to its outer surface. We now assert that this charge q will distribute itself on this surface so that all points of the conductor, including *those on the surface and those inside*, have the same potential.

Consider any two points A and B in or on the conductor. If they were not at the same potential, the charge carriers in the conductor near the point of lower potential would tend to move toward the point of higher potential. We have assumed, however, that a steady-state situation, in which such currents do not exist, has been reached; thus all points, both on the surface and inside

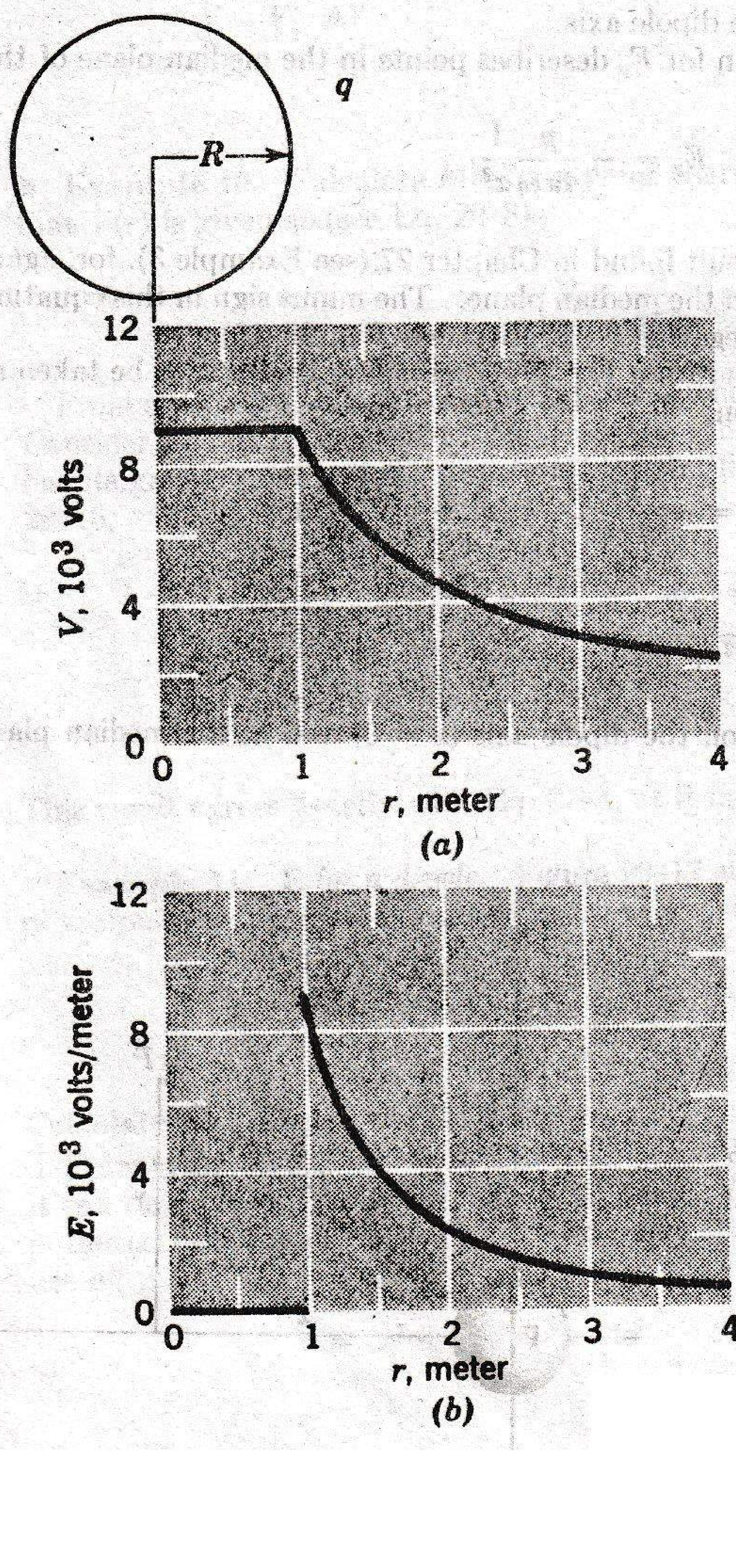


Fig. 29-18 (*a*) The potential and (*b*) the electric field strength, for points near a conducting spherical shell of 1.0 meter radius carrying a charge of $+1.0 \times 10^{-6}$ coul.

it, must have the same potential. Since the surface of the conductor is an equipotential surface, **E** for points on the surface must be at right angles to the surface.

We saw in Section 28-4 that a charge placed on an insulated conductor will spread over the surface until **E** equals zero for all points inside. We now have an alternative way of saying the same thing; the charge will move until all points of the conductor (surface points and interior points) are brought to the same potential, for if V is constant in the conductor then **E** is zero everywhere in the conductor ($E_l = -dV/dl$).

Figure 29-18*a* is a plot of potential against radial distance for an isolated spherical conducting shell of 1.0-meter radius carrying a positive charge of 1.0×10^{-6} coul. For points outside the shell $V(r)$ can be calculated from Eq. 29-8 because the charge q behaves, for such points, as if it were concentrated at the center of the sphere. Equation 29-8 is correct right up to the surface of the shell. Now let us push the test charge through the surface, assuming that there is a small hole, and into the interior. No extra work is needed because no electrical forces act on the test charge once it is inside the shell. Thus the potential everywhere inside is the same as that on the surface, as Fig. 29-18*a* shows.

Figure 29-18*b* shows the electric field strength for this same sphere. Note that E equals zero inside. The lower of these curves can be derived from the upper by differentiation with respect to r, using Eq. 29-16; the upper can be derived from the lower by integration with respect to r, using Eq. 29-6.

Figure 29-18 holds without change if the conductor is a solid sphere rather than a spherical shell as we have assumed. It is constructive to compare Fig. 29-18*b* (conducting shell or sphere) with Fig. 28-9, which holds for a *nonconducting* sphere. The student should try to understand the difference between these two figures, bearing in mind that in the first the charge lies on the surface whereas in the second it was assumed to have been spread uniformly throughout the volume of the sphere.

Finally we note that, as a general rule, the charge density tends to be high on isolated conducting surfaces whose radii of curvature are small, and conversely. For example, the charge density tends to be relatively high on sharp points and relatively low on plane regions on a conducting surface. The electric field strength E at points immediately above a charged surface is proportional to the charge density σ so that E may also reach very high values near sharp points. Glow discharges from sharp points during thunderstorms are a familiar example. The lightning rod acts in this way to neutralize charged clouds and thus prevent lightning strokes.

We can examine the qualitative relationship between σ and the curvature of the surface in a particular case by considering two spheres of different radii connected by a very long fine wire; see Fig. 29-19. Suppose that this entire assembly is raised

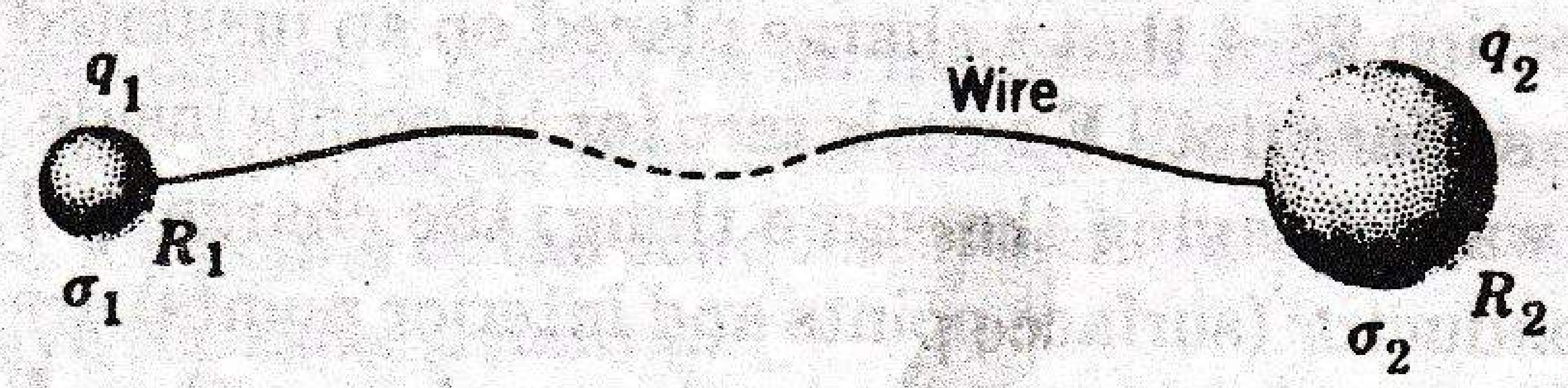


Fig. 29–19 Two spheres connected by a long fine wire.

to some arbitrary potential V. The (equal) potentials of the two spheres are, from Eq. 29-8,*

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{R_1} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{R_2},$$

which yields

$$\frac{q_1}{q_2} = \frac{R_1}{R_2}, \tag{29-18}$$

where q_1 is the charge on the sphere of radius R_1 and q_2 is the charge on the sphere of radius R_2.

The *surface charge densities* for each sphere are given by

$$\sigma_1 = \frac{q_1}{4\pi {R_1}^2} \quad \text{and} \quad \sigma_2 = \frac{q_2}{4\pi {R_2}^2}.$$

Dividing gives

$$\frac{\sigma_1}{\sigma_2} = \frac{q_1}{q_2}\frac{{R_2}^2}{{R_1}^2}.$$

Combining with Eq. 29-18 yields

$$\frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1},$$

which is consistent with our qualitative statement above. Note that the larger sphere has the larger total charge but the smaller charge density.

The fact that σ, and thus $\mathbf{E}$, can become very large near sharp points is important in the design of high-voltage equipment. *Corona discharge* can result from such points if the conducting object is raised to high potential and surrounded by air. Normally air is thought of as a nonconductor. However, it contains a small number of ions produced, for example, by the cosmic rays. A positively charged conductor will attract negative ions from the surrounding air and thus will slowly neutralize itself.

* Equation 29-8 holds only for an *isolated* point charge or spherically symmetric charge distribution. The spheres must be assumed to be so far apart that the charge on either one has a negligible effect on the distribution of charge on the other.

If the charged conductor has sharp points, the value of **E** in the air near the points can be very high. If the value is high enough, the ions, as they are drawn toward the conductor, will receive such large accelerations that, by collision with air molecules, they will produce vast additional numbers of ions. The air is thus made much more conducting, and the discharge of the conductor by this corona discharge may be very rapid indeed. The air surrounding sharp conducting points may even glow visibly because of light emitted from the air molecules during these collisions.

29-9 The Electrostatic Generator

The electrostatic generator was conceived by Lord Kelvin in 1890 and put into useful practice in essentially its modern form by R. J. Van de Graaff in 1931. It is a device for producing electric potential differences of the order of several millions of volts. Its chief application in physics is the use of this potential difference to accelerate charged particles to high energies. Beams of energetic particles made in this way can be used in many different "atom-smashing" experiments. The technique is to let a charged particle "fall" through a potential difference V, gaining kinetic energy as it does so.

Let a particle of (positive) charge q move in a vacuum under the influence of an electric field from one position A to another position B whose electric potential is lower by V. The electric potential energy of the system is reduced by qV because this is the work that an external agent would have to do to restore the system to its original condition. This decrease in potential energy appears as kinetic energy of the particle, or

$$K = qV. \tag{29-19}$$

K is in joules if q is in coulombs and V in volts. If the particle is an electron or a proton, q will be the quantum of charge e.

If we adopt the quantum of charge e as a unit in place of the coulomb, we arrive at another unit for energy, the *electron volt*, which is used extensively in atomic and nuclear physics. By substituting into Eq. 29-19,

$$\begin{aligned} 1 \text{ electron volt} &= (1 \text{ quantum of charge})(1 \text{ volt}) \\ &= (1.60 \times 10^{-19} \text{ coul})(1.00 \text{ volt}) \\ &= 1.60 \times 10^{-19} \text{ joule.} \end{aligned}$$

The electron volt can be used interchangeably with any other energy unit. Thus a 10-gm object moving at 1000 cm/sec can be said to have a kinetic energy of 3.1×10^{18} ev. Most physicists would prefer to express this result as 0.50 joule, the electron volt being inconveniently small. In atomic and nuclear problems, however, the electron volt (ev) and its multiples the Mev ($= 10^6$ ev), the Bev ($= 10^9$ ev) and the Gev ($= 10^{12}$ ev) are the usual units of choice.

▶ **Example 12.** *The electrostatic generator.* Figure 29–20, which illustrates the basic operating principle of the electrostatic generator, shows a small sphere of radius r placed inside a large spherical shell of radius R. The two spheres carry charges q and Q, respectively. Calculate their potential difference.

The potential of the large sphere is caused in part by its own charge and in part because it lies in the field set up by the charge q on the small sphere. From Eq. 29–8,

$$V_R = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{R} + \frac{q}{R}\right).$$

The potential of the small sphere is caused in part by its own charge and in part because it is inside the large sphere; see Fig. 29–18*a*. From Eq. 29–8,

$$V_r = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{Q}{R}\right).$$

The potential difference is

$$V_r - V_R = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right).$$

Thus, assuming q is positive, the inner sphere will always be higher in potential than the outer sphere. If the spheres are connected by a fine wire, the charge q will flow *entirely* to the outer sphere, regardless of the charge Q that may already be present.

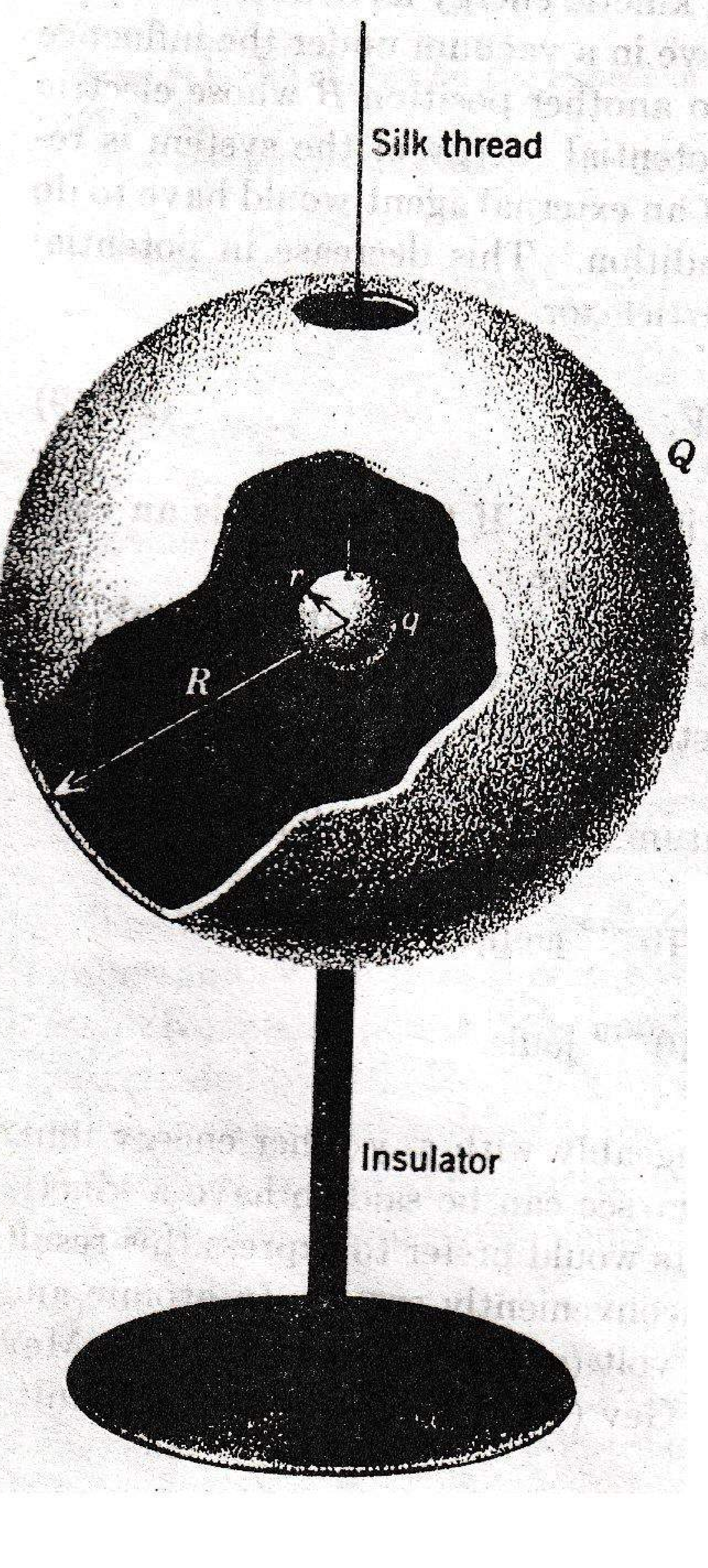


Fig. 29–20 Example 12. A small charged sphere of radius r is suspended inside a charged spherical shell whose outer surface has radius R.

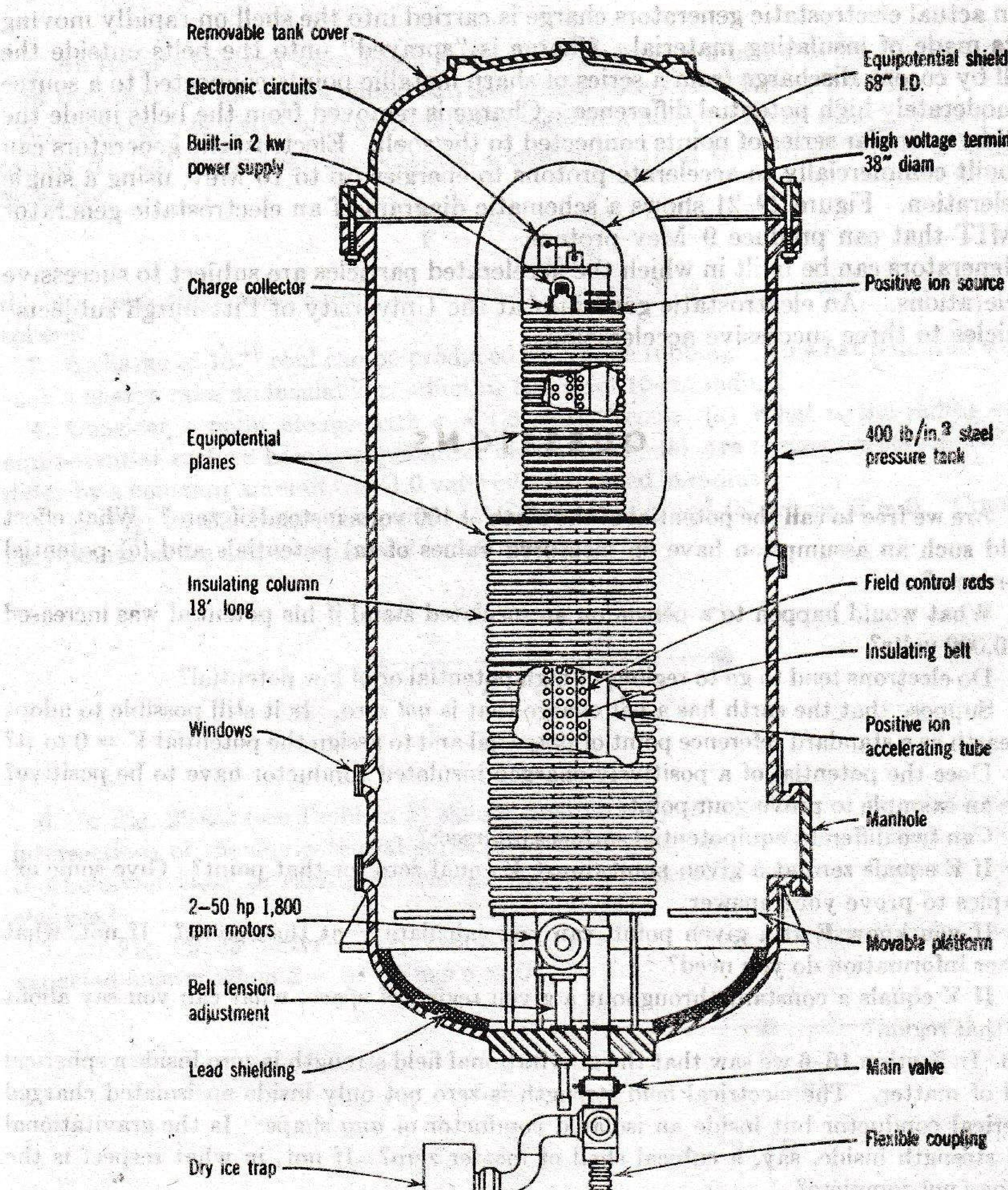


Fig. 29–21 An electrostatic generator at MIT capable of producing 9-Mev protons. The proton beam is accelerated vertically downward, being deflected into a horizontal plane by the analyzing magnet shown at the bottom. (Courtesy of J. G. Trump.)

From another point of view, we note that since the spheres when electrically connected form a single conductor at electrostatic equilibrium there can be only a single potential. This means that $V_r - V_R = 0$, which can occur only if $q = 0$.

In actual electrostatic generators charge is carried into the shell on rapidly moving belts made of insulating material. Charge is "sprayed" onto the belts outside the shell by corona discharge from a series of sharp metallic points connected to a source of moderately high potential difference. Charge is removed from the belts inside the shell by a similar series of points connected to the shell. Electrostatic generators can be built commercially to accelerate protons to energies up to 10 Mev, using a single acceleration. Figure 29–21 shows a schematic diagram of an electrostatic generator at MIT that can produce 9–Mev protons.

Generators can be built in which the accelerated particles are subject to successive accelerations. An electrostatic generator at the University of Pittsburgh subjects particles to three successive accelerations.

QUESTIONS

1. Are we free to call the potential of the earth +100 volts instead of zero? What effect would such an assumption have on measured values of (*a*) potentials and (*b*) potential differences?

2. What would happen to a person on an insulated stand if his potential was increased by 10,000 volts?

3. Do electrons tend to go to regions of high potential or of low potential?

4. Suppose that the earth has a net charge that is *not* zero. Is it still possible to adopt the earth as a standard reference point of potential and to assign the potential $V = 0$ to it?

5. Does the potential of a positively charged insulated conductor have to be positive? Give an example to prove your point.

6. Can two different equipotential surfaces intersect?

7. If **E** equals zero at a given point, must V equal zero for that point? Give some examples to prove your answer.

8. If you know **E** at a given point, can you calculate V at that point? If not, what further information do you need?

9. If V equals a constant throughout a given region of space, what can you say about **E** in that region?

10. In Section 16–6 we saw that the gravitational field strength is zero inside a spherical shell of matter. The electrical field strength is zero not only inside an isolated charged spherical conductor but inside an isolated conductor of *any* shape. Is the gravitational field strength inside, say, a cubical shell of matter zero? If not, in what respect is the analogy not complete?

11. How can you insure that the electric potential in a given region of space will have a constant value?

12. An isolated conducting spherical shell carries a negative charge. What will happen if a positively charged metal object is placed in contact with the shell interior? Assume that the positive charge is (*a*) less than, (*b*) equal to, and (*c*) greater than the negative charge in magnitude.

13. An uncharged metal sphere suspended by a silk thread is placed in a uniform external electric field **E**. What is the magnitude of the electric field for points inside the sphere? Is your answer changed if the sphere carries a charge?

14. A charge is placed on an insulated conductor in the form of a perfect cube. What will be the relative charge density at various points on the cube (surfaces, edges, corners); what will happen to the charge if the cube is in air?

PROBLEMS

1. An infinite charged sheet has a surface charge density σ of 1.0×10^{-7} coul/meter^2. How far apart are the equipotential surfaces whose potentials differ by 5.0 volts?

2. A charge q is distributed uniformly throughout a nonconducting spherical volume of radius R. (a) Show that the potential a distance a from the center, where $a < R$, is given by

$$V = \frac{q(3R^2 - a^2)}{8\pi\epsilon_0 R^3}.$$

(b) Is it reasonable that, according to this expression, V is not zero at the center of the sphere?

3. A charge of 10^{-8} coul can be produced by simple rubbing. To what potential would such a charge raise an insulated conducting sphere of 10-cm radius?

4. Consider a point charge with $q = 1.5 \times 10^{-8}$ coul. (a) What is the radius of an equipotential surface having a potential of 30 volts? (b) Are surfaces whose potentials differ by a constant amount (say 1.0 volt) evenly spaced in radius?

5. In Fig. 29–22, locate the points (a) where $V = 0$ and (b) where $\mathbf{E} = 0$. Consider only points on the axis and choose $d = 1.0$ meter.

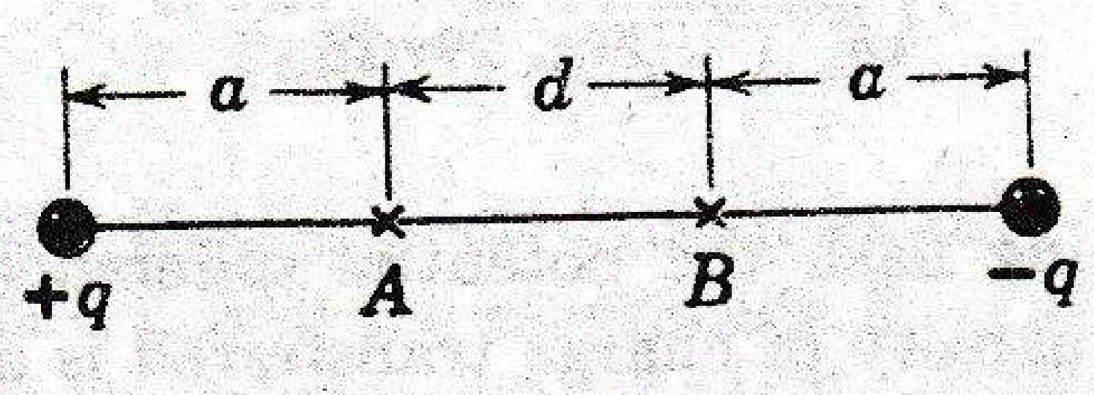


Fig. 29–22

6. In Fig. 29–22 (see Problem 5) sketch qualitatively (a) the lines of force and (b) the intersections of the equipotential surfaces with the plane of the figure. (Hint: Consider the behavior close to each point charge and at considerable distances from the pair of charges.)

7. In Fig. 29–23 derive an expression for $V_A - V_B$. Does your result reduce to the expected answer when $d = 0$? When $q = 0$?

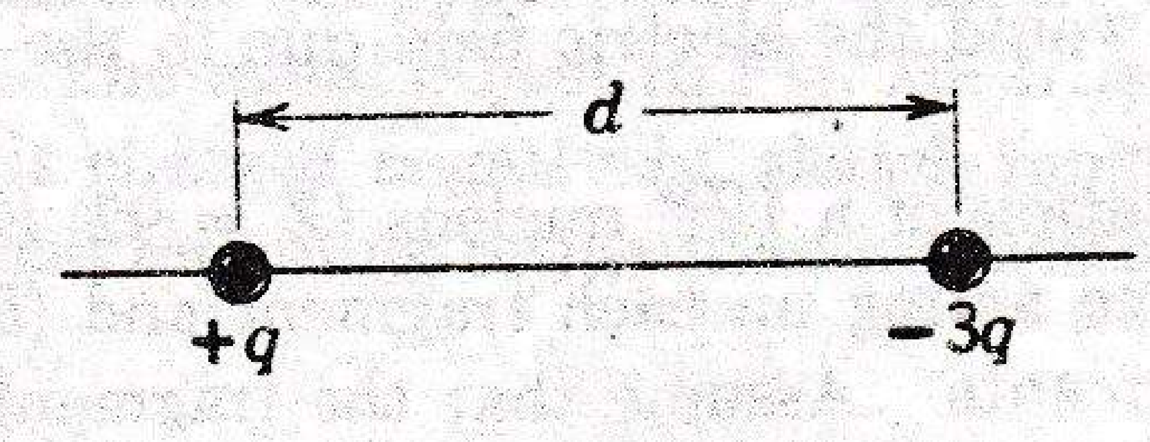


Fig. 29–23

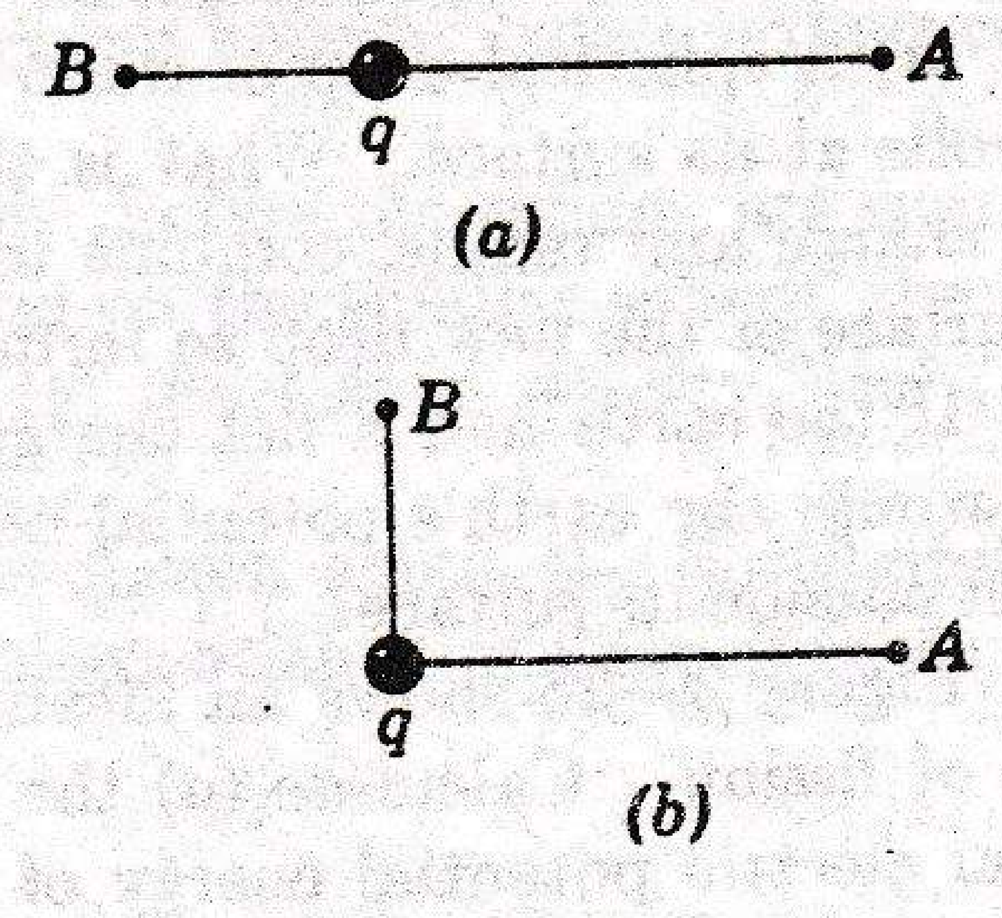


Fig. 29–24

8. A point charge has $q = +1.0 \times 10^{-6}$ coul. Consider point A which is 2.0 meters distant and point B which is 1.0 meter distant in a direction diametrically opposite, as in Fig. 29–24a. (a) What is the potential difference $V_A - V_B$? (b) Repeat if points A and B are located as in Fig. 29–24b.

9. Calculate the dipole moment of a water molecule under the assumption that all ten electrons in the molecule circulate symmetrically about the oxygen atom, that the OH distance is 0.96×10^{-8} cm, and that the angle between the two OH bonds is 104°. Compare with the value quoted on p. 719; see Fig. 29–11.

10. For the charge configuration of Fig. 29–25, show that $V(r)$ for points on the vertical axis, assuming $r \gg a$, is given by

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{2qa}{r^2}\right).$$

Is this an expected result? (Hint: The charge configuration can be viewed as the sum of an isolated charge and a dipole.)

