Faraday's Law

CHAPTER 35

35–1 Faraday's Experiments

For some physical laws it is hard to find experiments that lead in a direct and convincing way to the formulation of the law. Gauss's law, for example, emerged only slowly as the common factor with whose aid all electrostatic experiments could be interpreted and correlated. In Chapter 28 we found it best to state Gauss's law first and then to show that the underlying experiments were consistent with it.

Faraday's *law of electromagnetic induction*, which is one of the basic equations of electromagnetism (see Table 38–3), is different in that there are a number of simple experiments from which the law can be—and was—deduced directly. Such experiments were carried out by Michael Faraday in England in 1831 and by Joseph Henry in the United States at about the same time.

Figure 35–1 shows the terminals of a coil connected to a galvanometer. Normally we would not expect this instrument to deflect because there seems to be no electromotive force in this circuit; but if we push a bar magnet toward the coil, with its north pole facing the coil, a remarkable thing happens. *While the magnet is moving,* the galvanometer deflects, showing that a current has been set up in the coil. If the magnet is held stationary with respect to the coil, the galvanometer does not deflect. If the magnet is moved away from the coil, the galvanometer again deflects, but in the opposite direction, which means that the current in the coil is in the opposite direction. If we use the south pole end of a magnet instead of the north pole end, the experiment works as described but the deflections are reversed. Further experimentation shows that *what matters is the relative motion of the*

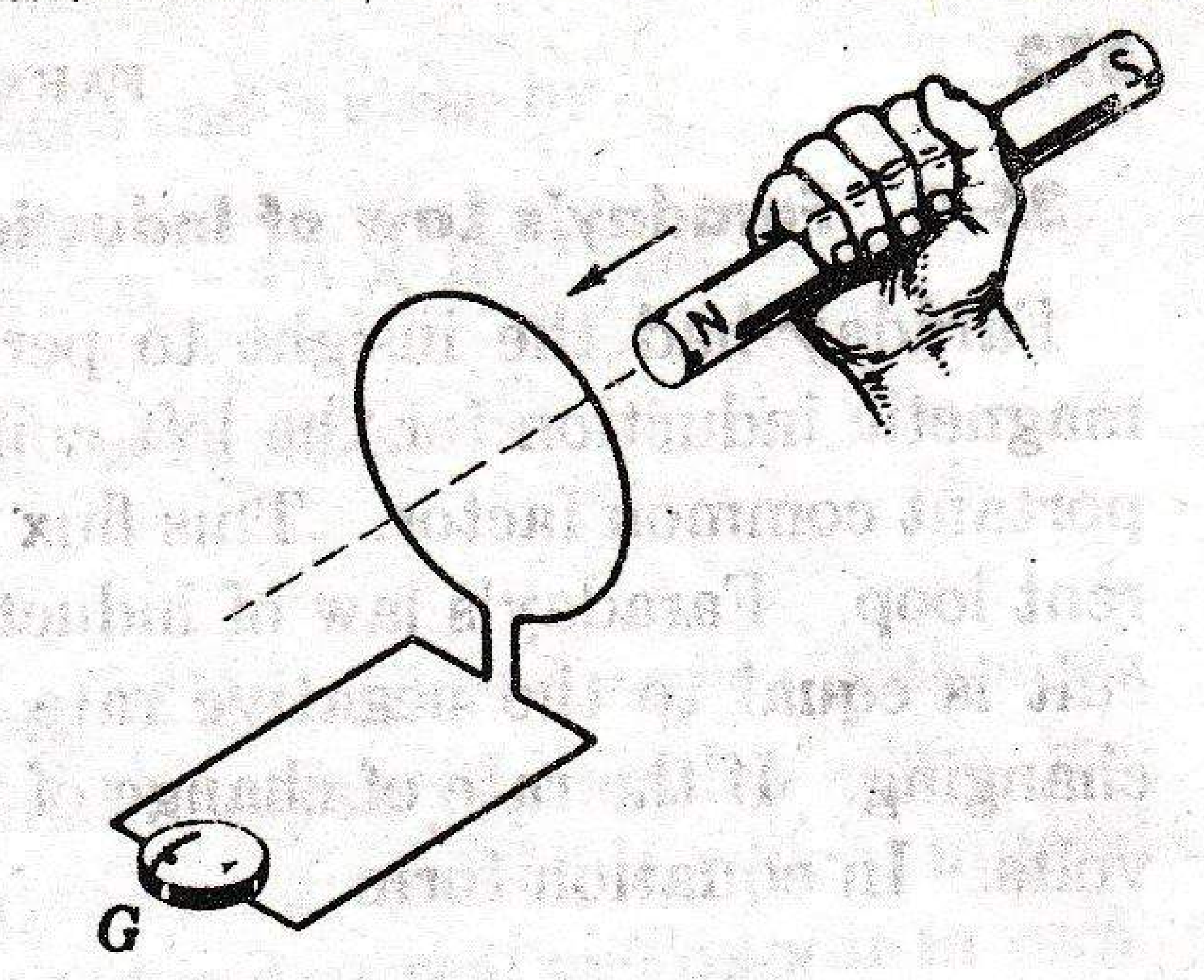


Fig. 35-1 Galvanometer G deflects while the magnet is moving with respect to the coil.

magnet and the coil. It makes no difference whether the magnet is moved toward the coil or the coil toward the magnet.

The current that appears in this experiment is called an *induced current* and is said to be set up by an *induced electromotive force.* Faraday was able to deduce from experiments like this the law that gives their magnitude and direction. Such emfs are very important in practice. The chances are good that the lights in the room in which you are reading this book are operated from an induced emf produced in a commercial electric generator.

In another experiment the apparatus of Fig. 35–2 is used. The coils are placed close together but at rest with respect to each other. When the switch S is closed, thus setting up a steady current in the right-hand coil, the galvanometer deflects momentarily; when the switch is opened, thus interrupting this current, the galvanometer again deflects momentarily, but in the opposite direction. No gross objects are moving in this experiment. In Faraday's words:

> When the contact was made, there was a sudden and very slight effect at the galvanometer, and there was also a similar slight effect when the contact with the battery was broken. But whilst the voltaic current was continuing to pass through the one helix, no galvanometrical appearances nor any effect like induction upon the other helix could be perceived, although the active power of the battery was proved to be very great. . . .

Experiment shows that there will be an induced emf in the left coil of Fig. 35–2 whenever the current in the right coil is *changing.* It is the *rate at which the current is changing* and *not the size of the current* that is significant.

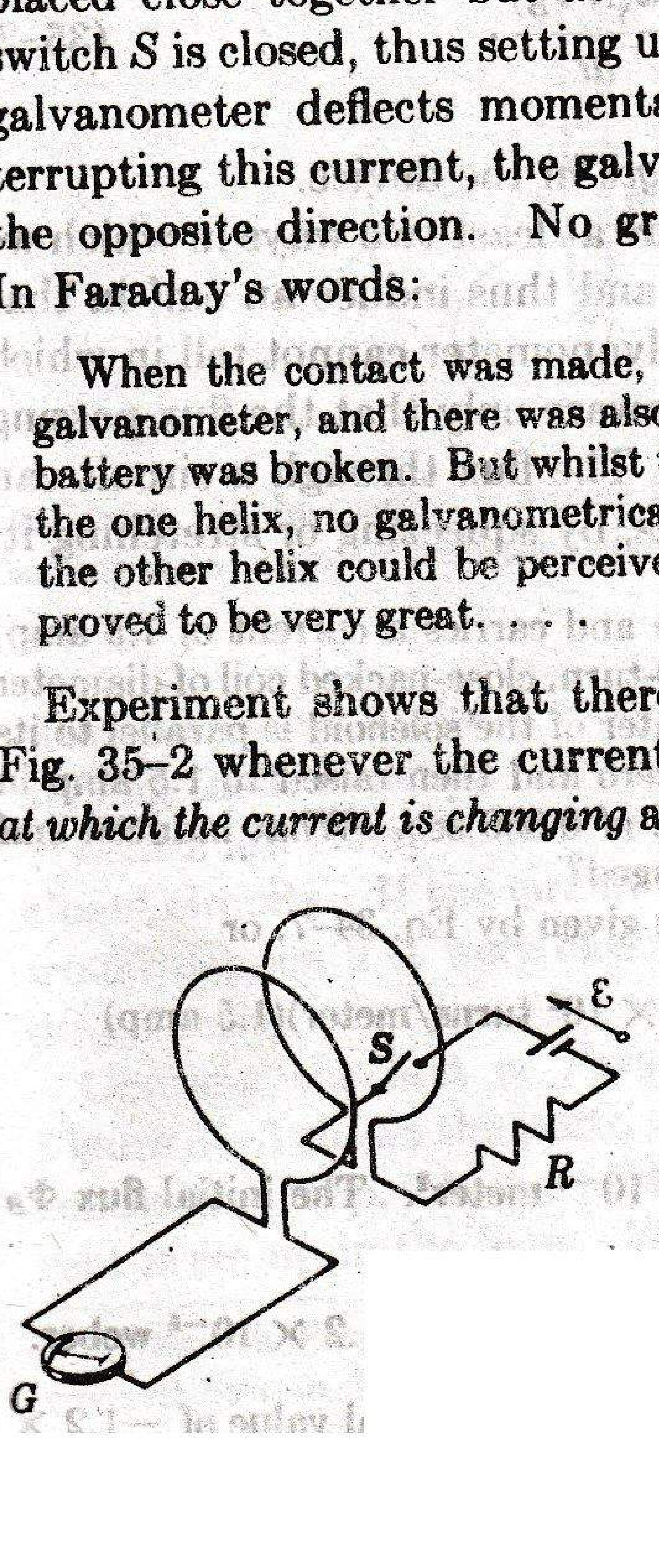


Fig. 35-2 Galvanometer G deflects momentarily when switch S is closed or opened.

35–2 Faraday's Law of Induction

Faraday had the insight to perceive that the change in the flux Φ_B of magnetic induction for the left coil in the preceding experiments is the important common factor. This flux may be set up by a bar magnet or a current loop. Faraday's law of induction says that the induced emf $\mathcal{E}$ in a circuit is equal to the negative rate at which the flux through the circuit is changing. If the rate of change of flux is in webers/sec, the emf $\mathcal{E}$ will be in volts. In equation form

$$\mathcal{E} = -\frac{d\Phi_B}{dt}. \qquad (35\text{–}1)$$

This equation is called *Faraday's law of induction.* The minus sign is an indication of the *direction* of the induced emf, a matter we discuss further in Section 35–3.

If Eq. 35–1 is applied to a coil of N turns, an emf appears in every turn and these emfs are to be added. If the coil is so tightly wound that each turn can be said to occupy the same region of space, the flux through each turn will then be the same. The flux through each turn is also the same for (ideal) toroids and solenoids (see Section 34–5). The induced emf in all such devices is given by

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -\frac{d(N\Phi_B)}{dt}, \qquad (35\text{–}2)$$

where $N\Phi_B$ measures the so-called *flux linkages* in the device.

Figures 35–1 and 35–2 suggest that there are at least two ways in which we can make the flux through a circuit change and thus induce an emf in that circuit. The coil that is connected to the galvanometer cannot tell in which of these experiments it is participating; it is aware only that the flux passing through its cross-sectional area is changing. The flux through a circuit can also be changed by changing its shape, that is, by squeezing or stretching it.

▶ **Example 1.** A long solenoid has 200 turns/cm and carries a current of 1.5 amp; its diameter is 3.0 cm. At its center we place a 100-turn, close-packed coil of diameter 2.0 cm. This coil is arranged so that **B** at the center of the solenoid is parallel to its axis. The current in the solenoid is reduced to zero and then raised to 1.5 amp in the other direction at a steady rate over a period of 0.050 sec. What induced emf appears in the coil while the current is being changed?

The induction B at the center of the solenoid is given by Eq. 34–7, or

$$B = \mu_0 ni = (4\pi \times 10^{-7}\text{ weber/amp-m})(200 \times 10^2\text{ turns/meter})(1.5\text{ amp})$$

$$= 3.8 \times 10^{-2}\text{ weber/meter}^2.$$

The area of the coil (not of the solenoid) is 3.1×10^{-4} meter². The initial flux Φ_B through each turn of the coil is given by

$$\Phi_B = BA = (3.8 \times 10^{-2}\text{ weber/meter}^2)(3.1 \times 10^{-4}\text{ meter}^2) = 1.2 \times 10^{-5}\text{ weber}.$$

The flux goes from an initial value of 1.2×10^{-5} weber to a final value of -1.2×10^{-5} weber. The *change* in flux $\Delta\Phi_B$ for each turn of the coil during the 0.050-sec period is thus twice the initial value. The induced emf is given by

$$\mathcal{E} = -\frac{N\Delta\Phi_B}{\Delta t} = -\frac{(100)(2 \times 1.2 \times 10^{-5}\text{ weber})}{0.050\text{ sec}} = -4.8 \times 10^{-2}\text{ volt} = -48\text{ mv.}$$

The minus sign deals with the *direction* of the emf, as we explain below. ◀

35–3 Lenz's Law

So far we have not specified the directions of the induced emfs. Although these directions can be found from a formal analysis of Faraday's law, we prefer to find them from the conservation-of-energy principle which, in this context, takes the form of Lenz's law: *The induced current will appear in such a direction that it opposes the change that produced it.** The minus sign in Faraday's law suggests this opposition. In mechanics the energy principle often allows us to draw conclusions about mechanical systems without analyzing them in detail. We use the same approach here.

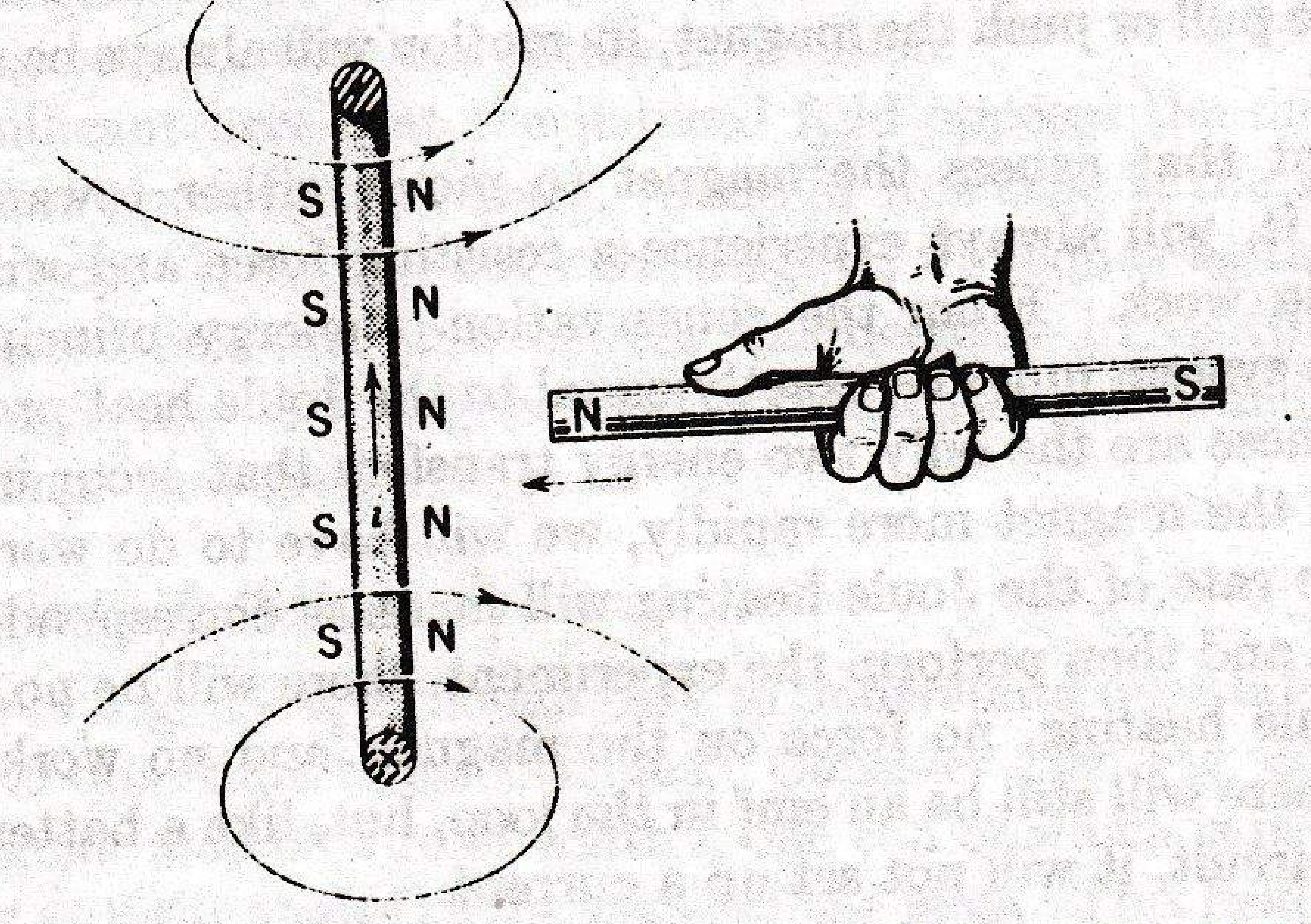


Fig. 35–3 If the magnet is moved toward the loop, the induced current points as shown, setting up a magnetic field that opposes the motion of the magnet.

Lenz's law refers to induced *currents*, which means that it applies only to closed circuits. If the circuit is open, we can usually think in terms of what would happen if it were closed and in this way find the direction of the induced emf.

Consider the first of Faraday's experiments described in Section 35–1. Figure 35–3 shows the north pole of a magnet and a cross section of a nearby conducting loop. As we push the magnet toward the loop, an induced current is set up in the loop. What is its direction?

* Heinrich Friedrich Lenz (1804–1865) deduced this law in 1834. Faraday also discovered how to determine the directions of induced emfs, but he did not express his results as succinctly as Lenz.

A current loop sets up a magnetic field at distant points like that of a magnetic dipole, one face of the loop being a north pole, the opposite face being a south pole. The north pole, as for bar magnets, is that face *from* which the lines of **B** emerge. If, as Lenz's law predicts, the loop in Fig. 35–3 is to oppose the motion of the magnet toward it, the face of the loop toward the magnet must become a north pole. The two north poles—one of the current loop and one of the magnet—will repel each other. The right-hand rule shows that for the magnetic field set up by the loop to emerge from the right face of the loop the induced current must be as shown. The current will be counterclockwise as we sight along the magnet toward the loop.

When we push the magnet toward the loop (or the loop toward the magnet), an induced current appears. In terms of Lenz's law this pushing is the "change" that produces the induced current, and, according to this law, the induced current will oppose the "push." If we pull the magnet away from the coil, the induced current will oppose the "pull" by creating a *south* pole on the right-hand face of the loop of Fig. 35–3. To make the right-hand face a south pole, the current must be opposite to that shown in Fig. 35–3. Whether we pull or push the magnet, its motion will always be automatically opposed.

