Electromotive Force and Circuits

CHAPTER 32

32-1

32-1 Electromotive Force

There exist in nature certain devices such as batteries and electric generators which are able to maintain a potential difference between two points to which they are attached. Such devices are called seats of *electromotive force* (abbr. emf). In this chapter we do not discuss their internal construction or detailed mode of action but confine ourselves to describing their gross electrical characteristics and to exploring their usefulness in electric circuits.

Figure 32–1*a* shows a seat of emf B, represented by a battery, connected to a resistor R. The seat of emf maintains its upper terminal positive and its lower terminal negative, as shown by the + and − signs. In the circuit external to B positive charge carriers would be driven in the direction shown by the arrows marked i. In other words, a clockwise current would be set up.

An emf is represented by an arrow which is placed next to the seat and points in the direction in which the seat, acting alone, would cause a positive charge carrier to move in the external circuit. A small circle is drawn on the tail of an emf arrow so that it will not be confused with a current arrow.

A seat of emf must be able to do work on charge carriers that enter it. In the circuit of Fig. 32–1*a*, for example, the seat acts to move positive charges from a point of low potential (the negative terminal) through the seat to a point of high potential (the positive terminal). This reminds us of a pump, which can cause water to move from a place of low gravitational potential to a place of high potential.

In Fig. 32–1*a* a charge dq passes through *any* cross section of the circuit in time dt. In particular, this charge enters the seat of emf $\mathcal{E}$ at its low-potential

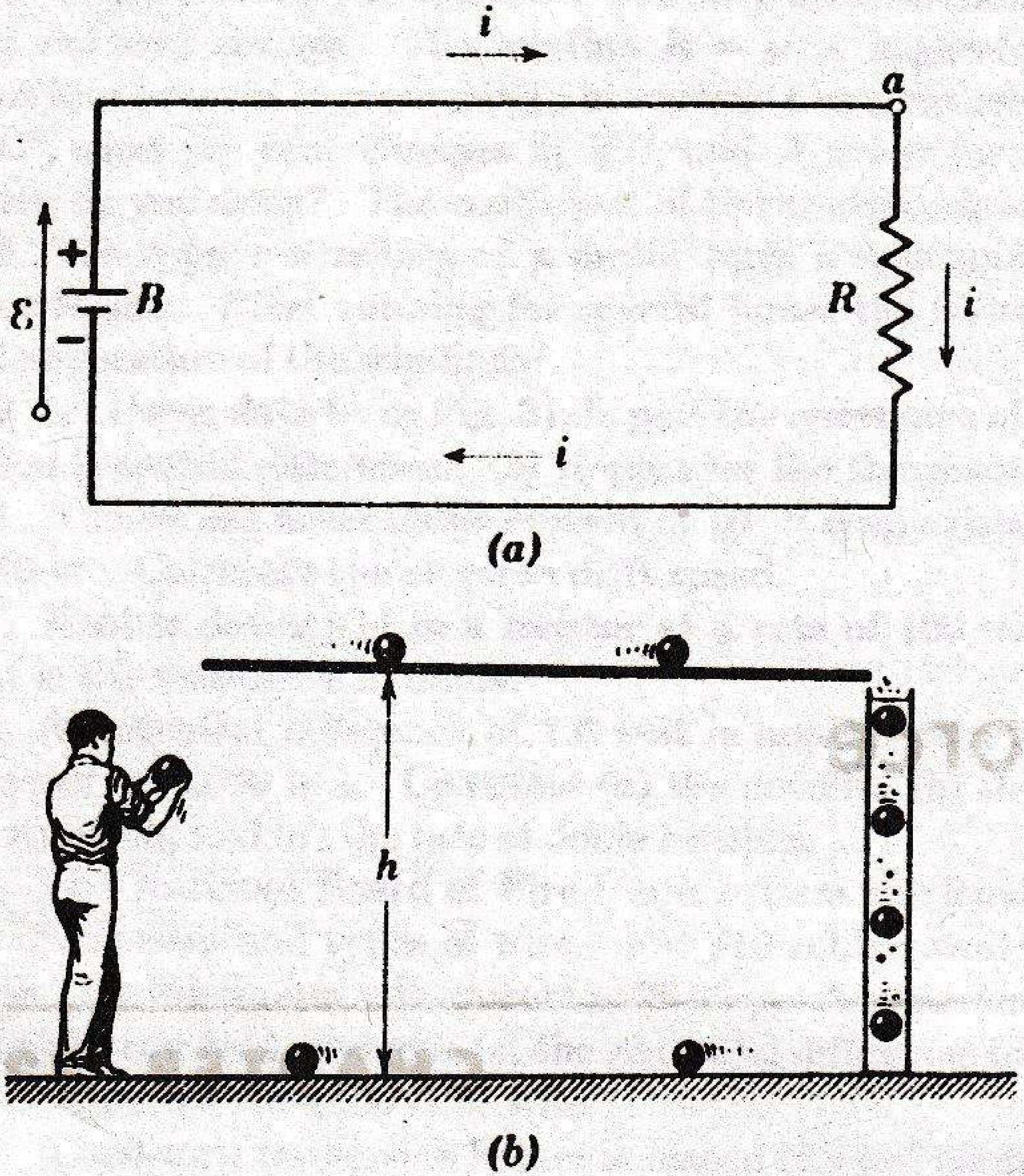


Fig. 32–1 (*a*) A simple electric circuit and (*b*) its gravitational analog.

end and leaves at its high-potential end. The seat must do an amount of work dW on the (positive) charge carriers to force them to go to the point of higher potential. The emf $\mathcal{E}$ of the seat is defined from

$$\mathcal{E} = dW/dq. \tag{32–1}$$

The unit of emf is the joule/coul (see Eq. 29–1) which is the *volt*. We might be inclined to say that a battery has an emf of 1 volt if it maintains a difference of potential of 1 volt between its terminals. This is true only under certain conditions, which we describe in Section 32–4.

If a seat of emf does work on a charge carrier, energy must be transferred within the seat. In a battery, for example, chemical energy is transferred into electrical energy. Thus we can describe a seat of emf as a device in which chemical, mechanical, or some other form of energy is changed (reversibly) into electrical energy. The chemical energy provided by the battery in Fig. 32–1a is stored in the electric and the magnetic * fields that surround the circuit. This stored energy does not increase because it is being drained away, by transfer to Joule heat in the resistor, at the same rate at which it is supplied. The electric and magnetic fields play an intermediary role in the energy transfer process, acting as a storage reservoir.

Figure 32–1b shows a gravitational analog of Fig. 32–1a. In the top figure the seat of emf B does work on the charge carriers. This energy, stored temporarily as electromagnetic field energy, appears eventually as Joule heat in resistor R. In the lower figure the man, in lifting the bowling balls from

* A current in a wire is surrounded by a magnetic field, and this field, like the electric field, can also be viewed as a site of stored energy (see Section 36–4).

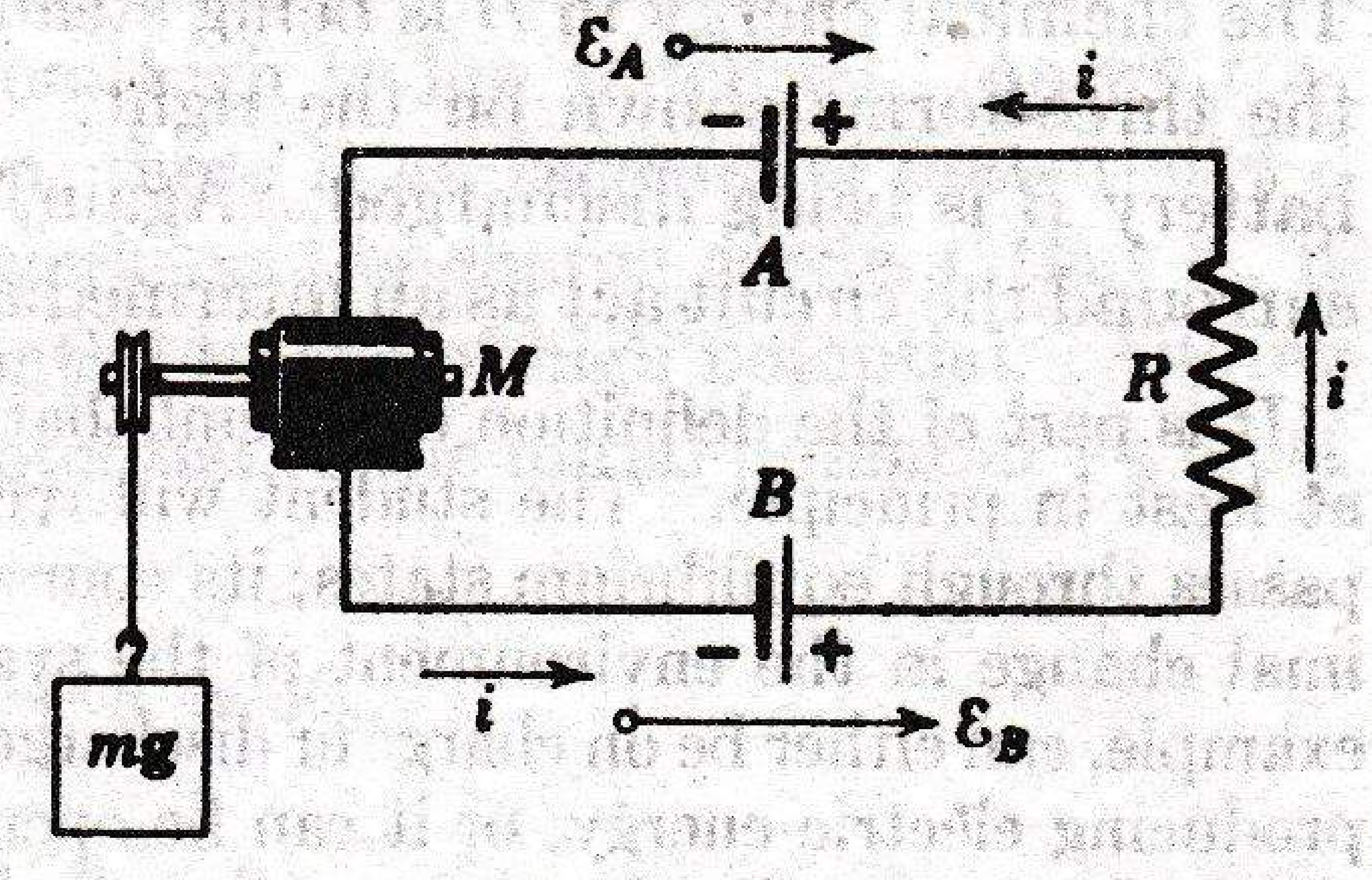
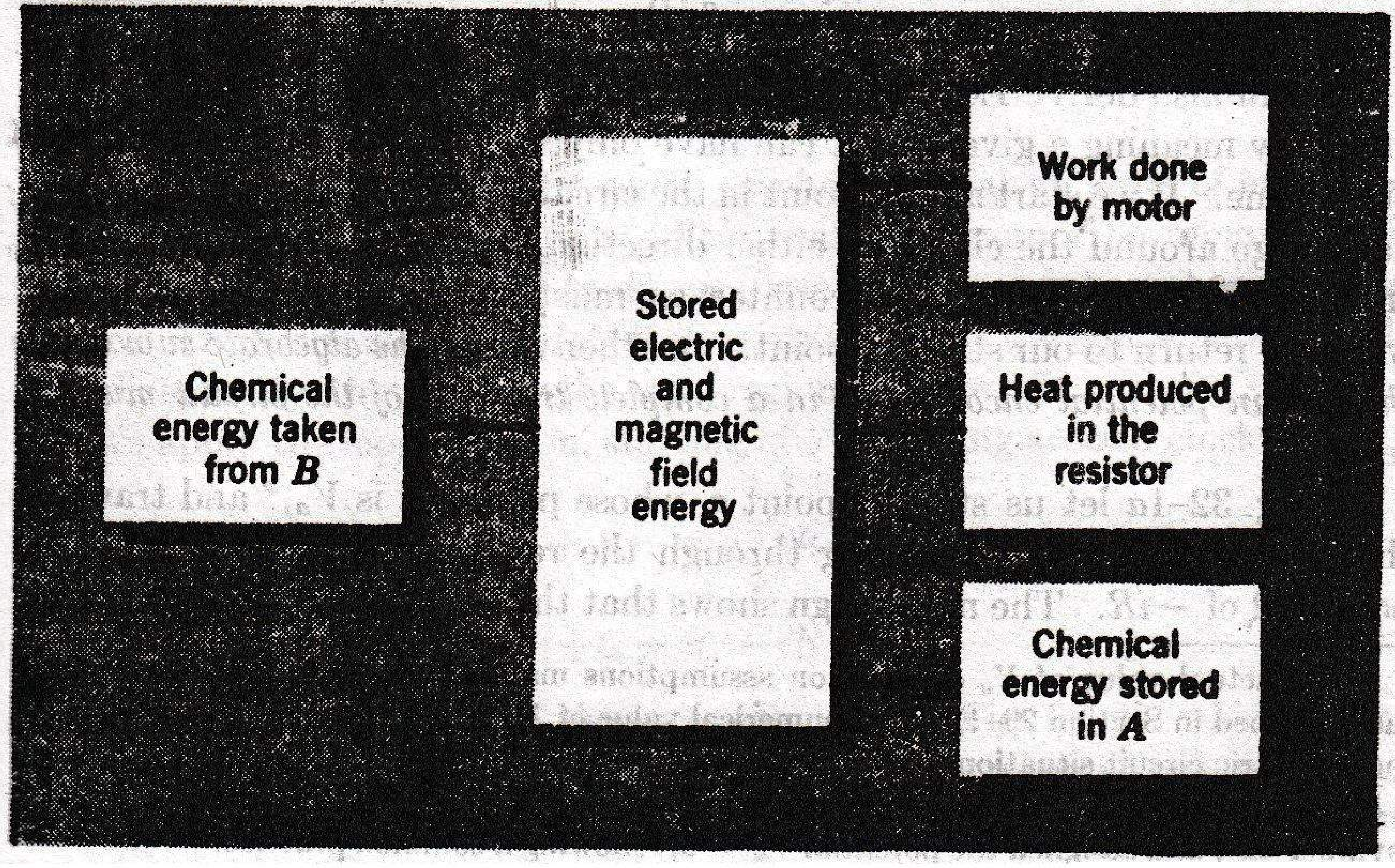


Fig. 32-2 Two batteries, a resistor, and a motor, connected in a single-loop circuit. It is given that $\mathcal{E}_B > \mathcal{E}_A$.

the floor to the shelf, does work on them. This energy is stored temporarily as gravitational field energy. The balls roll slowly and uniformly along the shelf, dropping from the right end into a cylinder full of viscous oil. They sink to the bottom at constant speed, are removed by a trapdoor mechanism not shown, and roll back along the floor to the left. The energy put into the system by the man appears eventually as heat in the viscous fluid. The energy supplied by the man comes from his own internal (chemical) energy. The circulation of charges in Fig. 32–1*a* will stop eventually if battery *B* is not charged; the circulation of bowling balls in Fig. 32–1*b* will stop eventually if the man does not replenish his store of internal energy by eating.

Figure 32–2 shows a circuit containing two (ideal) batteries, *A* and *B*, a resistor *R*, and an (ideal) electric motor employed in lifting a weight. The batteries are connected so that they tend to send charges around the circuit in opposite directions; the actual direction of the current is determined by *B*, which supplies the larger potential difference. The energy transfers in this circuit are



The chemical energy in B is being steadily depleted, the energy appearing in the three forms shown on the right. Battery A is being "charged" while battery B is being discharged. Again, the electric and magnetic fields that surround the circuit act as an intermediary.

It is part of the definition of an emf that the energy transfer process be *reversible*, at least in principle. The student will recall that a reversible process is one that passes through equilibrium states; its course can be reversed by making an infinitesimal change in the environment of the system; see Section 25-2. A battery, for example, can either be on charge or discharge; a generator can be driven mechanically, producing electric energy, or it can be operated backward as a motor. The (reversible) energy transfers here are

$$\text{electrical} \rightleftharpoons \text{chemical}$$

and

$$\text{electrical} \rightleftharpoons \text{mechanical}.$$

Joule heating is an electric energy transfer that is *not* reversible. We can easily heat a conductor by supplying electric energy to it, but it is not possible to set up a current in a closed copper loop by heating the loop uniformly. Because of this lack of reversibility, we do not associate an emf with Joule heating.

32-2 Calculating the Current

In a time dt an amount of energy given by $i^2R\,dt$ will appear in the resistor of Fig. 32-1a as Joule heat. During this same time a charge dq ($= i\,dt$) will have moved through the seat of emf, and the seat will have done work on this charge (see Eq. 32-1) given by

$$dW = \mathcal{E}dq = \mathcal{E}i\,dt.$$

From the conservation of energy principle, the work done by the seat must equal the Joule heat, or

$$\mathcal{E}i\,dt = i^2R\,dt.$$

Solving for i, we obtain

$$i = \mathcal{E}/R. \tag{32-2}$$

We can also derive Eq. 32-2 by considering that if electric potential is to have any meaning a given point can have only one value of potential at any given time. If we start at any point in the circuit of Fig. 32-1a and, in imagination, go around the circuit in either direction, adding up algebraically the changes in potential that we encounter, we must arrive at the same potential when we return to our starting point. In other words, *the algebraic sum of the changes in potential encountered in a complete traversal of the circuit must be zero.*

In Fig. 32-1a let us start at point a, whose potential is V_a,* and traverse the circuit clockwise. In going through the resistor, there is a change in potential of $-iR$. The minus sign shows that the top of the resistor is higher

* The actual value of V_a depends on assumptions made in the definition of potential (as described in Section 29-1). The numerical value of V_a is not important because, as in most electric circuit situations, we are concerned here with *differences* of potential. Point a in Fig. 32-1a (or any other single point in that figure) could be connected to ground (symbol ⏚) and assigned the potential $V_a = 0$, following a common practice.

in potential than the bottom, which must be true, because positive charge carriers move of their own accord from high to low potential. As we traverse the battery from bottom to top, there is an *increase* of potential $+\mathcal{E}$ because the battery does (positive) work on the charge carriers, that is, it moves them from a point of low potential to one of high potential. Adding the algebraic sum of the changes in potential to the initial potential V_a must yield the identical value V_a, or

$$V_a - iR + \mathcal{E} = V_a.$$

We write this as

$$-iR + \mathcal{E} = 0,$$

which is independent of the value of V_a and which asserts explicitly that the algebraic sum of the potential changes for a complete circuit traversal is zero. This relation leads directly to Eq. 32-2.

These two ways to find the current in single-loop circuits, based on the conservation of energy and on the concept of potential, are completely equivalent because potential differences are defined in terms of work and energy (Section 29-1). The statement that the sum of the changes in potential encountered in making a complete loop is zero is called *Kirchhoff's second rule;* for brevity we call it the *loop theorem.* It must always be borne in mind that this theorem is simply a particular way of stating the law of conservation of energy for electric circuits.

To prepare for the study of more complex circuits, let us examine the rules for finding potential differences; these rules follow from the previous discussion. They are not meant to be memorized but rather to be so thoroughly understood that it becomes trivial to re-derive them on each application.

1. If a resistor i traversed in the direction of the current, the change in potential is $-iR$; in the opposite direction it is $+iR$.

2. If a seat of emf is traversed in the direction of the emf, the change in potential is $+\mathcal{E}$; in the opposite direction it is $-\mathcal{E}$.

32-3 Other Single-Loop Circuits

Figure 32-3*a* shows a circuit which emphasizes that all seats of emf have an intrinsic internal resistance r. This resistance cannot be removed—although we would usually like to do so—because it is an inherent part of the device. The figure shows the internal resistance r and the emf separately, although, actually, they occupy the same region of space.

If we apply the loop theorem, starting at b and going around clockwise, we obtain

$$V_b + \mathcal{E} - ir - iR = V_b$$

or

$$+\mathcal{E} - ir - iR = 0.$$

The student should compare these equations with Fig. 32-2*b*, which shows the changes in potential graphically. In writing these equations, note that we traversed r and R in the direction of the current and $\mathcal{E}$ in the direction of the

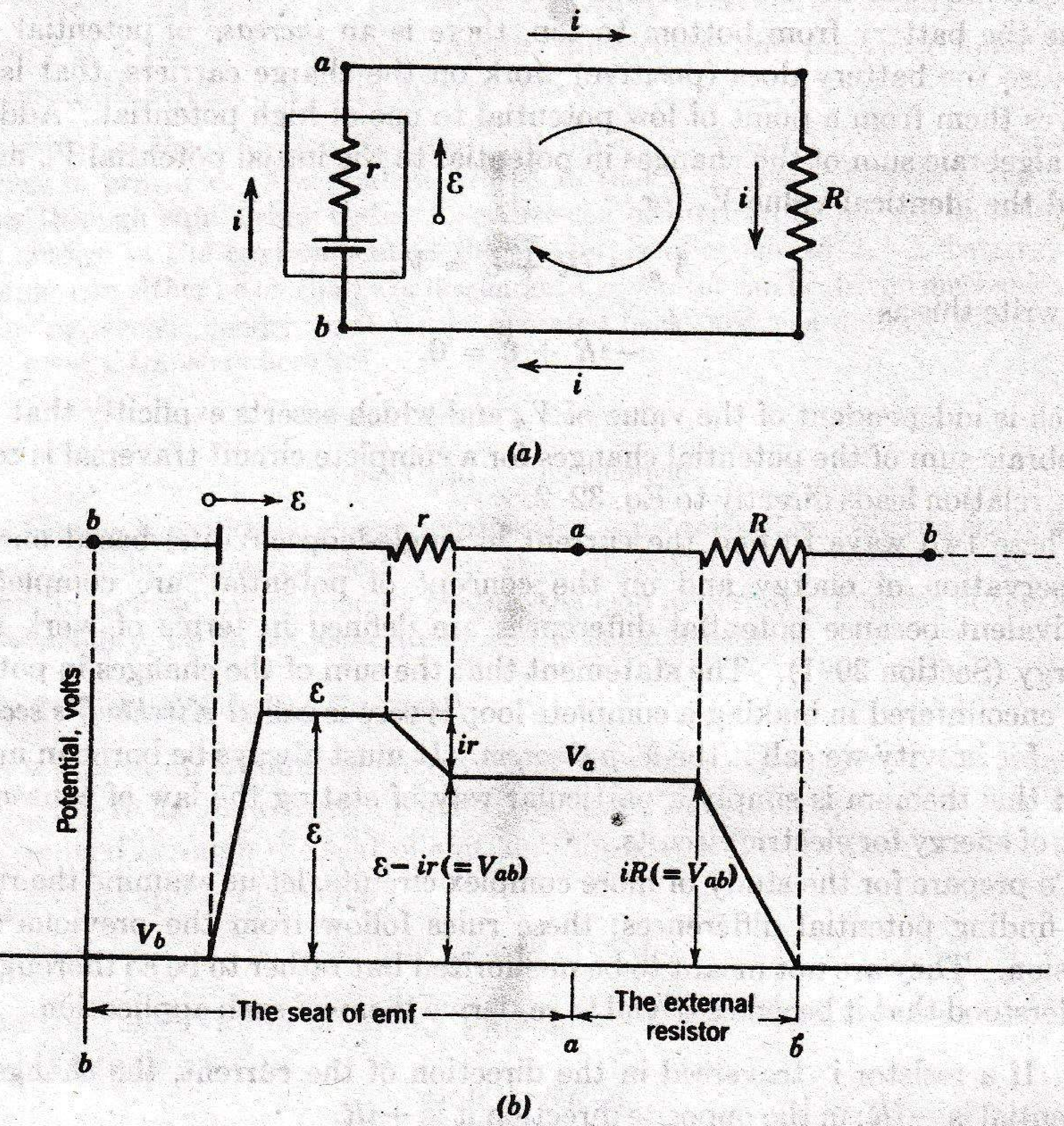


Fig. 32-3 A single-loop circuit. The rectangular block is a seat of emf with internal resistance r. (b) The same circuit is drawn for convenience as a straight line. Directly below are shown the changes in potential that one encounters in traversing the circuit clockwise from point b.

emf. The same equation follows if we start at any other point in the circuit or if we traverse the circuit in a counterclockwise direction. Solving for i gives

$$i = \frac{\mathcal{E}}{R + r}. \tag{32-3}$$

▶ **Example 1.** *Resistors in series.* Resistors in series are connected so that there is only one conducting path through them, as in Fig. 32-4. What is the equivalent resistance R of this series combination? The equivalent resistance is the single resistance R which, substituted for the series combination between the terminals ab, will leave the current i unchanged.

Applying the loop theorem (going clockwise from a) yields

$$-iR_1 - iR_2 - iR_3 + \mathcal{E} = 0$$

or

$$i = \frac{\mathcal{E}}{R_1 + R_2 + R_3}.$$

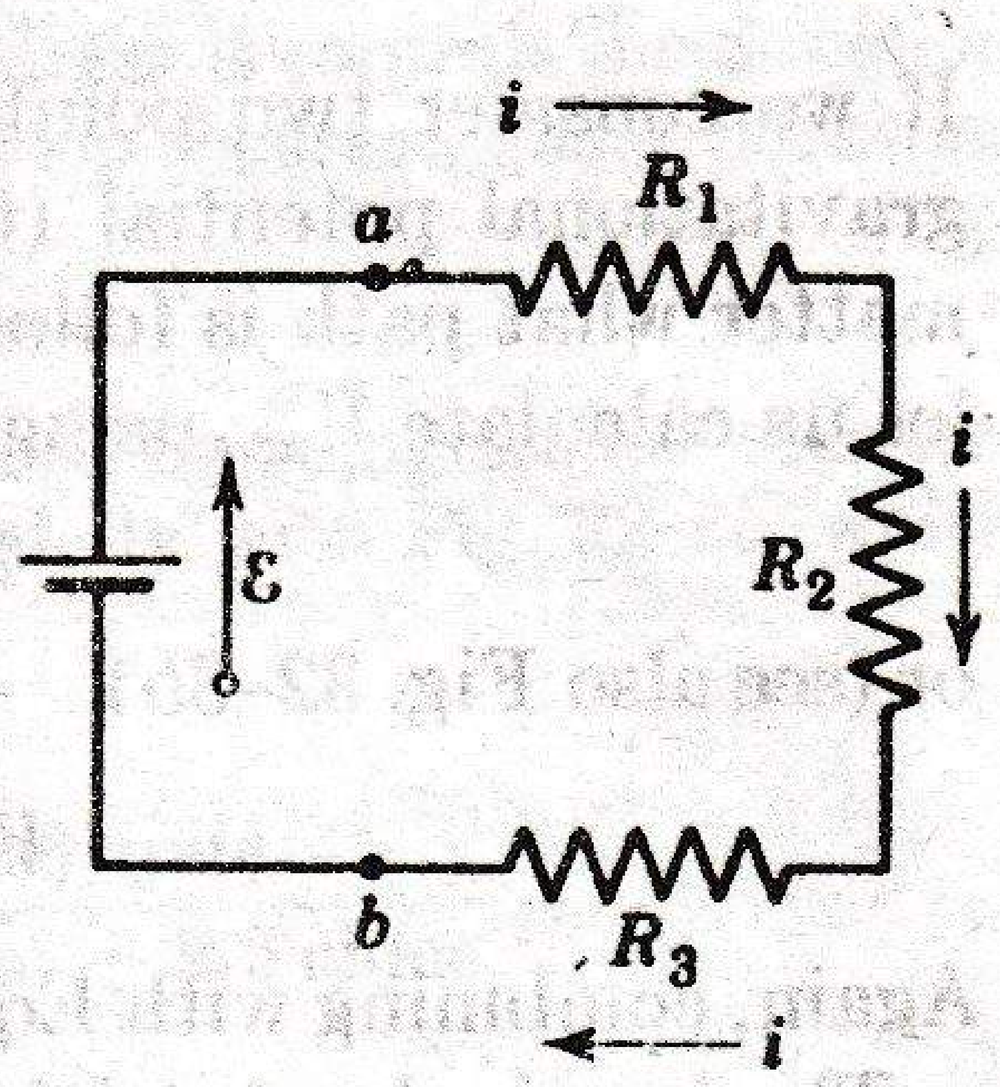


Fig. 32-4 Example 1. Three resistors are connected in series between terminals a and b.

For the equivalent resistance R

$$i = \frac{\mathcal{E}}{R}$$

or

$$R = R_1 + R_2 + R_3. \tag{32-4}$$

The extension to more than three resistors is clear. ◀

32-4 Potential Differences

We often want to compute the potential difference between two points in a circuit. In Fig. 32-3a for example, what is the relationship between the potential difference V_{ab} $(= V_a - V_b)$ between points b and a and the fixed circuit parameters $\mathcal{E}$, r, and R? To find this relationship, let us start from point b and traverse the circuit to point a, passing through resistor R against the current. If V_b and V_a are the potentials at b and a, respectively, we have

$$V_b + iR = V_a$$

because we experience an increase in potential in traversing a resistor against the current arrow. We rewrite this relation as

$$V_{ab} = V_a - V_b = +iR,$$

which tells us that V_{ab}, the potential difference between points a and b, has the magnitude iR and that point a is more positive than point b. Combining this last equation with Eq. 32-3 yields

$$V_{ab} = \mathcal{E}\frac{R}{R + r}. \tag{32-5}$$

To sum up: To find the potential difference between any two points in a circuit start at one point and traverse the circuit to the other, following any path, and add up algebraically the potential changes encountered. This algebraic sum will be the potential difference. This procedure is similar to that for finding the current in a closed loop, except that here the potential differences are added up over part of a loop and not over the whole loop.

The potential difference between any two points can have only one value; thus we must obtain the same answer for all paths that connect these points.

If we consider two points on the side of a hill, the measured difference in gravitational potential (that is, in altitude) between them is the same no matter what path is followed in going from one to the other. In Fig. 32–3a let us calculate V_{ab}, using a path passing through the seat of emf. We have

$$V_b + \mathcal{E} - ir = V_a$$

or (see also Fig. 32–3b)

$$V_{ab} = V_a - V_b = +\mathcal{E} - ir.$$

Again, combining with Eq. 32–3 leads to Eq. 32–5.

The terminal potential difference of the battery V_{ab}, as Eq. 32–5 shows, is less than $\mathcal{E}$ unless the battery has no internal resistance ($r = 0$) or if it is on open circuit ($R = \infty$); then V_{ab} is equal to $\mathcal{E}$. Thus the emf of a device is equal to its terminal potential difference *when on open circuit.*

▶ **Example 2.** In Fig. 32–5a let $\mathcal{E}_1$ and $\mathcal{E}_2$ be 2.0 volts and 4.0 volts, respectively; let the resistances r_1, r_2, and R be 1.0 ohm, 2.0 ohms, and 5.0 ohms, respectively. What is the current?

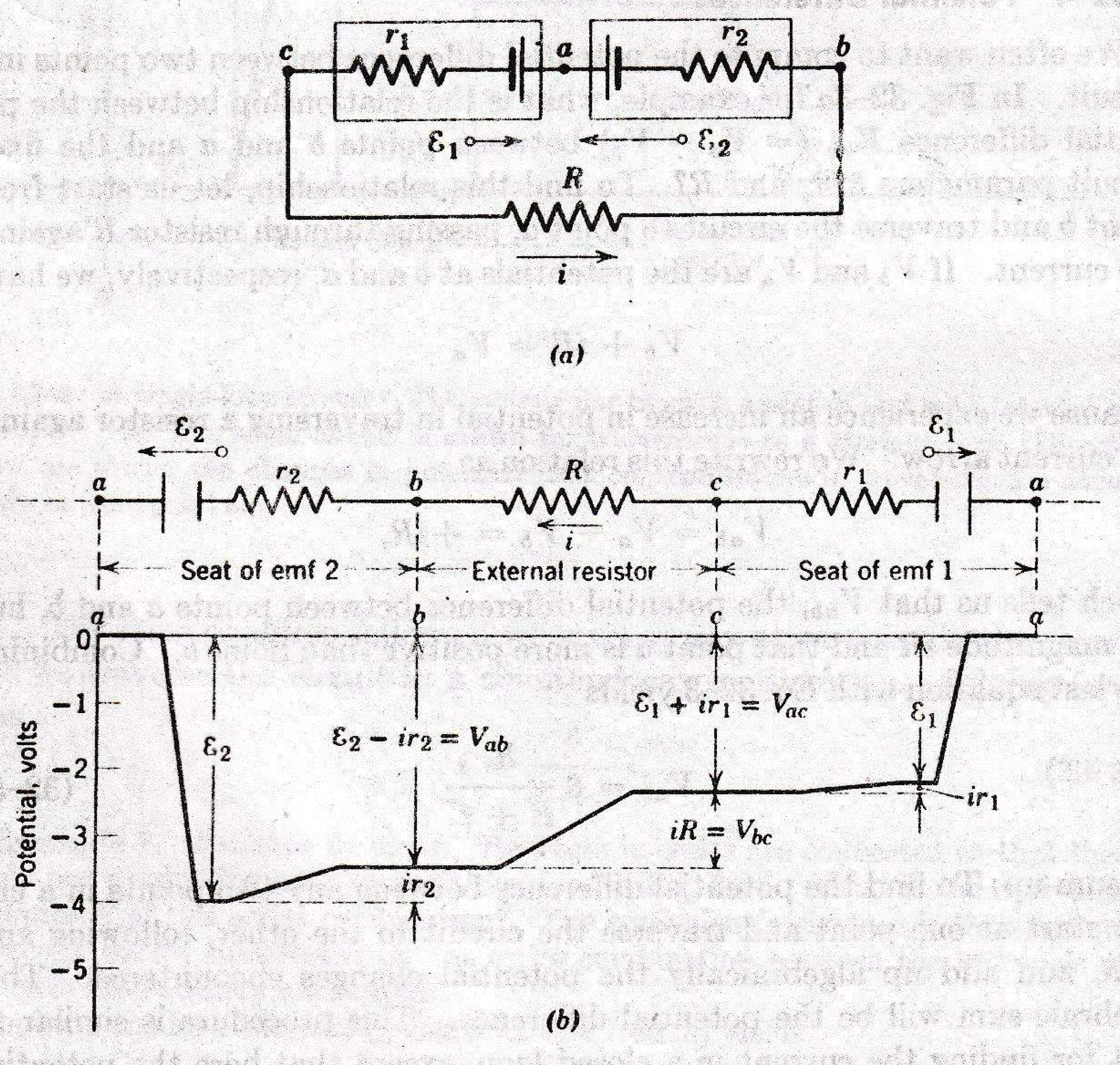


Fig. 32–5 (a) A single-loop circuit. (b) The same circuit is shown schematically as a straight line, the potential differences encountered in traversing the circuit clockwise from point a being displayed directly below. In the lower figure the potential of point a was assumed to be zero for convenience.

Emfs $\mathcal{E}_1$ and $\mathcal{E}_2$ oppose each other, but because $\mathcal{E}_2$ is larger it controls the direction of the current. Thus i will be counterclockwise. The loop theorem, going clockwise from a, yields

$$-\mathcal{E}_2 + ir_2 + iR + ir_1 + \mathcal{E}_1 = 0.$$

The student should check that the same result is obtained by going around counterclockwise. He should also compare this equation carefully with Fig. 32-5*b*, which shows the potential changes graphically.

Solving for i yields

$$i = \frac{\mathcal{E}_2 - \mathcal{E}_1}{R + r_1 + r_2} = \frac{4.0 \text{ volts} - 2.0 \text{ volts}}{5.0 \text{ ohms} + 1.0 \text{ ohm} + 2.0 \text{ ohms}}$$

$$= 0.25 \text{ amp}.$$

It is not necessary to know in advance what the actual direction of the current is. To show this, let us assume that the current in Fig. 32-5*a* is clockwise, an assumption that we know is incorrect. The loop theorem then yields (going clockwise from a)

$$-\mathcal{E}_2 - ir_2 - iR - ir_1 + \mathcal{E}_1 = 0$$

or

$$i = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R + r_1 + r_2}.$$

Substituting numerical values (see above) yields -0.25 amp for the current. The minus sign tells us that the current is in the opposite direction from the one we have assumed.

In more complex circuit problems involving many loops and branches it is often impossible to know in advance the correct directions for the currents in all parts of the circuit. We can assume directions for the currents at random. Those currents for which positive numerical values are obtained will have the correct directions; those for which negative values are obtained will be exactly opposite to the assumed directions. In all cases the numerical values will be correct.

Example 3. What is the potential difference (*a*) between points b and a in Fig. 32-5*a*? (*b*) Between points a and c?

(*a*) For points a and b we start at b and traverse the circuit to a, obtaining

$$V_{ab} \,(= V_a - V_b) = -ir_2 + \mathcal{E}_2 = -(0.25 \text{ amp})(2.0 \text{ ohms}) + 4.0 \text{ volts}$$

$$= +3.5 \text{ volts}.$$

Thus a is more positive than b and the potential difference (3.5 volts) is *less than* the emf (4.0 volts); see Fig. 32-5*b*.

(*b*) For points c and a, we start at c and traverse the circuit to a, obtaining

$$V_{ac} \,(= V_a - V_c) = +\mathcal{E}_1 + ir_1 = +2.0 \text{ volts} + (0.25 \text{ amp})(1.0 \text{ ohm})$$

$$= +2.25 \text{ volts}.$$

This tells us that a is at a higher potential than c. The terminal potential difference of $\mathcal{E}_1$ (2.25 volts) is *larger than* the emf (2.0 volts); see Fig. 32-5*b*. Charge is being forced through $\mathcal{E}_1$ in a direction opposite to the one in which it would send charge if it were acting by itself; if $\mathcal{E}_1$ is a storage battery, it is being charged at the expense of $\mathcal{E}_2$.

Let us test the first result by proceeding from b to a along a different path, namely, through R, r_1, and $\mathcal{E}_1$. We have

$$V_{ab} = iR + ir_1 + \mathcal{E}_1 = (0.25 \text{ amp})(5.0 \text{ ohms}) + (0.25 \text{ amp})(1.0 \text{ ohm}) + 2.0 \text{ volts} = +3.5 \text{ volts},$$

which is the same as the earlier result. ◀

32-5 Multiloop Circuits

Figure 32-6 shows a circuit containing two loops. For simplicity, we have neglected the internal resistances of the batteries. There are two *junctions*, *b* and *d*, and three *branches* connecting these junctions. The branches are the left branch *bad*, the right branch *bcd*, and the central branch *bd*. If the emfs and the resistances are given, what are the currents in the various branches?

We label the currents in the branches as i_1, i_2, and i_3, as shown. Current i_1 has the same value for any cross section of the left branch from *b* to *d*. Similarly, i_2 has the same value everywhere in the right branch and i_3 in the central branch. The directions of the currents have been chosen arbitrarily. The careful reader will note that i_3 must point in a direction opposite to the one we have shown. We have deliberately drawn it in wrong to show how the formal mathematical procedures will always indicate this to us.

The three currents i_1, i_2, and i_3 carry charge either toward junction *d* or away from it. Charge does not accumulate at junction *d*, nor does it drain away from this junction because the circuit is in a steady-state condition. Thus charge must be removed from the junction by the currents at the same rate that it is brought into it. If we arbitrarily call a current approaching the junction positive and the one leaving the junction negative, then

$$i_1 + i_3 - i_2 = 0.$$

This equation suggests a general principle for the solution of multiloop circuits: *At any junction the algebraic sum of the currents must be zero.* This *junction theorem* is also known as *Kirchhoff's first rule.* Note that it is simply a statement of the conservation of charge. Thus our basic tools for solving circuits are (*a*) the conservation of energy (see p. 793) and (*b*) the conservation of charge.

For the circuit of Fig. 32-6, the junction theorem yields only one relationship among the three unknowns. Applying the theorem at junction *b* leads to exactly the same equation, as the student should verify. To solve for the three unknowns, we need two more independent equations; they can be found from the loop theorem.

In single-loop circuits there is only one conducting loop around which to apply the loop theorem, and the current is the same in all parts of this loop.

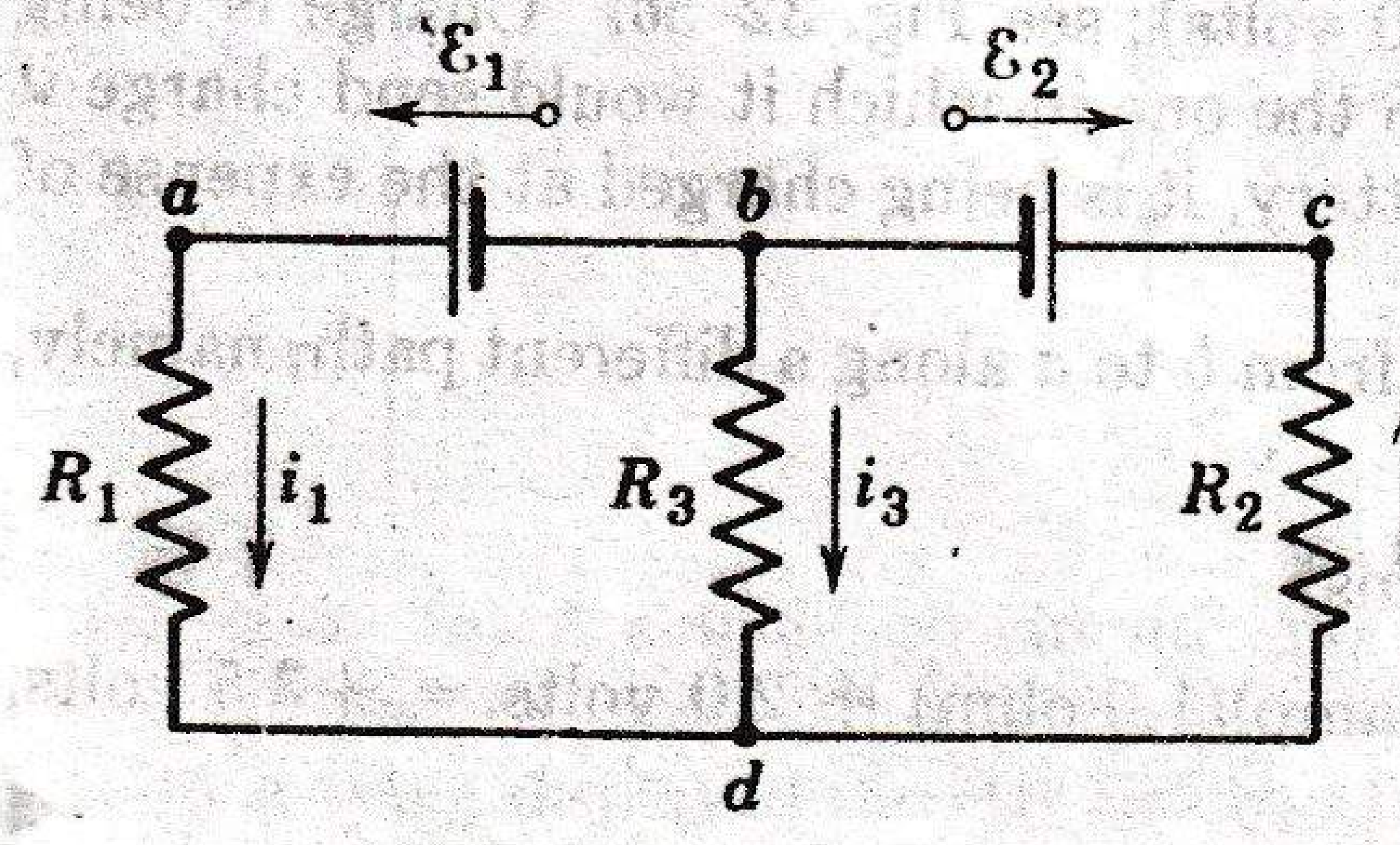


Fig. 32-6 A multiloop circuit.

In multiloop circuits there is more than one loop, and the current in general will not be the same in all parts of any given loop.

If we traverse the left loop of Fig. 32-6 in a counterclockwise direction, the loop theorem gives

$$\mathcal{E}_1 - i_1R_1 + i_3R_3 = 0. \tag{32-6}$$

The right loop gives

$$-i_3R_3 - i_2R_2 - \mathcal{E}_2 = 0. \tag{32-7}$$

These two equations, together with the relation derived earlier with the junction theorem, are the three simultaneous equations needed to solve for the unknowns i_1, i_2, and i_3. Doing so yields

$$i_1 = \frac{\mathcal{E}_1(R_2 + R_3) - \mathcal{E}_2R_3}{R_1R_2 + R_2R_3 + R_1R_3}, \tag{32-8a}$$

$$i_2 = \frac{\mathcal{E}_1R_3 - \mathcal{E}_2(R_1 + R_3)}{R_1R_2 + R_2R_3 + R_1R_3}, \tag{32-8b}$$

and

$$i_3 = \frac{-\mathcal{E}_1R_2 - \mathcal{E}_2R_1}{R_1R_2 + R_2R_3 + R_1R_3}. \tag{32-8c}$$

The student should supply the missing steps. Equation 32-8c shows that no matter what numerical values are given to the emfs and to the resistances the current i_3 will always have a negative value. This means that it will always point up in Fig. 32-6 rather than down, as we deliberately assumed. The currents i_1 and i_2 may be in either direction, depending on the particular numerical values given.

The student should verify that Eqs. 32-8 reduce to sensible conclusions in special cases. For $R_3 = \infty$, for example, we find

$$i_1 = i_2 = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2} \quad \text{and} \quad i_3 = 0.$$

What do these equations reduce to for $R_2 = \infty$?

The loop theorem can be applied to a large loop consisting of the entire circuit *abcda* of Fig. 32-6. This fact might suggest that there are more equations than we need, for there are only three unknowns and we already have three equations written in terms of them. However, the loop theorem yields for this loop

$$-i_1R_1 - i_2R_2 - \mathcal{E}_2 + \mathcal{E}_1 = 0,$$

which is nothing more than the sum of Eqs. 32-6 and 32-7. Thus this large loop does not yield another *independent* equation. It will never be found in solving multiloop circuits that there are more independent equations than variables.

▶ **Example 4.** *Resistors in parallel.* Figure 32–7 shows three resistors connected across the same seat of emf. Resistances across which the identical potential difference is applied are said to be in parallel. What is the equivalent resistance R of this parallel combination? The equivalent resistance is that single resistance which, substituted for the parallel combination between terminals ab, would leave the current i unchanged.

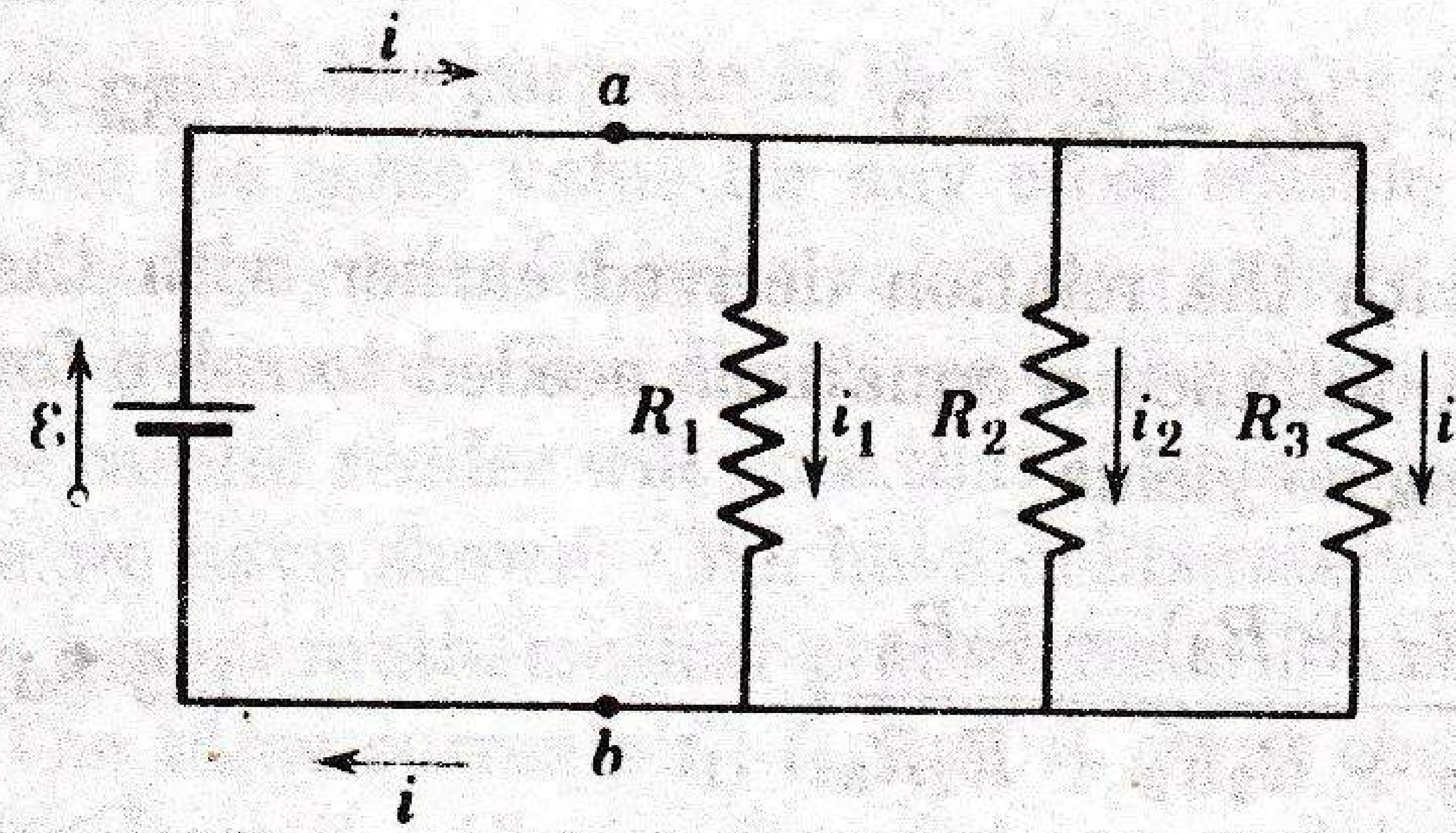


Fig. 32–7 Example 4. Three resistors are connected in parallel betweeen terminals a and b.

The currents in the three branches are

$$i_1 = \frac{V}{R_1}, \qquad i_2 = \frac{V}{R_2}, \qquad \text{and} \qquad i_3 = \frac{V}{R_3},$$

where V is the potential difference that appears between points a and b. The total current i is found by applying the junction theorem to junction a, or

$$i = i_1 + i_2 + i_3 = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right).$$

If the equivalent resistance is used instead of the parallel combination, we have

$$i = \frac{V}{R}.$$

Combining these two equations gives

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \qquad (32\text{–}9)$$

This formula can easily be extended to more than three resistances. Note that the equivalent resistance of a parallel combination is less than any of the resistances that make it up. ◀

32–6 Measuring Currents and Potential Differences

A meter to measure currents is called an *ammeter* (or a *milliammeter* or *microammeter*, depending on the size of the current to be measured). To determine the current in a wire, it is necessary to break or cut the wire and to insert the ammeter, so that the current to be measured passes through the meter (see Fig. 32–8).*

It is essential that the resistance R_A of the ammeter be *small* compared to other resistances in the circuit. Otherwise the act of inserting the meter will in itself change

* The meter must be connected so that the direction of current through it (assuming positive charge carriers) is *into* the meter terminal marked +. Otherwise the meter will deflect in a direction opposite to that intended.

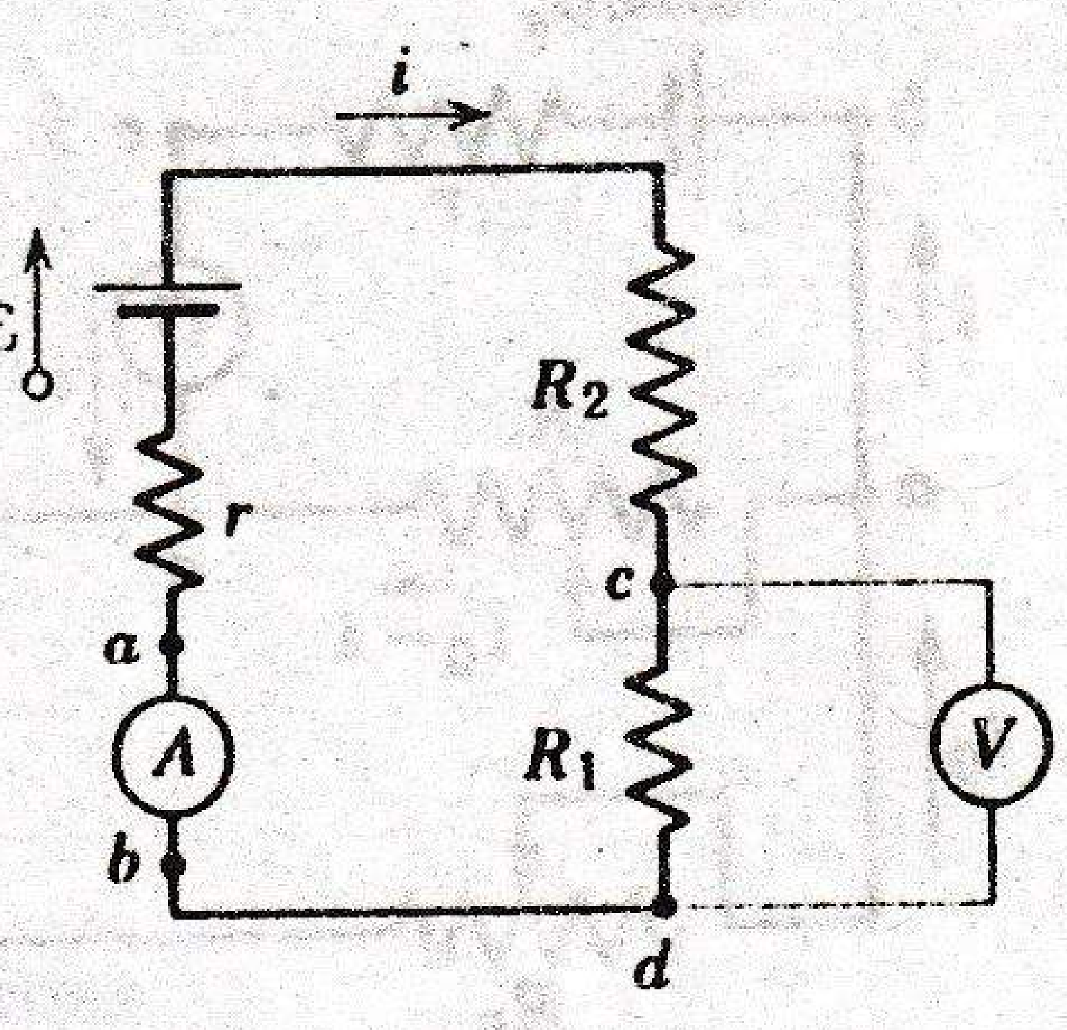


Fig. 32-8 An ammeter (A) is connected to read the current in a circuit, and a voltmeter (V) is connected to read the potential difference across resistor R_1.

the current to be measured. An ideal ammeter would have zero resistance. In the circuit of Fig. 32-8 the required condition, assuming that the voltmeter is not connected, is

$$R_A \ll r + R_1 + R_2.$$

A meter to measure potential differences is called a *voltmeter* (or a *millivoltmeter* or *microvoltmeter*). To find the potential difference between two points in a circuit, it is necessary to connect one of the voltmeter terminals to each of the circuit points, without breaking the circuit (see Fig. 32-8).*

It is essential that the resistance of the voltmeter R_V be *large* compared to any circuit resistance across which the voltmeter is connected. Otherwise the meter will itself constitute an important circuit element and will alter the circuit current and the potential difference to be measured. An ideal voltmeter would have an infinite resistance. In Fig. 32-8 the required condition is

$$R_V \gg R_1$$

In measuring potential difference in electronic circuits, where the effective circuit resistance may be of the order of 10^6 ohms or higher, it becomes necessary to use a *vacuum-tube voltmeter*, which is an electron-tube device designed specifically to have an extremely high effective resistance between its input terminals.