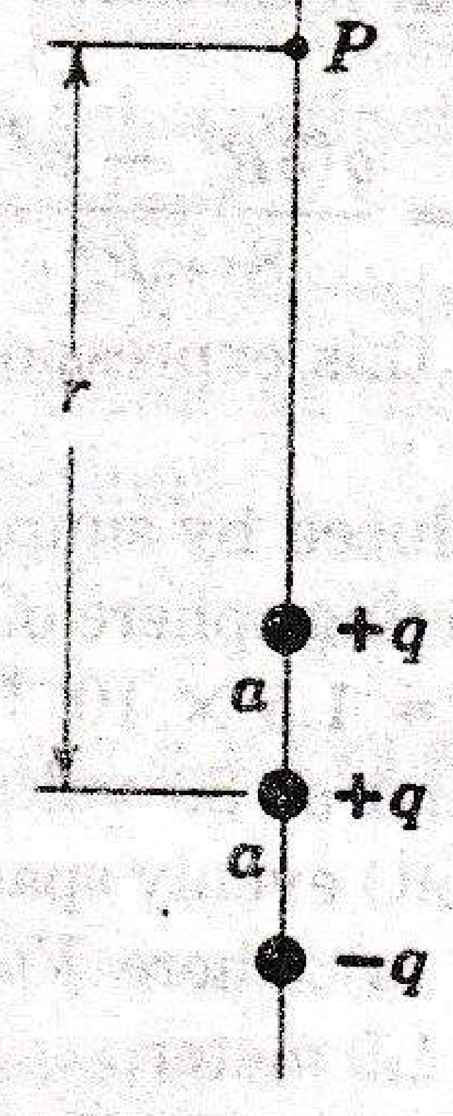


Fig. 29–25

11. In a typical lightning flash the potential difference between discharge points is about 10^9 volts and the quantity of charge transferred is about 30 coul. How much ice would it melt at 0°C if all the energy released could be used for this purpose?

12. Calculate (*a*) the electric potential established by the nucleus of a hydrogen atom at the mean distance of the circulating electron ($r = 5.3 \times 10^{-11}$ meter), (*b*) the electric potential energy of the atom when the electron is at this radius, and (*c*) the kinetic energy of the electron, assuming it to be moving in a circular orbit of this radius centered on the nucleus. (*d*) How much energy is required to ionize the hydrogen atom? Express all energies in electron volts.

13. What is the electric potential energy of the charge configuration of Fig. 29–7? Use the numerical values of Example 5.

14. (*a*) A spherical drop of water carrying a charge of 3×10^{-6} coul has a potential of 500 volts at its surface. What is the radius of the drop? (*b*) If two such drops, of the same charge and radius, combine to form a single spherical drop, what is the potential at the surface of the new drop so formed?

15. If the earth had a net charge equivalent to 1 electron/meter^2 of surface area, (*a*) what would the earth's potential be? (*b*) What would the electric field due to the earth be just outside its surface?

16. Figure 29–26 shows an idealized representation of a U^{238} nucleus ($Z = 92$) on the verge of fission. Calculate (*a*) the repulsive force acting on each fragment and (*b*) the mutual electric potential energy of the two fragments. Assume that the fragments are

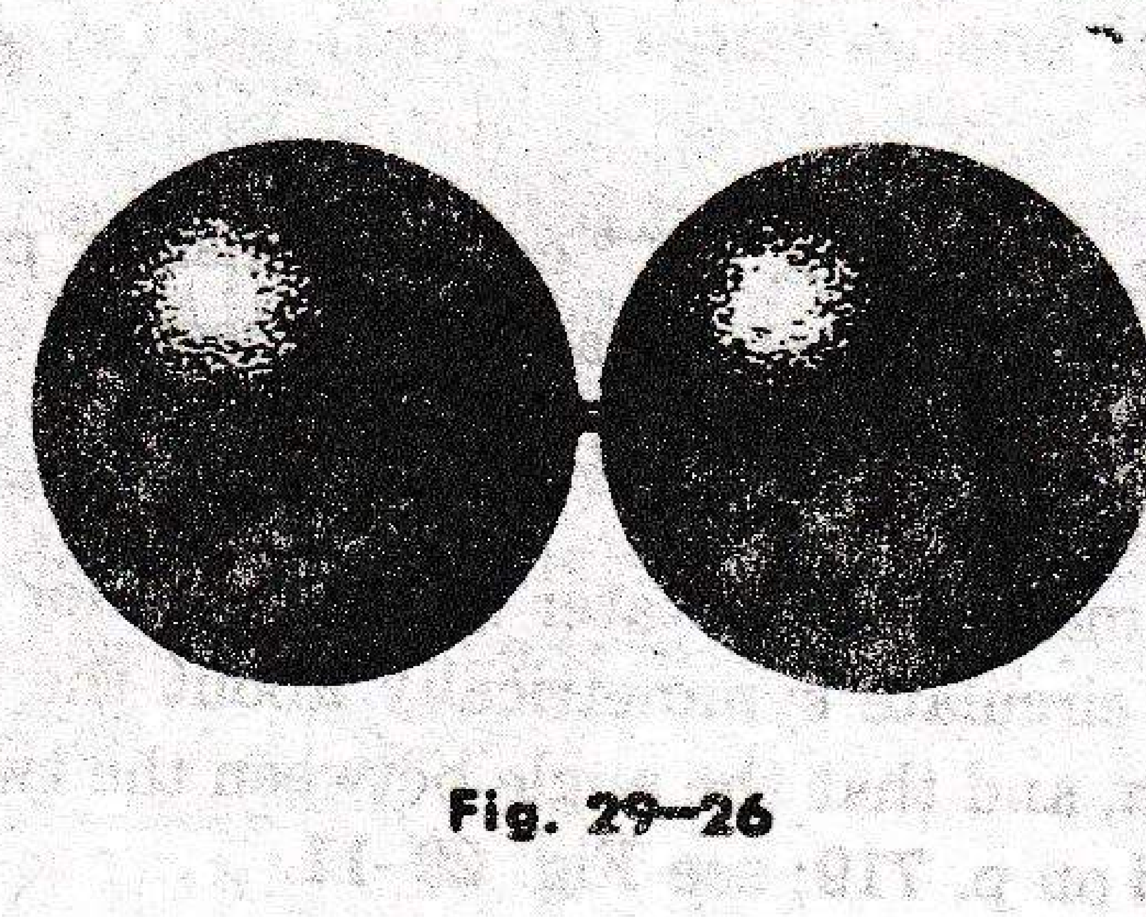


Fig. 29–26

equal in size and charge, spherical, and just touching. The radius of the initially spherical U^{238} nucleus is 8.0×10^{-15} meter. Assume that the material out of which nuclei are made has a constant density.

17. In the Millikan oil drop experiment (see Fig. 27-24) an electric field of 1.92×10^5 nt/coul is maintained at balance across two plates separated by 1.50 cm. Find the potential difference between the plates.

18. (*a*) Show that the electric potential at a point on the axis of a ring of charge of radius a, computed directly from Eq. 29-10, is given by

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2 + a^2}}.$$

(*b*) From this result derive an expression for E at axial points; compare with the direct calculation of E in Example 5, Chapter 27.

19. In Example 6 the potential at an axial point for a charged disk was shown to be

$$V = \frac{\sigma}{2\epsilon_0} (\sqrt{a^2 + r^2} - r).$$

From this result show that E for axial points is given by

$$E = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{r}{\sqrt{a^2 + r^2}}\right).$$

Does this expression for E reduce to an expected result for (*a*) $r \gg a$ and (*b*) for $r = 0$?

20. (*a*) Starting from Eq. 29-11, find the magnitude E_r of the radial component of the electric field due to a dipole. (*b*) For what values of θ is E_r zero?

21. Can a conducting sphere 10 cm in radius hold a charge of 4×10^{-6} coul in air without breakdown? The dielectric strength (minimum field required to produce breakdown) of air at 1 atm is 3×10^6 volts/meter.

22. A Geiger counter has a metal cylinder 2.0 cm in diameter along whose axis is stretched a wire 0.005 in. in diameter. If 850 volts are applied between them, what is the electric field strength at the surface of (*a*) the wire and (*b*) the cylinder?

23. Two metal spheres are 3.0 cm in radius and carry charges of $+1.0 \times 10^{-8}$ coul and -3.0×10^{-8} coul, respectively, assumed to be uniformly distributed. If their centers are 2.0 meters apart, calculate (*a*) the potential of the point halfway between their centers and (*b*) the potential of each sphere.

24. In Fig. 29-19 let $R_1 = 1.0$ cm and $R_2 = 2.0$ cm. Before the spheres are connected by the fine wire, a charge of 2.0×10^{-7} coul is placed on the smaller sphere, the larger sphere being uncharged. Calculate (*a*) the charge, (*b*) the charge density, and (*c*) the potential for each sphere after they are connected.

25. The metal object in Fig. 29-27 is a figure of revolution about the horizontal axis. If it is charged negatively, sketch roughly a few equipotentials and lines of force. Use physical reasoning rather than mathematical analysis.

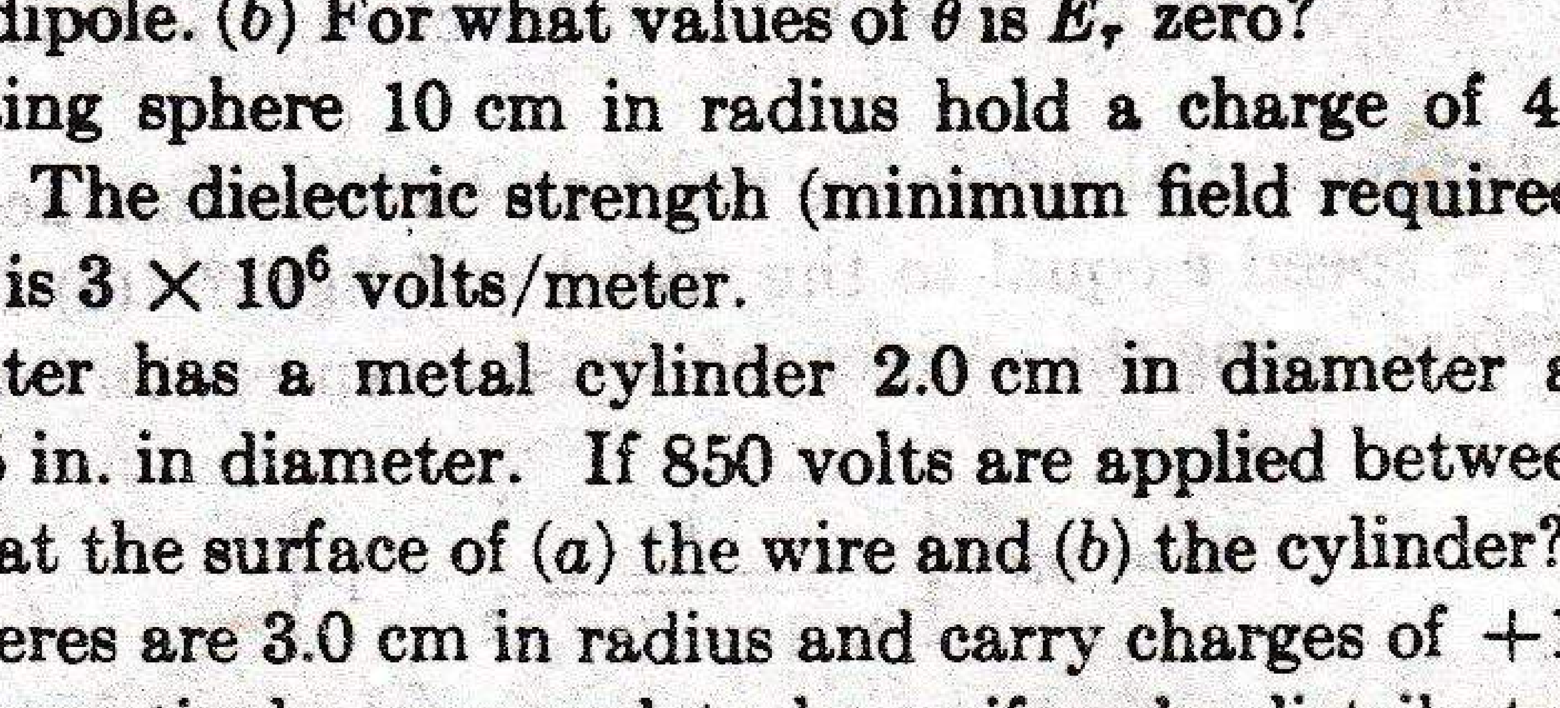


Fig. 29-27

26. Devise an arrangement of three point charges, separated by finite distances, that has zero electric potential energy.

27. Derive an expression for the work required to put the four charges together as indicated in Fig. 29–28.

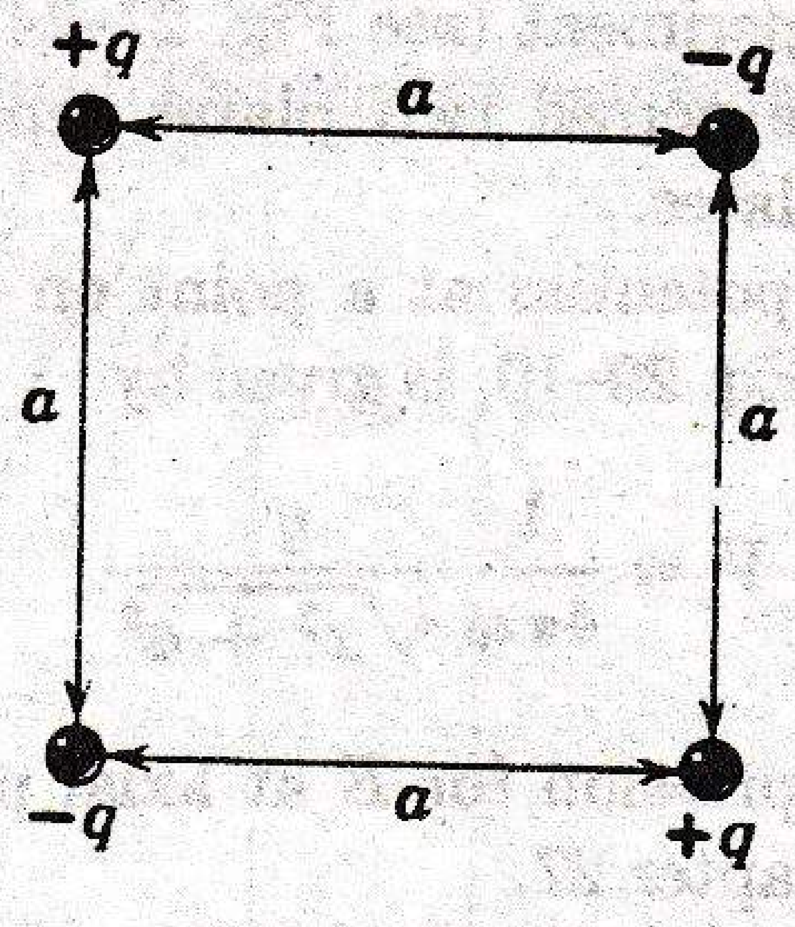


Fig. 29–28

28. A gold nucleus contains a positive charge equal to that of 79 protons. An α-particle ($Z = 2$) has a kinetic energy K at points far from this charge and is traveling directly toward the charge. The particle just touches the surface of the charge (assumed spherical) and is reversed in direction. Calculate K, assuming a nuclear radius of 5.0×10^{-15} meter. The actual α-particle energy used in the experiments of Rutherford and his collaborators was 5.0 Mev. What do you conclude?

29. What is the potential gradient, in volts/meter, at a distance of 10^{-12} meter from the center of the gold nucleus? What is the gradient at the nuclear surface?

30. For the spheres of Fig. 29–19, what is the ratio of electric field strengths at the surface?

31 (*a*) Through what potential difference must an electron fall, according to Newtonian mechanics, to acquire a speed v equal to the speed c of light? (*b*) Newtonian mechanics fails as $v \rightarrow c$. Therefore, using the correct relativistic expression for the kinetic energy

$$K = mc^2\left[\frac{1}{\sqrt{1-(v/c)^2}} - 1\right]$$

in place of the Newtonian expression $K = \frac{1}{2}mv^2$, determine the actual electron speed acquired in falling through the potential difference computed in (*a*). Express this speed as an appropriate fraction of the speed of light.

32. Two insulated concentric conducting spheres of radii R_1 and R_2 carry charges q_1 and q_2, respectively. Derive expressions for $E(r)$ and $V(r)$, where r is the distance from the center of the spheres. Plot $E(r)$ and $V(r)$ from $r = 0$ to $r = 4.0$ meters for $R_1 = 0.50$ meter, $R_2 = 1.0$ meter, $q_1 = +2.0 \times 10^{-6}$ coul, and $q_2 = +1.0 \times 10^{-6}$ coul. Compare with Fig. 29–18.

33. Let the potential difference between the shell of an electrostatic generator and the point at which charges are sprayed onto the moving belt be 3.0×10^6 volts. If the belt transfers charge to the shell at the rate of 3.0×10^{-3} coul/sec, what power must be provided to drive the belt, considering only electrical forces?

34. (*a*) How much charge is required to raise an isolated metallic sphere of 1.0-meter radius to a potential of 1.0×10^6 volts? Repeat for a sphere of 1.0-cm radius. (*b*) Why use a large sphere in an electrostatic generator since the same potential can be achieved for a smaller charge with a small sphere?

35. An alpha particle is accelerated through a potential difference of one million volts in an electrostatic generator. (*a*) What kinetic energy does it acquire? (*b*) What kinetic energy would a proton acquire under these same circumstances? (*c*) Which particle would acquire the greater speed, starting from rest?

Capacitors and Dielectrics

CHAPTER 30

30–1 Capacitance

In Section 29–3 we showed that the potential of a charged conducting sphere, assumed to be completely isolated with no other bodies (conducting or nonconducting) nearby, is given by

$$V_+' = \frac{1}{4\pi\epsilon_0}\frac{q}{R}, \tag{30–1}$$

in which q is the charge on the sphere and R is the sphere radius. The subscript on V indicates that we assume the charge to be positive. We represent this potential in Fig. 30–1 by the line marked V_+'. The line marked V_∞ in that figure represents the potential of an infinitely distant reference position; it has been assigned the value zero, following the usual convention.

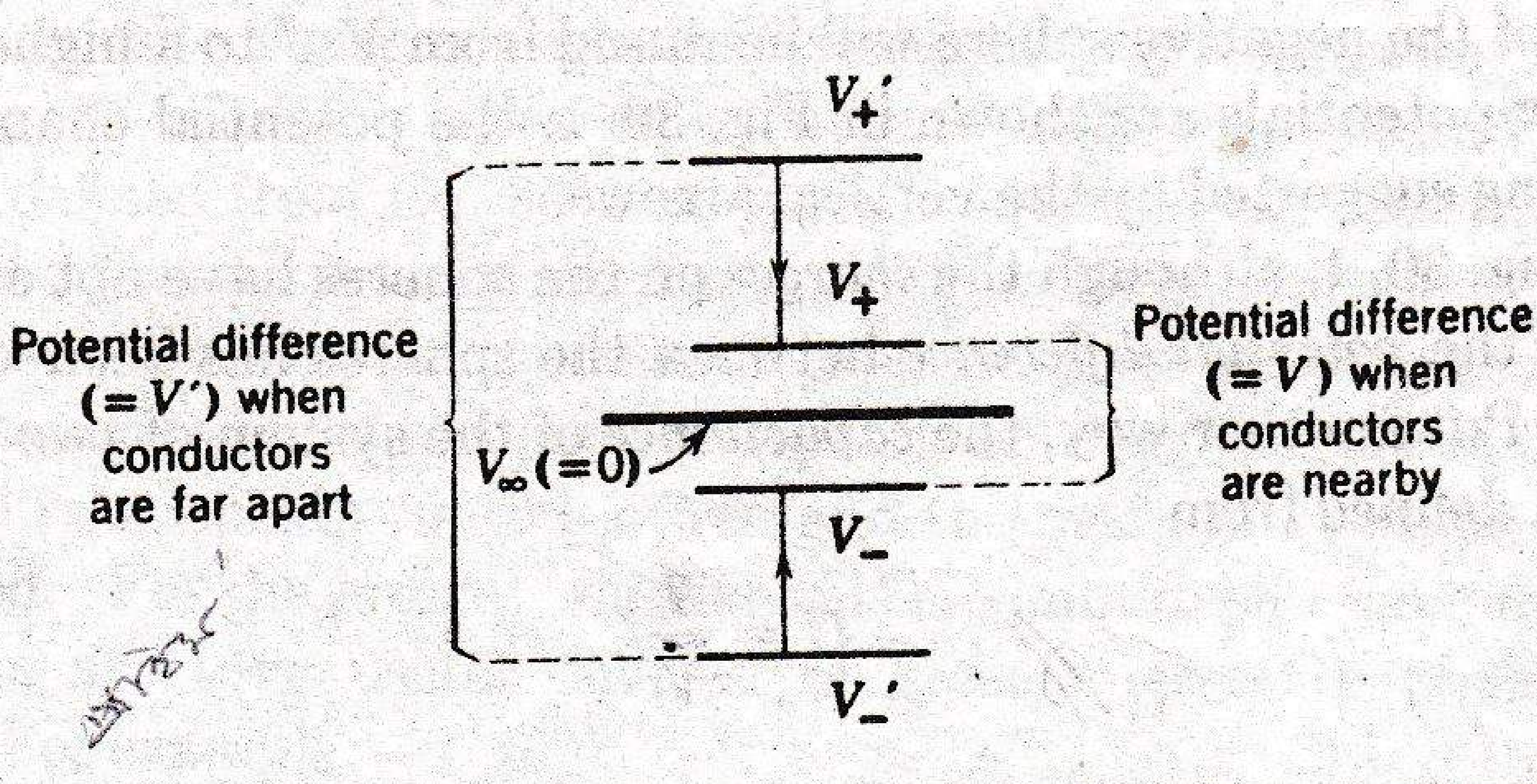


Fig. 30–1 The potential difference between two conductors that carry constant, equal, and opposite charges is reduced as the conductors are brought closer together.

Let us now imagine a second sphere of radius R, carrying a negative charge $-q$ and located a large distance ($\gg R$) from the first sphere so that each may still be considered to be electrically isolated. The potential of the second sphere is given by

$$V_-' = -\frac{1}{4\pi\epsilon_0}\frac{q}{R}, \tag{30-2}$$

and this quantity also is represented in Fig. 30–1.

The potential difference V' between the two spheres is

$$V' = V_+' - V_-' = \frac{1}{4\pi\epsilon_0}\frac{2q}{R}.$$

This shows that V', the potential difference, and q, the magnitude of the charge on either sphere, are proportional to each other. We may rewrite the equation as

$$q = (2\pi\epsilon_0 R)V' = C'V', \tag{30-3}$$

in which the proportionality constant in parentheses is called the *capacitance* of the two spheres and assigned the symbol C'.

Let us move the two spheres close together. The presence of each will now spoil the spherical symmetry of the lines of force emanating from the other. Lines from a given sphere, which, for large sphere separations, radiated uniformly in all directions to infinity, now terminate, in part, on the other sphere. Under these conditions Eqs. 30–1 and 30–2 no longer apply, since they were derived (see Section 29–3) on the assumption that spherical symmetry existed, permitting the useful application of Gauss's law.

A positive charge brought near to an isolated object serves to raise the potential of that object and a negative charge serves to lower it, as the student can see by considering the work required to move a positive test charge from infinity to points near such charges. Thus the potential of the positively charged sphere will be lowered by the presence nearby of the negatively charged sphere, from V_+' to some lower value V_+. Similarly the potential of the negative sphere will be raised from V_-' to a higher value V_-. These new potentials are shown in Fig. 30–1, the potential changes for each sphere being suggested by the vertical arrows.

From Fig. 30–1, although the *charges* on the spheres have not changed, it is clear that the *potential difference* between the spheres has been considerably reduced. Put another way, the *capacitance* of the system of two spheres (see Eq. 30–3), defined from

$$C = \frac{q}{V}, \tag{30-4}$$

has been made considerably larger than its initial value C' by bringing the spheres closer together.

It is also possible to use Eq. 30–4 to define the capacitance of a single isolated conductor such as a sphere. In such cases one may imagine that the second "plate," carrying an equal and opposite charge, is a conducting sphere of very large—essentially infinite—radius centered about the conductor. The potential of this infinitely distant sphere, according to the usual convention for potential measurements, is zero. The capacitance of an isolated sphere of radius R is given from Eqs. 30–4 and 30–1 as

$$C = \frac{q}{V} = 4\pi\epsilon_0 R.$$

The mks unit of capacitance that follows from Eq. 30–4 is the coul/volt. A special unit, the *farad*, is used to represent it. It is named in honor of Michael Faraday who, among other contributions, developed the concept of capacitance. Thus

$$1 \text{ farad} = 1 \text{ coul/volt}.$$

The submultiples of the farad, the *microfarad* ($1\ \mu f = 10^{-6}$ farad) and the *micromicrofarad* ($1\ \mu\mu f = 10^{-12}$ farad), are more convenient units in practice.

An analogy can be made between a capacitor carrying a charge q and a rigid container of volume $\mathcal{V}$ containing μ moles of an ideal gas.

The gas pressure p is directly proportional to μ for a fixed temperature, according to the ideal gas law (Eq. 23–2)

$$\mu = \left(\frac{\mathcal{V}}{RT}\right) p.$$

For the capacitor (Eq. 30–4)

$$q = (C)V.$$

Comparison shows that the capacitance C of the capacitor, assuming a fixed temperature, is analogous to the volume $\mathcal{V}$ of the container.

Note that any amount of charge can be put on the capacitor, and any mass of gas can be put in the container, up to certain limits. These correspond to electrical breakdown ("arcing over") for the capacitor and to rupture of the walls for the container.

Figure 30–2 shows a more general case of two nearby conductors, which are now permitted to be of any shape, carrying equal and opposite charges. Such an arrangement is called a *capacitor*, the conductors being called *plates*. The equal and opposite charges might be established by connecting the plates momentarily to opposite poles of a battery. The capacitance C of any capacitor is defined from Eq. 30–4 in which we remind the student that V is the *potential difference between the plates* and q is the magnitude of the *charge on either plate;* q must not be taken as the net charge of the capacitor, which is zero. The capacitance of a capacitor depends on the geometry of each plate, their spatial relationship to each other, and the medium in which the plates are immersed. For the present, we take this medium to be a vacuum.

Capacitors are very useful devices, of great interest to physicists and engineers. For example:

1. In this book we stress the importance of *fields* to the understanding of natural phenomena. A capacitor can be used to establish desired electric field configurations for various purposes. In Section 27–5 we described the deflection of an electron beam in a uniform field set up by a capacitor, al-

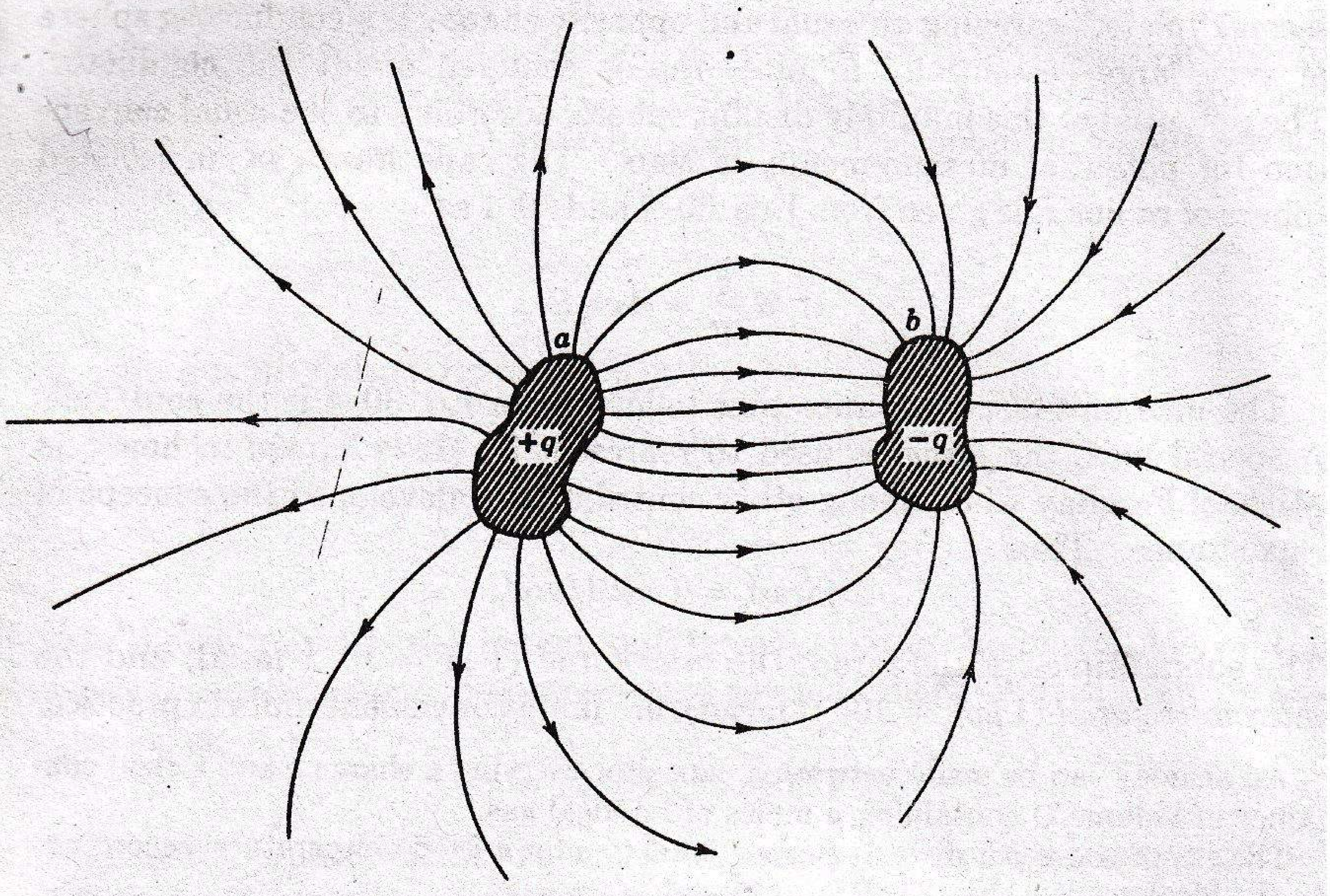


Fig. 30-2 Two insulated conductors carrying equal and opposite charges form a capacitor.

though we did not use this term in that section. In later sections we discuss the behavior of dielectric materials when placed in an electric field (provided conveniently by a capacitor) and we shall see how the laws of electromagnetism can be generalized to take the presence of dielectric bodies more readily into account.

2. A second important concept stressed in this book is *energy*. By analyzing a charged capacitor we show that electric energy may be considered to be stored in the electric field between the plates and indeed in any electric field, however generated. Because capacitors can confine strong electric fields to small volumes, they can serve as useful devices for storing energy. In many electron synchrotrons, which are cyclotron-like devices for accelerating electrons, energy accumulated and stored in a large bank of capacitors over a relatively long period of time is made available intermittently to accelerate the electrons by discharging the capacitor in a much shorter time. Many researches and devices in plasma physics also make use of bursts of energy stored in this way.

3. The electronic age could not exist without capacitors. They are used, in conjunction with other devices, to reduce voltage fluctuations in electronic power supplies, to transmit pulsed signals, to generate or detect electromagnetic oscillations at radio frequencies, and to provide time delays. In most of these applications the potential difference between the plates will not be constant, as we assume in this chapter, but will vary with time, often in a sinusoidal or a pulsed fashion. In later chapters we consider some aspects of the capacitor used as a circuit element.