The agent that causes the magnet to move, either toward the coil or away from it, will always experience a resisting force and will thus be required to do work. From the conservation-of-energy principle this work done on the system must be exactly equal to the Joule heat produced in the coil, since these are the only two energy transfers that occur in the system. If we move the magnet more rapidly, we will have to do work at a faster rate and the rate of the Joule heating will increase correspondingly. If we cut the loop and then perform the experiment, there will be no induced current, no Joule heating, no force on the magnet, and no work required to move it. There will still be an emf in the loop, but, like a battery connected to an open circuit, it will not set up a current.

If the current in Fig. 35–3 were in the *opposite* direction to that shown, the face of the loop toward the magnet would be a south pole, which would pull the bar magnet toward the loop. We would only need to push the magnet slightly to start the process and then the action would be self-perpetuating. The magnet would accelerate toward the loop, increasing its kinetic energy all the time. At the same time Joule heat would appear in the loop at a rate that would increase with time. This would indeed be a something-for-nothing situation! Needless to say, it does not occur.

Let us apply Lenz's law to Fig. 35–3 in a different way. Figure 35–4 shows the lines of **B** for the bar magnet.* On this point of view the "change" is the increase in Φ_B through the loop caused by bringing the magnet nearer. The induced current opposes this change by setting up a field that tends to oppose the increase in flux caused by the moving magnet. Thus the field

* There are two fields of **B** in this problem—one connected with the current loop and one with the bar magnet. The student must always be certain which one is meant.

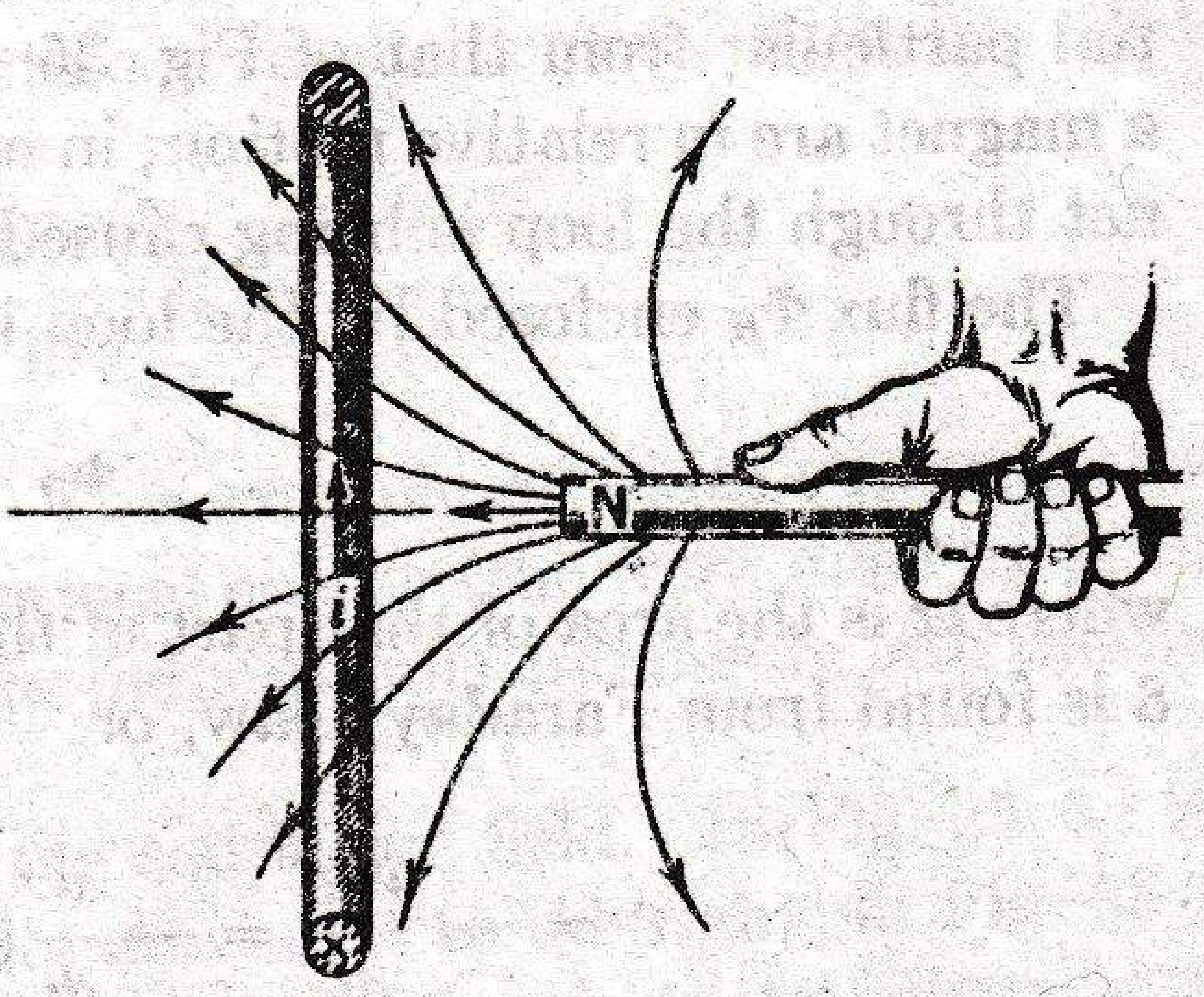


Fig. 35-4 In moving the magnet toward the loop, we increase Φ_B through the loop.

due to the induced current must point from left to right through the plane of the coil, in agreement with our earlier conclusion.

It is not significant here that the induced field opposes the magnet *field* but rather that it opposes the *change*, which in this case is the *increase* in Φ_B through the loop. If we withdraw the magnet, we reduce Φ_B through the loop. The induced field will now oppose this decrease in Φ_B (that is, the change) by *re-enforcing* the magnet field. In each case the induced field opposes the change that gives rise to it.

35-4 Induction—A Quantitative Study

The example of Fig. 35-4, although easy to understand qualitatively, does not lend itself to quantitative calculations. Consider then Fig. 35-5, which shows a rectangular loop of wire of width l, one end of which is in a uniform field **B** pointing at right angles to the plane of the loop. This field of **B** may be produced in the gap of a large electromagnet like that of Fig. 33-2. The dashed lines show the assumed limits of the magnetic field. The experiment consists in pulling the loop to the right at a constant speed v.

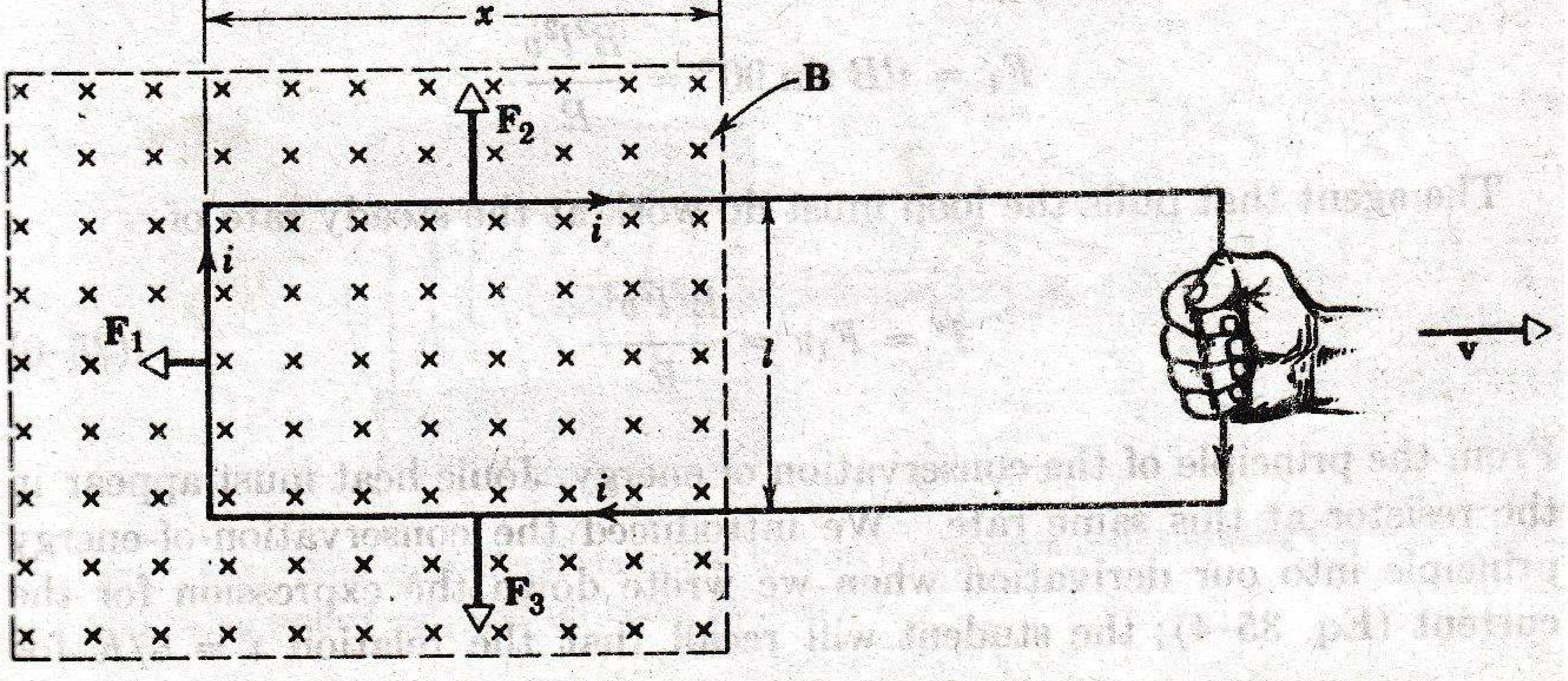


Fig. 35-5 A rectangular loop is pulled out of a magnetic field with velocity v.

Note that the situation described by Fig. 35–5 does not differ in any essential particular from that of Fig. 35–4. In each case a conducting loop and a magnet are in relative motion; in each case the flux of the field of the magnet through the loop is being caused to change with time.

The flux Φ_B enclosed by the loop in Fig. 35–5 is

$$\Phi_B = Blx,$$

where lx is the area of that part of the loop in which B is not zero. The emf $\mathcal{E}$ is found from Faraday's law, or

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(Blx) = -Bl\frac{dx}{dt} = Blv, \qquad (35\text{–}3)$$

where we have set $-dx/dt$ equal to the speed v at which the loop is pulled out of the magnetic field. Note that the only dimension of the loop that enters into Eq. 35–3 is the length l of the left end conductor. As we shall see later, the induced emf in Fig. 35–5 may be regarded as localized here. An induced emf such as this, produced by pulling a conductor through a magnetic field, is sometimes called a *motional emf.*

The emf Blv sets up a current in the loop given by

$$i = \frac{\mathcal{E}}{R} = \frac{Blv}{R}, \qquad (35\text{–}4)$$

where R is the loop resistance. From Lenz's law, this current (and thus $\mathcal{E}$) must be clockwise in Fig. 35–5; it opposes the "change" (the decrease in Φ_B) by setting up a field that is parallel to the external field within the loop.

The current in the loop will cause forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ to act on the three conductors, in accord with Eq. 33–6*a*, or

$$\mathbf{F} = i\mathbf{l} \times \mathbf{B}. \qquad (35\text{–}5)$$

Because $\mathbf{F}_2$ and $\mathbf{F}_3$ are equal and opposite, they cancel each other; $\mathbf{F}_1$, which is the force that opposes our effort to move the loop, is given in magnitude from Eqs. 35–5 and 35–4 as

$$F_1 = ilB \sin 90° = \frac{B^2l^2v}{R}.$$

The agent that pulls the loop must do work at the steady rate of

$$P = F_1v = \frac{B^2l^2v^2}{R}. \qquad (35\text{–}6)$$

From the principle of the conservation of energy, Joule heat must appear in the resistor at this same rate. We introduced the conservation-of-energy principle into our derivation when we wrote down the expression for the current (Eq. 35–4); the student will recall that the relation $i = \mathcal{E}/R$ for single-loop circuits is a direct consequence of this principle. Thus we should be able to write down the expression for the rate of Joule heating in the loop with the expectation that we will obtain a result identical with Eq. 35-6. Recalling Eq. 35-4, we put

$$P_J = i^2R = \left(\frac{Blv}{R}\right)^2 R = \frac{B^2l^2v^2}{R},$$

which is indeed the expected result. This example provides a quantitative illustration of the conversion of mechanical energy (the work done by an external agent) into electrical energy (the induced emf) and finally into thermal energy (the Joule heating).

Figure 35-6 shows a side view of the coil in the field. In Fig. 35-6*a* the coil is stationary; in Fig. 35-6*b* we are moving it to the right; in Fig. 35-6*c* we are moving it to the left. The lines of induction in these figures represent the *resultant field* produced by the vector addition of the field $\mathbf{B}_0$ due to the magnet and the field $\mathbf{B}_i$ due to the induced current, if any, in the coil.

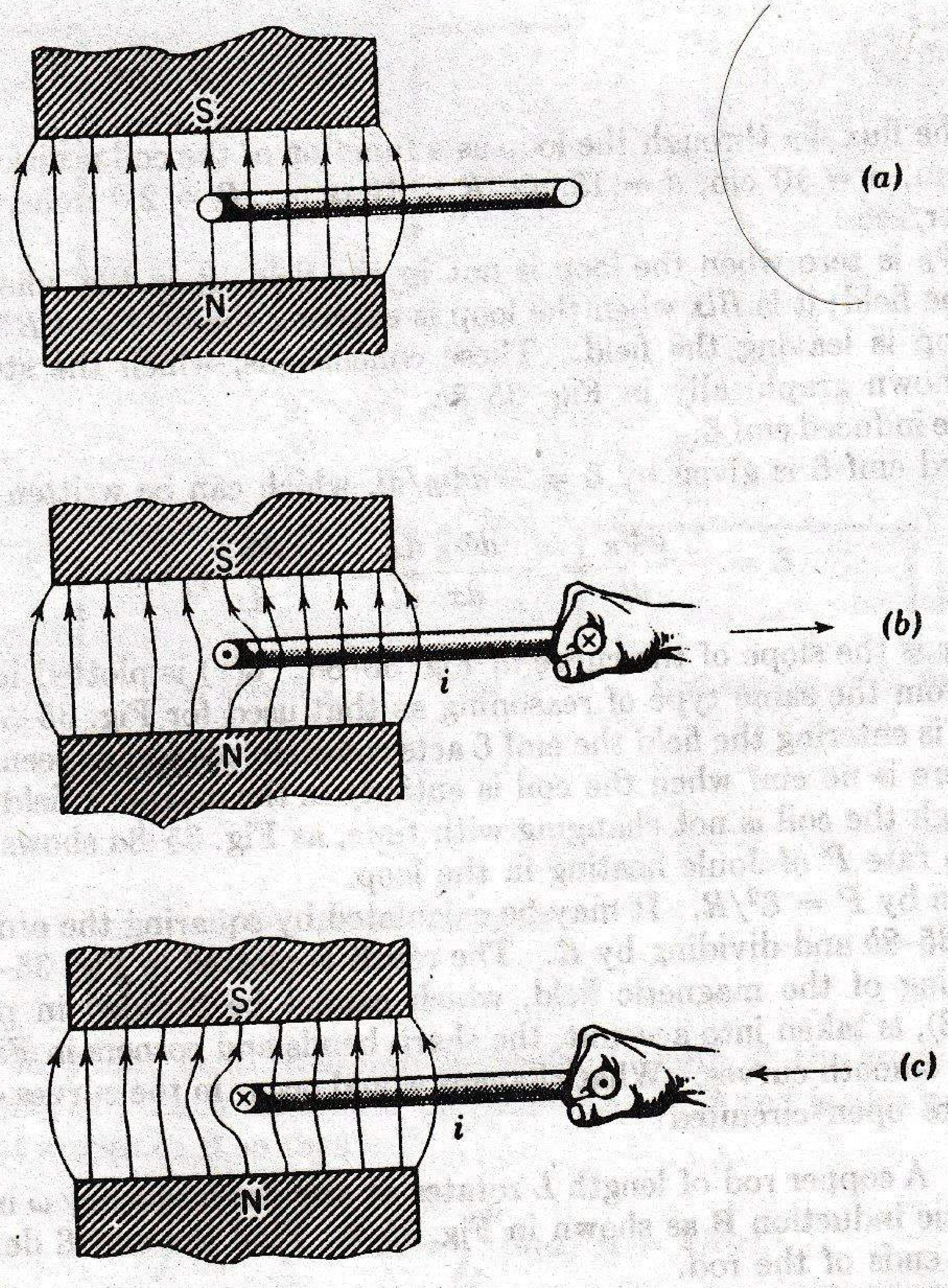


Fig. 35-6 Side view of a rectangular loop in a magnetic field showing the loop (*a*) at rest, (*b*) being pulled out, and (*c*) being pushed in.

These lines suggest convincingly that the agent moving the coil always experiences an opposing force.

▶ **Example 2.** Figure 35–7 shows a rectangular loop of resistance R, width l, and length a being pulled at constant speed v through a region of thickness d in which a uniform field of induction **B** is set up by a magnet.