32-7 The Potentiometer

Figure 32-9 shows the rudiments of a *potentiometer*, which is a device for measuring an unknown emf $\mathcal{E}_x$. The currents and emfs are marked as shown. Applying the loop theorem to loop *abcd* yields

$$-\mathcal{E}_x - ir + (i_0 - i)R = 0,$$

where $i_0 - i$, by application of the junction theorem at a, is the current in resistor R. Solving for i yields

$$i = \frac{i_0 R - \mathcal{E}_x,}{R + r}$$

in which R is a variable resistor. This relation shows that if R is adjusted to have the value R_x where

$$i_0 R_x = \mathcal{E}_x, \qquad (32\text{-}10)$$

the current i in the branch *abcd* becomes zero. To *balance* the potentiometer in this way, R must be adjusted manually until the sensitive meter G reads zero.

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* The voltmeter terminal marked + must be connected to the point of higher potential. Otherwise the meter will deflect in a direction opposite to that intended.

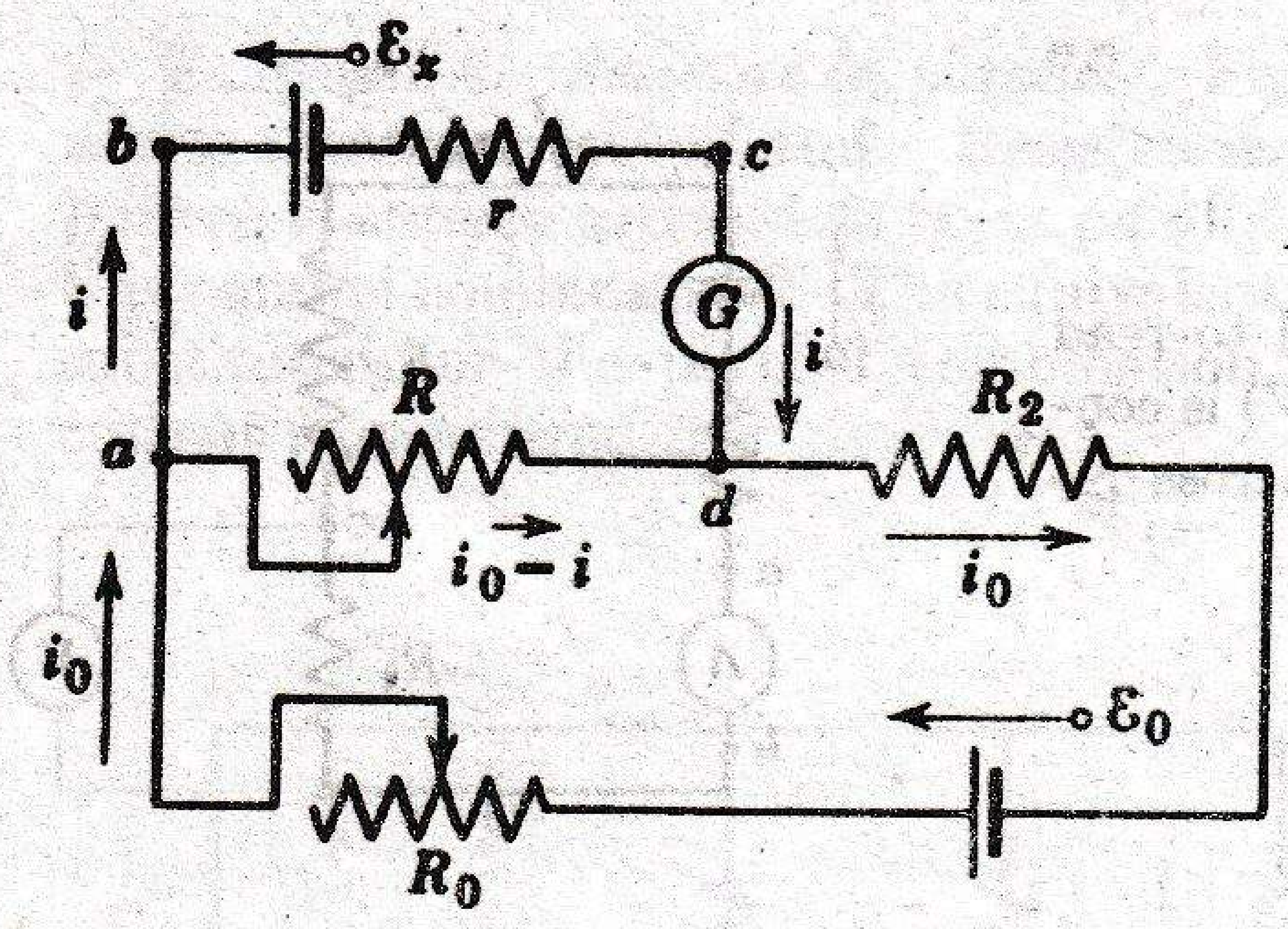


Fig. 32–9 Elements of a potentiometer

The emf can be obtained from Eq. 32–10 if the current i_0 is known. However, it is standard practice to replace $\mathcal{E}_x$ by a known standard emf $\mathcal{E}_s$, and once again to adjust R to the zero-current condition. This yields, assuming the current i_0 remains unchanged,

$$i_0 R_s = \mathcal{E}_s.$$

Combining the last two equations yields

$$\mathcal{E}_x = \mathcal{E}_s \frac{R_x}{R_s}, \tag{32–11}$$

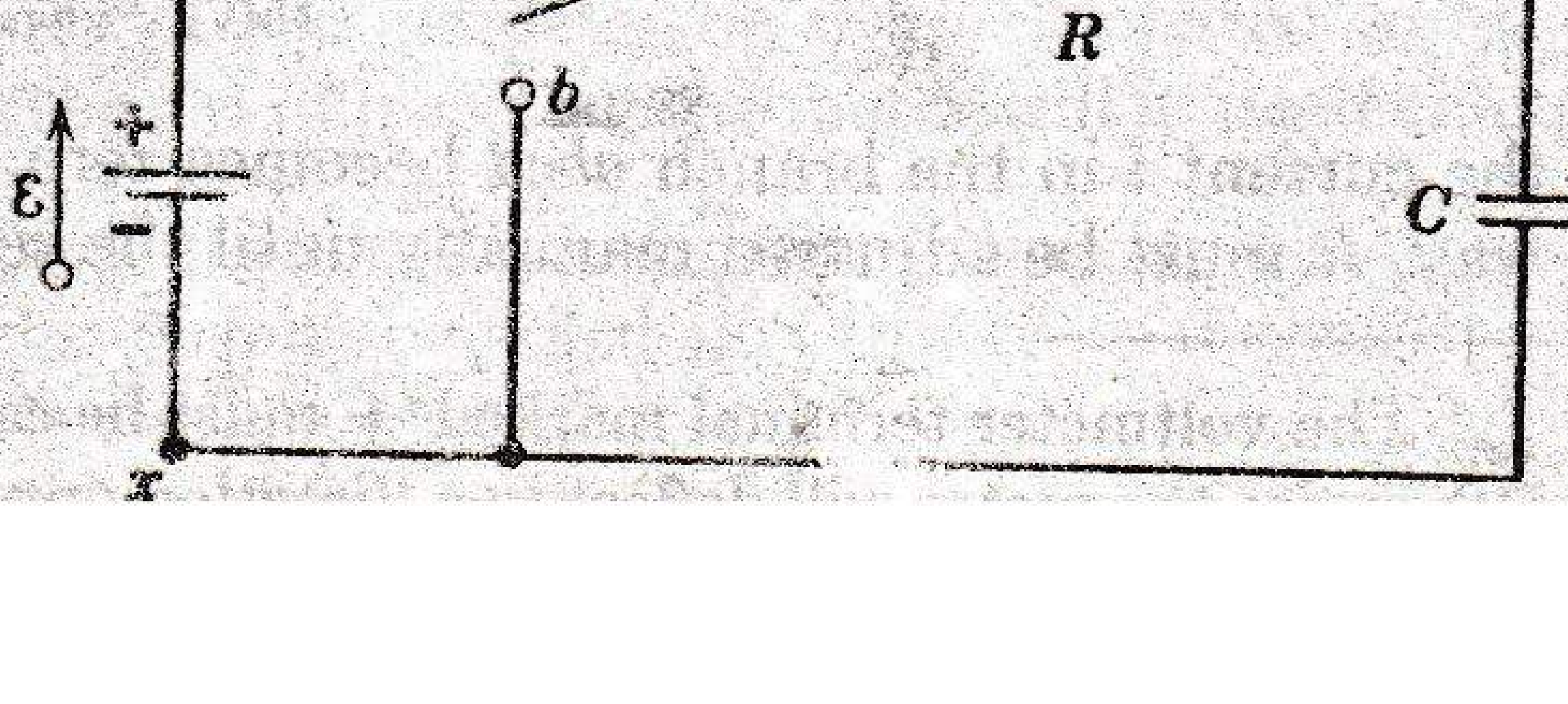
which allows us to compare emfs with precision. Note that the internal resistance r of the emf plays no role. In practice, potentiometers are conveniently packaged units, containing a *standard cell* which, after calibration at the National Bureau of Standards or elsewhere, serves as a convenient known standard seat of emf $\mathcal{E}_s$. Switching arrangements for replacing the unknown emf by the standard and arrangements for ascertaining that the current i_0 remains constant are also incorporated.

32–8 *RC* Circuits

The preceding sections dealt with circuits in which the circuit elements were resistors and in which the currents did not vary with time. Here we introduce the capacitor as a circuit element, which will lead us to the concept of time-varying currents. In Fig. 32–10 let switch S be thrown to position a. What current is set up in the single-loop circuit so formed? Let us apply conservation of energy principles.

In time dt a charge dq ($= i\,dt$) moves through any cross section of the circuit. The work done by the seat of emf ($= \mathcal{E}\,dq$; see Eq. 32–1) must equal the energy that appears as Joule heat in the resistor during time

Fig. 32–10 An *RC* circuit.



dt ($= i^2R\,dt$) plus the increase in the amount of energy U that is stored in the capacitor [$= dU = d(q^2/2C)$; see Eq. 30-25]. In equation form

$$\mathcal{E}\,dq = i^2R\,dt + d\left(\frac{q^2}{2C}\right)$$

or

$$\mathcal{E}\,dq = i^2R\,dt + \frac{q}{C}\,dq.$$

Dividing by dt yields

$$\mathcal{E}\frac{dq}{dt} = i^2R + \frac{q}{C}\frac{dq}{dt}.$$

But dq/dt is simply i, so that this equation becomes

$$\mathcal{E} = iR + \frac{q}{C}. \tag{32-12}$$

This equation also follows from the loop theorem, as it must, since the loop theorem was derived from the conservation of energy principle. Starting from point x and traversing the circuit clockwise, we experience an increase in potential in going through the seat of emf and decreases in potential in traversing the resistor and the capacitor, or

$$\mathcal{E} - iR - \frac{q}{C} = 0,$$

which is identical with Eq. 32-12.

We cannot immediately solve Eq. 32-12 because it contains two variables, q and i, which, however, are related by

$$i = \frac{dq}{dt}. \tag{32-13}$$

Substituting for i into Eq. 32-12 gives

$$\mathcal{E} = R\frac{dq}{dt} + \frac{q}{C}. \tag{32-14}$$

Our task now is to find the function $q(t)$ that satisfies this *differential equation.* Although this particular equation is not difficult to solve, we choose to avoid mathematical complexity by simply presenting the solution, which is

$$q = C\mathcal{E}(1 - e^{-t/RC}). \tag{32-15}$$

We can easily test whether this function $q(t)$ is really a solution of Eq. 32-14 by substituting it into that equation and seeing whether an identity results. Differentiating Eq. 32-15 with respect to time yields

$$\frac{dq}{dt}(= i) = \frac{\mathcal{E}}{R}e^{-t/RC}. \tag{32-16}$$

Substituting q (Eq. 32-15) and dq/dt (Eq. 32-16) into Eq. 32-14 yields an identity, as the student should verify. Thus Eq. 32-15 is a solution of Eq. 32-14.

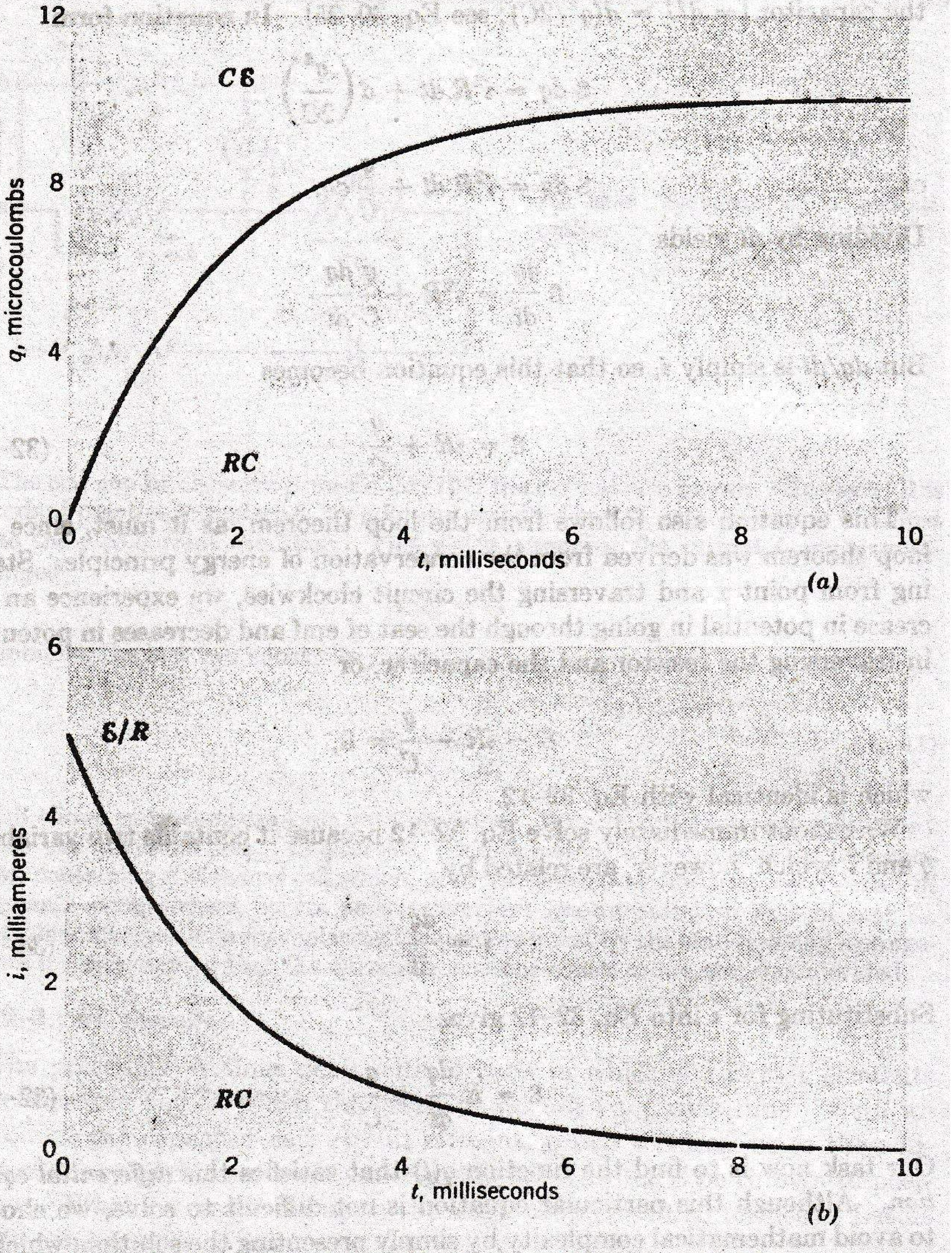


Fig. 32–11 If, in Fig. 32–10, we assume that $R = 2000$ ohms, $C = 1.0\ \mu f$, and $\mathcal{E} = 10$ volts, then (*a*) shows the variation of q with t during the charging process and (*b*) the variations of i with t. The time constant is $RC = 2.0 \times 10^{-3}$ sec.

Figure 32–11 shows some plots of Eqs. 32–15 and 32–16 for a particular case. Study of these plots and of the corresponding equations shows that (*a*) at $t = 0$, $q = 0$ and $i = \mathcal{E}/R$, and (*b*) as $t \to \infty$, $q \to C\mathcal{E}$ and $i \to 0$: that is, the current is initially $\mathcal{E}/R$ and finally zero; the charge on the capacitor plates is initially zero and finally $C\mathcal{E}$.

The quantity RC in Eqs. 32–15 and 32–16 has the dimensions of time (since the exponent must be dimensionless) and is called the *capacitative time constant* of the circuit. It is the time at which the charge on the capacitor has increased to within a factor of $(1 - e^{-1})$ (= 63%) of its equilibrium value. To show this, we put $t = RC$ in Eq. 32–15 to obtain

$$q = C\mathcal{E}(1 - e^{-1}) = 0.63C\mathcal{E}.$$

Since $C\mathcal{E}$ is the equilibrium charge on the capacitor, corresponding to $t \to \infty$, the foregoing statement follows.

▶ **Example 5.** After how many time constants will the energy stored in the capacitor in Fig. 32–10 reach one-half its equilibrium value?

The energy is given by Eq. 30–25, or

$$U = \frac{1}{2C}q^2,$$

the equilibrium energy U_∞ being $(1/2C)(C\mathcal{E})^2$. From Eq. 32–15, we can write the energy as

$$U = \frac{1}{2C}(C\mathcal{E})^2(1 - e^{-t/RC})^2$$

or

$$U = U_\infty(1 - e^{-t/RC})^2.$$

Putting $U = \frac{1}{2}U_\infty$ yields

$$\tfrac{1}{2} = (1 - e^{-t/RC})^2$$

and solving this relation for t yields finally

$$t = 1.22\,RC = 1.22 \text{ time constants.}$$ ◀

Figure 32–11 shows that if a resistance is included in the circuit the rate of increase of the charge of a capacitor toward its final equilibrium value is *delayed* in a way measured by the time constant RC. With no resistor present ($RC = 0$), the charge would rise immediately to its equilibrium value. Although we have shown that this time delay follows from an application of the loop theorem to RC circuits, it is important that the student develop a physical understanding of the causes of the delay.

When switch S in Fig. 32–10 is closed on a, the resistor experiences instantaneously an applied potential difference of $\mathcal{E}$, and an initial current of $\mathcal{E}/R$ is set up. Initially, the capacitor experiences no potential difference because its initial charge is zero, the potential difference always being given by q/C. The flow of charge thr ugh the resistor starts to charge the capacitor, which has several effects. First, the existence of a capacitor charge means that there must now be a potential difference ($= q/C$) across the capacitor; this, in turn, means that the potential difference across the resistor must decrease by this amount, since the sum of the two potential differences must always equal $\mathcal{E}$. This decrease in the potential difference across R means that the charging current is reduced. Thus the charge of the capacitor builds up and the charging current decreases until the capacitor is fully charged. At this point the full emf $\mathcal{E}$ is applied to the capacitor, there being no potential drop ($i = 0$) across the resistor. This is precisely the reverse of the initial situation. The student should review the derivations of Eqs. 32–15 and 32–16 and should study Fig. 32–11 with the qualitative arguments of this paragraph in mind.

Assume now that the switch S in Fig. 32–10 has been in position a for a time t such that $t \gg RC$. The capacitor is then fully charged for all practical purposes. The switch S is then thrown to position b. How do the charge of the capacitor and the current vary with time?

With the switch S closed on b, there is no emf in the circuit and Eq. 32–12 for the circuit, with $\mathcal{E} = 0$, becomes simply

$$iR + \frac{q}{C} = 0. \tag{32–17}$$

Putting $i = dq/dt$ allows us to write, as the differential equation of the circuit (compare Eq. 32–14),

$$R\frac{dq}{dt} + \frac{q}{C} = 0. \tag{32–18a}$$

The solution is

$$q = q_0e^{-t/RC}, \tag{32–18b}$$

as the student may readily verify by substitution, q_0 bein the initial charge on the capacitor. The capacitative time constant RC appears in this expression for capacitor discharge as well as in that for the charging process (Eq. 32–15). We see that at a time such that $t = RC$ the capacitor charge is reduced to q_0e^{-1}, which is 37% of the initial charge q_0.

The current during discharge follows from differentiating Eq. 32–18b, or

$$i = \frac{dq}{dt} = -\frac{q_0}{RC}e^{-t/RC} \tag{32–19}$$

The negative sign shows that the current is in the direction opposite to that shown in Fig. 32–10. This is as it should be, since the capacitor is discharging rather than charging. Since $q_0 = C\mathcal{E}$, we can write Eq. 32–19 as

$$i = -\frac{\mathcal{E}}{R}e^{-t/RC},$$

in which $\mathcal{E}/R$ appears as the initial current, corresponding to $t = 0$. This is reasonable because the initial potential difference for the fully charged capacitor is $\mathcal{E}$.

The behavior of the RC circuit of Fig. 32–10 during charge and discharge can be studied with a cathode-ray oscilloscope. This familiar laboratory device can display on its fluorescent screen plots of the variation of potential with time. Figure 32–12 shows the circuit of Fig. 32–10 with connections made to display (*a*) the potential difference V_C across the capacitor and (*b*) the potential difference V_R across the resistor as functions of time. V_C and V_R are given by

$$V_C = \left(\frac{1}{C}\right)q$$

and

$$V_R = (R)i,$$

the former being proportional to the charge and the latter to the current.

Figure 32–13 shows oscillograph plots of V_C and V_R that result when, in effect, switch S in Fig. 32–10 is thrown regularly back and forth between positions *a* and *b*, being left in each position for a time equal to several time constants. Intervals during which the charge is building up are labeled *ch* and those during which it is decaying are labeled *dis*.

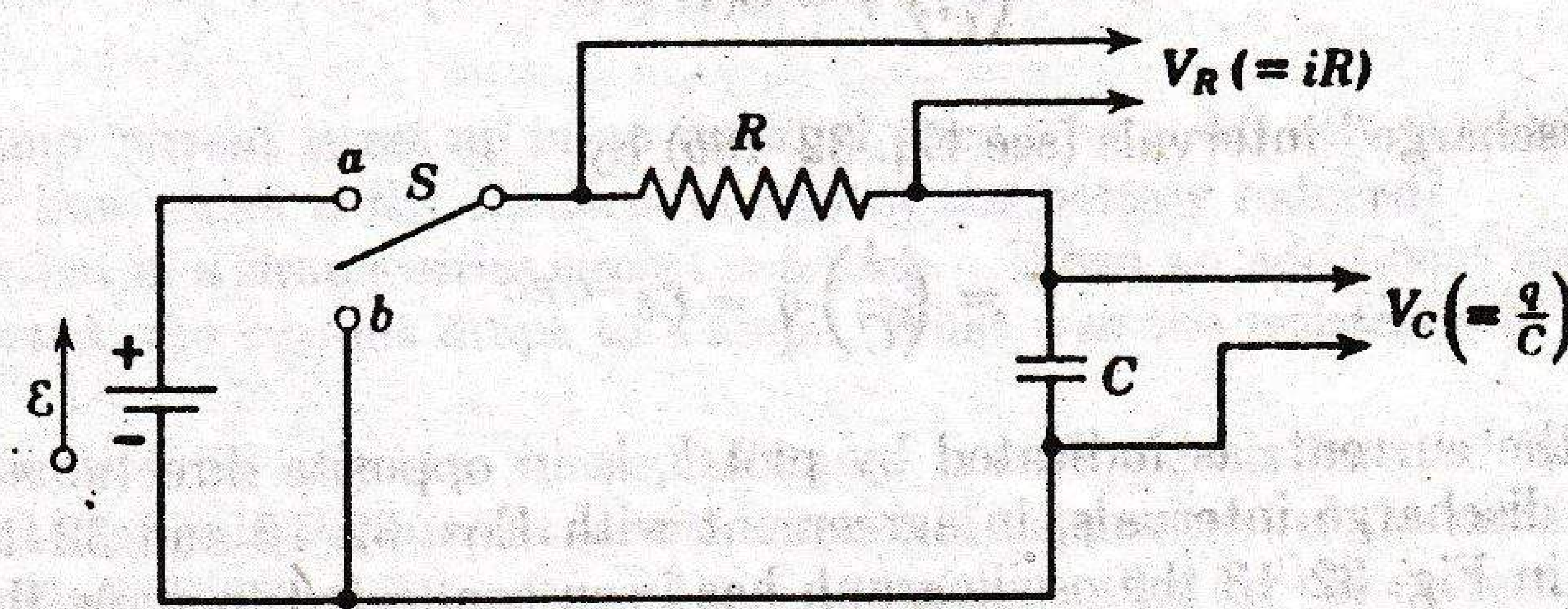


Fig. 32-12 The circuit of Fig. 32-10 with connections made to display the potential variations across the resistor and the capacitor on a cathode ray oscilloscope.

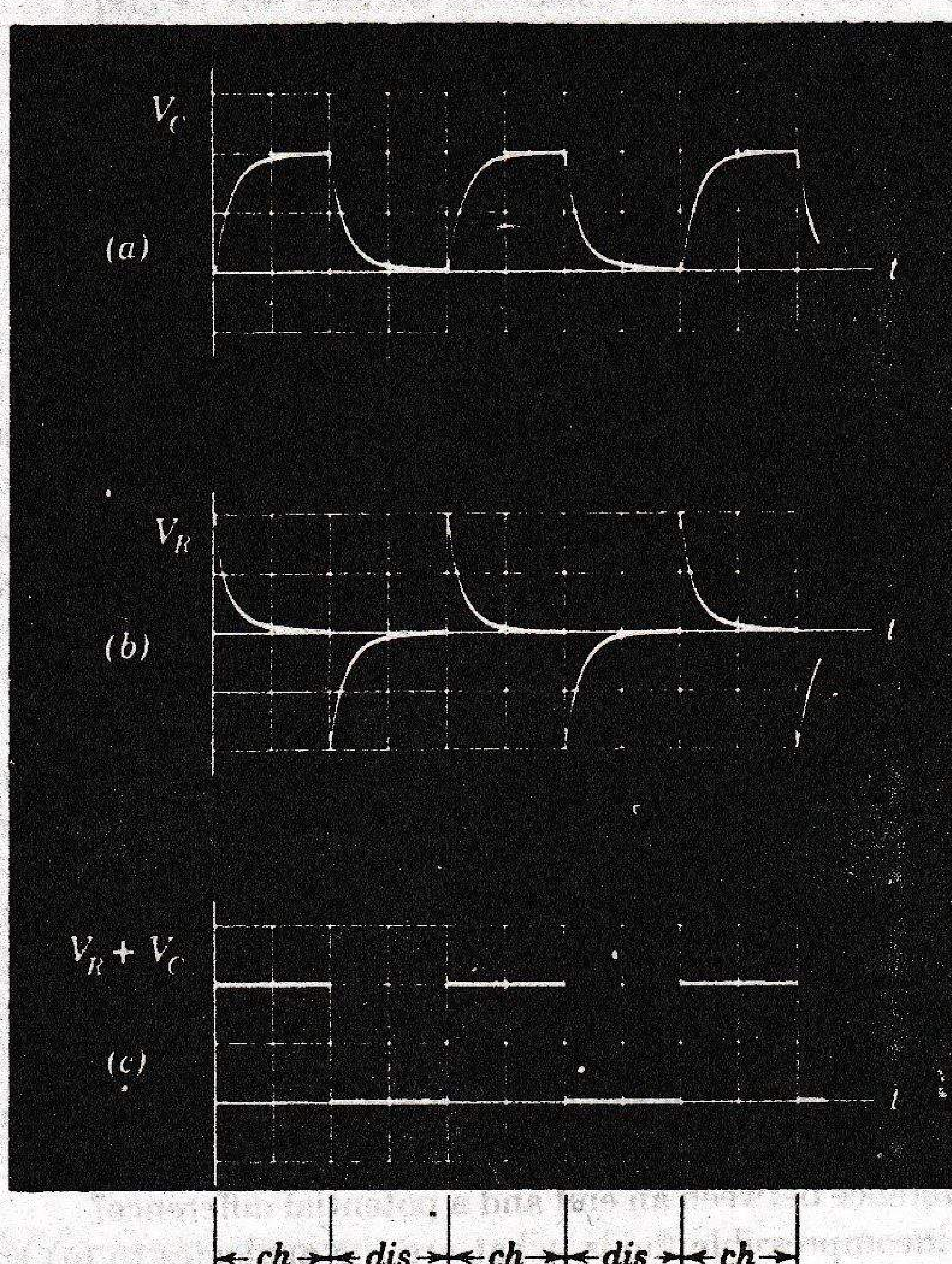


Fig. 32-13 In Fig. 32-10 switch S is thrown periodically, by electronic means, between positions a and b. The variations with time of the potential differences across (*a*) the capacitor, and (*b*) the resistor are shown, as displayed on a cathode ray oscilloscope. (*c*) The appearance of the screen when the oscilloscope is connected to display the sum of V_R and V_C. (Courtesy E. K. Hege, Rensselaer Polytechnic Institute.)

The "charge" intervals in plot a (see Eq. 32-15) are represented by

$$V = \left(\frac{1}{C}\right) q = \mathcal{E}(1 - e^{-t/RC})$$

and the "discharge" intervals (see Eq. 32-18b) by

$$V = \left(\frac{1}{C}\right) q = \mathcal{E}e^{-t/RC}.$$

Note that the current, as indicated by plot b, is in opposite directions during the charge and discharge intervals, in agreement with Eqs. 32-16 and 32-19.

In plot c in Fig. 32-13 the oscillograph has been connected to show the algebraic sum of plots a and b. According to the loop theorem this sum should equal $\mathcal{E}$ during the charge intervals and should be zero during the discharge intervals, when the battery is no longer in the circuit, that is,

$$V_R + V_C = \mathcal{E} \text{ during charge (see Eq. 32-12)}$$

$$V_R + V_C = 0 \text{ during discharge (see Eq. 32-17).}$$

Plot c is in exact agreement with this expectation.

QUESTIONS

1. Does the direction of the emf provided by a battery depend on the direction of current flow through the battery?
2. Figure 32-1b shows a gravitational analog of a single-loop electric circuit. Is the source of "gravitational emf" in this figure reversible as far as energy exchanges are concerned?
3. Discuss in detail the statement that the energy method and the loop theorem method for solving circuits are perfectly equivalent.
4. It is possible to generate a 10,000-volt potential difference by rubbing a pocket comb with wool. Why is this large voltage not dangerous when the much lower voltage provided by an ordinary electric outlet is very dangerous?
5. Devise a method for measuring the emf and the internal resistance of a battery.
6. A 25-watt, 110-volt bulb glows at normal brightness when connected across a bank of batteries. A 500-watt, 110-volt bulb glows only dimly when connected across the same bank. Explain.
7. Under what circumstances can the terminal potential difference of a battery exceed its emf?
8. What is the difference between an emf and a potential difference?
9. In the flow of incompressible fluids, what are the analogies to (a) the loop theorem and (b) the junction theorem?
10. Compare and contrast the formulas for the effective values of (a) capacitors and (b) resistors, in series and in parallel.
11. Does the time required for the charge on a capacitor in an RC circuit to build up to a given fraction of its equilibrium value depend on the value of the applied emf?
12. Devise a method whereby an RC circuit can be used to measure very high resistances.

PROBLEMS

1. A 5.0-amp current is set up in an external circuit by a 6.0-volt storage battery for 6.0 min. By how much is the chemical energy of the battery reduced?

2. The current in a simple series circuit is 5 amp. When an additional resistance of 2 ohms is inserted, the current drops to 4 amp. What was the resistance of the original circuit?

3. In Example 2 an ammeter whose resistance is 0.05 ohm is inserted in the circuit. What per cent change in the current results because of the presence of the meter?

4. In Fig. 32–3*a* put $\mathcal{E} = 2.0$ volts and $r = 100$ ohms. Plot (*a*) the current, and (*b*) the potential difference across R, as functions of R over the range 0 to 500 ohms. Make both plots on the same graph. (*c*) Make a third plot by multiplying together, for each value of R, the two curves plotted. What is the physical significance of this plot?

5. (*a*) In the circuit of Fig. 32–3*a* show that the power delivered to R as Joule heat is a maximum when R is equal to the internal resistance r of the battery. (*b*) Show that this maximum power is $P = \mathcal{E}^2/4r$.

6. Heat is to be generated in a 0.10-ohm resistor at the rate of 10 watts by connecting it to a battery whose emf is 1.5 volts. (*a*) What is the internal resistance of the battery? (*b*) What potential difference exists across the resistor?

7. (*a*) In Fig. 32–14 what value must R have if the current in the circuit is to be 0.001 amp? Take $\mathcal{E}_1 = 2.0$ volts, $\mathcal{E}_2 = 3.0$ volts, and $r_1 = r_2 = 3.0$ ohms. (*b*) What is the rate of Joule heating in R?

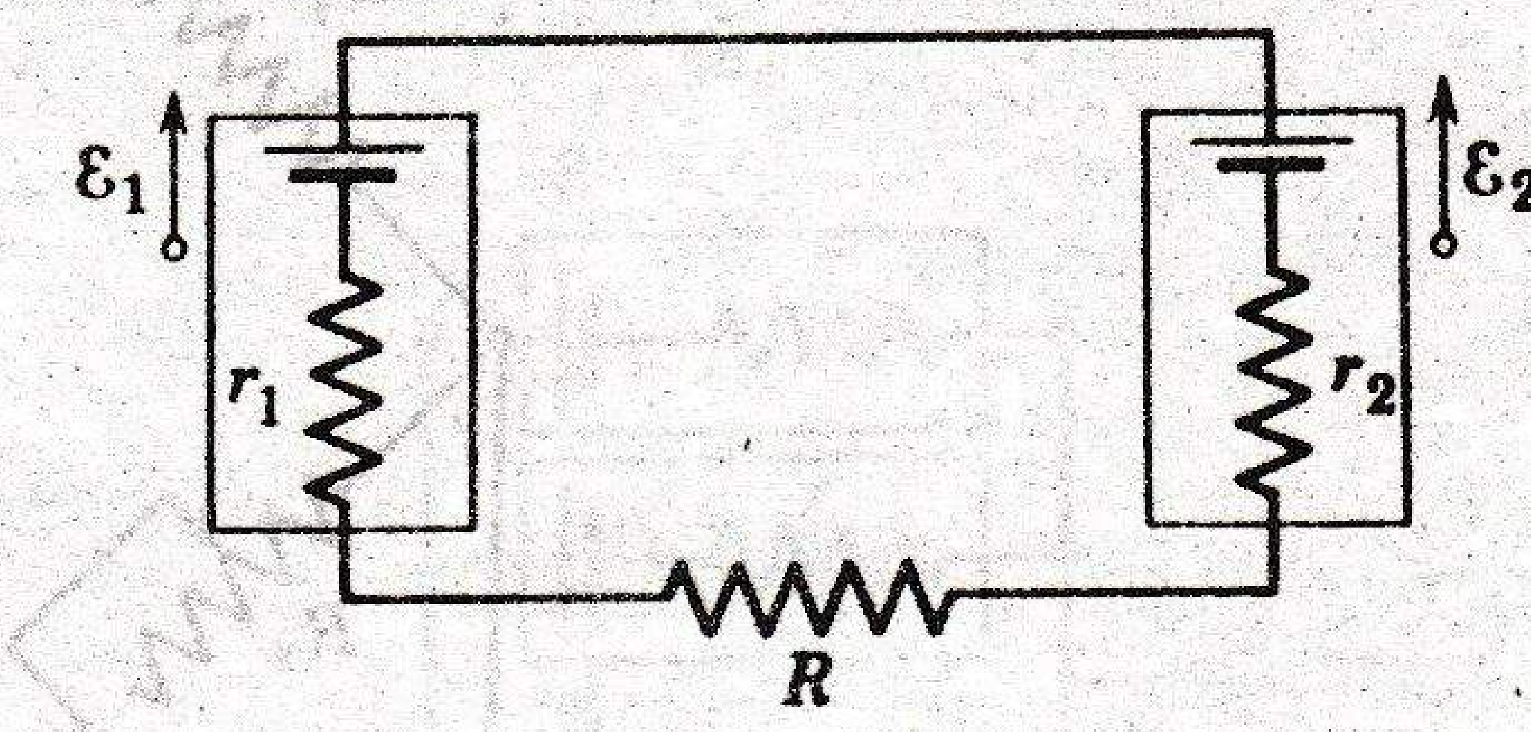


Fig. 32–14

8. A wire of resistance 5.0 ohms is connected to a battery whose emf $\mathcal{E}$ is 2.0 volts and whose internal resistance is 1.0 ohm. In 2.0 min (*a*) how much energy is transferred from chemical to electric form? (*b*) How much energy appears in the wire as Joule heat? (*c*) Account for the difference between (*a*) and (*b*).

9. In Fig. 32–6 calculate the potential difference between points c and d by as many paths as possible. Assume that $\mathcal{E}_1 = 4.0$ volts, $\mathcal{E}_2 = 1.0$ volt, $R_1 = R_2 = 10$ ohms, and $R_3 = 5$ ohms.

10. In Fig. 32–5 calculate the potential difference between a and c by considering a path that contains R and $\mathcal{E}_2$.

11. (*a*) In Fig. 32–15 what is the equivalent resistance of the network shown? (*b*) What are the currents in each resistor? Put $R_1 = 100$ ohms, $R_2 = R_3 = 50$ ohms, $R_4 = 75$ ohms, and $\mathcal{E} = 6.0$ volts.

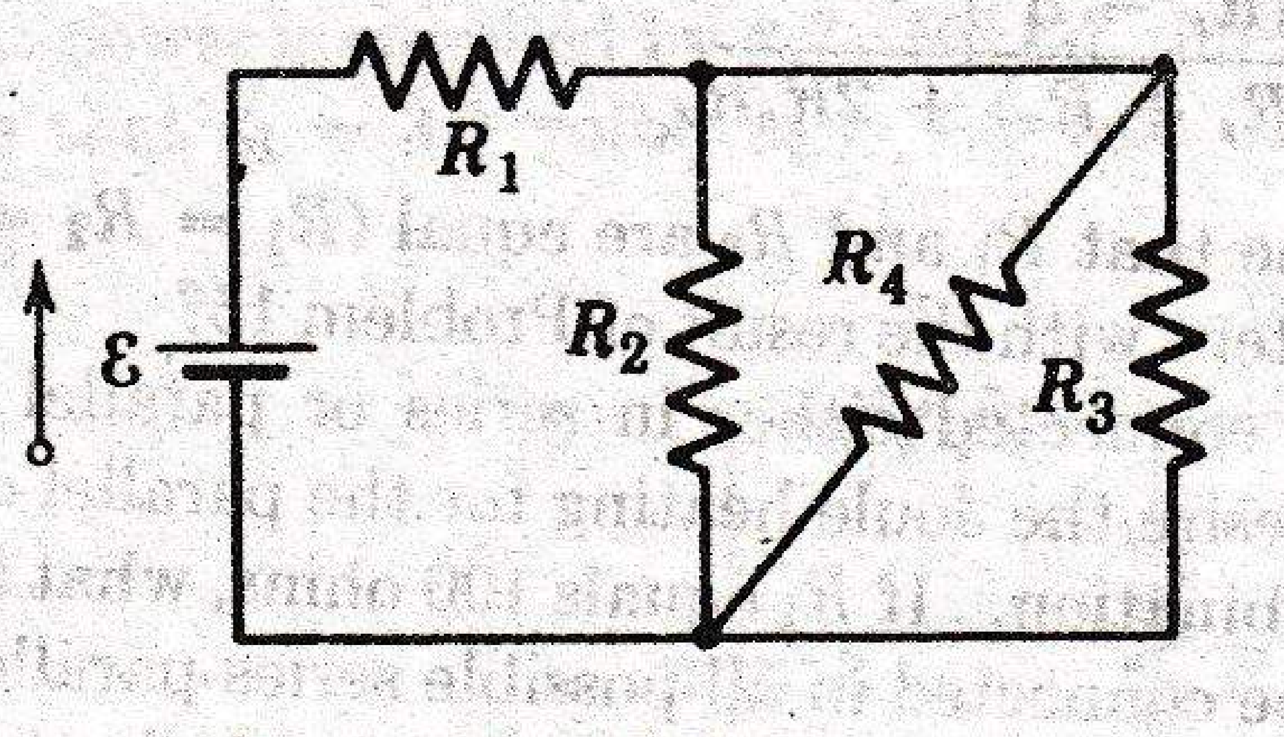


Fig. 32–15

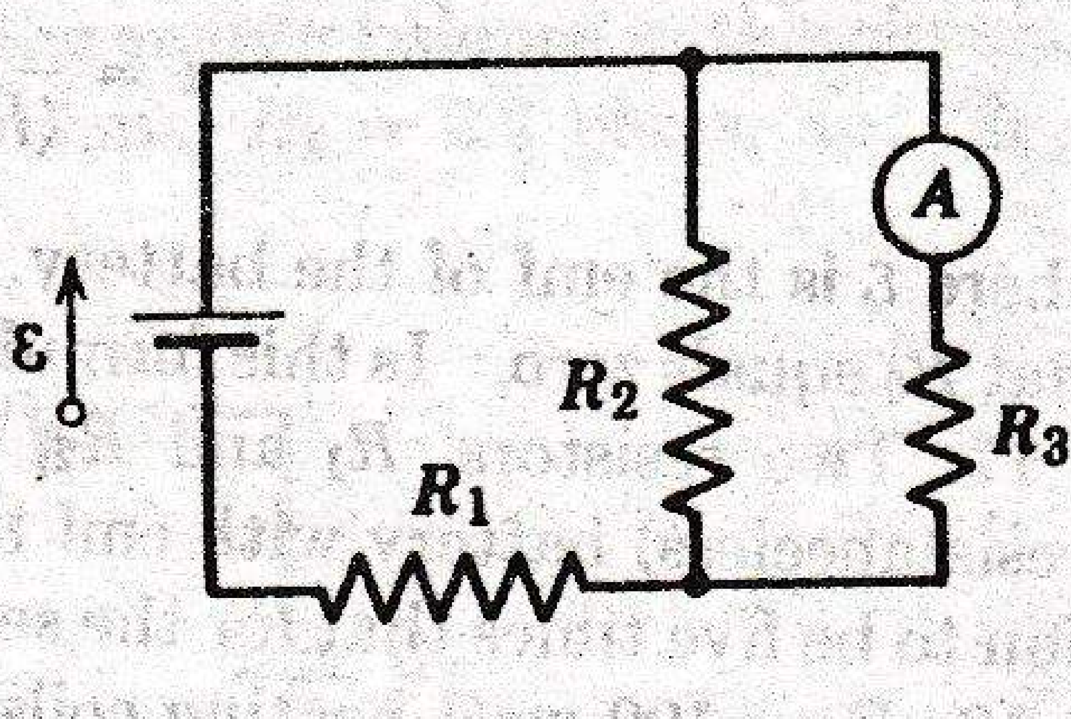


Fig. 32–16

12. In Fig. 32–16 imagine an ammeter inserted in the branch containing R_3. (*a*) What will it read, assuming $\mathcal{E} = 5.0$ volts, $R_1 = 2.0$ ohms, $R_2 = 4.0$ ohms, and $R_3 = 6.0$ ohms? (*b*) The ammeter and the source of emf are now physically interchanged. Show that the

ammeter reading remains unchanged. This reciprocity relationship holds for any circuit that contains only one source of emf.

13. Two batteries of emf $\mathcal{E}$ and internal resistance r are connected in parallel across a resistor R as in Fig. 32–20b. (a) For what value of R is the power delivered to the resistor a maximum? (b) What is the maximum power?

14. By using only two resistance coils—singly, in series, or in parallel—a student is able to obtain resistances of 3, 4, 12, and 16 ohms. What are the separate resistances of the coils?

15. *The Wheatstone bridge.* In Fig. 32–17 R_s is to be adjusted in value until points a and b are brought to exactly the same potential. (One tests for this condition by momentarily connecting a sensitive meter between a and b; if these points are at the same potential, the meter will not deflect.) Show that when this adjustment is made the following relation holds:

$$R_x = R_s \frac{R_2}{R_1}.$$

Unknown resistors (R_x) can be measured in terms of standards (R_s) using this device, which is called a Wheatstone bridge.

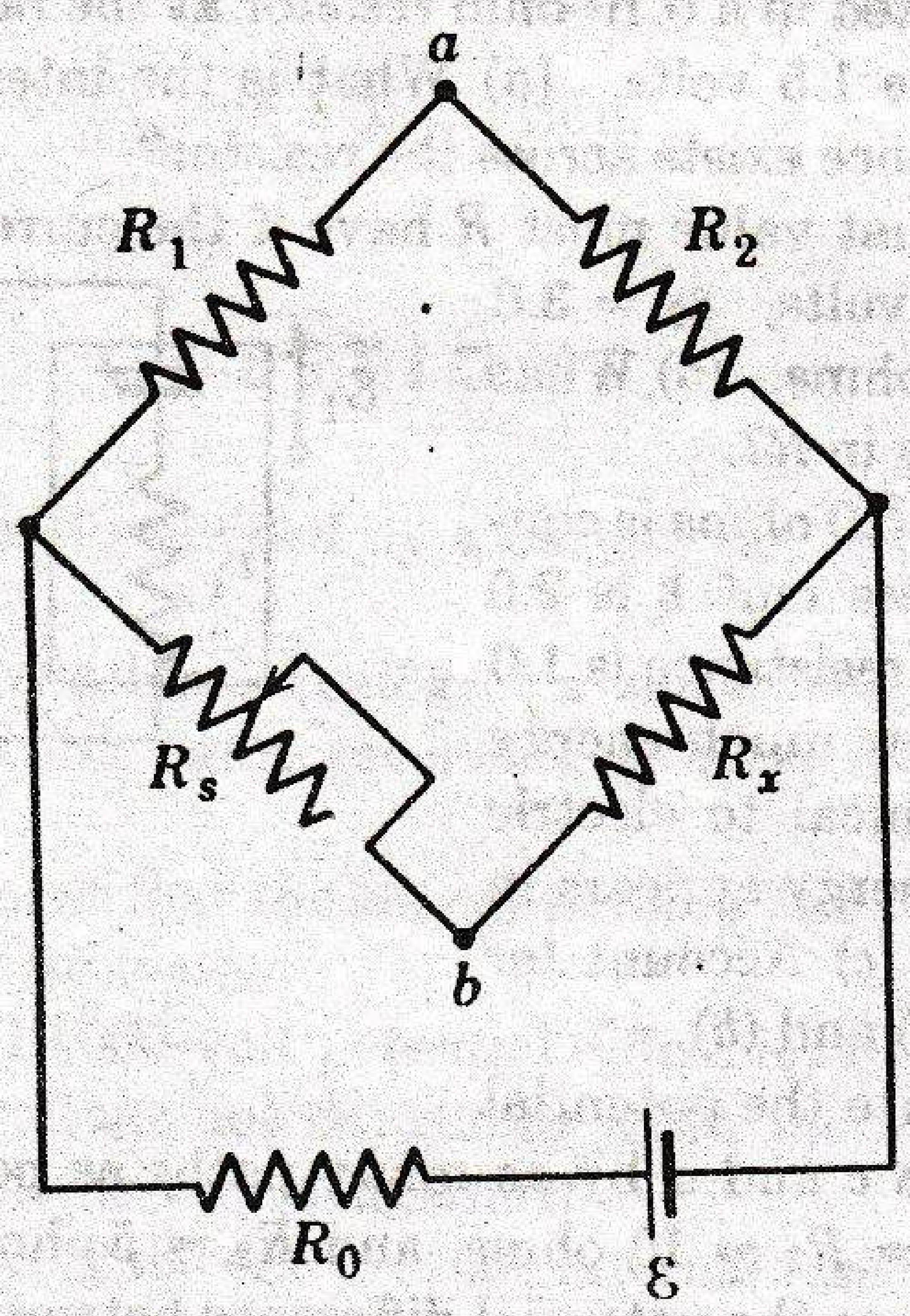


Fig. 32–17

16. If points a and b in Fig. 32–17 are connected by a wire of resistance r, show that the current in the wire is

$$i = \frac{\mathcal{E}(R_s - R_x)}{(R + 2r)(R_s + R_x) + 2R_sR_x},$$

where $\mathcal{E}$ is the emf of the battery. Assume that R_1 and R_2 are equal ($R_1 = R_2 = R$) and that R_0 equals zero. Is this formula consistent with the result of Problem 15?

17. Two resistors, R_1 and R_2, may be connected either in series or parallel across a (resistanceless) battery with emf $\mathcal{E}$. We desire the Joule heating for the parallel combination to be five times that for the series combination. If R_1 equals 100 ohms, what is R_2?

18. Four 100-watt heating coils are to be connected in all possible series-parallel combinations and plugged into a 100-volt line. What different rates of heat dissipation are possible?

19. What is the equivalent resistance between the terminal points x and y of the circuits shown in (a) Fig. 32–18a, (b) Fig. 32–18b, and (c) Fig. 32–18c? Assume that the resistance of each resistor is 10 ohms.

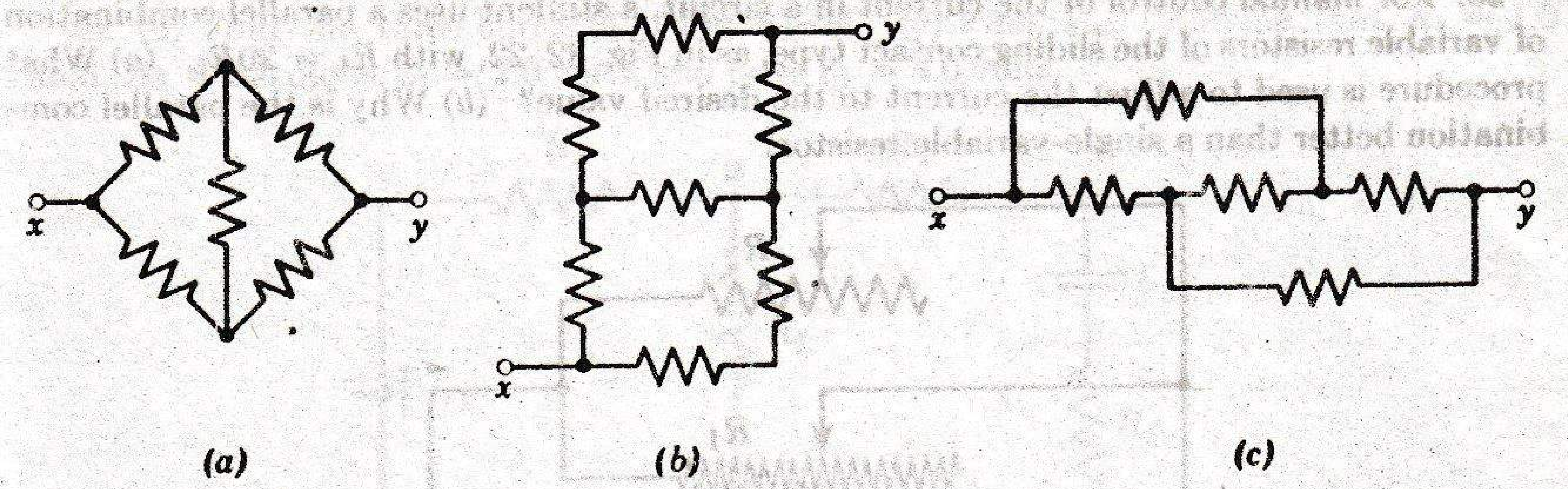


Fig. 32-18

20. In Fig. 32-19 find the current in each resistor and the potential difference between a and b. Put $\mathcal{E}_1 = 6.0$ volts, $\mathcal{E}_2 = 5.0$ volts, $\mathcal{E}_3 = 4.0$ volts, $R_1 = 100$ ohms, and $R_2 = 50$ ohms.