30-2 Calculating Capacitance

Figure 30-3 shows a *parallel-plate* capacitor formed of two parallel conducting plates of area A separated by a distance d. If we connect each plate to the terminal of a battery, a charge $+q$ will appear on one plate and a charge $-q$ on the other. If d is small compared with the plate dimensions, the electric field strength **E** between the plates will be uniform, which means that the lines of force will be parallel and evenly spaced. The laws of electromagnetism (see Problem 20, Chapter 35) require that there be some "fringing" of the lines at the edges of the plates, but for small enough d it can be neglected for our present purpose.

We can calculate the capacitance of this device using Gauss's law. Figure 30-3 shows (dashed lines) a Gaussian surface of height h closed by plane caps of area A that are the shape and size of the capacitor plates. The flux of **E** is zero for the part of the Gaussian surface that lies inside the top capacitor plate because the electric field inside a conductor carrying a static charge is zero. The flux of **E** through the wall of the Gaussian surface is zero because, to the extent that the fringing of the lines of force can be neglected, **E** lies in the wall.

This leaves only the face of the Gaussian surface that lies between the plates. Here **E** is constant and the flux Φ_E is simply EA. Gauss's law gives

$$\epsilon_0 \Phi_E = \epsilon_0 EA = q. \tag{30-5}$$

The work required to carry a test charge q_0 from one plate to the other can be expressed either as $q_0 V$ (see Eq. 29-1) or as the product of a force $q_0 E$ times a distance d or $q_0 Ed$. These expressions must be equal, or

$$V = Ed. \tag{30-6}$$

More formally, Eq. 30-6 is a special case of the general relation (Eq. 29-5; see also Example 1, Chapter 29)

$$V = -\int \mathbf{E} \cdot d\mathbf{l},$$

where V is the difference in potential between the plates. The integral may

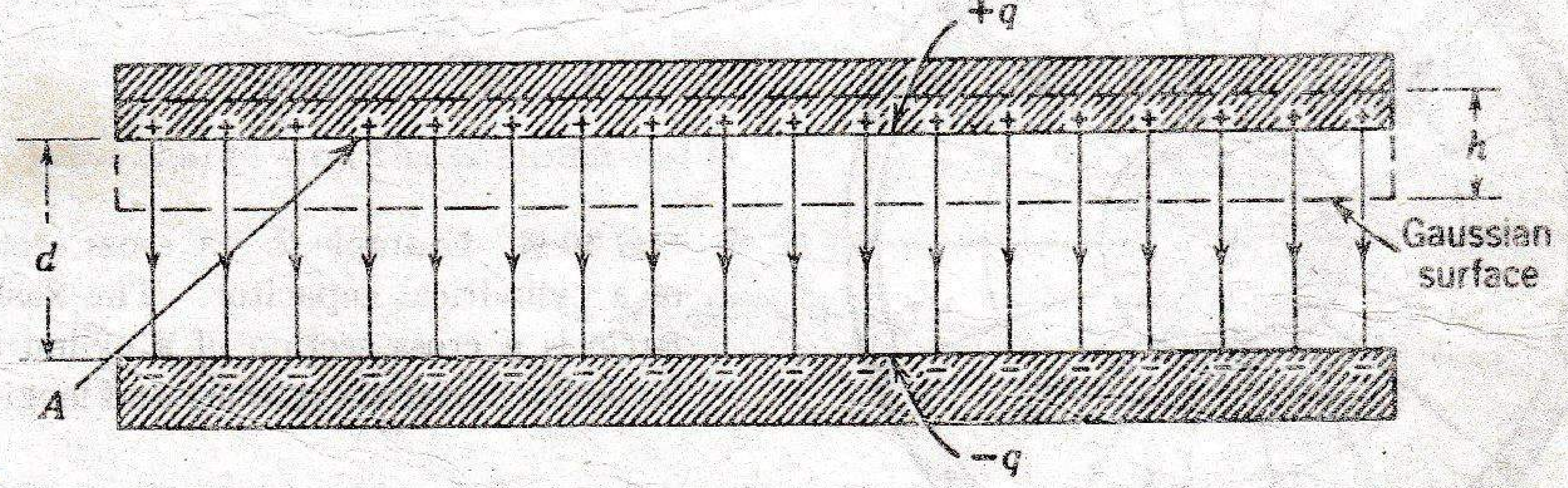


Fig. 30-3 A parallel-plate capacitor with plates of area A. The dashed line represents a Gaussian surface whose height is h and whose top and bottom caps are the same shape and size as the capacitor plates.

be taken over any path that starts on one plate and ends on the other because each plate is an equipotential surface and the electrostatic force is path independent. Although the simplest path between the plates is a perpendicular straight line, Eq. 30-6 follows no matter what path of integration we choose.

If we substitute Eqs. 30-5 and 30-6 into the relation $C = q/V$, we obtain

$$C = \frac{q}{V} = \frac{\epsilon_0 EA}{Ed} = \frac{\epsilon_0 A}{d}. \tag{30-7}$$

Equation 30-7 holds only for capacitors of the parallel-plate type; different formulas hold for capacitors of different geometry.

In Section 26-4 we stated that ϵ_0, which we first met in connection with Coulomb's law, was not measured in terms of that law because of experimental difficulties. Equation 30-7 suggests that ϵ_0 might be measured by building a capacitor of accurately known plate area and plate spacing and determining its capacitance experimentally by measuring q and V in the relation $C = q/V$. Thus Eq. 30-7 can be solved for ϵ_0 and a numerical value found in terms of the measured quantities A, d, and C; ϵ_0 has been measured accurately in this way.

Example 1. The parallel plates of an air-filled capacitor are everywhere 1.0 mm apart. What must the plate area be if the capacitance is to be 1.0 farad?

From Eq. 30-7

$$A = \frac{dC}{\epsilon_0} = \frac{(1.0 \times 10^{-3}\ \text{meter})(1.0\ \text{farad})}{8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2} = 1.1 \times 10^8\ \text{meter}^2.$$

This is the area of a square sheet more than 6 miles on edge; the farad is indeed a large unit.

Example 2. *A cylindrical capacitor.* A cylindrical capacitor consists of two coaxial cylinders (Fig. 30-4) of radius a and b and length l. What is the capacitance of this device? Assume that the capacitor is very long (that is, that $l \gg b$) so that fringing of the lines of force at the ends can be ignored for the purpose of calculating the capacitance.

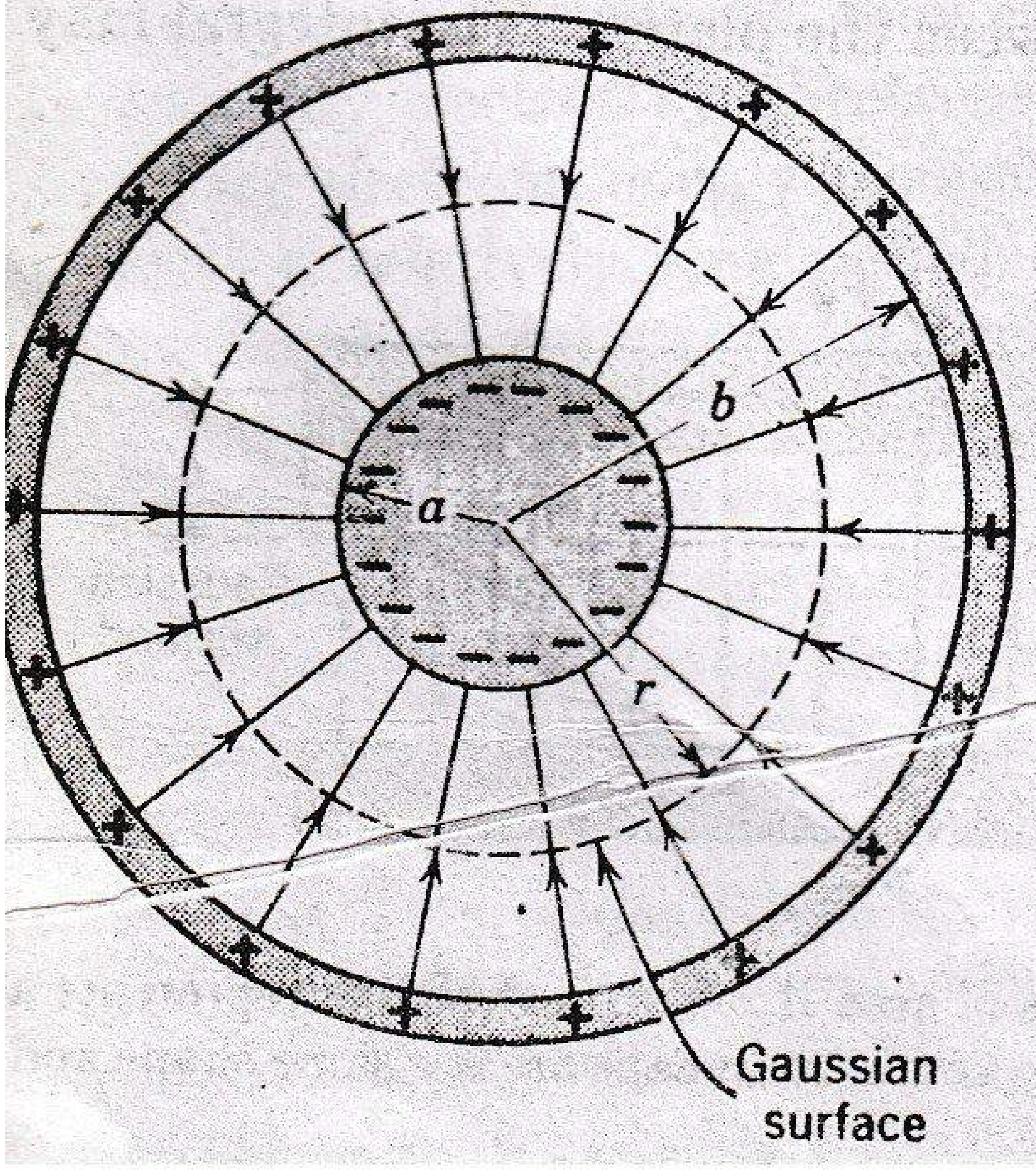


Fig. 30-4 Example 2. A cross section of a cylindrical capacitor. The dashed circle is a cross section of a cylindrical Gaussian surface of radius r and length l.

As a Gaussian surface construct a coaxial cylinder of radius r and length l, closed by plane caps. Gauss's law

$$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = q$$

gives

$$\epsilon_0 E(2\pi r)(l) = q,$$

the flux being entirely through the cylindrical surface and not through the end caps. Solving for E yields

$$E = \frac{q}{2\pi\epsilon_0 r l}.$$

The potential difference between the plates is given by Eq. 29–5 [note that **E** and $d\mathbf{l}$ ($= d\mathbf{r}$) point in opposite directions] or

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b E\, dr = \int_a^b \frac{q}{2\pi\epsilon_0 l} \frac{dr}{r} = \frac{q}{2\pi\epsilon_0 l} \ln \frac{b}{a}.$$

Finally, the capacitance is given by

$$C = \frac{q}{V} = \frac{2\pi\epsilon_0 l}{\ln(b/a)}.$$

Like the relation for the parallel-plate capacitor (Eq. 30–7), this relation also depends only on geometrical factors.

Example 3. *Capacitors in parallel.* Figure 30–5 shows three capacitors connected in parallel. What single capacitance C is equivalent to this combination? "Equivalent" means that if the parallel combination and the single capacitance were each in a box with wires a and b connected to terminals, it would not be possible to distinguish the two by electrical measurements external to the box.

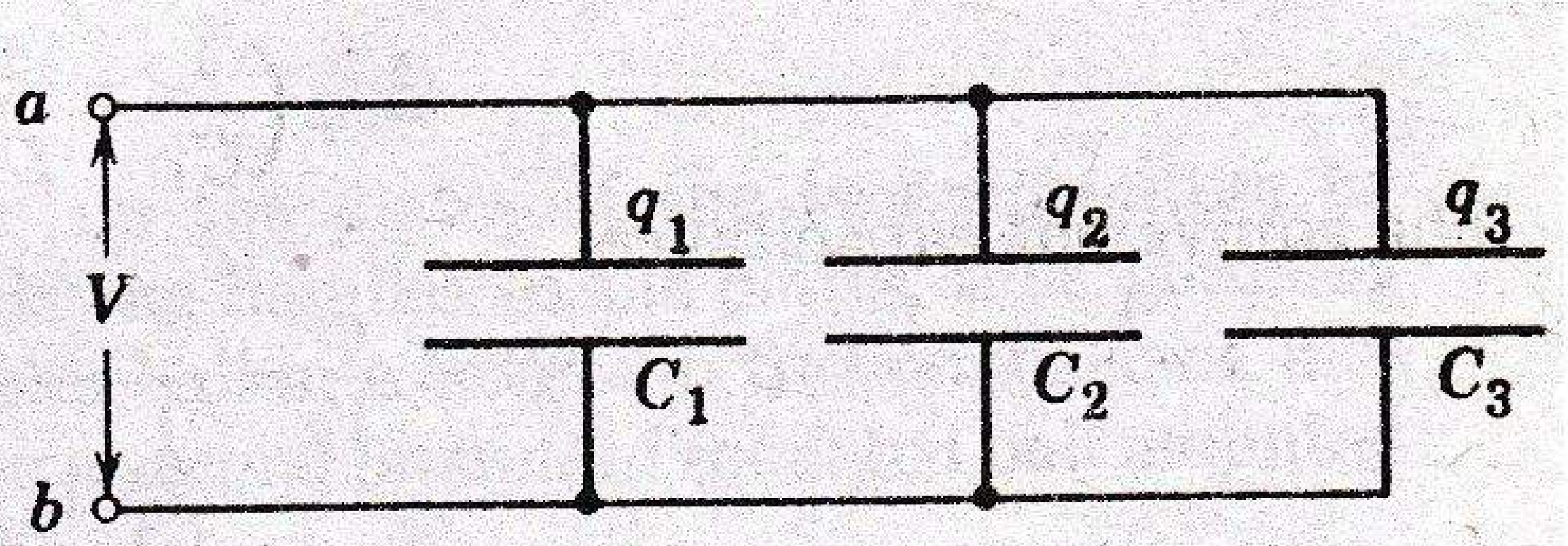


Fig. 30–5 Example 3. Three capacitors in parallel.

The potential difference across each capacitor in Fig. 30–5 will be the same. This follows because all of the upper plates are connected together and to terminal a, whereas all of the lower plates are connected together and to terminal b. Applying the relation $q = CV$ to each capacitor yields

$$q_1 = C_1 V; \quad q_2 = C_2 V; \quad \text{and} \quad q_3 = C_3 V.$$

The total charge q on the combination is

$$q = q_1 + q_2 + q_3$$
$$= (C_1 + C_2 + C_3)V.$$

The equivalent capacitance C is

$$C = \frac{q}{V} = C_1 + C_2 + C_3.$$

This result can easily be extended to any number of parallel-connected capacitors.

Example 4. *Capacitors in series.* Figure 30–6 shows three capacitors connected in series. What single capacitance C is "equivalent" (see Example 3) to this combination?

For capacitors connected as shown, the magnitude q of the charge on each plate must be the same. This is true because the net charge on the part of the circuit enclosed by the dashed line in Fig. 30–6 must be zero; that is, the charge present on these plates initially is zero and connecting a battery between a and b will only produce a charge separation, the *net* charge on these plates still being zero. Assuming that neither C_1 nor C_2 "sparks over," there is no way for charge to enter or leave the region enclosed by the dashed line.

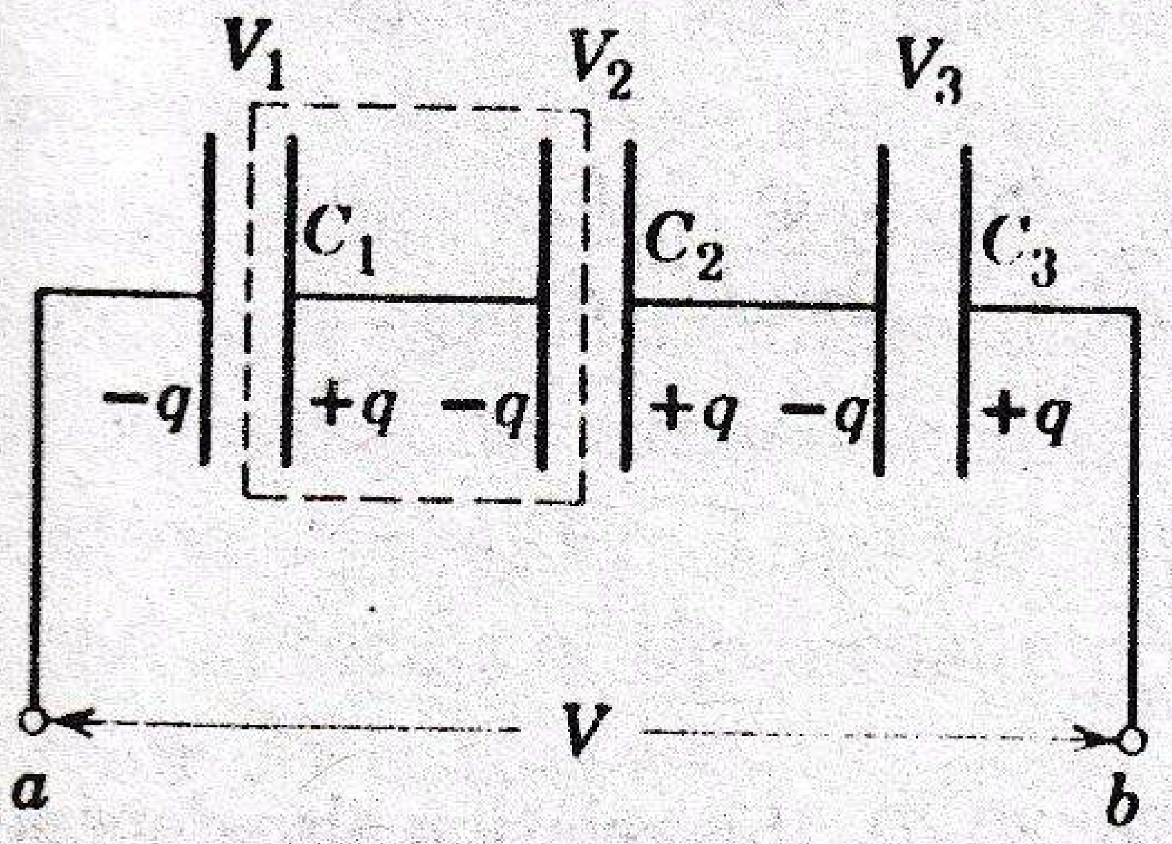


Fig. 30–6 Example 4. Three capacitors in series.

Applying the relation $q = CV$ to each capacitor yields

$$V_1 = q/C_1; \qquad V_2 = q/C_2; \qquad \text{and} \qquad V_3 = q/C_3.$$

The potential difference for the series combination is

$$V = V_1 + V_2 + V_3$$
$$= q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right).$$

The equivalent capacitance

$$C = \frac{q}{V} = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3}},$$

or

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

The equivalent series capacitance is always less than the smallest capacitance in the chain. ◀

30–3 Parallel-Plate Capacitor with Dielectric

Equation 30–7 holds only for a parallel-plate capacitor with its plates in a vacuum. Michael Faraday, in 1837, first investigated the effect of filling the space between the plates with a dielectric, say mica or oil. In Faraday's words:

> The question may be stated thus: suppose A an electrified plate of metal suspended in air, and B and C two exactly similar plates, placed parallel to and on each side of A at equal distances and insulated; A will then induce equally toward B and C [that is, equal charges will appear on these plates]. If in this position of the plates some other dielectric than air, as shell-lac, be introduced between A and

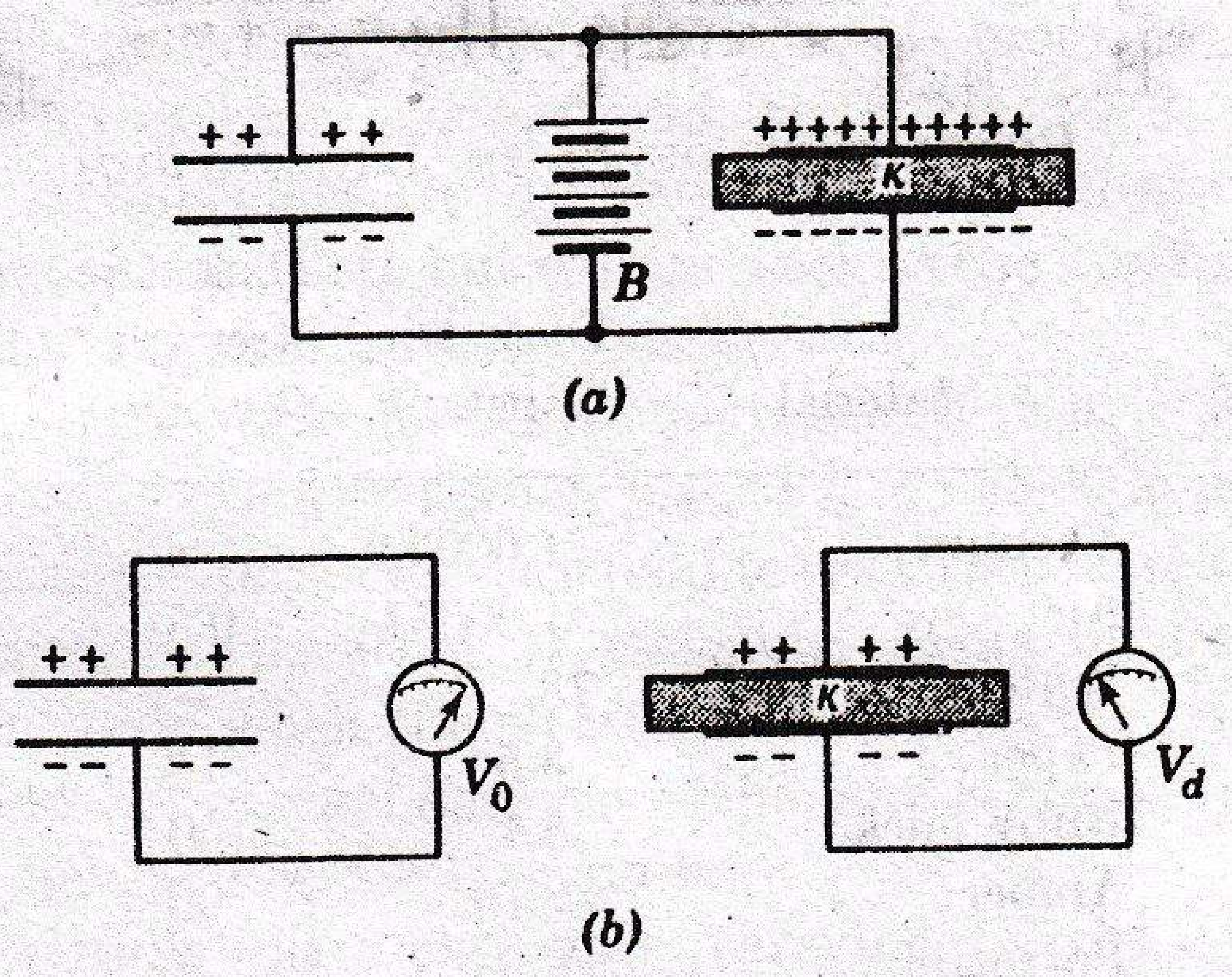


Fig. 30–7 (*a*) Battery *B* supplies the same potential difference to each capacitor; the one on the right has the higher charge. (*b*) Both capacitors carry the same charge; the one on the right has the lower potential difference, as indicated by the meter readings.

> *C*, will the induction between them remain the same? Will the relation of *C* and *B* to *A* be unaltered, notwithstanding the difference of the dielectrics interposed between them?

Faraday answered this question by constructing two identical capacitors, in one of which he placed a dielectric, the other containing air at normal pressure. When both capacitors were charged to the *same potential difference*, Faraday found by experiment that *the charge on the one containing the dielectric was greater than that on the other*; see Fig. 30–7*a*.

Faraday measured the relative charges on the plates of the two capacitors by touching a metal ball (fitted with an insulating handle) to the plates, thus sampling the charge quantitatively. He then put this ball in a Coulomb torsion balance and measured the force of Coulomb repulsion on a second (standard) charged ball mounted on the balance arm.

Since q is larger, for the same V, if a dielectric is present, it follows from the relation $C = q/V$ that *the capacitance of a capacitor increases if a dielectric is placed between the plates*. The ratio of the capacitance with the dielectric * to that without is called the *dielectric constant* κ of the material; see Table 30–1.

Instead of maintaining the two capacitors at the same potential difference, we can place the *same charge* on them, as in Fig. 30–7*b*. Experiment then shows that the potential difference V_d between the plates of the right-hand capacitor is smaller than that for the left-hand capacitor by the factor $1/\kappa$, or

$$V_d = V_0/\kappa.$$

We are led once again to conclude, from the relation $C = q/V$, that the effect of the dielectric is to increase the capacitance by a factor κ.

* Assumed to fill completely the space between the plates.

Table 30–1

PROPERTIES OF SOME DIELECTRICS

Material	Dielectric Constant	Dielectric Strength * (kv/mm)
Vacuum	1.00000	∞
Air	1.00054	0.8
Water	78	—
Paper	3.5	14
Ruby mica	5.4	160
Amber	2.7	90
Porcelain	6.5	4
Fused quartz	3.8	8
Pyrex glass	4.5	13
Bakelite	4.8	12
Polyethylene	2.3	50
Polystyrene	2.6	25
Teflon	2.1	60
Neoprene	6.9	12
Pyranol oil	4.5	12
Titanium dioxide	100	6

* This is the maximum potential gradient that may exist in the dielectric without the occurrence of electrical breakdown. Dielectrics are often placed between conducting plates to permit a higher potential difference to be applied between them than would be possible with air as the dielectric.

For a parallel-plate capacitor we can write, as an experimental result,

$$C = \frac{\kappa \epsilon_0 A}{d}. \tag{30–8}$$

Equation 30–7 is a special case of this relation found by putting $\kappa = 1$, corresponding to a vacuum between the plates. Experiment shows that the capacitance of *all* types of capacitor is increased by the factor κ if the space between the plates is filled with a dielectric. Thus the capacitance of any capacitor can be written as

$$C = \kappa \epsilon_0 L,$$

where L depends on the geometry and has the dimensions of a length. For a parallel-plate capacitor (see Eq. 30–7) L is A/d; for a cylindrical capacitor (see Example 2) it is $2\pi l/\ln{(b/a)}$.

30–4 Dielectrics—An Atomic View

We now seek to understand, in atomic terms, what happens when a dielectric is placed in an electric field. There are two possibilities. The molecules of some dielectrics, like water, have permanent electric dipole moments.

In such materials (called *polar*) the electric dipole moments $\mathbf{p}$ tend to align themselves with an external electric field, as in Fig. 30-8; see also Section 27-6. Because the molecules are in constant thermal agitation, the degree of alignment will not be complete but will increase as the applied electric field is increased or as the temperature is decreased.

Whether or not the molecules have permanent electric dipole moments, they acquire them by *induction* when placed in an electric field. In Section 29-5 we saw that the external electric field tends to separate the negative and the positive charge in the atom or molecule. This *induced electric dipole moment* is present only when the electric field is present. It is proportional to the electric field (for normal field strengths) and is created already lined up with the electric field as Fig. 29-12 suggests.

Let us use a parallel-plate capacitor, carrying a fixed charge q and not connected to a battery (see Fig. 30-7*b*), to provide a uniform external electric field $\mathbf{E}_0$ into which we place a dielectric slab. The over-all effect of alignment and induction is to separate the center of positive charge of the entire slab slightly from the center of negative charge. The slab, as a whole, although remaining electrically neutral, becomes *polarized*, as Fig. 30-9*b* suggests. The net effect is a pile-up of positive charge on the right face of the slab and of negative charge on the left face; within the slab no excess charge appears in any given volume element. Since the slab as a whole remains neutral, the positive *induced surface charge* must be equal in magnitude to the negative induced surface charge. Note that in this process electrons in the dielectric

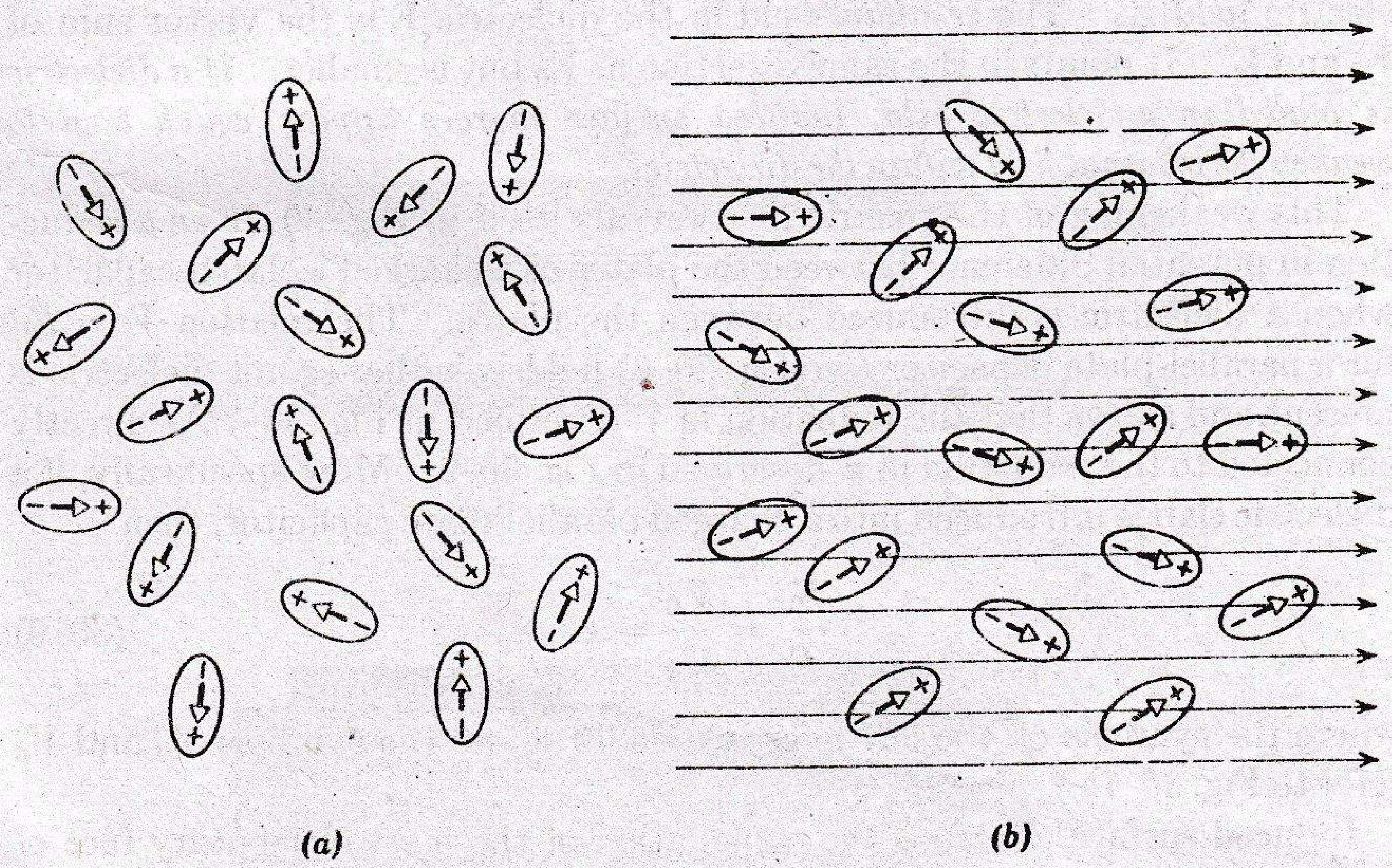


Fig. 30-8 (*a*) Molecules with a permanent electric dipole moment, showing their random orientation in the absence of an external electric field. (*b*) An electric field is applied, producing partial alignment of the dipoles. Thermal agitation prevents complete alignment.