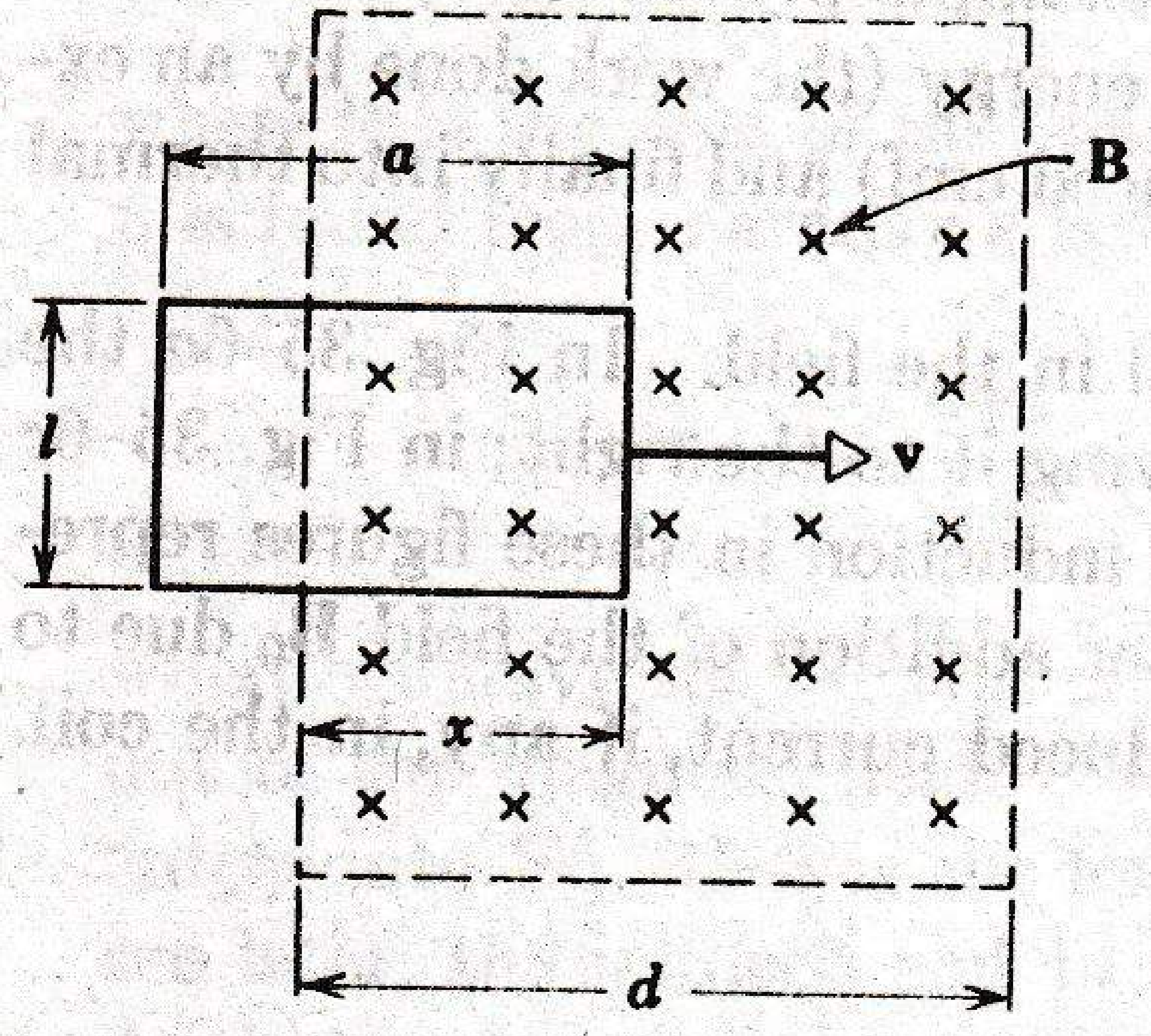


Fig. 35–7 Example 2. A rectangular loop is caused to move with a velocity **v** through a magnetic field. The position of the loop is measured by x, the distance between the effective left edge of the field **B** and the right end of the loop.

(*a*) Plot the flux Φ_B through the loop as a function of the coil position x. Assume that $l = 4$ cm, $a = 10$ cm, $d = 15$ cm, $R = 16$ ohms, $B = 2.0$ webers/meter², and $v = 1.0$ meter/sec.

The flux Φ_B is zero when the loop is not in the field; it is Bla when the loop is entirely in the field; it is Blx when the loop is entering the field and $Bl[a - (x - d)]$ when the loop is leaving the field. These conclusions, which the student should verify, are shown graphically in Fig. 35–8*a*.

(*b*) Plot the induced emf $\mathcal{E}$.

The induced emf $\mathcal{E}$ is given by $\mathcal{E} = -d\Phi_B/dt$, which can be written as

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d\Phi_B}{dx}\frac{dx}{dt} = -\frac{d\Phi_B}{dx}v,$$

where $d\Phi_B/dx$ is the slope of the curve of Fig. 35–8*a*. $\mathcal{E}(x)$ is plotted in Fig. 35–8*b*. Lenz's law, from the same type of reasoning as that used for Fig. 35–5, shows that when the coil is entering the field the emf $\mathcal{E}$ acts counterclockwise as seen from above. Note that there is no emf when the coil is entirely in the magnetic field because the flux Φ_B through the coil is not changing with time, as Fig. 35–8*a* shows.

(*c*) Plot the rate P of Joule heating in the loop.

This is given by $P = \mathcal{E}^2/R$. It may be calculated by squaring the ordinate of the curve of Fig. 35–8*b* and dividing by R. The result is plotted in Fig. 35–8*c*.

If the fringing of the magnetic field, which cannot be avoided in practice (see Problem 34–28), is taken into account, the sharp bends and corners in Fig. 35–8 will be replaced by smooth curves. What changes would occur in the curves of Fig. 35–8 if the coil were open circuited?

Example 3. A copper rod of length L rotates at angular frequency ω in a uniform field of magnetic induction **B** as shown in Fig. 35–9. Find the emf $\mathcal{E}$ developed between the two ends of the rod.

If a wire of length dl is moved at velocity **v** at right angles to a field **B**, a motional emf $d\mathcal{E}$ will be developed (see Eq. 35–3) given by

$$d\mathcal{E} = Bv\,dl.$$

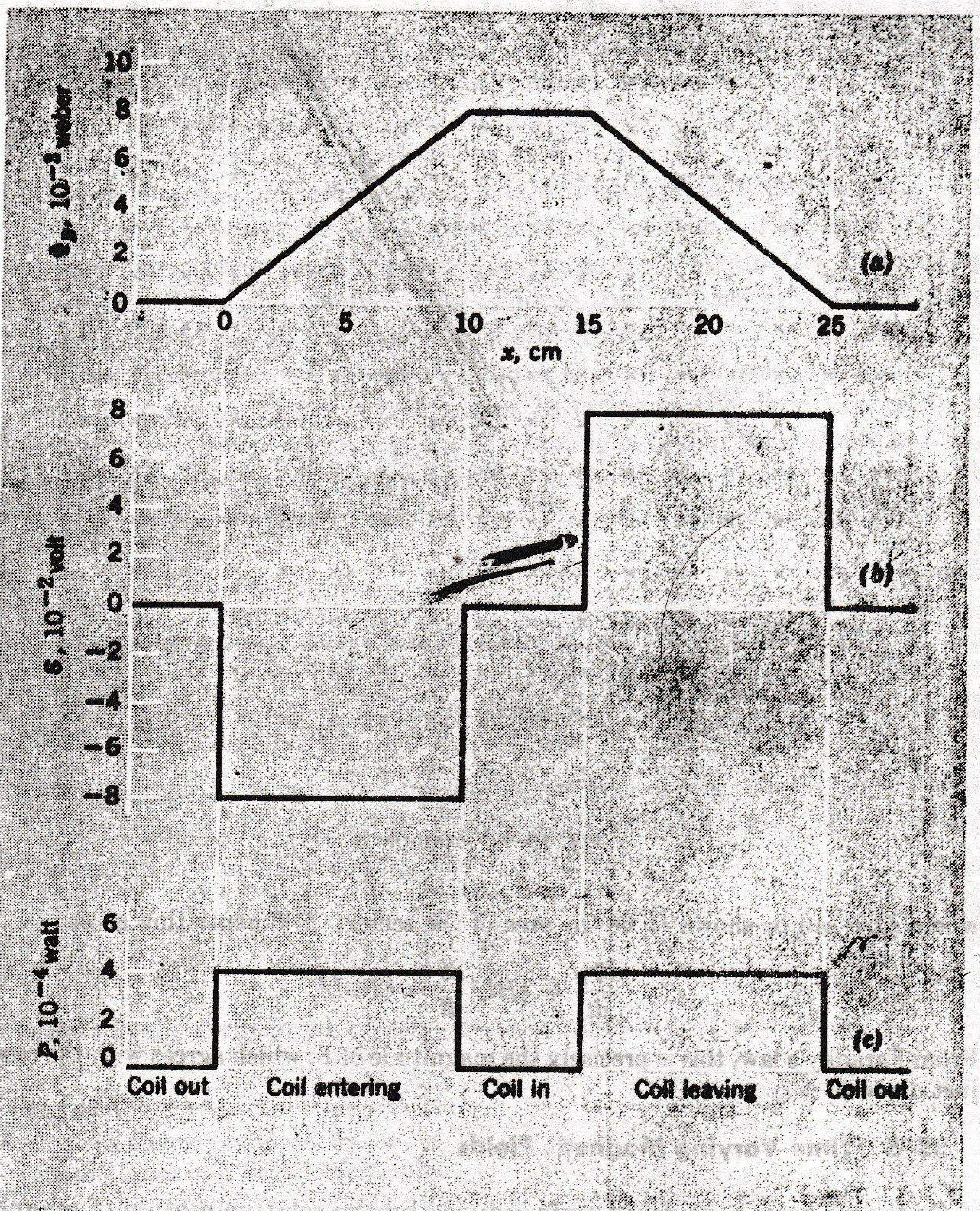


Fig. 35–8 Example 2.

The rod of Fig. 35–9 may be divided into elements of length dl, the linear speed v of each element being ωl. Each element is perpendicular to **B** and is also moving in a direction at right angles to **B** so that

$$\mathcal{E} = \int d\mathcal{E} = \int_0^L Bv\, dl = \int_0^L B(\omega l)\, dl = \tfrac{1}{2}B\omega L^2.$$

For a second approach, consider that at any instant the flux enclosed by the sector aOb in Fig. 35–9 is given by

$$\Phi_B = BA = B(\tfrac{1}{2}L^2\theta),$$

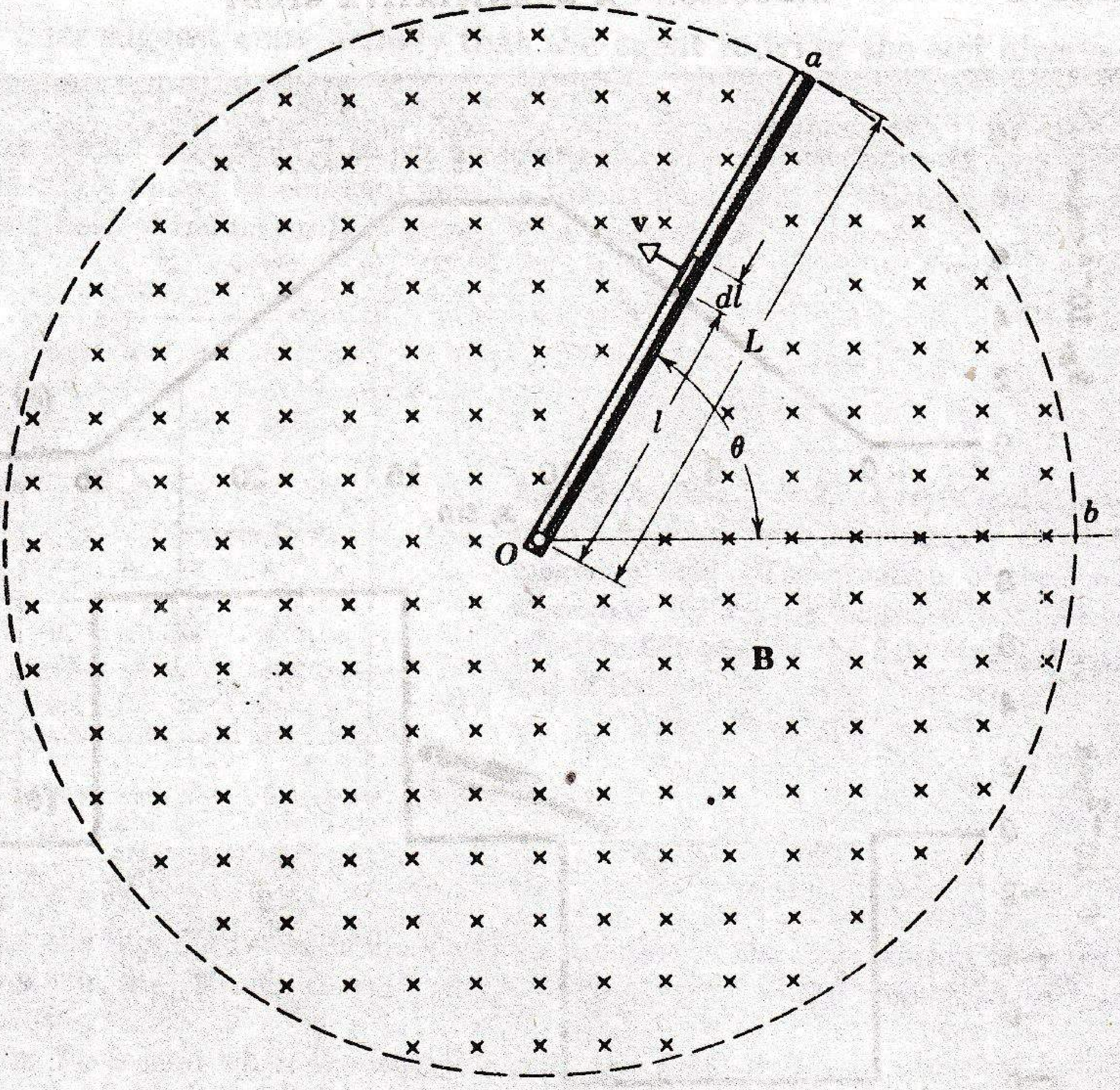


Fig. 35–9 Example 3.

where $\frac{1}{2}L^2\theta$ can be shown to be the area of the sector. Differentiating gives

$$\frac{d\Phi_B}{dt} = \tfrac{1}{2}BL^2\frac{d\theta}{dt} = \tfrac{1}{2}B\omega L^2.$$

From Faraday's law, this is precisely the magnitude of $\mathcal{E}$, which agrees with the result just derived. ◀

35–5 Time-Varying Magnetic Fields

So far we have considered emfs induced by the relative motion of magnets and coils. In this section we assume that there is no physical motion of gross objects but that the magnetic field may vary with time. If a conducting loop is placed in such a time-varying field, the flux through the loop will change and an induced emf will appear in the loop. This emf will set the charge carriers in motion, that is, it will induce a current.

From a microscopic point of view we can say, equally well, that the changing flux of **B** sets up an induced electric field **E** at various points around the loop. These induced electric fields are just as real as electric fields set up by static charges and will exert a force **F** on a test charge q_0 given by $\mathbf{F} = q_0\mathbf{E}$. Thus we can restate Faraday's law of induction in a loose but informative way as: *A changing magnetic field produces an electric field.*

To fix these ideas, consider Fig. 35-10, which shows a uniform field of induction **B** at right angles to the plane of the page. We assume that **B** is increasing in magnitude at the same constant rate dB/dt at every point. This could be done by causing the current in the windings of the electromagnet that establishes the field to increase with time in the proper way.

The circle of arbitrary radius r shown in Fig. 35-10 encloses, at any instant, a flux Φ_B. Because this flux is changing with time, an induced emf given by $\mathcal{E} = -d\Phi_B/dt$ will appear around the loop. The electric fields **E** induced at various points of the loop must, from symmetry, be tangent to the loop. Thus the electric lines of force that are set up by the changing magnetic field are in this case concentric circles.

If we consider a test charge q_0 moving around the circle of Fig. 35-10, the work W done on it per revolution is, in terms of the definition of an emf, simply $\mathcal{E}q_0$. From another point of view, it is $(q_0E)(2\pi r)$, where q_0E is the force that acts on the charge and $2\pi r$ is the distance over which the force acts. Setting the two expressions for W equal and canceling q_0 yields

$$\mathcal{E} = E2\pi r. \tag{35-7}$$

In a more general case than that of Fig. 35-10 we must write

$$\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l}. \tag{35-8}$$

If this integral is evaluated for the conditions of Fig. 35-10, we obtain Eq. 35-7 at once. If Eq. 35-8 is combined with Eq. 35-1 ($\mathcal{E} = -d\Phi_B/dt$), Faraday's law of induction can be written as

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}, \tag{35-9}$$

which is the form in which this law is expressed in Table 38-3.

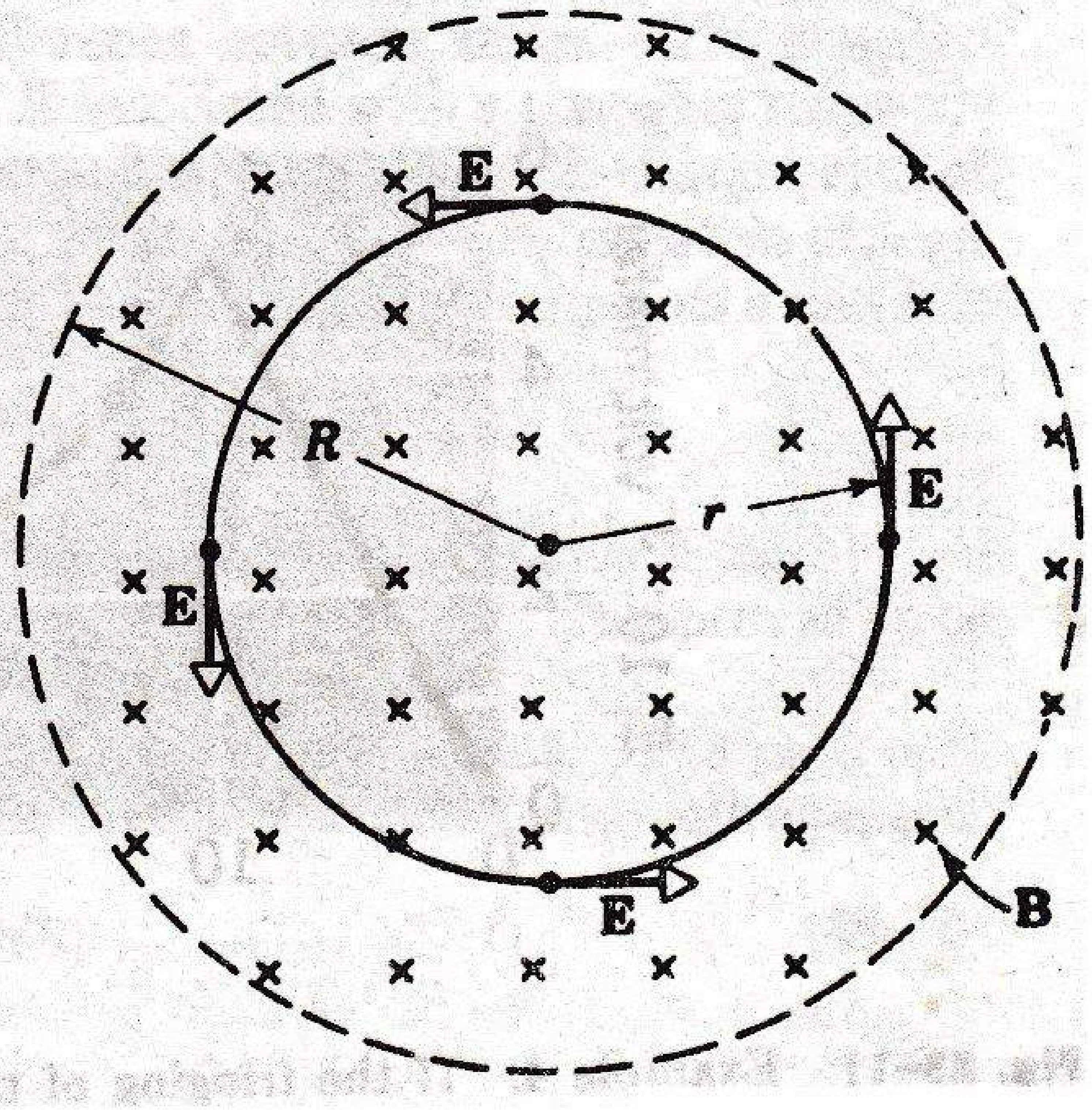


Fig. 35-10 The induced electric fields at four points produced by an increasing magnetic field. We assume that B has symmetry about an axis perpendicular to the page through the center of the circle of radius r. This can be arranged by assuming that the magnetic induction decreases slightly along a radius starting from the center of the figure, the value of B being constant for a particular value of r. (The magnetic field cannot end abruptly at radius R—unless a special arrangement of currents is provided—but must approach zero gradually. This "fringing" does not change any of the arguments of this section.