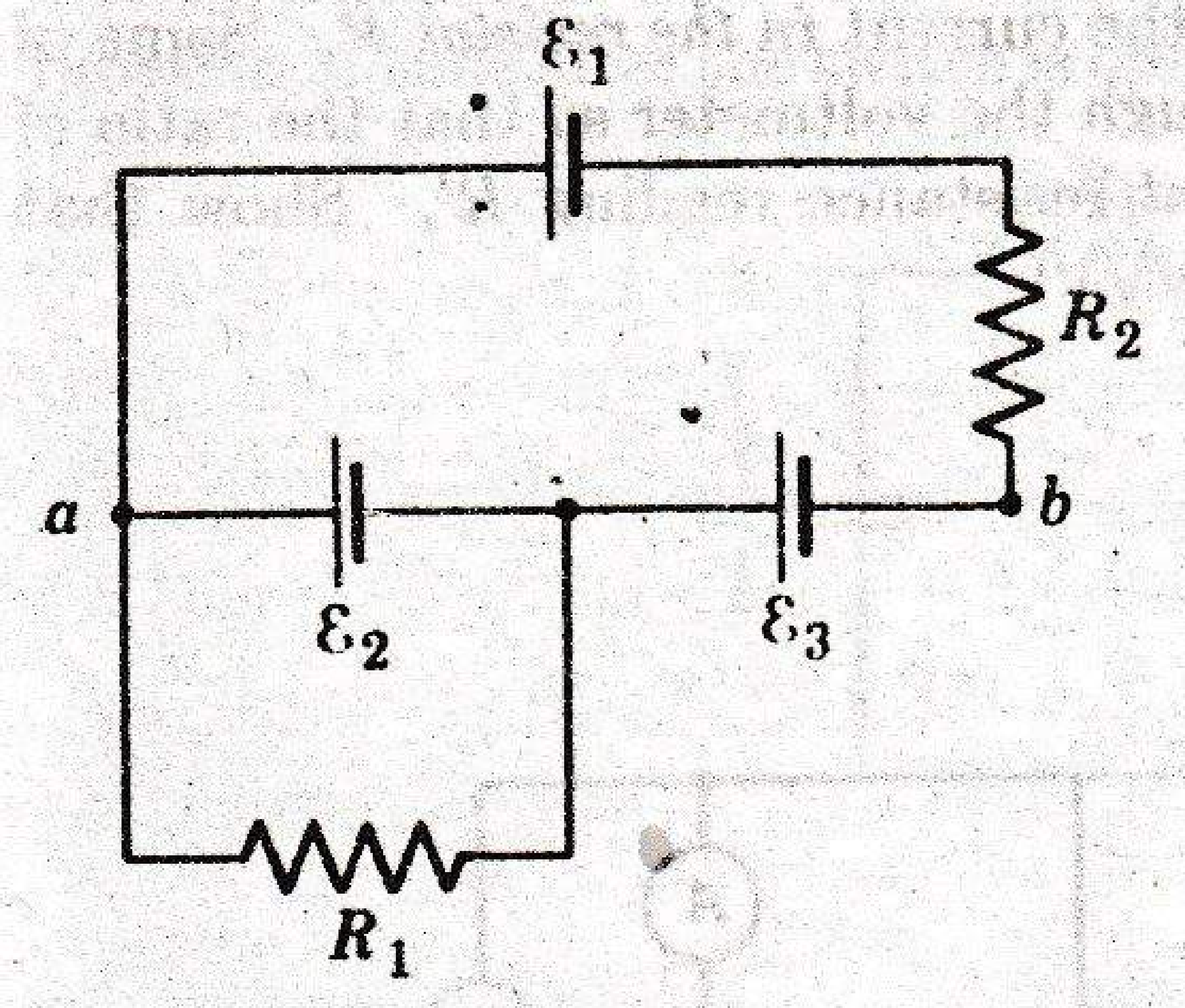


Fig. 32-19

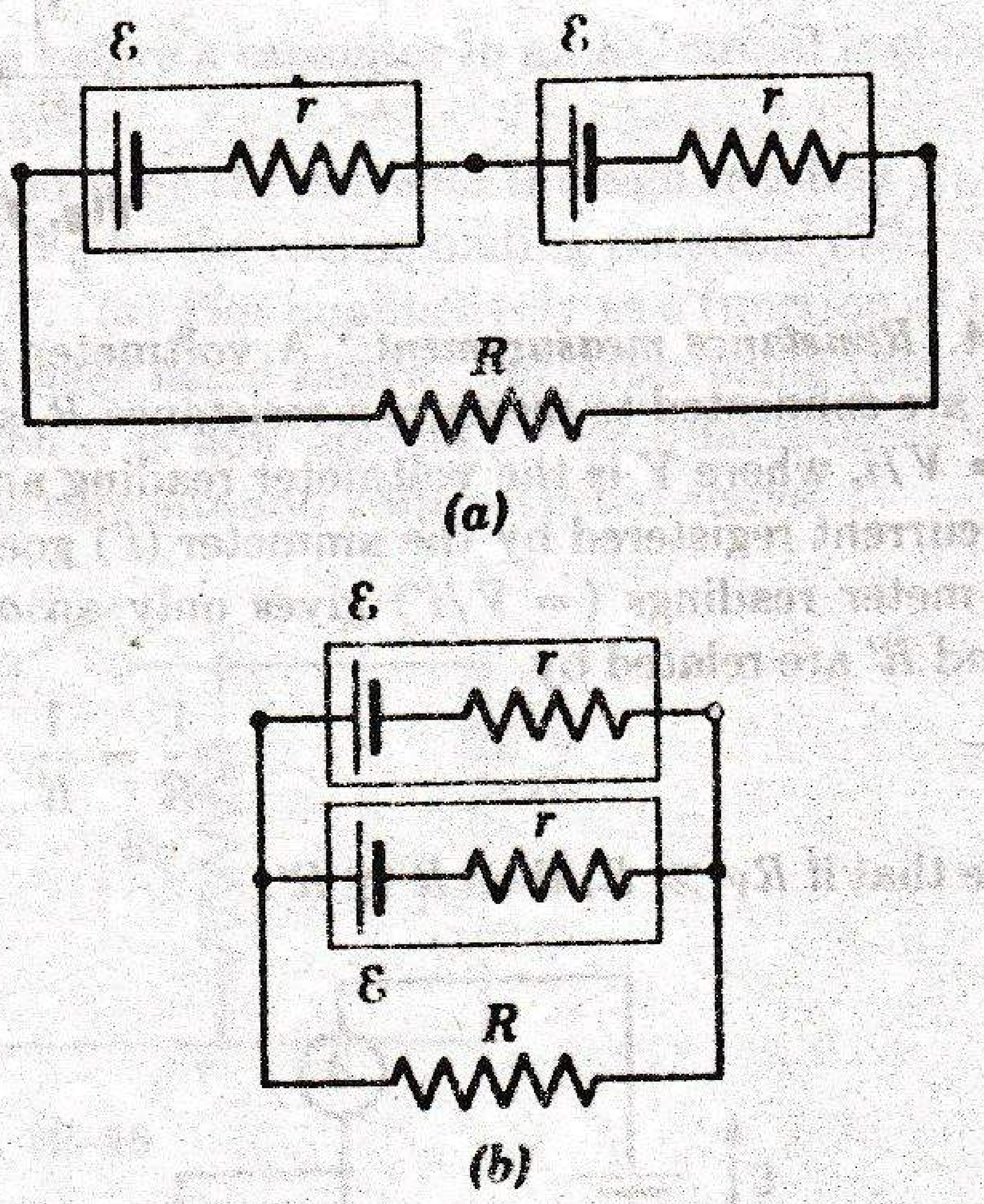


Fig. 32-20

21. You are given two batteries of emf $\mathcal{E}$ and internal resistance r. They may be connected either in series or in parallel and are used to establish a current in a resistor R, as in Fig. 32-20. Derive expressions for the current in R for both methods of connection. Which connection yields the larger current if (a) $R > r$ and if (b) $R < r$?

22. (a) In Fig. 32-21 what power appears as Joule heat in R_1? In R_2? In R_3? (b) What power is supplied by $\mathcal{E}_1$? By $\mathcal{E}_2$? (c) Discuss the energy balance in this circuit. Assume that $\mathcal{E}_1 = 3.0$ volts, $\mathcal{E}_2 = 1.0$ volt, $R_1 = 5.0$ ohms, $R_2 = 2.0$ ohms, and $R_3 = 4.0$ ohms.

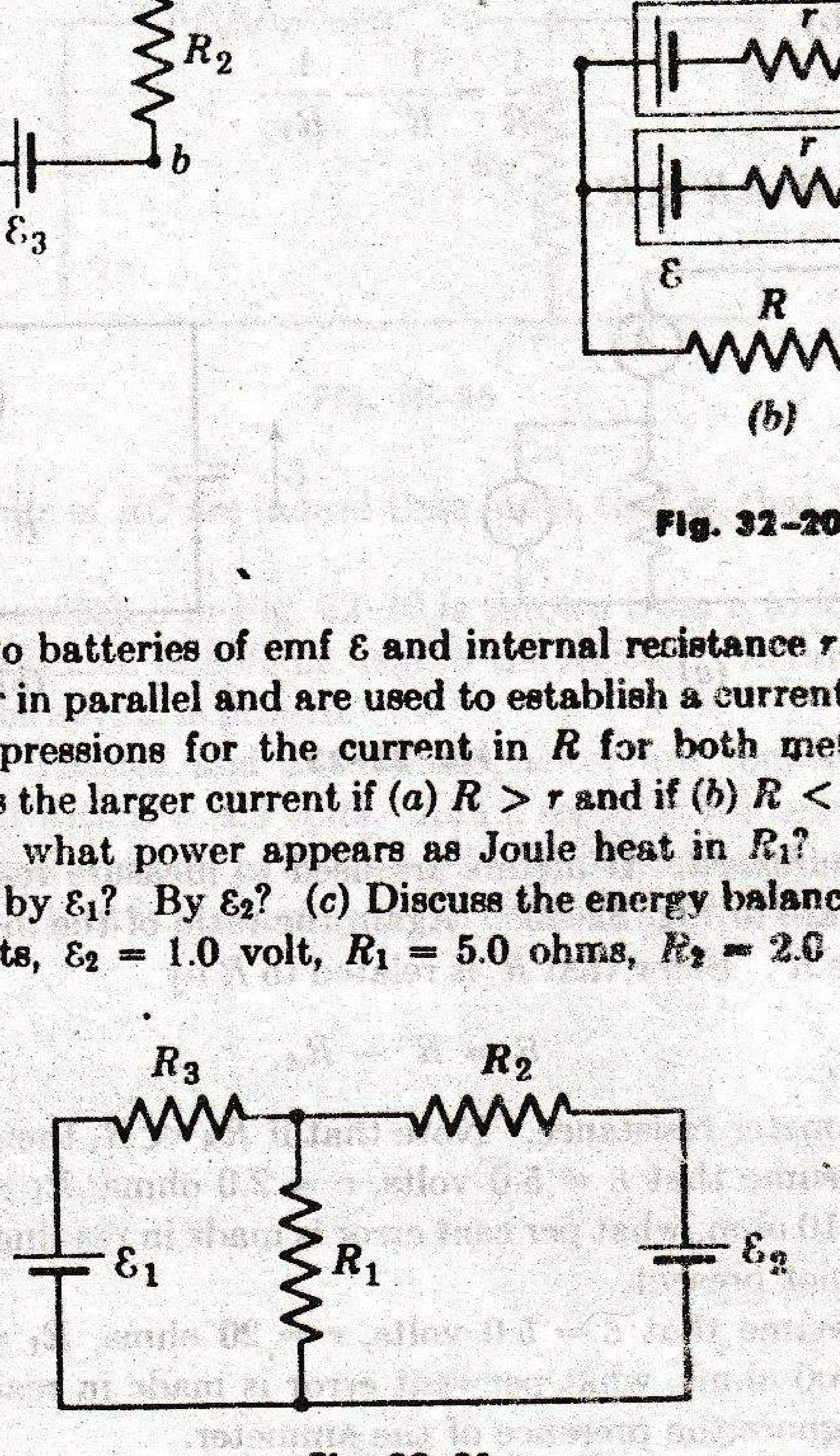


Fig. 32-21

23. For manual control of the current in a circuit, a student uses a parallel combination of variable resistors of the sliding contact type, as in Fig. 32–22, with $R_1 = 20R_2$. (*a*) What procedure is used to adjust the current to the desired value? (*b*) Why is the parallel combination better than a single-variable resistor?

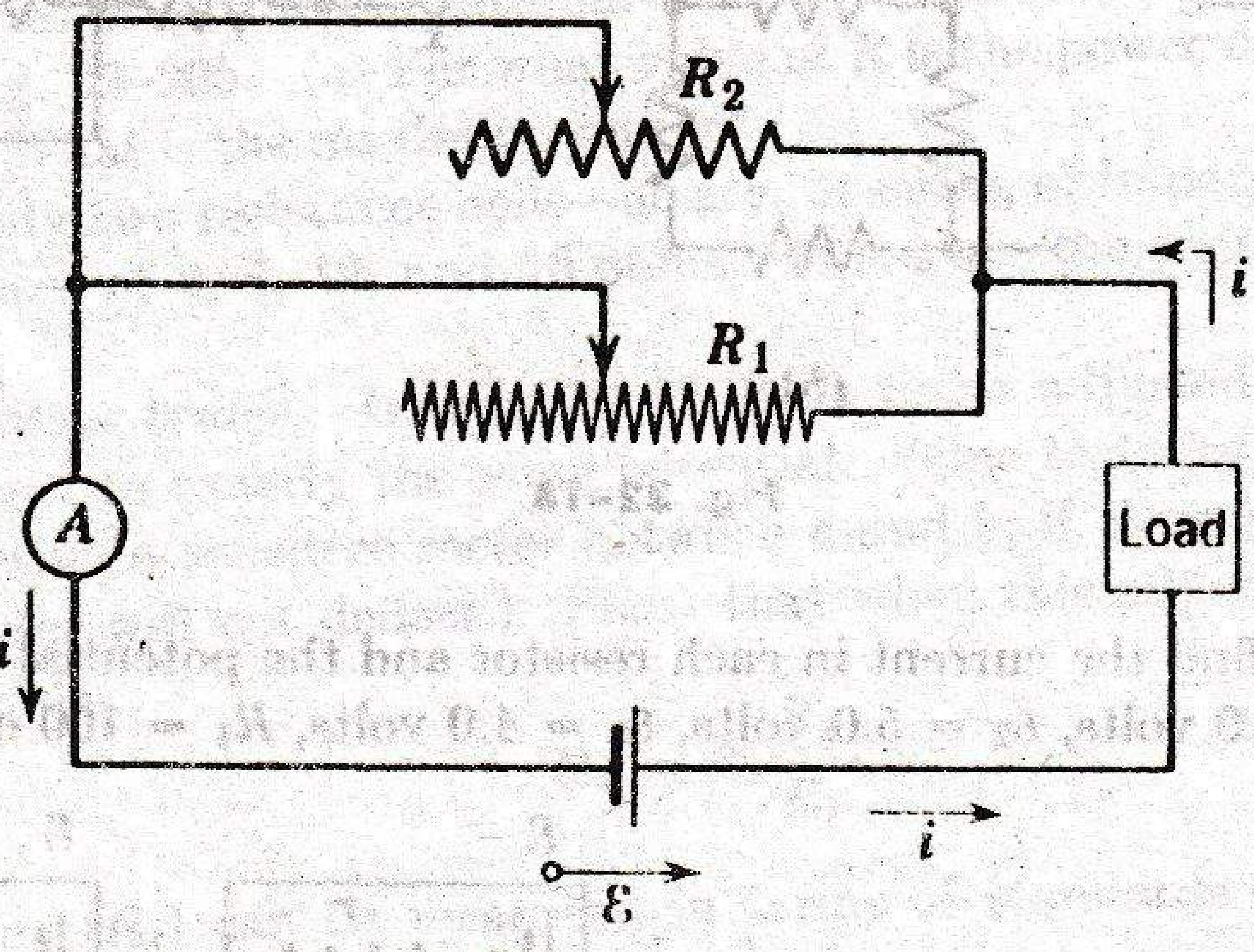


Fig. 32–22

24. *Resistance measurement.* A voltmeter (resistance R_V) and an ammeter (resistance R_A) are connected to measure a resistance R, as in Fig. 32–23*a*. The resistance is given by $R = V/i$, where V is the voltmeter reading and i is the current *in the resistor R*. Some of the current registered by the ammeter (i') goes through the voltmeter so that the ratio of the meter readings ($= V/i'$) gives only an *apparent* resistance reading R'. Show that R and R' are related by

$$\frac{1}{R} = \frac{1}{R'} - \frac{1}{R_V}.$$

Note that if $R_V \gg R$, then $R \cong R'$.

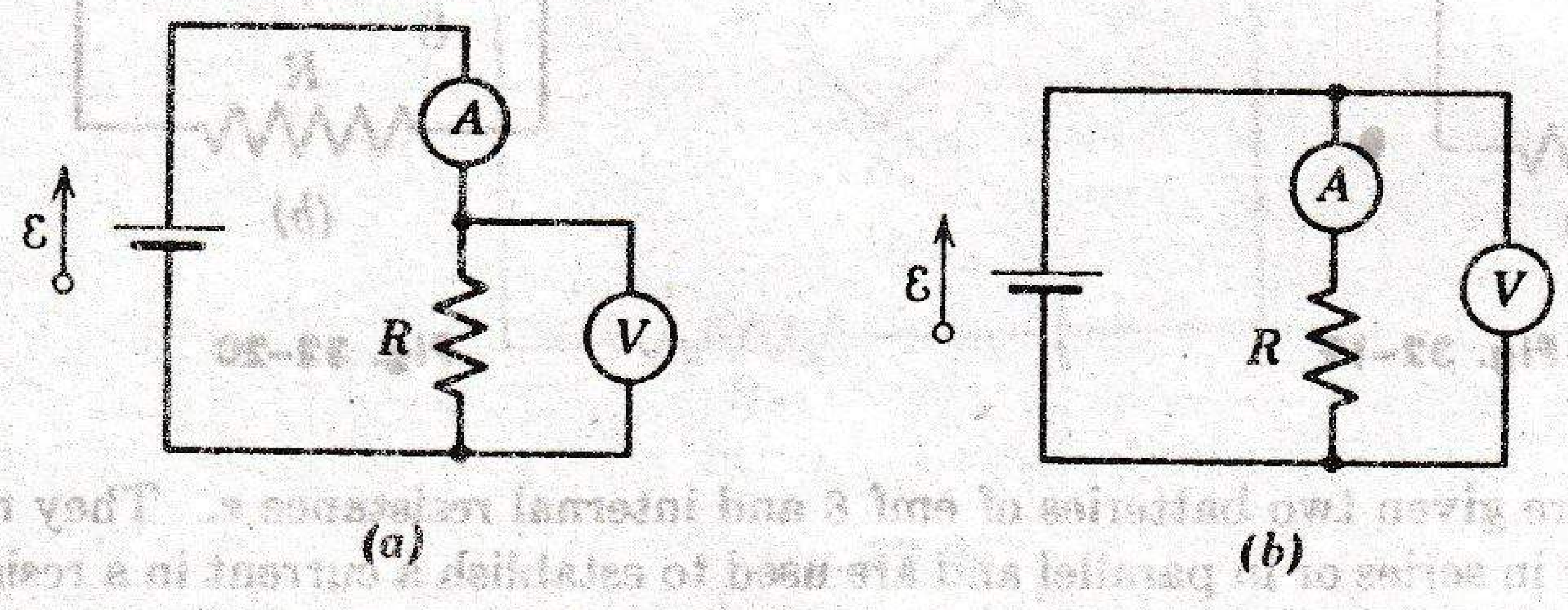


Fig. 32–23

25. *Resistance measurement.* If meters are used to measure resistance, they may also be connected as they are in Fig. 32–23*b*. Again the ratio of the meter readings gives only an apparent resistance R'. Show that R' is related to R by

$$R = R' - R_A,$$

in which R_A is the ammeter resistance. Note that if $R_A \ll R$, then $R \cong R'$.

26. In Fig. 32–8 assume that $\mathcal{E} = 5.0$ volts, $r = 2.0$ ohms, $R_1 = 5.0$ ohms, and $R_2 = 4.0$ ohms. If $R_A = 0.10$ ohm, what per cent error is made in reading the current? Assume that the voltmeter is not present.

27. In Fig. 32–8 assume that $\mathcal{E} = 5.0$ volts, $r = 20$ ohms, $R_1 = 50$ ohms, and $R_2 = 40$ ohms. If $R_v = 1000$ ohms, what per cent error is made in reading the potential differences across R_1? Ignore the presence of the ammeter.

28. (*a*) Find the three currents in Fig. 32–24. (*b*) Find V_{ab}. Assume that $R_1 = 1.0$ ohm, $R_2 = 2.0$ ohms, $\mathcal{E}_1 = 2.0$ volts, and $\mathcal{E}_2 = \mathcal{E}_3 = 4.0$ volts.

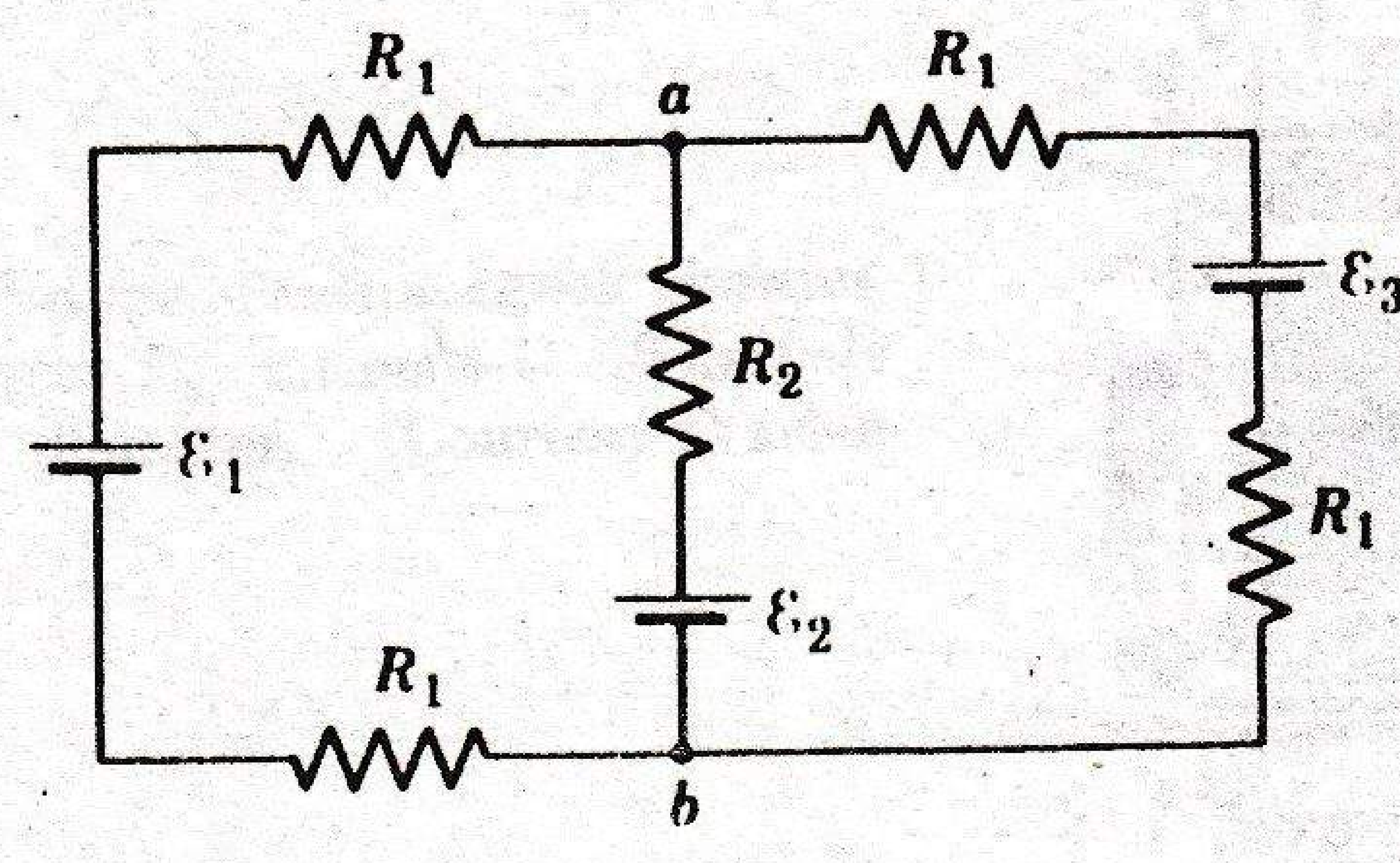


Fig. 32–24

29. How many time constants must elapse before a capacitor in an *RC* circuit is charged to within 1.0 per cent of its equilibrium charge?

30. In the circuit of Fig. 32–25 let i_1, i_2, and i_3 be the currents through resistors R_1, R_2, and R_3, respectively, and let V_1, V_2, V_3, and V_C be the corresponding potential differences across the resistors and across the capacitor *C*. (*a*) Plot qualitatively as a function of time after switch *S* is closed the currents and voltages listed above. (*b*) After being closed for a large number of time constants, the switch *S* is now opened. Plot qualitatively as a function of time after the switch is opened the currents and voltages listed above.

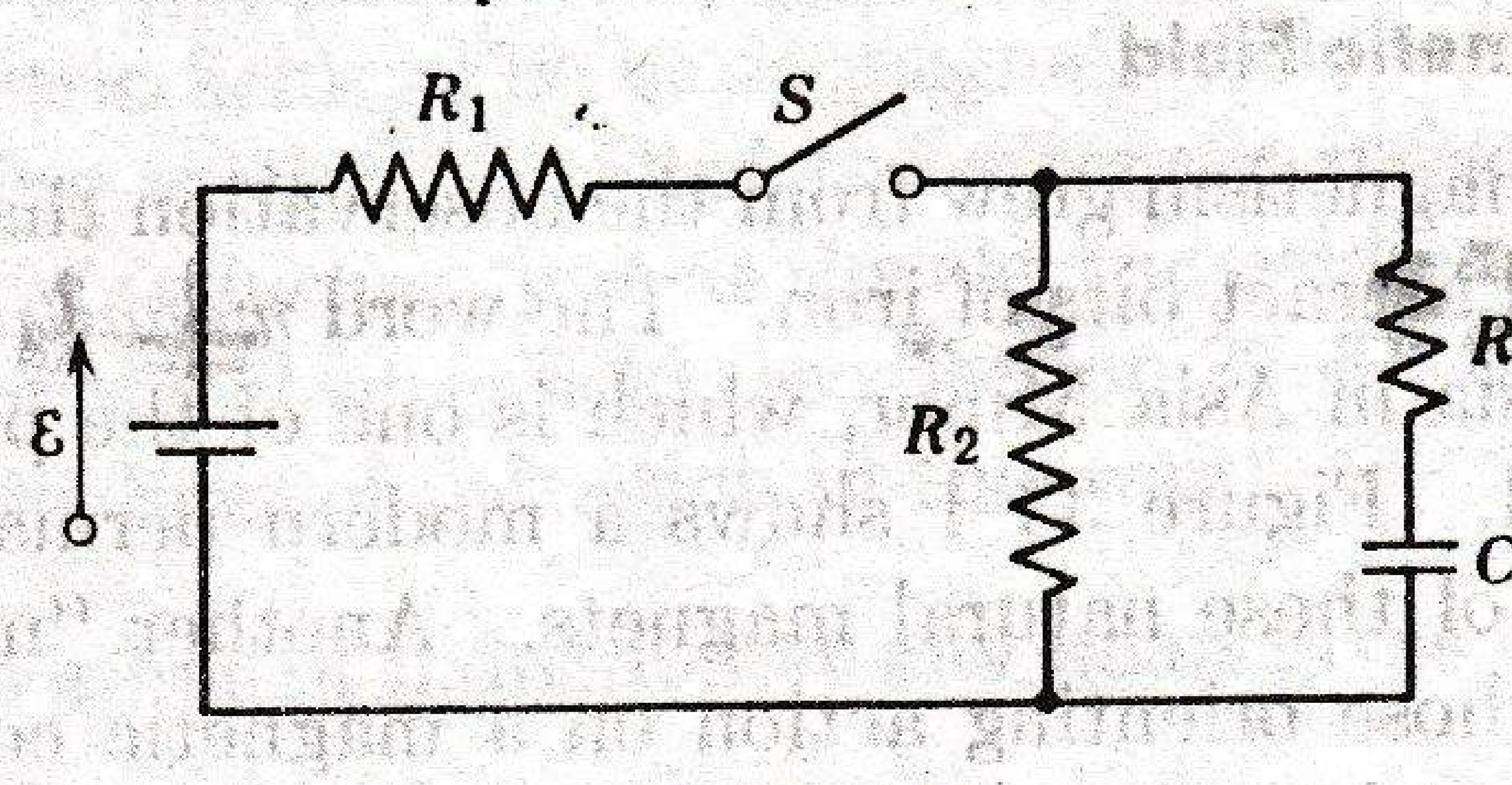


Fig. 32–25

31. Show that the units of *RC* are indeed time units, that is, that 1 ohm × 1 farad = 1 sec.

32. Prove that when switch *S* in Fig. 32–10 is thrown from *a* to *b* all the energy stored in the capacitor is transformed into Joule heat in the resistor. Assume that the capacitor is fully charged before the switch is thrown.

33. A 3.0×10^6-ohm resistor and a 1.0-μf capacitor are connected in a single-loop circuit with a seat of emf with $\mathcal{E} = 4.0$ volts. At 1.0 sec after the connection is made, what are the rates at which (*a*) the charge of the capacitor is increasing, (*b*) energy is being stored in the capacitor, (*c*) Joule heat is appearing in the resistor, and (*d*) energy is being delivered by the seat of emf?

The Magnetic Field

CHAPTER 33

33–1 The Magnetic Field

The science of magnetism grew from the observation that certain "stones" (magnetite) would attract bits of iron. The word *magnetism* comes from the district of Magnesia in Asia Minor, which is one of the places at which the stones were found. Figure 33–1 shows a modern permanent magnet, the lineal descendant of these natural magnets. Another "natural magnet" is the earth itself, whose orienting action on a magnetic compass needle has been known since ancient times.

In 1820 Oersted first discovered that a current in a wire can also produce magnetic effects, namely, that it can change the orientation of a compass needle. We pointed out in Section 26–1 how this important discovery linked the then separate sciences of magnetism and electricity. The magnetic ef-

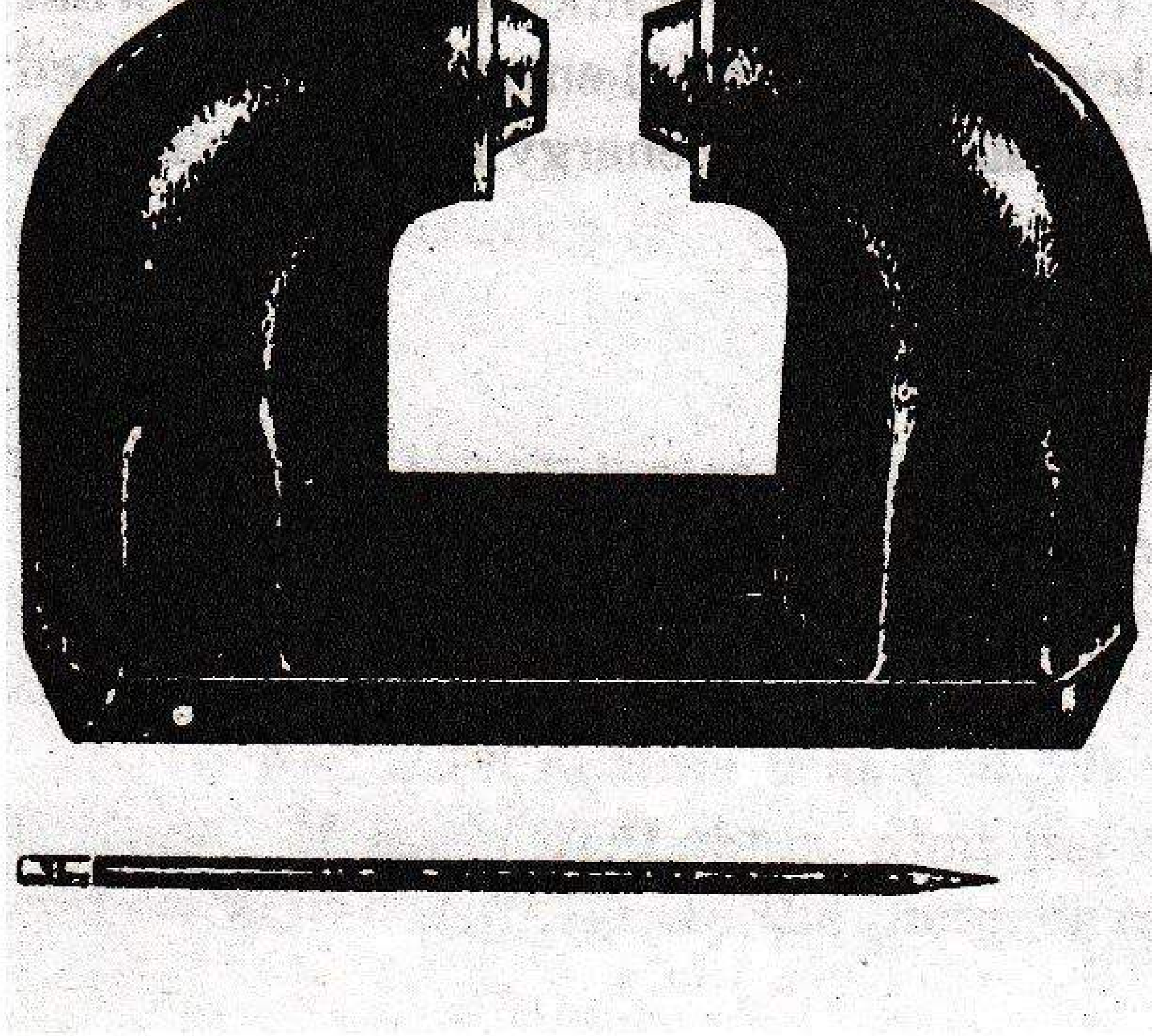


Fig. 33–1 A permanent magnet. Lines of magnetic induction leave the north pole face, marked N, and enter the south pole face on the other side of the air gap.

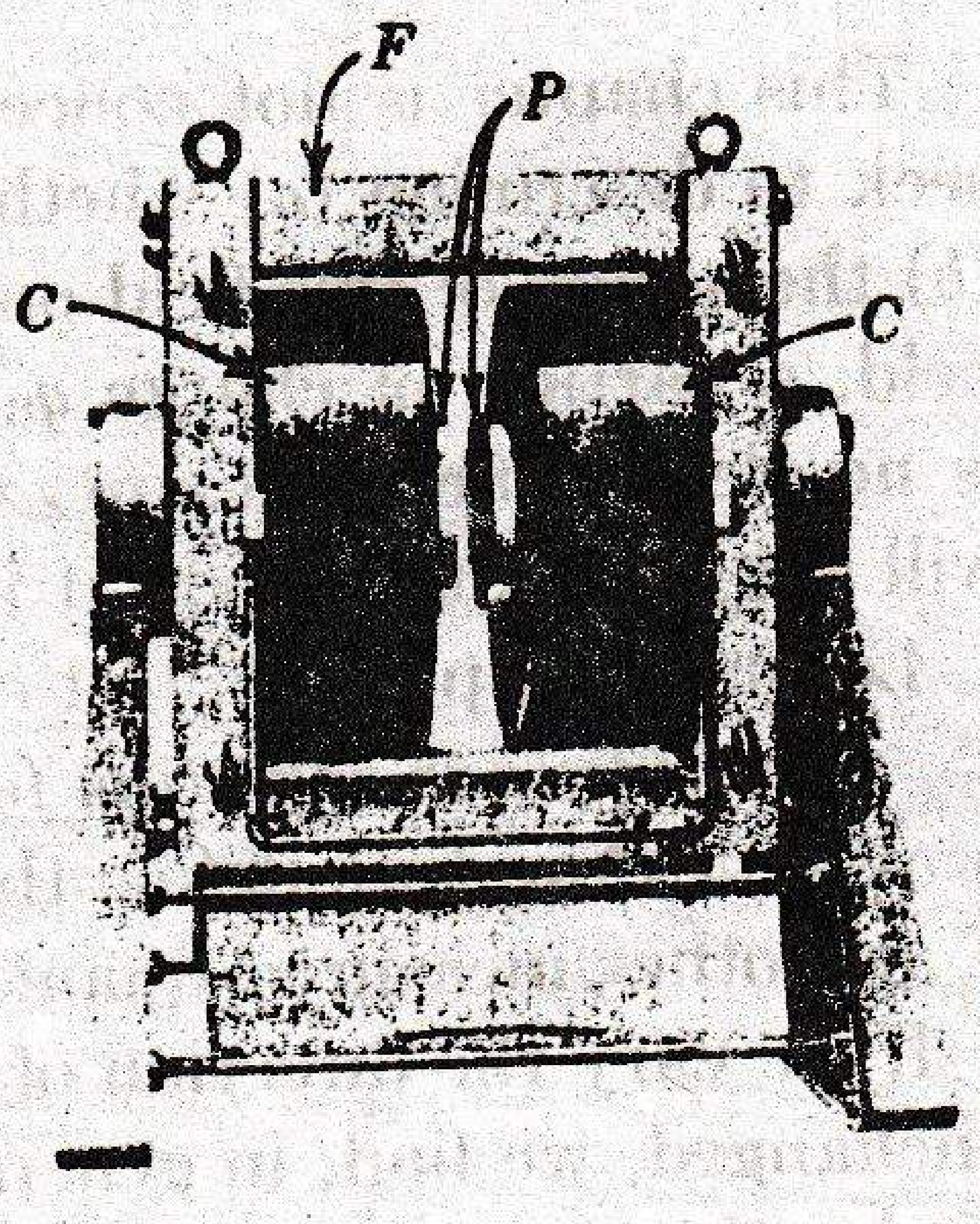


Fig. 33-2 A research-type electromagnet showing iron frame *F*, pole faces *P*, and coils *C*. The pole faces are 12 in. in diameter. (Courtesy Varian Associates.)

fect of a current in a wire can be intensified by forming the wire into a coil of many turns and by providing an iron core. Figure 33-2 shows how this is done in a large electromagnet of a type commonly used for research involving magnetism.

We define the space around a magnet or a current-carrying conductor as the site of a *magnetic field*, just as we defined the space near a charged rod as the site of an electric field. The basic magnetic field vector **B**, which we define in the following section, is called the *magnetic induction:* * it can be represented by *lines of induction*, just as the electric field was represented by lines of force. As for the electric field (see Section 27-3), the magnetic field vector is related to its lines of induction in this way:

1. The tangent to a line of induction at any point gives the *direction* of **B** at that point.

2. The lines of induction are drawn so that the number of lines per unit cross-sectional area is proportional to the *magnitude* of the magnetic field vector **B**. Where the lines are close together B is large and where they are far apart B is small.

As for the electric field, the field vector **B** is of fundamental importance, the lines of induction simply giving a graphic representation of the way **B** varies throughout a certain region of space.

The *flux* Φ_B for a magnetic field can be defined in exact analogy with the flux Φ_E for the electric field, namely

$$\Phi_B = \int \mathbf{B} \cdot d\mathbf{S}, \tag{33-1}$$

in which the integral is taken over the surface (closed or open) for which Φ_B is defined.

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* *Magnetic field strength* would be a more suitable name for **B**, but it has been usurped for historical reasons by another vector connected with the magnetic field (see Section 37-7).

33-2 The Definition of B

This chapter is not concerned with the *causes* of the magnetic field; we seek to determine (*a*) whether a magnetic field exists at a given point and (*b*) the action of this field on charges moving through it. As for the electric field, a particle of charge q_0 serves as a test body. We assume that there is no electric field present, which means that, neglecting gravity, no force will act on the test body if it is placed *at rest* at the point in question.

Let us fire a positive test charge with arbitrary velocity **v** through a point *P*. *If a sideways deflecting force* **F** *acts on it, we assert that a magnetic field is present at P* and we define the magnetic induction **B** of this field in terms of **F** and other measured quantities.

If we vary the direction of **v** through point *P*, keeping the magnitude of **v** unchanged, we find, in general, that although **F** will always remain at right angles to **v** its magnitude *F* will change. For a particular orientation of **v** (and also for the opposite orientation $-\mathbf{v}$) the force **F** becomes zero. We define this direction as the direction of **B**, the specification of the sense of **B** (that is, the way it points along this line) being left to the more complete definition of **B** that we give below.

Having found the *direction* of **B**, we are now able to orient **v** so that the test charge moves at right angles to **B**. We will find that the force **F** is now a maximum, and we define the *magnitude* of **B** from the measured magnitude of this maximum force $F_\perp$, or

$$B = \frac{F_\perp}{q_0 v}. \tag{33-2}$$

Let us regard this definition of **B** (in which we have specified its magnitude and direction, but not its sense) as preliminary to the complete vector definition that we now give: *If a positive test charge q_0 is fired with velocity* **v** *through a point P and if a (sideways) force* **F** *acts on the moving charge, a magnetic induction* **B** *is present at point P, where* **B** *is the vector that satisfies the relation*

$$\mathbf{F} = q_0\mathbf{v} \times \mathbf{B}, \tag{33-3a}$$

v, q_0, and **F** being measured quantities. The magnitude of the magnetic deflecting force **F**, according to the rules for vector products, is given by *

$$F = q_0 v B \sin\theta, \tag{33-3b}$$

in which θ is the angle between **v** and **B**.

Figure 33-3 shows the relations among the vectors. We see that **F**, being at right angles to the plane formed by **v** and **B**, will always be at right angles to **v** (and also to **B**) and thus will always be a sideways force. Equation 33-3*a* is consistent with the observed facts that (*a*) the magnetic force vanishes as $v \to 0$, (*b*) the magnetic force vanishes if **v** is either parallel or antiparallel to the direction of **B** (in these cases $\theta = 0$ or 180° and $\mathbf{v} \times \mathbf{B} = 0$), and (*c*) if **v** is at right angles to **B** ($\theta = 90°$), the deflecting force has its maximum value, given by Eq. 33-2, that is, $F_\perp = q_0 v B$.

* The student may wish to review Section 2-4, which deals with vector products.

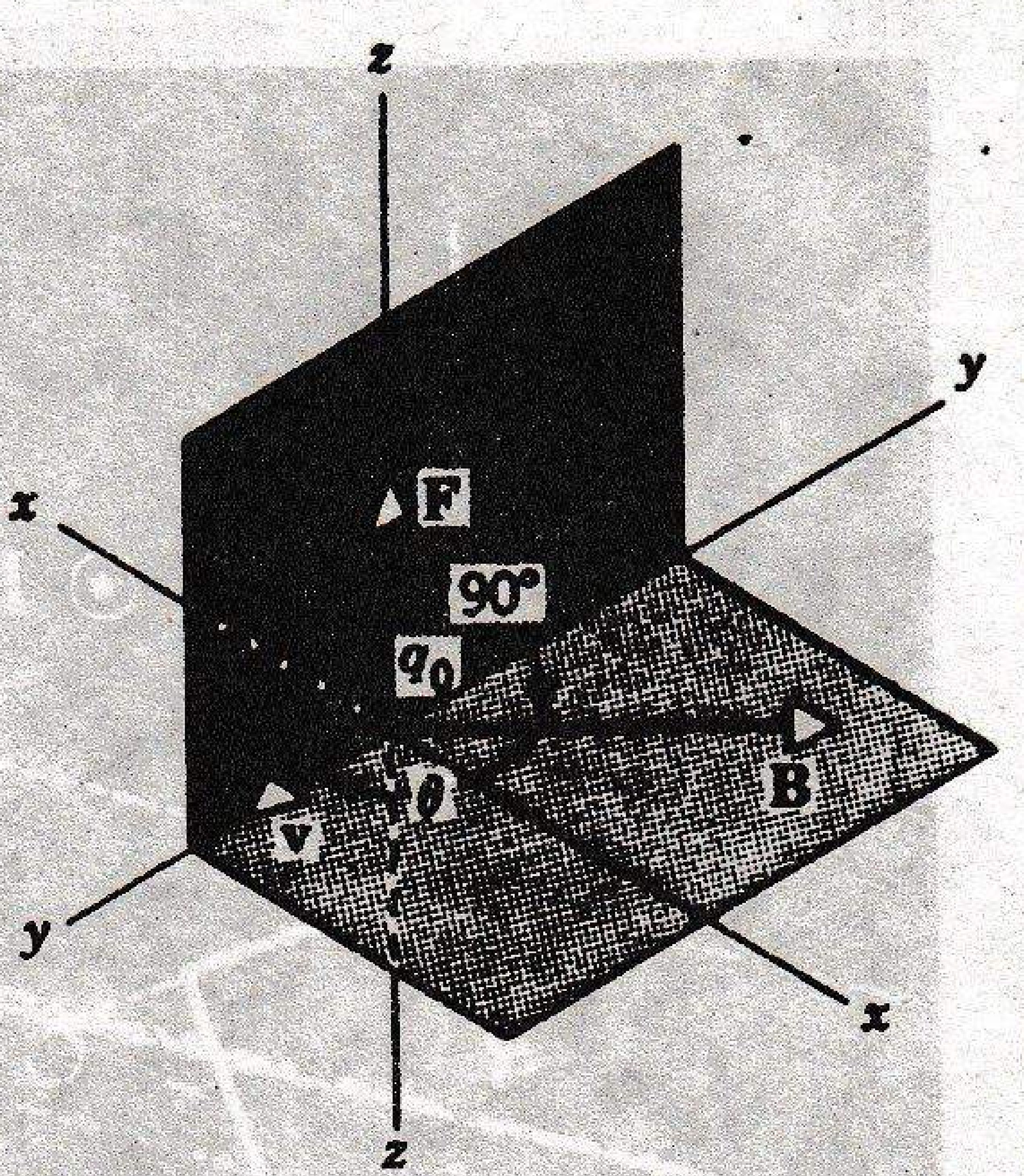

Fig. 33-3 Illustrating $\mathbf{F} = q_0\mathbf{v} \times \mathbf{B}$ (Eq. 33-3*a*).

This definition of **B** is similar in spirit, although more complex, than the definition of the electric field strength **E**, which we can cast into this form: *If a positive test charge q_0 is placed at point P and if an (electric) force* **F** *acts on the stationary charge, an electric field* **E** *is present at P, where* **E** *is the vector satisfying the relation*

$$\mathbf{F} = q_0\mathbf{E},$$

q_0 and **F** being measured quantities. In defining **E**, the only characteristic direction to appear is that of the electric force $\mathbf{F}_E$ which acts on the positive test body; the direction of **E** is taken to be that of $\mathbf{F}_E$. In defining **B**, *two* characteristic directions appear, those of **v** and of the magnetic force $\mathbf{F}_B$; they prove always to be at right angles.

In Fig. 33-4 a positive and a negative electron are created at point *P* in a bubble chamber. A magnetic field is perpendicular to the chamber, pointing out of the plane of the figure (symbol ⊙).* The relation $\mathbf{F} = q_0\mathbf{v} \times \mathbf{B}$ (Eq. 33-3*a*) shows that the deflecting forces acting on the two particles are as indicated in the figure. These deflecting forces would make the tracks deflect as shown.

The unit of **B** that follows from Eq. 33-3 is the (nt/coul)/(meter/sec). This is given the special name weber/meter², or tesla. Recalling that a coul/sec is an ampere,

$$1 \text{ weber/meter}^2 = \frac{1 \text{ nt}}{\text{coul (meter/sec)}} = \frac{1 \text{ nt}}{\text{amp-m}}$$

An earlier unit for **B**, still in common use, is the *gauss;* the relationship is

$$1 \text{ weber/meter}^2 = 10^4 \text{ gauss}.$$

The *weber* is used to measure Φ_B, the flux of **B**; see Eq. 33-1.

The fact that the magnetic force is always at right angles to the direction of motion means that (for steady magnetic fields) the work done by this

* The symbol ⊗ indicates a vector into the page, the × being thought of as the tail of an arrow; the symbol ⊙ indicates a vector out of the page, the dot being thought of as the tip of an arrow.

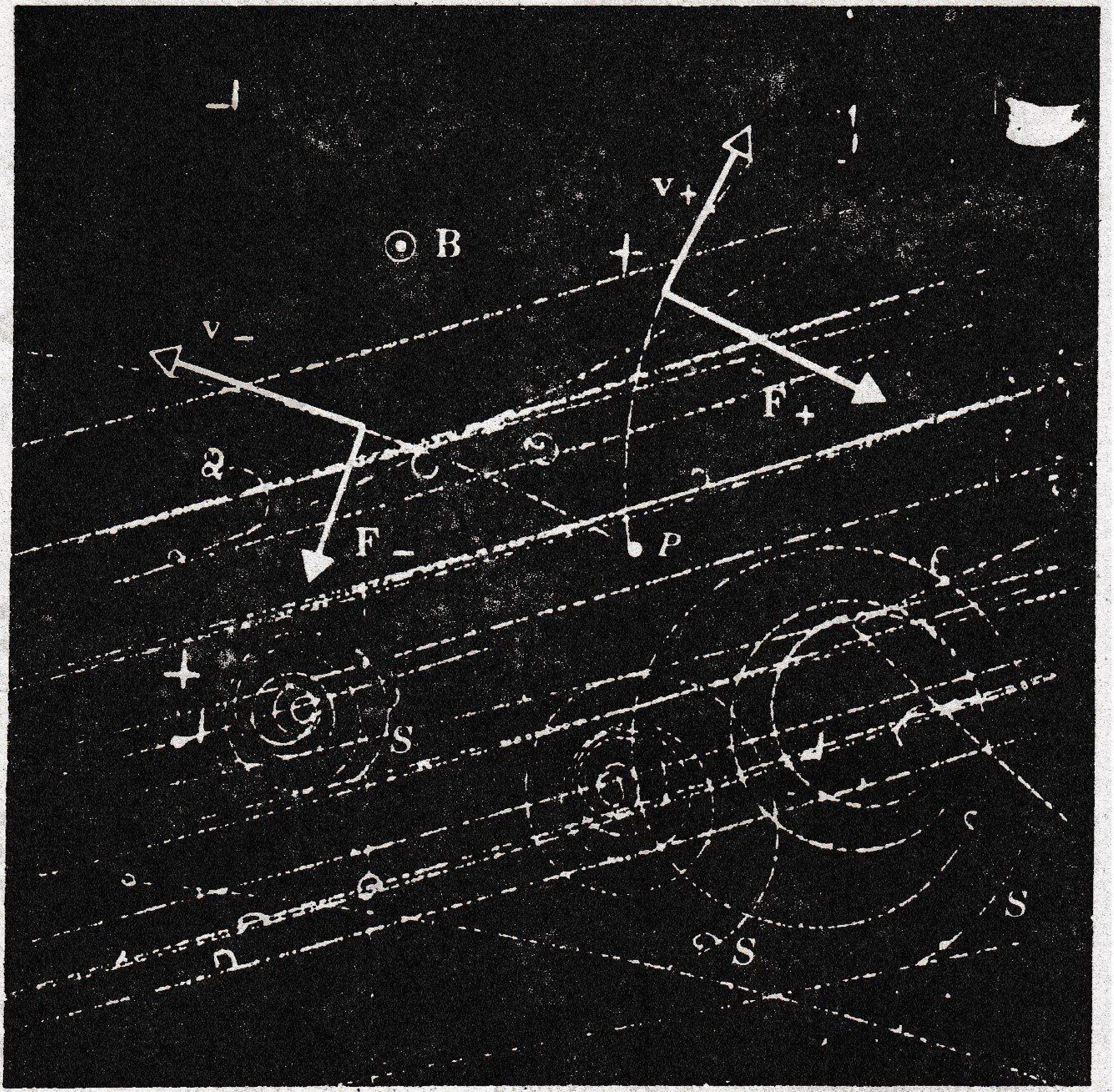

Fig. 33–4 A *bubble chamber* is a device for rendering visible, by means of small bubbles, the tracks of charged particles that pass through the chamber. The figure is a photograph taken with such a chamber immersed in a field of magnetic induction B and exposed to radiations from a large cyclotron-like accelerator. The curved V at point P is formed by a positive and a negative electron, which deflect in opposite directions in the magnetic field. The spirals S are the tracks of three low-energy electrons. (Courtesy E. O. Lawrence Radiation Laboratory, University of California.)

force on the particle is zero. For an element of the path of the particle of length $d\mathbf{l}$, this work dW is $\mathbf{F}_B \cdot d\mathbf{l}$; dW is zero because $\mathbf{F}_B$ and $d\mathbf{l}$ are always at right angles. Thus a static magnetic field cannot change the kinetic energy of a moving charge; it can only deflect it sideways.

If a charged particle moves through a region in which both an electric field and a magnetic field are present, the resultant force is found by combining Eqs. 27–2 and 33–3*a*, or

$$\mathbf{F} = q_0\mathbf{E} + q_0\mathbf{v} \times \mathbf{B}. \qquad (33\text{–}4)$$

This is sometimes called the *Lorentz relation* in tribute to H. A. Lorentz who did so much to develop and clarify the concepts of the electric and magnetic fields.

▶ **Example 1.** A uniform field of magnetic induction **B** points horizontally from south to north; its magnitude is 1.5 webers/meter². If a 5.0-Mev proton moves vertically downward through this field, what force will act on it?

The kinetic energy of the proton is

$$K = (5.0 \times 10^6 \text{ ev})(1.6 \times 10^{-19} \text{ joule/ev}) = 8.0 \times 10^{-13} \text{ joule.}$$

Its speed can be found from the relation $K = \frac{1}{2}mv^2$, or

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{(2)(8.0 \times 10^{-13} \text{ joule})}{1.7 \times 10^{-27} \text{ kg}}} = 3.1 \times 10^7 \text{ meters/sec.}$$

Equation 33-3*b* gives

$$F = qvB \sin\theta = (1.6 \times 10^{-19} \text{ coul})(3.1 \times 10^7 \text{ meters/sec})(1.5 \text{ webers/meter}^2)(\sin 90°)$$
$$= 7.4 \times 10^{-12} \text{ nt.}$$

The student can show that this force is about 4×10^{14} times greater than the weight of the proton.

The relation $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ shows that the *direction* of the deflecting force is to the east. If the particle had been negatively charged, the deflection would have been to the west. This is predicted automatically by Eq. 33-3*a* if we substitute $-e$ for q_0. ◀

33-3 Magnetic Force on a Current

A current is an assembly of moving charges. Because a magnetic field exerts a sideways force on a moving charge, we expect that it will also exert a sideways force on a wire carrying a current. Figure 33-5 shows a length l of wire carrying a current i and placed in a field of magnetic induction **B**. For simplicity we have oriented the wire so that the current density vector **j** is at right angles to **B**.