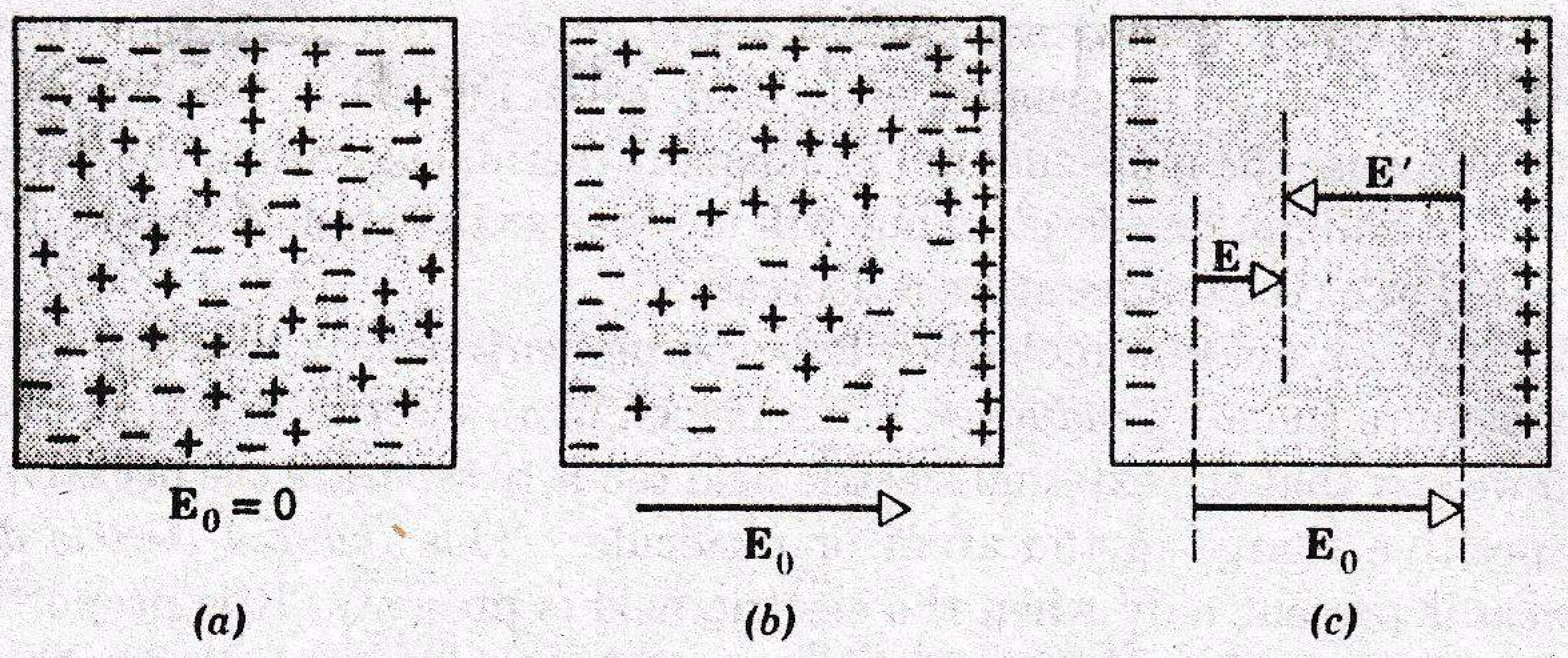


Fig. 30–9 (*a*) A dielectric slab, showing the random distribution of plus and minus charges. (*b*) An external field $\mathbf{E}_0$, established by putting the slab between the plates of a parallel-plate capacitor (not shown), separates the center of plus charge in the slab slightly from the center of minus charge, resulting in the appearance of surface charges. No *net* charge exists in any volume element located in the *interior* of the slab. (*c*) The surface charges set up a field $\mathbf{E}'$ which opposes the external field $\mathbf{E}_0$ associated with the charges on the capacitor plates. The resultant field $\mathbf{E}$ ($= \mathbf{E}_0 + \mathbf{E}'$) in the dielectric is thus less than $\mathbf{E}_0$.

are displaced from their equilibrium positions by distances that are considerably less than an atomic diameter. There is no transfer of charge over macroscopic distances such as occurs when a current is set up in a conductor.

Figure 30–9*c* shows that the induced surface charges will always appear in such a way that the electric field set up *by them* ($\mathbf{E}'$) opposes the external electric field $\mathbf{E}_0$. The *resultant* field in the dielectric $\mathbf{E}$ is the vector sum of $\mathbf{E}_0$ and $\mathbf{E}'$. It points in the same direction as $\mathbf{E}_0$ but is smaller. *If a dielectric is placed in an electric field, induced surface charges appear which tend to weaken the original field within the dielectric.*

This weakening of the electric field reveals itself in Fig. 30–7*b* as a reduction in potential difference between the plates of a charged isolated capacitor when a dielectric is introduced between the plates. The relation $V = Ed$ for a parallel-plate capacitor (see Eq. 30–6) holds whether or not dielectric is present and shows that the reduction in V described in Fig. 30–7*b* is directly connected to the reduction in E described in Fig. 30–9. More specifically, if a dielectric slab is introduced into a charged parallel-plate capacitor, then

$$\frac{E_0}{E} = \frac{V_0}{V_d} = \kappa \qquad (30\text{–}9)$$

where the symbols on the left refer to Fig. 30–9 and the symbols V_0 and V_d refer to Fig. 30–7*b*.*

Induced surface charge is the explanation of the most elementary fact of static electricity, namely, that a charged rod will attract uncharged bits of

* Equation 30–9 does not hold if the battery remains connected while the dielectric slab is introduced. In this case V (hence E) could not change. Instead, the charge q on the capacitor plates would increase by a factor κ, as Fig. 30–7*a* suggests.

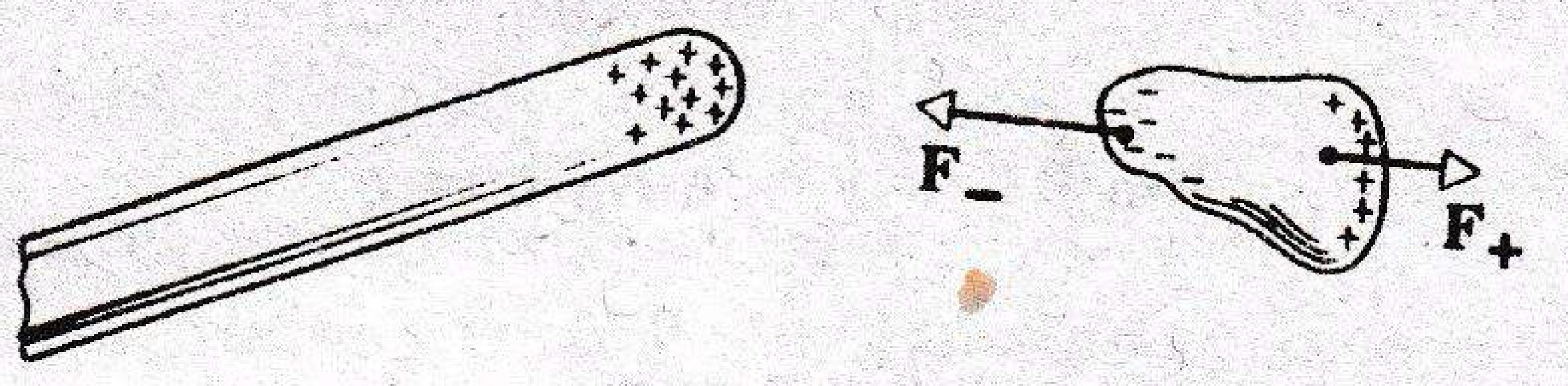


Fig. 30–10 A charged rod attracts an uncharged piece of paper because unbalanced forces act on the induced surface charges.

paper, etc. Figure 30–10 shows a bit of paper in the field of a charged rod. Surface charges appear on the paper as shown. The negatively charged end of the paper will be pulled toward the rod and the positively charged end will be repelled. These two forces do not have the same magnitude because the negative end, being closer to the rod, is in a stronger field and experiences a stronger force. The net effect is an attraction. A dielectric body in a *uniform* electric field will not experience a net force.

30–5 Dielectrics and Gauss's Law

So far our use of Gauss's law has been confined to situations in which no dielectric was present. Now let us apply this law to a parallel-plate capacitor filled with a dielectric of dielectric constant κ.

Figure 30–11 shows the capacitor both with and without the dielectric. It is assumed that the charge q on the plates is the same in each case. Gaussian surfaces have been drawn after the fashion of Fig. 30–3.

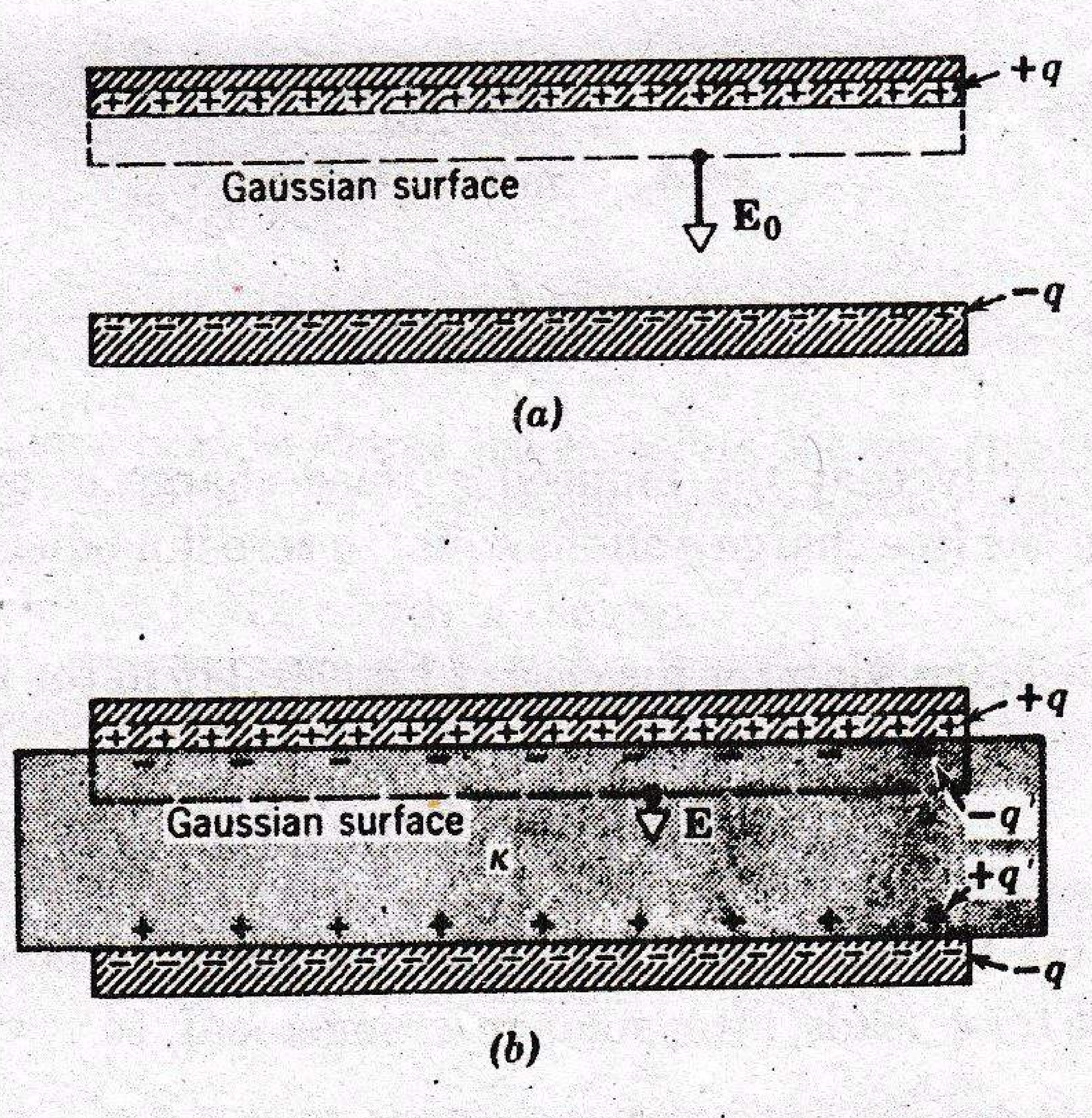


Fig. 30–11 A parallel-plate capacitor (*a*) without and (*b*) with a dielectric. The charge q on the plates is assumed to be the same in each case.

If no dielectric is present (Fig. 30–11*a*), Gauss's law (see Eq. 30–5) gives

$$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 E_0 A = q$$

or

$$E_0 = \frac{q}{\epsilon_0 A}. \tag{30–10}$$

If the dielectric is present (Fig. 30–11*b*), Gauss's law gives

$$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 E A = q - q'$$

or

$$E = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A}, \tag{30–11}$$

in which $-q'$, the *induced surface charge*, must be distinguished from q, the so-called *free charge* on the plates. These two charges, both of which lie within the Gaussian surface, are opposite in sign; $q - q'$ is the *net* charge within the Gaussian surface.

Equation 30–9 shows that in Fig. 30–11

$$E = \frac{E_0}{\kappa}.$$

Combining this with Eq. 39–10, we have

$$E = \frac{E_0}{\kappa} = \frac{q}{\kappa \epsilon_0 A}. \tag{30–12}$$

Inserting this in Eq. 30–11 yields

$$\frac{q}{\kappa \epsilon_0 A} = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A} \tag{30–13a}$$

or

$$q' = q\left(1 - \frac{1}{\kappa}\right). \tag{30–13b}$$

This shows correctly that the induced surface charge q' is always less in magnitude than the free charge q and is equal to zero if no dielectric is present, that is, if $\kappa = 1$.

Now we write Gauss's law for the case of Fig. 30–11*b* in the form

$$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = q - q', \tag{30–14}$$

$q - q'$ again being the net charge within the Gaussian surface. Substituting from Eq. 30–13*b* for q' leads, after some rearrangement, to

$$\epsilon_0 \oint \kappa \mathbf{E} \cdot d\mathbf{S} = q. \tag{30–15}$$

This important relation, although derived for a parallel-plate capacitor, is true generally and is the form in which Gauss's law is usually written when dielectrics are present. Note the following:

1. The flux integral now contains a factor κ.
2. The charge q contained within the Gaussian surface is taken to be the *free charge only*. Induced surface charge is deliberately ignored on the right side of this equation, having been taken into account by the introduction of κ on the left side. Equations 30-14 and 30-15 are completely equivalent formulations.

Example 5. Figure 30-12 shows a dielectric slab of thickness b and dielectric constant κ placed between the plates of a parallel-plate capacitor of plate area A and separation d. A potential difference V_0 is applied with no dielectric present. The battery is then disconnected and the dielectric slab inserted. Assume that $A = 100$ cm^2, $d = 1.0$ cm, $b = 0.50$ cm, $\kappa = 7.0$, and $V_0 = 100$ volts and (*a*) calculate the capacitance C_0 before the slab is inserted.

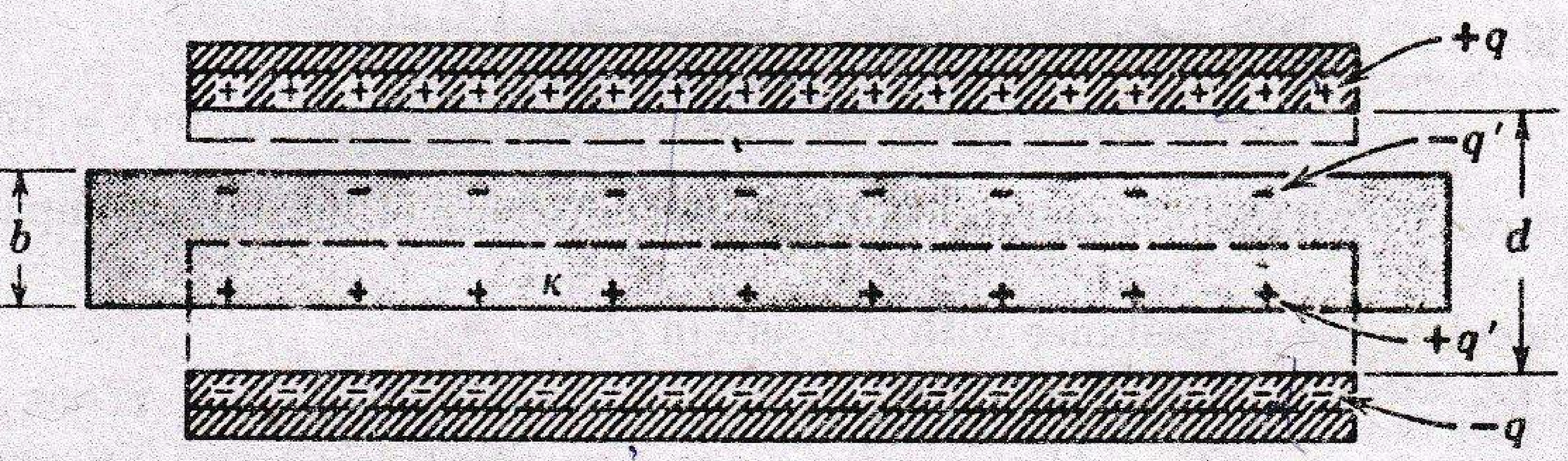


Fig. 30-12 Example 5. A parallel-plate capacitor containing a dielectric slab.

From Eq. 30-7, C_0 is found:

$$C_0 = \frac{\epsilon_0 A}{d} = \frac{(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(10^{-2}\ \text{meter}^2)}{10^{-2}\ \text{meter}} = 8.9\ \mu\mu\text{f}.$$

(*b*) Calculate the free charge q.
From Eq. 30-4,

$$q = C_0 V_0 = (8.9 \times 10^{-12}\ \text{farad})(100\ \text{volts}) = 8.9 \times 10^{-10}\ \text{coul}.$$

Because of the technique used to charge the capacitor, the free charge remains unchanged as the slab is introduced. If the charging battery had *not* been disconnected, this would not be the case.

(*c*) Calculate the electric field strength in the gap.

Applying Gauss's law in the form given in Eq. 30-15 to the Gaussian surface of Fig. 30-12 (upper plate) yields

$$\epsilon_0 \oint \kappa \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 E_0 A = q,$$

or $$E_0 = \frac{q}{\epsilon_0 A} = \frac{8.9 \times 10^{-10}\ \text{coul}}{(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(10^{-2}\ \text{meter}^2)} = 1.0 \times 10^4\ \text{volts/meter}.$$

Note that we put $\kappa = 1$ here because the surface over which we evaluate flux integral does not pass through any dielectric. Note too that E_0 remains unchanged when the slab is introduced; this derivation takes no specific account of the presence of the dielectric.

(*d*) Calculate the electric field strength in the dielectric.

Applying Eq. 30–15 to the Gaussian surface of Fig. 30–12 (lower plate) yields

$$\epsilon_0 \oint \kappa \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 \kappa E A = q.$$

Note that κ appears here because the surface cuts through the dielectric and that only the free charge q appears on the right. Thus we have

$$E = \frac{q}{\kappa \epsilon_0 A} = \frac{E_0}{\kappa} = \frac{1.0 \times 10^4 \text{ volts/meter}}{7.0} = 0.14 \times 10^4 \text{ volts/meter}.$$

(*e*) Calculate the potential difference between the plates.

Applying Eq. 29–5 to a straight perpendicular path from the lower plate (L) to the upper one (U) yields

$$V = -\int_L^U \mathbf{E} \cdot d\mathbf{l} = -\int_L^U E \cos 180° \, dl = \int_L^U E \, dl = E_0(d - b) + Eb.$$

Numerically

$$V = (1.0 \times 10^4 \text{ volts/meter})(5 \times 10^{-3} \text{ meter}) + (0.14 \times 10^4 \text{ volts/meter})(5 \times 10^{-3} \text{ meter}) = 57 \text{ volts}.$$

This contrasts with the original applied potential difference of 100 volts; compare Fig. 30–7*b*.

(*f*) Calculate the capacitance with the slab in place.

From Eq. 30–4,

$$C = \frac{q}{V} = \frac{8.9 \times 10^{-10} \text{ coul}}{57 \text{ volts}} = 16 \,\mu\mu\text{f}.$$

When the dielectric slab is introduced, the potential difference drops from 100 to 57 volts and the capacitance rises from 8.9 to 16 $\mu\mu$f, a factor of 1.8. If the dielectric slab had filled the capacitor, the capacitance would have risen by a factor of κ (= 7.0) to 62 $\mu\mu$f. ◀

30–6 Three Electric Vectors

For all situations that we encounter in this book our discussion of the behavior of dielectrics in an electric field is adequate. However, the problems that we treat are simple ones, such as that of a rectangular slab placed at right angles to a uniform external electric field. For more difficult problems, such as that of finding **E** at the center of a dielectric ellipsoid placed in a (possibly nonuniform) external electric field, it greatly simplifies the labor and leads to deeper insight if we introduce a new formalism. We do so largely so that students who take a second course in electromagnetism will have some familiarity with the concepts.

Let us rewrite Eq. 30–13*a*, which applies to a parallel-plate capacitor containing a dielectric, as

$$\frac{q}{A} = \epsilon_0 \left(\frac{q}{\kappa \epsilon_0 A} \right) + \frac{q'}{A}. \tag{30–16}$$

The quantity in parentheses (see Eq. 30–12) is simply the electric field strength E in the dielectric. The last term in Eq. 30–16 is the *induced surface charge per unit area*. We call it the *electric polarization* P, or

$$P = \frac{q'}{A}. \tag{30–17}$$

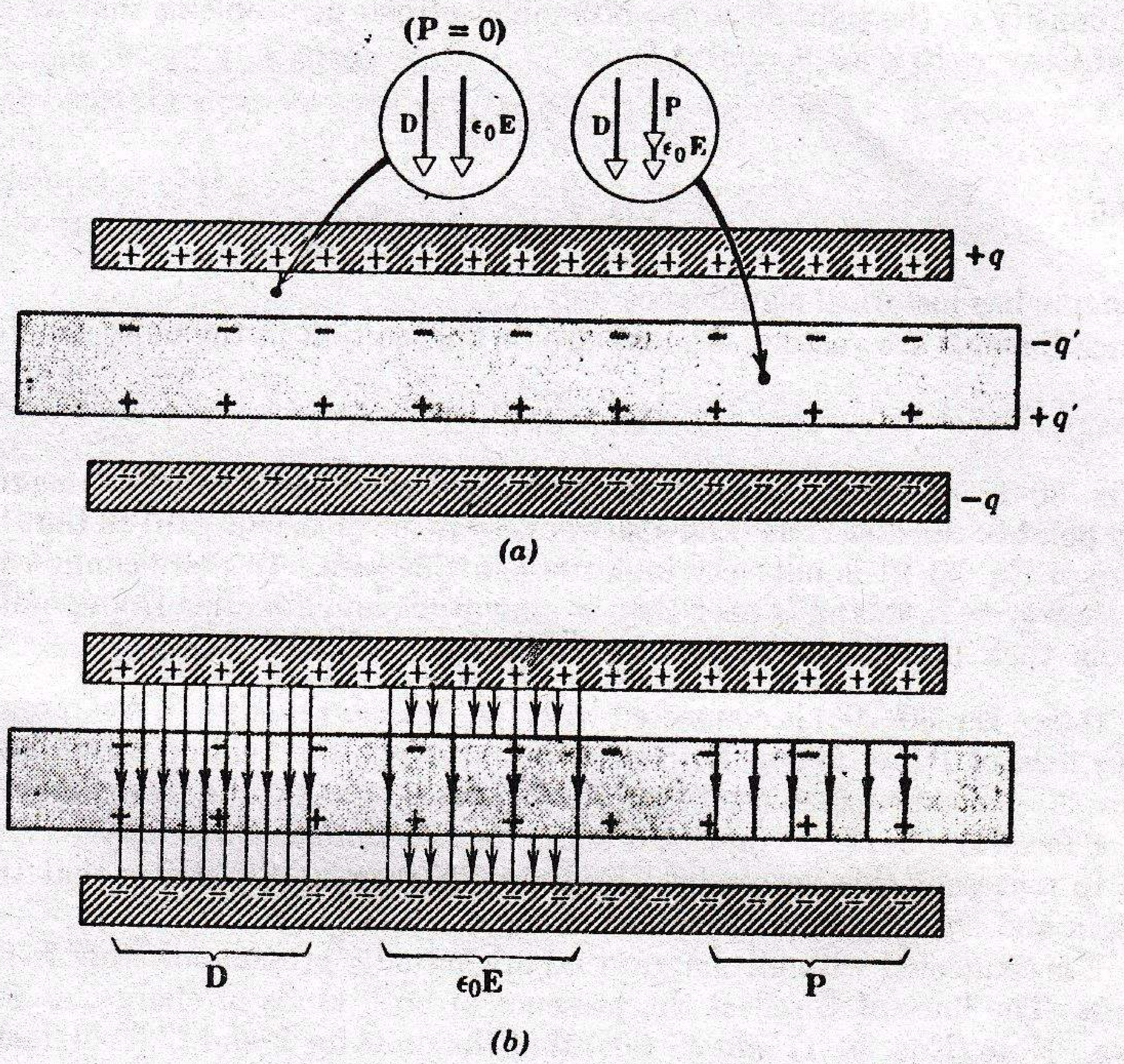


Fig. 30-13 (*a*) Showing **D**, ϵ_0**E**, and **P** in the dielectric (*upper right*) and in the gap (*upper left*) for a parallel-plate capacitor. (*b*) Showing samples of the lines associated with **D** (free charge), ϵ_0**E** (all charges), and **P** (polarization charge).

The name is suitable because the induced surface charge q' appears when the dielectric is polarized.

The electric polarization P can be defined in an equivalent way by multiplying the numerator and denominator in Eq. 30-17 by d, the thickness of the dielectric slab in Fig. 30-11,

$$P = \frac{q'd}{Ad}. \tag{30-18}$$

The numerator is the product $q'd$ of the magnitude of the (equal and opposite) polarization charges by their separation. It is thus the induced electric dipole moment of the dielectric slab. Since the denominator Ad is the volume of the slab, we see that the electric polarization can also be defined as the induced electric dipole moment per unit volume in the dielectric. This definition suggests that since the electric dipole moment is a vector the electric polarization is also a vector, its magnitude being P. The direction of **P** is from the negative induced charge to the positive induced charge, as for any dipole. In Fig. 30-13, which shows a capacitor with a dielectric slab filling half the space between the plates, **P** points down.

We can now rewrite Eq. 30-16 as

$$\frac{q}{A} = \epsilon_0 E + P. \tag{30-19}$$

The quantity on the right occurs so often in electrostatic problems that we give it the special name *electric displacement* D, or

$$D = \epsilon_0 E + P \tag{30–20a}$$

in which

$$D = \frac{q}{A}. \tag{30–20b}$$

The name has historical significance only.

Since **E** and **P** are vectors, **D** must also be one, so that in the more general case we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}. \tag{30–21}$$

In Fig. 30–13 all three vectors point down and each has a constant magnitude for every point in the dielectric (and also at every point in the air gap) so that the vector nature of Eq. 30–21 is not very important in this case. In more complicated problems, however, **E**, **P**, and **D** may vary in magnitude and direction from point to point.

From their definitions we see the following:

1. **D** (see Eq. 30–20*b*) is connected with the *free charge* only. We can represent the vector field of **D** by *lines of* **D**, just as we represent the field of **E** by lines of force. Figure 30–13*b* shows that the lines of **D** begin and end on the free charges.
2. **P** (see Eq. 30–17) is connected with the *polarization charge* only. It is also possible to represent this vector field by lines. Figure 30–13*b* shows that the lines of **P** begin and end on the polarization charges.
3. **E** is connected with all charges that are actually present, whether free or polarization. The lines of **E** reflect the presence of both kinds of charge, as Fig. 30–13*b* shows. Note (Eqs. 30–17 and 30–20*b*) that the units for **P** and **D** (coul/meter^2) differ from those of **E** (nt/coul).

The electric field vector **E**, which is what determines the force that acts on a suitably placed test charge, remains of fundamental interest. **D** and **P** are auxiliary vectors useful as aids in the solution of problems more complex than that of Fig. 30–13.

The vectors **D** and **P** can both be expressed in terms of **E** alone. A convenient starting point is the identity

$$\frac{q}{A} = \kappa\epsilon_0 \left(\frac{q}{\kappa\epsilon_0 A}\right).$$

Comparison with Eqs. 30–12 and 30–20*b* shows that this, extended to vector form, can be written as

$$\mathbf{D} = \kappa\epsilon_0 \mathbf{E}. \tag{30–22}$$

We can also write the polarization (see Eqs. 30–17 and 30–13*b*) as

$$P = \frac{q'}{A} = \frac{q}{A}\left(1 - \frac{1}{\kappa}\right).$$

Since $q/A = D$, we can rewrite this, using Eq. 30–22 and casting the result into vector form, as

$$\mathbf{P} = \epsilon_0(\kappa - 1)\mathbf{E}. \tag{30–23}$$

This shows clearly that in a vacuum ($\kappa = 1$) the polarization vector **P** is zero.*

---

* Certain waxes, when polarized in their molten state, retain a permanent polarization after solidifying, even though the external polarizing field is removed. *Electrets*, manufactured in this way, are the electrostatic analog of permanent magnets in that they possess a gross permanent electric dipole moment. Materials from which electrets can be constructed are called *ferroelectric*. Electrets do not obey Eq. 30–23 because they have a nonvanishing value of **P** even though **E** = 0.

Equations 30-22 and 30-23 show that for isotropic materials, to which a single dielectric constant κ can be assigned, **D** and **P** both point in the direction of **E** at any given point.

The definition of **D** given by Eq. 30-22 allows us to write Eq. 30-15, that is, Gauss's law in the presence of a dielectric, simply as

$$\oint \mathbf{D} \cdot d\mathbf{S} = q, \tag{30-24}$$

where, as before, q represents the free charge only, the induced surface charges being excluded.

▶ **Example 6.** In Figure 30-13, using data from Example 5, calculate E, D, and P: (a) in the dielectric and (b) in the air gap.

(a) The electric field in the dielectric is calculated in Example 5 to be 1.43×10^3 volts/meter. From Eq. 30-22,

$$D = \kappa\epsilon_0 E$$

$$= (7.0)(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(1.43 \times 10^3\ \text{volts/meter})$$

$$= 8.9 \times 10^{-8}\ \text{coul/meter}^2$$

and, from Eq. 30-23,

$$P = \epsilon_0(\kappa - 1)E$$

$$= (8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(7.0 - 1)(1.43 \times 10^3\ \text{volts/meter})$$

$$= 7.5 \times 10^{-8}\ \text{coul/meter}^2.$$

(b) The electric field E_0 in the air gap is calculated in Example 5 to be 1.00×10^4 volts/meter. From Eq. 30-22,

$$D_0 = \kappa\epsilon_0 E_0$$

$$= (1)(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(1.00 \times 10^4\ \text{volts/meter})$$

$$= 8.9 \times 10^{-8}\ \text{coul/meter}^2$$

and, from Eq. 30-23, recalling that $\kappa = 1$ in the air gap,

$$P_0 = \epsilon_0(\kappa - 1)E_0 = 0.$$

Note that **P** vanishes outside the dielectric, **D** has the same value in the dielectric and in the gap, and **E** has different values in the dielectric and in the gap. The student should verify that Eq. 30-21 ($\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$) is correct both in the gap and in the dielectric.