▶ **Example 4.** Let B in Fig. 35–10 be increasing at the rate dB/dt. Let R be the radius of the cylindrical region in which the magnetic field is assumed to exist. What is the magnitude of the electric field **E** at any radius r? Assume that $dB/dt = 0.10$ weber/m²-sec and $R = 10$ cm.

(*a*) For $r < R$, the flux Φ_B through the loop is

$$\Phi_B = B(\pi r^2).$$

Substituting into Faraday's law (Eq. 35–9),

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}$$

yields

$$(E)(2\pi r) = -\frac{d\Phi_B}{dt} = -(\pi r^2)\frac{dB}{dt}.$$

Solving for E yields

$$E = -\tfrac{1}{2}r\frac{dB}{dt}.$$

The minus sign is retained to suggest that the induced electric field **E** acts to *oppose* the change of the magnetic field. Note that $E(r)$ depends on dB/dt and not on B. Substituting numerical values, assuming $r = 5$ cm, yields, for the magnitude of E,

$$E = \tfrac{1}{2}r\frac{dB}{dt} = (\tfrac{1}{2})(5\times 10^{-2}\text{ meter})\left(\frac{0.10\text{ weber}}{\text{m}^2\text{-sec}}\right) = 2.5\times 10^{-3}\text{ volt/meter}.$$

(*b*) For $r > R$ the flux through the loop is

$$\Phi_B = \int \mathbf{B}\cdot d\mathbf{S} = B(\pi R^2).$$

This equation is true because $\mathbf{B}\cdot d\mathbf{S}$ is zero for those points of the loop that lie outside the effective boundary of the magnetic field.

From Faraday's law (Eq. 35–9),

$$(E)(2\pi r) = -\frac{d\Phi_B}{dt} = -(\pi R^2)\frac{dB}{dt}.$$

Solving for E yields

$$E = -\frac{1}{2}\frac{R^2}{r}\frac{dB}{dt}.$$

These two expressions for $E(r)$ yield the same result, as they must, for $r = R$. Figure 35–11 is a plot of the magnitude of $E(r)$ for the numerical values given. ◀

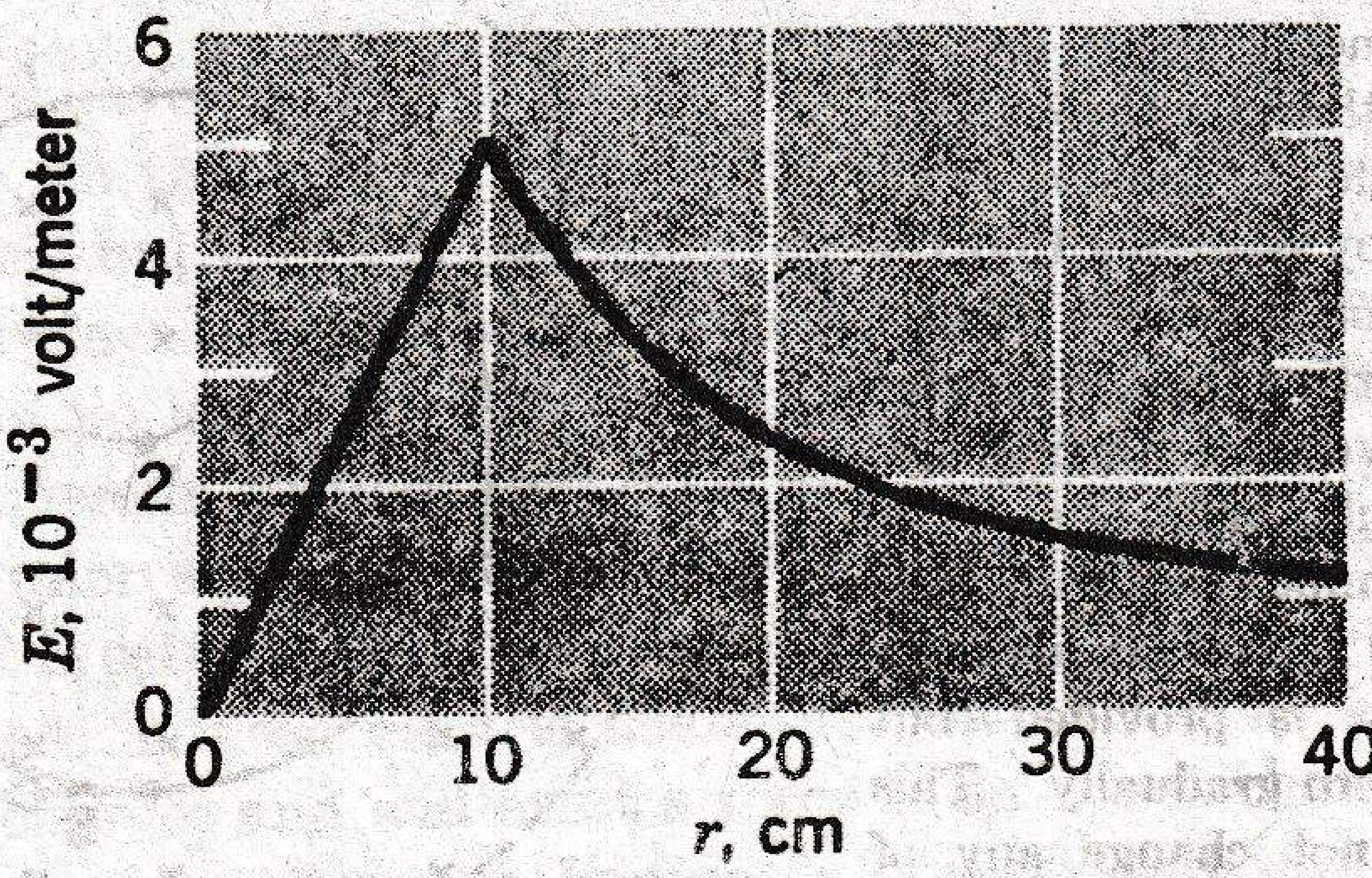


Fig. 35–11 Example 4. If the fringing of the field in Fig. 35–10 were to be taken into account, the result would be a rounding of the sharp cusp at $r = R$ (= 10 cm).

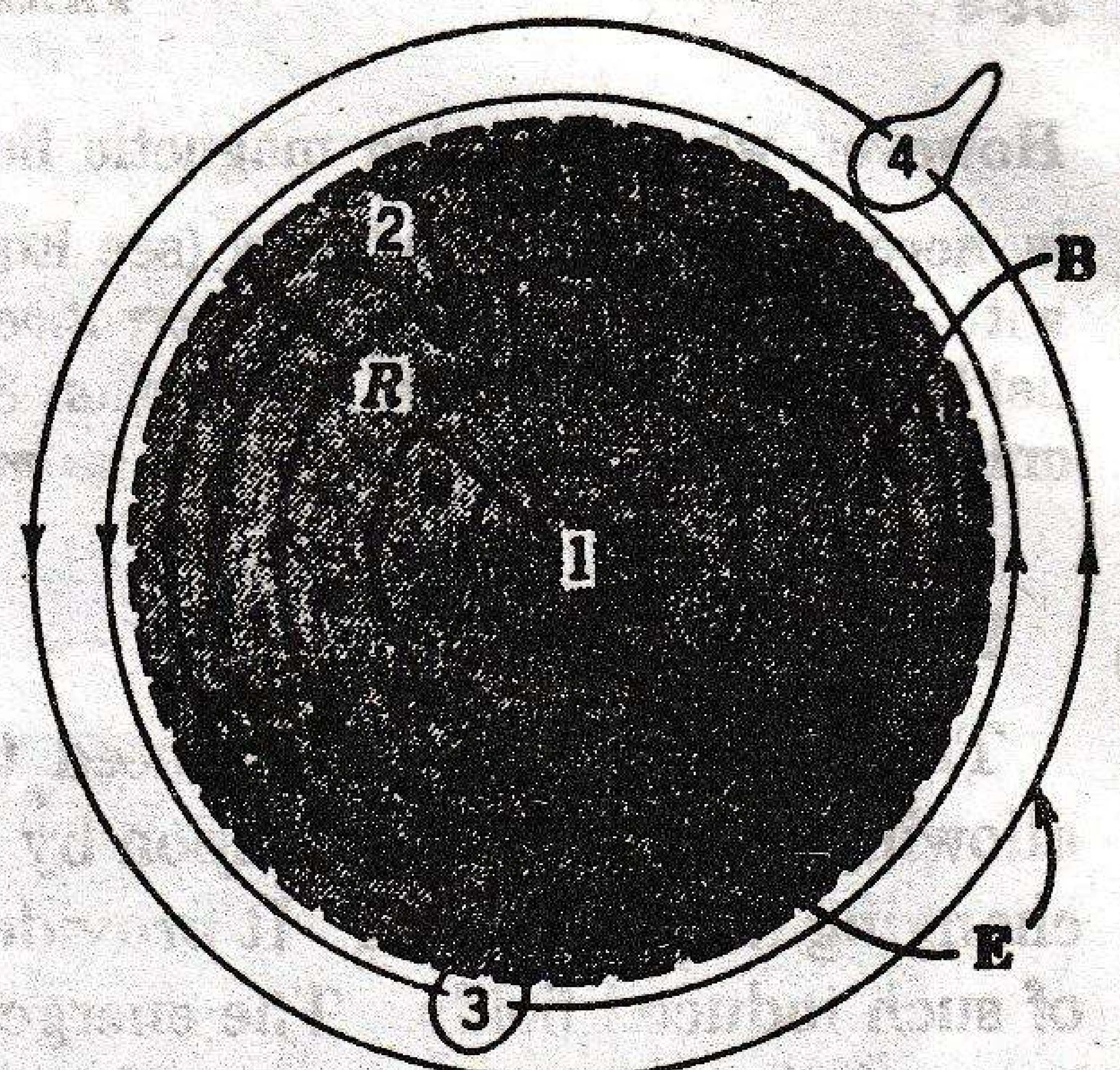


Fig. 35-12 Showing the circular lines of E from an increasing magnetic field. The four loops are imaginary paths around which an emf can be calculated.

In applying Lenz's law to Fig. 35-10, imagine that a circular conducting loop is placed concentrically in the field. Since Φ_B through this loop is increasing, the induced current in the loop will tend to oppose this "change" by setting up a magnetic field of its own that points up within the loop. Thus the induced current i must be counterclockwise, which means that the lines of the induced electric field **E**, which is responsible for the current, must also be counterclockwise. If the magnetic field in Fig. 35-10 were *decreasing* with time, the induced current and the lines of force of the induced electric field **E** would be clockwise, again opposing the *change* in Φ_B.

Figure 35-12 shows four of many possible loops to which Faraday's law may be applied. For loops 1 and 2, the induced emf $\mathcal{E}$ is the same because these loops lie entirely within the changing magnetic field and thus have the same value of $d\Phi_B/dt$. Note that even though the emf $\mathcal{E}$ $(= \oint \mathbf{E}\cdot d\mathbf{l})$ is the same for these two loops, the distribution of electric fields **E** around the perimeter of each loop, as indicated by the electric lines of force, is different. For loop 3 the emf is less because Φ_B and $d\Phi_B/dt$ for this loop are less, and for loop 4 the induced emf is zero.

The induced electric fields that are set up by the induction process are not associated with charges but with a changing magnetic flux. Although both kinds of electric fields exert forces on charges, there is a difference between them. The simplest manifestation of this difference is that lines of **E** associated with a changing magnetic flux can form closed loops (see Fig. 35-12); lines of **E** associated with charges cannot but can always be drawn to start on a positive charge and end on a negative charge.

Equation 29-5, which defined the potential difference between two points a and b, is

$$V_b - V_a = \frac{W_{ab}}{q_0} = -\int_a^b \mathbf{E}\cdot d\mathbf{l}.$$

We have insisted that if potential is to have any useful meaning this integral (and W_{ab}) must have the same value for every path connecting a with b. This proved to be true for every case examined in earlier chapters.

An interesting special case comes up if a and b are the same point. The path connecting them is now a closed loop; V_a must be identical with V_b and this equation reduces to

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0. \qquad (35\text{-}10)$$

However, when changing magnetic flux is present, $\oint \mathbf{E} \cdot d\mathbf{l}$ is precisely *not* zero but is, according to Faraday's law (see Eq. 35–9), $-d\Phi_B/dt$. Electric fields associated with stationary charges are *conservative*, but those associated with changing magnetic fields are *nonconservative;* see Section 8–2. Electric potential, which can be defined only for a conservative force, *has no meaning for electric fields produced by induction.*

35–6 The Betatron

The betatron is a device used to accelerate electrons to high speeds by allowing them to be acted upon by induced electric fields that are set up by a changing magnetic flux. It provides an excellent illustration of the "reality" of such induced fields. The energetic electrons can be used for fundamental research in physics or to produce penetrating X-rays which are useful in cancer therapy and in industry.

Figure 35–13 shows the 100-Mev betatron at the General Electric Company. At this energy the electron speed is $0.999986c$, where c is the speed of light, so that relativistic mechanics must certainly be used in the analysis of its operation. Figure 35–14 shows a vertical cross section through the central part of the betatron to which the man in Fig. 35–13 is pointing.

The magnetic field in the betatron has several functions: (*a*) it guides the electrons in a circular path; (*b*) it accelerates the electrons in this path;

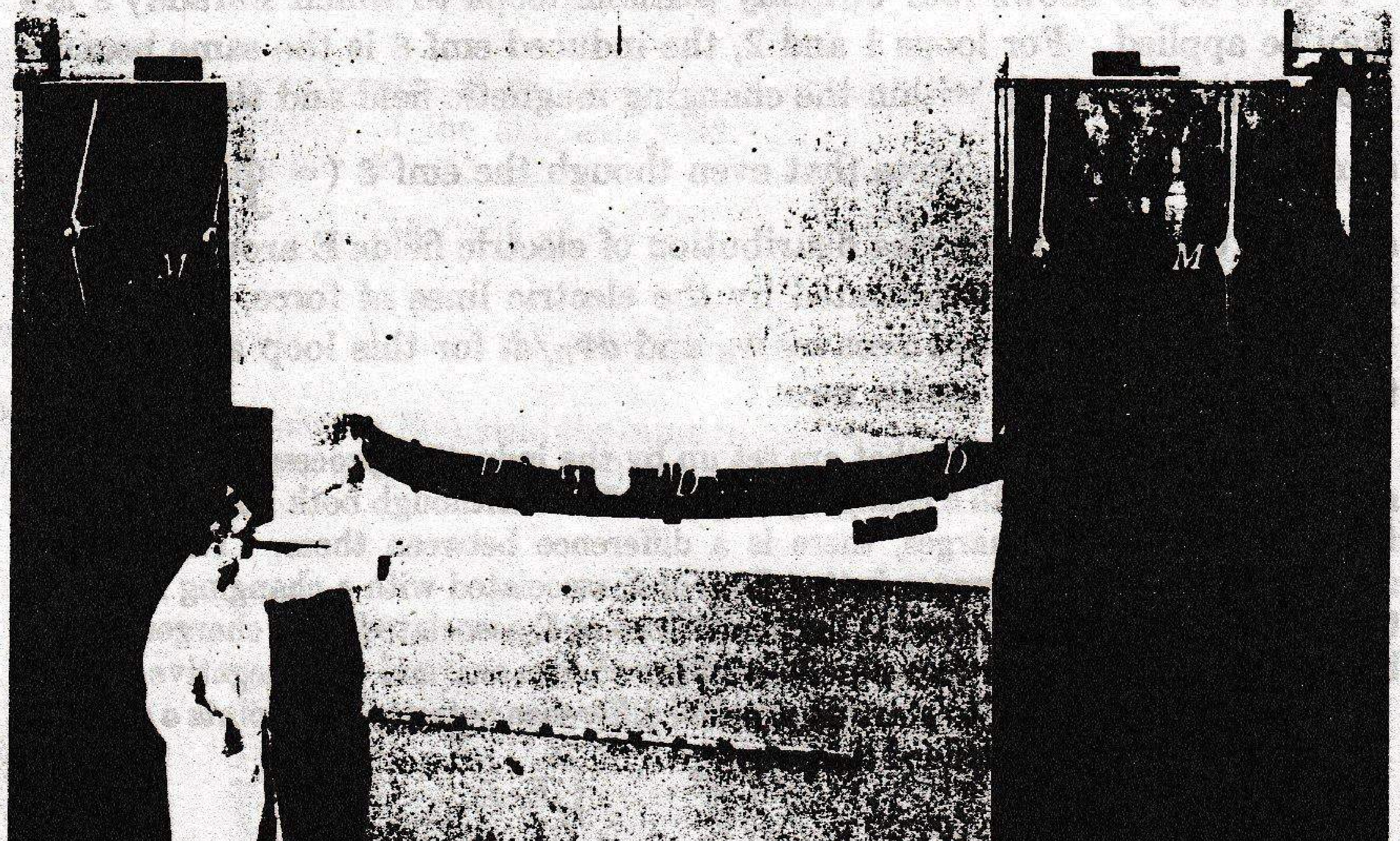


Fig. 35–13 A 100-Mev betatron. *M* shows the magnet, *C* the magnetizing coils, and *D* the region in which the "doughnut" is located. (Courtesy General Electric Company.)