The current i in a metal wire is carried by the free (or conduction) electrons, n being the number of such electrons per unit volume of the wire. The magnitude of the average force on one such electron is given by Eq. 33-3*b*, or, since $\theta = 90°$,

$$F' = q_0 vB \sin\theta = ev_d B$$

where v_d is the drift speed. From the relation $v_d = j/ne$ (Eq. 31-5),

$$F' = e\left(\frac{j}{ne}\right)B = \frac{jB}{n}.$$

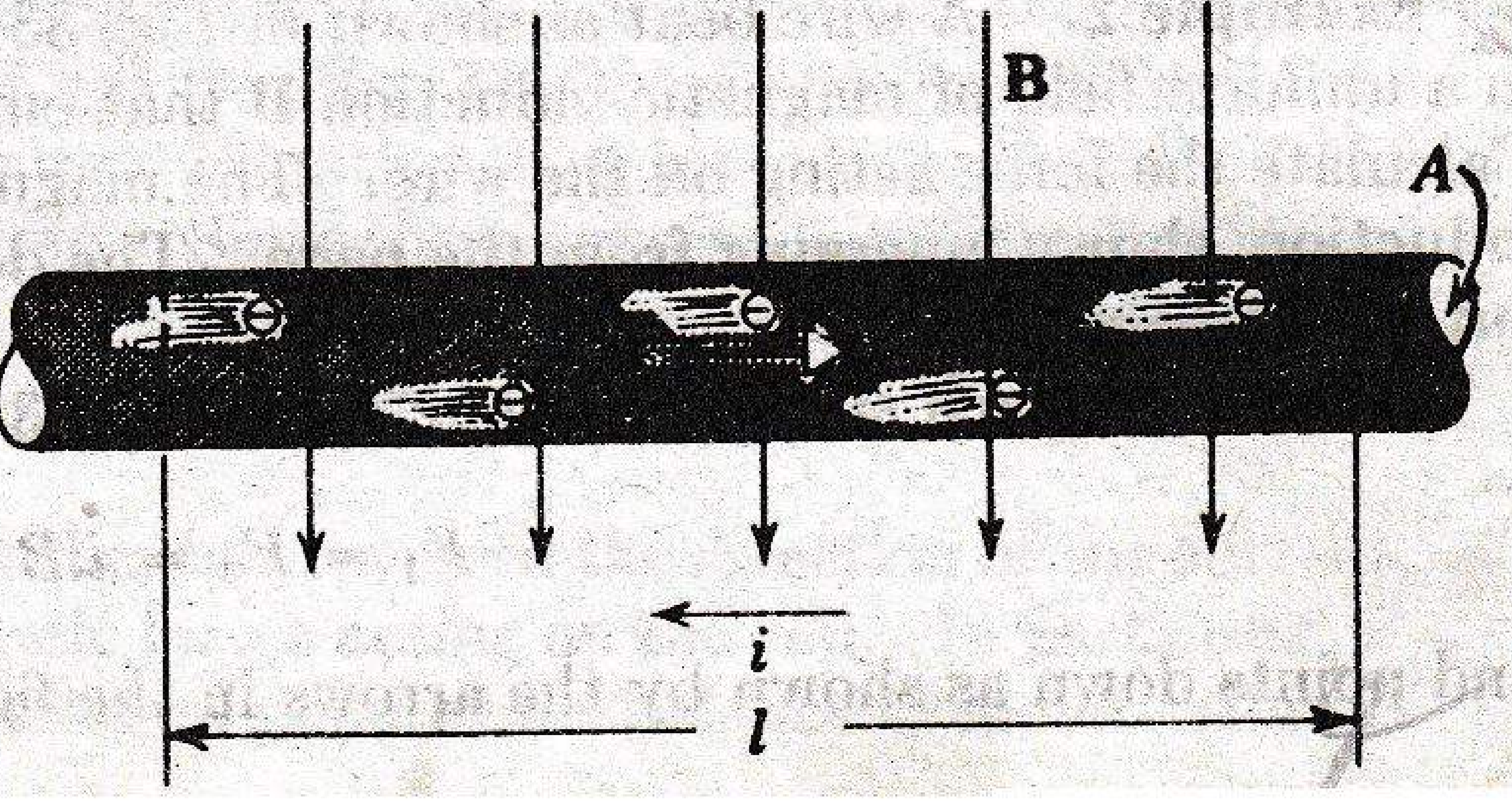

Fig. 33-5 A wire carrying a current i is placed at right angles to a field of magnetic induction **B**.

The length l of the wire contains nAl free electrons, Al being the volume of the section of wire of cross section A that we are considering. The total force on the free electrons in the wire, and thus on the wire itself, is

$$F = (nAl)F' = nAl\frac{jB}{n}.$$

Since jA is the current i in the wire, we have

$$F = ilB. \tag{33-5}$$

The negative charges, which move to the right in the wire of Fig. 33-5, are equivalent to positive charges moving to the left, that is, in the direction of the current arrow. For such a positive charge the velocity $\mathbf{v}$ would point to the left and the force on the wire, given by Eq. 33-3a ($\mathbf{F} = q_0\mathbf{v} \times \mathbf{B}$) points up, out of the page. This same conclusion follows if we consider the actual negative charge carriers for which $\mathbf{v}$ points to the right but q_0 has a negative sign. Thus by measuring the sideways magnetic force on a wire carrying a current and placed in a magnetic field we cannot tell whether the current carriers are negative charges moving in a given direction or positive charges moving in the opposite direction.

Equation 33-5 holds only if the wire is at right angles to $\mathbf{B}$. We can express the more general situation in vector form as

$$\mathbf{F} = i\mathbf{l} \times \mathbf{B}, \tag{33-6a}$$

where $\mathbf{l}$ is a (displacement) vector that points along the (straight) wire in the direction of the current. Equation 33-6a is equivalent to the relation $\mathbf{F} = q_0\mathbf{v} \times \mathbf{B}$ (Eq. 33-3a); either can be taken as a defining equation for $\mathbf{B}$. The student should note that the vector $\mathbf{l}$ in Fig. 33-5 points to the left and that the magnetic force $\mathbf{F}$ ($= i\mathbf{l} \times \mathbf{B}$) points up, out of the page. This agrees with the conclusion obtained by analyzing the forces that act on the individual charge carriers.

If we consider a differential element of a conductor of length dl, the force $d\mathbf{F}$ acting on it can be found, by analogy with Eq. 33-6a, from

$$d\mathbf{F} = i\,d\mathbf{l} \times \mathbf{B}. \tag{33-6b}$$

By integrating this formula in an appropriate way we can find the force $\mathbf{F}$ on a nonlinear conductor.

▶ **Example 2.** A wire bent as shown in Fig. 33-6 carries a current i and is placed in a uniform field of magnetic induction $\mathbf{B}$ that emerges from the plane of the figure. Calculate the force acting on the wire. The magnetic field is represented by lines of induction, shown emerging from the page. The dots show that the sense of $\mathbf{B}$ is up, out of the page.

The force on each straight section, from Eq. 33-6a, has the magnitude

$$F_1 = F_3 = ilB$$

and points down as shown by the arrows in the figure.

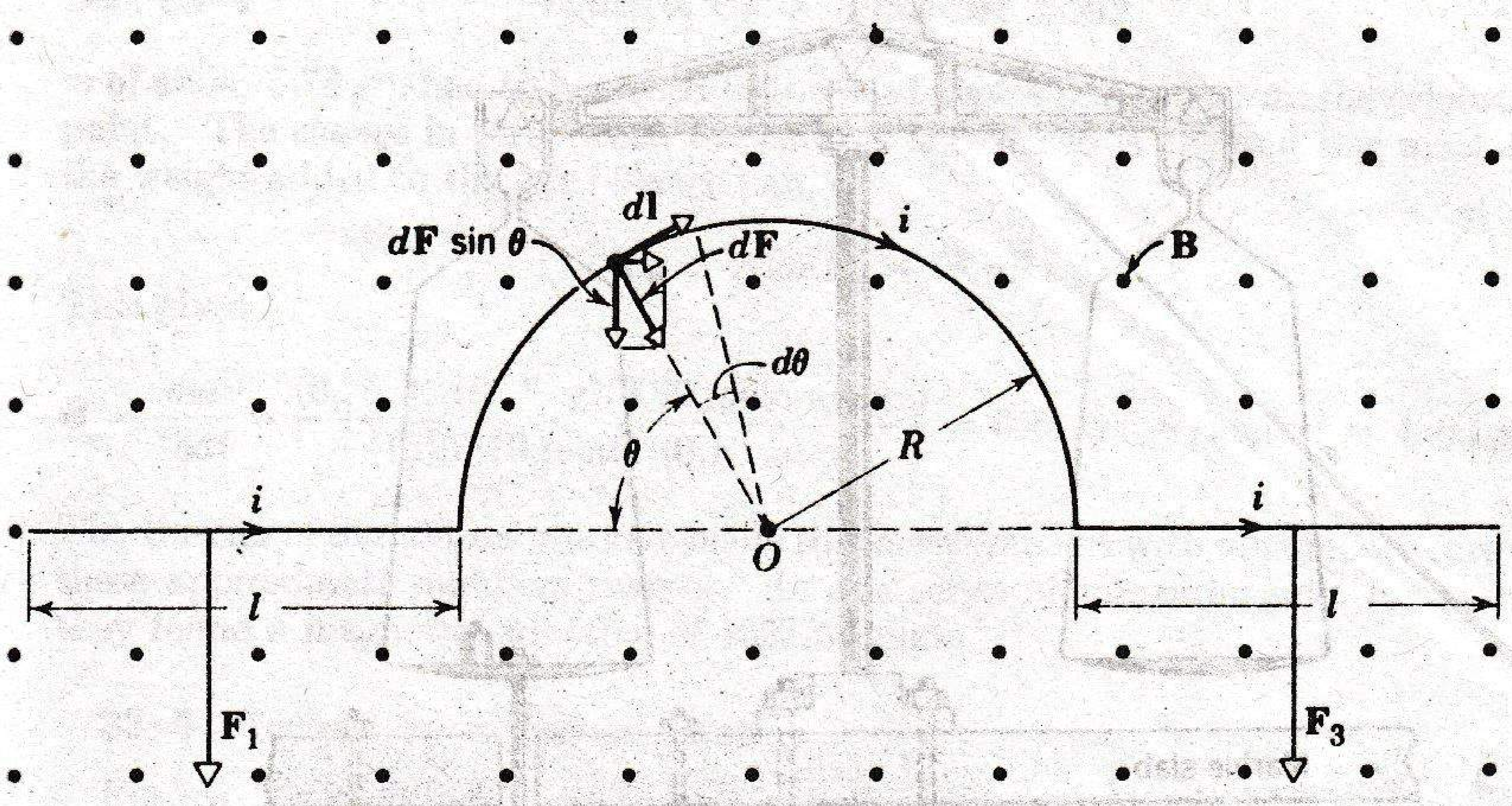

Fig. 33-6 Example 2.

A segment of wire of length dl on the arc has a force dF on it whose magnitude is

$$dF = iB\,dl = iB(R\,d\theta)$$

and whose direction is radially toward O, the center of the arc. Only the downward component of this force is effective, the horizontal component being canceled by an oppositely directed component associated with the corresponding arc segment on the other side of O. Thus the total force on the semicircle of wire about O points down and is

$$F_2 = \int_0^\pi dF \sin\theta = \int_0^\pi (iBR\,d\theta)\sin\theta = iBR\int_0^\pi \sin\theta\,d\theta = 2iBR.$$

The resultant force on the whole wire is

$$F = F_1 + F_2 + F_3 = 2ilB + 2iBR = 2iB(l + R).$$

Notice that this force is the same as that acting on a straight wire of length $2l + 2R$. ◀

Figure 33-7 shows the arrangement used by Thomas, Driscoll, and Hipple at the National Bureau of Standards to measure the magnetic induction provided by a laboratory magnet such as that of Fig. 33-2. The rectangle is a coil of nine turns whose width a and length b are about 10 cm and 70 cm, respectively. The lower end of the coil is placed in the field of magnetic induction **B** and the upper end is hung from the arm of a balance; **B** enters the plane of the figure at right angles.

An accurately known current i of about 0.10 amp is set up in the coil in the direction shown, and weights are placed in the right-hand balance pan until the system is balanced. The magnetic force **F** ($= i\mathbf{l} \times \mathbf{B}$; see Eq. 33-6$a$) on the bottom leg of the coil points upward, as shown in the figure. Equation 33-5 also shows that the force *on each wire* at the bottom of the coil is iaB. Since there are nine wires, the total force on the bottom leg of the coil is $9iaB$. The forces on the vertical sides of the coil ($= i\mathbf{l} \times \mathbf{B}$) are sideways; because they are equal and opposite, they cancel and produce no effect.

After balancing the system, the experimenters reversed the direction of the current, which changed the sign of all the magnetic forces acting on the coil. In particular, **F** then pointed downward, which caused the rest point of the balance to move. A mass

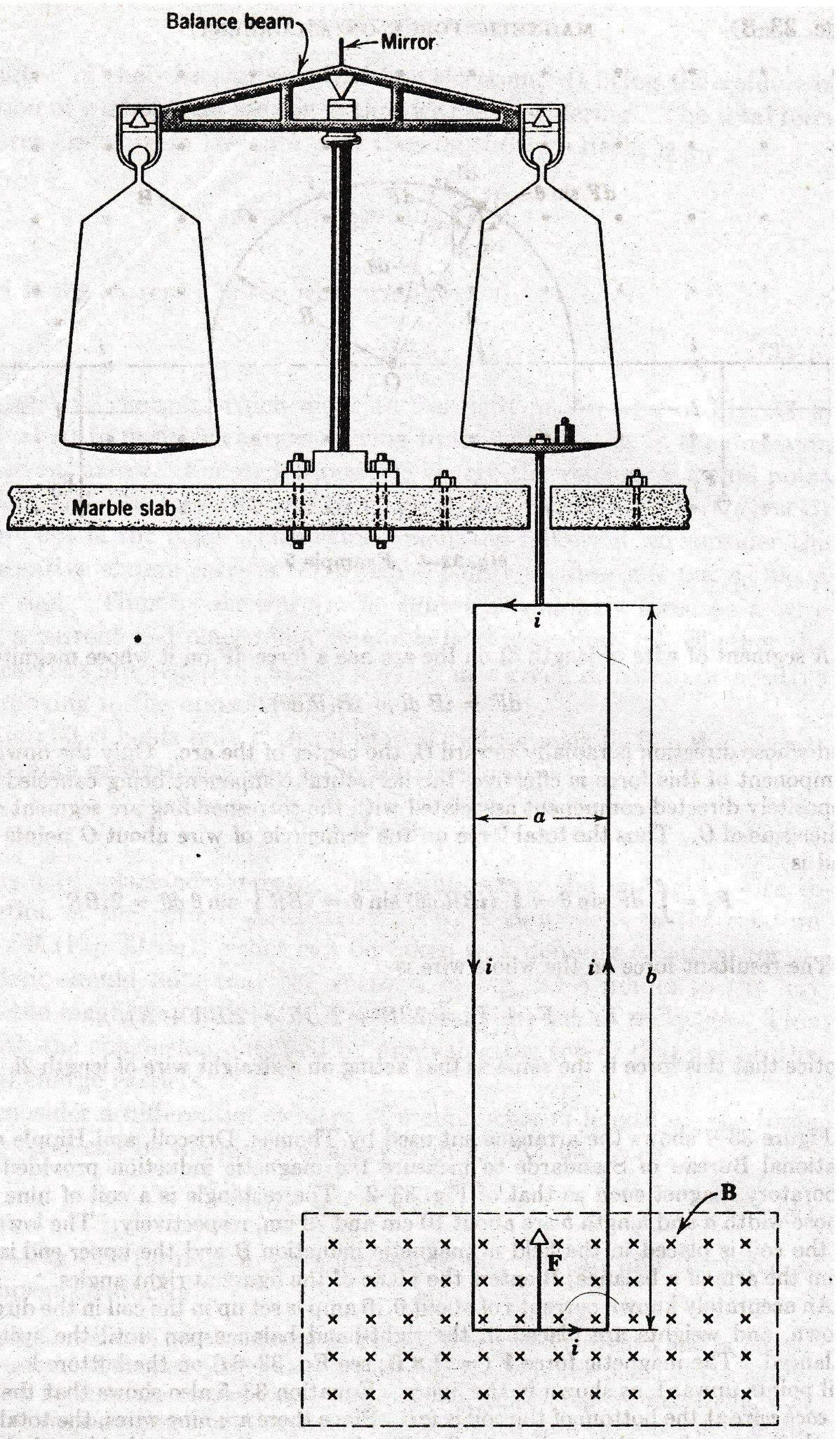

Fig. 33-7 Apparatus used to measure **B**. The zero point of the balance is observed by means of a light beam reflected from the mirror attached to the balance beam.

m of about 8.78 gm had to be added to the left balance pan to restore the original rest point. The *change* in force when the current is reversed is $2F$, and this must equal the weight added to the left balance pan, or

$$mg = 2(9iaB) = 18iaB.$$

This gives

$$B = \frac{mg}{18ai} = \frac{(8.78 \times 10^{-3}\ \text{kg})(9.80\ \text{meters/sec}^2)}{(18)(0.10\ \text{meter})(0.10\ \text{amp})} = 0.48\ \text{weber/meter}^2 = 4800\ \text{gauss}.$$

The Bureau of Standards' workers made this measurement with much more care than these approximate numbers suggest. In one series of measurements, for example, they found a magnetic induction of 4697.55 gauss.

33-4 Torque on a Current Loop

Figure 33-8 shows a rectangular loop of wire of length a and width b placed in a uniform field of induction $\mathbf{B}$, with sides 1 and 3 always normal to the field direction. The normal nn' to the plane of the loop makes an angle θ with the direction of $\mathbf{B}$.

Assume the current to be as shown in the figure. Wires must be provided to lead the current into the loop and out of it. If these wires are twisted tightly together, there will be no net magnetic force on the twisted pair because the currents in the two wires are in opposite directions. Thus the lead wires may be ignored. Also, some way of supporting the loop must be provided. Let us imagine it to be suspended from a long string attached to the loop at its center of mass. In this way the loop will be free to turn, through a small angle at least, about any axis through the center of mass.

The net force on the loop is the resultant of the forces on the four sides of the loop. On side 2 the vector $\mathbf{l}$ points in the direction of the current and has the magnitude b. The angle between $\mathbf{l}$ and $\mathbf{B}$ for side 2 (see Fig. 33-8b) is

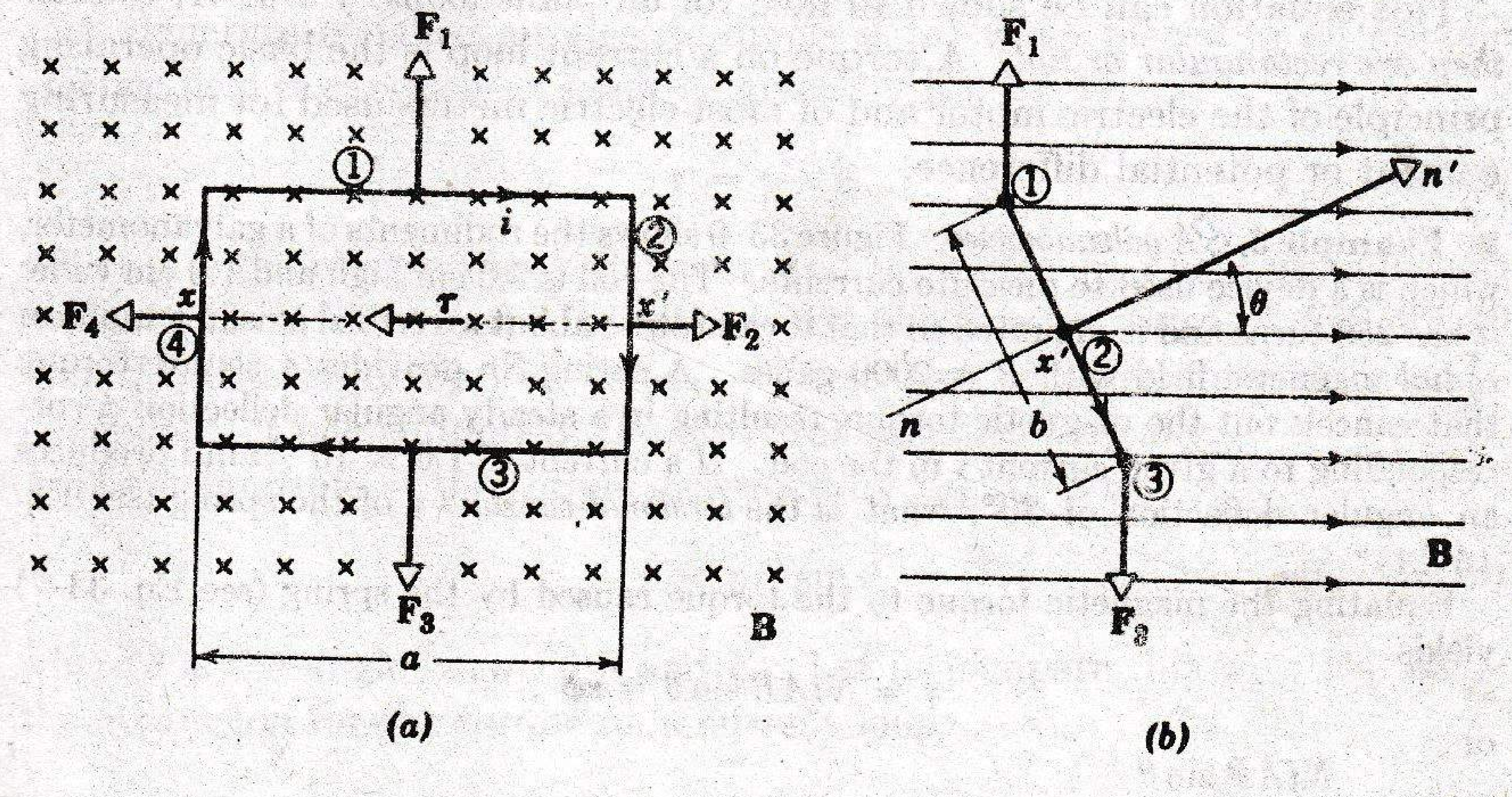

Fig. 33-8 A rectangular coil carrying a current i is placed in a uniform external magnetic field.

$90° - \theta$. Thus the magnitude of the force on this side is

$$F_2 = ibB \sin (90° - \theta) = ibB \cos \theta.$$

From the relation $\mathbf{F} = i\mathbf{l} \times \mathbf{B}$ (Eq. 33-6*a*), we find the direction of $\mathbf{F}_2$ to be out of the plane of Fig. 33-8*b*. The student can show that the force $\mathbf{F}_4$ on side 4 has the same magnitude as $\mathbf{F}_2$ but points in the opposite direction. Thus $\mathbf{F}_2$ and $\mathbf{F}_4$, taken together, have no effect on the motion of the loop. The net force they provide is zero, and, since they have the same line of action, the net torque due to these forces is also zero.

The common magnitude of $\mathbf{F}_1$ and $\mathbf{F}_3$ is iaB. These forces, too, are oppositely directed so that they do not tend to move the coil bodily. As Fig. 33-8*b* shows, however, they do *not* have the same line of action if the coil is in the position shown; there is a net torque, which tends to rotate the coil clockwise about the line xx'. The coil can be supported on a rigid axis that lies along xx', with no loss of its freedom of motion. This torque can be represented in Fig. 33-8*b* by a vector pointing into the figure at point x' or in Fig. 33-8*a* by a vector pointing along the xx' axis from right to left.

The magnitude of the torque $\boldsymbol{\tau}'$ is found by calculating the torque caused by $\mathbf{F}_1$ about axis xx' and doubling it, for $\mathbf{F}_3$ exerts the same torque about this axis that $\mathbf{F}_1$ does. Thus

$$\tau' = 2(iaB)\left(\frac{b}{2}\right)(\sin \theta) = iabB \sin \theta.$$

This torque acts on every turn of the coil. If there are N turns, the torque on the entire coil is

$$\tau = N\tau' = NiabB \sin \theta = NiAB \sin \theta, \qquad (33\text{-}7)$$

in which A, the area of the coil, is substituted for ab.

This equation can be shown to hold for *all plane loops of area A, whether they are rectangular or not.* A torque on a current loop is the basic operating principle of the electric motor and of most electric meters used for measuring current or potential difference.

▶ **Example 3.** *A galvanometer.* Figure 33-9 shows the rudiments of a galvanometer, which is a device used to measure currents. The coil is 2.0 cm high and 1.0 cm wide; it has 250 turns and is mounted so that it can rotate about a vertical axis in a uniform *radial* magnetic field with $B = 2000$ gauss. A spring Sp provides a countertorque that cancels out the magnetic torque, resulting in a steady angular deflection ϕ corresponding to a given current i in the coil. If a current of 1.0×10^{-4} amp produces an angular deflection of 30°, what is the *torsional constant* κ of the spring (see Eq. 15-21)?

Equating the magnetic torque to the torque caused by the spring (see Eq. 33-7) yields

$$\tau = NiAB \sin \theta = \kappa\phi$$

or

$$\kappa = \frac{NiAB \sin \theta}{\phi}$$

$$= \frac{(250)(1.0 \times 10^{-4}\text{ amp})(2.0 \times 10^{-4}\text{ meter}^2)(0.20\text{ weber/meter}^2)(\sin 90°)}{30°}$$

$$= 3.3 \times 10^{-8}\text{ nt-m/deg.}$$

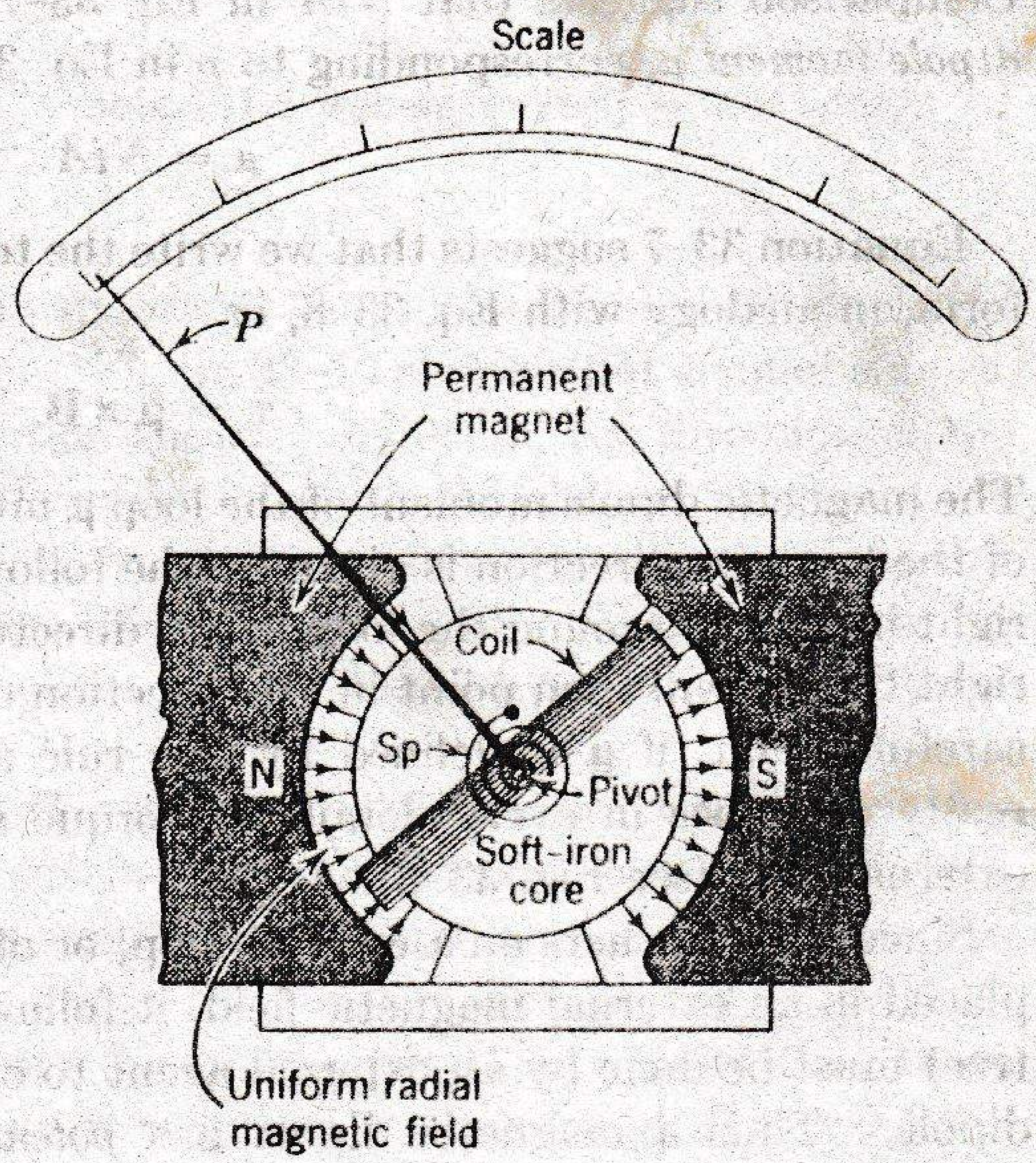

Fig. 33-9 Example 3. The elements of a galvanometer, showing the coil, the helical spring Sp, and pointer P.

Note that the normal to the plane of the coil (that is, the pointer P) is always at right angles to the (radial) magnetic field so that $\theta = 90°$. ◀

A current loop orienting itself in an external magnetic field reminds us of the action of a compass needle in such a field. One face of the loop behaves like the north pole of the needle; * the other face behaves like the south pole. Compass needles, bar magnets, and current loops can all be regarded as *magnetic dipoles*. We show this here for the current loop, reasoning entirely by analogy with *electric* dipoles.

A structure is called an electric dipole if (*a*) when placed in an *external* electric field it experiences a torque given by Eq. 27-11,

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}, \tag{33-8}$$

where $\mathbf{p}$ is the electric dipole moment, and (*b*) it sets up a field of its own at distant points, described qualitatively by the lines of force of Fig. 29-10 and quantitatively by Eq. 29-11. These two requirements are not independent; if one is fulfilled, the other follows automatically.

The magnitude of the torque described by Eq. 33-8 is

$$\tau = pE \sin\theta, \tag{33-9}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{E}$. Let us compare this with Eq. 33-7, the expression for the torque on a current loop:

$$\tau = (NiA)B \sin\theta. \tag{33-7}$$

* The north pole of a compass needle is the end that points toward the geographic north.

In each case the appropriate field (E or B) appears, as does a term $\sin\theta$. Comparison suggests that NiA in Eq. 33–7 can be taken as the *magnetic dipole moment* μ, corresponding to p in Eq. 33–9, or

$$\mu = NiA. \tag{33–10}$$

Equation 33–7 suggests that we write the torque on a current loop in vector form, in analogy with Eq. 33–8, or

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}. \tag{33–11}$$

The magnetic dipole moment of the loop $\boldsymbol{\mu}$ must be taken to lie along the axis of the loop; its direction is given by the following rule: Let the fingers of the right hand curl around the loop in the direction of the current; the extended right thumb will then point in the direction of $\boldsymbol{\mu}$. The student should check carefully that, if $\boldsymbol{\mu}$ is defined by this rule and Eq. 33–10, Eq. 33–11 correctly describes in every detail the torque acting on a current loop in an external field (see Fig. 33–8).

Since a torque acts on a current loop, or other magnetic dipole, when it is placed in an external magnetic field, it follows that work (positive or negative) must be done by an external agent to change the orientation of such a dipole. Thus a magnetic dipole has *potential energy* associated with its orientation in an external magnetic field. This energy may be taken to be zero for any arbitrary position of the dipole. By analogy with the assumption made for electric dipoles in Section 27–6, we assume that the magnetic energy U is zero when $\boldsymbol{\mu}$ and $\mathbf{B}$ are at right angles, that is, when $\theta = 90°$. This choice of a zero-energy configuration for U is arbitrary because we are interested only in the *changes* in energy that occur when the dipole is rotated.

The magnetic potential energy in any position θ is defined as the work that an external agent must do to turn the dipole from its zero-energy position ($\theta = 90°$) to the given position θ. Thus

$$U = \int_{90°}^{\theta} \tau\, d\theta = \int_{90°}^{\theta} NiAB \sin\theta\, d\theta = \mu B \int_{90°}^{\theta} \sin\theta\, d\theta = -\mu B \cos\theta,$$

in which Eq. 33–7 is used to substitute for τ. In vector symbolism this relation can be written as

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}, \tag{33–12}$$

which is in perfect correspondence with Eq. 27–13, the expression for the energy of an *electric* dipole in an external *electric* field,

$$U = -\mathbf{p} \cdot \mathbf{E}.$$

▶ **Example 4.** A circular coil of N turns has an effective radius a and carries a current i. How much work is required to turn it in an external magnetic field $\mathbf{B}$ from a position in which θ equals zero to one in which θ equals 180°? Assume that $N = 100$, $a = 5.0$ cm, $i = 0.10$ amp, and $B = 1.5$ webers/meter^2.

The work required is the difference in energy between the two positions, or, from Eq. 33–12,

$$W = U_{\theta=180°} - U_{\theta=0} = (-\mu B \cos 180°) - (-\mu B \cos 0) = 2\mu B.$$

But $\mu = NiA$, so that

$$W = 2NiAB = 2Ni(\pi a^2)B$$

$$= (2)(100)(0.10 \text{ amp})(\pi)(5 \times 10^{-2} \text{ meter})^2(1.5 \text{ webers/meter}^2) = 0.24 \text{ joule.}$$ ◀

33-5 The Hall Effect

In 1879 E. H. Hall devised an experiment that gives the sign of the charge carriers in a conductor; see p. 771. Figure 33-10 shows a flat strip of copper, carrying a current i in the direction shown. As usual, the direction of the current arrow, labeled i, is the direction in which the charge carriers would move *if* they were positive. The current arrow can represent either positive charges moving down (as in Fig. 33-10*a*) or negative charges moving up (as in Fig. 33-10*b*). The Hall effect can be used to decide between these two possibilities.

A field of magnetic induction **B** is set up at right angles to the strip by placing the strip between the polefaces of an electromagnet. This field exerts a deflecting force **F** on the strip (given by $i\mathbf{l} \times \mathbf{B}$), which points to the right in the figure. Since the sideways force on the strip is due to the sideways forces on the charge carriers (given by $q\mathbf{v} \times \mathbf{B}$), it follows that these carriers, whether they are positive or negative, will tend to drift toward the right in Fig. 33-10 as they drift along the strip, producing a *transverse Hall potential difference* V_{xy} between points such as x and y. The sign of the charge carriers is determined by the sign of this Hall potential difference. If the carriers are positive, y will be at a higher potential than x; if they are negative, y will be at a lower potential than x. Experiment shows that in metals the charge carriers are negative.

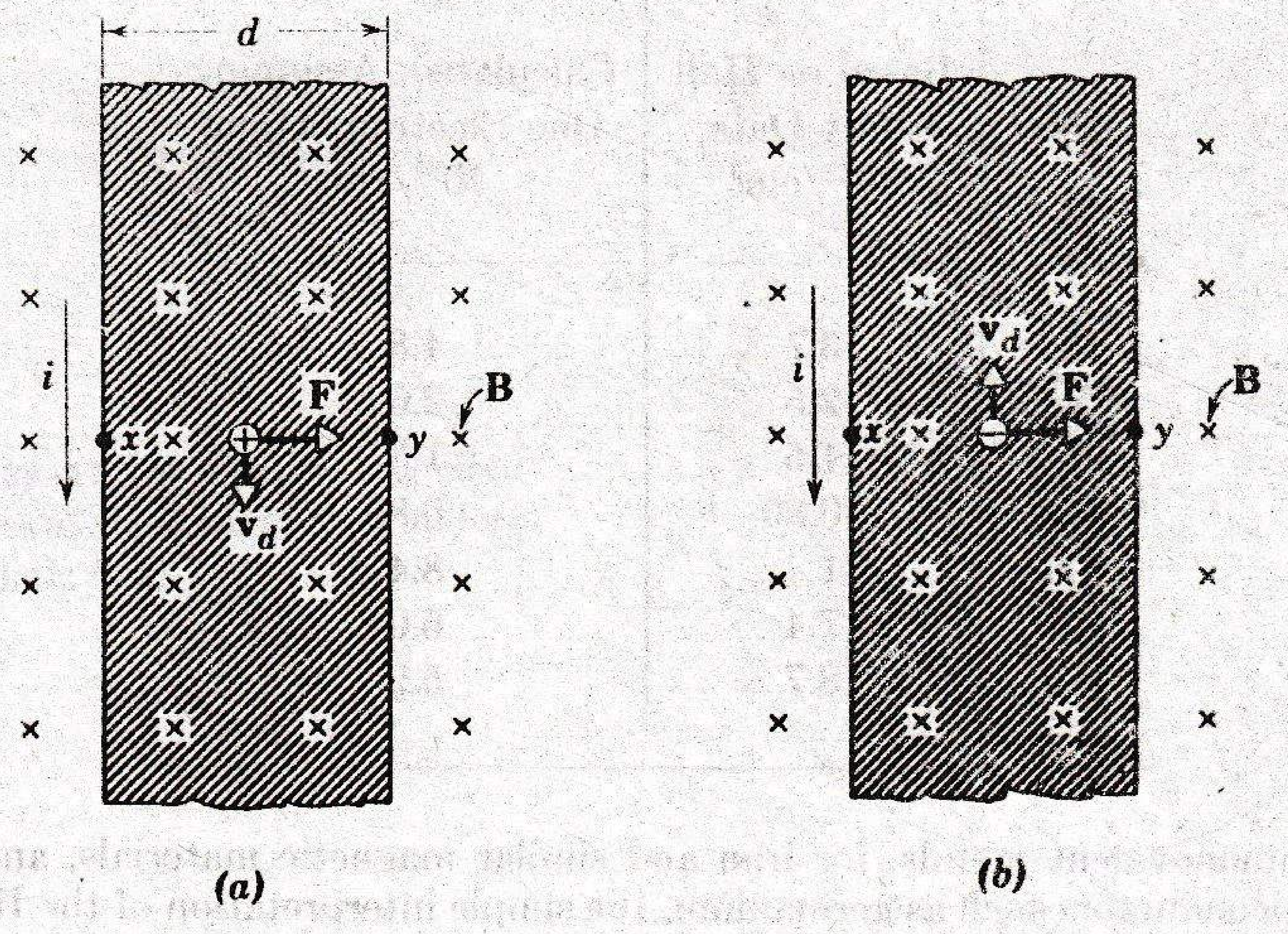

Fig. 33-10 A current i is set up in a copper strip placed in a field of magnetic induction **B**, assuming (*a*) positive carriers and (*b*) negative carriers.

To analyze the Hall effect quantitatively, let us use the *free-electron* model of a metal, the same model used in Section 31–4 to explain resistivity. The charge carriers can be assumed to move along the conductor with a certain constant drift speed v_d. The magnetic deflecting force that causes the moving charge carriers to drift toward the right edge of the strip is given by $q\mathbf{v}_d \times \mathbf{B}$ (see Eq. 33–3a).

The charge carriers do not build up without limit on the right edge of the strip because the displacement of charge gives rise to a transverse *Hall electric field* $\mathbf{E}_H$, which acts, inside the conductor, to oppose the sideways drift of the carriers. This Hall electric field is another manifestation of the Hall potential difference and is related to it by

$$E_H = V_{xy}/d.$$

Eventually an equilibrium is reached in which the sideways magnetic deflecting force on the charge carriers is just canceled by the oppositely directed electric force $q\mathbf{E}_H$ caused by the Hall electric field, or

$$q\mathbf{E}_H + q\mathbf{v}_d \times \mathbf{B} = 0,$$

which can be written

$$\mathbf{E}_H = -\mathbf{v}_d \times \mathbf{B}. \tag{33-13}$$

This equation shows explicitly that if $\mathbf{E}_H$ and $\mathbf{B}$ are measured $\mathbf{v}_d$ can be determined both in magnitude and direction; given the direction of $\mathbf{v}_d$, the sign of the charge carriers follows at once, as Fig. 33–10 shows.

The number of charge carriers per unit volume (n) can also be found from Hall effect measurements. If we write Eq. 33–13 in terms of magnitudes, for the case in which $\mathbf{v}_d$ and $\mathbf{B}$ are at right angles, we obtain $E_H = v_d B$. Combining this with Eq. 31–5 ($v_d = j/ne$) leads to

$$E_H = \frac{j}{ne} B \quad \text{or} \quad n = \frac{jB}{eE_H}. \tag{33-14}$$

The agreement between experiment and Eq. 33–14 is rather good for monovalent metals, as Table 33–1 shows.

Table 33–1

NUMBER OF CONDUCTION ELECTRONS PER UNIT VOLUME

Metal	Based on Hall Effect Data, 10^{22}/cm^3	Calculated, Assuming One Electron/Atom, 10^{22}/cm^3
Li	3.7	4.8
Na	2.5	2.6
K	1.5	1.3
Cs	0.80	0.85
Cu	11	8.4
Ag	7.4	6.0
Au	8.7	5.9

For nonmonovalent metals, for iron and similar magnetic materials, and for so-called semiconductors such as germanium, the simple interpretation of the Hall effect in terms of the free-electron model is not valid. A theoretical interpretation of the

Hall effect based on modern quantum physics gives a reasonable agreement with experiment in all cases.

▶ **Example 5.** A copper strip 2.0 cm wide and 1.0 mm thick is placed in a magnetic field with $B = 1.5$ webers/meter^2, as in Fig. 33–10. If a current of 200 amp is set up in the strip, what Hall potential difference appears across the strip?

From Eq. 33–14,

$$E_H = \frac{jB}{ne},$$

but

$$E_H = \frac{V_{xy}}{d} \quad \text{and} \quad j = \frac{i}{A} = \frac{i}{dh},$$

where h is the thickness of the strip. Combining these equations gives

$$V_{xy} = \frac{iB}{neh} = \frac{(200\ \text{amp})(1.5\ \text{webers/meter}^2)}{(8.4 \times 10^{28}/\text{meter}^3)(1.6 \times 10^{-19}\ \text{coul})(1.0 \times 10^{-3}\ \text{meter})}$$

$$= 2.2 \times 10^{-5}\ \text{volt} = 22\mu\text{v}.$$

These potential differences are not large. See p. 681 for the calculation of n. ◀

33–6 Circulating Charges

Figure 33–11 shows a negatively charged particle introduced with velocity **v** into a uniform field of magnetic induction **B**. We assume that **v** is at right angles to **B** and thus lies entirely in the plane of the figure. The relation $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ (Eq. 33–3a) shows that the particle will experience a sideways deflecting force of magnitude qvB. This force will lie in the plane of the figure, which means that the particle cannot leave this plane.

This reminds us of a stone held by a rope and whirled in a horizontal circle on a smooth surface. Here, too, a force of constant magnitude, the tension in the rope, acts in a plane and at right angles to the velocity. The charged particle, like the stone, also moves with constant speed in a circular

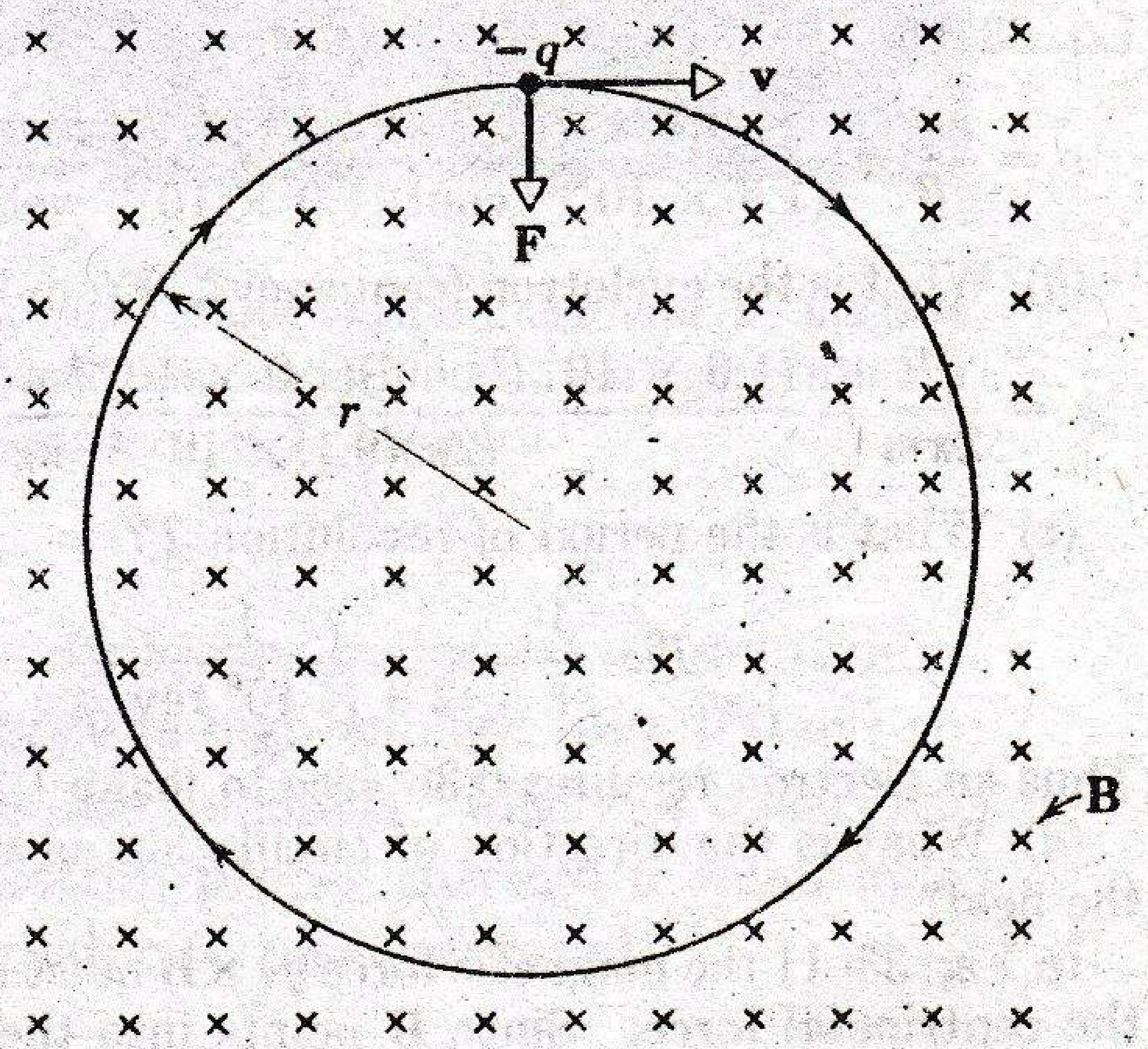

Fig. 33–11 A charge $-q$ circulates at right angles to a uniform magnetic field.

path. From Newton's second law we have

$$qvB = \frac{mv^2}{r} \quad \text{or} \quad r = \frac{mv}{qB}, \tag{33-15}$$

which gives the radius of the path. The three spirals in Fig. 33–4 show relatively low-energy electrons in a bubble chamber. The paths are not circles because the electrons lose energy by collisions in the chamber as they move.

The angular velocity ω is given by v/r or, from Eq. 33–15,

$$\omega = \frac{v}{r} = \frac{qB}{m}.$$

The frequency ν measured, say, in rev/sec, is given by

$$\nu = \frac{\omega}{2\pi} = \frac{qB}{2\pi m}. \tag{33-16}$$

Note that ν *does not depend on the speed of the particle.* Fast particles move in large circles (Eq. 33–15) and slow ones in small circles, but all require the same time T (the *period*) to complete one revolution in the field.

The frequency ν is a characteristic frequency for the charged particle in the field and may be compared to the characteristic frequency of a swinging pendulum in the earth's gravitational field or to the characteristic frequency of an oscillating mass-spring system. It is sometimes called the *cyclotron frequency* of the particle in the field because particles circulate at this frequency in the cyclotron.

▶ **Example 6.** A 10-ev electron is circulating in a plane at right angles to a uniform field of magnetic induction of 1.0×10^{-4} weber/meter² (= 1.0 gauss).

(*a*) What is its orbit radius?

The velocity of an electron whose kinetic energy is K can be found from

$$v = \sqrt{\frac{2K}{m}}.$$

The student should verify that this yields 1.9×10^6 meters/sec for v. Then, from Eq. 33–15,

$$r = \frac{mv}{qB} = \frac{(9.1 \times 10^{-31}\ \text{kg})(1.9 \times 10^6\ \text{meters/sec})}{(1.6 \times 10^{-19}\ \text{coul})(1.0 \times 10^{-4}\ \text{weber/meter}^2)} = 0.11\ \text{meter} = 11\ \text{cm}.$$

(*b*) What is the cyclotron frequency? From Eq. 33–16,

$$\nu = \frac{qB}{2\pi m} = \frac{(1.6 \times 10^{-19}\ \text{coul})(1.0 \times 10^{-4}\ \text{weber/meter}^2)}{(2\pi)(9.1 \times 10^{-31}\ \text{kg})} = 2.8 \times 10^6\ \text{rev/sec}.$$

(*c*) What is the period of revolution T?

$$T = \frac{1}{\nu} = \frac{1}{2.8 \times 10^6\ \text{rev/sec}} = 3.6 \times 10^{-7}\ \text{sec}.$$

Thus an electron requires 0.36 μsec to make 1 revolution in a 1.0-gauss field.

(*d*) What is the direction of circulation as viewed by an observer sighting along the field?

In Fig. 33–11 the magnetic force $q\mathbf{v} \times \mathbf{B}$ must point radially inward, since it provides the centripetal force. Since **B** points into the plane of the paper, **v** would have to point to the left at the position shown in the figure if the charge q were positive. However, the charge is an electron, with $q = -e$, which means that **v** must point to the right. Thus the charge circulates clockwise as viewed by an observer sighting in the direction of **B**. ◀

33-7 The Cyclotron

The cyclotron, first put into operation by Ernest Lawrence (1902–1958) in 1932, accelerates charged particles, such as protons or deuterons,* to high energies so that they can be used in atom-smashing experiments. Figure 33-12 shows the University of Pittsburgh cyclotron.

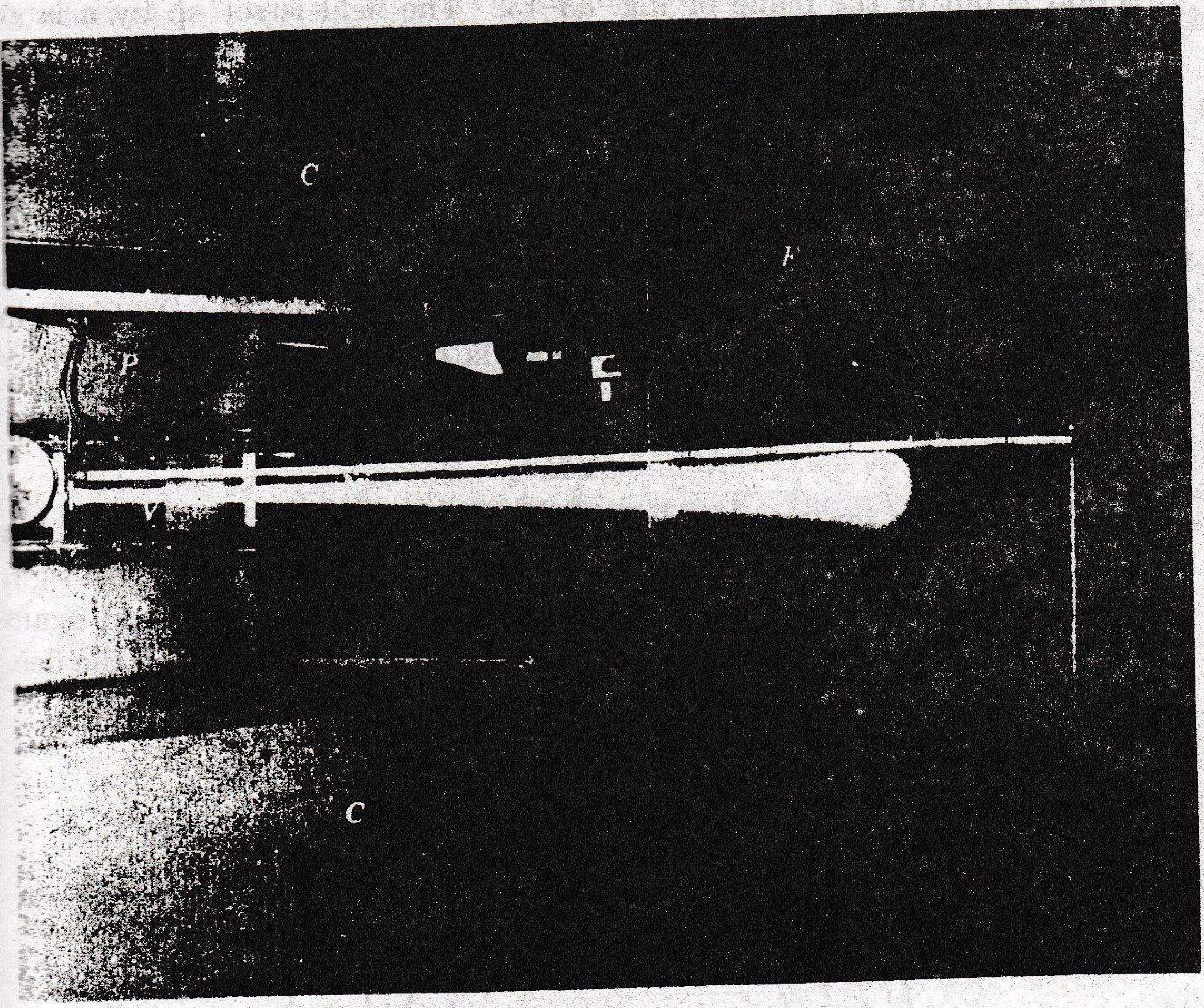

Fig. 33-12 The University of Pittsburgh cyclotron. Note vacuum chamber V, magnet frame F, magnetic pole faces P, magnet coils C, and the deuteron beam emerging into the air of the laboratory. The rule is 6 ft long. (Courtesy A. J. Allen.)

In an *ion source* at the center of the cyclotron molecules of deuterium are bombarded with electrons whose energy is high enough (say 100 ev) so that many positive ions are formed during the collisions. Many of these ions are free deuterons, which enter the cyclotron proper through a small hole in the wall of the ion source and are available to be accelerated.

* Deuterons are the nuclei of heavy hydrogen.

The cyclotron uses a modest potential difference for accelerating (say 10^5 volts), but it requires the ion to pass through this potential difference a number of times. To reach 10 Mev with 10^5 volts accelerating potential requires 100 passages. A magnetic field is used to bend the ions around so that they may pass again and again through the same accelerating potential.

Figure 33-13 is a top view of the part of the cyclotron that is inside the vacuum tank marked V in Fig. 33-12. The two D-shaped objects, called *dees*, are made of copper sheet and form part of an electric oscillator which establishes an accelerating potential difference across the gap between the dees. The direction of this potential difference is made to change sign some millions of times per second.

The dees are immersed in a magnetic field ($B \cong 1.6$ webers/meter^2) whose direction is out of the plane of Fig. 33-13. The field is set up by a large electromagnet, marked F in Fig. 33-12. Finally, the space in which the ions move is evacuated to a pressure of about 10^{-6} mm-Hg. If this were not done, the ions would continually collide with air molecules.

Suppose that a deuteron, emerging from the ion source, finds the dee that it is facing to be negative; it will accelerate toward this dee and will enter it. Once inside, it is screened from electrical forces by the metal walls of the dees. The magnetic field is not screened by the dees so that the ion bends in a circular path whose radius, which depends on the velocity, is given by Eq. 33-15, or

$$r = \frac{mv}{qB}.$$

After a time t_0 the ion emerges from the dee on the other side of the ion source. Let us assume that the accelerating potential has now changed sign. Thus the ion *again* faces a negative dee, is further accelerated, and again describes a semicircle, of somewhat larger radius (see Eq. 33-15), in the dee.