It can be shown from Maxwell's equations that no matter how complex the problem the component of **D** *normal* to the surface of the dielectric has the same value on each side of the surface. In this problem **D** itself is normal to the surface, there being no component but the normal one. It can also be shown that the component of **E** *tangential* to the dielectric surface has the same value on each side of the surface. This *boundary condition*, like the one for **D**, is trivial in this problem, both tangential components being zero. In more complex problems these boundary conditions on **D** and **E** are very important. Table 30-2 summarizes the properties of the electric vectors **E**, **D**, and **P**. ◀

Table 30–2

THREE ELECTRIC VECTORS

Name	Symbol	Associated with	Boundary Condition
Electric field strength	**E**	All charges	Tangential component continuous
Electric displacement	**D**	Free charges only	Normal component continuous
Polarization (electric dipole moment per unit volume)	**P**	Polarization charges only	Vanishes in a vacuum

Defining equation for **E**	$\mathbf{F} = q\mathbf{E}$	Eq. 27–2
General relation among the three vectors	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$	Eq. 30–21
Gauss's law when dielectric media are present	$\oint \mathbf{D}\cdot d\mathbf{S} = q$ (q = free charge only)	Eq. 30–24
Empirical relations for certain dielectric materials *	$\mathbf{D} = \kappa\epsilon_0\mathbf{E}$ $\mathbf{P} = (\kappa - 1)\epsilon_0\mathbf{E}$	Eq. 30–22 Eq. 30–23

* Generally true, with κ independent of **E**, except for certain materials called *ferroelectrics*; see footnote on page 758.

30–7 Energy Storage in an Electric Field

In Section 29–6 we saw that all charge configurations have a certain *electric potential energy U*, equal to the work W (which may be positive or negative) that must be done to assemble them from their individual components, originally assumed to be infinitely far apart and at rest. This potential energy reminds us of the potential energy stored in a compressed spring or the gravitational potential energy stored in, say, the earth-moon system.

For a simple example, work must be done to separate two equal and opposite charges. This energy is stored in the system and can be recovered if the charges are allowed to come together again. Similarly, a charged capacitor has stored in it an electrical potential energy U equal to the work W required to charge it. This energy can be recovered if the capacitor is allowed to discharge. We can visualize the work of charging by imagining that an external agent pulls electrons from the positive plate and pushes them onto the negative plate, thus bringing about the charge separation; normally the work of charging is done by a battery, at the expense of its store of chemical energy.

Suppose that at a time t a charge $q'(t)$ has been transferred from one plate to the other. The potential difference $V(t)$ between the plates at that moment will be $q'(t)/C$. If an extra increment of charge dq' is transferred, the small amount of additional work needed will be

$$dW = V\,dq = \left(\frac{q'}{C}\right) dq'.$$

If this process is continued until a total charge q has been transferred, the total work will be found from

$$W = \int dW = \int_0^q \frac{q'}{C}\,dq' = \frac{1}{2}\frac{q^2}{C}. \tag{30–25}$$

From the relation $q = CV$ we can also write this as

$$W\ (= U) = \tfrac{1}{2}CV^2. \tag{30–26}$$

It is reasonable to suppose that the energy stored in a capacitor resides in the electric field. As q or V in Eqs. 30–25 and 30–26 increase, for example, so does the electric field E; when q and V are zero, so is E.

In a parallel-plate capacitor, neglecting fringing, the electric field has the same value for all points between the plates. Thus the *energy density u*, which is the stored energy per unit volume, should also be uniform; u (see Eq. 30–26) is given by

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad},$$

where Ad is the volume between the plates. Substituting the relation $C = \kappa\epsilon_0 A/d$ (Eq. 30–8) leads to

$$u = \frac{\kappa\epsilon_0}{2}\left(\frac{V}{d}\right)^2.$$

However, V/d is the electric field strength E, so that

$$u = \tfrac{1}{2}\kappa\epsilon_0 E^2. \tag{30–27}$$

Although this equation was derived for the special case of a parallel-plate capacitor, it is true in general. *If an electric field* **E** *exists at any point in space, we can think of that point as the site of stored energy in amount, per unit volume, of* $\frac{1}{2}\kappa\epsilon_0 E^2$.

▶ **Example 7.** A capacitor C_1 is charged to a potential difference V_0. This charging battery is then removed and the capacitor is connected as in Fig. 30–14 to an uncharged capacitor C_2.

(*a*) What is the final potential difference V across the combination?

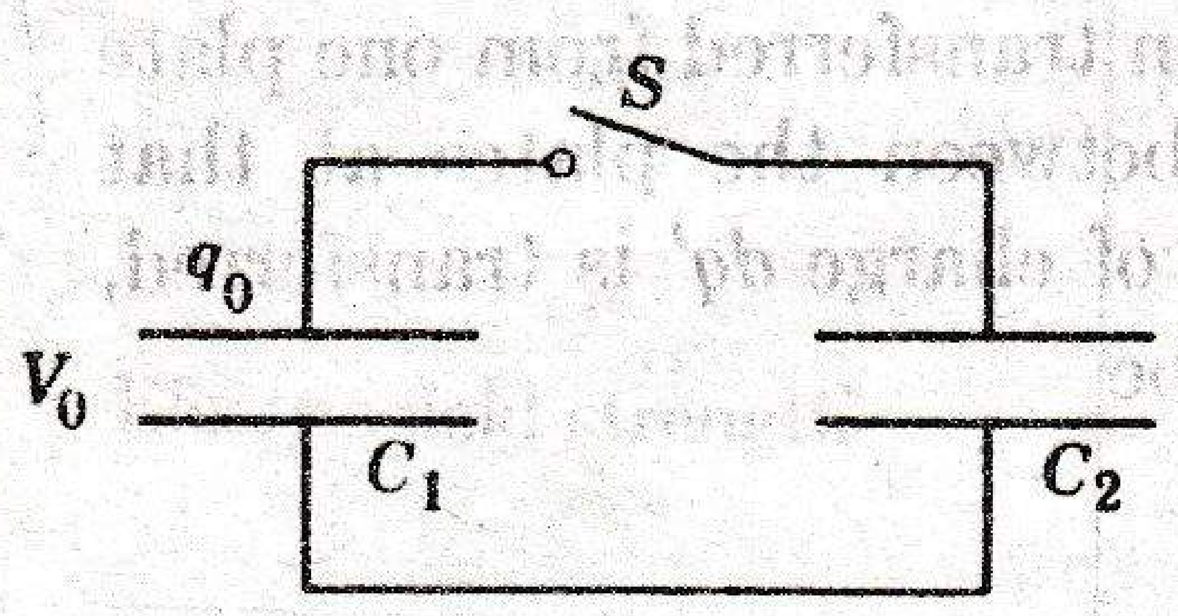


Fig. 30–14 Example 7. C_1 is charged and then connected to C_2 by closing switch S.

The original charge q_0 is now shared by the two capacitors. Thus

$$q_0 = q_1 + q_2.$$

Applying the relation $q = CV$ to each of these terms yields

$$C_1V_0 = C_1V + C_2V$$

or

$$V = V_0 \frac{C_1}{C_1 + C_2}.$$

This suggests a way to measure an unknown capacitance (C_2, say) in terms of a known one.

(*b*) What is the stored energy before and after the switch in Fig. 30–14 is thrown?

The initial stored energy is

$$U_0 = \tfrac{1}{2}C_1V_0^2.$$

The final stored energy is

$$U = \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}C_2V^2 = \tfrac{1}{2}(C_1 + C_2)\left(\frac{V_0C_1}{C_1 + C_2}\right)^2 = \left(\frac{C_1}{C_1 + C_2}\right)U_0.$$

Thus U is less than U_0! The "missing" energy appears as heat in the connecting wires as the charges move through them.

Example 8. A parallel-plate capacitor has plates with area A and separation d. A battery charges the plates to a potential difference V_0. The battery is then disconnected, and a dielectric slab of thickness d is introduced. Calculate the stored energy both before and after the slab is introduced and account for any difference.

The energy U_0 before introducing the slab is

$$U_0 = \tfrac{1}{2}C_0V_0^2.$$

After the slab is in place, we have

$$C = \kappa C_0 \quad \text{and} \quad V = V_0/\kappa$$

and thus

$$U = \tfrac{1}{2}CV^2 = \tfrac{1}{2}\kappa C_0\left(\frac{V_0}{\kappa}\right)^2 = \frac{1}{\kappa}U_0.$$

The energy after the slab is introduced is *less* by a factor $1/\kappa$. The "missing" energy would be apparent to the person who inserted the slab. He would feel a "tug" on the slab and would have to restrain it if he wished to insert the slab without acceleration. This means that he would have to do negative work on it, or, alternatively, that the condenser + slab system would do positive work on him. This positive work is

$$W = U_0 - U = \tfrac{1}{2}C_0V_0^2\left(1 - \frac{1}{\kappa}\right).$$

Figure 30-15 shows how the forces that do this work arise, in terms of attraction between the free charge on the plates and the induced surface charges that appear on the slab when it is introduced between the plates.

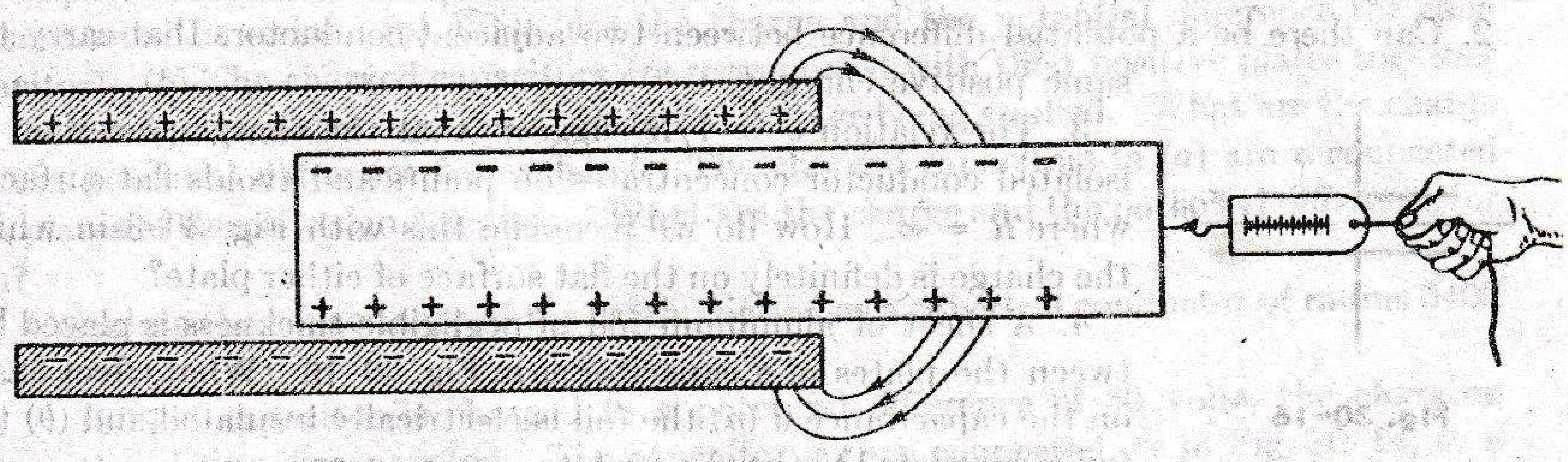


Fig. 30-15 When we introduce a dielectric slab into a charged capacitor, as shown, forces arise which tend to pull the slab into the capacitor.

Example 9. A conducting sphere of radius R, in a vacuum, carries a charge q. (*a*) Compute the total electrostatic energy stored in the surrounding space.

At any radius r from the center of the sphere (assuming $r > R$) E is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

The energy density at any radius r is found from Eq. 30-27, assuming $\kappa = 1$, or

$$u = \tfrac{1}{2}\epsilon_0 E^2 = \frac{q^2}{32\pi^2\epsilon_0 r^4}.$$

The energy dU that lies in a spherical shell between the radii r and $r + dr$ is

$$dU = (4\pi r^2)(dr)u = \frac{q^2}{8\pi\epsilon_0}\frac{dr}{r^2},$$

where $(4\pi r^2)(dr)$ is the volume of the spherical shell. The total energy U is found by integration, or

$$U = \int dU = \frac{q^2}{8\pi\epsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\epsilon_0 R}.$$

Note that this relation follows at once from Eq. 30-25 ($U = q^2/2C$), where C (see p. 651) is the capacitance ($= 4\pi\epsilon_0 R$) of an isolated sphere of radius R.

(*b*) What is the radius R_0 of a spherical surface such that half the stored energy lies within it?

In the equation just given we put

$$\tfrac{1}{2}U = \frac{q^2}{8\pi\epsilon_0}\int_R^{R_0}\frac{dr}{r^2}$$

or

$$\frac{q^2}{16\pi\epsilon_0 R} = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R_0}\right),$$

which yields, after some rearrangement,

$$R_0 = 2R.$$

QUESTIONS

1. A capacitor is connected across a battery. (*a*) Why does each plate receive a charge of exactly the same magnitude? (*b*) Is this true even if the plates are of different sizes?

2. Can there be a potential difference between two adjacent conductors that carry the same positive charge?

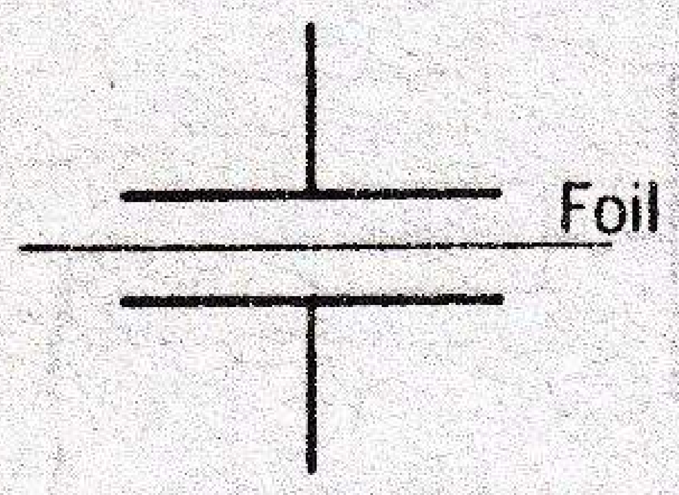


Fig. 30–16

3. The relation $\sigma \propto 1/R$ suggests that the charge placed on an isolated conductor concentrates on points and avoids flat surfaces, where $R = \infty$. How do we reconcile this with Fig. 30–3 in which the charge is definitely on the flat surface of either plate?

4. A sheet of aluminum foil of negligible thickness is placed between the plates of a capacitor as in Fig. 30–16. What effect has it on the capacitance if (*a*) the foil is electrically insulated and (*b*) the foil is connected to the upper plate.

5. Discuss similarities and differences when (*a*) a dielectric slab and (*b*) a conducting slab are inserted between the plates of a parallel-plate capacitor. Assume the slab thicknesses to be one-half the plate separation.

6. An oil-filled parallel-plate capacitor has been designed to have a capacitance C and to operate safely at or below a certain maximum potential difference V_m without arcing over. However, the designer did not do a good job and the capacitor occasionally arcs over. What can be done to redesign the capacitor, keeping C and V_m unchanged and using the same dielectric?

7. Would you expect the dielectric constant, for substances containing permanent molecular electric dipoles, to vary with temperature?

8. What is your estimate of the amount by which the center of positive charge and the center of negative charge are displaced in a situation like that of Fig. 30–9*b*? An intuitive guess is all that is called for.

9. For a given potential difference, does a capacitor store more or less charge with a dielectric than it does without a dielectric (vacuum)? Explain in terms of the microscopic picture of the situation.

10. An isolated conducting sphere is given a positive charge. Does its mass increase, decrease, or remain the same?

11. A dielectric slab is inserted in one end of a charged parallel-plate capacitor (the plates being horizontal and the charging battery having been disconnected) and then released. Describe what happens. Neglect friction.

12. A capacitor is charged by using a battery, which is then disconnected. A dielectric slab is then slipped between the plates. Describe qualitatively what happens to the charge, the capacitance, the potential difference, the electric field strength, and the stored energy.

13. While a capacitor remains connected to a battery, a dielectric slab is slipped between the plates. Describe qualitatively what happens to the charge, the capacitance, the potential difference, the electric field strength, and the stored energy. Is work required to insert the slab?

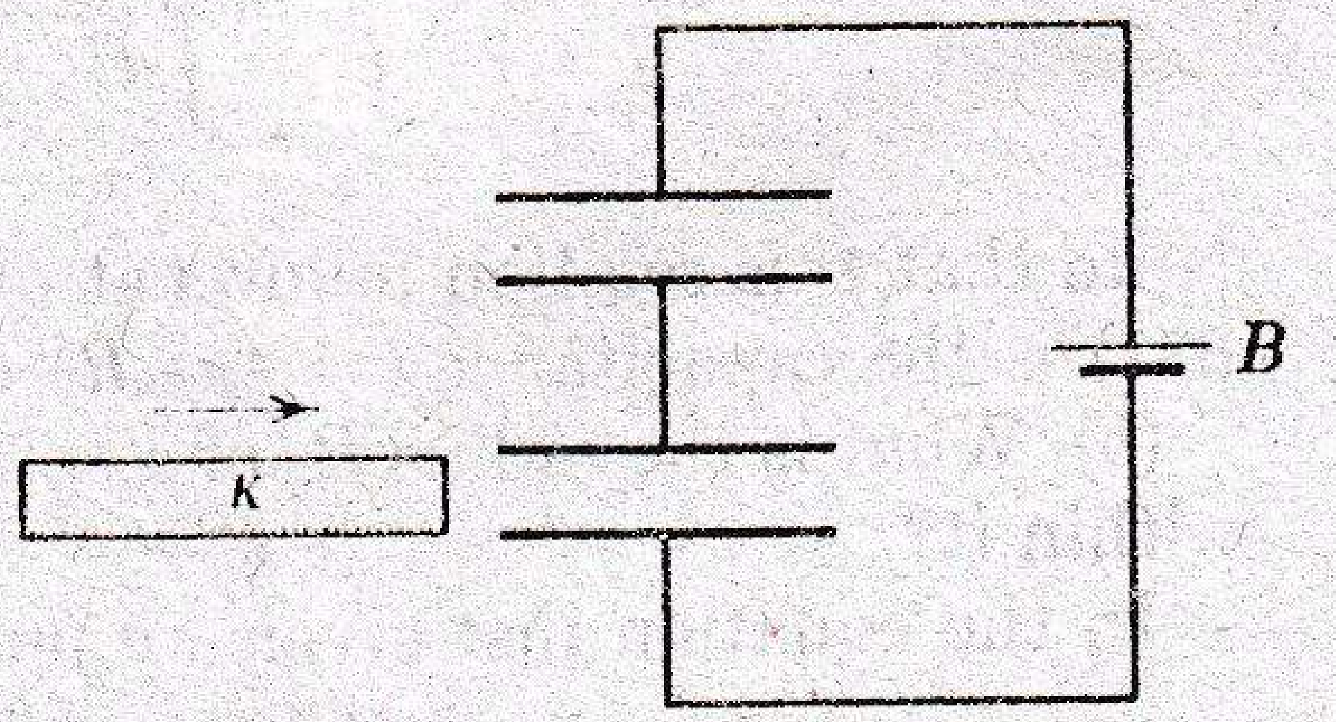


Fig. 30–17

14. Two identical capacitors are connected as shown in Fig. 30–17. A dielectric slab is slipped between the plates of one capacitor, the battery remaining connected. Describe qualitatively what happens to the charge, the capacitance, the potential difference, the electric field strength, and the stored energy for each capacitor.

15. Show that the dielectric constant of a conductor can be taken to be infinitely great.

PROBLEMS

1. A potential difference of 300 volts is applied to a 2.0-μf capacitor and an 8.0-μf capacitor connected in series. (*a*) What are the charge and the potential difference for each capacitor? (*b*) The charged capacitors are reconnected with their positive plates together and their negative plates together, no external voltage being applied. What are the charge and the potential difference for each? (*c*) The charged capacitors in (*a*) are reconnected with plates of *opposite* sign together. What are the charge and the potential difference for each?

2. Calculate the capacitance of the earth, viewed as a spherical conductor of radius 6400 km.

3. A 100-$\mu\mu f$ capacitor is charged to a potential difference of 50 volts, the charging battery then being disconnected. The capacitor is then connected, as in Fig. 30–14, to a second capacitor. If the measured potential difference drops to 35 volts, what is the capacitance of this second capacitor?

4. If we solve Eq. 30–7 for ϵ_0, we see that its mks units are farads/meter. Show that these units are equivalent to those obtained earlier for ϵ_0, namely coul^2/nt-m^2.

5. Figure 30–18 shows two capacitors in series, the rigid center section of length b being movable vertically. Show that the equivalent capacitance of the series combination is independent of the position of the center section and is given by

$$C = \frac{\epsilon_0 A}{a - b}.$$

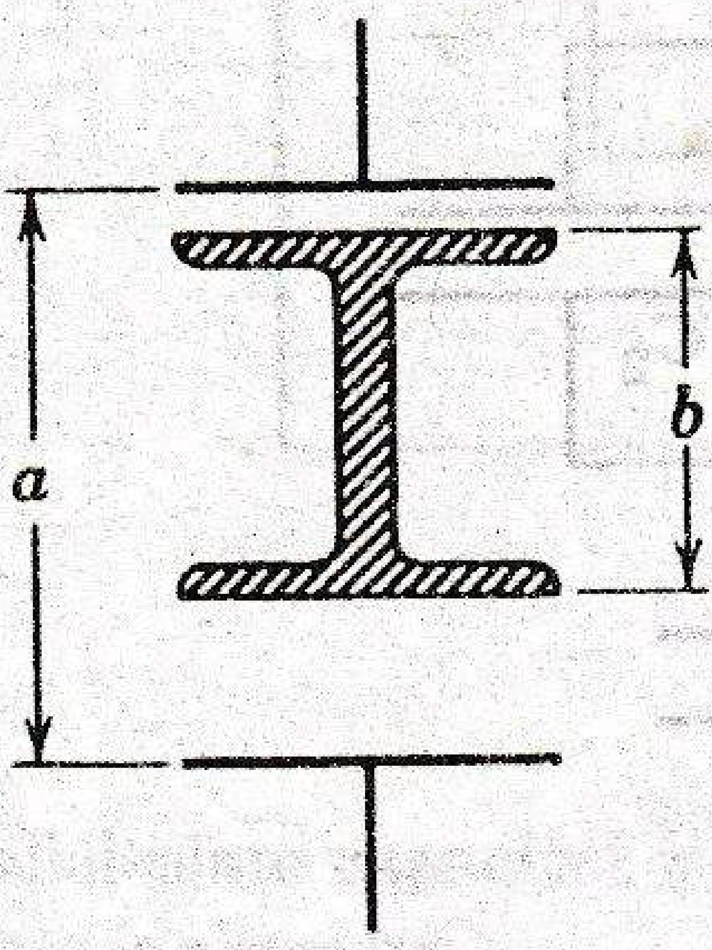


Fig. 30–18

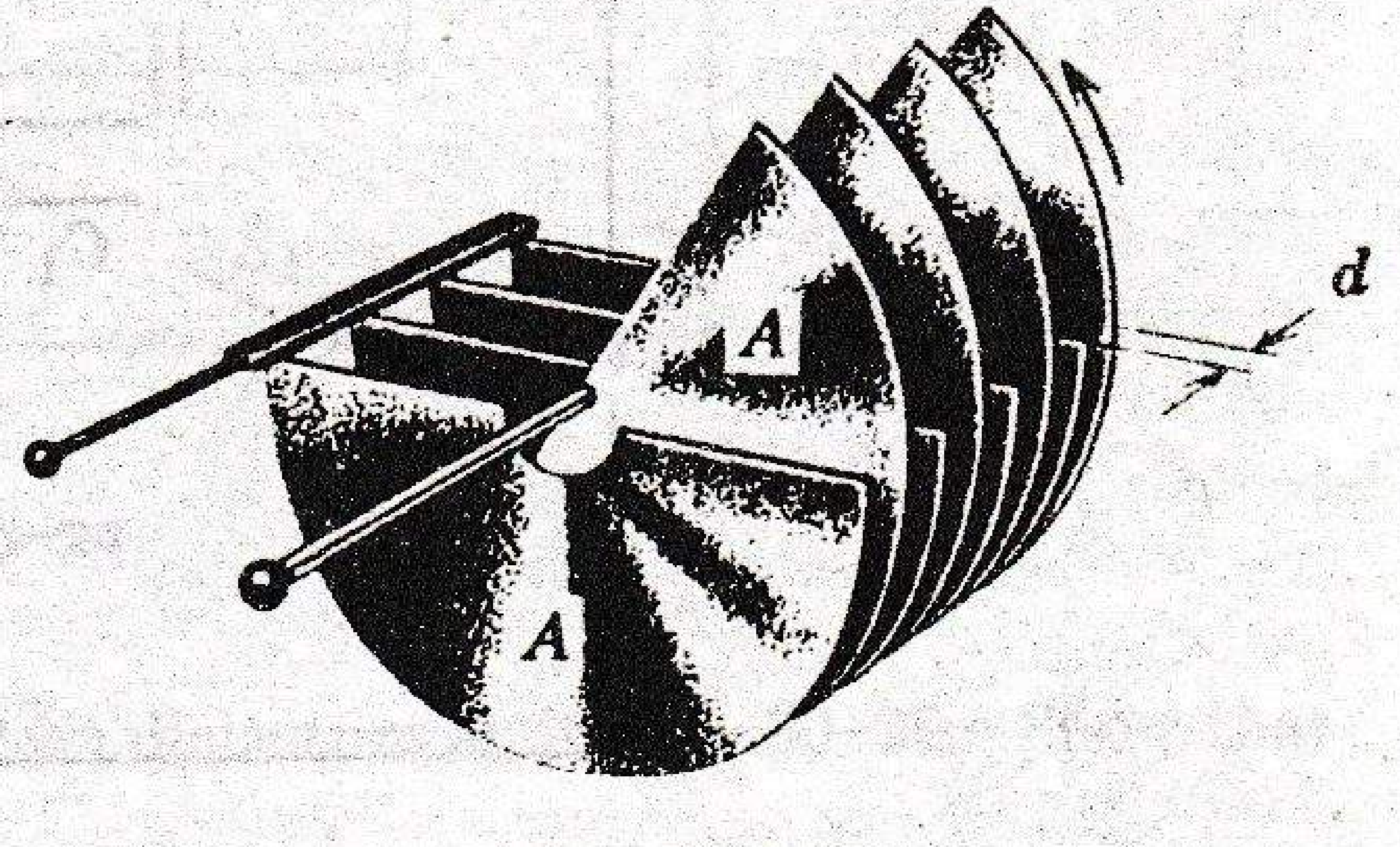


Fig. 30–19

6. In Fig. 30–19 a variable air capacitor of the type used in tuning radios is shown. Alternate plates are connected together, one group being fixed in position, the other group being capable of rotation. Consider a pile of n plates of alternate polarity, each having an area A and separated from adjacent plates by a distance d. Show that this capacitor has a maximum capacitance of

$$C = \frac{(n - 1)\epsilon_0 A}{d}.$$

7. A spherical capacitor consists of two concentric spherical shells of radii a and b, with $b > a$. Show that its capacitance is

$$C = 4\pi\epsilon_0 \frac{ab}{b - a}.$$

8. A capacitor has square plates, each of side a, making an angle of θ with each other as shown in Fig. 30–20. Show that for small θ the capacitance is given by

$$C = \frac{\epsilon_0 a^2}{d}\left(1 - \frac{a\theta}{2d}\right)$$

(Hint: The capacitor may be divided into differential strips which are effectively in parallel.)

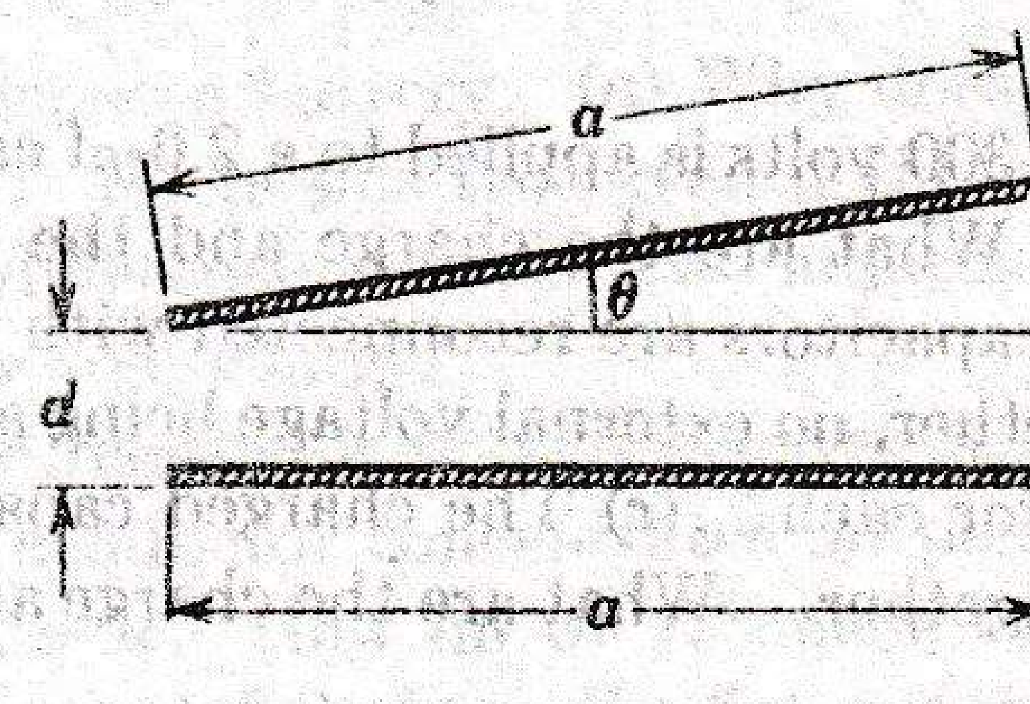


Fig. 30–20

9. Suppose that the two spherical shells of a spherical capacitor have their radii approximately equal. Under these conditions the device approximates a parallel plate capacitor with $b - a = d$. Show that the formula in Problem 7 does indeed reduce to Eq. 30–7 in this case.

10. A parallel-plate capacitor has circular plates of 8.0-cm radius and 1.0-mm separation. What charge will appear on the plates if a potential difference of 100 volts is applied?

11. In Fig. 30–21 find the equivalent capacitance of the combination. Assume that $C_1 = 10\ \mu f$, $C_2 = 5\ \mu f$, $C_3 = 4\ \mu f$, and $V = 100$ volts.

12. In Fig. 30–21 suppose that capacitor C_3 breaks down electrically, becoming equivalent to a conducting path. What *changes* in (*a*) the charge and (*b*) the potential difference occur for capacitor C_1?