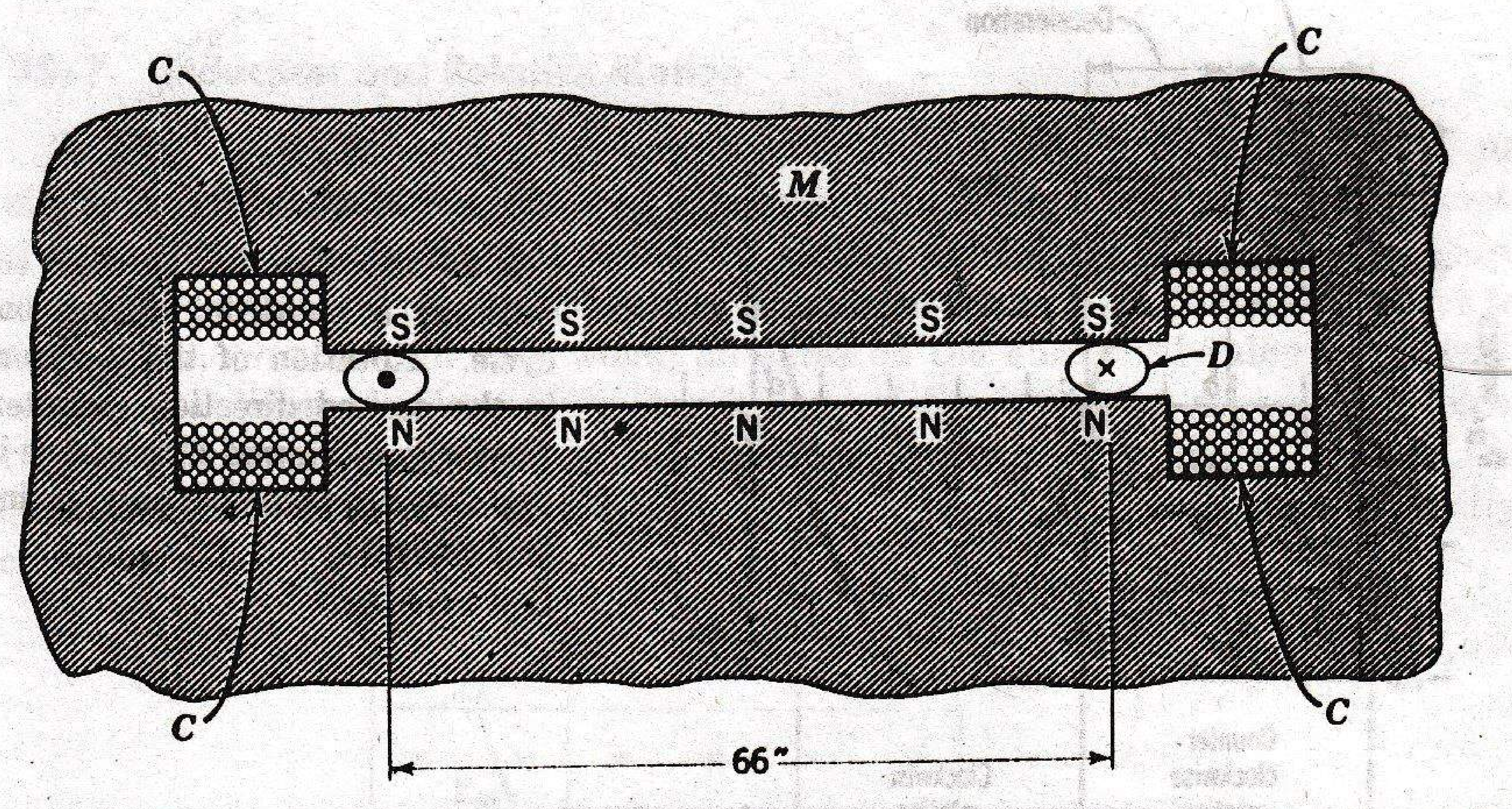


Fig. 35-14 Cross section of a betatron, showing magnet M, coils C, and "doughnut" D. Electrons emerge from the page at the left and enter it at the right.

(*c*) it keeps the radius of the orbit in which the electrons are moving a constant; (*d*) it introduces the electrons into the orbit initially and removes them from the orbit after they have reached full energy; and finally (*e*) it provides a restoring force that resists any tendency for the electrons to leave their orbit, either vertically or radially. It is remarkable that it is possible to do all these things by proper shaping and control of the magnetic field.

The object marked D in Fig. 35-14 is an evacuated glass "doughnut" inside which the electrons travel. Their orbit is a circle at right angles to the plane of the figure. The electrons emerge from the plane at the left (·) and enter it at the right (×). In the General Electric machine the radius of the electron path is 33 in. The coils C and the 130-ton steel magnet shown in Fig. 35-13 provide the magnetic flux that passes through the plane of this orbit.

The current in coils C is made to alter periodically, 60 times/sec, to produce a changing flux through the orbit, shown in Fig. 35-15. Here Φ_B is taken as positive when $\mathbf{B}$ is pointing up, as in Fig. 35-14. If the electrons are to circulate in the direction shown, they must do so during the positive half-cycle, marked ac in Fig. 35-15. The student should verify this (see Section 33-6). The electrons are accelerated by electric fields set up by the changing flux. The direction of these induced fields depends on the sign of $d\Phi_B/dt$ and must be chosen to accelerate, and not to decelerate, the electrons. Thus only half the positive half-cycle in Fig. 35-15 can be used for acceleration; it will prove to be ab.

The average value of $d\Phi_B/dt$ during the quarter-cycle ab is the slope of the dashed line, or

$$\overline{\frac{d\Phi_B}{dt}} = \frac{1.8 \text{ weber}}{4.2 \times 10^{-3} \text{ sec}} = 430 \text{ volts.}$$

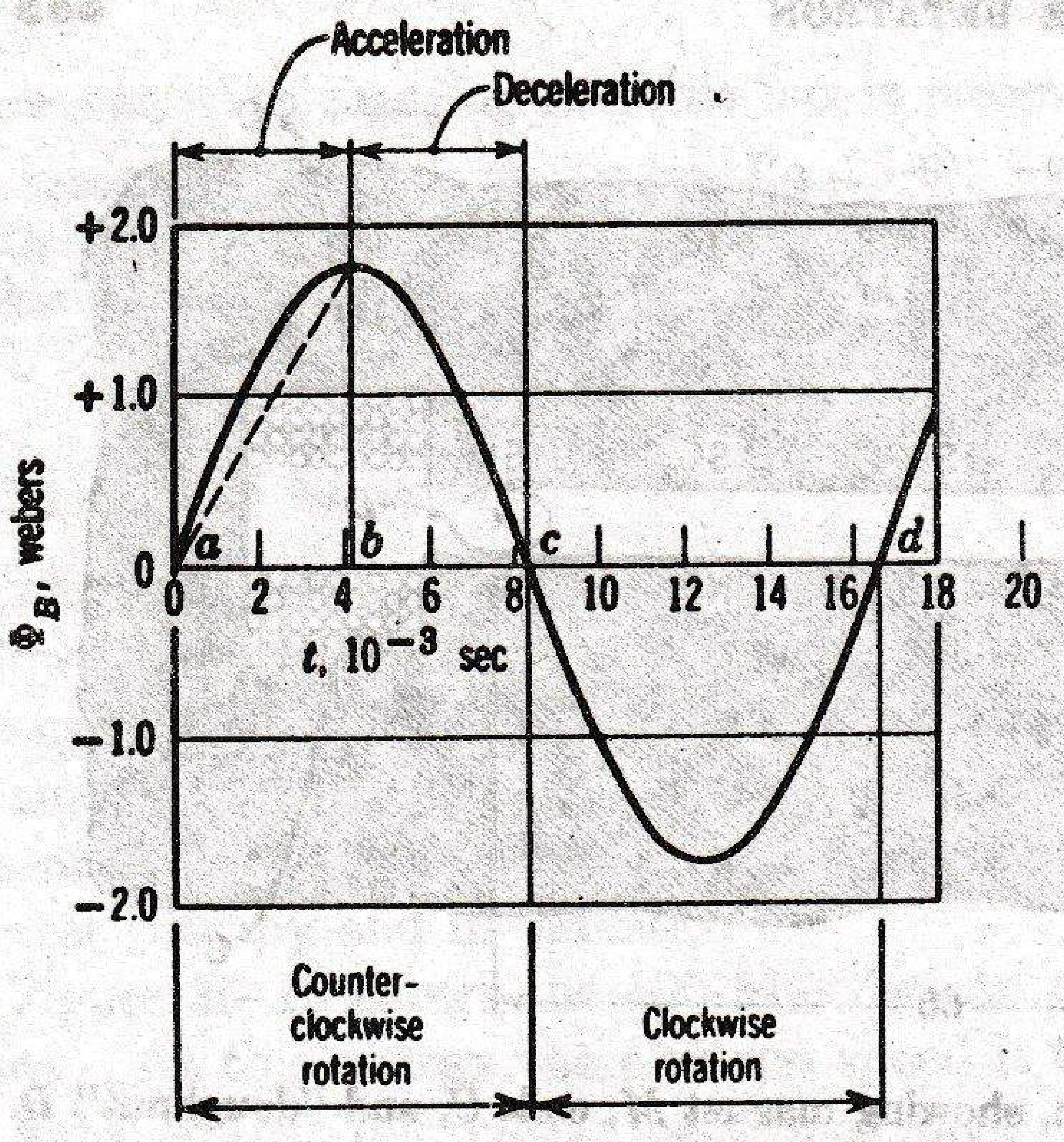


Fig. 35–15 The flux through the orbit of a betatron, during one cycle. Rotation of the electrons in the desired direction (counterclockwise as viewed from above in Fig. 35–14) is possible only during half-cycle *ac*.

From Faraday's law (Eq. 35–1), this is also the emf in volts. The electron will thus increase its energy by 430 ev every time it makes a trip around the orbit in the changing flux. If the electron gains only 430 ev of energy per revolution, it must make about 230,000 rev to gain its full 100 Mev. For an orbit radius of 33 in., this corresponds to a length of path of some 750 miles.

The betatron provides a good example of the fact that electric potential has no meaning for electric fields produced by induction. If a potential exists, it must be true that, as Eq. 35–10 shows, $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for any closed path. In the betatron, however, this integral, evaluated around the orbit, is precisely *not* zero but is, in our example, 430 volts. It must not be thought, of course, that the betatron violates the conservation of energy principle. The gain in kinetic energy of the circulating electron (430 ev/rev) must be supplied by an identifiable energy source. It comes, in fact, from the generator that energizes the magnet coils, thus providing the changing magnetic field. The energy is transmitted to the electron through the intermediary of this changing field.

▶ **Example 5.** In the betatron of Fig. 35–13, which is the "accelerating" quarter-cycle?

Let us assume that it is *ab* in Fig. 35–15, during which Φ_B through the orbit is *increasing*. If a conducting loop were placed to coincide with the orbit, an induced current would appear in the loop to oppose the tendency of Φ_B to increase. This means that a magnetic field would be set up that would oppose the field of the large magnet. Thus **E** would point outward at the right side of "doughnut" *D* in Fig. 35–14 and inward on the left side. The force $(-e\mathbf{E})$ acting on the electron is in the opposite direction to **E** because of the negative charge of the electron. Thus the tangential force acting on the electron is in the same direction as that at which it circulates in its orbit; this means that the speed of the electron will increase, as desired. The student should go through this same analysis carefully, assuming (incorrectly, as it will turn out) that the accelerating half-cycle is *bc* in Fig. 35–15 rather than *ab*.

35–7 Induction and Relative Motion

Faraday's law, in the form $\mathcal{E} = -d\Phi_B/dt$, gives correctly the induced emf no matter whether the change in Φ_B is produced by moving a coil, moving a magnet, changing the strength of a magnetic field, changing the shape of a conducting loop, or in other ways. However, observers who are in relative motion with respect to each other, even though they would all agree on the numerical value of the emf, would give different microscopic descriptions of the induction process. In electromagnetic systems, as well as in mechanical systems, it is important that the state of motion of the observer with respect to his environment be made perfectly clear.

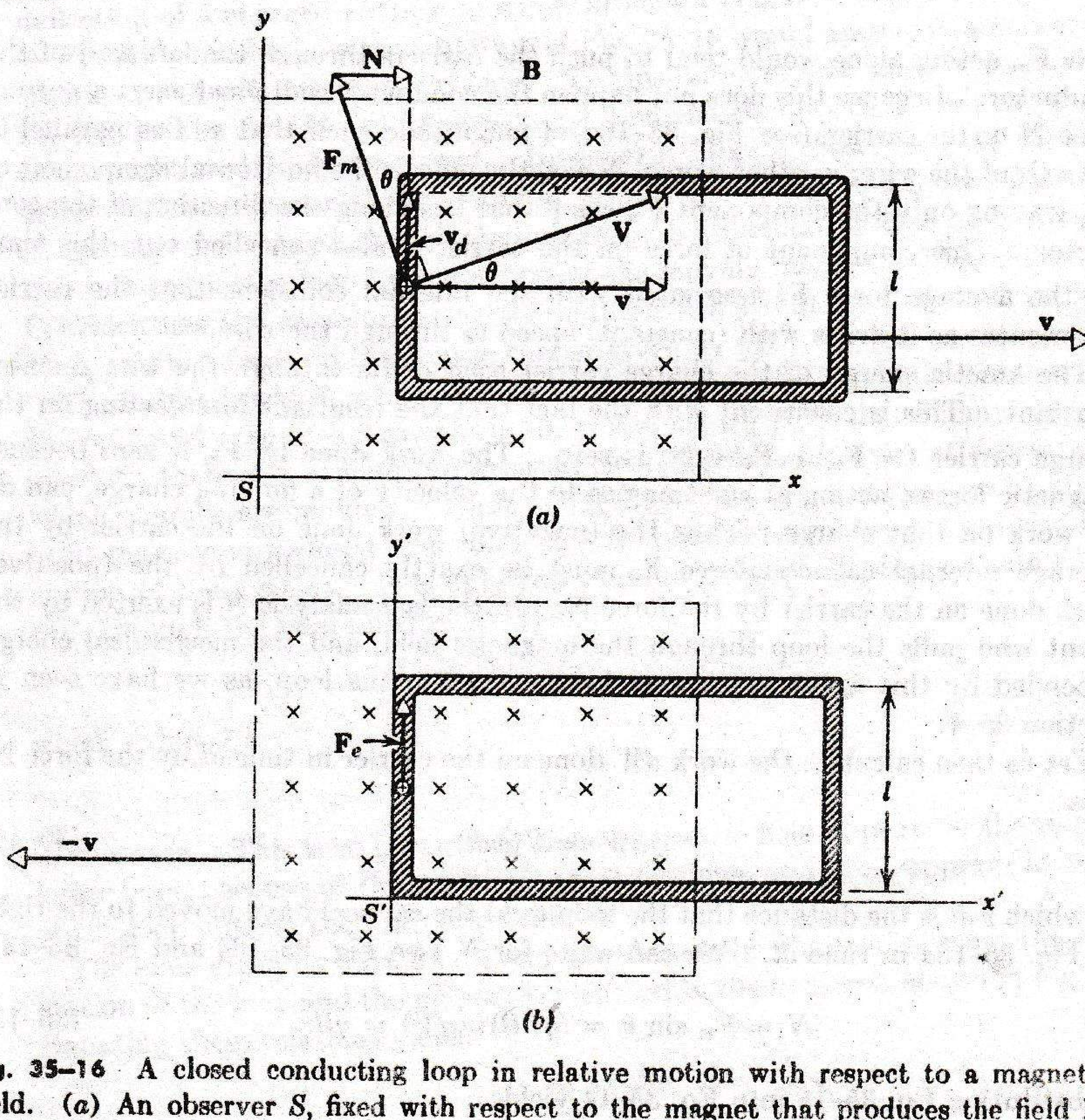


Fig. 35–16 A closed conducting loop in relative motion with respect to a magnetic field. (*a*) An observer *S*, fixed with respect to the magnet that produces the field **B**, sees the loop moving to the right. (*b*) An observer *S'*, fixed with respect to the loop, sees the magnet moving to the left.

Figure 35–16 shows a closed loop which is caused to move at velocity **v** with respect to a magnet that provides a uniform field **B** in the region shown. We consider first an observer, identified as *S*, who is *at rest with respect to the magnet* used to establish the field **B**; see Fig. 35–16*a*. The induced emf in this case is called a *motional emf* because the conducting loop is moving with respect to this observer.

Consider a positive charge carrier at the center of the left end of the conducting loop. To observer *S*, this charge, constrained to move to the right along with the loop, is a charge q moving, for an instant at least when the loop has just been set in motion, with a velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$ and as such it experiences a sideways magnetic deflecting force given by Eq. 33–3*a* ($\mathbf{F} = q\mathbf{v} \times \mathbf{B}$). This force causes the carriers to move upward along the conductor, so that they acquire a drift velocity $\mathbf{v}_d$ also, as shown in Fig. 35–16*a*.

The resultant equilibrium speed of the carrier when the velocity $\mathbf{v}$ of the loop has become constant is now $\mathbf{V}$, which we find by adding $\mathbf{v}$ and $\mathbf{v}_d$ vectorially in Fig. 35–16*a*. The magnetic deflecting force $\mathbf{F}_m$ is, as always, at right angles to the resultant velocity $\mathbf{V}$ of the carrier and is given by

$$\mathbf{F}_m = q\mathbf{V} \times \mathbf{B} \tag{35–13}$$

Now $\mathbf{F}_m$ acting alone would tend to push the carriers through the left wall of the conductor. Because this does not happen the conductor wall must exert a normal force $\mathbf{N}$ on the carriers (see Fig. 35–16*a*) of magnitude such that $\mathbf{v}_d$ lies parallel to the axis of the wire; in other words, $\mathbf{N}$ exactly cancels the horizontal component of $\mathbf{F}_m$, leaving only the component $\mathbf{F}_m \cos\theta$ that lies along the direction of the conductor. This component of force on the carrier is also cancelled out, this time by the average force $\overline{\mathbf{F}_i}$ associated with the internal collisions that the carrier experiences as it drifts with (constant) speed v_d through the wire.

The kinetic energy of the charge carrier as it drifts through the wire remains constant. This is consistent with the fact that the resultant force acting on the charge carrier ($= \mathbf{F}_m + \overline{\mathbf{F}_i} + \mathbf{N}$) is zero. The work done by $\mathbf{F}_m$ is zero because magnetic forces, acting at right angles to the velocity of a moving charge, can do no work on that charge. Thus the (negative) work done on the carrier by the average internal collision force $\mathbf{F}_i$ must be exactly cancelled by the (positive) work done on the carrier by the force $\mathbf{N}$. In the last analysis $\mathbf{N}$ is exerted by the agent who pulls the loop through the magnetic field, and the mechanical energy expended by this agent appears as heat energy in the loop, as we have seen in Section 35–4.