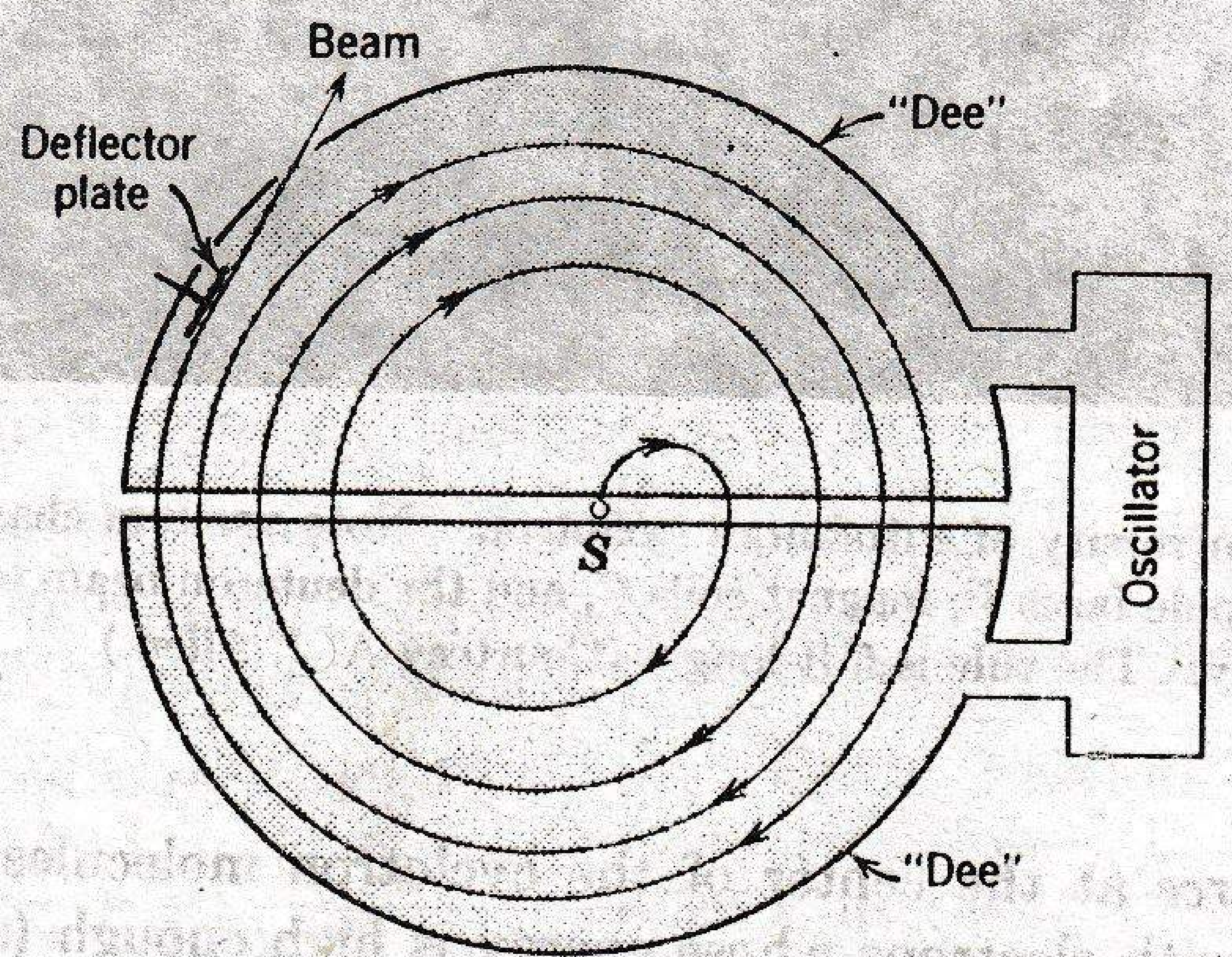

Fig. 33-13 The elements of a cyclotron showing the ion source S and the dees. The deflector plate, held at a suitable negative potential, deflects the particles out of the dee system.

The time of passage through this dee, however, is still t_0. This follows because the period of revolution T of an ion circulating in a magnetic field does not depend on the speed of the ion; see Eq. 33-16. This process goes on until the ion reaches the outer edge of one dee where it is pulled out of the system by a negatively charged deflector plate.

The key to the operation of the cyclotron is that the characteristic frequency ν at which the ion circulates in the field must be equal to the fixed frequency ν_0 of the electric oscillator, or

$$\nu = \nu_0.$$

This *resonance condition* says that if the energy of the circulating ion is to increase energy must be fed to it at a frequency ν_0 that is equal to the natural frequency ν at which the ion circulates in the field. In the same way we feed energy to a swing by pushing it at a frequency equal to the natural frequency of oscillation of the swing.

From Eq. 33-16 ($\nu = qB/2\pi m$), we can rewrite the resonance equation as

$$\frac{qB}{2\pi m} = \nu_0. \tag{33-17}$$

Once we have selected an ion to be accelerated, q/m is fixed; usually the oscillator is designed to work at a single frequency ν_0. We then "tune" the cyclotron by varying B until Eq. 33-17 is satisfied and an accelerated beam appears.

The *energy* of the particles produced in the cyclotron depends on the radius R of the dees. From Eq. 33-15 ($r = mv/qB$), the velocity of a particle circulating at this radius is given by

$$v = \frac{qBR}{m}.$$

The kinetic energy is then

$$K = \tfrac{1}{2}mv^2 = \frac{q^2B^2R^2}{2m}. \tag{33-18}$$

▶ **Example 7.** The University of Pittsburgh cyclotron has an oscillator frequency of 12×10^6 cycles/sec and a dee radius of 21 in. (*a*) What value of magnetic induction B is needed to accelerate deuterons?

From Eq. 33-17, $\nu_0 = qB/2\pi m$, so that

$$B = \frac{2\pi \nu_0 m}{q} = \frac{(2\pi)(12 \times 10^6/\text{sec})(3.3 \times 10^{-27}\ \text{kg})}{1.6 \times 10^{-19}\ \text{coul}} = 1.6\ \text{webers/meter}^2.$$

Note that the deuteron has the same charge as the proton but (very closely) twice the mass.

(*b*) What deuteron energy results?

From Eq. 33-18,

$$K = \frac{q^2B^2R^2}{2m} = \frac{(1.6 \times 10^{-19}\ \text{coul})^2(1.6\ \text{webers/meter}^2)^2(21 \times 0.0254\ \text{meter})^2}{(2)(3.3 \times 10^{-27}\ \text{kg})}$$

$$= (2.8 \times 10^{-12}\ \text{joule})\left(\frac{1\ \text{ev}}{1.6 \times 10^{-19}\ \text{joule}}\right) = 17\ \text{Mev}.$$ ◀

The cyclotron fails to operate at high energies because one of its assumptions, that the frequency of rotation of an ion circulating in a magnetic field is independent of its speed, is true only for speeds much less than that of light. As the particle speed increases, we must use the *relativistic mass* m in Eq. 33–16. The relativistic mass increases with velocity (Eq. 8–20) so that at high enough speeds ν decreases with velocity. Thus the ions get out of step with the electric oscillator, and eventually the energy of the circulating ion stops increasing.

Another difficulty associated with the acceleration of particles to high energies is that the size of the magnet that would be required to guide such particles in a circular path is very large. For a 30-Bev proton, for example, in a field of 15,000 gauss the radius of curvature is 65 meters. A magnet of the cyclotron type of this size (about 430 ft in diameter) would be prohibitively expensive. Incidentally, a 30-Bev proton has a speed equal to 0.99998 that of light.

Both the relativistic and the economic limitations have been removed by techniques that can be understood in terms of Eq. 33–17 in which m is now taken to be the relativistic mass, given by Eq. 8–13, or

$$m = \frac{m_0}{\sqrt{1 - (v/c)^2}},$$

v being the speed of the particle and c being that of light.

As the particle speed increases, the relativistic mass m also increases. To maintain the equality in Eq. 33–17, and thus insure resonance, one may decrease the oscillator frequency ν_0 as the particle (assumed to be a proton) accelerates in such a way that the product $\nu_0 m$ remains constant. Accelerators that use this technique are called *synchrocyclotrons*.

To remove the magnet cost limitation one can vary *both* B and ν_0 in a cyclic fashion in such a way that not only is Eq. 33–17 satisfied at all times but the particle orbit radius remains constant during the acceleration process. This permits the use of an annular (or ring-shaped) magnet, rather than the conventional cyclotron type, at great saving in cost. With the *two* variables B and ν_0 at our disposal, it is possible to preserve *two* equalities during the acceleration process, one being Eq. 33–17 and the other being the relation

$$v = \omega_0 R_0 = (2\pi\nu_0)R_0$$

in which R_0 is the desired (fixed) orbit radius. Accelerators that use this technique are called *synchrotrons*. Table 33–2 shows some characteristics of the accel-

Table 33–2

THE CERN PROTON SYNCHROTRON

Orbit diameter	560 ft
Vacuum chamber cross section	5.5 in. × 2.7 in.
Maximum magnetic field	1.4 webers/meter^2
Frequency range per cycle	7 mc/sec
Pulse rate	20/min
Maximum proton energy	28 Bev
Energy gain per cycle	54 kev
Distance traveled by a proton	5 × 10^4 miles
Protons per pulse	10^{11}
Cost	\$28 × 10^6

erator built at Geneva, Switzerland, by the European Council for Nuclear Research (CERN), and embodying these principles.

33-8 Thomson's Experiment

In 1897 J. J. Thomson, working at the Cavendish Laboratory in Cambridge, measured the ratio of the charge e of the electron to its mass m by observing its deflection in combined electric and magnetic fields. The discovery of the electron is usually said to date from this historic experiment, although H. A. Lorentz and P. Zeeman (1865–1943), during the previous year, had measured this same quantity for electrons bound in atoms, using a method entirely different from Thomson's.

In Fig. 33-14, which is a modernized version of Thomson's apparatus, electrons are emitted from hot filament F and accelerated by an applied potential difference V. They then enter a region in which they move at right angles to an electric field **E** and a field of magnetic induction **B**; **E** and **B** are themselves at right angles to each other. The beam is made visible as a spot of light when it strikes fluorescent screen S. The entire region in which the electrons move is highly evacuated so that collisions with air molecules will not occur.

The resultant force on a charged particle moving through an electric and a magnetic field is given by Eq. 33-4, or

$$\mathbf{F} = q_0\mathbf{E} + q_0\mathbf{v} \times \mathbf{B}.$$

Study of Fig. 33-14 shows that the electric field deflects the particle upward and the magnetic field deflects it downward. If these deflecting forces are to

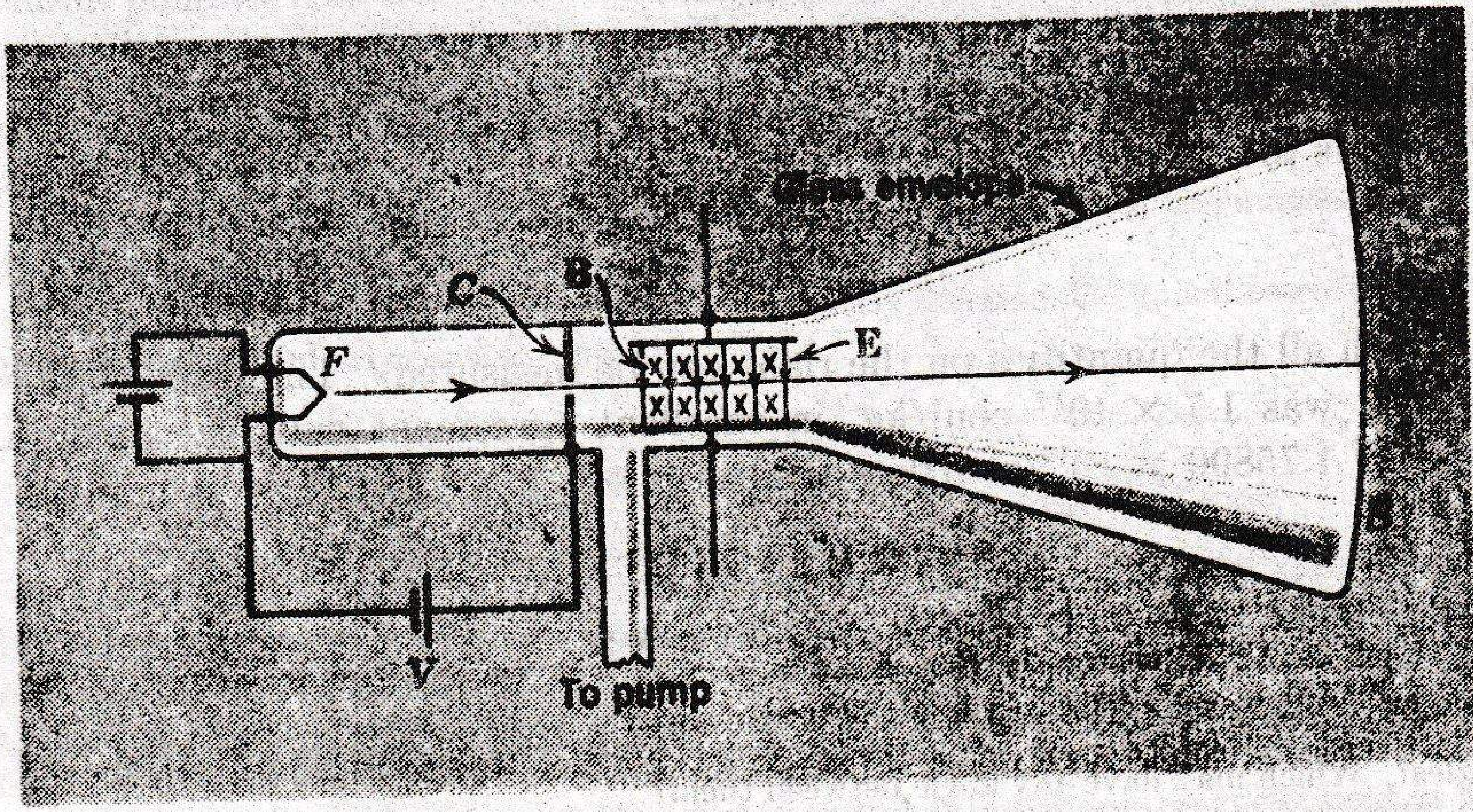

Fig. 33-14 Electrons from the heated filament F are accelerated by a potential difference V and pass through a hole in the screen C. After passing through a region in which perpendicular electric and magnetic fields are present, they strike the fluorescent screen S.

cancel (that is, if $\mathbf{F} = 0$), this equation, for this problem, reduces to

$$eE = evB$$

or

$$E = vB. \tag{33-19}$$

Thus for a given electron speed v the condition for zero deflection can be satisfied by adjusting E or B.

Thomson's procedure was (*a*) note the position of the undeflected beam spot, with **E** and **B** both equal to zero; (*b*) apply a fixed electric field **E**, measuring on the fluorescent screen the deflection so caused; and (*c*) apply a magnetic field and adjust its value until the beam deflection is restored to zero.

In Section 27–5 we saw that the deflection y of an electron in a purely electric field (step *b*), measured at the far edge of the deflecting plates, is given by Eq. 27–9, or, with small changes in notation,

$$y = \frac{eEl^2}{2mv^2},$$

where v is the electron speed and l is the length of the deflecting plates; y is not measurable directly, but it may be calculated from the measured displacement of the spot on the screen if the geometry of the apparatus is known. Thus y, E, and l are known; the ratio e/m and the velocity v are unknown. We cannot calculate e/m until we have found the velocity, which is the purpose of step *c* above.

If (step *c*) the electric force is set equal and opposite to the magnetic force, the net force is zero and we can write (Eq. 33–19)

$$v = \frac{E}{B}.$$

Substituting this equation into the equation for y and solving for the ratio e/m leads to

$$\frac{e}{m} = \frac{2yE}{B^2l^2}, \tag{33-20}$$

in which all the quantities on the right can be measured. Thomson's value for e/m was 1.7×10^{11} coul/kg, in excellent agreement with the modern value of 1.75890×10^{11} coul/kg.

QUESTIONS

1. Of the three vectors in the equation $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, which pairs are always at right angles? Which may have any angle between them?

2. Why do we not simply define the magnetic induction **B** to point in the direction of the magnetic force that acts on the moving charge?

3. Imagine that you are sitting in a room with your back to one wall and that an electron beam, traveling horizontally from the back wall toward the front wall, is deflected to your right. What is the direction of the field of magnetic induction that exists in the room?

4. If an electron is not deflected in passing through a certain region of space, can we be sure that there is no magnetic field in that region?

5. If a moving electron is deflected sideways in passing through a certain region of space, can we be sure that a magnetic field exists in that region?

6. A beam of protons is deflected sideways. Could this deflection be caused (*a*) by an electric field? (*b*) By a magnetic field? (*c*) If either could be responsible, how would you be able to tell which was present?

7. A conductor, even though it is carrying a current, has zero net charge. Why, then, does a magnetic field exert a force on it?

8. Equation 33–11 ($\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$) shows that there is no torque on a current loop in an external magnetic field if the angle between the axis of the loop and the field is (*a*) 0° or (*b*) 180°. Discuss the nature of the equilibrium (that is, is it stable, neutral, or unstable?) for these two positions.

9. In Example 4 we showed that the work required to turn a current loop end-for-end in an external magnetic field is $2\mu B$. Does this hold no matter what the original orientation of the loop was?

10. Imagine that the room in which you are seated is filled with a uniform magnetic field with **B** pointing vertically upward. A circular loop of wire has its plane horizontal. For what direction of current in the loop, as viewed from above, will the loop be in stable equilibrium with respect to forces and torques of magnetic origin?

11. A rectangular current loop is in an arbitrary orientation in an external magnetic field. Is any work required to rotate the loop about an axis perpendicular to its plane?

12. (*a*) In measuring Hall potential differences, why must we be careful that points x and y in Fig. 33–10 are exactly opposite to each other? (*b*) If one of the contacts is movable, what procedure might we follow in adjusting it to make sure that the two points are properly located?

13. A uniform magnetic field fills a certain cubical region of space. Can an electron be fired into this cube from the outside in such a way that it will travel in a closed circular path inside the cube?

14. Imagine the room in which you are seated to be filled with a uniform magnetic field with **B** pointing vertically downward. At the center of the room two electrons are suddenly projected horizontally with the same speed but in opposite directions. (*a*) Discuss their motions. (*b*) Discuss their motions if one particle is an electron and one a positron.

15. In Fig. 33–4, why are the low-energy electron tracks spirals? That is, why does the radius of curvature change in the constant magnetic field in which the chamber is immersed?

16. What are the primary functions of (*a*) the electric field and (*b*) the magnetic field in the cyclotron?

17. For Thomson's e/m experiment to work properly (Section 33–8), is it essential that the electrons have a fairly constant speed?

PROBLEMS

1. The electrons in the beam of a television tube have an energy of 12 kev. The tube is oriented so that the electrons move horizontally from south to north. The vertical component of the earth's magnetic field points down and has $B = 5.5 \times 10^{-5}$ weber/meter^2. (*a*) In what direction will the beam deflect? (*b*) What is the acceleration of a given electron? (*c*) How far will the beam deflect in moving 20 cm through the television tube?

2. In a nuclear experiment a 1.0-Mev proton moves in a uniform magnetic field in a circular path. What energy must (*a*) an alpha particle and (*b*) a deuteron have if they are to circulate in the same orbit?

3. A wire 1.0 meter long carries a current of 10 amp and makes an angle of 30° with a uniform magnetic field with $B = 1.5$ webers/meter^2. Calculate the magnitude and direction of the force on the wire.

4. A wire of 60 cm length and mass 10 gm is suspended by a pair of flexible leads in a magnetic field of induction 0.40 weber/meter^2. What are the magnitude and direction of the current required to remove the tension in the supporting leads? See Fig. 33–15.

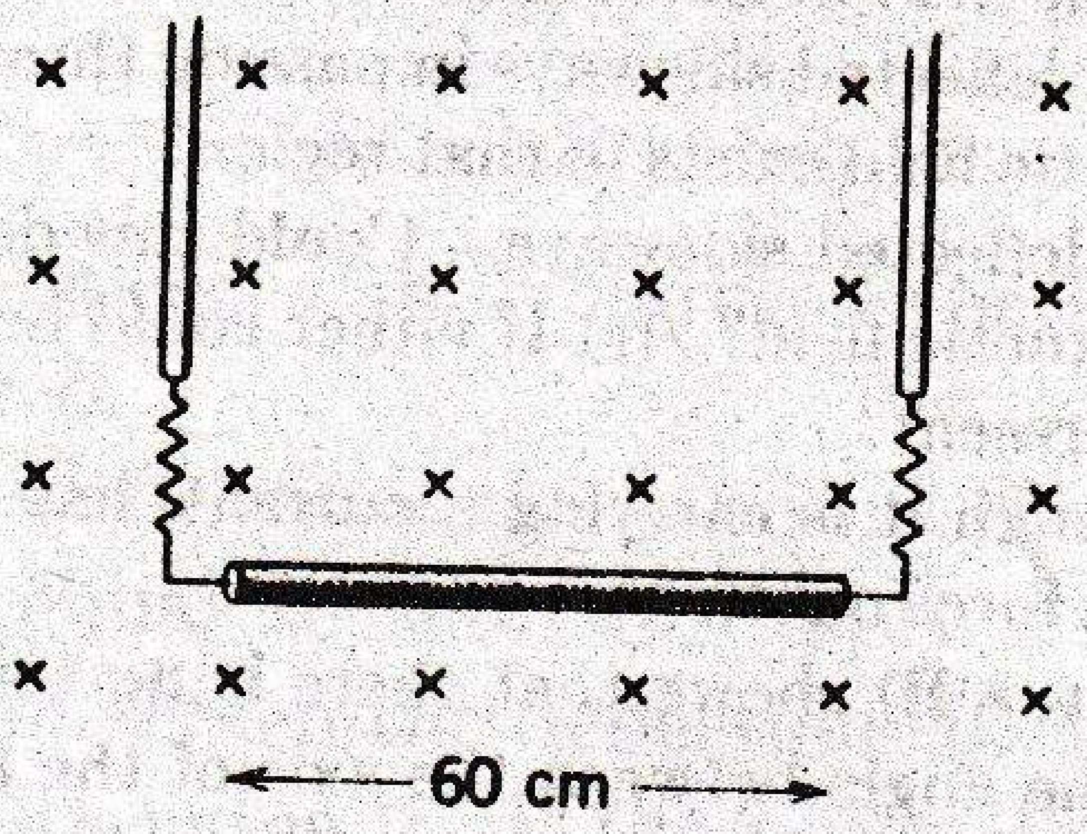

Fig. 33–15

5. Express magnetic induction B and magnetic flux Φ in terms of the fundamental dimensions M, L, T, and Q (mass, length, time, and charge).

6. A metal wire of mass m slides without friction on two rails spaced a distance d apart, as in Fig. 33–16. The track lies in a vertical uniform field of magnetic induction **B**. A *constant current* i flows from generator G along one rail, across the wire, and back down the other rail. Find the velocity (speed and direction) of the wire as a function of time, assuming it to be at rest at $t = 0$.

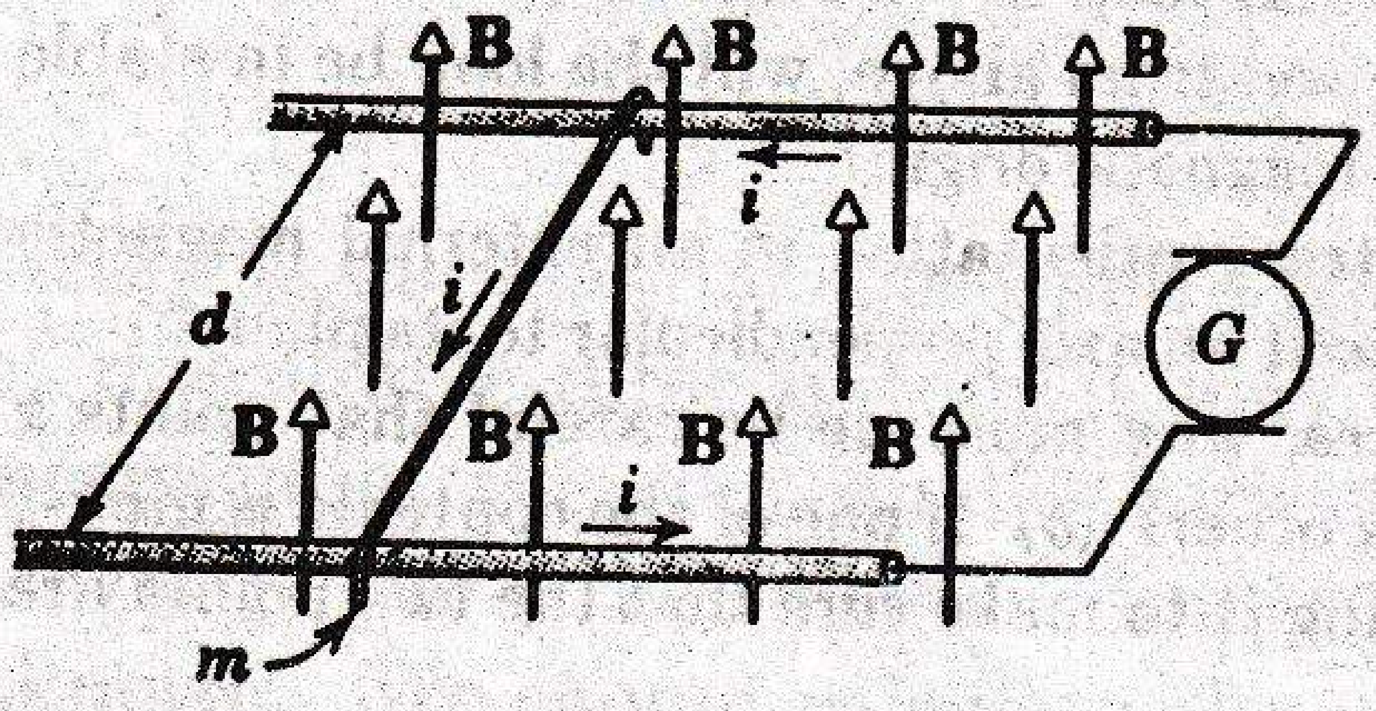

Fig. 33–16

7. A U-shaped wire of mass m and length l is immersed with its two ends in mercury (Fig. 33–17). The wire is in a homogeneous field of magnetic induction **B**. If a charge, that is, a current pulse $q = \int i\,dt$, is sent through the wire, the wire will jump up. Calculate, from the height h that the wire reaches, the size of the charge or current pulse, assuming that the time of the current pulse is very small in comparison with the time of flight. Make use of the fact that impulse of force equals $\int F\,dt$, which equals mv. (Hint: Try to relate $\int i\,dt$ to $\int F\,dt$.) Evaluate q for $B = 0.1$ weber/meter^2, $m = 10$ gm, $l = 20$ cm, and $h = 3$ meters.

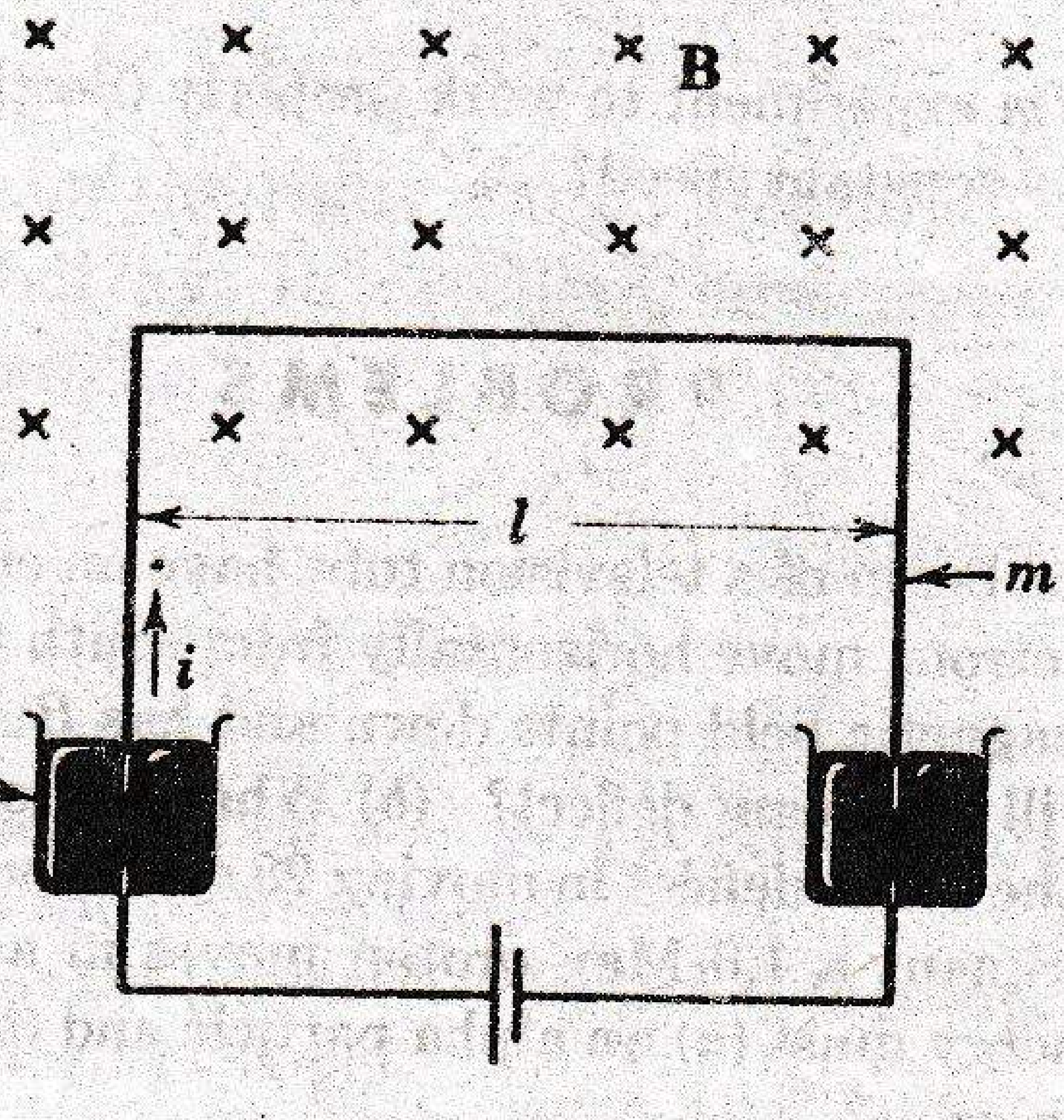

Fig. 33–17

8. Figure 33–18 shows a wire ring of radius a at right angles to the general direction of a radially symmetric diverging magnetic field. The magnetic induction at the ring is everywhere of the same magnitude B, and its direction at the ring is everywhere at an angle θ with a normal to the plane of the ring. The twisted lead wires have no effect on the problem. Find the magnitude and direction of the force the field exerts on the ring if the ring carries a current i as shown in the figure.

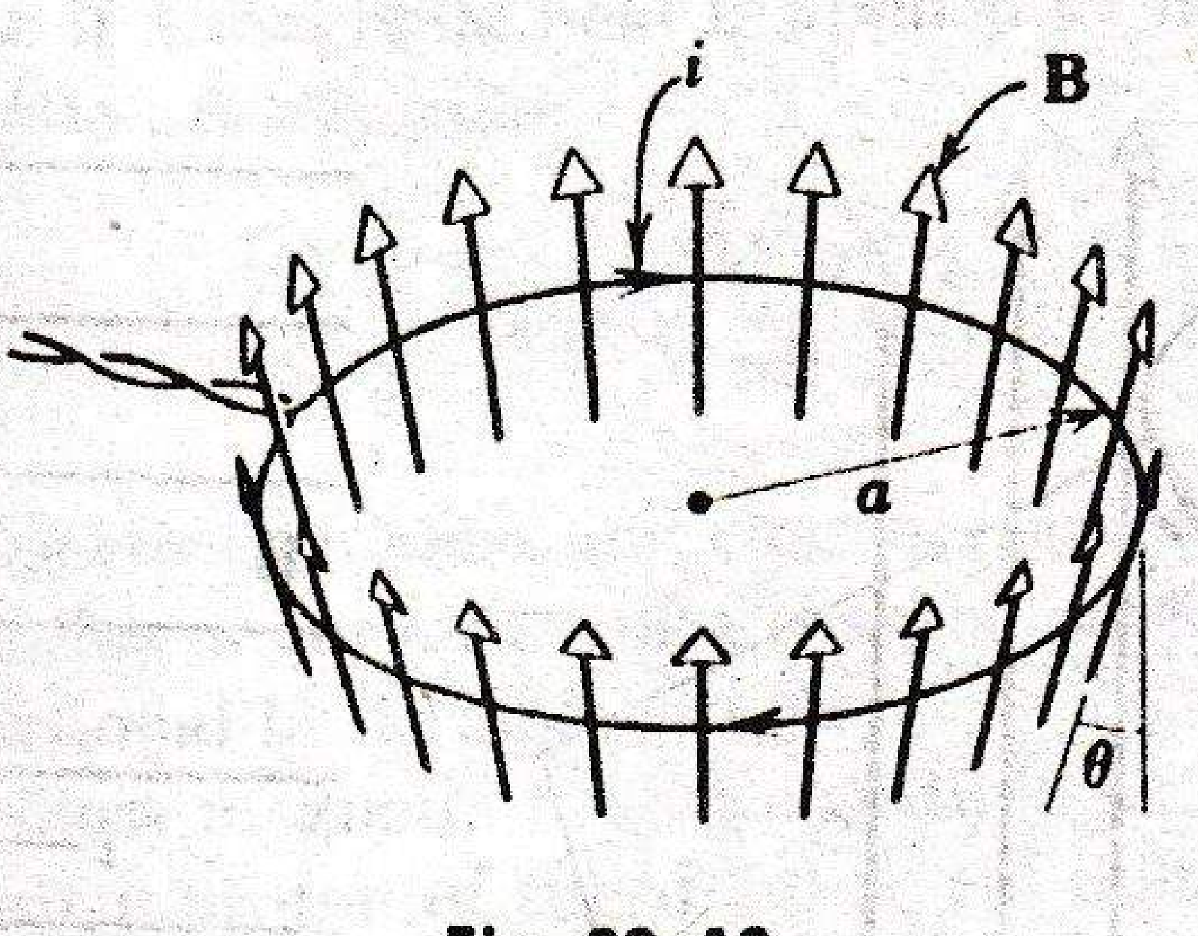

Fig. 33–18

9. A copper rod weighing 0.30 lb rests on two rails 1.0 ft apart and carries a current of 50 amp from one rail to the other. The coefficient of starting friction is 0.60. What is the smallest magnetic field that would cause the bar to slide and what is its direction?

10. Figure 33–19 shows a wire of arbitrary shape carrying a current i between points a and b. The wire lies in a plane at right angles to a uniform field of magnetic induction **B**. Prove that the force on the wire is the same as that on a straight wire carrying a current i directly from a to b. (Hint: Replace the wire by a series of "steps" parallel and perpendicular to the straight line joining a and b.)

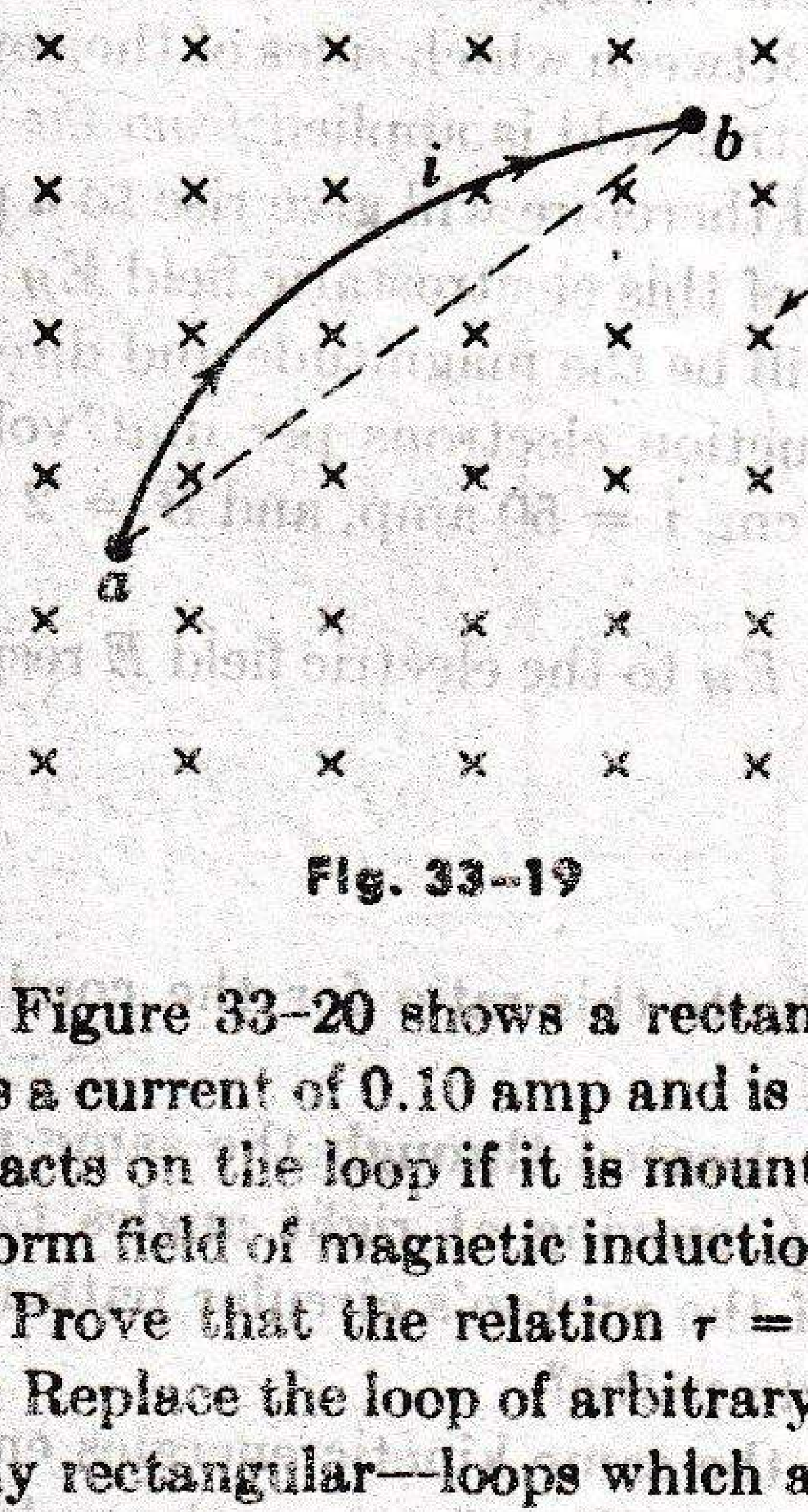

Fig. 33–19

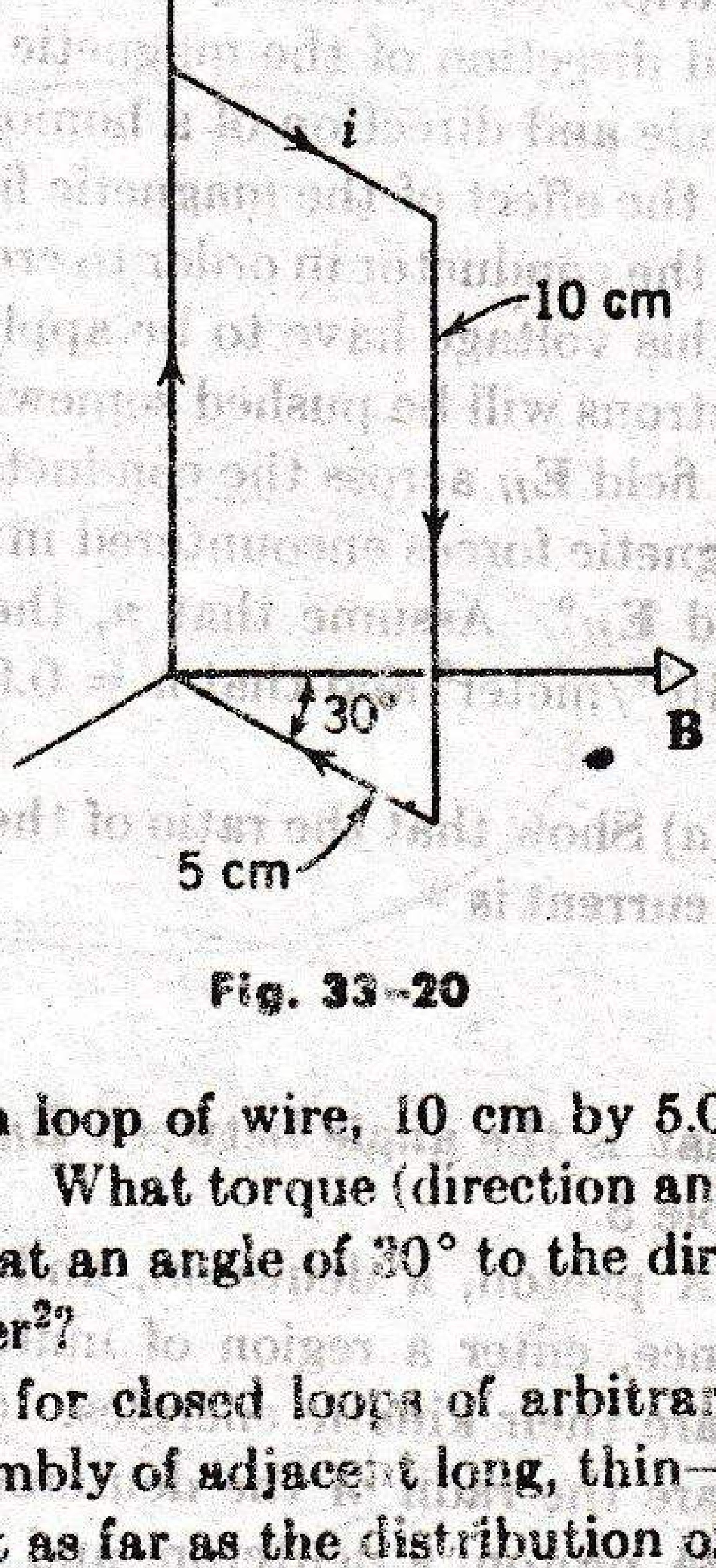

Fig. 33–20

11. Figure 33–20 shows a rectangular twenty-turn loop of wire, 10 cm by 5.0 cm. It carries a current of 0.10 amp and is hinged at one side. What torque (direction and magnitude) acts on the loop if it is mounted with its plane at an angle of 30° to the direction of a uniform field of magnetic induction 0.50 weber/meter²?

12. Prove that the relation $\tau = NAiB \sin\theta$ holds for closed loops of arbitrary shape. (Hint: Replace the loop of arbitrary shape by an assembly of adjacent long, thin—approximately rectangular—loops which are equivalent to it as far as the distribution of current is concerned.)

13. A length L of wire carries a current i. Show that if the wire is formed into a circular coil the maximum torque in a given magnetic field is developed when the coil has *one* turn only and the maximum torque has the value

$$\tau = \frac{1}{4\pi} L^2 iB.$$

14. Figure 33-21 shows a wooden cylinder with a mass m of 0.25 kg, a radius R, and a length l of 0.1 meter with N equal to ten turns of wire wrapped around it longitudinally, so that the plane of the wire loop contains the axis of the cylinder. What is the least current through the loop that will prevent the cylinder from rolling down an inclined plane whose surface is inclined at an angle θ to the horizontal, in the presence of a vertical field of magnetic induction 0.5 weber/meter^2, if the plane of the windings is parallel to the inclined plane?

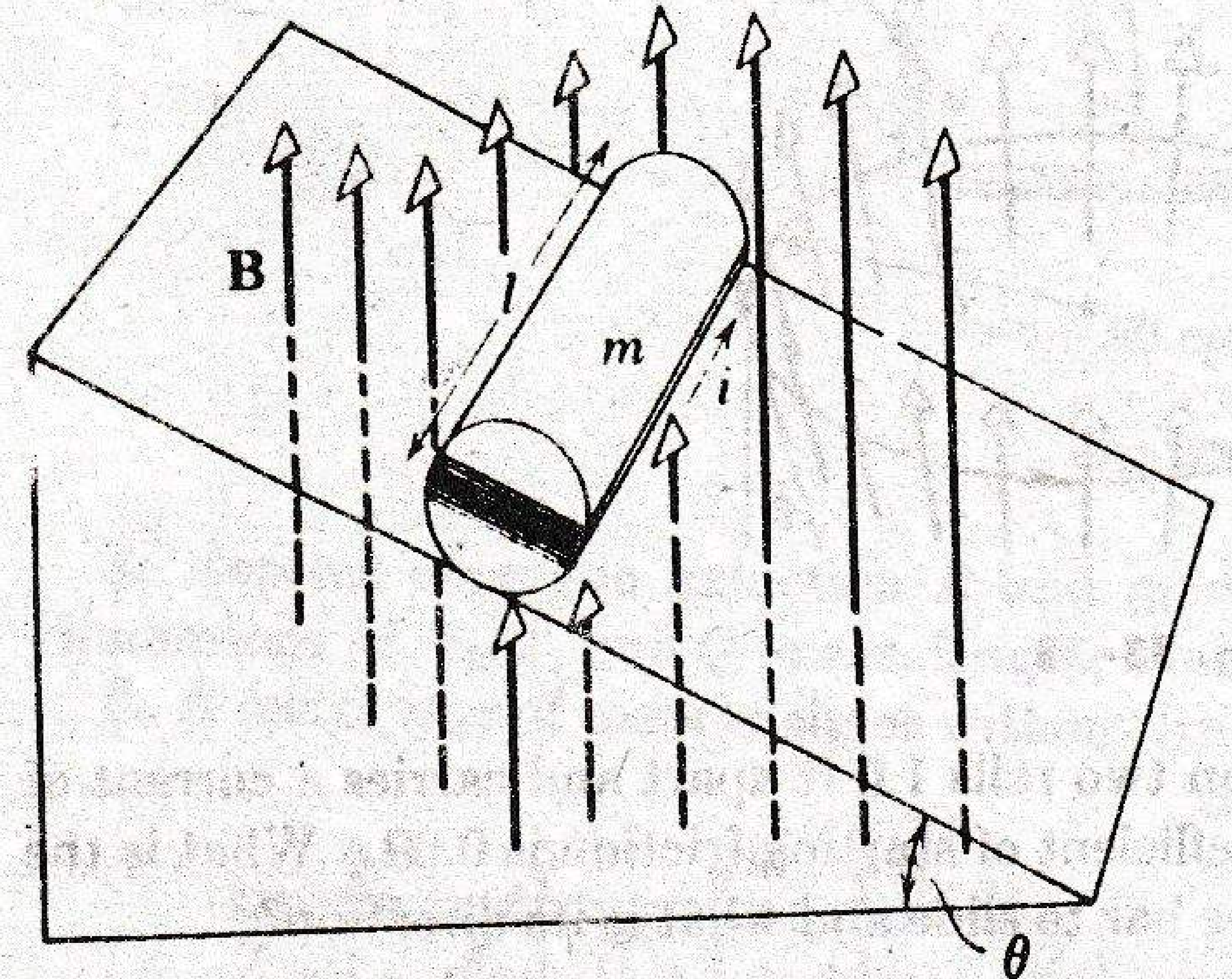

Fig. 33-21

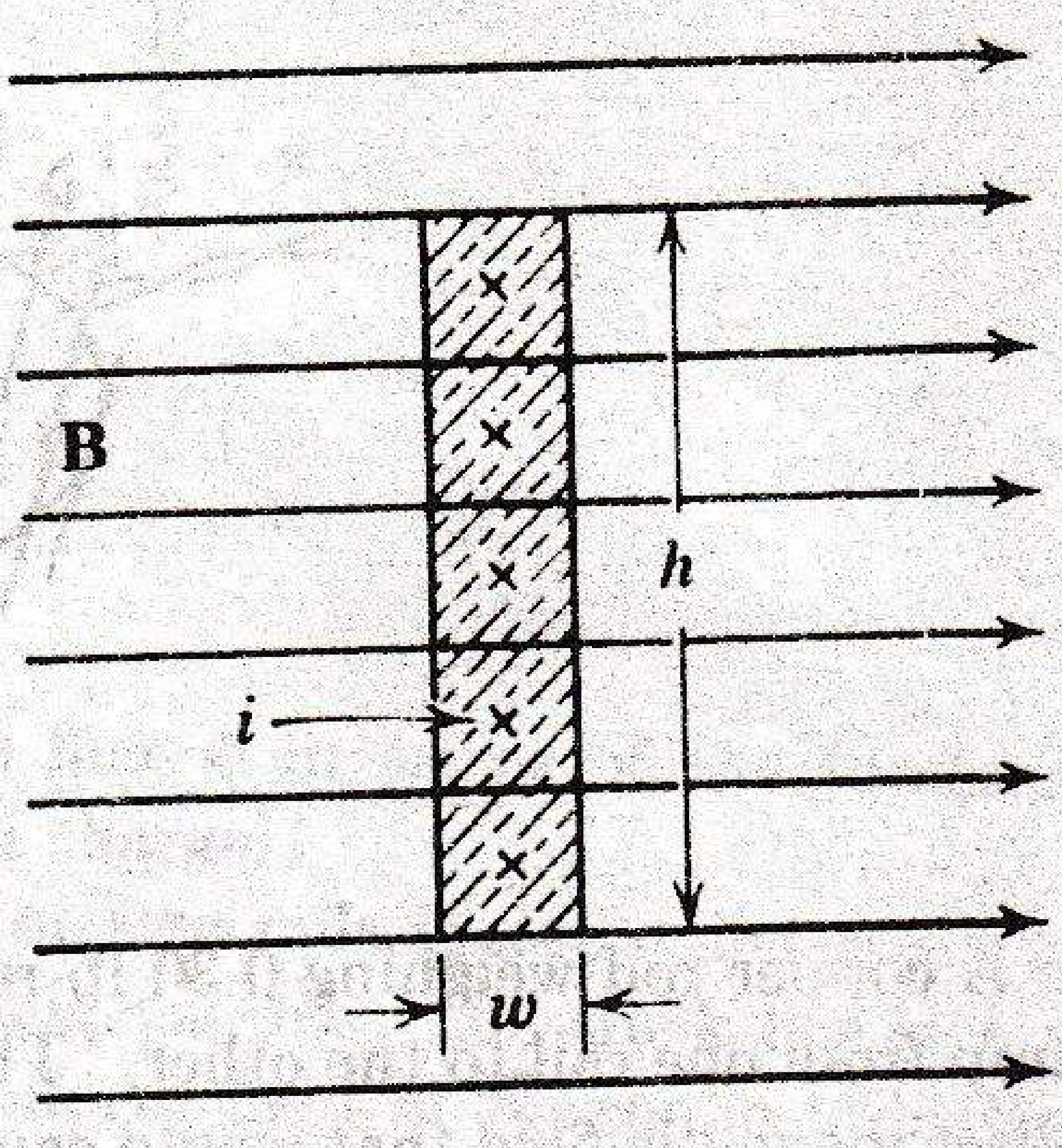

Fig. 33-22

15. A current i, indicated by the crosses in Fig. 33-22, is established in a strip of copper of height h and width w. A uniform field of magnetic induction **B** is applied at right angles to the strip. (*a*) Calculate the drift speed v_d for the electrons. (*b*) What are the magnitude and direction of the magnetic force **F** acting on the electrons? (*c*) What would the magnitude and direction of a homogeneous electric field **E** have to be in order to counterbalance the effect of the magnetic field? (*d*) What is the voltage V necessary between two sides of the conductor in order to create this field **E**? Between which sides of the conductor would this voltage have to be applied? (*e*) If no electric field is applied *from the outside*, the electrons will be pushed somewhat to one side and therefore will give rise to a uniform electric field $\mathbf{E}_H$ across the conductor until the forces of this electrostatic field $\mathbf{E}_H$ balance the magnetic forces encountered in part (*b*). What will be the magnitude and direction of the field $\mathbf{E}_H$? Assume that n, the number of conduction electrons per unit volume, is 1.1×10^{29}/meter^3 and that $h = 0.02$ meter, $w = 0.1$ cm, $i = 50$ amp, and $B = 2$ webers/meter^2.

16. (*a*) Show that the ratio of the Hall electric field E_H to the electric field E responsible for the current is

$$\frac{E_H}{E} = \frac{B}{ne\rho}.$$

(*b*) What is the angle between $\mathbf{E}_H$ and **E**? (*c*) Evaluate this ratio for the conditions of Example 5.

17. A proton, a deuteron, and an α-particle, accelerated through the same potential difference, enter a region of uniform magnetic field, moving at right angles to **B**. (*a*) Compare their kinetic energies. (*b*) If the radius of the proton's circular path is 10 cm, what are the radii of the deuteron and the α-particle paths?

18. A proton, a deuteron, and an α-particle with the same kinetic energies enter a region of uniform magnetic field, moving at right angles to **B**. Compare the radii of their circular paths.

19. An α-particle travels in a circular path of radius 0.45 meter in a magnetic field with $B = 1.2$ webers/meter^2. Calculate (*a*) its speed, (*b*) its period of revolution, (*c*) its kinetic energy, and (*d*) the potential difference through which it would have to be accelerated to achieve this energy.

20. An electron is accelerated through 15,000 volts and is then allowed to circulate at right angles to a uniform magnetic field with $B = 250$ gauss. What is its path radius?

21. Electrons are observed to be ejected in various directions with negligible speed from the negative plate of a parallel-plate capacitor when the plate is illuminated by light of a certain wavelength (photoelectric effect). The plates are separated by a distance d and a potential difference V is maintained between them. Show that none of these electrons will reach the positive plate if a magnetic field is applied at right angles to the electric field and the magnetic induction has a value

$$B > \left(\frac{2Vm}{ed^2}\right)^{1/2},$$

in which m and e are the electron mass and charge, respectively.

22. Show that the radius of curvature of a charged particle moving at right angles to a magnetic field is proportional to its momentum.

23. What is the smallest magnetic field (magnitude and direction) that can be set up at the equator to permit a proton of speed 1.0×10^7 meters/sec to circulate around the earth?

24. A deuteron in a large cyclotron is moving in a magnetic field with $B = 1.5$ webers/meter^2 and an orbit radius of 2.0 meters. Because of a grazing collision with a target, the deuteron breaks up, with a negligible loss of kinetic energy, into a proton and a neutron. Discuss the subsequent motions of each. Assume that the deuteron energy is shared equally by the proton and neutron at breakup.

25. A 2-kev positron is projected into a uniform field of induction **B** of 0.10 weber/meter^2 with its velocity vector making an angle of 89° with **B**. Convince yourself that the path will be a helix, its axis being the direction of **B**. Find the period, the pitch p, and the radius r of the helix; see Fig. 33-23.