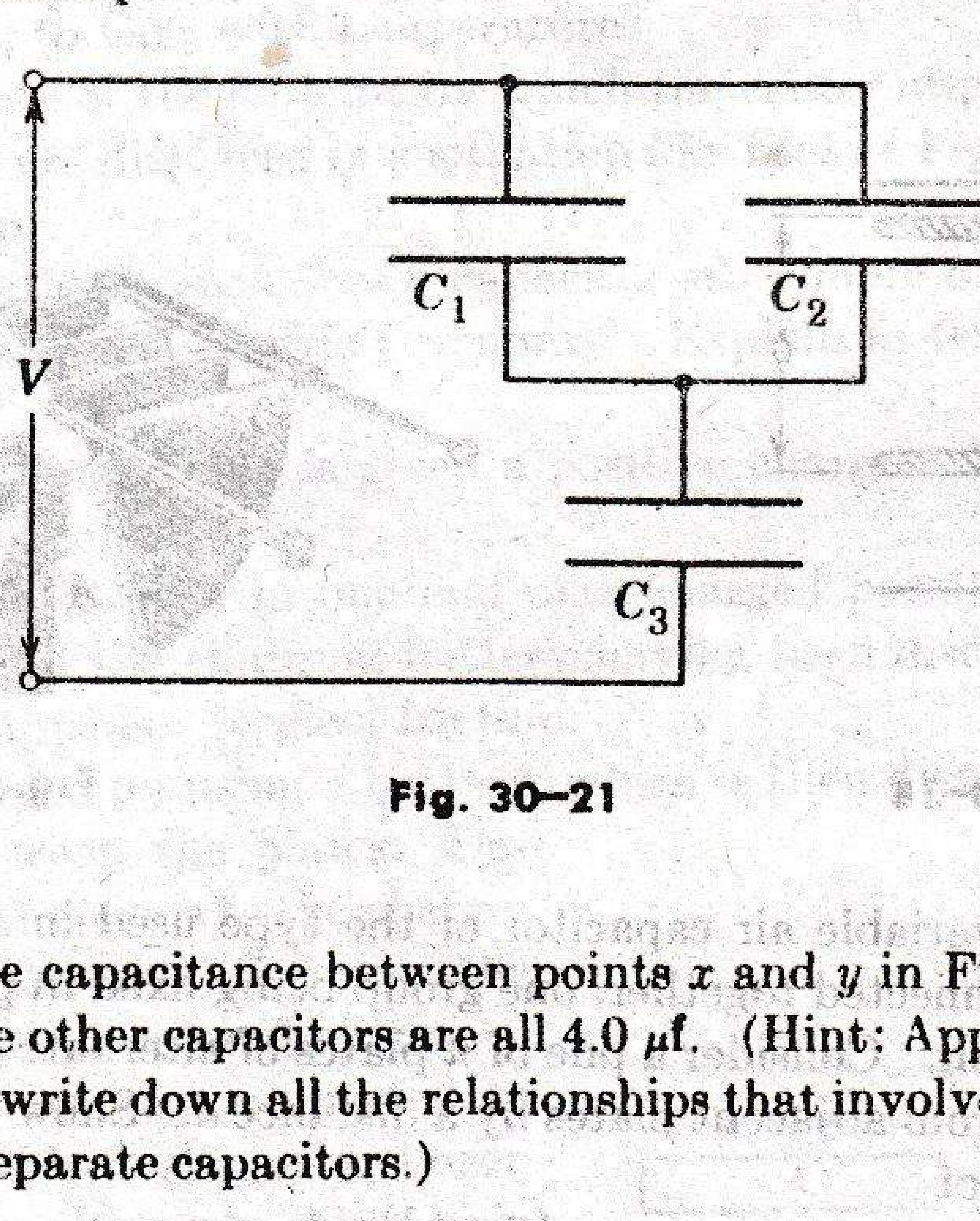


Fig. 30–21

13. Find the effective capacitance between points x and y in Fig. 30–22. Assume that $C_2 = 10\ \mu f$ and that the other capacitors are all 4.0 μf. (Hint: Apply a potential difference V between x and y and write down all the relationships that involve the charges and potential differences for the separate capacitors.)

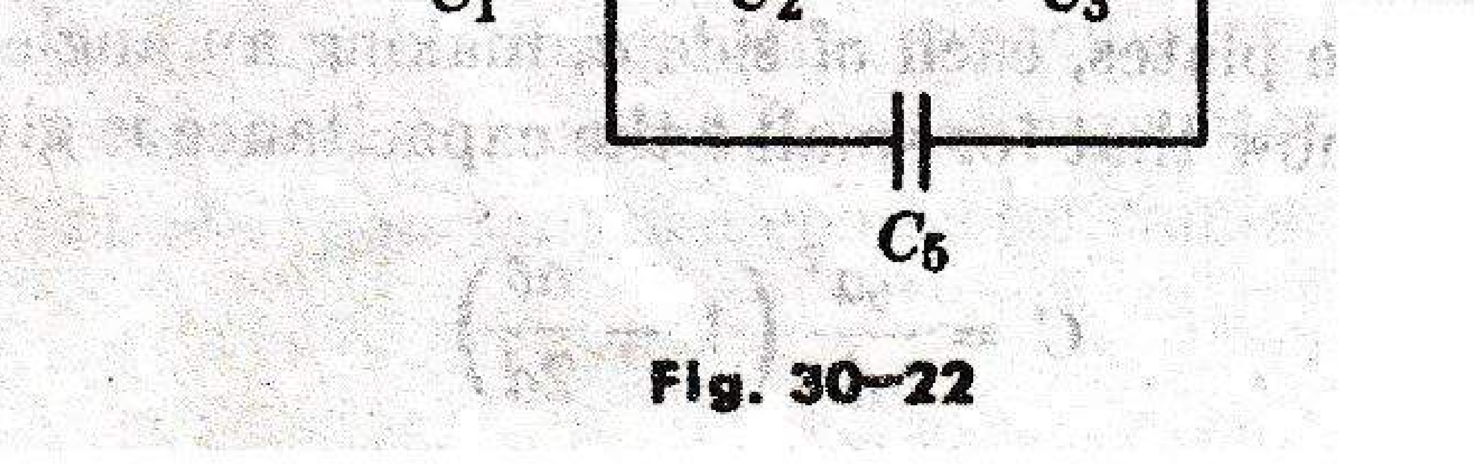


Fig. 30–22

14. If you have available several 2.0-μf capacitors, each capable of withstanding 200 volts without breakdown, how would you assemble a combination having an equivalent capacitance of (*a*) 0.40 μf or of (*b*) 1.2 μf, each capable of withstanding 1000 volts?

15. In Fig. 30–23 find the equivalent capacitance of the combination. Assume that $C_1 = 10\ \mu f$, $C_2 = 5\ \mu f$, $C_3 = 4\ \mu f$, and $V = 100$ volts.

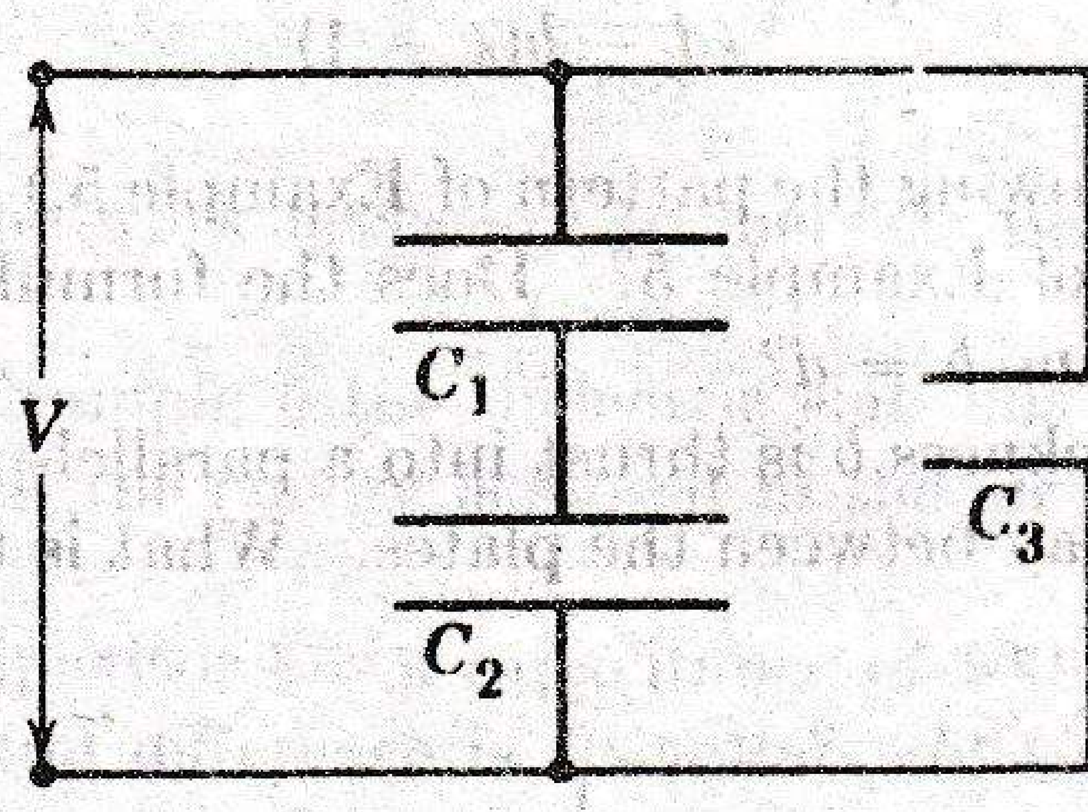


Fig. 30–23

16. In Fig. 30–24 the battery *B* supplies 12 volts. (*a*) Find the charge on each capacitor when switch S_1 is closed and (*b*) when switch S_2 is also closed. Take $C_1 = 1\ \mu f$, $C_2 = 2\ \mu f$, $C_3 = 3\ \mu f$, and $C_4 = 4\ \mu f$.

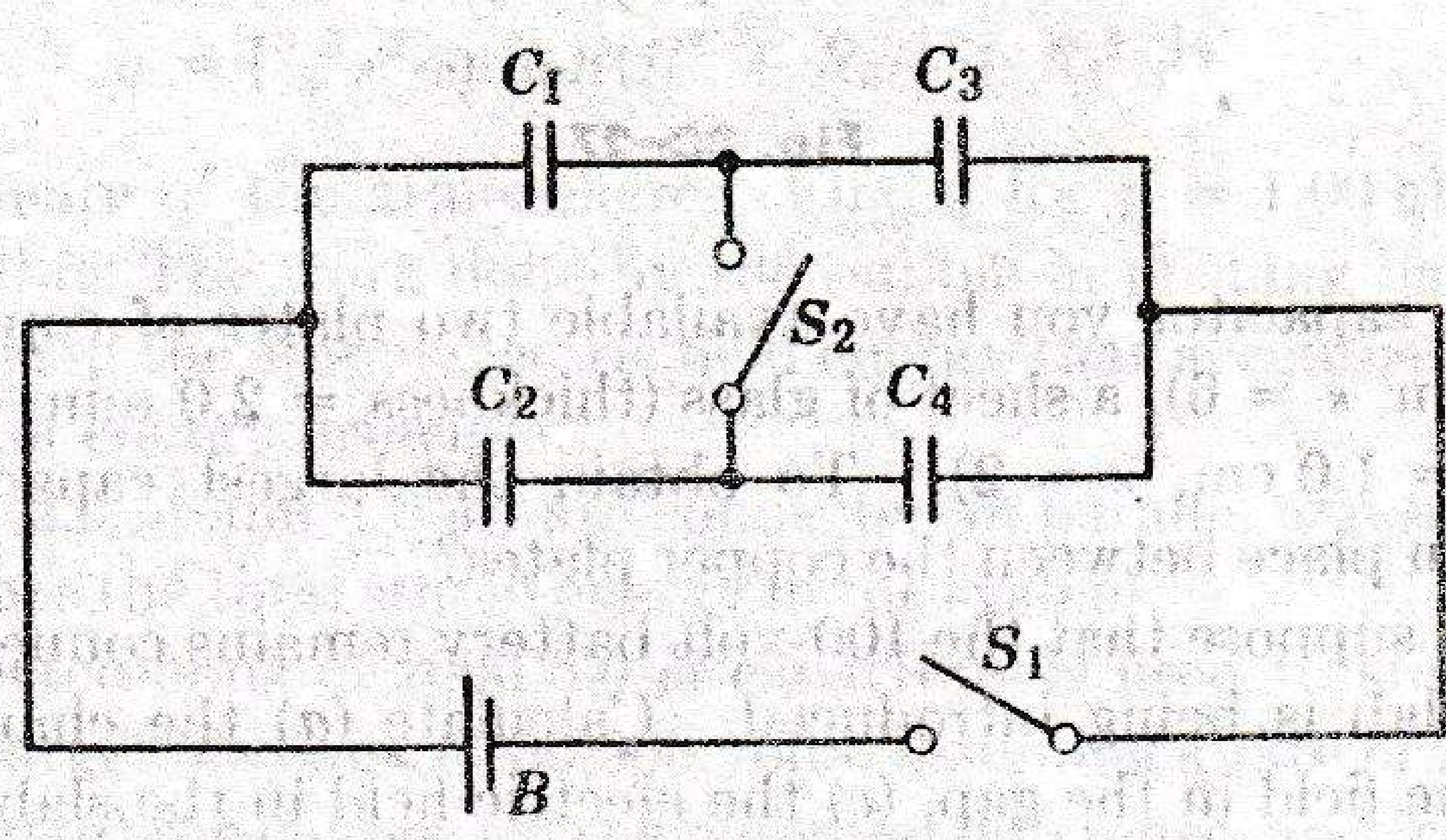


Fig. 30–24

17. A parallel-plate capacitor is filled with two dielectrics as in Fig. 30–25. Show that the capacitance is given by

$$C = \frac{\epsilon_0 A}{d}\left(\frac{\kappa_1 + \kappa_2}{2}\right).$$

Check this formula for all the limiting cases that you can think of. (Hint: Can you justify regarding this arrangement as two capacitors in parallel?)

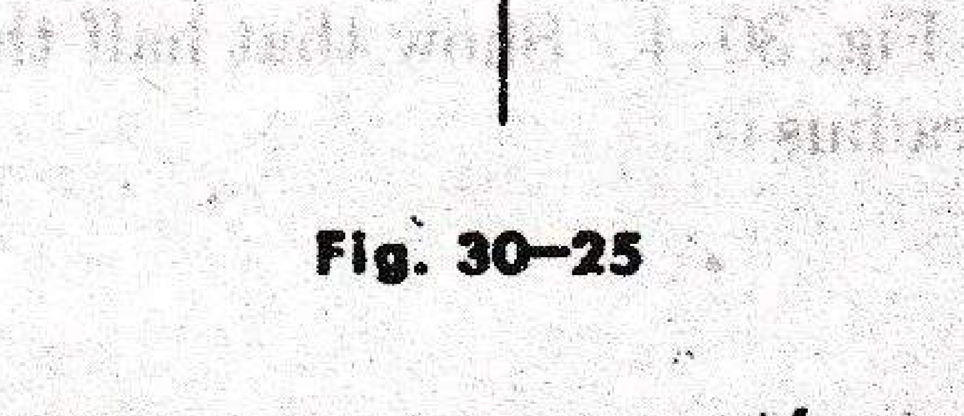


Fig. 30–25

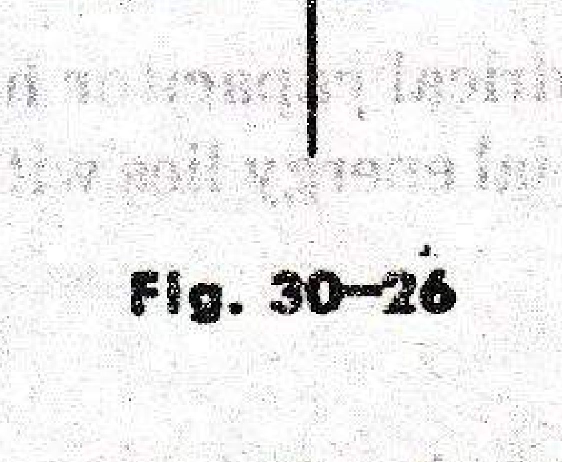


Fig. 30–26

18. A parallel-plate capacitor is filled with two dielectrics as in Fig. 30–26. Show that the capacitance is given by

$$C = \frac{2\epsilon_0 A}{d}\left(\frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right).$$

Check this formula for all the limiting cases that you can think of. (Hint: Can you justify regarding this arrangement as two capacitors in series?)

19. A dielectric slab of thickness b is inserted between the plates of a parallel-plate capacitor of plate separation d. Show that the capacitance is given by

$$C = \frac{\kappa\epsilon_0 A}{\kappa d - b(\kappa - 1)}.$$

(Hint: Derive the formula following the pattern of Example 5.) Does this formula predict the correct numerical result of Example 5? Does the formula seem reasonable for the special cases of $b = 0$, $\kappa = 1$, and $b = d$?

20. A slab of copper of thickness b is thrust into a parallel-plate capacitor as shown in Fig. 30–27; it is exactly halfway between the plates. What is the capacitance before and after the slab is introduced?

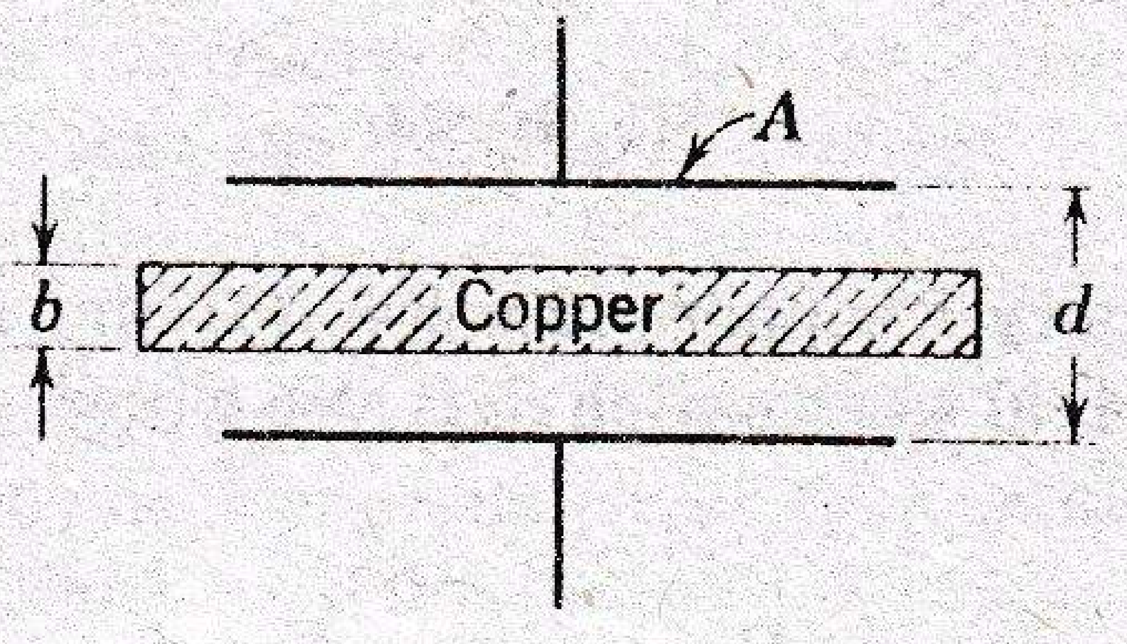


Fig. 30–27

21. For making a capacitor you have available two plates of copper, a sheet of mica (thickness = 0.10 mm, $\kappa = 6$), a sheet of glass (thickness = 2.0 mm, $\kappa = 7$), and a slab of paraffin (thickness = 1.0 cm, $\kappa = 2$). To obtain the largest capacitance, which sheet (or sheets) should you place between the copper plates?

22. In Example 5, suppose that the 100-volt battery remains connected during the time that the dielectric slab is being introduced. Calculate (*a*) the charge on the capacitor plates, (*b*) the electric field in the gap, (*c*) the electric field in the slab, and (*d*) the capacitance. For all of these quantities give the numerical values before and after the slab is introduced. Contrast your results with those of Example 5 by constructing a tabular listing.

23. A parallel-plate capacitor has a capacitance of 100 $\mu\mu$f, a plate area of 100 cm^2, and a mica dielectric. At 50 volts potential difference, calculate (*a*) E in the mica, (*b*) the free charge on the plates, and (*c*) the induced surface charge.

24. Two parallel plates of area 100 cm^2 are each given equal but opposite charges of 8.9×10^{-7} coul. Within the dielectric material filling the space between the plates the electric field strength is 1.4×10^6 volts/meter. (*a*) Find the dielectric constant of the material. (*b*) Determine the magitude of the charge induced on each dielectric surface.

25. Hard rubber has a dielectric constant of 2.8 and a dielectric strength of 18×10^6 volts/meter. If it is used as the dielectric material in a parallel-plate capacitor, what minimum area may the plates of the capacitor have in order that the capacitance be 7.0×10^{-2} μf and that the capacitor be able to withstand a potential difference of 4000 volts?

26. A cylindrical capacitor has radii a and b as in Fig. 30–4. Show that half the stored electric potential energy lies within a cylinder whose radius is

$$r = \sqrt{ab}.$$

27 An isolated metal sphere whose diameter is 10 cm has a potential of 8000 volts. What is the energy density at the surface of the sphere?

28. A parallel-plate capacitor has plates of area A and separation d and is charged to a potential difference V. The charging battery is then disconnected and the plates are pulled

apart until their separation is $2d$. Derive expressions in terms of A, d, and V for (*a*) the new potential difference, (*b*) the initial and the final stored energy, and (*c*) the work required to separate the plates.

29. Show that the plates of a parallel-plate capacitor attract each other with a force given by

$$F = \frac{q^2}{2\epsilon_0 A}.$$

Prove this by calculating the work necessary to increase the plate separation from x to $x + dx$.

30. In the capacitor of Example 5 the dielectric slab fills half the space between the plates. (*a*) What per cent of the energy is stored in the air gaps? (*b*) What per cent is stored in the slab?

31. A parallel-plate air capacitor has a capacitance of 100 $\mu\mu$f. (*a*) What is the stored energy if the applied potential difference is 50 volts? (*b*) Can you calculate the energy density for points between the plates?

32. For the capacitors of Problem 1, compute the energy stored for the three different connections of parts (*a*), (*b*), and (*c*). Compare your answers and explain any differences.

33. A charge q is placed on the surface of an originally uncharged soap bubble of radius R_0. Because of the mutual repulsion of the charged surface, the radius is increased to a somewhat larger value R. Show that

$$q = [\tfrac{32}{3}\pi^2\epsilon_0 p R_0 R(R^2 + R_0 R + R_0^2)]^{1/2}$$

in which p is the pressure of the atmosphere. Find q for p = 1.00 atm, R_0 = 2.00 cm, and R = 2.10 cm. (Hint: The work done by the bubble in pushing back the atmosphere must equal the decrease in the stored electric field energy that accompanies the expansion, from the conservation of energy principle.)

34. Two capacitors (2.0 μf and 4.0 μf) are connected in parallel across a 300-volt potential difference. Calculate the total stored energy in the system.

35. A parallel-connected bank of 2000 5.0-μf capacitors is used to store electric energy. What does it cost to charge this bank to 50,000 volts, assuming a rate of 2¢/kw-hr?

36. In Fig. 30–21 find (*a*) the charge, (*b*) the potential difference, and (*c*) the stored energy for each capacitor. Assume the numerical values of Problem 11.

37. In Fig. 30–23 find (*a*) the charge, (*b*) the potential difference, and (*c*) the stored energy for each capacitor. Assume the numerical values of Problem 15.

Current and Resistance

CHAPTER 31

31–1 Current and Current Density

The free electrons in an isolated metallic conductor, such as a length of copper wire, are in random motion like the molecules of a gas confined to a container. They have no net directed motion along the wire. If a hypothetical plane is passed through the wire, the rate at which electrons pass through it from right to left is the same as the rate at which they pass through from left to right; the *net* rate is zero.

If the ends of the wire are connected to a battery, an electric field will be set up at every point within the wire. If the potential difference maintained by the battery is 10 volts and if the wire (assumed uniform) is 5 meters long, the strength of this field at every point will be 2 volts/meter. This field **E** will act on the electrons and will give them a resultant motion in the direction of $-\mathbf{E}$. We say that an *electric current* i is established; if a net charge q passes through any cross section of the conductor in time t the current, assumed constant, is

$$i = q/t. \tag{31-1}$$

The appropriate mks units are amperes for i, coulombs for q, and seconds for t. The student will recall (Section 26–4) that Eq. 31–1 is the defining equation for the coulomb and that we have not yet given an operational definition of the ampere; we do so in Section 34–4.

If the rate of flow of charge with time is not constant, the current varies with time and is given by the differential limit of Eq. 31–1, or

$$i = dq/dt. \tag{31-2}$$

In the rest of this chapter we consider only constant currents.

The current i is the same for all cross sections of a conductor, even though the cross-sectional area may be different at different points. In the same way the rate at which water (assumed incompressible) flows past any cross section of a pipe is the same even if the cross section varies. The water flows faster where the pipe is smaller and slower where it is larger, so that the volume rate, measured perhaps in gal/min, remains unchanged. This constancy of the electric current follows because charge must be conserved; it does not pile up steadily or drain away steadily from any point in the conductor under the assumed steady-state conditions. In the language of Section 18-3 there are no "sources" or "sinks" of charge.

The existence of an electric field inside a conductor does not contradict Section 28-4, in which we asserted that **E** equals zero inside a conductor. In that section, which dealt with a state in which all net motion of charge had stopped (*electrostatics*), we assumed that the conductor was insulated and that no potential difference was deliberately maintained between any two points on it, as by a battery. In this chapter, which deals with charges in motion, we relax this restriction.

The electric field that acts on the electrons in a conductor does not produce a *net* acceleration because the electrons keep colliding with the atoms (strictly, ions) that make up the conductor. This array of ions, coupled together by strong spring-like forces of electric origin, is called the *lattice* (see Fig. 21-5). The over-all effect of these collisions is to transfer kinetic energy from the accelerating electrons into vibrational energy of the lattice. The electrons acquire a constant average *drift speed* v_d in the direction $-\mathbf{E}$. The analogy is to a marble rolling down a long flight of stairs and not to a marble falling freely from the same height. In the first case the acceleration caused by the (gravitational) field is effectively canceled by the decelerating effects of collisions with the stair treads so that, under proper conditions, the marble rolls down the stairs with zero average acceleration, that is, at constant average speed.

Although in metals the charge carriers are electrons, in electrolytes or in gaseous conductors they may also be positive or negative ions or both. A convention for labeling the directions of currents is needed because charges of opposite sign move in opposite directions in a given field. A positive charge moving in one direction is equivalent in nearly all external effects to a negative charge moving in the opposite direction. Hence, for simplicity and algebraic consistency, *we assume that all charge carriers are positive and we draw the current arrows in the direction that such charges would move.* If the charge carriers are negative, they simply move opposite to the direction of the current arrow (see Fig. 31-1). When we encounter a case (as in the *Hall effect;* see Section 33-5) in which the sign of the charge carriers makes a difference in the external effects, we will disregard the convention and take the actual situation into account.

Current i is a characteristic of a particular conductor. It is a macroscopic quantity, like the mass of an object, the volume of an object, or the length of a rod. A related microscopic quantity is the current density $\mathbf{j}$. It is a

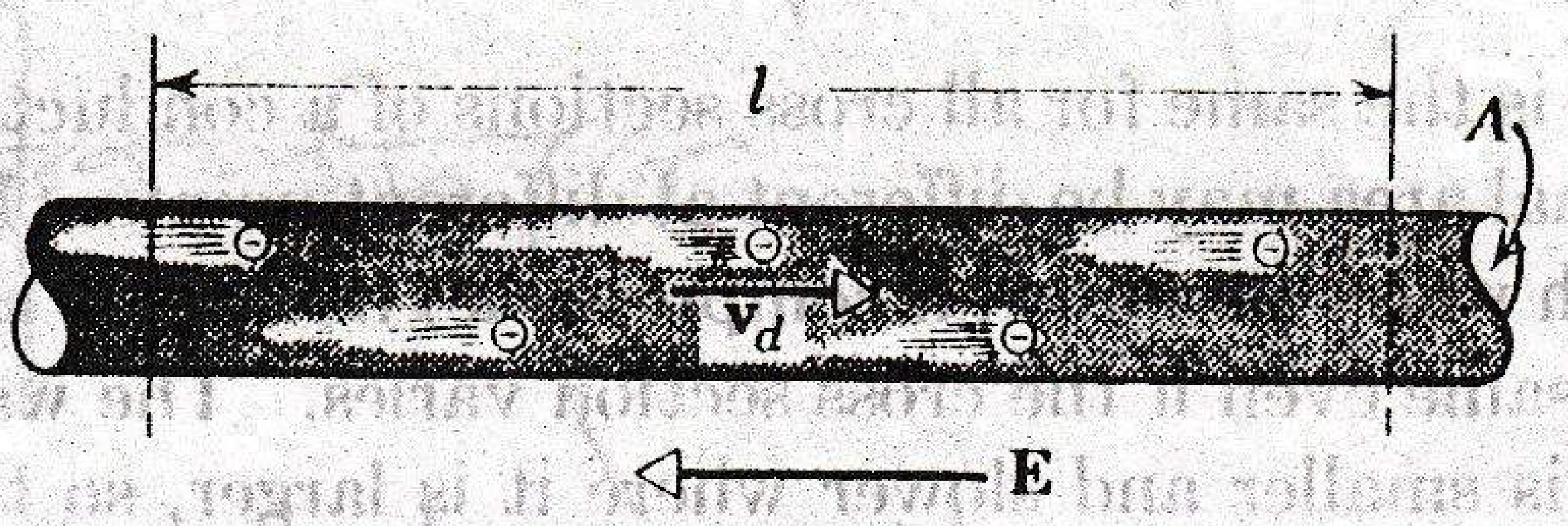


Fig. 31–1 Electrons drift in a direction opposite to the electric field in a conductor.

vector and is characteristic of a point inside a conductor rather than of the conductor as a whole. If the current is distributed uniformly across a conductor of cross-sectional area A, the magnitude of the current density for all points on that cross section is

$$j = i/A. \tag{31-3}$$

The vector $\mathbf{j}$ at any point is oriented in the direction that a positive charge carrier would move at that point. An electron at that point would move in the direction $-\mathbf{j}$.

The general relationship between $\mathbf{j}$ and i is that, for a particular surface in a conductor, i is the flux of the vector $\mathbf{j}$ over that surface, or

$$i = \int \mathbf{j} \cdot d\mathbf{S}, \tag{31-4}$$

where $d\mathbf{S}$ is an element of surface area and the integral is taken over the surface in question. Equation 31–3 (written as $i = jA$) is a special case of this relationship in which the surface of integration is a cross section of the conductor and in which $\mathbf{j}$ is constant over this surface and at right angles to it. However, Eq. 31–4 may be applied to any surface through which we wish to know the current. Equation 31–4 shows clearly that i is a scalar because the integrand $\mathbf{j} \cdot d\mathbf{S}$ is a scalar.

The arrow often associated with the current in a wire does not indicate that current is a vector but merely shows the *sense* of charge flow. Positive charge carriers either move in a certain direction along the wire or in the opposite direction, these two possibilities being represented by + or − in algebraic equations. Note that (*a*) the current in a wire remains unchanged if the wire is bent, tied into a knot, or otherwise distorted, and (*b*) the arrows representing the sense of currents do not in any way obey the laws of vector addition.

The drift speed v_d of charge carriers in a conductor can be computed from the current density j. Figure 31–1 shows the conduction electrons in a wire moving to the right at an assumed constant drift speed v_d. The number of conduction electrons in the wire is nAl where n is the number of conduction electrons per unit volume and Al is the volume of the wire. A charge of magnitude

$$q = (nAl)$$

passes out of the wire, through its right end, in a time t given by

$$t = \frac{l}{v_d}.$$

The current i is given by

$$i = \frac{q}{t} = \frac{nAle}{l/v_d} = nAev_d.$$

Solving for v_d and recalling that $j = i/A$ (Eq. 31-3) yields

$$v_d = \frac{i}{nAe} = \frac{j}{ne}. \qquad (31\text{-}5)$$

Example 1. An aluminum wire whose diameter is 0.10 in. is welded end to end to a copper wire with a diameter of 0.064 in. The composite wire carries a steady current of 10 amp. What is the current density in each wire?