Let us then calculate the work dW done on the carrier in time dt by the force $\mathbf{N}$; it is

$$dW = N(v\,dt) \tag{35–14}$$

in which $v\,dt$ is the distance that the loop (and the carrier) have moved to the right in Fig. 35–16*a* in time dt. We can write for N (see Fig. 35–16*a* and Eq. 35–13).

$$N = F_m \sin\theta = (qVB)(v_d/V) = qBv_d \tag{35–15}$$

Substituting Eq. 35–15 into Eq. 35–14 yields

$$\begin{aligned} dW &= (qBv_d)(v\,dt) \\ &= (qBv)(v_d\,dt) = qBv\,dl \end{aligned} \tag{35–16}$$

in which $dl(= v_d\,dt)$ is the distance the carrier drifts along the conductor in time dt.

The work done on the carrier as it makes a complete circuit of the loop is found by integrating Eq. 35–16 around the loop and is

$$W = \oint dW = qBvl \tag{35–17}$$

This follows because work contributions for the top and the bottom of the loops are opposite in sign and cancel and no work is done in those portions of the loop that lie outside the magnetic field.

An agent that does work on charge carriers, thus establishing a current in a closed conducting loop, can be viewed as an emf. We can write, making use of Eq. 35-17,

$$\mathcal{E} = \frac{W}{q} = \frac{qBvl}{q} = Blv, \tag{35-18}$$

which is, of course, the same result that we derived from Faraday's law of induction; see Eq. 35-3. Thus a motional emf is intimately connected with the sideways deflection of a charged particle moving through a magnetic field.

We now consider how the situation of Fig. 35-16 would appear to an observer S' who is *at rest with respect to the loop.* To this observer, the magnet is moving to the left in Fig. 35-16*b* with velocity $-\mathbf{v}$, and the charge q is at rest as far as its left-to-right motion is concerned. However S', like S, observes that the charge drifts clockwise around the loop and he measures the same emf $\mathcal{E}$ that S measures. S' accounts for this, at the microscopic level, by postulating that an electric field $\mathbf{E}$ is induced in the loop by the action of the moving magnet. This induced field $\mathbf{E}$, which has the same origin as the induced fields that we discussed in Section 35-5, exerts a force on the charge carrier given by $q\mathbf{E}$.

The induced field $\mathbf{E}$, which exists in the end of the loop only, is associated with an emf $\mathcal{E}$ and generates a current in the closed loop. Note that, in any closed loop in which there is a current, an internal electric field must exist at every point at which charges are moving. These electric fields however are set up by the emf, as in the case of a closed loop connected to a battery, and are *not* induced by the motion of the magnet. It is only this induced field $\mathbf{E}$ that we associate with the emf, through the relation (Eq. 35-8)

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l}$$

which reduced to

$$\mathcal{E} = El \tag{35-19}$$

in this case. This is so because no *induced* electric field appears in the upper and lower bars, because of the nature of their motions, and none appears in the part of the loop outside the magnetic field.

The emfs given by Eqs. 35-19 and 35-18 must be identical because the relative motion of the loop and the magnet is identical in the two cases shown in Fig. 35-16. Equating these relations yields

$$El = Blv,$$

or

$$E = vB. \tag{35-20a}$$

In Fig. 35-16*b* the vector $\mathbf{E}$ points upward along the axis of the left end of the conducting loop because this is the direction in which positive charges are observed to drift. The directions of $\mathbf{v}$ and $\mathbf{B}$ are clearly shown in this figure. We see, then, that Eq. 35-20*a* is consistent with the more general vector relation

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \tag{35-20b}$$

We have not proved Eq. 35–20*b* except for the special case of Fig. 35–16; nevertheless it proves to be true in general, that is, no matter what the angle between **v** and **B**.

An experiment carried out in 1926 by the German physicist Wilhelm Wien (1864–1928) gives concrete support to our descriptions of the effect of the motion of an observer on the nature of the electric and magnetic fields that he observes. The radiations emitted from atoms vary slightly in wavelength if the atom is immersed in either a magnetic or an electric field. The atom can thus be used as a probe to examine the nature of such fields. Wien fired a beam of atoms with velocity **v** through a magnetic field **B**. The radiations emitted were identical in the distribution of their wavelengths with those that would have been emitted by a *resting* atom immersed in (*a*) the magnetic field **B** *and* (*b*) an induced electric field **E** given by Eq. 35–20*b*.

We interpret Eq. 35-20*b* in the following way: Observer *S* fixed with respect to the magnet is aware only of a magnetic field. The force to him arises from the motion of the charges through **B**. Observer *S'* fixed on the charge carrier is aware of an electric field **E** also and attributes the force on the charge (at rest with respect to him initially) to the electric field. *S* says the force is of purely magnetic origin and *S'* says the force is of purely electric origin. From the point of view of *S*, the induced emf is given by $\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$. From the point of view of *S'*, the same induced emf is given by $\oint \mathbf{E} \cdot d\mathbf{l}$, where **E** is the (induced) electric vector that he observes at points along the circuit.

For a third observer *S''* who judges that both the magnet and the loop are moving, the force tending to move charges around the loop is neither purely electric nor purely magnetic, but a bit of each. In summary, in the equation

$$\mathbf{F}/q = \mathbf{E} + \mathbf{v} \times \mathbf{B},$$

different observers form different assessments of **E**, **B**, and **v** but, when these are combined, all observers form the same assessment of $\mathbf{F}/q$ and all obtain the same value for the induced emf in the loop (this depends only on the relative motion). That is, the total force is the same for all observers, but each observer forms a different estimate of the separate electric and magnetic forces contributing to the same total force.

▶ **Example 7.** In Fig. 35–16 assume that $B = 2.0$ webers/meter², $l = 10$ cm, and $v = 1.0$ meter/sec. Calculate (*a*) the induced electric field observed by *S'*, and (*b*) the emf induced in the loop.

(*a*) The electric field, which is apparent only to observer *S'*, is associated with the moving magnet and is given in magnitude (see Eq. 35–20*a*) by

$$\begin{aligned} E &= vB \\ &= (1.0 \text{ meter/sec})(2.0 \text{ webers/meter}^2) \\ &= 2.0 \text{ volt/meter.} \end{aligned}$$

(*b*) Observer *S* would calculate the induced (motional) emf from

$$\begin{aligned} \mathcal{E} &= Blv \\ &= (2.0 \text{ webers/meters}^2)(1.0 \times 10^{-1} \text{ meter})(1.0 \text{ meter/sec}) \\ &= 0.20 \text{ volt.} \end{aligned}$$

Observer S' would not regard the emf as motional and would use the relationship

$$\mathcal{E} = El$$

$$= (2.0\ \text{volt/meter})(1.0 \times 10^{-1}\ \text{meter})$$

$$= 0.20\ \text{volt}.$$

As must be the case, both observers agree as to the numerical value of the emf. ◀

QUESTIONS

1. The north pole of a magnet is moved away from a metallic ring, as in Fig. 35-17. In the part of the ring farthest from the reader, which way does the current point?

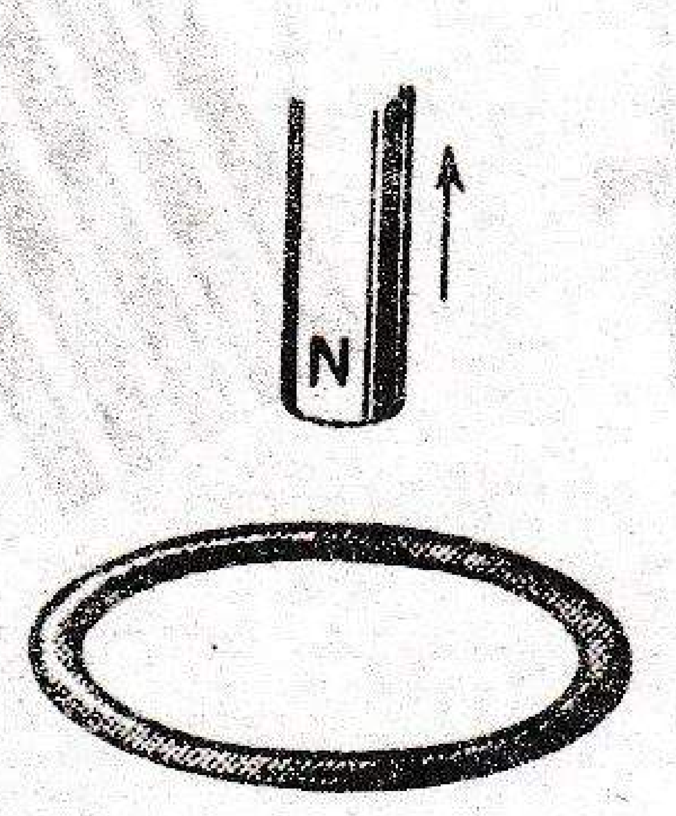


Fig. 35-17

2. *Eddy currents.* A sheet of copper is placed in a magnetic field as shown in Fig. 35-18. If we attempt to pull it out of the field or push it further in, an automatic resisting force appears. Explain its origin. (Hint: Currents, called eddy currents, are induced in the sheet in such a way as to oppose the motion.)

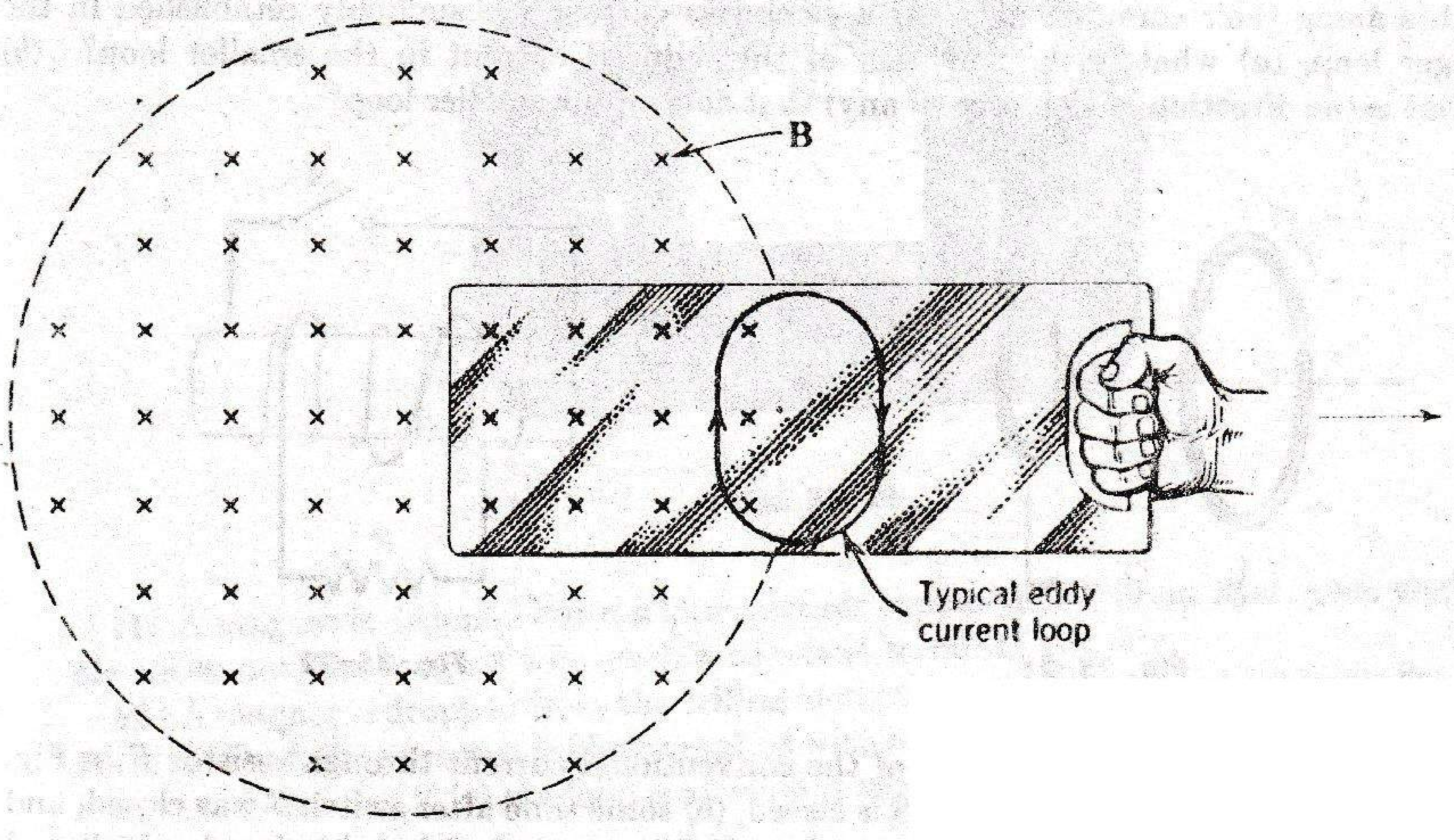


Fig. 35-18

3. *Electromagnetic shielding.* Consider a conducting sheet lying in a plane perpendicular to a magnetic field **B**, as shown in Fig. 35–19. (*a*) If **B** suddenly changes, the full *change* in **B** is not immediately detected in region P. Explain. (*b*) If the resistivity of the sheet is zero, the *change* is not ever detected at P. Explain. (*c*) If **B** changes periodically at high frequency and the conductor is made of a material of low resistivity, the region near P is almost completely shielded from the *changes* in flux. Explain. (*d*) Is such a conductor useful as a shield from *static* magnetic fields? Explain.

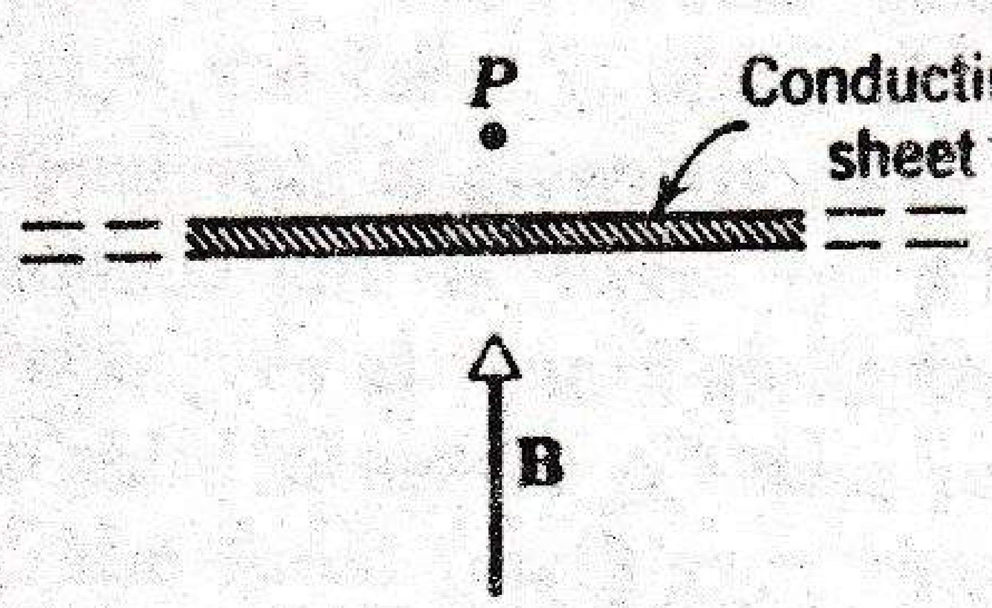


Fig. 35–19

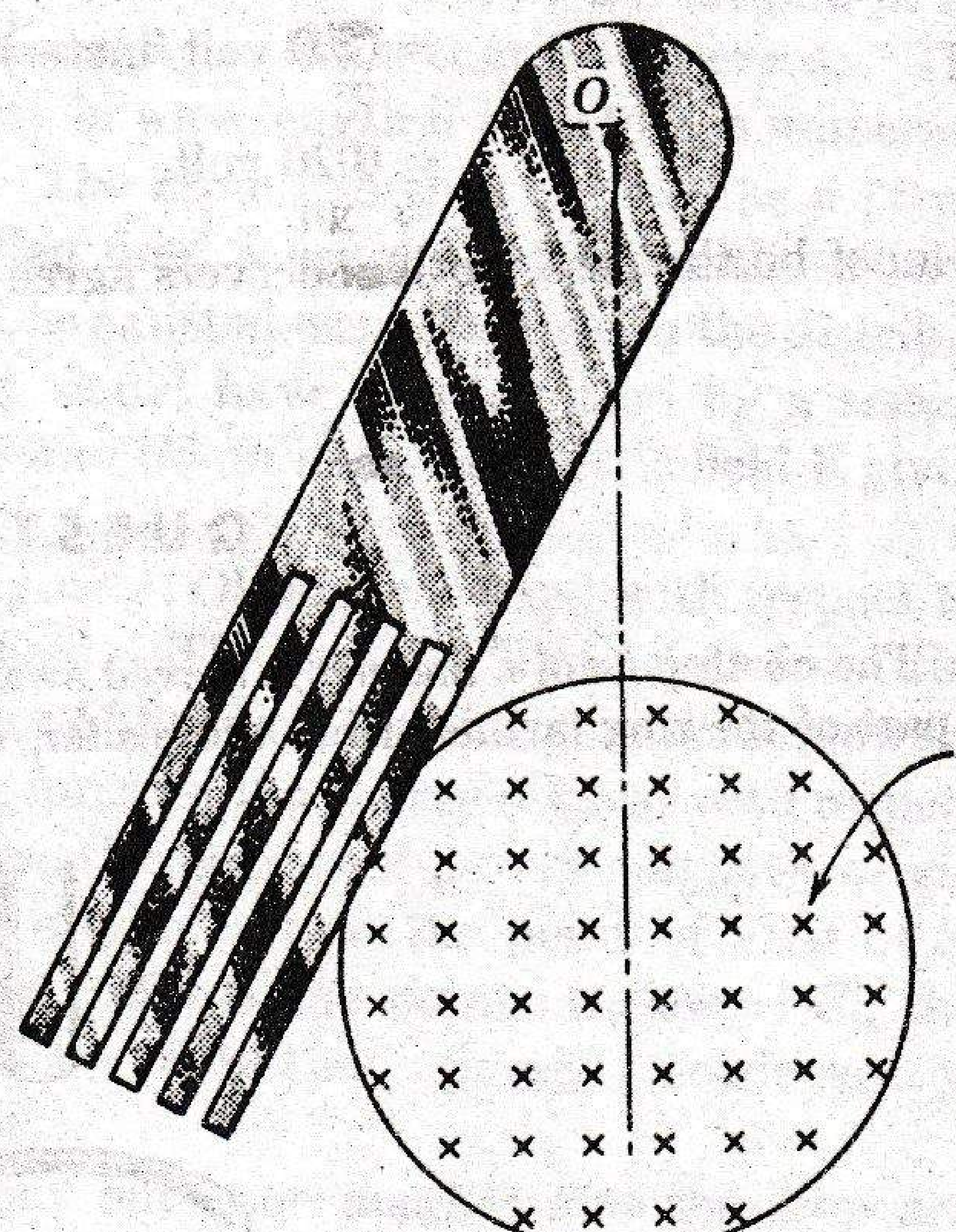


Fig. 35–20

4. *Magnetic damping.* A strip of copper is mounted as a pendulum about O in Fig. 35–20. It is free to swing through a magnetic field normal to the page. If the strip has slots cut in it as shown, it can swing freely through the field. If a strip without slots is substituted, the vibratory motion is strongly damped. Explain. (Hint: Use Lenz's law; consider the paths that the charge carriers in the strip must follow if they are to oppose the motion.)