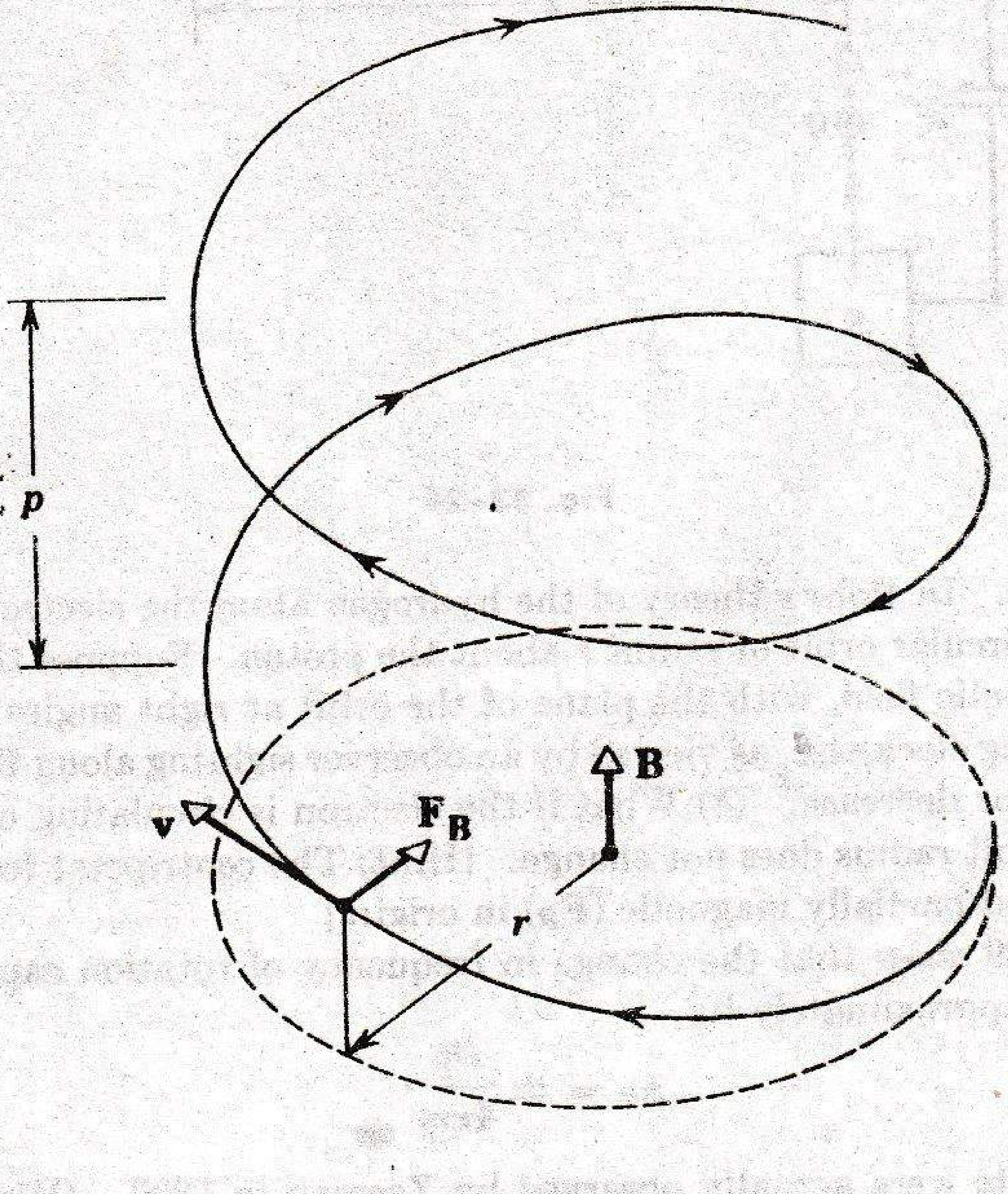

Fig. 33-23

26. (*a*) In a magnetic field with $B = 0.50$ weber/meter^2, for what path radius will an electron circulate at 0.1 the speed of light? (*b*) What will its kinetic energy be?

27. *Time-of-flight spectrometer.* S. A. Goudsmit has devised a method for measuring accurately the masses of heavy ions by timing their period of circulation in a known magnetic field. A singly charged ion of iodine makes 7 rev in a field of 4.5×10^{-2} weber/meter^2 in about 1.29×10^{-3} sec. What (approximately) is its mass in kilograms? Actually, the mass measurements are carried out to much greater accuracy than these approximate data suggest.

28. *Mass spectrometer.* Figure 33–24 shows an arrangement used by Dempster to measure the masses of ions. An ion of mass M and charge $+q$ is produced essentially at rest in source S, a chamber in which a gas discharge is taking place. The ion is accelerated by potential difference V and allowed to enter a field of magnetic induction **B**. In the field it moves in a semicircle, striking a photographic plate at distance x from the entry slit and being recorded. Show that the mass M is given by

$$M = \frac{B^2 q}{8V} x^2.$$

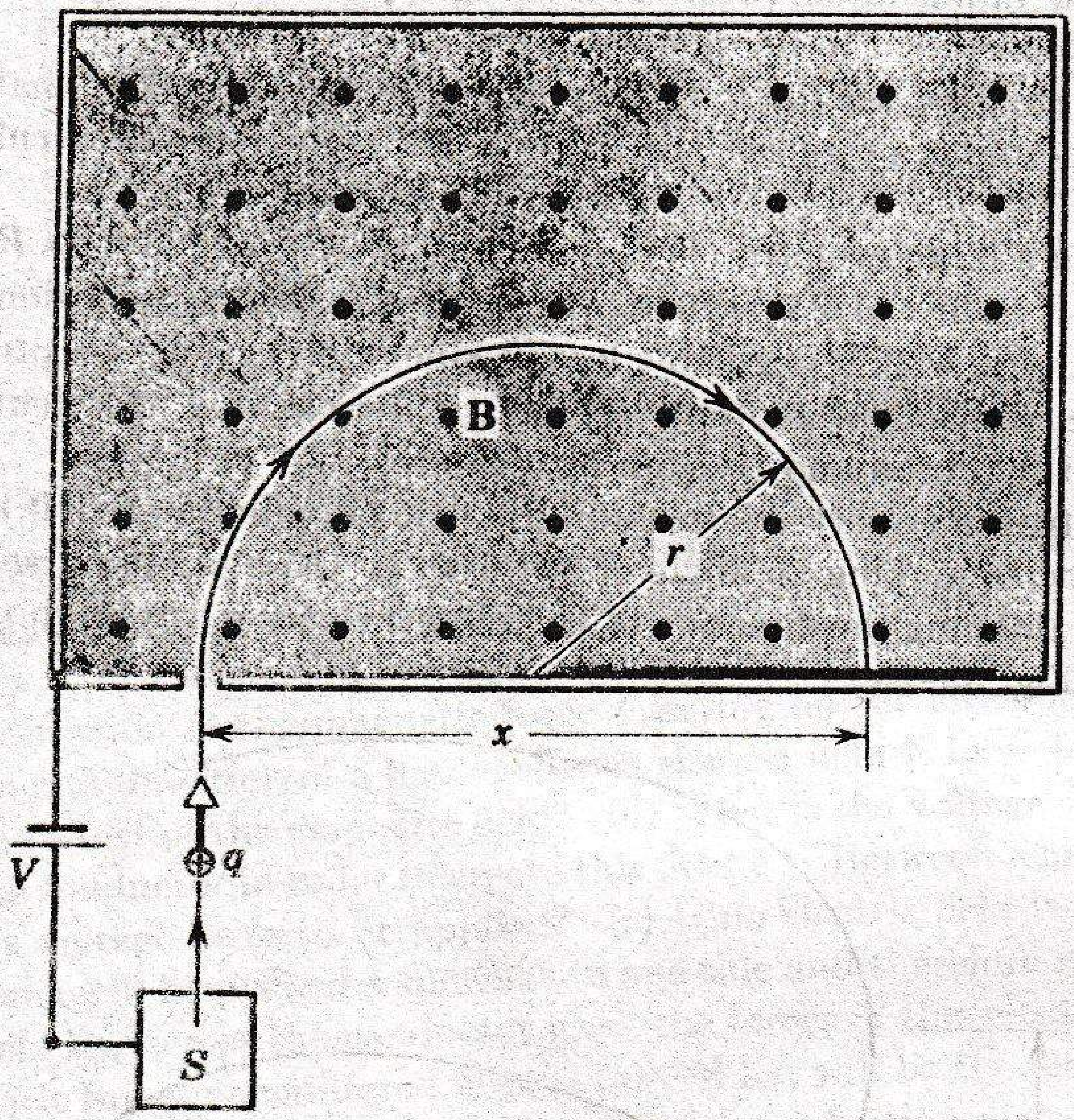

Fig. 33–24

29. *Zeeman effect.* In Bohr's theory of the hydrogen atom the electron can be thought of as moving in a circular orbit of radius r about the proton. Suppose that such an atom is placed in a magnetic field, with the plane of the orbit at right angles to **B**. (*a*) If the electron is circulating clockwise, as viewed by an observer sighting along **B**, will the angular frequency increase or decrease? (*b*) What if the electron is circulating counterclockwise? Assume that the orbit radius does not change. [Hint: The centripetal force is now partially electric ($\mathbf{F}_E$) and partially magnetic ($\mathbf{F}_B$) in origin.]

30. In Problem 29 show that the change in frequency of rotation caused by the magnetic field is given approximately by

$$\Delta\nu = \pm \frac{Be}{4\pi m}.$$

Such frequency shifts were actually observed by Zeeman in 1896. (Hint: Calculate the frequency of rotation without the magnetic field and also with it. Subtract, bearing in mind that because the effect of the magnetic field is very small some—but not all—terms containing B can be set equal to zero with little error.)

31. The University of Pittsburgh cyclotron is normally adjusted to accelerate deuterons. (*a*) What energy of protons could it produce, using the same oscillator frequency as that used for deuterons? (*b*) What magnetic induction would be required? (*c*) What energy of protons could be produced if the magnetic induction was left at the value used for deuterons? (*d*) What oscillator frequency would then be required? (*e*) Answer the same questions for α-particles.

32. Estimate the total path length traversed by a deuteron in the University of Pittsburgh cyclotron during the acceleration process. Assume an accelerating potential between the dees of 80,000 volts.

33. In a synchrocyclotron producing 400-Mev protons, what must the ratio be of the oscillator frequency at the beginning of an accelerating cycle to that at the end? Such a proton has a speed of $0.70c$, where c is the speed of light.

34. A 10-kev electron moving horizontally enters a region of space in which there is a downward-directed electric field of magnitude 100 volts/cm. (*a*) What are the magnitude and direction of the (smallest) field of magnetic induction that will allow the electron to continue to move horizontally? Ignore gravitational forces, which are rather small. (*b*) Is it possible for a proton to pass through this combination of fields undeflected? If so, under what circumstances?

35. An electric field of 1500 volts/meter and a magnetic field of 0.40 weber/meter^2 act on a moving electron to produce no force. (*a*) Calculate the minimum electron speed v. (*b*) Draw the vectors **E**, **B**, and **v**.

Ampère's Law

CHAPTER 34

34-1 Ampère's Law

One class of problems involving magnetic fields, dealt with in Chapter 33, concerns the forces exerted by a magnetic field on a moving charge or on a current-carrying conductor and the torque exerted on a magnetic dipole. A second class concerns the *production* of a magnetic field by a current-carrying conductor or by moving charges. This chapter deals with problems of this second class.

The discovery that currents produce magnetic effects was made by Oersted in 1820. Figure 34-1, which shows a wire surrounded by a number of small magnets, illustrates a modification of his experiment. If there is no current in the wire, all the magnets are aligned with the earth's magnetic field. When a strong current is present, the magnets point so as to suggest that the magnetic lines of induction form closed circles around the wire. This view is strengthened by the experiment of Fig. 34-2, which shows iron filings on a horizontal glass plate, through the center of which a current-carrying conductor passes.

Today we write the quantitative relationship between current i and the magnetic field $\mathbf{B}$ as

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i, \tag{34-1}$$

which is known as *Ampère's law*. Ampère, being an advocate of the action-at-a-distance point of view, did not formulate his results in terms of fields; this was first done by Maxwell. Ampère's law, including an important ex-

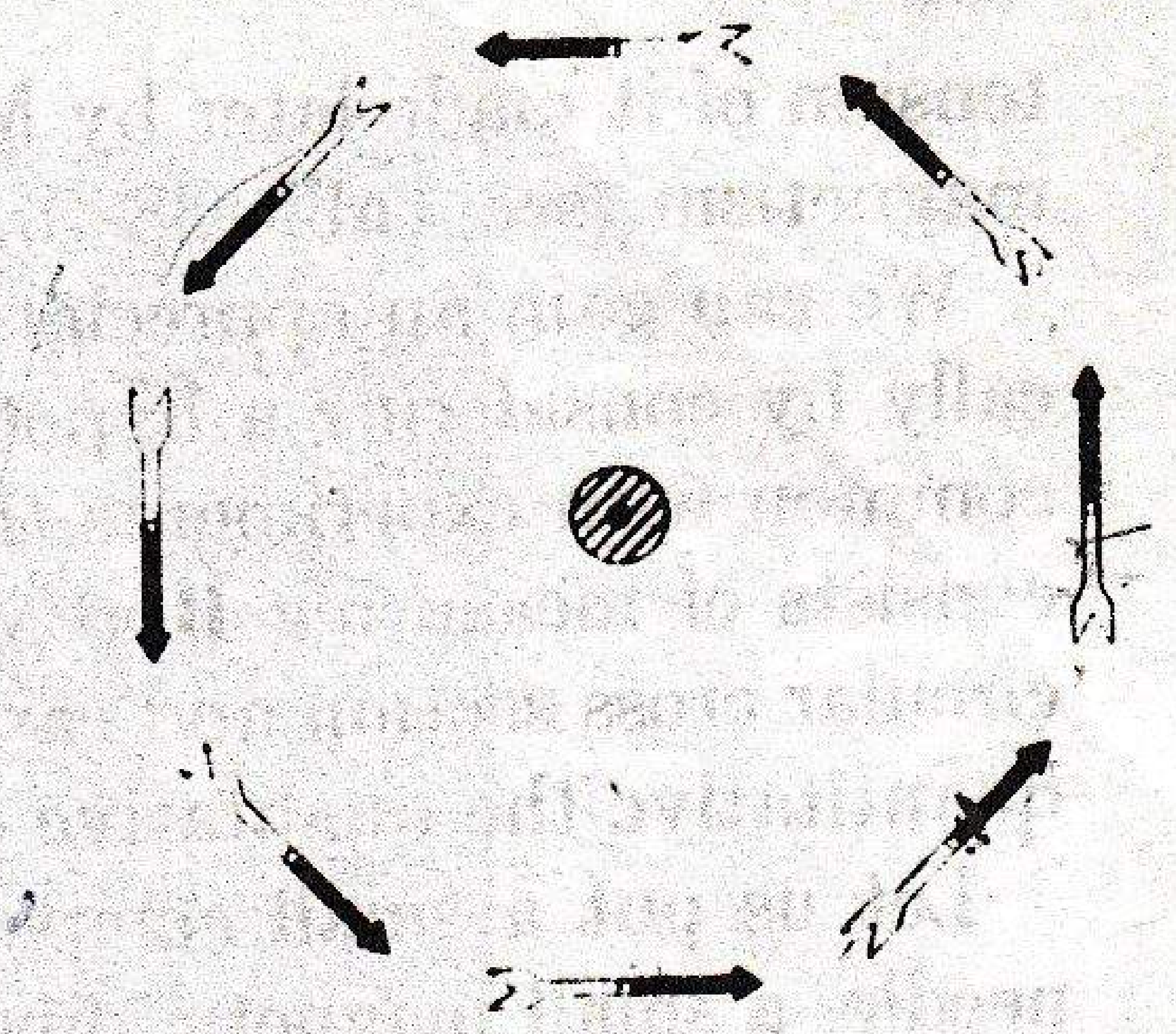

Fig. 34–1 An array of compass needles near a wire carrying a strong current. The black ends of the compass needles are their north poles. The dot shows the current emerging from the page. As usual, the direction of a current is taken as the direction of flow of positive charge.

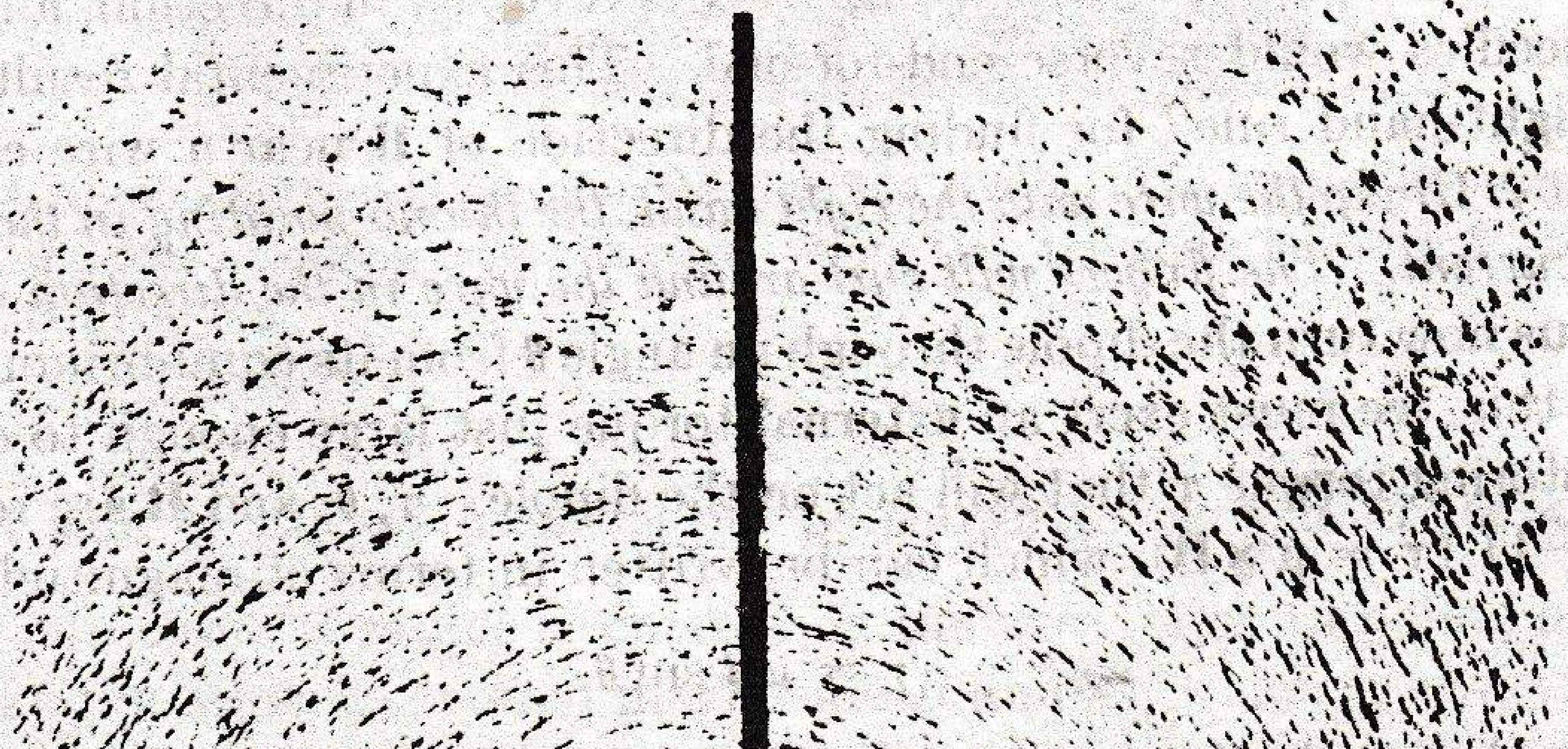

Fig. 34–2 Iron filings around a wire carrying a strong current. (Courtesy Physical Science Study Committee.)

tension of it made later by Maxwell, is one of the basic equations of electromagnetism (see Table 38–3).

We can gain an appreciation of the way Ampère's law developed historically by considering a hypothetical experiment which has, in fact, much in common with experiments that were actually carried out. The experiment consists of measuring **B** at various distances r from a long straight wire of circular cross section and carrying a current i. This can be done by making quantitative the qualitative observation of Fig. 34–1.

Let us put a small compass needle a distance r from the wire. Such a needle, a small magnetic dipole, tends to line up with an external magnetic field, with its north pole pointing in the direction of **B**. Figure 34–1 makes it clear that **B** at the site of the dipole is tangent to a circle of radius r centered on the wire.

If the current in the wire of Fig. 34–1 is reversed in direction, all the compass needles would reverse end-for-end. This experimental result leads to the "right-hand rule" for finding the direction of **B** near a wire carrying a current i: *Grasp the wire with the right hand, the thumb pointing in the direction of the current. The fingers will curl around the wire in the direction of* **B**.

Let us now turn the dipole through an angle θ from its equilibrium position. To do this, we must exert an external torque just large enough to overcome the restoring torque $\boldsymbol{\tau}$ that will act on the dipole. $\boldsymbol{\tau}$, θ, and **B** are related by Eq. 33–11 ($\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$), which can be written in terms of magnitudes as

$$\tau = \mu B \sin\theta \tag{34–2}$$

and in which μ is the magnitude of the magnetic moment of the dipole, θ being the angle between the vectors $\boldsymbol{\mu}$ and **B**. Even though we may not know the value of μ for the compass needle, we may take it to be a constant, independent of the position or orientation of the needle. Thus by measuring τ and θ in Eq. 34–2 we can obtain a *relative* measure of B for various distances r and for various currents i in the wire. The experimental results can be described by the proportionality

$$B \propto \frac{i}{r}. \tag{34–3}$$

We can convert this proportionality into an equality by inserting a proportionality constant. As for Coulomb's law, and for similar reasons (see Section 26–4), we do not write this constant simply as k but in a more complex form, namely $\mu_0/2\pi$, in which μ_0 is called the *permeability constant.** Equation 34–3 then becomes

$$B = \frac{\mu_0 i}{2\pi r}, \tag{34–4}$$

which we choose to write in the form

$$(B)(2\pi r) = \mu_0 i. \tag{34–5}$$

* μ_0 has no connection with the dipole moment μ that appears in Eq. 34–2.

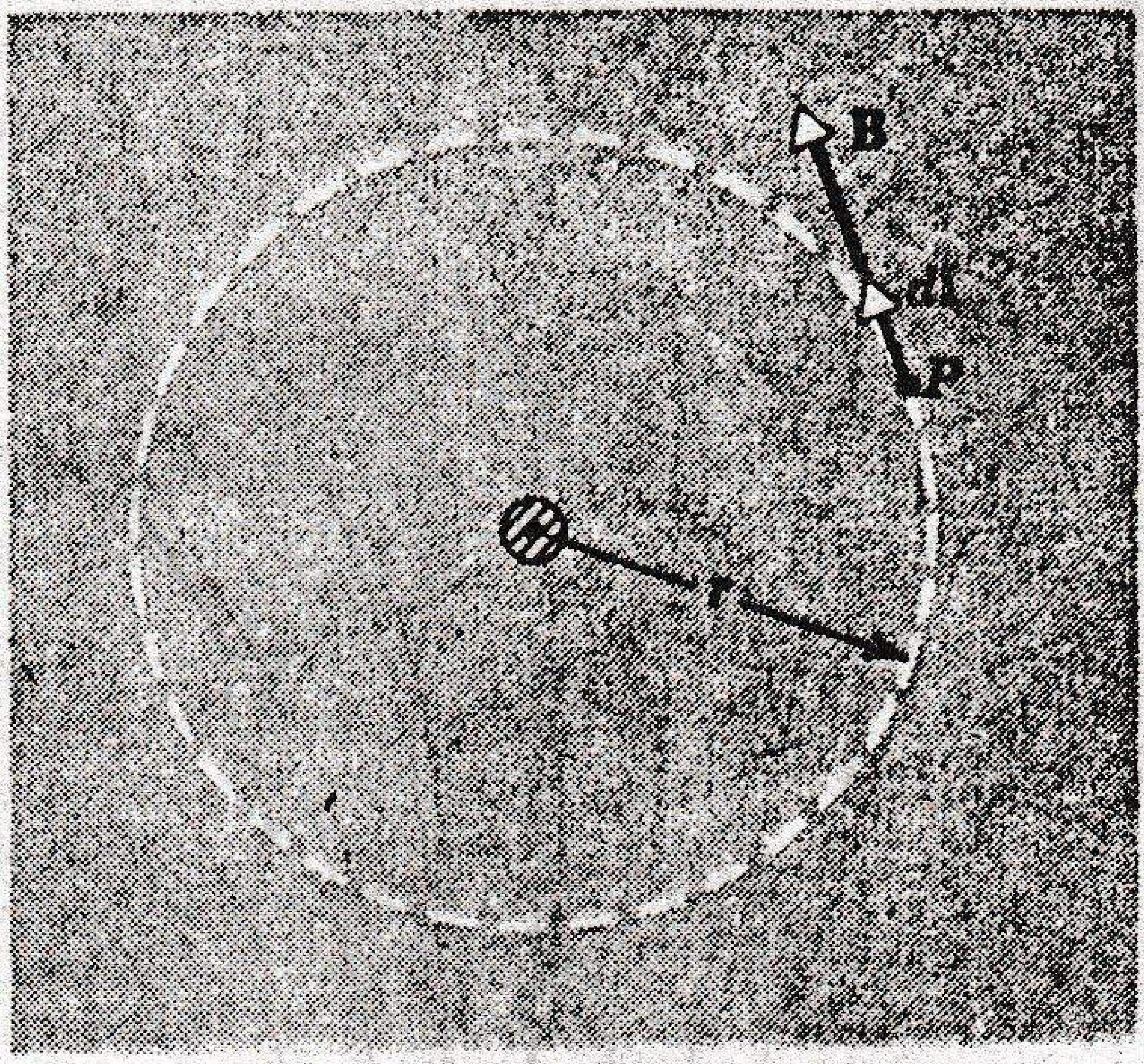

Fig. 34-3 A circular path of integration surrounding a wire. The central dot suggests a current i in the wire emerging from the page. Note that the angle between **B** and $d\mathbf{l}$ is zero so that $\mathbf{B}\cdot d\mathbf{l} = B\,dl$.

The left side of Eq. 34-5 can easily be shown to be $\oint \mathbf{B}\cdot d\mathbf{l}$ for a path consisting of a circle of radius r centered on the wire. For all points on this circle **B** has the same (constant) magnitude B and $d\mathbf{l}$, which is always tangent to the path of integration, points in the same direction as **B**, as Fig. 34-3 shows. Thus

$$\oint \mathbf{B}\cdot d\mathbf{l} = \oint B\,dl = B\oint dl = (B)(2\pi r),$$

$\oint dl$ being simply the circumference of the circle. *In this special case*, therefore, we can write the experimentally observed connection between the field and the current as

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i, \tag{34-1}$$

which is Ampère's law. A host of other experiments suggests that Eq. 34-1 is true in general * for *any* magnetic field configuration, for *any* assembly of currents, and for *any* path of integration.

In applying Ampère's law in the general case, we construct a *closed linear path* in the magnetic field as shown in Fig. 34-4. This path is divided into elements of length $d\mathbf{l}$, and for each element the quantity $\mathbf{B}\cdot d\mathbf{l}$ is evaluated. Recall that $\mathbf{B}\cdot d\mathbf{l}$ has the magnitude $B\,dl\cos\theta$ and can be interpreted as the product of dl and the component of **B** ($= B\cos\theta$) parallel to $d\mathbf{l}$. The inte-

* We must modify Eq. 34-1 if a time-varying electric field is present within the path of integration. In this chapter we assume that if electric fields are present, they are constant in magnitude and direction.

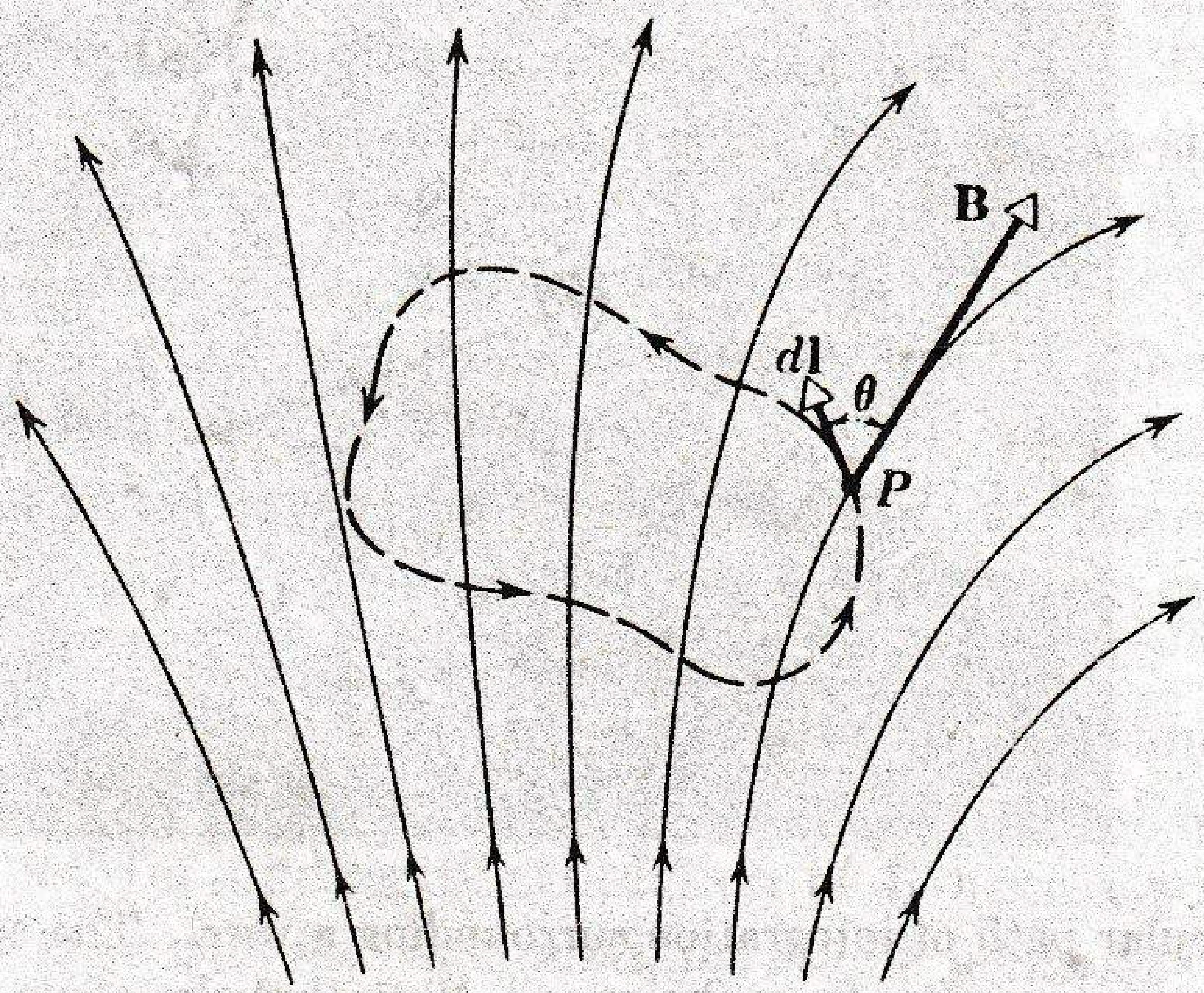

Fig. 34-4 A path of integration in a magnetic field.

gral is the sum of the quantities $\mathbf{B}\cdot d\mathbf{l}$ for all path elements in the complete loop. The term i on the right of Eq. 34–1 is the *net* current that passes through the area bounded by the closed path.

The permeability constant in Ampère's law has an assigned value of

$$\mu_0 = 4\pi \times 10^{-7} \text{ weber/amp-meter}$$

Both this and the permittivity constant (ϵ_0) occur in electromagnetic formulas when the mks system of units is used.

The student may wonder why ϵ_0 in Coulomb's law is a measured quantity, whereas μ_0 in Ampère's law is an assigned quantity. The answer is that the ampere, which is the mks unit for the current i in Ampère's law, is defined by a laboratory technique (the *current balance*) that involves forces exerted by magnetic fields and in which this same constant μ_0 appears. In effect, as we show in detail in Section 34–4, the size of the current that we agree to define as one ampere is adjusted so that μ_0 may have exactly the value assigned to it above. In Coulomb's law, on the other hand, the quantities $\mathbf{F}$, q, and r are measured in ways in which the constant ϵ_0 plays no role. This constant must then take on the particular value that makes the left side of Coulomb's law equal to the right side; no arbitrary assignment is possible.

34–2 B Near a Long Wire

We have seen that the lines of magnetic induction for a long straight wire carrying a current i are concentric circles centered on the wire and that B at a distance r from the wire is given by Eq. 34–4:

$$B = \frac{\mu_0 i}{2\pi r}. \tag{34-4}$$

We may regard this as an experimental result consistent with, and readily derivable from, Ampère's law.

It is interesting to compare Eq. 34–4 with the expression for the electric field near a long line of charge, or

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}. \tag{28-7}$$

In each case there are multiplying constants, namely $\mu_0/2\pi$ and $1/2\pi\epsilon_0$, and factors describing the device responsible for the field, namely i and λ. Finally, each field varies as $1/r$.

Equation 28–7 may be derived from Gauss's law by relating the electric field at a Gaussian surface to the net charge within this surface. The (surface) integral in Gauss's law is evaluated for a closed cylindrical surface to which the lines of **E** are everywhere perpendicular.

Equation 34–4 may be derived from Ampère's law by relating the magnetic field at a path of integration to the net current that pierces this path. The (line) integral in Ampère's law is evaluated for a closed circular path to which the lines of **B** are everywhere tangent.

▶ **Example 1** Derive an expression for **B** at a distance r from the center of a long cylindrical wire of radius R, where $r < R$. The wire carries a current i_0, distributed uniformly over the cross section of the wire.

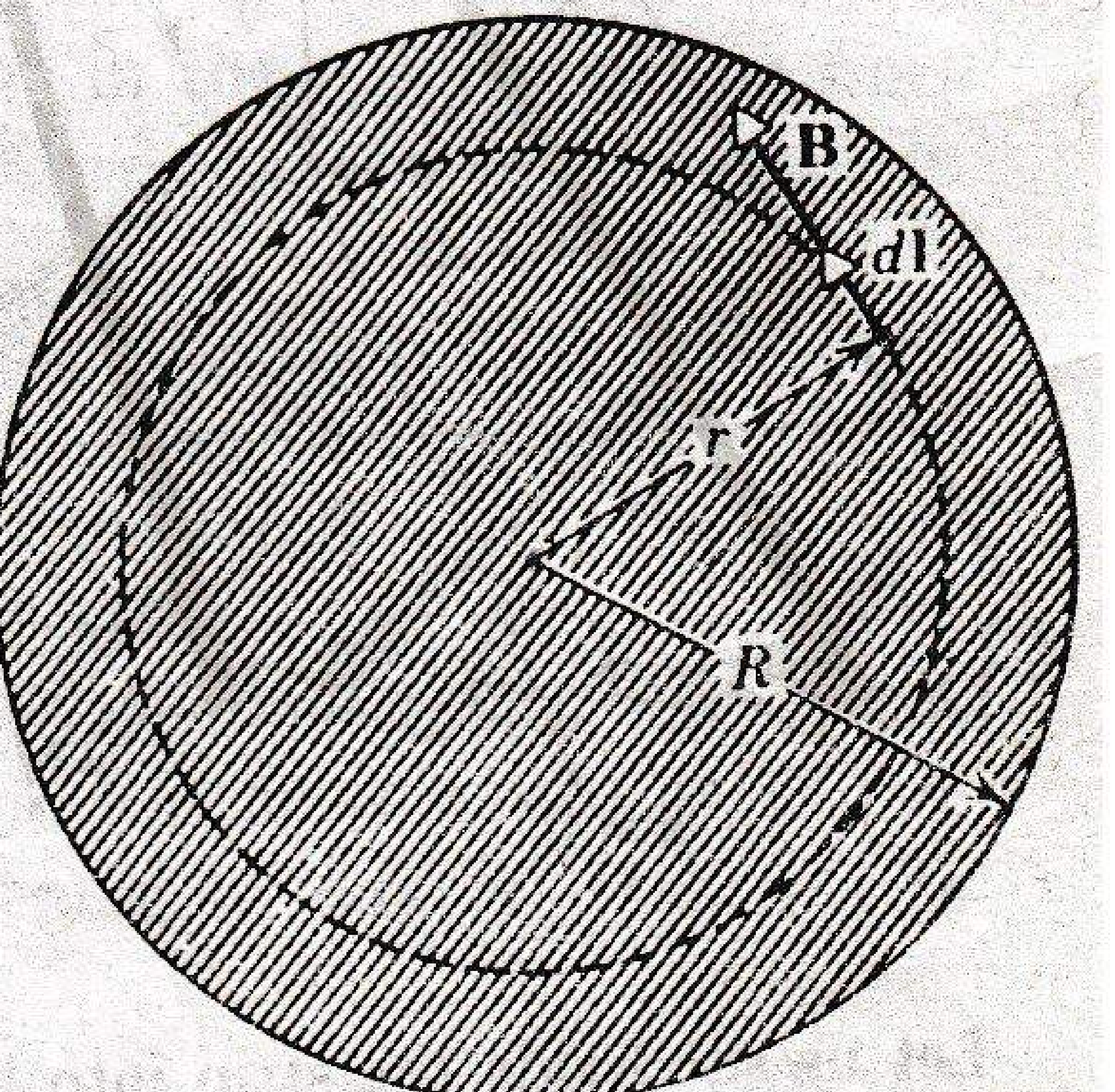

Fig. 34–5 Example 1. A circular path of integration inside a wire. A current i_0, distributed uniformly over the cross section of the wire, emerges from the page.

Figure 34–5 shows a circular path of integration inside the wire. Symmetry suggests that **B** is tangent to the path as shown. Ampère's law,

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i,$$

gives

$$(B)(2\pi r) = \mu_0 i_0 \frac{\pi r^2}{\pi R^2},$$

since only the fraction of the current that passes through the path of integration is included in the factor i on the right. Solving for B and dropping the subscript on the current yields

$$B = \frac{\mu_0 i r}{2\pi R^2}.$$

At the surface of the wire ($r = R$) this equation reduces to the same expression as that found by putting $r = R$ in Eq. 34–4 ($B = \mu_0 i/2\pi R$).

Example 2. Figure 34-6 shows a flat strip of copper of width a and negligible thickness carrying a current i. Find the magnetic field at a distance R from the center of the strip, at right angles to the strip.

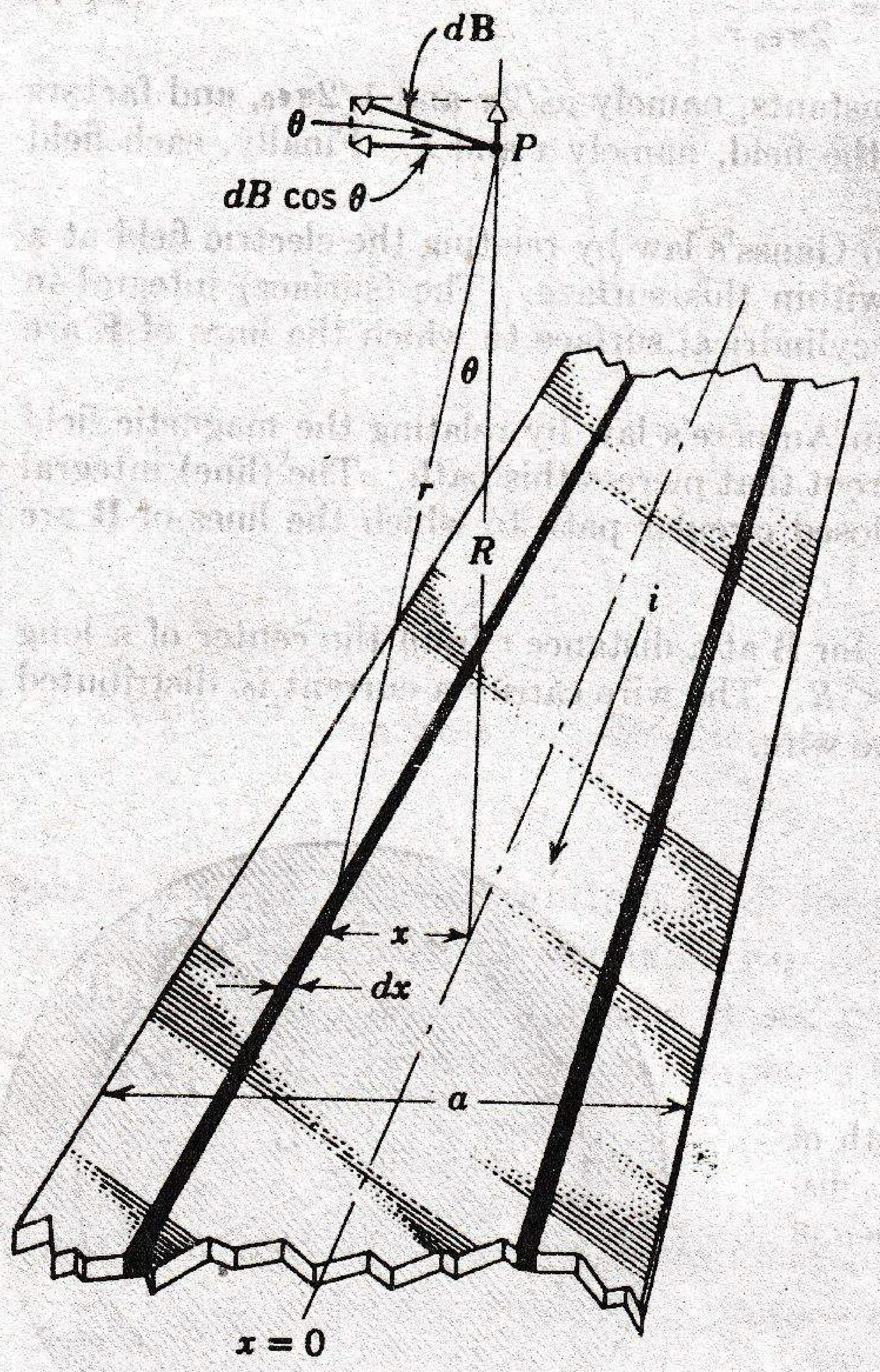

Fig. 34-6 Example 2. A flat strip of width a carries a current i.

Let us subdivide the strip into long infinitesimal filaments of width dx, each of which may be treated as a wire carrying a current di given by $i(dx/a)$. The field contribution dB at point P in Fig. 34-6 is given, for the element shown, by the differential form of Eq. 34-4, or

$$dB = \frac{\mu_0}{2\pi}\frac{di}{r} = \frac{\mu_0}{2\pi}\frac{i(dx/a)}{R \sec\theta}.$$

Note that the vector $\mathbf{dB}$ is at right angles to the line marked r.

Only the horizontal component of $\mathbf{dB}$, namely $dB \cos\theta$, is effective, the vertical component being canceled by the contribution associated with a symmetrically located filament on the other side of the origin. Thus B at point P is given by the (scalar) integral

$$B = \int dB \cos\theta = \int \frac{\mu_0 i(dx/a)}{2\pi R \sec\theta} \cos\theta$$

$$= \frac{\mu_0 i}{2\pi a R}\int \frac{dx}{\sec^2\theta}.$$

The variables x and θ are not independent, being related by

$$x = R \tan \theta$$

or

$$dx = R \sec^2 \theta \, d\theta.$$

Bearing in mind that the limits on θ are $\pm\tan^{-1}(a/2R)$ and eliminating dx from this expression for B, we find

$$B = \frac{\mu_0 i}{2\pi a R} \int \frac{R \sec^2 \theta \, d\theta}{\sec^2 \theta}$$

$$= \frac{\mu_0 i}{2\pi a} \int_{-\tan^{-1} a/2R}^{+\tan^{-1} a/2R} d\theta = \frac{\mu_0 i}{\pi a} \tan^{-1} \frac{a}{2R}.$$

At points far from the strip, $a/2R$ is a small angle, for which $\tan^{-1}\alpha \cong \alpha$. Thus we have, as an approximate result,

$$B \cong \frac{\mu_0 i}{\pi a}\left(\frac{a}{2R}\right) = \frac{\mu_0}{2\pi} \frac{i}{R}.$$

This result is expected because at distant points the strip cannot be distinguished from a cylindrical wire (see Eq. 34-4).

34-3 Magnetic Lines of Induction

Figure 34-7 shows the lines of magnetic induction representing the field of **B** near a long straight wire. Note the increase in the spacing of the lines with increasing distance from the wire. This represents the $1/r$ decrease in B predicted by Eq. 34-4.

Figure 34-8 shows the resultant lines of magnetic induction associated with a current in a wire that is oriented at right angles to a uniform *external* field $\mathbf{B}_e$. At any point the resultant magnetic induction **B** will be the vector sum of $\mathbf{B}_e$ and $\mathbf{B}_i$, where $\mathbf{B}_i$ is the magnetic induction set up by the current in the wire. The fields $\mathbf{B}_e$ and $\mathbf{B}_i$ tend to cancel above the wire and to reenforce each other below the wire. At point P in Fig. 34-8 $\mathbf{B}_e$ and $\mathbf{B}_i$ cancel exactly. Very near the wire the field is represented by circular lines and is essentially $\mathbf{B}_i$.

Michael Faraday, who originated the concept of lines of induction, endowed them with more reality than they are currently given. He imagined

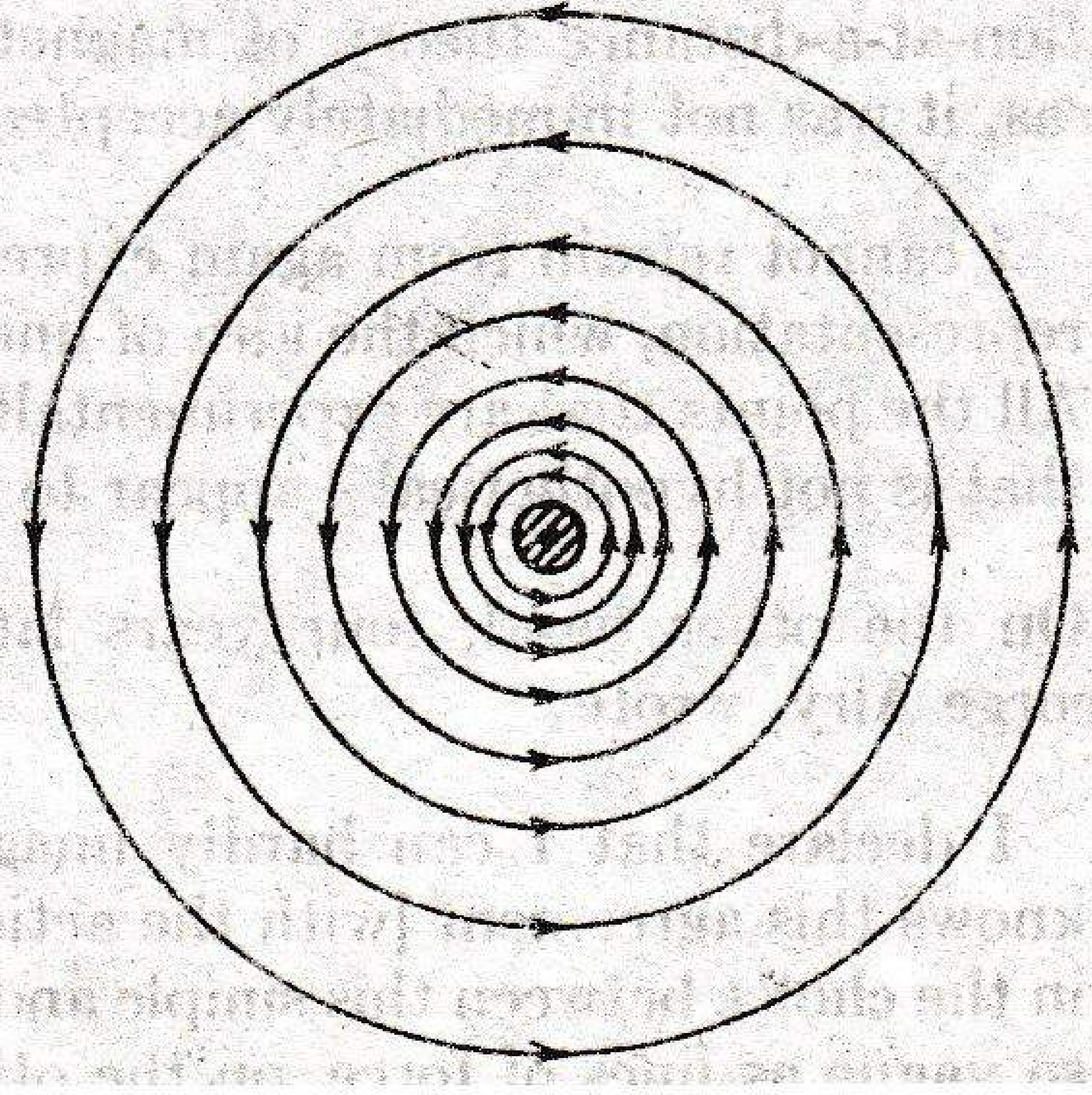

Fig. 34-7 Lines of **B** near a long cylindrical wire. A current i, suggested by the central dot, emerges from the page.

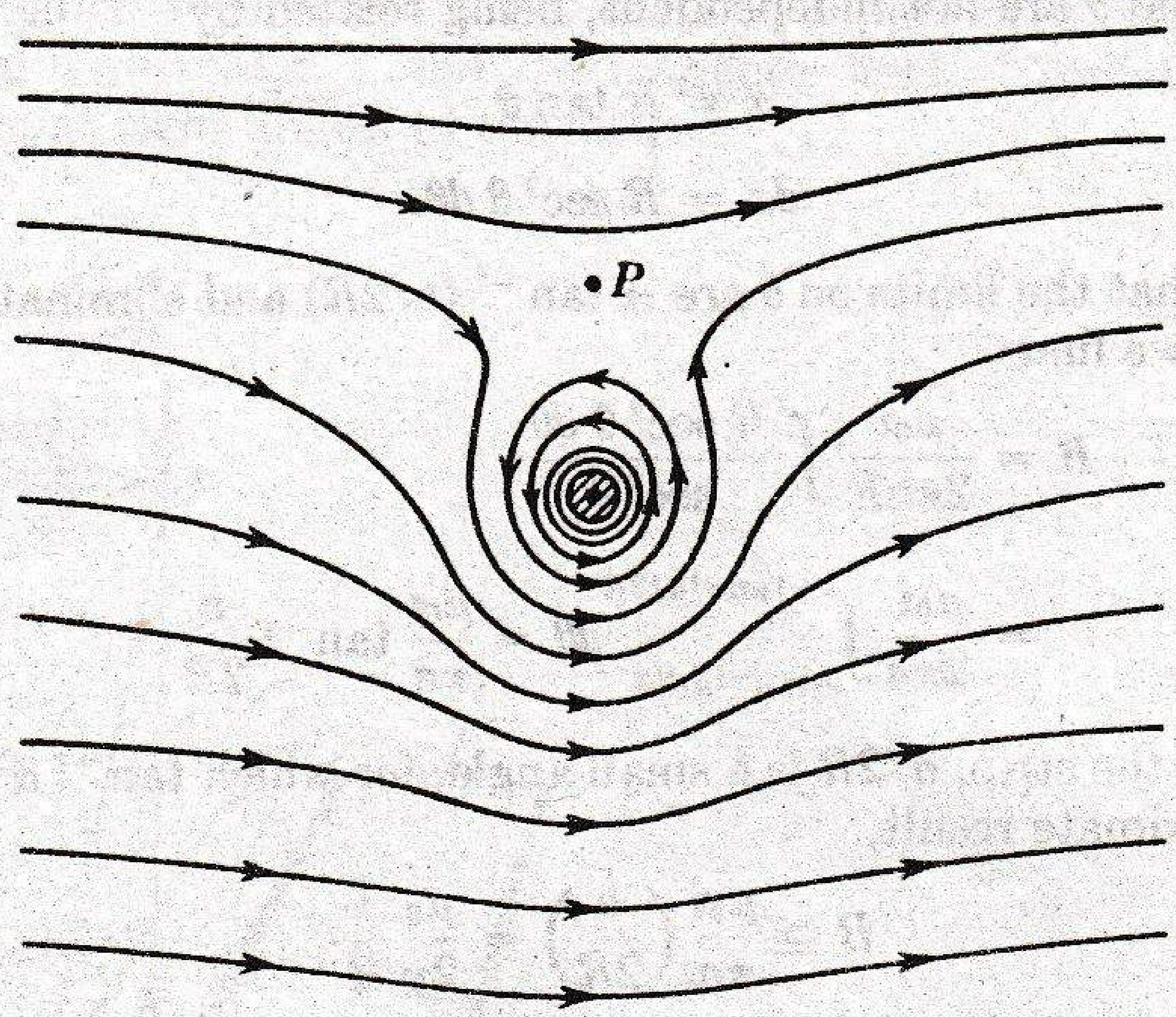

Fig. 34–8 Lines of **B** near a long current-carrying wire immersed in a uniform external field $\mathbf{B}_e$ that points to the right. The current i is emerging from the page.

that, like stretched rubber bands, they represent the site of mechanical forces. On this picture can we not visualize that the wire in Fig. 34–8 will be pushed up? Today we use lines of induction largely for purposes of visualization. For quantitative calculations we use the field vectors, describing the force on the wire in Fig. 34–8, for example, from the relation $\mathbf{F} = i\mathbf{l} \times \mathbf{B}$.

In applying this relation to Fig. 34–8, we recall that **B** is always the *external field* in which the wire is immersed; that is, it is $\mathbf{B}_e$ and thus points to the right. Since **l** points out of the page, the magnetic force on the wire ($= i\mathbf{l} \times \mathbf{B}_e$) does indeed point up. It is necessary to use only the external field in such calculations because the field set up by the current in the wire cannot exert a force on the wire, just as the gravitational field of the earth cannot exert a force on the earth itself but only on another body. In Fig. 34–7, for example, there is no magnetic force on the wire because no external magnetic field is present.

Faraday's idea of lines of induction was instrumental in overthrowing the older action-at-a-distance theory of magnetic (and electric) attraction. Like many new ideas, it was not immediately accepted. In 1851, for example, Faraday wrote:

> I cannot refrain from again expressing my conviction of the truthfulness of the representation, which the idea of lines of force affords in regard to magnetic action. All the points that are experimentally established in regard to that action—i.e., all that is not hypothetical—appear to be well and truly represented by it.

On the other hand, four years later another well-known British scientist, Sir George Airy, wrote:

> I declare that I can hardly imagine anyone who practically and numerically knows this agreement [with the action-at-a-distance theory] to hesitate an instant in the choice between this simple and precise action, on the one hand, and anything so vague as lines of force, on the other hand.