The current is distributed uniformly over the cross section of each conductor, except near the junction, which means that the current density may be taken as constant for all points within each wire. The cross-sectional area of the aluminum wire is 0.0079 in.^{2} Thus, from Eq. 31-3,

$$j_{Al} = \frac{i}{A} = \frac{10 \text{ amp}}{0.0079 \text{ in.}^2} = 1300 \text{ amp/in.}^2$$

The cross-sectional area of the copper wire is 0.0032 in.^{2} Thus

$$j_{Cu} = \frac{i}{A} = \frac{10 \text{ amp}}{0.0032 \text{ in.}^2} = 3100 \text{ amp/in.}^2$$

The fact that the wires are of different materials does not enter into consideration here.

Example 2. What is v_d for the copper wire in Example 1?

We can write the current density for the copper wire as 480 amp/cm^2. To compute n we start from the fact that there is one free electron per atom in copper. The number of atoms per unit volume is dN_0/M where d is the density, N_0 is Avogadro's number, and M is the atomic weight. The number of free electrons per unit volume is then

$$n = \frac{dN_0}{M} = \frac{(9.0 \text{ gm/cm}^3)(6.0 \times 10^{23} \text{ atoms/mole})(1 \text{ electron/atom})}{64 \text{ gm/mole}}$$

$$= 8.4 \times 10^{22} \text{ electrons/cm}^3.$$

Finally, v_d is, from Eq. 31-5,

$$v_d = \frac{j}{ne} = \frac{480 \text{ amp/cm}^2}{(8.4 \times 10^{22} \text{ electrons/cm}^3)(1.6 \times 10^{-19} \text{ coul/electron})}$$

$$= 3.6 \times 10^{-2} \text{ cm/sec}.$$

It takes 28 sec for the electrons in this wire to drift 1.0 cm. Would you have guessed that v_d was so low? The drift speed of electrons must not be confused with the speed at which changes in the electric field configuration travel along wires, a speed which approaches that of light. When pressure is applied to one end of a long water-filled tube, a *pressure wave* travels rapidly along the tube. The speed at which *water* moves through the tube is much lower, however.

31–2 Resistance, Resistivity, and Conductivity

If the same potential difference is applied between the ends of a rod of copper and of a rod of wood, very different currents result. The characteristic of the conductor that enters here is its *resistance*. We define the resistance of a conductor (often called a *resistor;* symbol -VVVV-) between two points by applying a potential difference V between those points, measuring the current i, and dividing:

$$R = V/i. \tag{31-6}$$

If V is in volts and i in amperes, the resistance R will be in *ohms*.

The flow of charge through a conductor is often compared with the flow of water through a pipe, which occurs because there is a difference in pressure between the ends of the pipe, established perhaps by a pump. This pressure difference can be compared with the potential difference established between the ends of a resistor by a battery. The flow of water (ft^3/sec, say) is compared with the current (coul/sec or amp). The rate of flow of water for a given pressure difference is determined by the nature of the pipe. Is it long or short? Is it narrow or wide? Is it empty or filled, perhaps with gravel? These characteristics of the pipe are analogous to the resistance of a conductor.

Primary standards of resistance, kept at the National Bureau of Standards, are spools of wire whose resistances have been accurately measured. Because resistance varies with temperature, these standards, when used, are placed in an oil bath at a controlled temperature. They are made of a special alloy, called *manganin*, for which the change of resistance with temperature is very small. They are carefully annealed to eliminate strains, which also affect the resistance. These primary standard resistors are used chiefly to calibrate secondary standards for other laboratories.

Operationally, the primary resistance standards are not measured by using Eq. 31–6 but are measured in an indirect way which involves magnetic fields. Equation 31–6 is, in fact, used to measure V, by setting up an accurately known current i (using a *current balance;* see Section 34–4) in an accurately known resistance R. This operational procedure for potential difference is the one normally used in place of the conceptual definition introduced in Section 29–1, in which one measures the work per unit charge required to move a test charge between two points.

Related to resistance is the *resistivity* ρ, which is a characteristic of a material rather than of a particular specimen of a material; it is defined, for isotropic materials,* from

$$\rho = \frac{E}{j}. \tag{31-7}$$

The resistivity of copper is 1.7×10^{-8} ohm-m; that of fused quartz is about 10^{16} ohm-m. Few physical properties are measurable over such a range of values; Table 31–1 lists some values for common metals.

* These are materials whose properties (electrical in this case) do not vary with direction in the material.

Table 31-1

PROPERTIES OF METALS AS CONDUCTORS

	Resistivity (at 20°C), ohm-m	Temperature Coefficient of Resistivity,* α, per C°	Density, gm/cm^3	Melting Point, °C
Aluminum	2.8×10^{-8}	3.9×10^{-3}	2.7	659
Copper	1.7×10^{-8}	3.9×10^{-3}	8.9	1080
Carbon (amorphous)	3.5×10^{-5}	-5×10^{-4}	1.9	3500
Iron	1.0×10^{-7}	5.0×10^{-3}	7.8	1530
Manganin	4.4×10^{-7}	1×10^{-5}	8.4	910
Nickel	6.8×10^{-8}	6×10^{-3}	8.9	1450
Silver	1.6×10^{-8}	3.8×10^{-3}	10.5	960
Steel	1.8×10^{-7}	3×10^{-3}	7.7	1510
Wolfram (tungsten)	5.6×10^{-8}	4.5×10^{-3}	19	3400

* This quantity, defined from

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT} \tag{31-8}$$

is the fractional change in resistivity ($d\rho/\rho$) per unit change in temperature. It varies with temperature, the values here referring to 20°C. For copper ($\alpha = 3.9 \times 10^{-3}$/C°) the resistivity increases by 0.39 per cent for a temperature increase of 1 C° near 20°C. Note that α for carbon is negative, which means that the resistivity *decreases* with increasing temperature.

Consider a cylindrical conductor, of cross-sectional area A and length l, carrying a steady current i. Let us apply a potential difference V between its ends. If the cylinder cross sections at each end are equipotential surfaces, the electric field strength and the current density will be constant for all points in the cylinder and will have the values

$$E = \frac{V}{l} \quad \text{and} \quad j = \frac{i}{A}.$$

The resistivity ρ may then be written as

$$\rho = \frac{E}{j} = \frac{V/l}{i/A}.$$

But V/i is the resistance R which leads to

$$R = \rho\frac{l}{A}. \tag{31-9}$$

V, i, and R are *macroscopic* quantities, applying to a particular body or extended region. The corresponding *microscopic* quantities are **E**, **j**, and ρ; they have values at every point in a body. The macroscopic quantities are related to each other by Eq. 31–6 ($V = iR$) and the microscopic quantities by Eq. 31–7, which can be written in vector form as $\mathbf{E} = \mathbf{j}\rho$.

The macroscopic quantities can be found by integrating over the microscopic quantities, using relations already given, namely

$$i = \int \mathbf{j} \cdot d\mathbf{S} \tag{31-4}$$

and

$$V_{ab} = -\int_a^b \mathbf{E} \cdot d\mathbf{l}. \tag{29-5}$$

The integral in Eq. 31–4 is a surface integral, carried out over any cross section of the conductor. The integral in Eq. 29–5 is a line integral carried out along an arbitrary line drawn along the conductor, connecting any two equipotential surfaces, identified by a and b. For a long wire connected to a battery equipotential surface a might be chosen as a cross section of the wire near the positive battery terminal and b might be a cross section near the negative terminal.

The resistance of a conductor between a and b can be expressed in microscopic terms by dividing the two equations, or

$$R = \frac{V_{ab}}{i} = \frac{-\int_a^b \mathbf{E} \cdot d\mathbf{l}}{\int \mathbf{j} \cdot d\mathbf{S}}.$$

If the conductor is a long cylinder of cross section A and length l and if points a and b are its ends, the foregoing equation for R (see Eq. 31–7) reduces to

$$R = \frac{El}{jA} = \rho \frac{l}{A},$$

which is Eq. 31–9.

The macroscopic quantities V, i, and R are of primary interest when we are making electrical measurements on real conducting objects. They are the quantities that one reads on meters. The microscopic quantities **E**, **j**, and ρ are of primary importance when we are concerned with the fundamental behavior of matter (rather than of specimens of matter), as we usually are in the research area of *solid state physics*. Thus Section 31–4 deals appropriately with an atomic view of the *resistivity* of a metal and not of the *resistance* of a metallic specimen. The microscopic quantities are also important when we are interested in the interior behavior of irregularly shaped conducting objects.

Example 3. A rectangular carbon block has dimensions 1.0 cm × 1.0 cm × 50 cm. (*a*) What is the resistance measured between the two square ends? (*b*) Between two opposing rectangular faces? The resistivity of carbon at 20°C is 3.5×10^{-5} ohm-m.

(*a*) The area of a square end is 1.0 cm² or 1.0×10^{-4} meter². Equation 31–9 gives for the resistance between the square ends:

$$R = \rho \frac{l}{A} = \frac{(3.5 \times 10^{-5}\text{ ohm-m})(0.50\text{ meter})}{1.0 \times 10^{-4}\text{ meter}^2} = 0.18\text{ ohm}.$$

(*b*) For the resistance between opposing rectangular faces (area = 5.0×10^{-3} meter²), we have

$$R = \rho \frac{l}{A} = \frac{(3.5 \times 10^{-5}\text{ ohm-m})(10^{-2}\text{ meter})}{5.0 \times 10^{-3}\text{ meter}^2} = 7.0 \times 10^{-5}\text{ ohm}.$$

Thus a given conductor can have a number of resistances, depending on how the potential difference is applied to it. The ratio of resistances for these two cases is 2600. We assume in each that the potential difference is applied to the block in such a way that the surfaces between which the resistance is desired are equipotential. Otherwise Eq. 31-9 would not be valid. ◀

Figure 31-2 shows (solid curve) how the resistivity of copper varies with temperature. Sometimes, for practical use, such data are expressed in equation form. If we are interested in only a limited range of temperatures extending, say, from 0 to 500°C, we can fit a straight line to the curve of Fig. 31-2, making it pass through two arbitrarily selected points; see the dashed line. We choose the point labeled T_0, ρ_0 in the figure as a reference point, T_0 being 0°C in this case and ρ_0 being 1.56×10^{-8} ohm-m. The resistivity ρ at any temperature T can be found from the empirical equation of the dashed straight line in Fig. 31-2, which is

$$\rho = \rho_0[1 + \bar{\alpha}(T - T_0)]. \tag{31-10}$$

This relation shows correctly that $\rho \rightarrow \rho_0$ as $T \rightarrow T_0$.

If we solve Eq. 31-10 for $\bar{\alpha}$, we obtain

$$\bar{\alpha} = \frac{1}{\rho_0}\frac{\rho - \rho_0}{T - T_0}.$$

Comparison with Eq. 31-8 shows that $\bar{\alpha}$ is a *mean temperature coefficient of resistivity* for a selected pair of temperatures rather than the temperature coefficient of resistivity at a particular temperature, which is the definition of α. For most practical purposes Eq. 31-10 gives results that are within the acceptable range of accuracy.

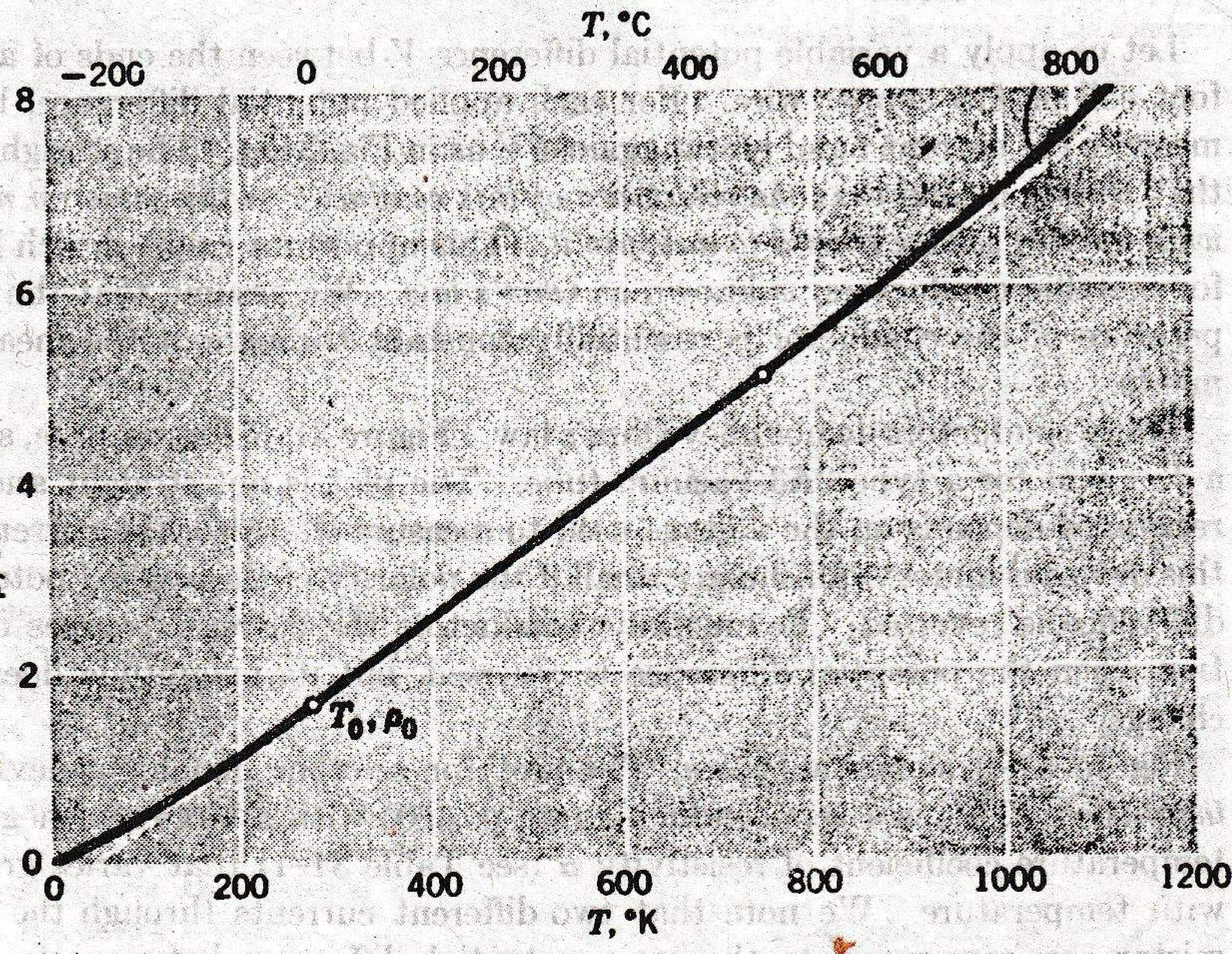


Fig. 31-2 The resistivity of copper as a function of temperature. The dashed line is an approximation chosen to fit the curve at the two circled points. The point marked T_0, ρ_0 is chosen as a reference point.

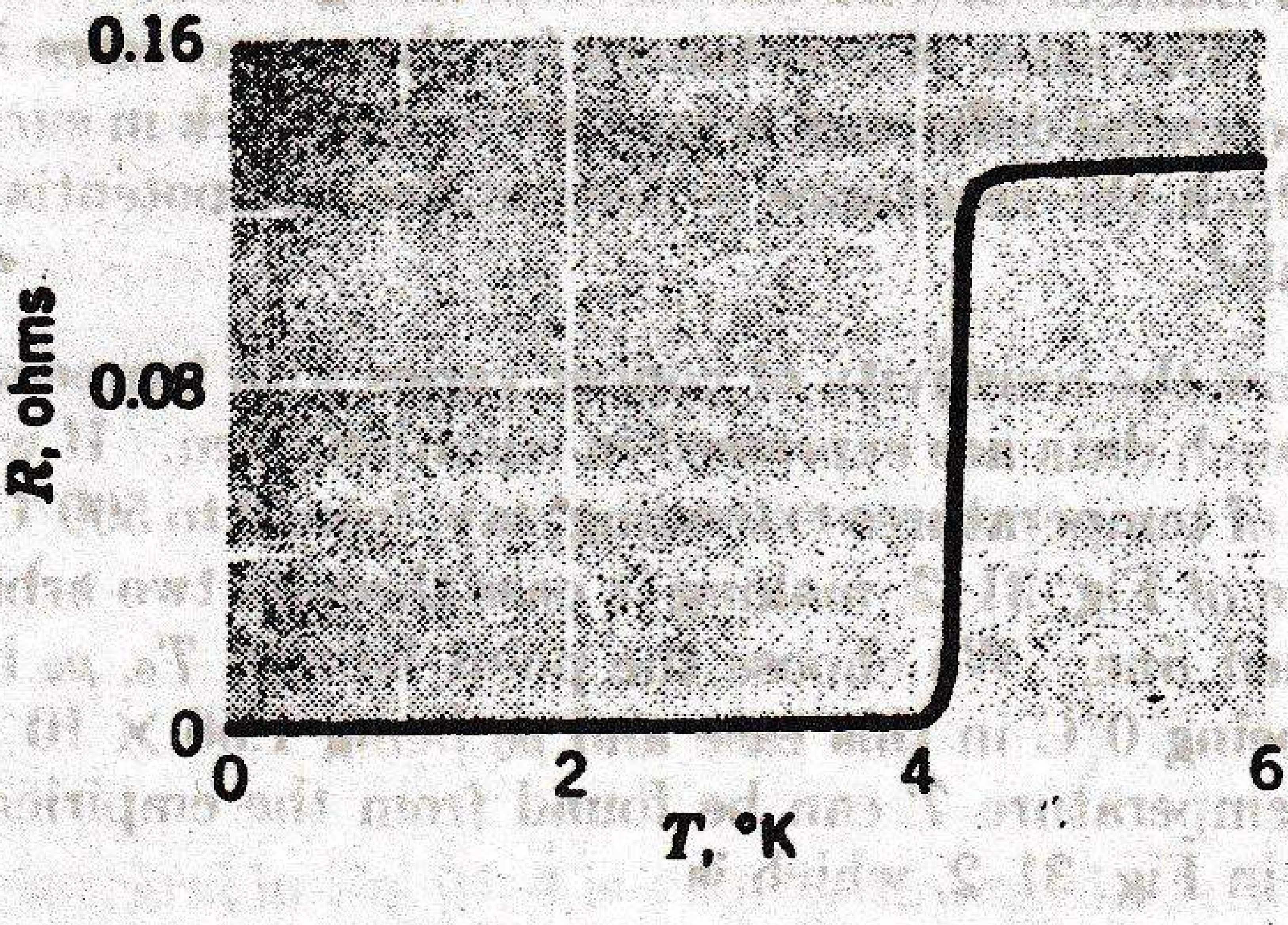


Fig. 31–3 The resistance of mercury disappears below about 4°K.

The curve of Fig. 31–2 does not go to zero at the absolute zero of temperature, even though it appears to do so, the residual resistivity at this temperature being 0.02×10^{-8} ohm-m. For many substances the resistance *does* become zero at some low temperature. Figure 31–3 shows the resistance of a specimen of mercury for temperatures below 6°K. In the space of about 0.05 K° the resistance drops abruptly to an immeasurably low value. This phenomenon, called *superconductivity*,* was discovered by Kamerlingh Onnes in the Netherlands in 1911. The resistance of materials in the superconducting state seems to be truly zero; currents, once established in closed superconducting circuits, persist for weeks without diminution, even though there is no battery in the circuit. If the temperature is raised slightly above the superconducting point, such currents drop rapidly to zero.

31–3 Ohm's Law

Let us apply a variable potential difference V between the ends of a 100-foot coil of #18 copper wire. For each applied potential difference, let us measure the current i and plot it against V as in Fig. 31–4. The straight line that results means that *the resistance of this conductor is the same no matter what applied voltage is used to measure it.* This important result, which holds for metallic conductors, is known as *Ohm's law.* We assume that the temperature of the conductor is essentially constant throughout the measurements.

Many conductors do not obey Ohm's law. Figure 31–5, for example, shows a V–i plot for a type 2A3 vacuum tube. The plot is not straight and the resistance depends on the voltage used to measure it. Also, the current for this device is almost vanishingly small if the polarity of the applied potential difference is reversed. For metallic conductors the current reverses direction when the potential difference is reversed, but its magnitude does not change.

Figure 31–6 shows a typical V–i plot for another nonohmic device, a *thermistor*. This is a semiconductor (see p. 650) with a large and negative temperature coefficient of resistivity α (see Table 31–1) that varies greatly with temperature. We note that two different currents through the thermistor can correspond to the same potential difference between its ends.

* See "Superconductivity" by B. T. Matthias, *Scientific American*, p. 92, November 1957.

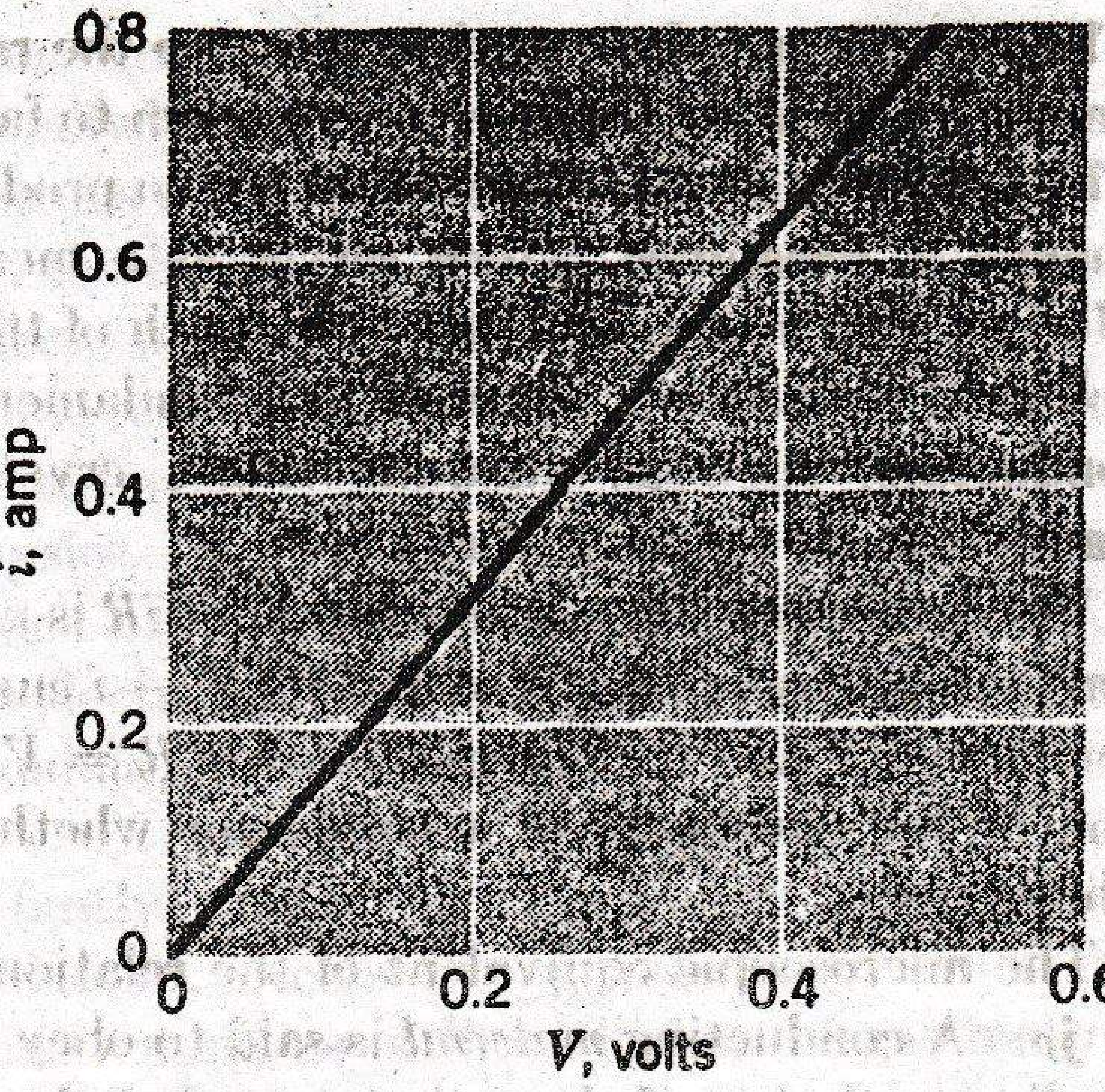


Fig. 31-4 The current in a particular copper conductor as a function of potential difference. This conductor obeys Ohm's law.

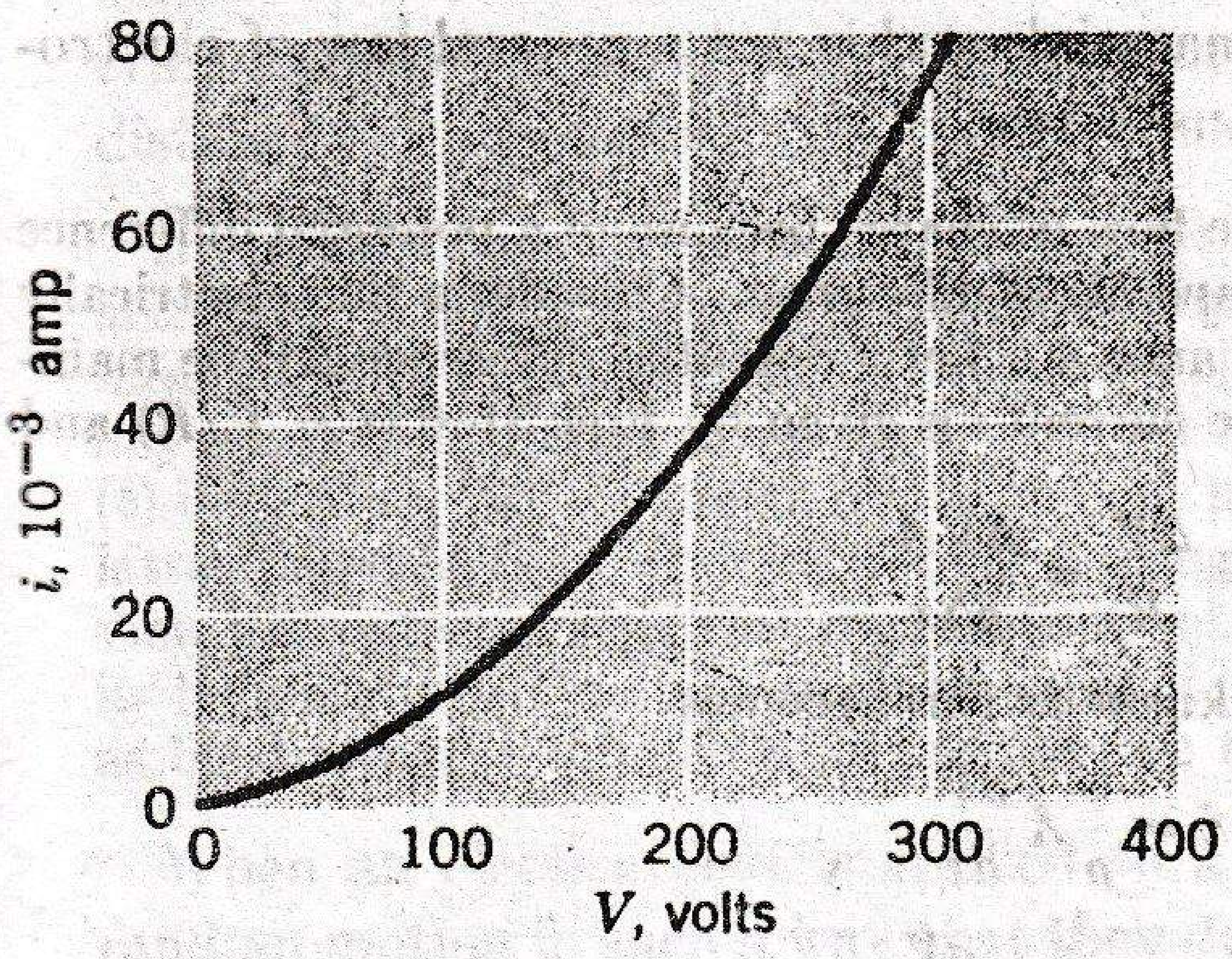


Fig. 31-5 The current in a type 2A3 vacuum tube as a function of potential difference. This conductor does not obey Ohm's law.

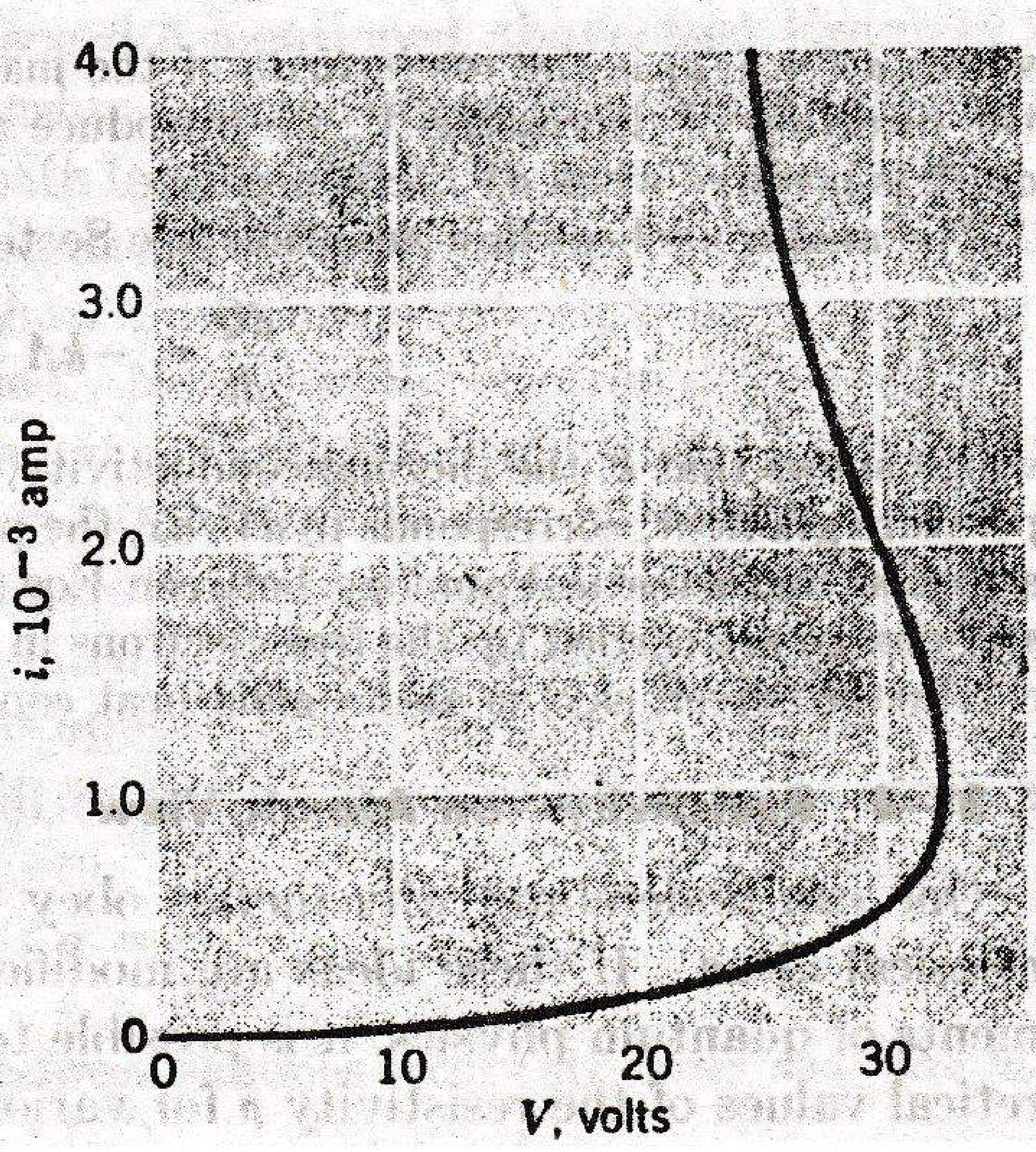


Fig. 31-6 A plot of current as a function of potential difference in a Western Electric 1-B thermistor. The curve shows how the voltage across the thermistor varies as the current through it is increased. The shape of the curve can be accounted for in terms of the large negative temperature coefficient of resistivity of the material of which the device is made.