5. Two conducting loops face each other a distance d apart (Fig. 35–21). An observer sights along their common axis. If a clockwise current i is suddenly established in the larger loop, (*a*) what is the direction of the induced current in the smaller loop? (*b*) What is the direction of the force (if any) that acts on the smaller loop?

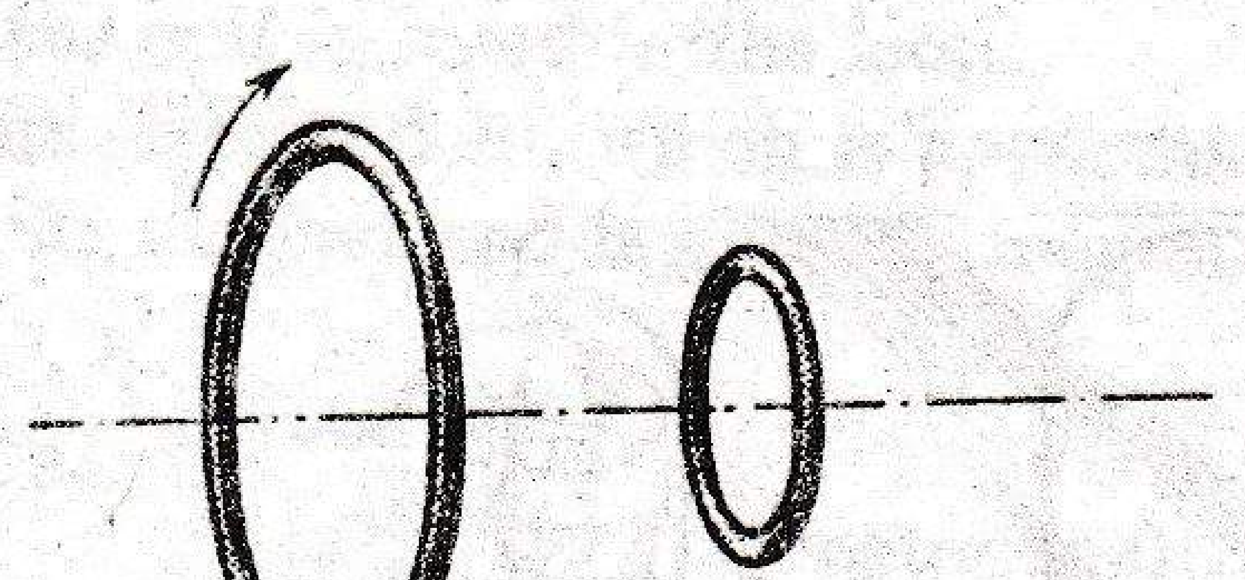


Fig. 35–21

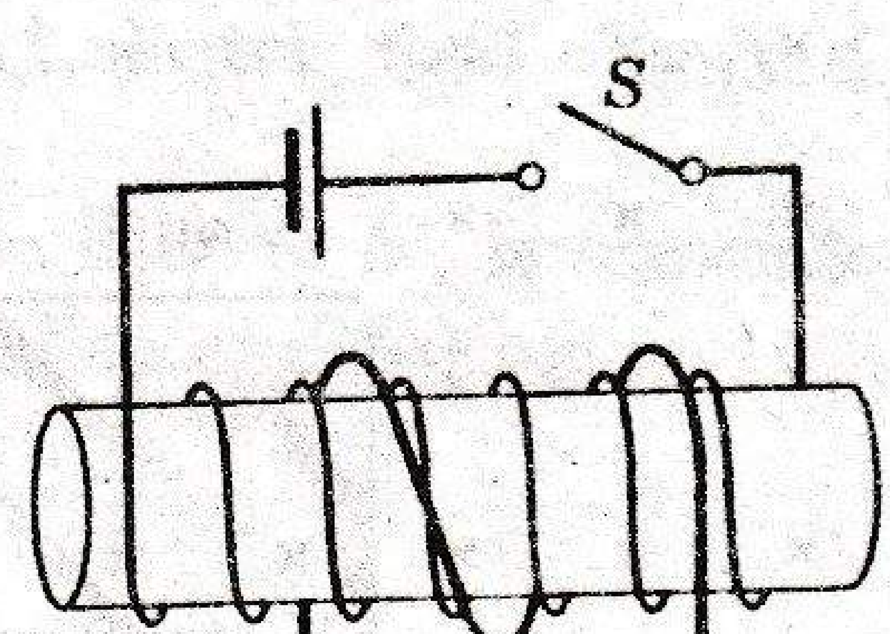


Fig. 35–22

6. What is the direction, if any, of the conventional current through resistor R in Fig. 35–22 (*a*) immediately after switch S is closed, (*b*) some time after switch S was closed, and (*c*) immediately after switch S is opened. (*d*) When switch S is held closed, which end of the coil acts as a north pole?

7. A current-carrying solenoid is moved toward a conducting loop as in Fig. 35–23. What is the direction of circulation of current in the loop as we sight toward it as shown?

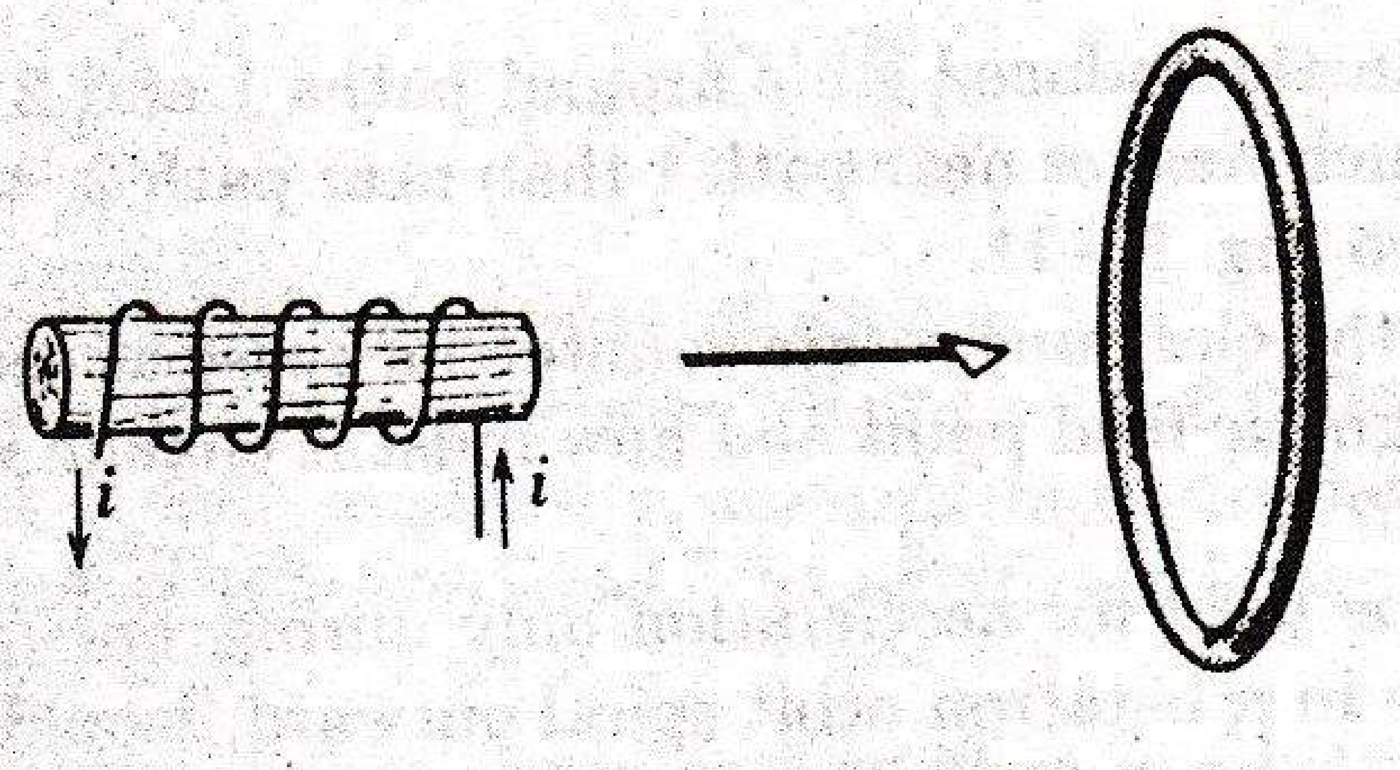


Fig. 35–23

8. If the resistance R in the left-hand circuit of Fig. 35–24 is increased, what is the direction of the induced current in the right-hand circuit?

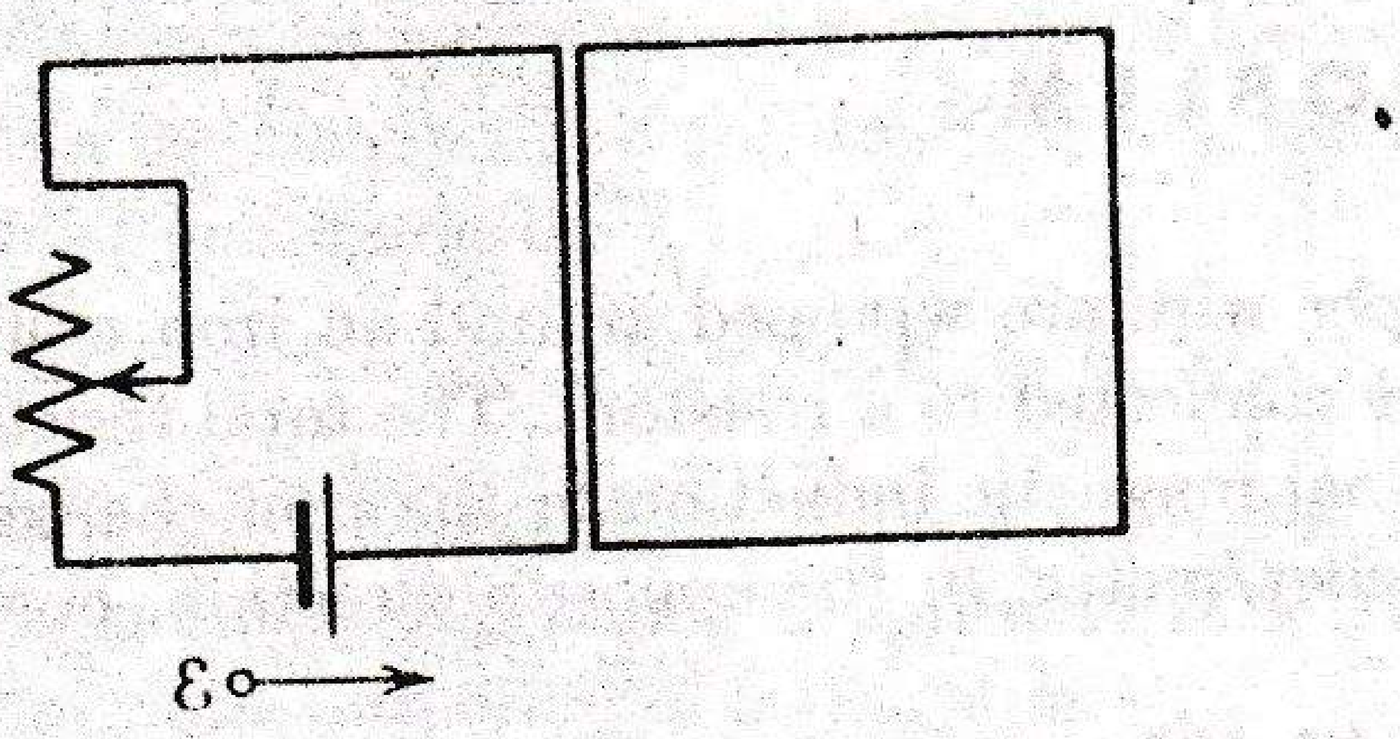


Fig. 35–24

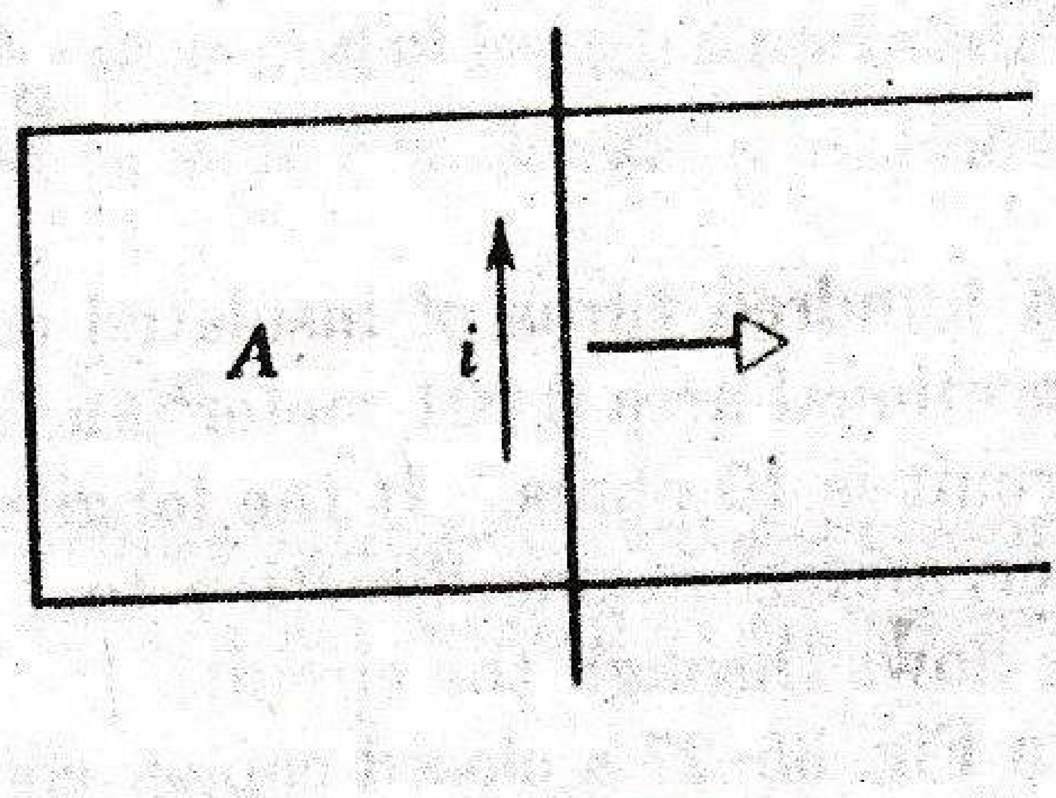


Fig. 35–25

9. In Fig. 35–25 the movable wire is moved to the right, causing an induced current as shown. What is the direction of **B** in region A?

10. A loop, shown in Fig. 35–26, is removed from the magnet by pulling it vertically upward. (*a*) What is the direction of the induced current? (*b*) Is a force required to remove the loop? (*c*) Does the total amount of Joule heat produced in removing the loop depend on the time taken to remove it?

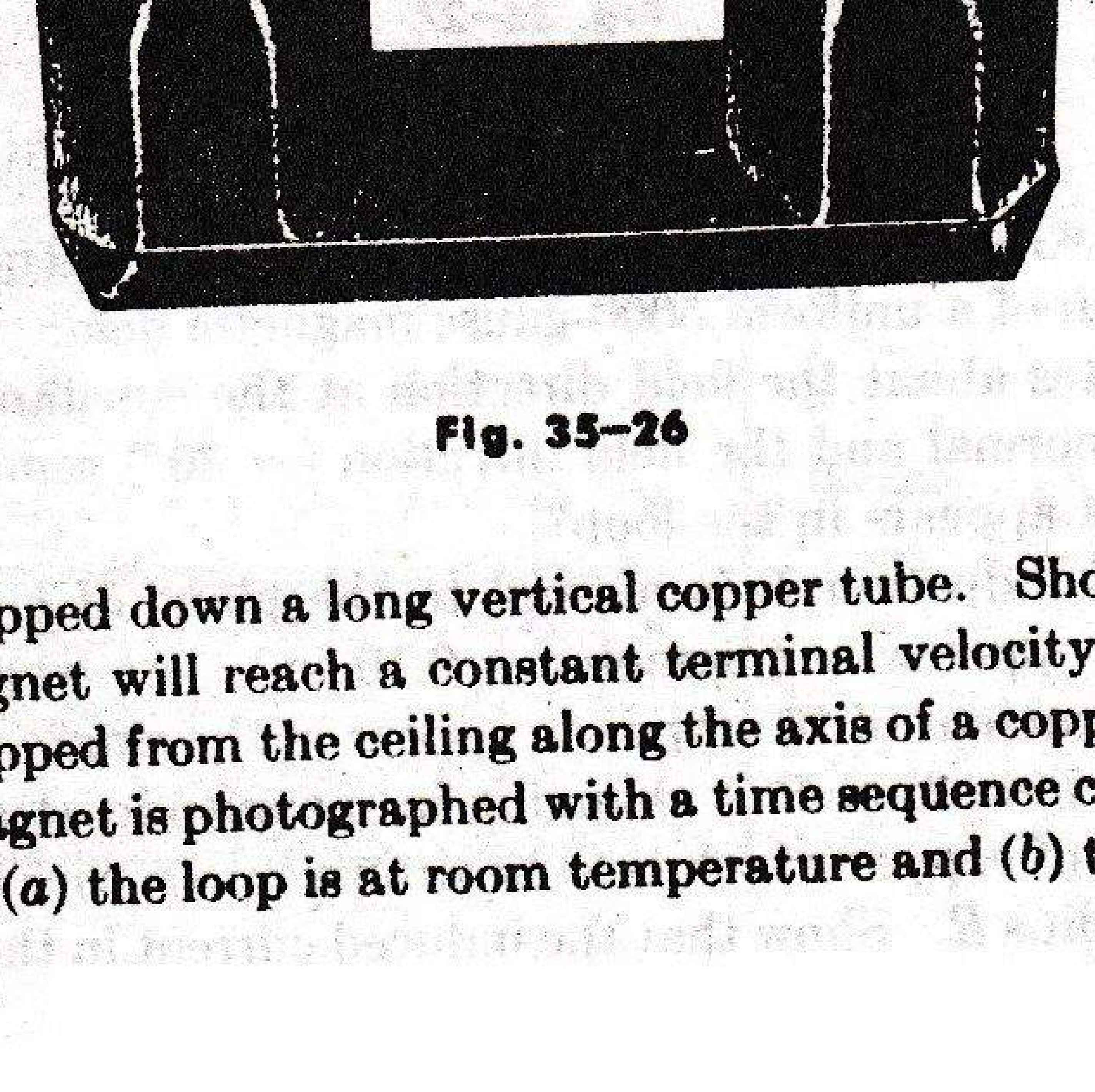


Fig. 35–26

11. A magnet is dropped down a long vertical copper tube. Show that, even neglecting air resistance, the magnet will reach a constant terminal velocity.