34-4 Two Parallel Conductors

Figure 34-9 shows two long parallel wires separated by a distance d and carrying currents i_a and i_b. It is an experimental fact, noted by Ampère only one week after word of Oersted's experiments reached Paris, that two such conductors attract each other.

Some of Ampère's colleagues thought that in view of Oersted's experiment this attraction between two conductors was an obvious result and did not need to be proved. They reasoned that if wire a and wire b each exert forces on a compass needle they should exert forces on each other. This conclusion is wrong. When he heard it, Arago, a contemporary of Ampère, drew two iron keys from his pocket and replied, "Each of these keys attracts a magnet. Do you believe that they therefore also attract each other?"

Wire a in Fig. 34-9 will produce a field of induction $\mathbf{B}_a$ at all nearby points. The magnitude of $\mathbf{B}_a$, due to the current i_a, at the site of the second wire is, from Eq. 34-4,

$$B_a = \frac{\mu_0 i_a}{2\pi d}.$$

The right-hand rule shows that the direction of $\mathbf{B}_a$ at wire b is down, as shown in the figure.

Wire b, which carries a current i_b, finds itself immersed in an *external* field of magnetic induction $\mathbf{B}_a$. A length l of this wire will experience a sideways magnetic force ($= i\mathbf{l} \times \mathbf{B}$) whose magnitude is

$$F_b = i_b l B_a = \frac{\mu_0 l i_b i_a}{2\pi d}. \qquad (34\text{-}6)$$

The vector rule of signs tells us that $\mathbf{F}_b$ lies in the plane of the wires and points to the left in Fig. 34-9.

We could have started with wire b, computed the field of induction which it produces at the site of wire a, and then computed the force on wire a. The

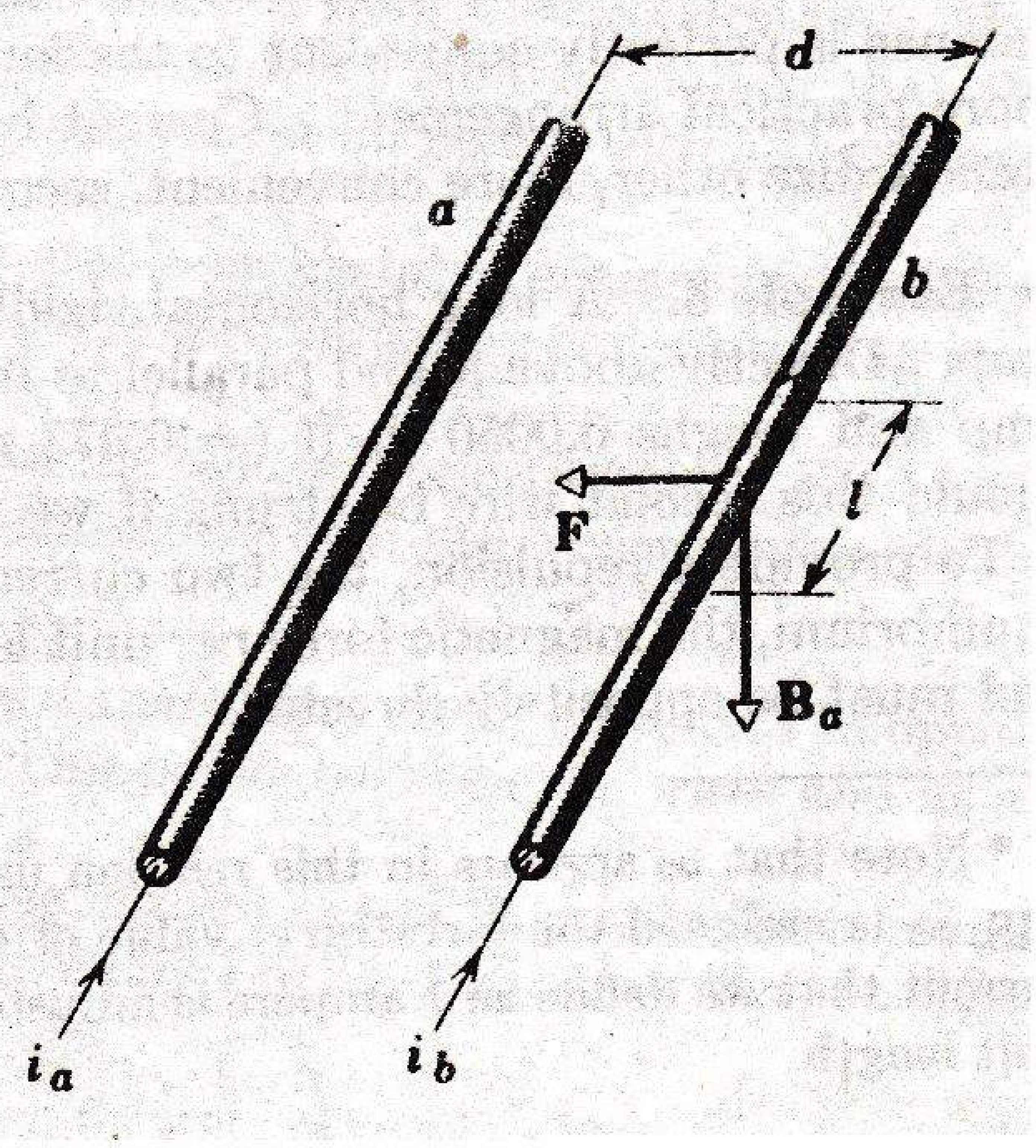

Fig. 34-9 Two parallel wires that carry parallel currents attract each other.

force on wire a would, for parallel currents, point to the right. The forces that the two wires exert on each other are equal and opposite, as they must be according to Newton's law of action and reaction. For antiparallel currents the two wires repel each other.

This discussion reminds us of our discussion of the electric field between two point charges in Section 27–1. There we saw that the charges act on each other through the intermediary of the electric field. The conductors in Fig. 34–9 act on each other through the intermediary of the *magnetic* field. We think in terms of

$$\text{current} \rightleftharpoons \text{field}$$

and not, as in the action-at-a-distance point of view, in terms of

$$\text{current} \rightleftharpoons \text{current}.$$

The attraction between long parallel wires is used to define the ampere. Suppose that the wires are 1 meter apart (d = 1.0 meter) and that the two currents are equal ($i_a = i_b = i$). If this common current is adjusted until, by measurement, the force of attraction per unit length between the wires is 2×10^{-7} nt/meter, the current is defined to be 1 *ampere*. From Eq. 34–6,

$$\frac{F}{l} = \frac{\mu_0 i^2}{2\pi d} = \frac{(4\pi \times 10^{-7}\ \text{weber/amp-m})(1\ \text{amp})^2}{(2\pi)(1\ \text{meter})}$$
$$= 2 \times 10^{-7}\ \text{nt/meter}$$

as expected.*

At the National Bureau of Standards primary measurements of current are made with a *current balance*. This consists of a carefully wound coil placed between two other coils, as in Fig. 34–10. The outer pair of coils is fastened to the table, and the inner one is hung from the arm of a balance. The coils are so connected that the current to be measured exists, as a common current, in all three of them.

The coils exert forces on one another—just as the parallel wires of Fig. 34–9 do—which can be measured by loading weights on the balance pan. The current is defined in terms of this measured force and the dimensions of the coils. The current balance is perfectly equivalent to the long parallel wires of Fig. 34–9 but is a much more practical arrangement. Current balance measurements are used primarily to standardize other, more convenient, secondary methods of measuring currents.

▶ **Example 3.** A long horizontal rigidly supported wire carries a current i_a of 100 amp. Directly above it and parallel to it is a fine wire that carries a current i_b of 20 amp and weighs 0.0050 lb/ft (= 0.073 nt/meter). How far above the lower wire should this second wire be strung if we hope to support it by magnetic repulsion?

To provide a repulsion, the two currents must point in opposite directions. For equilibrium, the magnetic force per unit length must equal the weight per unit length and must be oppositely directed.

---

* Note that μ_0 appears in this relation used to define the ampere. As stated on page 848, μ_0 is assigned the (arbitrary) value of $4\pi \times 10^{-7}$ weber/amp-m, and the size of the current that we define as 1 ampere is adjusted to give the required force of attraction per unit length.

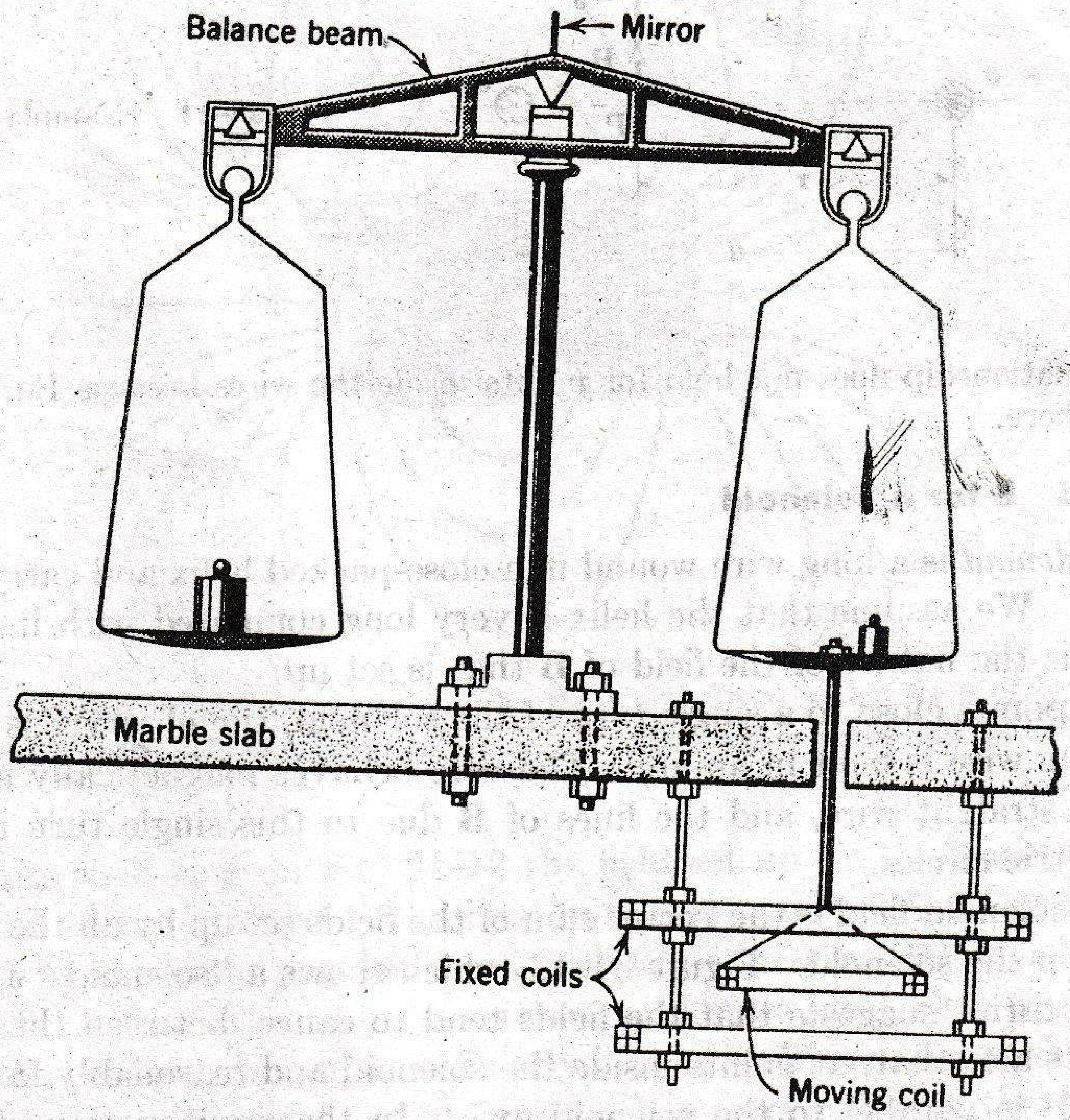

Fig. 34–10 A current balance.

Solving Eq. 34–6 for d yields

$$d = \frac{\mu_0 i_a i_b}{2\pi(F/l)} = \frac{(4\pi \times 10^{-7}\ \text{weber/amp-m})(100\ \text{amp})(20\ \text{amp})}{(2\pi)(0.073\ \text{nt/meter})}$$

$$= 5.5 \times 10^{-3}\ \text{meter} = 5.5\ \text{mm}.$$

We assume that the wire diameters are much smaller than their separation. This assumption is necessary because in deriving Eq. 34–6 we tacitly assumed that the magnetic induction produced by one wire is uniform for all points within the second wire.

Is the equilibrium of the suspended wire stable or unstable against vertical displacements? This can be tested by displacing the wire vertically and examining how the forces on the wire change.

Suppose that the fine wire is suspended *below* the rigidly supported wire. How may it be made to "float"? Is the equilibrium against vertical displacements stable or unstable?

Example 4. Two parallel wires a distance d apart carry equal currents i in opposite directions. Find the magnetic induction for points between the wires and at a distance x from one wire.

Study of Fig. 34–11 shows that $\mathbf{B}_a$ due to the current i_a and $\mathbf{B}_b$ due to the current i_b point in the same direction at P. Each is given by Eq. 34–4 ($B = \mu_0 i/2\pi r$) so that

$$B = B_a + B_b = \frac{\mu_0 i}{2\pi}\left(\frac{1}{x} + \frac{1}{d - x}\right).$$

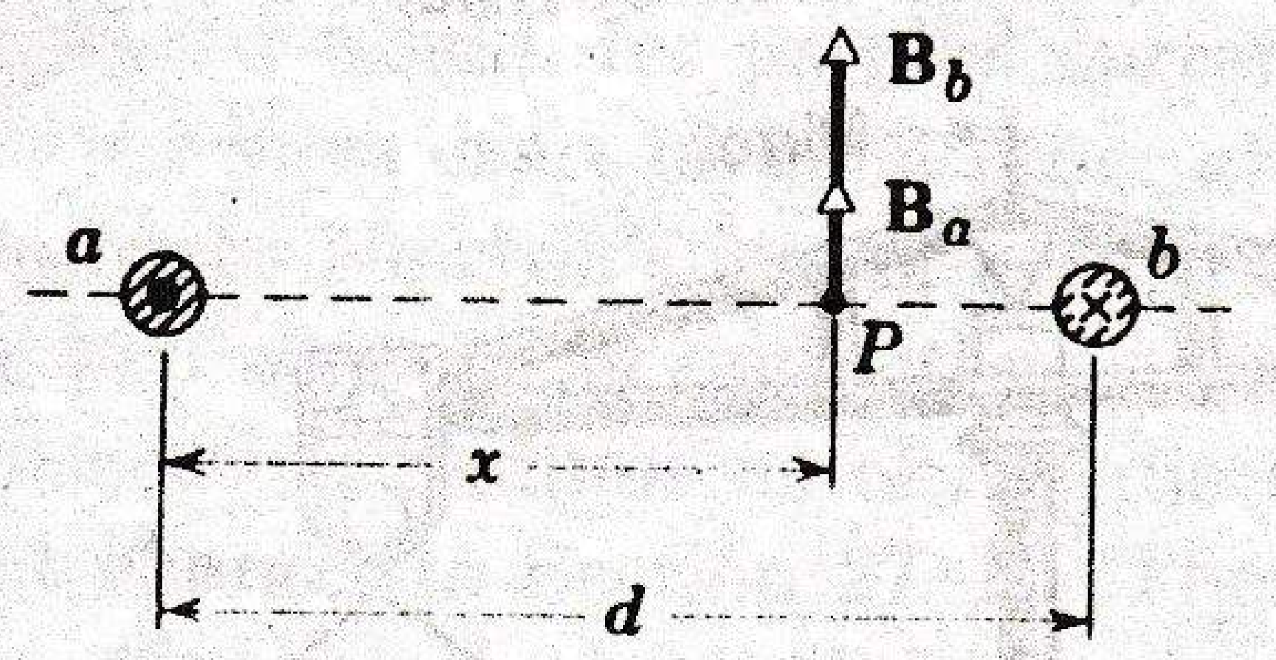

Fig. 34-11 Example 4.

This relationship does not hold for points inside the wires because Eq. 34-4 is not valid there. ◀

34-5 B for a Solenoid

A *solenoid* is a long wire wound in a close-packed helix and carrying a current i. We assume that the helix is very long compared with its diameter. What is the nature of the field of **B** that is set up?

For points close to a single turn of the solenoid, the observer is not aware that the wire is bent in an arc. The wire behaves magnetically almost like a long straight wire, and the lines of **B** due to this single turn are almost concentric circles.

The solenoid field is the vector sum of the fields set up by all the turns that make up the solenoid. Figure 34-12, which shows a "solenoid" with widely spaced turns, suggests that the fields tend to cancel between the wires. It suggests also that, at points inside the solenoid and reasonably far from the wires, **B** is parallel to the solenoid axis. In the limiting case of adjacent square tightly packed wires, the solenoid becomes essentially a cylindrical current sheet and the requirements of symmetry then make the statement just given necessarily true. We assume that it is true in what follows.

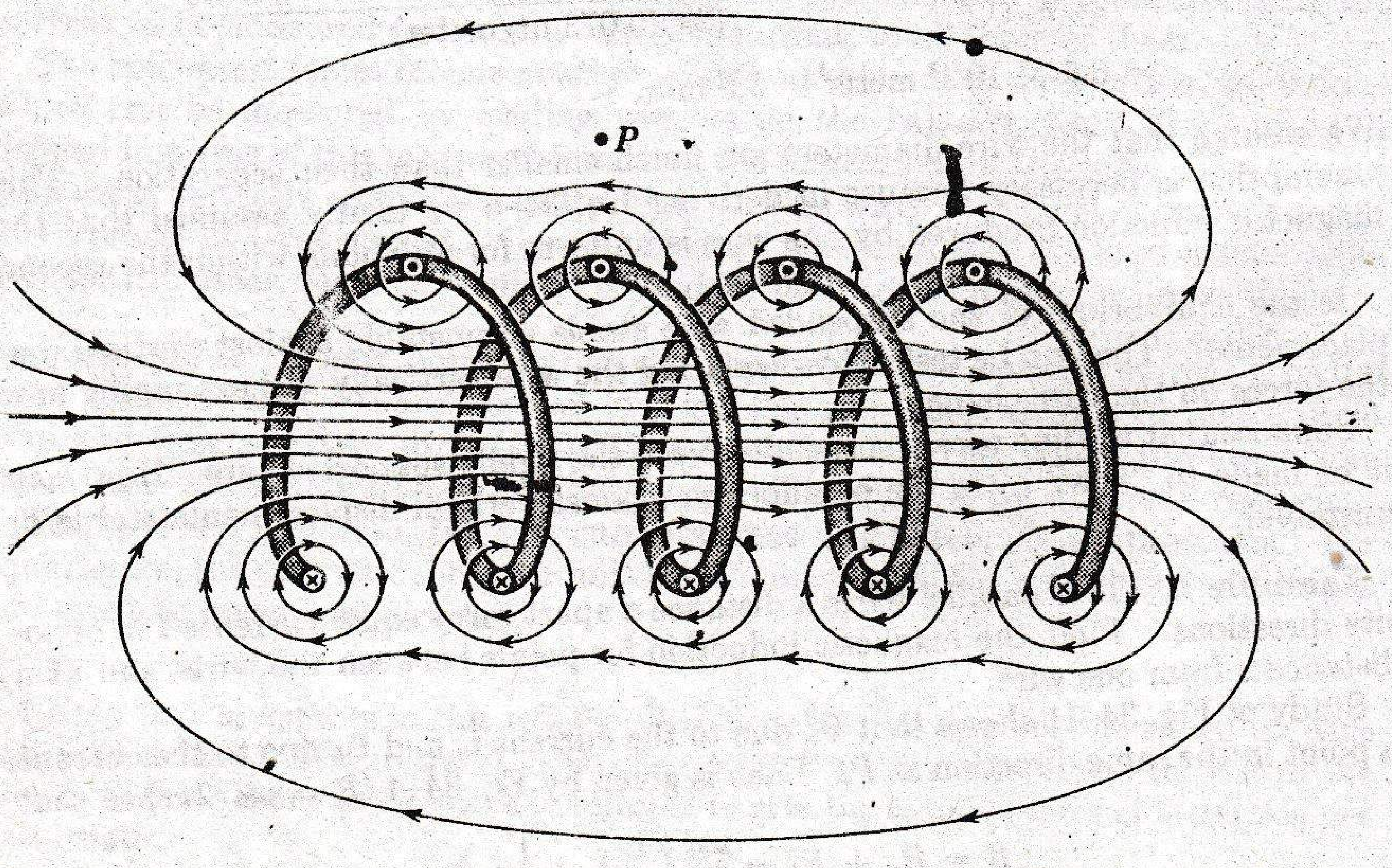

Fig. 34-12 A loosely wound solenoid.

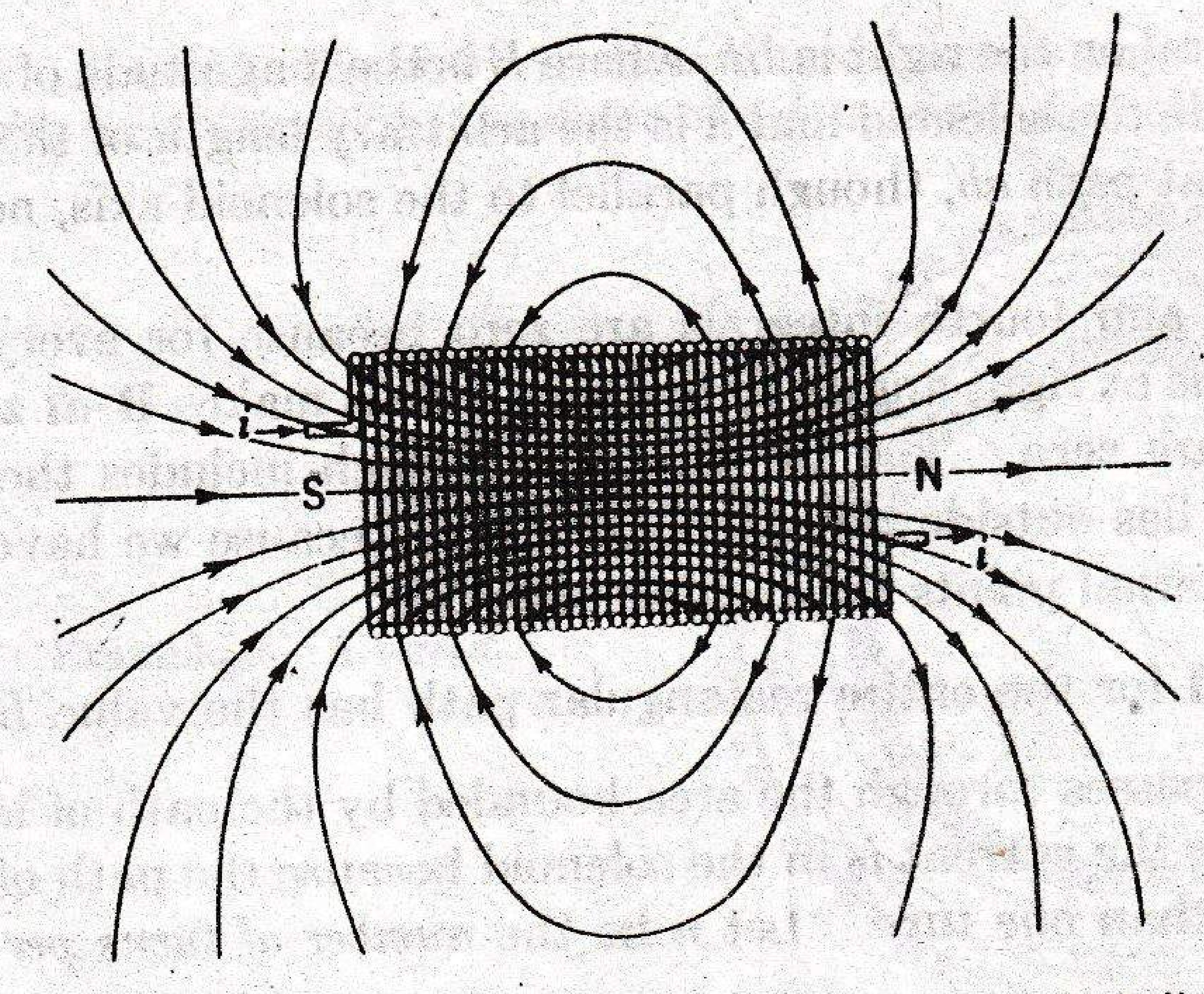

Fig. 34–13 A solenoid of finite length. The right end, from which lines of **B** emerge, behaves like the north pole of a compass needle. The left end behaves like the south pole.

For points such as P in Fig. 34–12 the field set up by the upper part of the solenoid turns (marked ⊙) points to the left and tends to cancel the field set up by the lower part of the solenoid turns (marked ⊗), which points to the right. As the solenoid becomes more and more ideal, that is, as it approaches the configuration of an infinitely long cylindrical current sheet, the field of induction at outside points approaches zero. Taking the external field to be zero is not a bad assumption for a practical solenoid if its length is much greater than its diameter and if we consider only external points near the central region of the solenoid, that is, away from the ends. Figure 34–13 shows the lines of induction for a real solenoid, which is far from ideal in that the length is not much greater than the diameter. Even here the spacing of the lines of induction in the central plane shows that the external field is much weaker than the internal field.

Let us apply Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i,$$

to the rectangular path *abcd* in the ideal solenoid of Fig. 34–14. We write the integral $\oint \mathbf{B} \cdot d\mathbf{l}$ as the sum of four integrals, one for each path segment:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B} \cdot d\mathbf{l} + \int_b^c \mathbf{B} \cdot d\mathbf{l} + \int_c^d \mathbf{B} \cdot d\mathbf{l} + \int_d^a \mathbf{B} \cdot d\mathbf{l}.$$

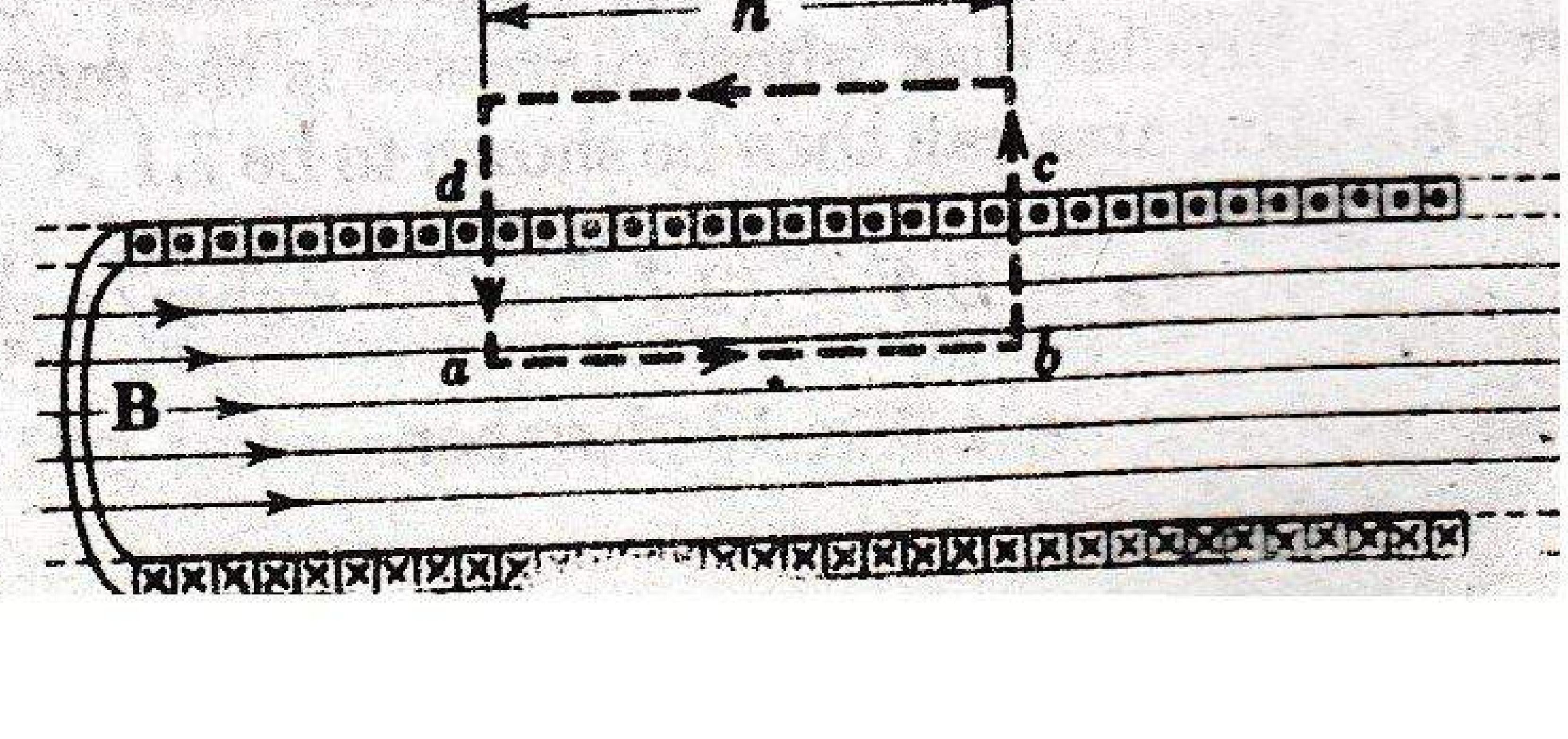

Fig. 34–14 A section of an ideal solenoid, made of adjacent square turns, equivalent to an infinitely long cylindrical current sheet.

The first integral on the right is Bh, where B is the magnitude of the magnetic induction inside the solenoid and h is the arbitrary length of the path from a to b. Note that path ab, though parallel to the solenoid axis, need not coincide with it.

The second and fourth integrals are zero because for every element of these paths **B** is at right angles to the path. This makes $\mathbf{B}\cdot d\mathbf{l}$ zero and thus the integrals are zero. The third integral, which includes the part of the rectangle that lies outside the solenoid, is zero because we have taken **B** as zero for all external points for an ideal solenoid.

Thus $\oint \mathbf{B}\cdot d\mathbf{l}$ for the entire rectangular path has the value Bh. The net current i that passes through the area bounded by the path of integration is not the same as the current i_0 in the solenoid because the path of integration encloses more than one turn. Let n be the *number of turns per unit length;* then

$$i = i_0(nh).$$

Ampère's law then becomes

$$Bh = \mu_0 i_0 nh$$

or

$$B = \mu_0 i_0 n. \tag{34-7}$$

Although Eq. 34–7 was derived for an infinitely long ideal solenoid, it holds quite well for actual solenoids for internal points near the center of the solenoid. It shows that B does not depend on the diameter or the length of the solenoid and that B is constant over the solenoid cross section. A solenoid is a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor is a practical way to set up a known uniform electric field.

▶ **Example 5.** A solenoid is 1.0 meter long and 3.0 cm in mean diameter. It has five layers of windings of 850 turns each and carries a current of 5.0 amp.

(*a*) What is B at its center? From Eq. 34–7,

$$B = \mu_0 i_0 n = (4\pi \times 10^{-7}\text{ weber/amp-m})(5.0\text{ amp})(5 \times 850\text{ turns/meter})$$
$$= 2.7 \times 10^{-2}\text{ weber/meter}^2.$$

We can use Eq. 34–7 even if the solenoid has more than one layer of windings because the diameter of the windings does not enter.

(*b*) What is the magnetic flux Φ_B for a cross section of the solenoid at its center? To the extent that **B** is constant, we can calculate the flux from

$$\Phi_B = \int \mathbf{B}\cdot d\mathbf{S} = BA,$$

where A is the effective cross-sectional area. Let us assume that A represents the area of a circular disk whose diameter is the mean diameter of the windings (3.0 cm). The effective area can then be shown to be 7.1×10^{-4} meter², and

$$\Phi_B = BA = (2.7 \times 10^{-2}\text{ weber/meter}^2)(7.1 \times 10^{-4}\text{ meter}^2)$$
$$= 1.9 \times 10^{-5}\text{ weber}.$$

Example 6. *A toroid.* Figure 34–15 shows a toroid, which may be described as a solenoid of finite length bent into the shape of a doughnut. Calculate **B** at interior points.

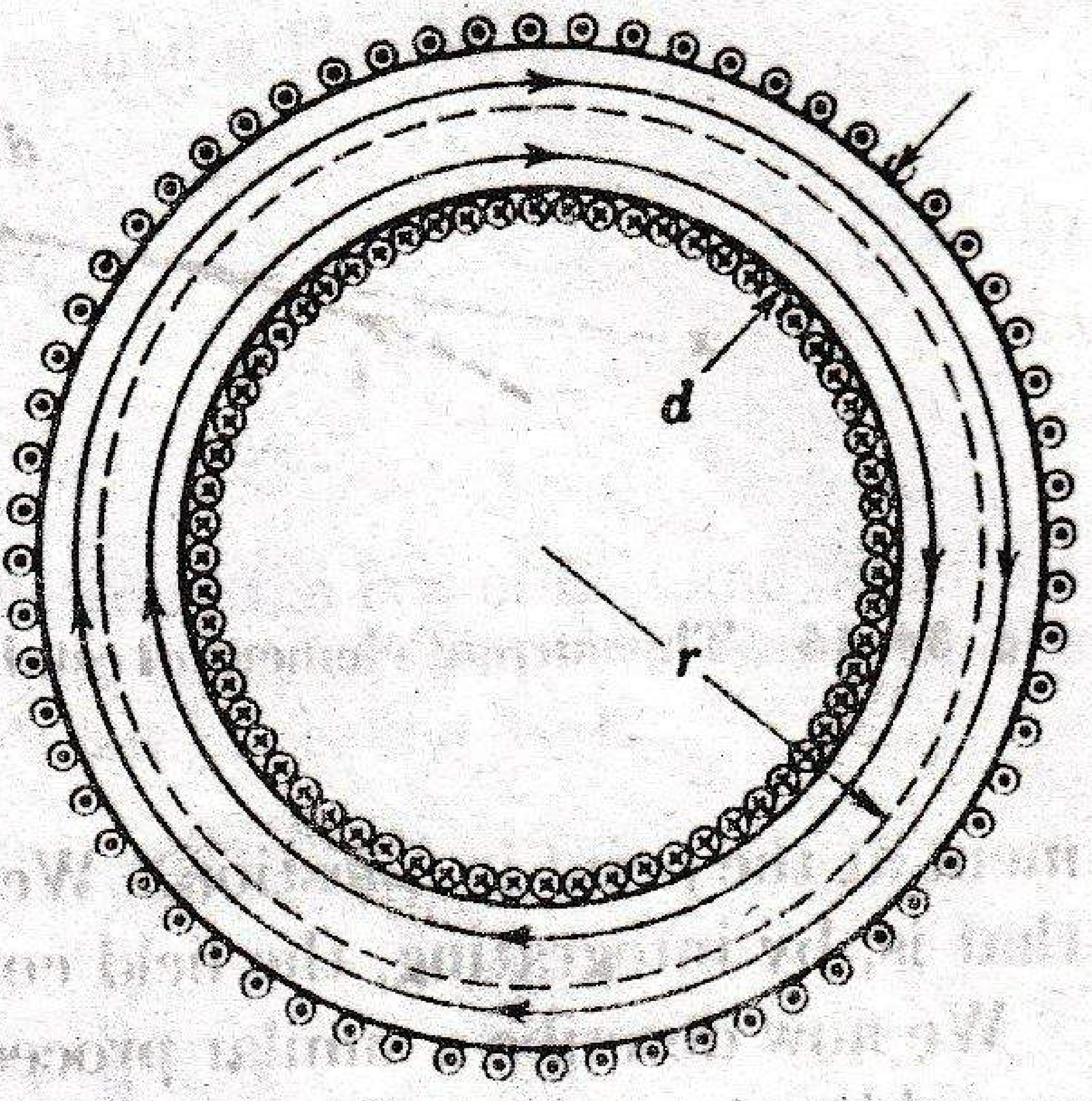

Fig. 34–15 Example 6. A toroid.

From symmetry the lines of **B** form concentric circles inside the toroid, as shown in the figure. Let us apply Ampère's law to the circular path of integration of radius r:

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i$$

or

$$(B)(2\pi r) = \mu_0 i_0 N,$$

where i_0 is the current in the toroid windings and N is the total number of turns. This gives

$$B = \frac{\mu_0}{2\pi}\frac{i_0 N}{r}.$$

In contrast to the solenoid, B is not constant over the cross section of a toroid. Show from Ampère's law that B equals zero for points outside an ideal toroid. ◀

34–6 The Biot-Savart Law

Ampère's law can be used to calculate magnetic fields only if the symmetry of the current distribution is high enough to permit the easy evaluation of the line integral $\oint \mathbf{B}\cdot d\mathbf{l}$. This requirement limits the usefulness of the law in practical problems. The law does not fail; it simply becomes difficult to apply in a useful way.

Similarly, in electrostatics, Gauss's law can be used to calculate electric fields only if the symmetry of the charge distribution is high enough to permit the easy evaluation of the surface integral $\oint \mathbf{E}\cdot d\mathbf{S}$. We can, for example, use Gauss's law to find the electric field due to a long uniformly charged rod but we cannot apply it usefully to an electric dipole, for the symmetry is not high enough in this case.

To compute **E** at a given point for an *arbitrary* charge distribution, we divided the distribution into *charge elements dq* and (see Section 27–4) we used Coulomb's law to calculate the field contribution $d\mathbf{E}$ due to each ele-

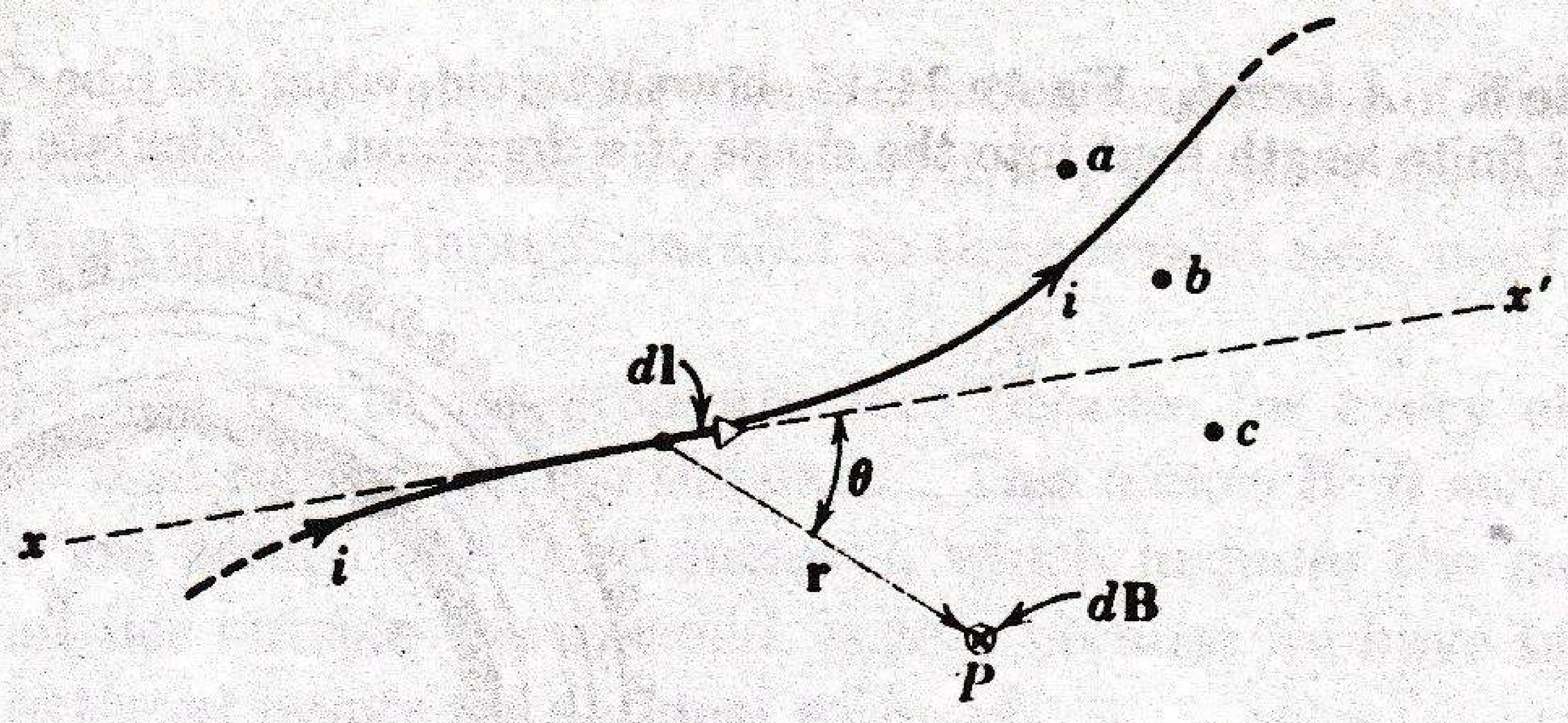

Fig. 34–16 The current element $d\mathbf{l}$ establishes a magnetic field contribution $d\mathbf{B}$ at point P.

ment at the point in question. We found the field **E** at that point by adding, that is, by integrating, the field contributions $d\mathbf{E}$ for the entire distribution.

We now describe a similar procedure for computing **B** at any point due to an arbitrary current distribution. We divide the current distribution into *current elements* and, using the law of Biot and Savart (which we describe below), we calculate the field contribution $d\mathbf{B}$ due to each current element at the point in question. We find the field **B** at that point by integrating the field contributions for the entire distribution.

Figure 34–16 shows an arbitrary current distribution consisting of a current i in a curved wire. The figure also shows a typical current element; it is a length $d\mathbf{l}$ of the conductor carrying a current i. Its direction is that of the tangent to the conductor (dashed line). A current element cannot exist as an isolated entity because a way must be provided to lead the current into the element at one end and out of it at the other. Nevertheless, we can think of an actual circuit as made up of a large number of current elements placed end to end.

Let P be the point at which we want to know the magnetic induction $d\mathbf{B}$ associated with the current element. According to the Biot-Savart law, $d\mathbf{B}$ is given in magnitude by

$$dB = \frac{\mu_0 i}{4\pi} \frac{dl \sin\theta}{r^2}, \tag{34–8}$$

where **r** is a displacement vector from the element to P and θ is the angle between this vector and $d\mathbf{l}$. The direction of $d\mathbf{B}$ is that of the vector $d\mathbf{l} \times \mathbf{r}$. In Fig. 34–16, for example, $d\mathbf{B}$ at point P for the current element shown is directed into the page at right angles to the plane of the figure. Note that Eq. 34–8, being an inverse square law that describes the magnetic induction due to a current element, may be viewed as the magnetic equivalent of Coulomb's law, which is an inverse square law that describes the electric field due to a charge element.

The law of Biot and Savart may be written in vector form as

$$d\mathbf{B} = \frac{\mu_0 i}{4\pi} \frac{d\mathbf{l} \times \mathbf{r}}{r^3}. \tag{34–9}$$

This formulation reduces to that of Eq. 34-8 when expressed in terms of magnitudes; it also gives complete information about the direction of $d\mathbf{B}$, namely that it is the same as the direction of the vector $d\mathbf{l} \times \mathbf{r}$.

The resultant field at P is found by integrating Eq. 34-9, or

$$\mathbf{B} = \int d\mathbf{B}, \tag{34-10}$$

where the integral is a vector integral.

▶ **Example 7.** *A long straight wire.* We illustrate the law of Biot and Savart by applying it to find $\mathbf{B}$ due to a current i in a long straight wire. We discussed this problem at length in connection with Ampère's law in Section 34-1.

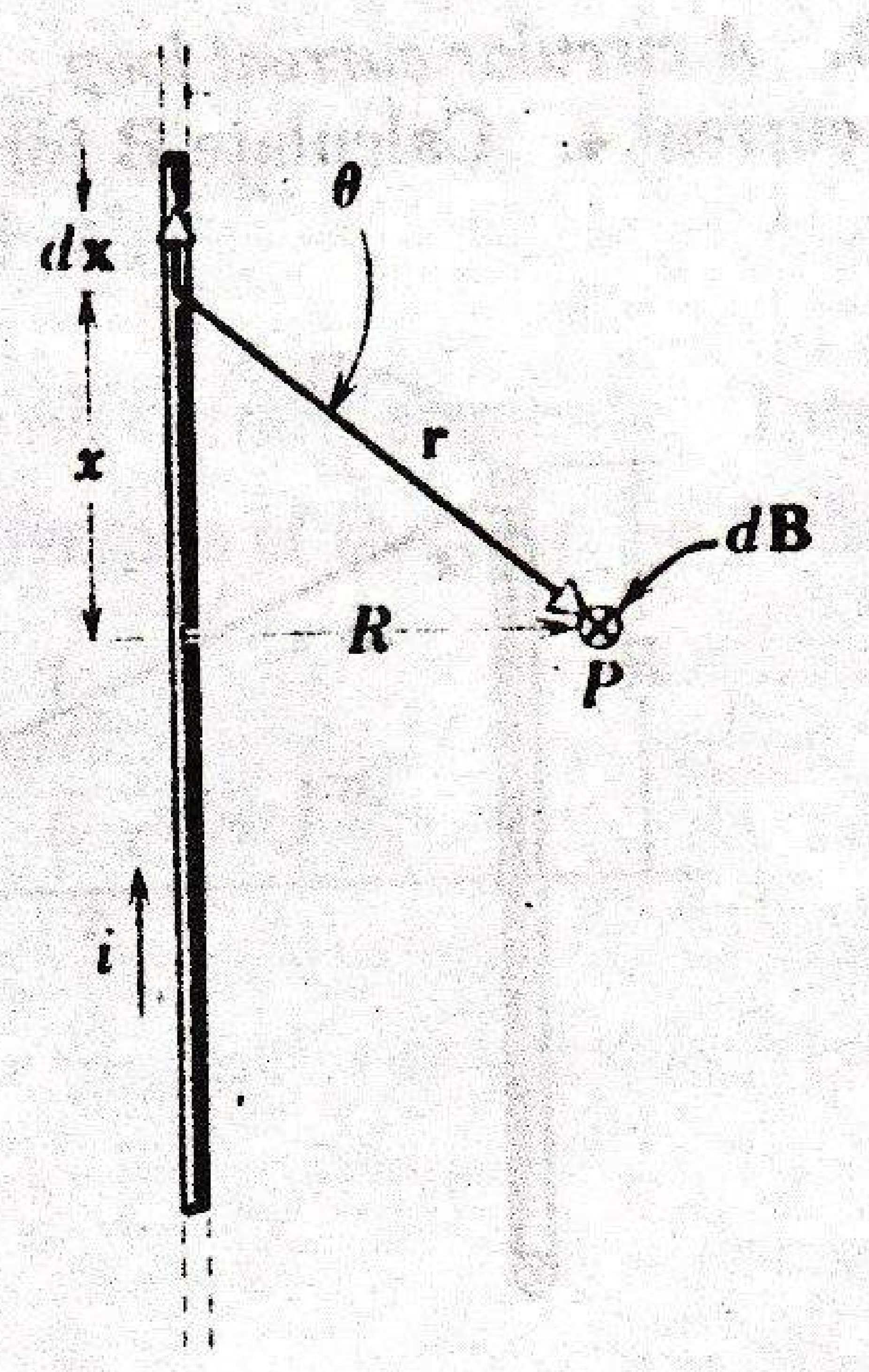

Fig. 34-17 Example 7.

Figure 34-17, a side view of the wire, shows a typical current element dx. The magnitude of the contribution $d\mathbf{B}$ of this element to the magnetic field at P is found from Eq. 34-8, or

$$dB = \frac{\mu_0 i}{4\pi}\frac{dx \sin\theta}{r^2}.$$

The directions of the contributions $d\mathbf{B}$ at point P for all elements are the same, namely, into the plane of the figure at right angles to the page. Thus the vector integral of Eq. 34-10 reduces to a scalar integral, or

$$B = \int dB = \frac{\mu_0 i}{4\pi}\int_{x=-\infty}^{x=+\infty}\frac{\sin\theta\, dx}{r^2}.$$

Now, x, θ, and r are not independent, being related (see Fig. 34-17) by

$$r = \sqrt{x^2 + R^2}$$

and

$$\sin\theta\,[= \sin(\pi - \theta)] = \frac{R}{\sqrt{x^2 + R^2}}.$$

so that the expression for B becomes

$$B = \frac{\mu_0 i}{4\pi} \int_{-\infty}^{+\infty} \frac{R\,dx}{(x^2 + R^2)^{3/2}}$$

$$= \frac{\mu_0 i}{4\pi R} \left| \frac{x}{(x^2 + R^2)^{1/2}} \right|_{x=-\infty}^{x=+\infty}$$

$$= \frac{\mu_0}{2\pi} \frac{i}{R}.$$

This is the result that we arrived at earlier for this problem (see Eq. 34-4). The law of Biot and Savart will always yield results that are consistent with Ampère's law and with experiment.

This problem reminds us of its electrostatic equivalent. We derived an expression for **E** due to a long charged rod, using Gauss's law (Section 28-6); we also solved this problem by integration methods, using Coulomb's law (Section 27-4).

Example 8. *A circular current loop.* Figure 34-18 shows a circular loop of radius R carrying a current i. Calculate **B** for points on the axis.

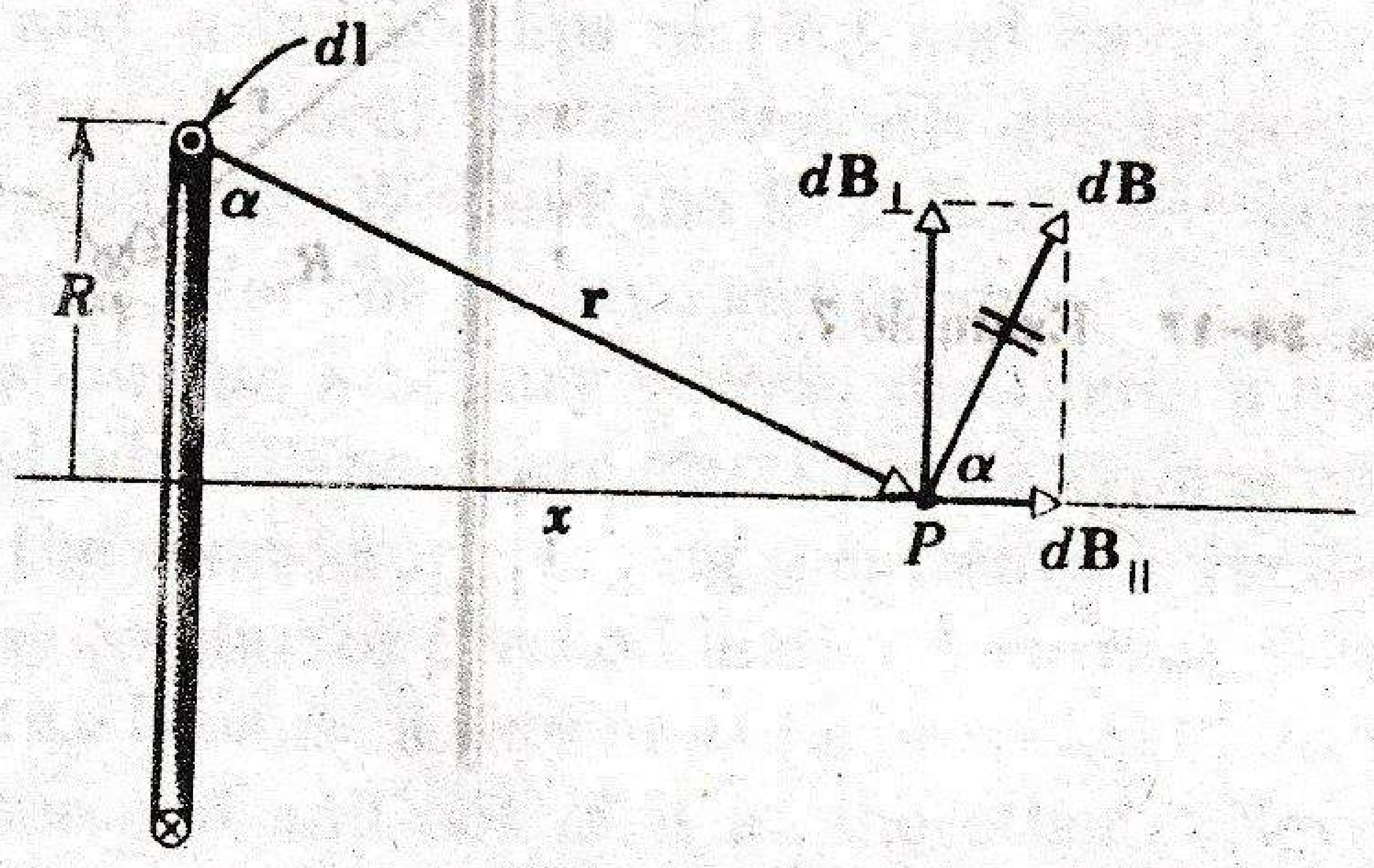

Fig. 34-18 Example 8. A ring of radius R carrying a current i.

The vector $d\mathbf{l}$ for a current element at the top of the loop points perpendicularly out of the page. The angle θ between $d\mathbf{l}$ and $\mathbf{r}$ is 90°, and the plane formed by $d\mathbf{l}$ and $\mathbf{r}$ is normal to the page. The vector $d\mathbf{B}$ for this element is at right angles to this plane and thus lies in the plane of the figure and at right angles to $\mathbf{r}$, as the figure shows.

Let us resolve $d\mathbf{B}$ into two components, one, $d\mathbf{B}_{\parallel}$, along the axis of the loop and another, $d\mathbf{B}_{\perp}$, at right angles to the axis. Only $d\mathbf{B}_{\parallel}$ contributes to the total induction $\mathbf{B}$ at point P. This follows because the components $d\mathbf{B}_{\parallel}$ for all current elements lie on the axis and add directly; however, the components $d\mathbf{B}_{\perp}$ point in different directions perpendicular to the axis, and their resultant for the complete loop is zero, from symmetry. Thus

$$B = \int dB_{\parallel},$$

where the integral is a simple scalar integration over the current elements.