Thermistors are often used to measure the rate of energy flow in microwave beams by allowing the microwave beam to fall on the thermistor and heat it. The relatively small temperature rise so produced results in a relatively large change in resistance, which serves as a measure of the microwave power. Modern electronics, and therefore much of the character of our present technological civilization, depends in a fundamental way on the fact that many conductors, such as vacuum tubes, crystal rectifiers, thermistors, and transistors, do *not* obey Ohm's law.

We stress that the relationship $V = iR$ is *not* a statement of Ohm's law. A conductor obeys this law only if its $V - i$ curve is linear, that is, if R is independent of V and i. The relationship $R = V/i$ remains as the general definition of the resistance of a conductor whether or not the conductor obeys Ohm's law.

The microscopic equivalent of the relationship $V = iR$ is Eq. 31-7, or $\mathbf{E} = \mathbf{j}\rho$. A conducting *material* is said to obey Ohm's law if a plot of E versus j is linear, that is, if the resistivity ρ is independent of E and j. Ohm's law is a specific property of certain materials and is not a general law of electromagnetism, for example, like Gauss's law.

A close analogy exists between the flow of charge because of a potential difference and the flow of heat because of a temperature difference. Consider a thin electrically conducting slab of thickness Δx and area A. Let a potential difference ΔV be maintained between opposing faces. The current i is given by Eqs. 31-6 ($i = V/R$) and 31-9 ($R = \rho l/A$), or

$$i = \frac{\Delta V}{R} = \frac{\Delta V A}{\rho \Delta x}.$$

In the limiting case of a slab of thickness dx this becomes

$$i = \frac{1}{\rho} A \frac{dV}{dx}$$

or

$$\frac{dq}{dt} = -\sigma A \frac{dV}{dx}, \tag{31-11}$$

where σ ($= 1/\rho$) is the *conductivity* of the material. Since positive charge flows in the direction of decreasing V, we introduce a minus sign into Eq. 31-11, that is, dq/dt is positive when dV/dx is negative.

The analogous heat flow equation (see Section 22-4) is

$$\frac{dQ}{dt} = -kA \frac{dT}{dx}, \tag{31-12}$$

which shows that k, the thermal conductivity, corresponds to σ and dT/dx, the temperature gradient, corresponds to dV/dx, the potential gradient. There is more than a formal mathematical analogy between Eqs. 31-11 and 31-12. Both heat energy and charge are carried by the free electrons in a metal; empirically, a good electrical conductor (silver, say) is also a good heat conductor and conversely.

31-4 Resistivity—an Atomic View

One can understand why metals obey Ohm's law on the basis of simple classical ideas. If these ideas are modified when necessary by the requirements of quantum physics, it is possible to go further and to calculate theoretical values of the resistivity ρ for various metals. These calculations are not simple, but when they have been carried out the agreement with the experimental value of ρ has usually been good.

In a metal the valence electrons are not attached to individual atoms but are free to move about within the lattice and are called *conduction electrons*. In copper there is one such electron per atom, the other 28 remaining bound to the copper nuclei to form ionic cores.

The speed distribution of conduction electrons can be described correctly only in terms of quantum physics. For our purposes, however, it suffices to consider only a suitably defined average speed $\bar{v}$; for copper $\bar{v} = 1.6 \times 10^8$ cm/sec. In the absence of an electron field, the directions in which the electrons move are completely random, like those of the molecules of a gas confined to a container.

The electrons collide constantly with the ionic cores of the conductor, that is, they interact with the lattice, often suffering sudden changes in speed and direction. These collisions remind us of the collisions of gas molecules confined to a container. As in the case of molecular collisions, we can describe electron-lattice collisions by a *mean free path* λ, where λ is the average distance that an electron travels between collisions.*

In an ideal metallic crystal at 0°K electron-lattice collisions would not occur, according to the predictions of quantum physics, that is, $\lambda \to \infty$ as $T \to 0$°K for ideal crystals. Collisions take place in actual crystals because (*a*) the ionic cores at any temperature T are vibrating about their equilibrium positions in a random way, (*b*) impurities, that is, foreign atoms, may be present, and (*c*) the crystal may contain lattice imperfections, such as rows of missing atoms and displaced atoms. On this view it is not surprising that the resistivity of a metal can be increased by (*a*) raising its temperature, (*b*) adding small amounts of impurities, and (*c*) straining it severely, as by drawing it through a die, to increase the number of lattice imperfections.

When an electric field is applied to a metal, the electrons modify their random motion in such a way that they drift slowly, in the opposite direction to that of the field, with an average drift speed v_d. This drift speed is much less than the effective average speed $\bar{v}$ mentioned above (see Example 2). Figure 31-7 suggests the relationship between these two speeds. The solid lines suggest a possible random path followed by an electron in the absence of an applied field; the electron proceeds from x to y, making six collisions on the way. The dashed curves show how this same event *might* have occurred if an electric field **E** had been applied. Note that the electron drifts steadily to the right, ending at y' rather than at y. In preparing Fig. 31-7, it has been assumed that the drift speed v_d is $0.02\bar{v}$; actually, it is more like $10^{-10}\bar{v}$, so that the "drift" exhibited in the figure is greatly exaggerated.

The drift speed v_d can be calculated in terms of the applied electric field E and of $\bar{v}$ and λ. When a field is applied to an electron in the metal it will experience a force eE which will impart to it an acceleration a given by Newton's second law,

$$a = \frac{eE}{m}.$$

* It can be shown that collisions between electrons occur only rarely and have little effect on the resistivity.

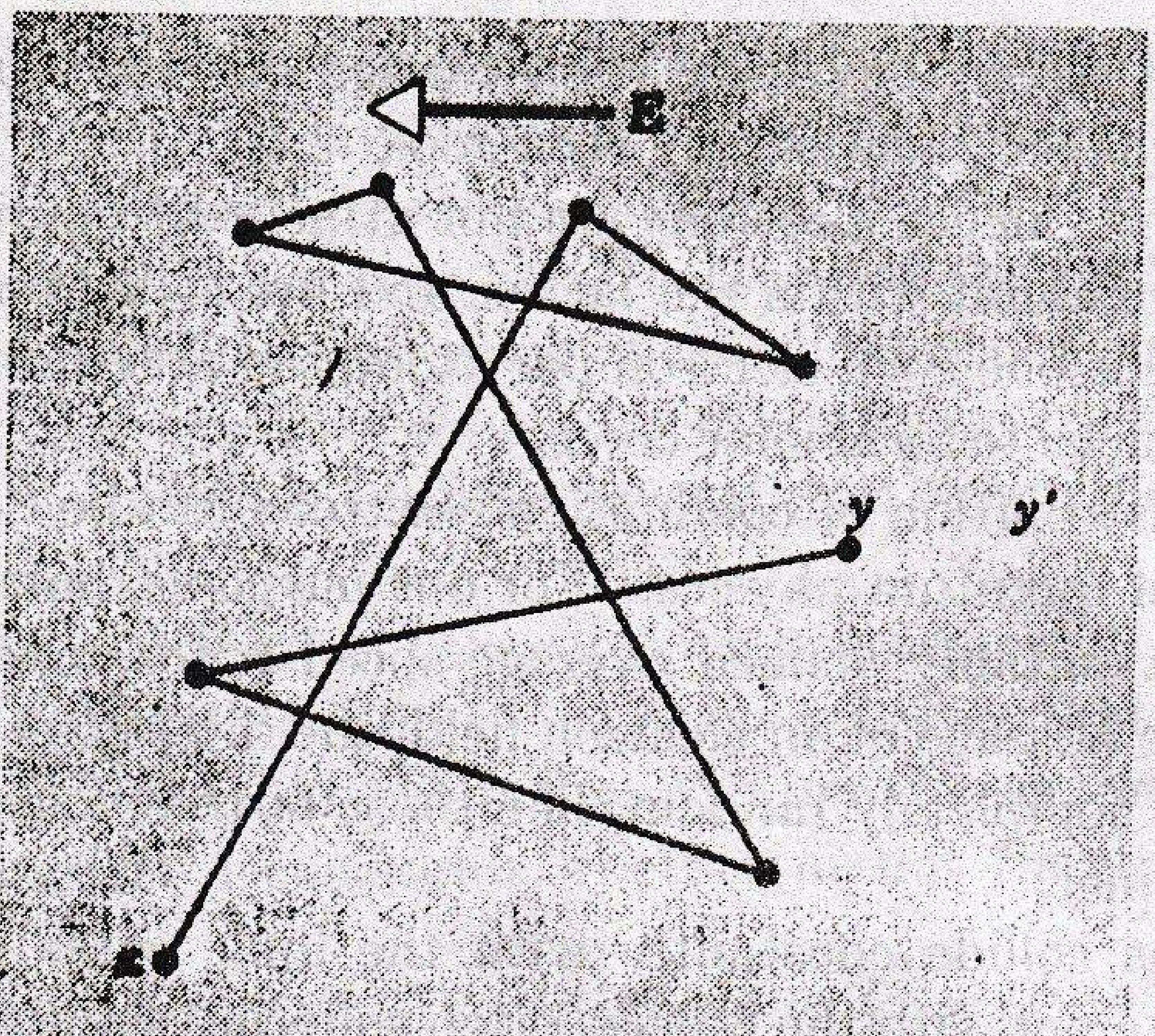


Fig. 31–7 The solid lines show an electron moving from x to y, making six collisions. The dashed curves show what the electron path *might* have been in the presence of an electric field **E**. Note the steady drift in the direction of $-\mathbf{E}$.

Consider an electron that has just collided with an ion core. The collision, in general, will momentarily destroy the tendency to drift and the electron will have a truly random direction after the collision. At its next collision the electron's velocity will have changed, on the average, by $a(l/\bar{v})$ where $l/\bar{v}$ is the mean time between collisions. We call this the drift speed v_d, or

$$v_d = a\left(\frac{\lambda}{\bar{v}}\right) = \frac{eE\lambda}{m\bar{v}}. \qquad (31\text{–}13)$$

The electron's motion through the conductor is analogous to the constant rate of fall of a stone in water. The gravitational force F_g on the stone is opposed by a viscous resisting force that is proportional to the velocity, or

$$F_g = mg = bv,$$

where b is a viscous coefficient (see Section 15–8). Thus the constant terminal speed of the stone is

$$v = \left(\frac{1}{b}\right) F_g.$$

We can rewrite Eq. 31–13 as

$$v_d = \left(\frac{\lambda}{m\bar{v}}\right) F_E,$$

where F_E ($= eE$) is the electrical force. Comparison of these equations shows that the equivalent "viscous coefficient" for the motion of an electron in a particular conductor is $\lambda/2m\bar{v}$. If λ is short, the conductor exhibits a greater "viscous effect" on the electron motion, and the drift speed v_d is proportionally lower.

We may express v_d in terms of the current density (Eq. 31–5) and combine with Eq. 31–13 to obtain

$$v_d = \frac{j}{ne} = \frac{eE\lambda}{m\bar{v}}.$$

Combining this with Eq. 31-7 ($\rho = E/j$) leads finally to

$$\rho = \frac{m\bar{v}}{ne^2\lambda}. \tag{31-14}$$

Equation 31-14 can be taken as a statement that metals obey Ohm's law *if* we can show that $\bar{v}$ and λ do not depend on the applied electric field E. In this case ρ will not depend on E, which (see Section 31-3) is the criterion that a material obey Ohm's law. The quantities $\bar{v}$ and λ depend on the speed distribution of the conduction electrons. We have seen that this distribution is affected only slightly by the application of even a relatively large electric field, since $\bar{v}$ is of the order of 10^8 cm/sec and v_d (see Example 1) only of the order of 10^{-2} cm/sec, a ratio of 10^{10}. We may be sure that whatever the values of $\bar{v}$ and λ are (for copper at 20°C, say) in the absence of a field they remain essentially unchanged when the field is applied. Thus the right side of Eq. 31-14 is independent of E and the material obeys Ohm's law. The numerical calculation of ρ from Eq. 31-14 is hampered by the difficulty of calculating λ, although the calculation has been carried out in a number of cases.

▶ **Example 4.** What are (*a*) the mean time τ between collisions and (*b*) the mean free path for free electrons in copper?

(*a*) From Eq. 31-14 (see also Example 2), we have

$$\tau = \frac{\lambda}{\bar{v}} = \frac{m}{ne^2\rho} = \frac{(9.1 \times 10^{-31}\ \text{kg})}{(8.4 \times 10^{28}/\text{meter}^3)(1.6 \times 10^{-19}\ \text{coul})^2(1.7 \times 10^{-8}\ \text{ohm-m})}$$

$$= 2.5 \times 10^{-14}\ \text{sec}.$$

(*b*) The mean free path is

$$\lambda = \tau\bar{v} = (2.5 \times 10^{-14}\ \text{sec})(1.6 \times 10^8\ \text{cm/sec}) = 4.0 \times 10^{-6}\ \text{cm}.$$

This is about 200 ionic diameters. ◀

31-5 Energy Transfers in an Electric Circuit

Figure 31-8 shows a circuit consisting of a battery B connected to a "black box." A steady current i exists in the connecting wires and a steady potential difference V_{ab} exists between the terminals a and b. The box might contain a resistor, a motor, or a storage battery, among other things.

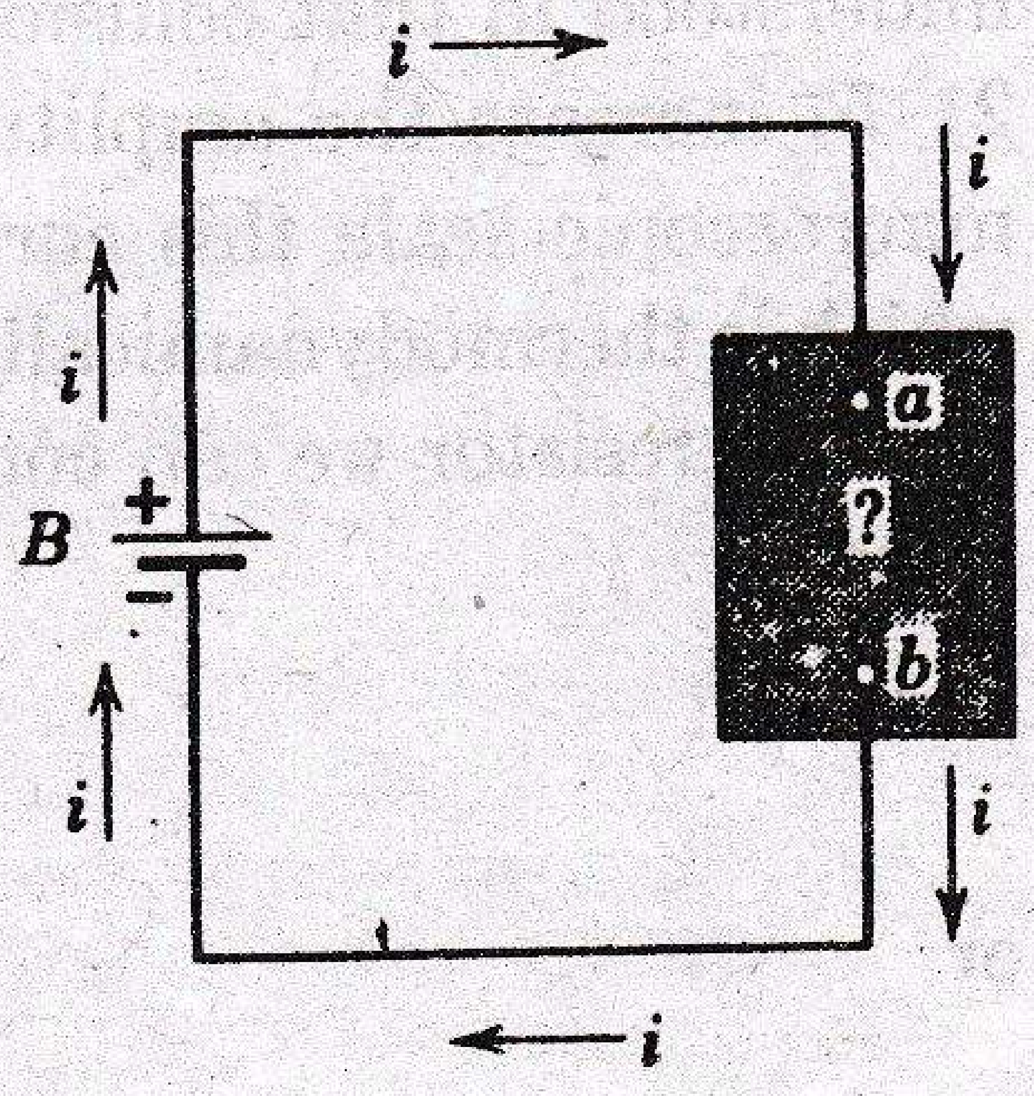


Fig. 31-8 A battery B sets up a current in a circuit containing a "black box."

Terminal a, connected to the positive battery terminal, is at a higher potential than terminal b. If a charge dq moves through the box from a to b, this charge will decrease its electric potential energy by $dq\,V_{ab}$ (see Section 29-6). The conservation-of-energy principle tells us that this energy is transferred in the box from electric potential energy to some other form. What that other form will be depends on what is in the box. In a time dt the energy dU transferred inside the box is then

$$dU = dq\,V_{ab} = i\,dt\,V_{ab}.$$

We find the *rate* of energy transfer P by dividing by the time, or

$$P = \frac{dU}{dt} = iV_{ab}. \qquad (31\text{-}15)$$

If the device in the box is a motor, the energy appears largely as mechanical work done by the motor; if the device is a storage battery that is being charged, the energy appears largely as stored chemical energy in this second battery.

If the device is a resistor, we assert that the energy appears as heat in the resistor. To see this, consider a stone of mass m that falls through a height h. It decreases its gravitational potential energy by mgh. If the stone falls in a vacuum or—for practical purposes—in air, this energy is transformed into kinetic energy of the stone. If the stone falls in water, however, its speed eventually becomes constant, which means that the kinetic energy no longer increases. The potential energy that is steadily being made available as the stone falls then appears as thermal energy in the stone and the surrounding water. It is the viscous, friction-like drag of the water on the surface of the stone that stops the stone from accelerating, and it is at this surface that thermal energy appears.

The course of the electrons through the resistor is much like that of the stone through water. The electrons travel with a constant drift speed v_d and thus do not gain kinetic energy. The electric potential energy that they lose is transferred to the resistor as heat. On a microscopic scale this can be understood in that collisions between the electrons and the lattice (see Fig. 21-5) increase the amplitude of the thermal vibrations of the lattice; on a macroscopic scale this corresponds to a temperature increase. This effect, which is thermodynamically irreversible, is called *Joule heating*.

For a resistor we can combine Eqs. 31-15 and 31-6 ($R = V/i$) and obtain either

$$P = i^2R \qquad (31\text{-}16)$$

or

$$P = \frac{V^2}{R}. \qquad (31\text{-}17)$$

Note that Eq. 31–15 applies to electrical energy transfer of *all* kinds; Eqs. 31–16 and 31–17 apply only to the transfer of electrical energy to heat energy in a resistor. Equations 31–16 and 31–17 are known as *Joule's law*. This law is a particular way of writing the conservation-of-energy principle for the special case in which electrical energy is transferred into heat energy.

The unit of power that follows from Eq. 31–15 is the volt-amp. It can be written as

$$1 \text{ volt-amp} = 1 \text{ volt-amp} \left(\frac{1 \text{ joule}}{1 \text{ volt} \times 1 \text{ coul}}\right)\left(\frac{1 \text{ coul}}{1 \text{ amp} \times 1 \text{ sec}}\right)$$

$$= 1 \text{ joule/sec}.$$

The first conversion factor in parenthesis comes from the definition of the volt (Eq. 29–1); the second comes from the definition of the coulomb. The joule/sec is such a common unit that it is given a special name of its own, the *watt;* see Section 7–7. Power is not an exclusively electrical concept, of course, and we can express in watts the power ($= \mathbf{F} \cdot \mathbf{v}$) expended by an agent that exerts a force $\mathbf{F}$ while it moves with a velocity $\mathbf{v}$.

▶ **Example 5.** You are given a 20-ft length of heating wire made of the special alloy Nichrome; it has a resistance of 24 ohms. Can you obtain more heat by winding one coil or by cutting the wire in two and winding two separate coils? In each case the coils are to be connected individually across a 110-volt line.

The power P for the single coil is given by Eq. 31–17:

$$P = \frac{V^2}{R} = \frac{(110 \text{ volts})^2}{24 \text{ ohms}} = 500 \text{ watts}.$$

The power for a coil of half the length is given by

$$P' = \frac{(110 \text{ volts})^2}{12 \text{ ohms}} = 1000 \text{ watts}.$$

There are two "half-coils," so that the total power obtained by cutting the wire in half is 2000 watts, or four times that for the single coil. This would seem to suggest that we could buy a 500-watt heating coil, cut it in half, and rewind it to obtain 2000 watts. Why is this not a practical idea? ◀

QUESTIONS

1. Name other physical quantities that, like current, are scalars having a sense represented by an arrow in a diagram.

2. What conclusions can you draw by applying Eq. 31–4 to a closed surface through which a number of wires pass in random directions, carrying steady currents of different sizes?

3. A potential difference V is applied to a circular cylinder of carbon by clamping it between circular copper electrodes, as in Fig. 31–9. Discuss the difficulty of calculating the resistance of the carbon cylinder, using the relation $R = \rho L/A$.

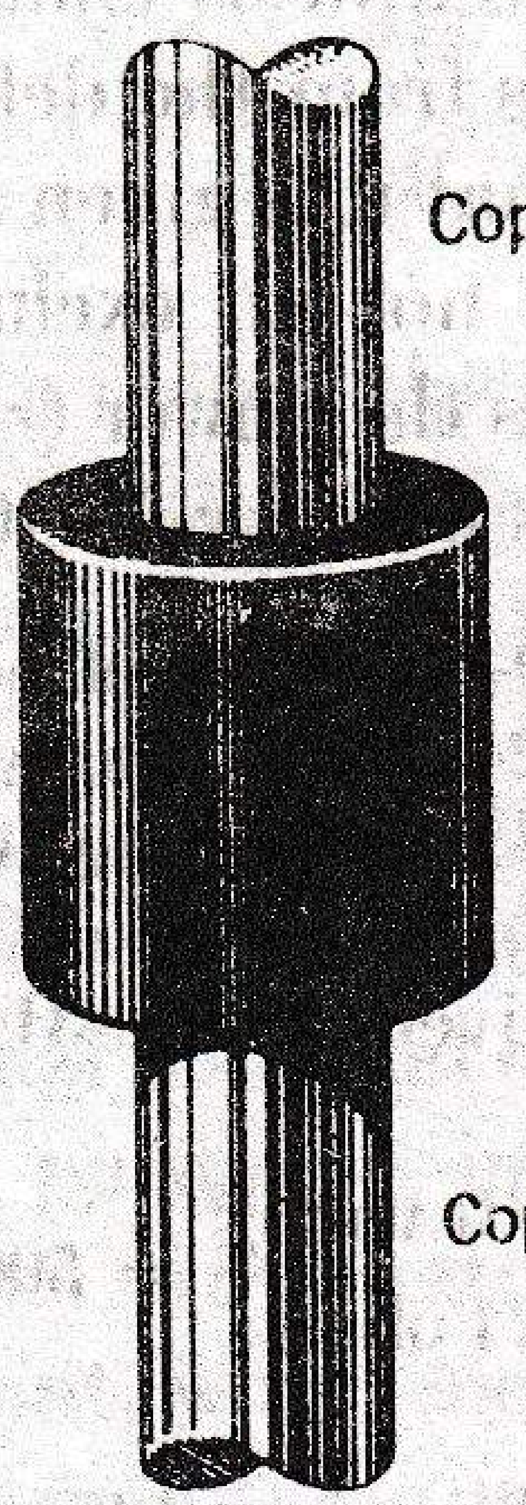


Fig. 31–9

4. How would you measure the resistance of a pretzel-shaped conductor? Give specific details to clarify the concept.

5. Discuss the difficulties of testing whether the filament of a light bulb obeys Ohm's law.

6. Does the relation $V = iR$ apply to nonohmic resistors?

7. The temperature coefficient of resistance of a thermistor is negative and varies greatly with temperature. Account qualitatively for the shape of the curve of i versus V for the thermistor of Fig. 31–6.

8. A potential difference V is applied to a copper wire of diameter d and length l. What is the effect on the electron drift speed of (*a*) doubling V, (*b*) doubling l, and (*c*) doubling d?

9. If the drift speeds of the electrons in a conductor under ordinary circumstances are so slow (see Example 2), why do the lights in a room turn on so quickly after the switch is closed?

10. Can you think of a way to measure the drift speed for electrons by timing their travel along a conductor?

11. Why are the dashed white lines in Fig. 31–7 curved slightly?

12. A current i enters the top of a copper sphere of radius R and leaves at a diametrically opposite point. Are all parts of the sphere equally effective in dissipating Joule heat?

13. What special characteristics must (*a*) heating wire and (*b*) fuse wire have?

14. Equation 31–16 ($P = i^2R$) seems to suggest that the rate of Joule heating in a resistor is reduced if the resistance is made less; Eq. 31–17 ($P = V^2/R$) seems to suggest just the opposite. How do you reconcile this apparent paradox?

15. Is the filament resistance lower or higher in a 500-watt light bulb than in a 100-watt bulb? Both bulbs are designed to operate on 110 volts.

16. Five wires of the same length and diameter are connected in turn between two points maintained at constant potential difference. Will heat be developed at the fastest rate in the wire of (*a*) the smallest or (*b*) the largest resistance?

PROBLEMS

1. A current of 5 amp exists in a 10-ohm resistance for 4 min. (*a*) How many coulombs and (*b*) how many electrons pass through any cross section of the resistor in this time?

2. A current is established in a gas discharge tube when a sufficiently high potential difference is applied across the two electrodes in the tube. The gas ionizes; electrons move toward the positive terminal and positive ions toward the negative terminal. What are the magnitude and sense of the current in a hydrogen discharge tube in which 3.1×10^{18} electrons and 1.1×10^{18} protons move past a cross-sectional area of the tube each second?

3. A copper wire and an iron wire of the same length have the same potential difference applied to them. (*a*) What must be the ratio of their radii if the current is to be the same? (*b*) Can the current density be made the same by suitable choices of the radii?

4. A current i enters one corner of a square sheet of copper and leaves at the opposite corner. Sketch arrows for various points within the square to represent the relative values of $\mathbf{j}$. Intuitive guesses rather than detailed mathematical analysis are called for.

5. The belt of an electrostatic generator is 50 cm wide and travels at 30 meters/sec. The belt carries charge into the sphere at a rate corresponding to 10^{-4} amp. Compute the surface charge density on the belt.

6. A square aluminum rod is 1.0 meter long and 5.0 mm on edge. (*a*) What is the resistance between its ends? (*b*) What must be the diameter of a circular 1.0-meter copper rod if its resistance is to be the same?

7. A wire with a resistance of 6.0 ohms is drawn out so that its new length is three times its original length. Find the resistance of the longer wire, assuming that the resistivity and density of the material are not changed during the drawing process.

8. A copper wire and an iron wire of equal length l and diameter d are joined and a potential difference V is applied between the ends of the composite wire. Calculate (*a*) the electric field strength in each wire, (*b*) the current density in each wire, and (*c*) the potential difference across each wire. Assume that l = 10 meters, d = 2.0 mm, and V = 100 volts.

9. A rod of a certain metal is 1.00 meter long and 0.550 cm in diameter. The resistance between its ends (at 20°C) is 2.87×10^{-3} ohm. A round disk is formed of this same material, 2.00 cm in diameter and 1.00 mm thick. (*a*) What is the resistance between the opposing round faces? (*b*) What is the material?

10. Steel trolley-car rail has a cross-sectional area of 7.1 in.^{2} What is the resistance of 10 miles of single track? The resistivity of the steel is 6.0×10^{-7} ohm-m.

11. (*a*) At what temperature would the resistance of a copper conductor be double its resistance at 0°C? (*b*) Does this same temperature hold for all copper conductors, regardless of shape or size?

12. It is desired to make a long cylindrical conductor whose temperature coefficient of resistivity at 20°C will be close to zero. (*a*) If such a conductor is made by assembling alternate disks of iron and carbon, what is the ratio of the thickness of a carbon disk to that of an iron disk? Assume that the temperature remains essentially the same in each disk. (*b*) What is the ratio of the rate of Joule heating in a carbon disk to that in an iron disk?

13. When a metal rod is heated, not only its resistance but also its length and its cross-sectional area change. The relation $R = \rho l/A$ suggests that all three factors should be taken into account in measuring ρ at various temperatures. If the temperature changes by 1.0 C°, what per cent changes in R, l, and A occur for a copper conductor. What conclusion do you draw? The coefficient of linear expansion is 1.7×10^{-5}/C°.

14. The copper windings of a motor have a resistance of 50 ohms at 20°C, when the motor is idle. After running for several hours the resistance rises to 58 ohms. What is the temperature of the windings?

15. (*a*) Using data from Fig. 31-5, plot the resistance of the vacuum tube as a function of applied potential difference. (*b*) Repeat for the thermistor of Fig. 31-6.

16. A small but measurable current of 10^{-10} amp exists in a copper wire whose diameter is 0.10 in. Calculate the electron drift speed.

17. Heat is developed in a resistor at a rate of 100 watts when the current is 3.0 amp. What is the resistance in ohms?

18. A potential difference of 1.0 volt is applied to a 100-ft length of #18 copper wire (diameter = 0.040 in.). Calculate (*a*) the current, (*b*) the current density, (*c*) the electric field strength, and (*d*) the rate of Joule heating.

19. The National Board of Fire Underwriters has fixed safe current-carrying capacities for various sizes and types of wire. For #10 rubber-coated copper wire (wire diameter = 0.10 in.) the maximum safe current is 25 amp. At this current, find (*a*) the current density, (*b*) the electric field strength, (*c*) the potential difference for 1000 ft of wire, and (*d*) the rate of Joule heating for 1000 ft of wire.

20. A 500-watt immersion heater is placed in a pot containing 2.0 liters of water at 20°C. (*a*) How long will it take to bring the water to boiling temperature, assuming that 80% of the available energy is absorbed by the water? (*b*) How much longer will it take to boil half the water away?

21. A nichrome heater dissipates 500 watts when the applied potential difference is 110 volts and the wire temperature is 800°C. How much power would it dissipate if the wire temperature were held to 200°C by immersion in a bath of cooling oil? The applied potential difference remains the same; $\bar{\alpha}$ for nichrome is about 4×10^{-4}/C°.

22. A beam of 16-Mev deuterons from a cyclotron falls on a copper block. The beam is equivalent to a current of 15×10^{-6} amp. (*a*) At what rate do deuterons strike the block? (*b*) At what rate is heat produced in the block?

23. A "500-watt" heating unit is designed to operate from a 115-volt line. (*a*) By what percentage will its heat output drop if the line voltage drops to 110 volts? Assume no change in resistance. (*b*) Taking the variation of resistance with temperature into account, would the actual heat output drop be larger or smaller than that calculated in (*a*)?

24. Show that p, the power per unit volume transformed into Joule heat in a resistor, can be written as

$$p = j^2\rho \quad \text{or} \quad p = E^2/\rho.$$