For the current element shown in Fig. 34-18 we have, from the Biot-Savart law (Eq. 34-8),

$$dB = \frac{\mu_0 i}{4\pi} \frac{dl \sin 90°}{r^2}.$$

We also have
$$dB_{\parallel} = dB \cos \alpha.$$

Combining gives
$$dB_{\parallel} = \frac{\mu_0 i \cos \alpha \, dl}{4\pi r^2}.$$

Figure 34-18 shows that r and α are not independent of each other. Let us express each in terms of a new variable x, the distance from the center of the loop to the point P. The relationships are

$$r = \sqrt{R^2 + x^2}$$

and
$$\cos \alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + x^2}}.$$

Substituting these values into the expression for $dB_{\parallel}$ gives

$$dB_{\parallel} = \frac{\mu_0 i R}{4\pi(R^2 + x^2)^{3/2}} \, dl.$$

Note that i, R, and x have the same values for all current elements. Integrating this equation, noting that $\int dl$ is simply the circumference of the loop ($= 2\pi R$), yields

$$B = \int dB_{\parallel} = \frac{\mu_0 i R}{4\pi(R^2 + x^2)^{3/2}} \int dl$$

$$= \frac{\mu_0 i R^2}{2(R^2 + x^2)^{3/2}}. \qquad (34\text{-}11)$$

◄

If we put $x \gg R$ in Example 8 so that points close to the loop are not considered, Eq. 34-11 reduces to

$$B = \frac{\mu_0 i R^2}{2x^3}.$$

Recalling that πR^2 is the area A of the loop and considering loops with N turns, we can write this equation as

$$B = \frac{\mu_0}{2\pi} \frac{(NiA)}{x^3} = \frac{\mu_0}{2\pi} \frac{\mu}{x^3},$$

where μ is the *magnetic dipole moment* of the current loop. This reminds us of the result derived in Problem 10, Chapter 27 [$E = (1/2\pi\epsilon_0)(p/x^3)$], which is the formula for the *electric* field strength on the axis of an *electric* dipole.

Thus we have shown in two ways that a current loop can be regarded as a magnetic dipole: It experiences a torque given by $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$ when placed in an *external* magnetic field (Eq. 33-11); it generates its own magnetic field given, for points on the axis, by the equation just developed.

Table 34-1 is a summary of the properties of electric and magnetic dipoles.

Table 34-1

SOME DIPOLE EQUATIONS

Property	Dipole Type	Equation
Torque in an external field	electric	$\tau = \mathbf{p} \times \mathbf{E}$
	magnetic	$\tau = \boldsymbol{\mu} \times \mathbf{B}$
Energy in an external field	electric	$U = -\mathbf{p} \cdot \mathbf{E}$
	magnetic	$U = -\boldsymbol{\mu} \cdot \mathbf{B}$
Field at distant points along axis	electric	$E = \frac{1}{2\pi\epsilon_0} \frac{p}{x^3}$
	magnetic	$B = \frac{\mu_0}{2\pi} \frac{\mu}{x^3}$
Field at distant points along perpendicular bisector	electric	$E = \frac{1}{4\pi\epsilon_0} \frac{p}{x^3}$
	magnetic	$B = \frac{\mu_0}{4\pi} \frac{\mu}{x^3}$

▶ **Example 9.** In the Bohr model of the hydrogen atom the electron circulates around the nucleus in a path of radius 5.1×10^{-11} meter at a frequency ν of 6.8×10^{15} rev/sec. (*a*) What value of B is set up at the center of the orbit?

The current is the rate at which charge passes any point on the orbit and is given by

$$i = e\nu = (1.6 \times 10^{-19} \text{ coul})(6.8 \times 10^{15}/\text{sec}) = 1.1 \times 10^{-3} \text{ amp}.$$

B at the center of the orbit is given by Eq. 34-11 with $x = 0$, or

$$B = \frac{\mu_0 i R^2}{2(R^2 + x^2)^{3/2}} = \frac{\mu_0 i}{2R}$$

$$= \frac{(4\pi \times 10^{-7} \text{ weber/amp-m})(1.1 \times 10^{-3} \text{ amp})}{(2)(5.1 \times 10^{-11} \text{ meter})}$$

$$= 14 \text{ webers/meter}^2.$$

(*b*) What is the equivalent magnetic dipole moment? From Eq. 33-10,

$$\mu = NiA = (1)(1.1 \times 10^{-3} \text{ amp})(\pi)(5.1 \times 10^{-11} \text{ meter})^2$$

$$= 9.0 \times 10^{-24} \text{ amp-m}^2.$$

QUESTIONS

1. Can the path of integration around which we apply Ampère's law pass *through* a conductor?

2. Suppose we set up a path of integration around a cable that contains twelve wires with different currents (some in opposite directions) in each wire. How do we calculate i in Ampère's law in such a case?

3. Apply Ampère's law qualitatively to the three paths shown in Fig. 34–19.

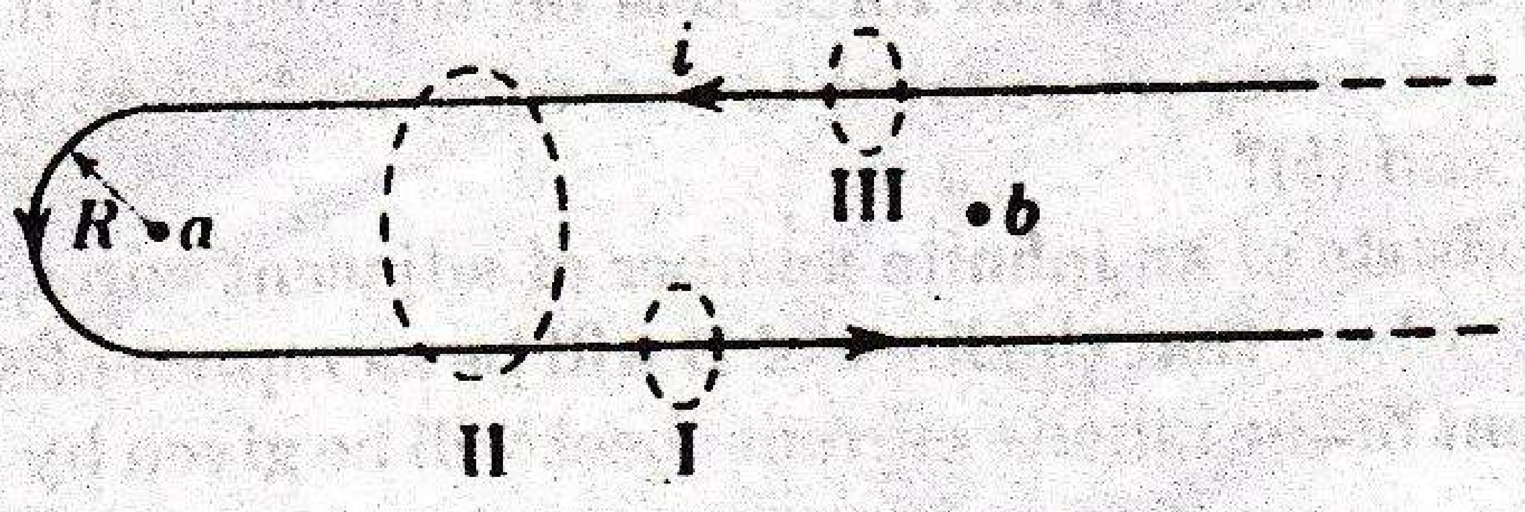

Fig. 34–19

4. Is **B** constant in magnitude for points that lie on a given line of induction?

5. Discuss and compare Gauss's law and Ampère's law.

6. A current is set up in a long copper pipe. Is there a magnetic field (*a*) inside and (*b*) outside the pipe?

7. Equation 34–4 ($B = \mu_0 i/2\pi r$) suggests that a strong magnetic field is set up at points near a long wire carrying a current. Since there is a current i and a magnetic field **B**, why is there not a force on the wire in accord with the equation $\mathbf{F} = i\mathbf{l} \times \mathbf{B}$?

8. In electronics, wires that carry equal but opposite currents are often twisted together to reduce their magnetic effect at distant points. Why is this effective?

9. A beam of 20-Mev protons emerges from a cyclotron. Is a magnetic field associated with these particles?

10. Test the "floating" wire of Example 3 for equilibrium under *horizontal* displacements. Consider that the wire floats above the rigidly supported wire and also below it. Summarize the equilibrium situation for both wire positions and for both vertical and horizontal displacements.

11. Explain qualitatively the forces of interaction between parallel wires carrying parallel or antiparallel currents in terms of Faraday's lines of induction representation.

12. Comment on this statement: "The magnetic induction outside a long solenoid cannot be zero, if only for the reason that the helical nature of the windings produces a field for external points like that of a straight wire along the solenoid axis."

13. A current is sent through a vertical spring from whose lower end a weight is hanging; what will happen?

14. Does Eq. 34–7 ($B = \mu_0 in$) hold for a solenoid of square cross section?

15. What is the direction of the magnetic fields at points a, b, and c in Fig. 34–16 set up by the particular current element shown?

16. In a circular loop of wire carrying a current i, is **B** uniform for all points within the loop?

17. Discuss analogies and differences between Coulomb's law and the Biot-Savart law.

18. Equation 34–9 gives the law of Biot and Savart in vector form. Write its electrostatic equivalent [that is, Eq. 27–6, or $dE = dq/(4\pi\epsilon_0 r^2)$] in vector form.

19. How might you measure the magnetic dipole moment of a compass needle?

20. What is the basis for saying that a current loop is a magnetic dipole?

PROBLEMS

1. A #10 bare copper wire (0.10 in. in diameter) can carry a current of 50 amp without overheating. For this current, what is B at the surface of the wire?

2. A surveyor is using a compass 20 ft below a power line in which there is a steady current of 100 amp. Will this interfere seriously with the compass reading? The horizontal component of the earth's magnetic field at the site is 0.2 gauss.

3. A long straight wire carries a current of 50 amp. An electron, traveling at 10^7 meters/sec, is 5.0 cm from the wire. What force acts on the electron if the electron velocity is directed (*a*) toward the wire, (*b*) parallel to the wire, and (*c*) at right angles to the directions defined by (*a*) and (*b*)?

4. A conductor consists of an infinite number of adjacent wires, each infinitely long and carrying a current i. Show that the lines of **B** will be as represented in Fig. 34–20 and that B for all points in front of the infinite current sheet will be given by

$$B = \tfrac{1}{2}\mu_0 n i,$$

where n is the number of conductors per unit length. Derive both by direct application of Ampère's law and by considering the problem as a limiting case of Example 2.

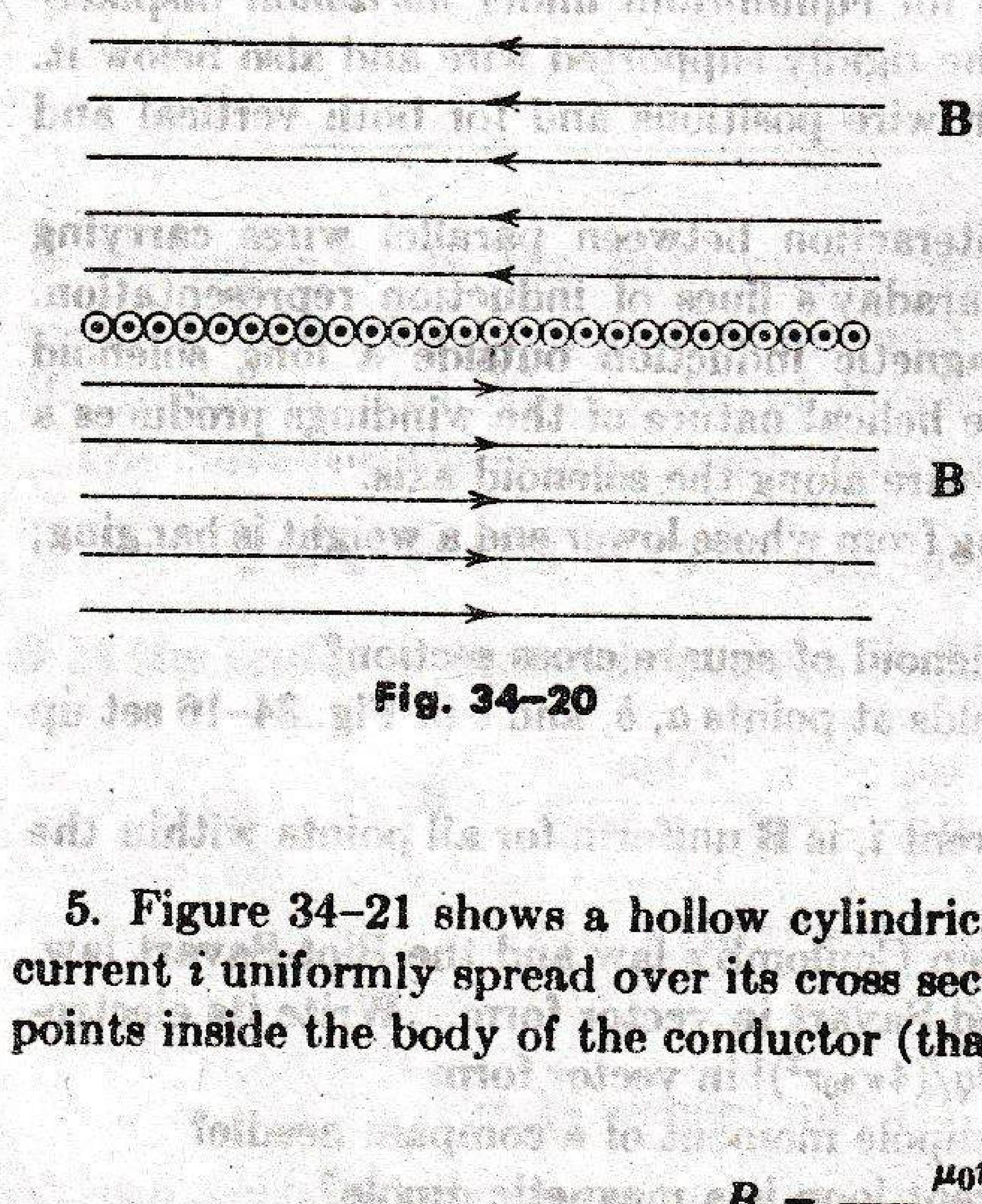

Fig. 34–20

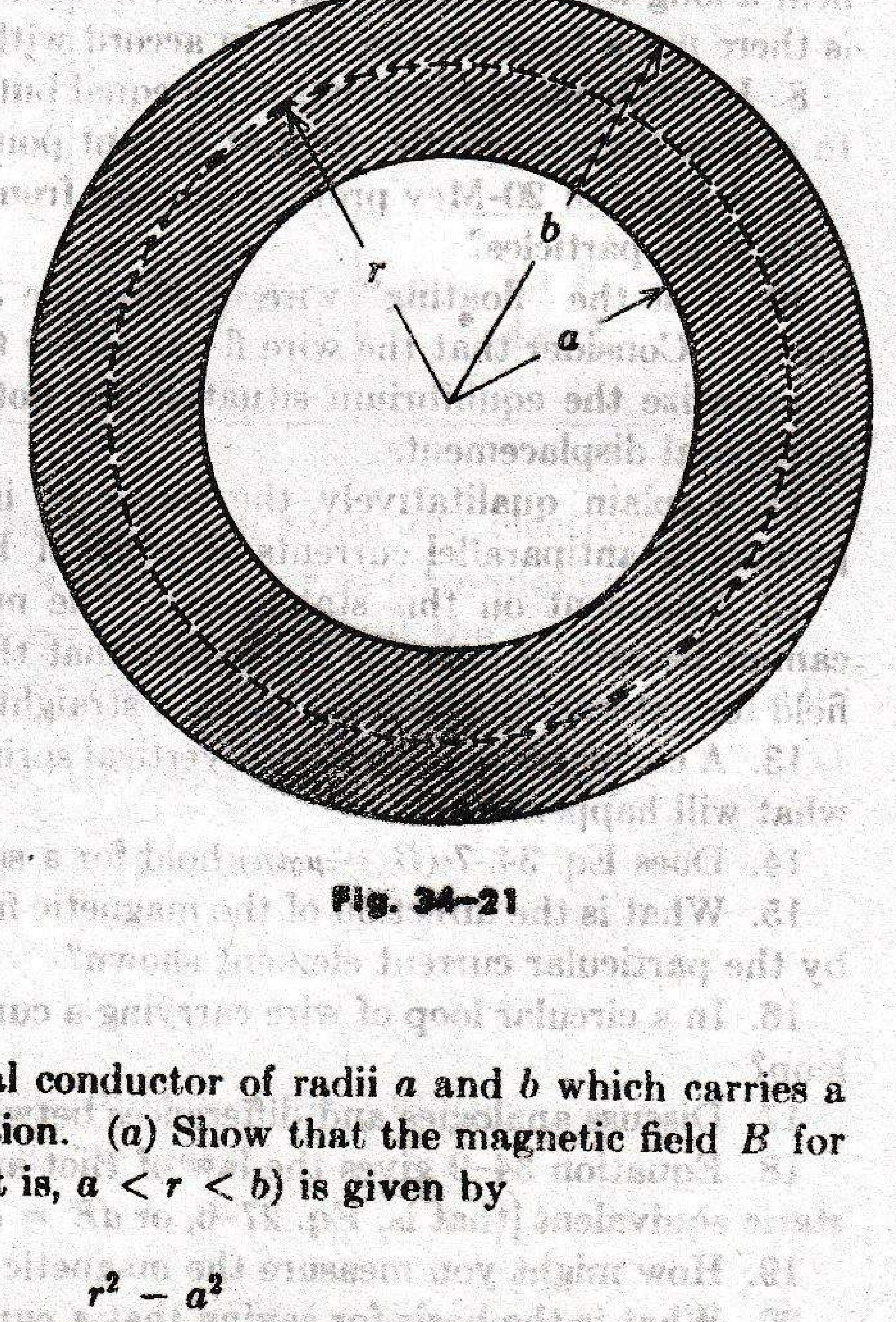

Fig. 34–21

5. Figure 34–21 shows a hollow cylindrical conductor of radii a and b which carries a current i uniformly spread over its cross section. (*a*) Show that the magnetic field B for points inside the body of the conductor (that is, $a < r < b$) is given by

$$B = \frac{\mu_0 i}{2\pi(b^2 - a^2)} \frac{r^2 - a^2}{r}.$$

Check this formula for the limiting case of $a = 0$. (*b*) Make a rough plot of the general behavior of $B(r)$ from $r = 0$ to $r \to \infty$.

6. A long coaxial cable consists of two concentric conductors with the dimensions shown in Fig. 34-22. There are equal and opposite currents i in the conductors. (a) Find the magnetic induction B at r within the inner conductor ($r < a$). (b) Find B between the two conductors ($a < r < b$). (c) Find B within the outer conductor ($b < r < c$). (d) Find B outside the cable ($r > c$).

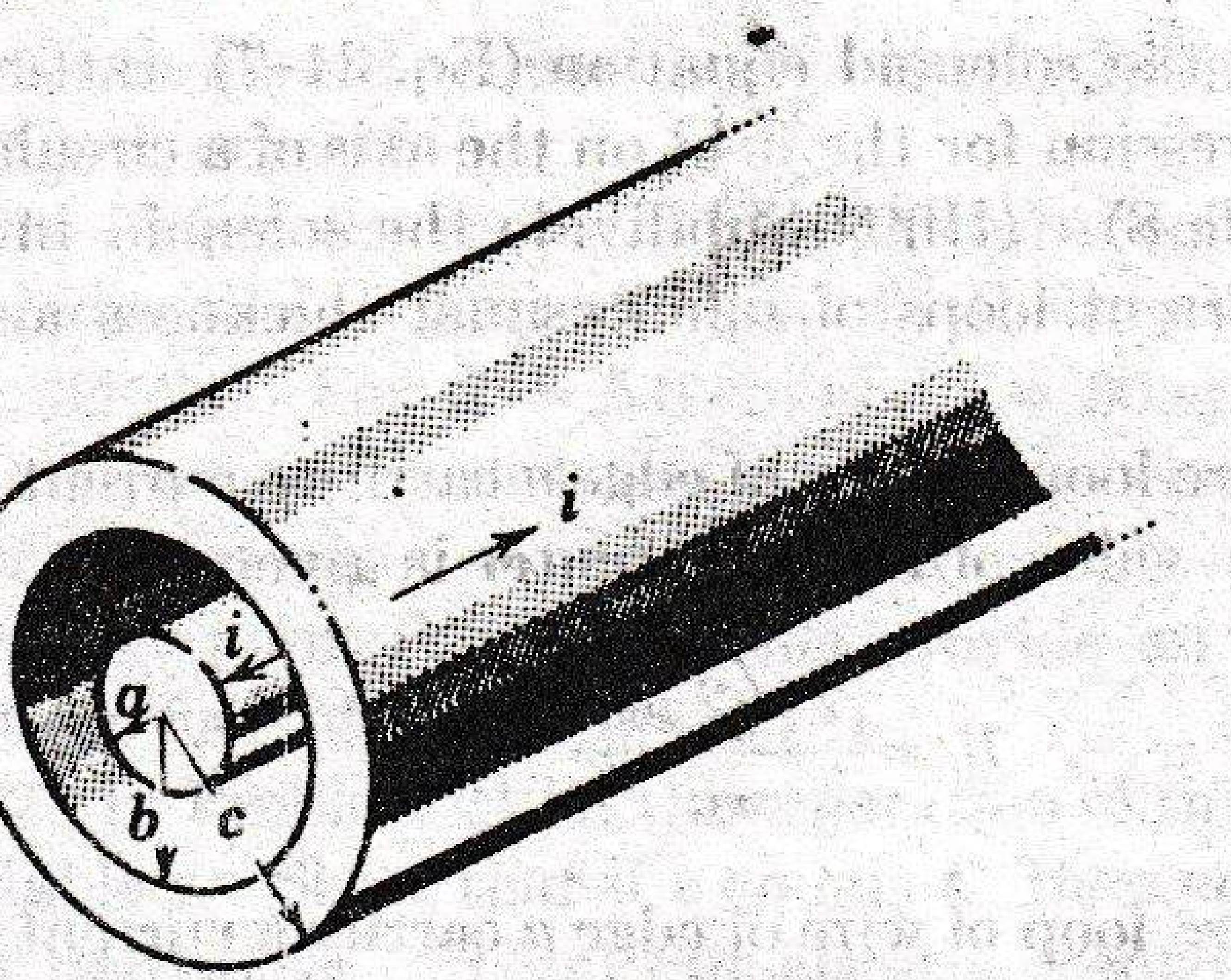

Fig. 34-22

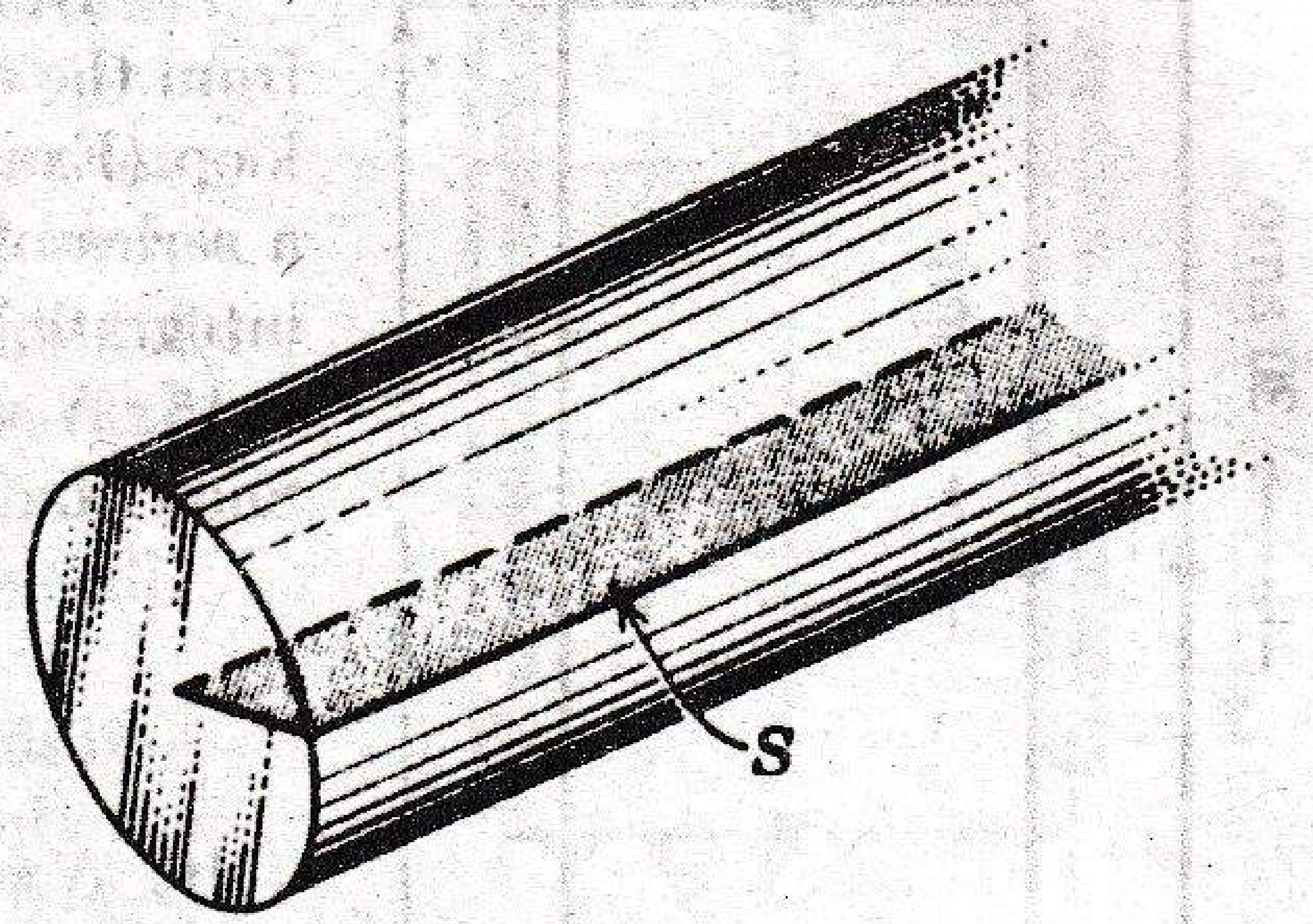

Fig. 34-23

7. A long copper wire carries a current of 10 amp. Calculate the magnetic flux per meter of wire for a plane surface S inside the wire, as in Fig. 34-23.

8. Two long, parallel #10 copper wires (diameter = 0.10 in.) carry currents of 10 amp in opposite directions. (a) If their centers are 2.0 cm apart, calculate the flux per meter that exists in the space between the axes of the wires. (b) What fraction of the flux in (a) lies inside the wires? (c) Repeat the calculation of (a) for parallel currents.

9. A long wire carrying a current of 100 amp is placed in a uniform external magnetic field of 50 gauss. The wire is at right angles to this external field. Locate the points at which the *resultant* magnetic field is zero.

10. Two long wires a distance d apart carry equal antiparallel currents i, as in Fig. 34-24. Show that B at point P, which is equidistant from the wires, is given by

$$B = \frac{2\mu_0 i d}{\pi(4R^2 + d^2)}.$$

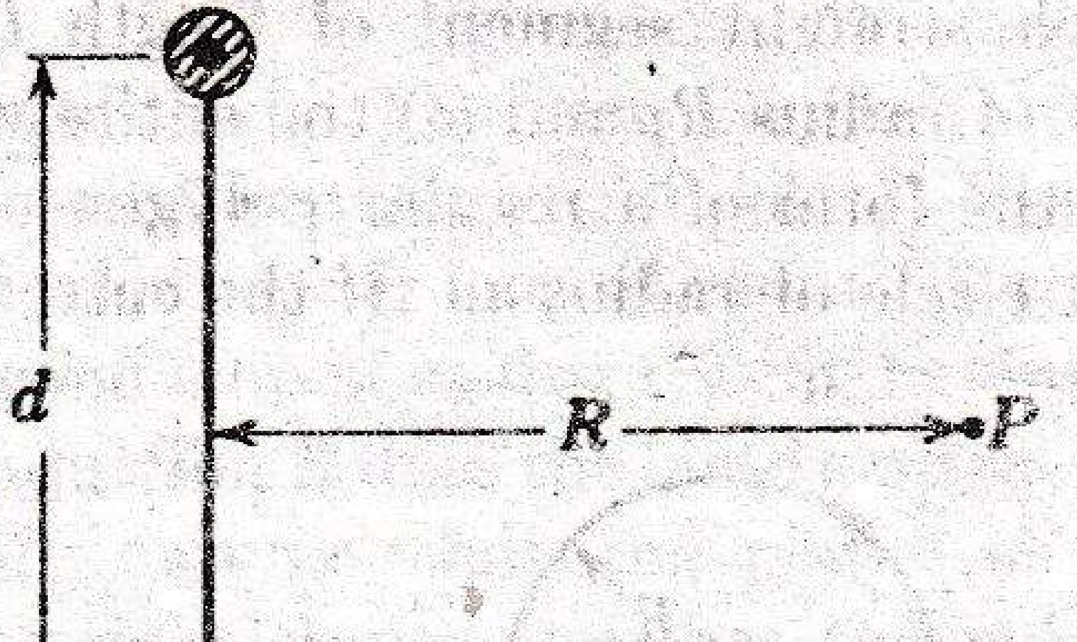

Fig. 34-24

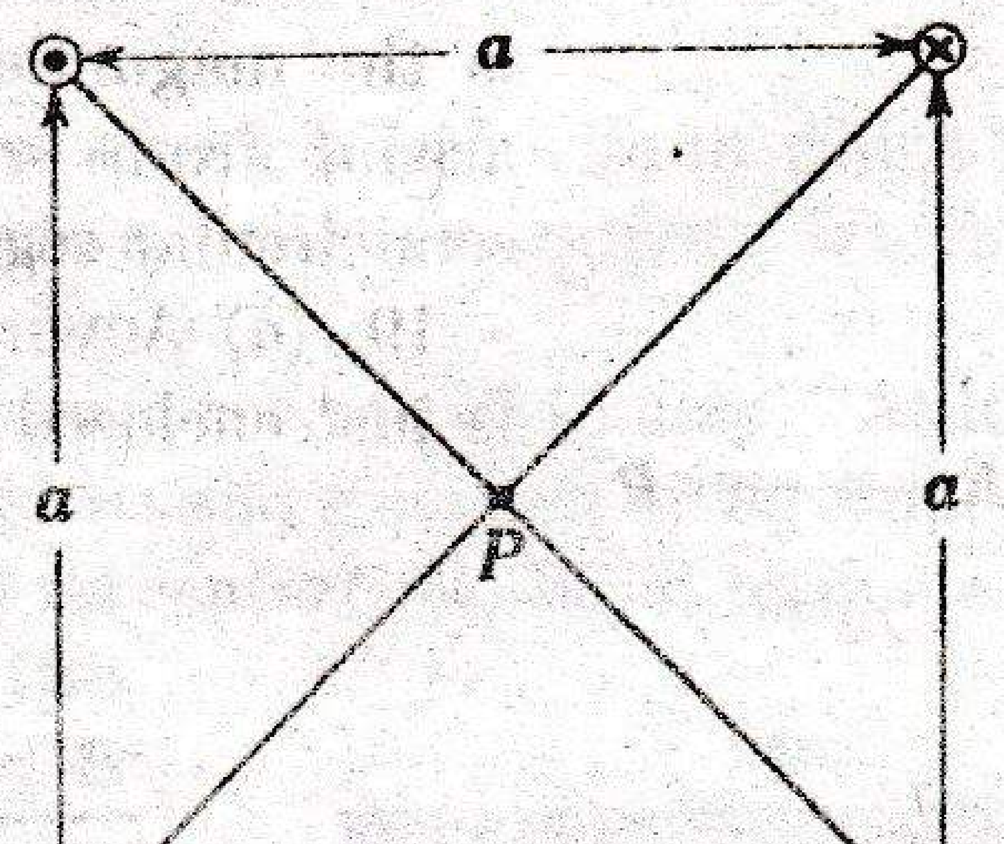

Fig. 34-25

11. Four long #10 copper wires are parallel to each other, their cross section forming a square 20 cm on edge. A 20-amp current is set up in each wire in the direction shown in Fig. 34-25. What are the magnitude and direction of **B** at the center of the square?

12. In Problem 11 what is the force per meter acting on the lower left wire, in magnitude and direction?

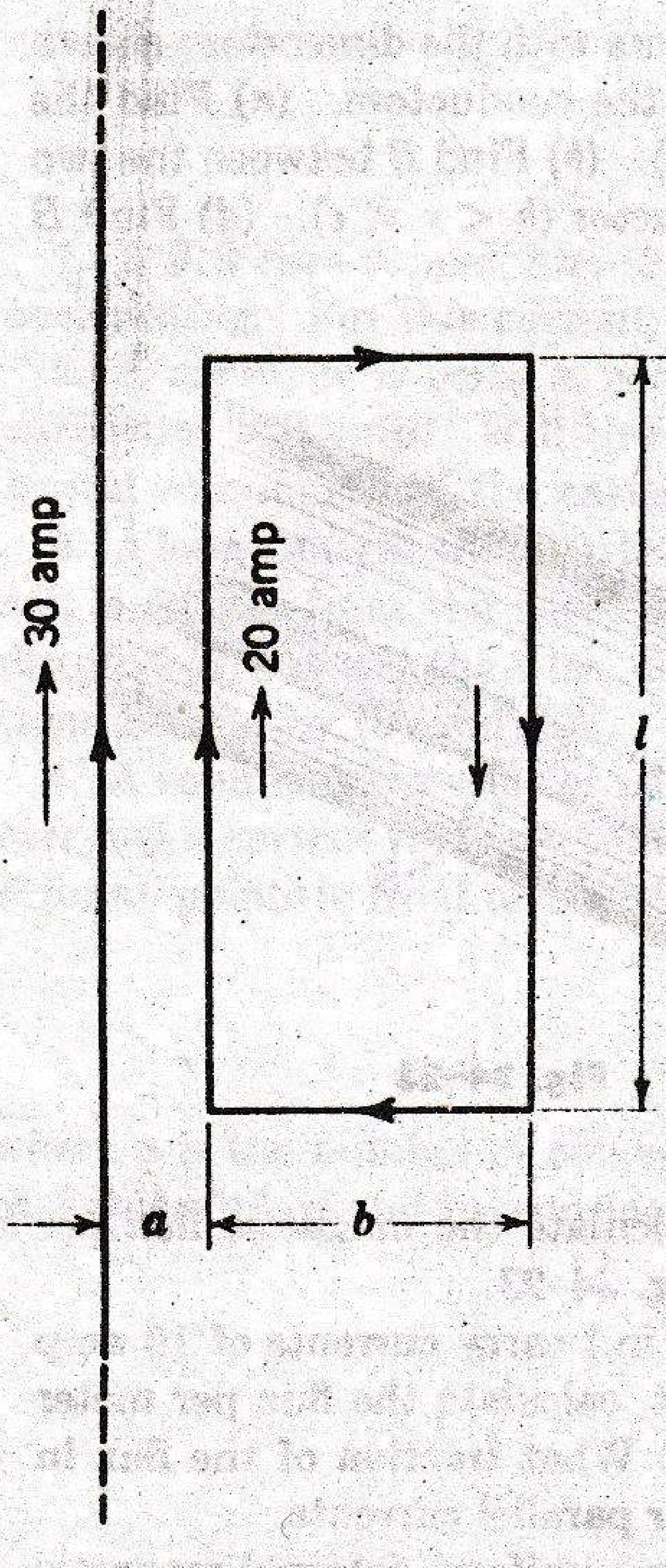

Fig. 34–26

13. Figure 34–26 shows a long wire carrying a current of 30 amp. The rectangular loop carries a current of 20 amp. Calculate the resultant force acting on the loop. Assume that $a = 1.0$ cm, $b = 8.0$ cm, and $l = 30$ cm.

14. Derive the solenoid equation (Eq. 34–7) starting from the expression for the field on the axis of a circular loop (Example 8). (Hint: Subdivide the solenoid into a series of current loops of infinitesimal thickness and integrate.)

15. A square loop of wire of edge a carries a current i. Show that the value of B at the center is given by

$$B = \frac{2\sqrt{2}\,\mu_0 i}{\pi a}.$$

16. A square loop of wire of edge a carries a current i. (*a*) Show that B for a point on the axis of the loop and a distance x from its center is given by

$$B = \frac{4\mu_0 i a^2}{\pi(4x^2 + a^2)(4x^2 + 2a^2)^{1/2}}.$$

(*b*) Does this reduce to the result of Problem 15 for $x = 0$? (*c*) Does the square loop behave like a dipole for points such that $x \gg a$? If so, what is its dipole moment?

17. A straight wire segment of length l carries a current i. (*a*) Show that the field of induction **B** to be associated with this segment, at a distance R from the segment along a perpendicular bisector (see Fig. 34–27), is given in magnitude by

$$B = \frac{\mu_0 i}{2\pi R}\,\frac{l}{(l^2 + 4R^2)^{1/2}}.$$

(*b*) Does this expression reduce to an expected result as $l \to \infty$?

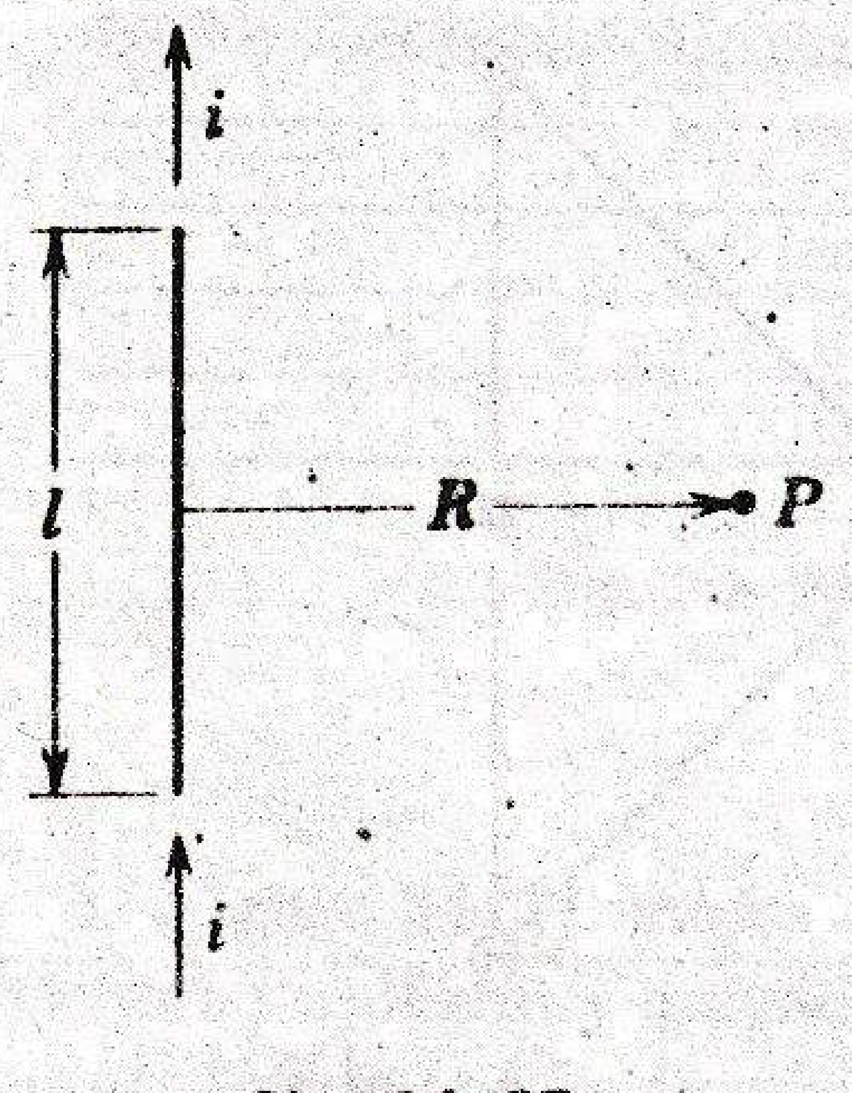

Fig. 34–27

18. The wire shown in Fig. 34–28 carries a current i. What is the magnetic induction at the center C of the semicircle arising from (*a*) each straight segment of length l, (*b*) the semicircular segment of radius R, and (*c*) the entire wire?

19. (*a*) A wire in the form of a regular polygon of n sides is just enclosed by a circle of radius a. If the current in this

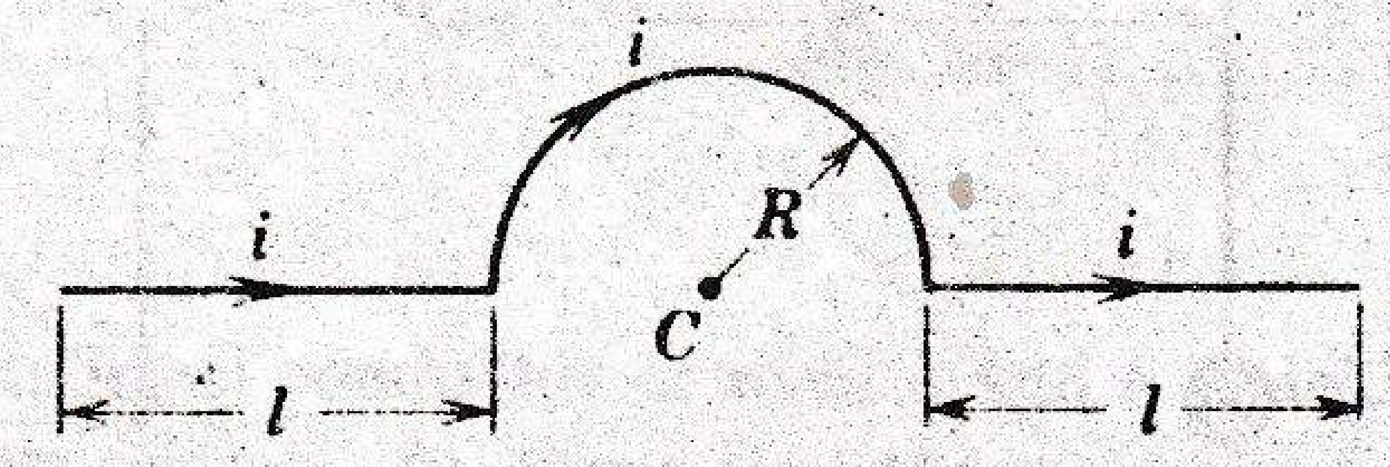

Fig. 34–28

wire is i, show that the magnetic induction at the center of the circle is given by

$$B = \frac{\mu_0 n i}{2\pi a}\tan(\pi/n).$$

(*b*) Show that as $n \to \infty$ this result approaches that of a circular loop.

20. (*a*) Show that B at the center of a rectangle of length l and width d, carrying a current i, is given by

$$B = \frac{2\mu_0 i}{\pi} \frac{(l^2 + d^2)^{1/2}}{ld}.$$

(*b*) What does B reduce to for $l \gg d$? Is this a result that you expect?

21. *Helmholtz coils.* Two 300-turn coils are arranged a distance apart equal to their radius, as in Fig. 34–29. For $R = 5.0$ cm and $i = 50$ amp, plot B as a function of distance x along the common axis over the range $x = -5$ cm to $x = +5$ cm, taking $x = 0$ at point P. (Such coils provide an especially uniform field of B near point P.)

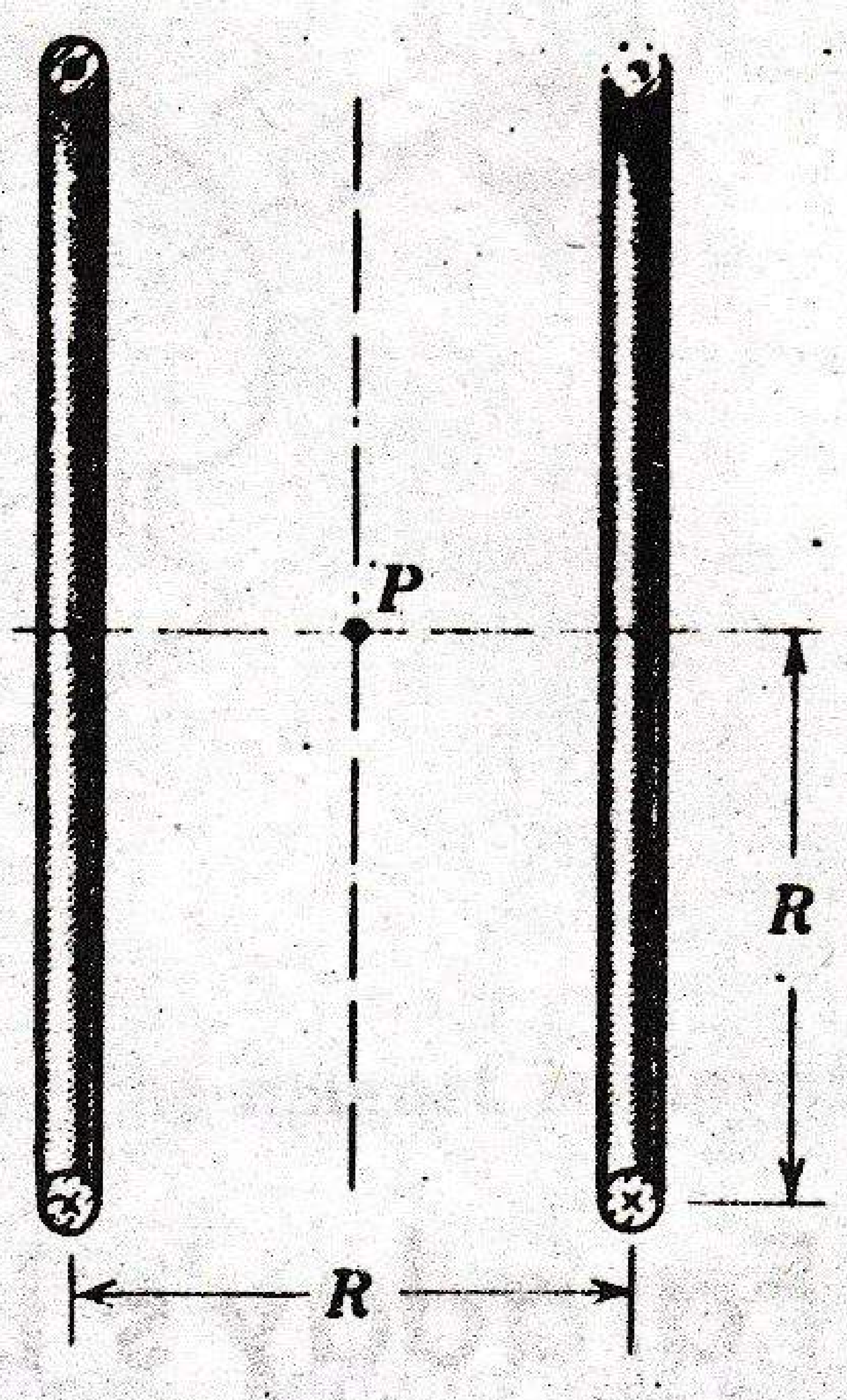

Fig. 34–29

22. In Problem 21 let the separation of the coils be a variable z. Show that if z equals R, then not only the first derivative (dB/dx) but also the second (d^2B/dx^2) of B is zero at point P. This accounts for the uniformity of B near point P for this particular coil separation.

23. A long "hairpin" is formed by bending a piece of wire as shown in Fig. 34–19. If a 10-amp current is set up, what are the direction and magnitude of **B** at point a? At point b? Take $R = 0.50$ cm.

24. Calculate **B** at point P in Fig. 34–30. Assume that $i = 10$ amp and $a = 8.0$ cm.

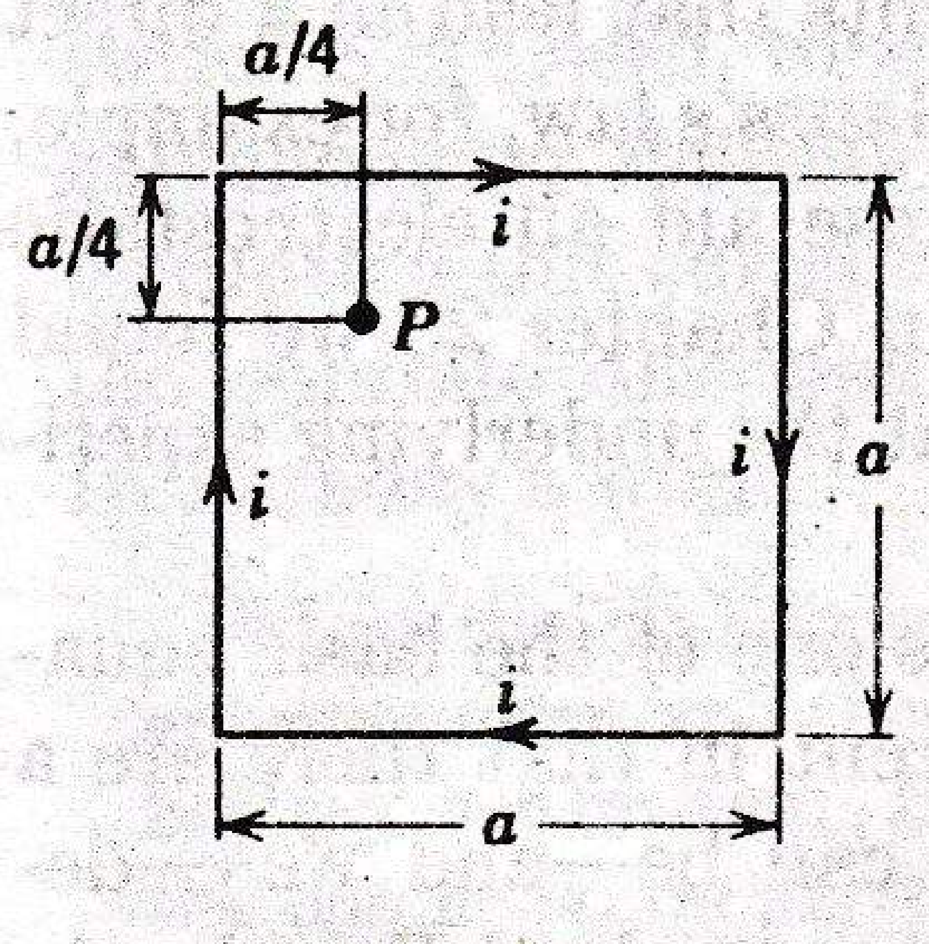

Fig. 34–30

25. A plastic disk of radius R has a charge q uniformly distributed over its surface. If the disk is rotated at an angular frequency ω about its axis, show that (*a*) the induction at the center of the disk is

$$B = \frac{\mu_0 \omega q}{2\pi R}$$

and (*b*) the magnetic dipole moment of the disk is

$$\mu = \frac{\omega q R^2}{4}.$$

(Hint: The rotating disk is equivalent to an array of current loops; see Example 8.)

26. You are given a length l of wire in which a current i may be established. The wire may be formed into a circle or a square. Which yields the larger value for B at the central point? See Problem 15.

27. A circular copper loop of radius 10 cm carries a current of 15 amp. At its center is placed a second loop of radius 1.0 cm, having 50 turns and a current of 1.0 amp. (*a*) What magnetic induction **B** does the large loop set up at its center? (*b*) What torque acts on the small loop? Assume that the planes of the two loops are at right angles and that the induction **B** provided by the large loop is essentially uniform throughout the volume occupied by the small loop.

28. Show that it is impossible for a uniform magnetic field **B** to drop abruptly to zero as one moves at right angles to it, as suggested by the horizontal arrow in Fig. 34–31 (see point a). In actual magnets fringing of the lines of force always occurs, which means that **B** approaches zero in a continuous and gradual way. (Hint: Apply Ampère's law to the rectangular path shown by the dashed lines.)

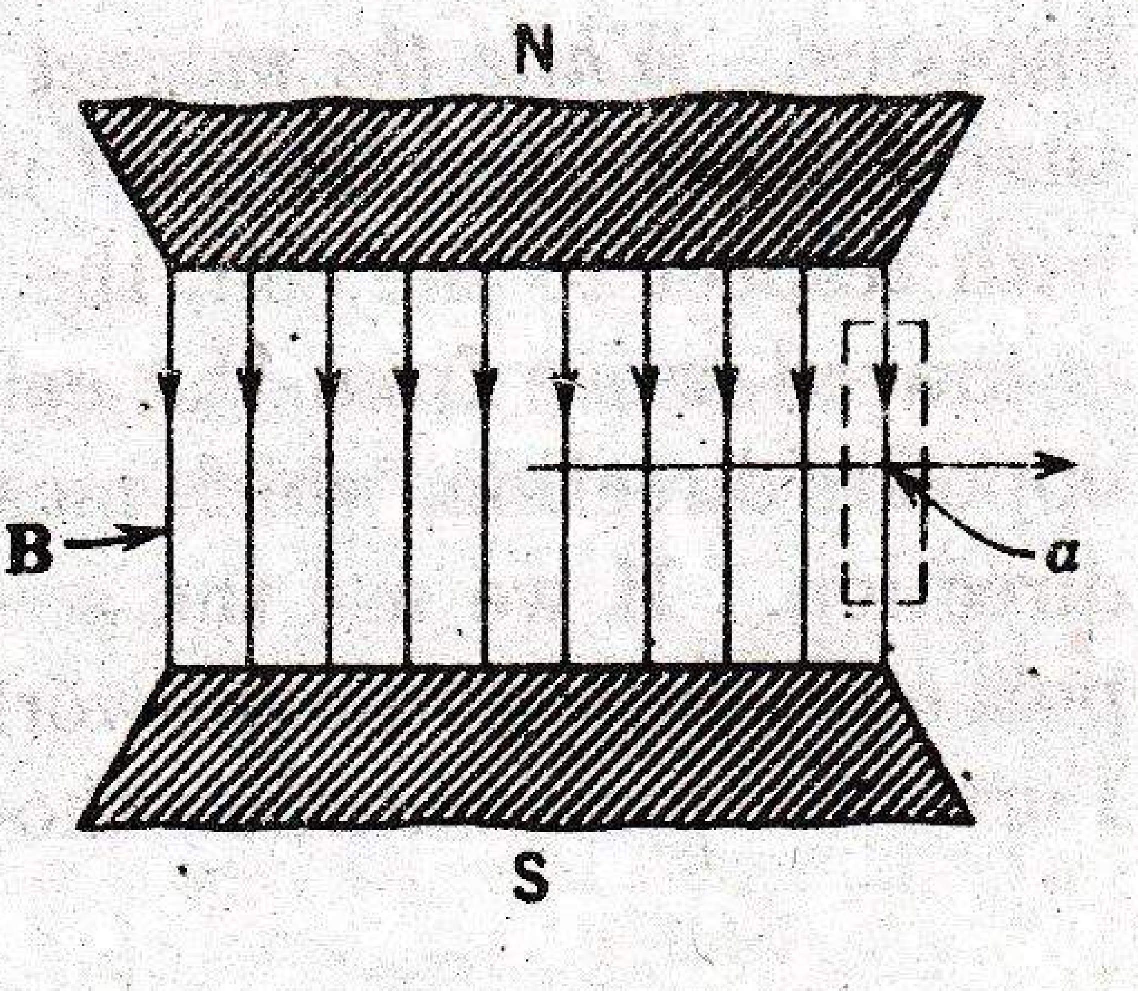

Fig. 34–31