12. A magnet is dropped from the ceiling along the axis of a copper loop lying flat on the floor. If the falling magnet is photographed with a time sequence camera, what differences, if any, will be noted if (*a*) the loop is at room temperature and (*b*) the loop is packed in dry ice?

13. A copper ring and a wooden ring of the same dimensions are placed so that there is the same changing magnetic flux through each. How do the induced electric fields in each ring compare?

14. In Fig. 35–12 how can the induced emfs around paths 1 and 2 be identical? The induced electric fields are much weaker near path 1 than near path 2, as the spacing of the lines of force shows. See also Fig. 35–11.

15. In a certain betatron the electrons rotate counterclockwise as seen from above. In what direction must the magnetic field point and how must it change with time while the electron is being accelerated?

16. Why can a betatron be used for acceleration only during one-quarter of a cycle?

17. To make the electrons in a betatron orbit spiral outward, would it be necessary to increase or to decrease the central flux? Assume that **B** at the orbit remains essentially unchanged.

18. A cyclotron is a so-called *resonance device*. Does a betatron depend on resonance?

PROBLEMS

1. A hundred turns of insulated copper wire are wrapped around an iron cylinder of cross-sectional area 0.001 meter^2 and are connected to a resistor. The total resistance in the circuit is 10 ohms. If the longitudinal magnetic induction in the iron changes from 1 weber/meter^2 in one direction to 1 weber/meter^2 in the opposite direction, how much charge flows through the circuit?

2. In Fig. 35–27 a closed copper coil with 100 turns and a total resistance of 5.0 ohms is placed *outside* a solenoid like that of Example 1. If the current in the solenoid is changed as in that example, what current appears in the coil?

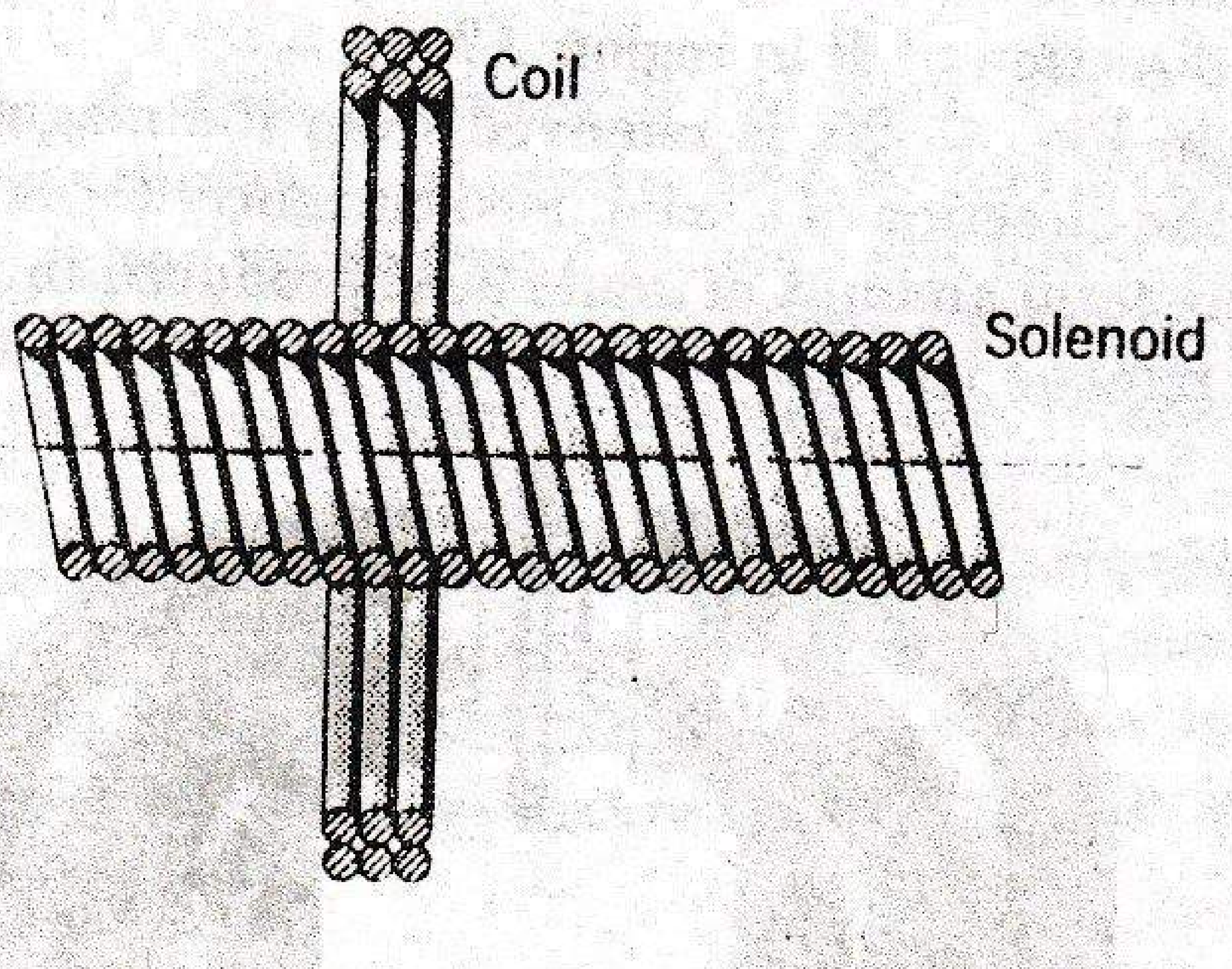


Fig. 35–27

3. A circular loop of wire 10 cm in diameter is placed with its normal making an angle of 30° with the direction of a uniform 5000-gauss magnetic field. The loop is "wobbled" so that its normal rotates about the field direction at the constant rate of 100 rev/min; the angle between the normal and the field direction (= 30°) remains unchanged during this process. What emf appears in the loop?

4. A uniform field of induction **B** is normal to the plane of a circular ring 10-cm in diameter made of #10 copper wire (diameter = 0.10 in.). At what rate must B change with time if an induced current of 10 amp is to appear in the ring?

5. A uniform field of induction **B** is changing in magnitude at a constant rate dB/dt. You are given a mass m of copper which is to be drawn into a wire of radius r and formed into a circular loop of radius R. Show that the induced current in the loop does not depend

on the size of the wire or of the loop and, assuming **B** perpendicular to the loop, is given by

$$i = \frac{m}{4\pi\rho\delta}\frac{dB}{dt},$$

where ρ is the resistivity and δ the density of copper.

6. You are given 50 cm of #18 copper wire (diameter = 0.040 in.). It is formed into a circular loop and placed at right angles to a uniform magnetic field that is increasing with time at the constant rate of 100 gauss/sec. At what rate is Joule heat generated in the loop?

7. A small bar magnet is pulled rapidly through a conducting loop, along its axis. Sketch qualitatively (*a*) the induced current and (*b*) the rate of Joule heating as a function of the position of the center of the magnet. Assume that the north pole of the magnet enters the loop first and that the magnet moves at constant speed. Plot the induced current as positive if it is clockwise as viewed along the path of the magnet.

8. *Alternating current generator*. A rectangular loop of N turns and of length a and width b is rotated at a frequency ν in a uniform field of induction **B**, as in Fig. 35-28. (*a*) Show that an induced emf given by

$$\mathcal{E} = 2\pi\nu NbaB \sin 2\pi\nu t = \mathcal{E}_0 \sin 2\pi\nu t$$

appears in the loop. This is the principle of the commercial alternating-current generator. (*b*) Design a loop that will produce an emf with $\mathcal{E}_0 = 150$ volts when rotated at 60 rev/sec in a field of magnetic induction of 5000-gauss.

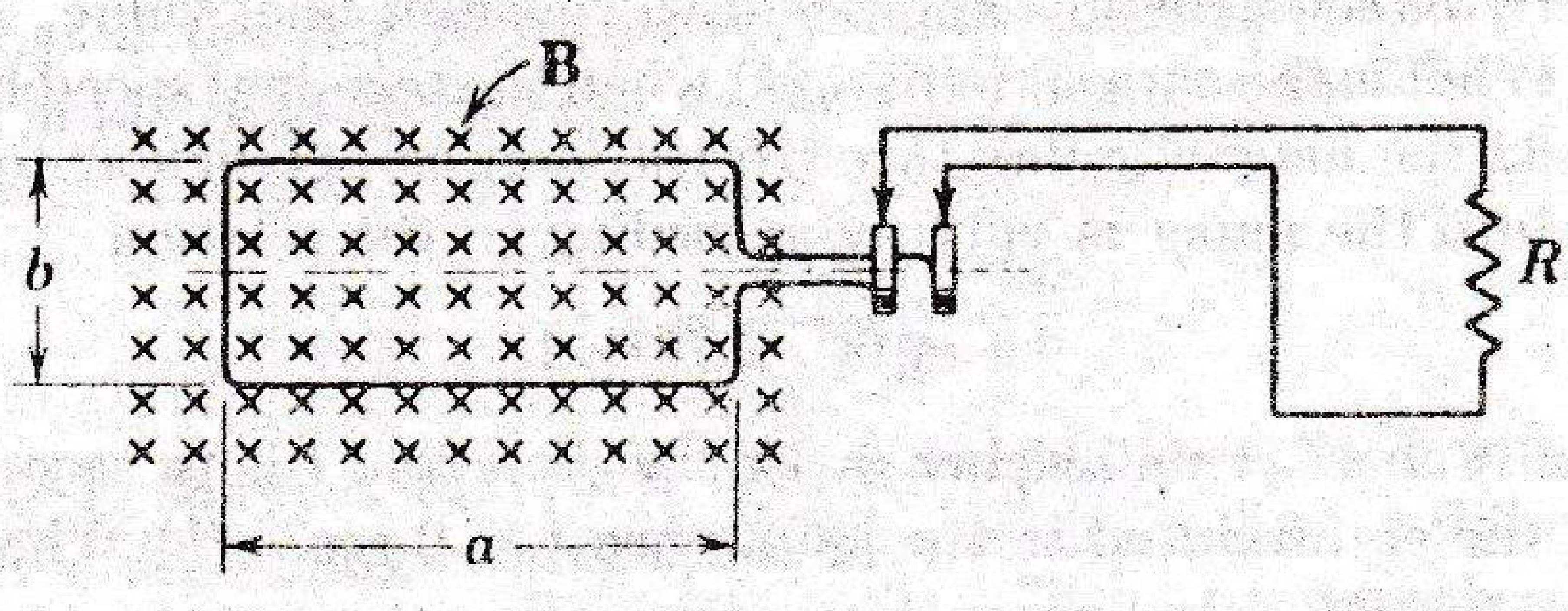


Fig. 35-28

9. A stiff wire bent into a semicircle of radius R is rotated with a frequency ν in a uniform field of induction **B**, as shown in Fig. 35-29. What are the amplitude and frequency of the induced voltage and of the induced current when the internal resistance of the meter M is R_M and the remainder of the circuit has negligible resistance?

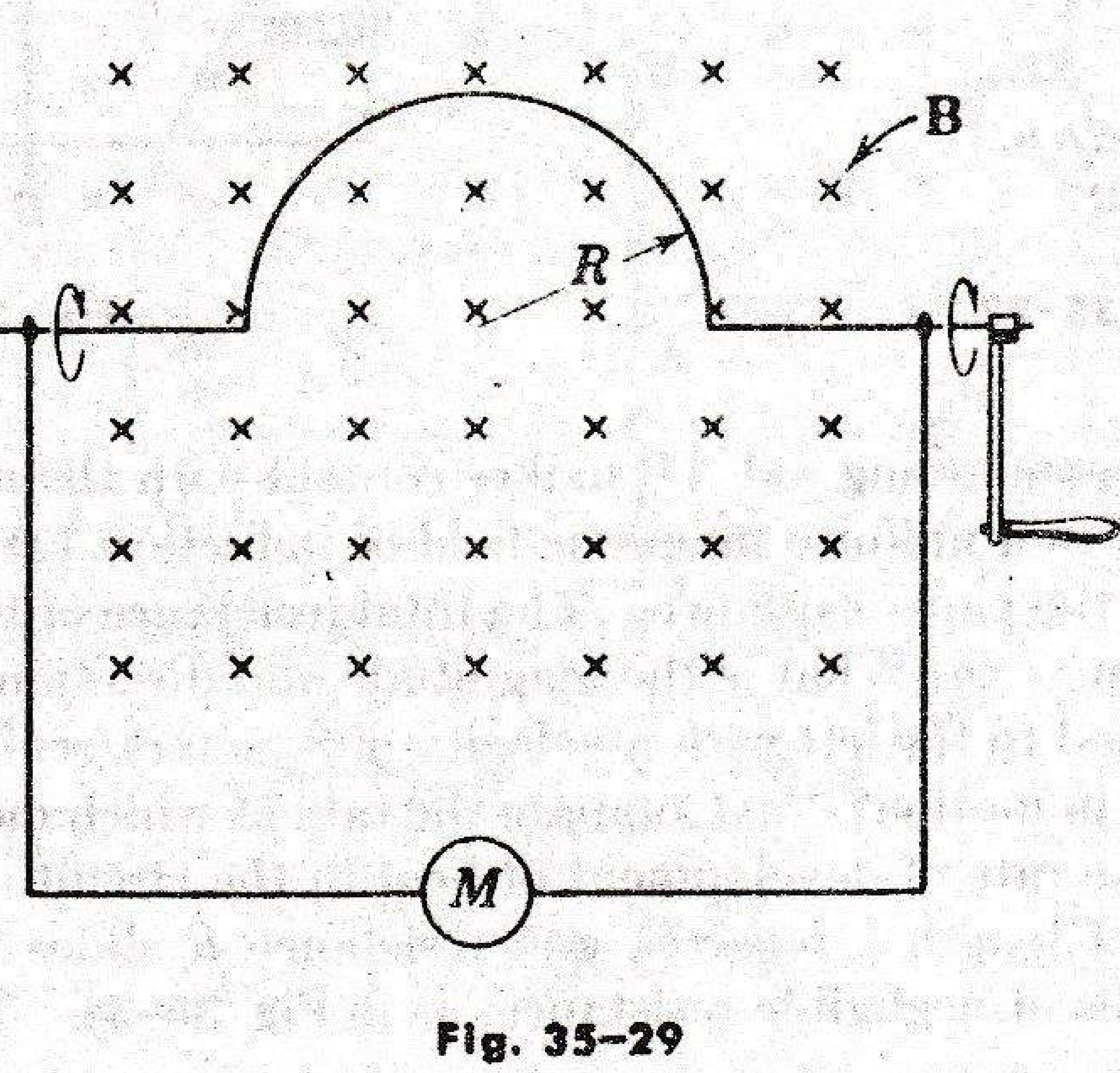


Fig. 35-29

10. A circular copper disk 10 cm in diameter rotates at 1800 rev/min about an axis through its center and at right angles to the disk. A uniform field of induction **B** of 10,000 gauss is perpendicular to the disk. What potential difference develops between the axis of the disk and its rim?

11. Figure 35-30 shows a copper rod moving with velocity **v** parallel to a long straight wire carrying a current i. Calculate the induced emf in the rod, assuming that $v = 5.0$ meters/sec, $i = 100$ amp, $a = 1.0$ cm, and $b = 20$ cm.

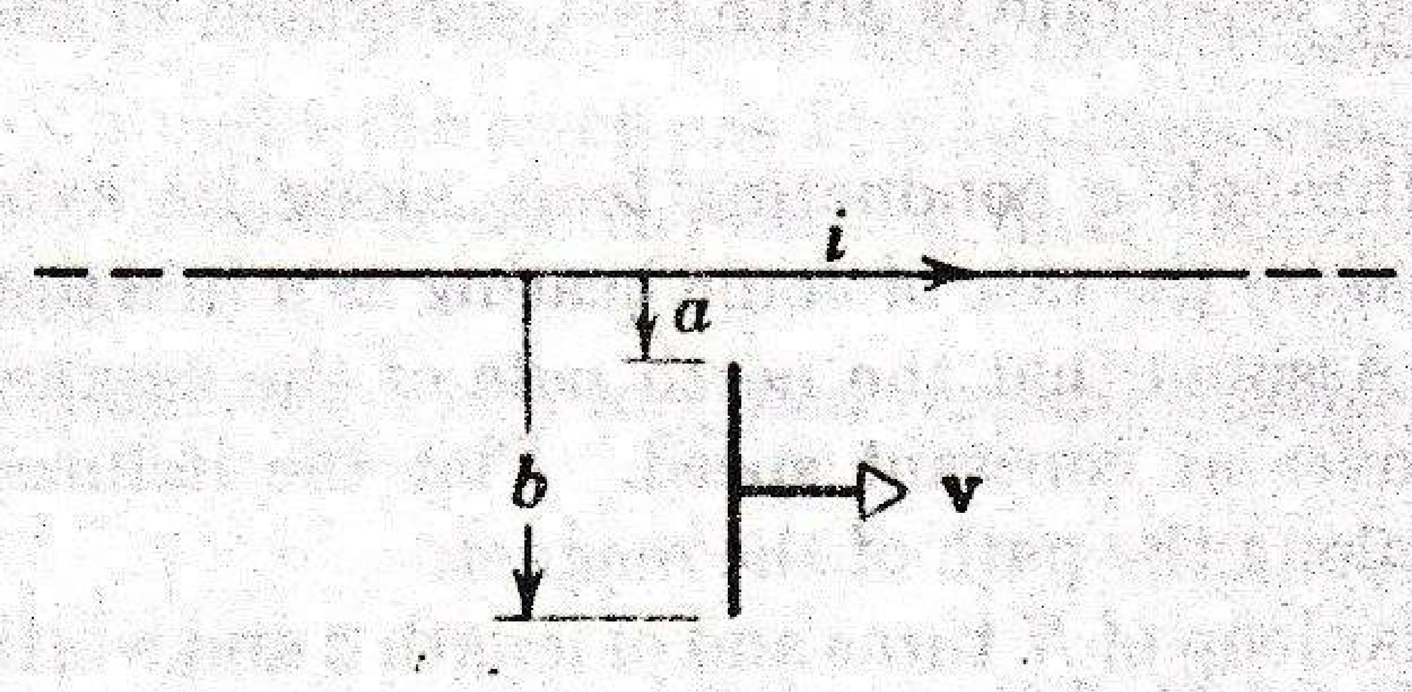


Fig. 35-30

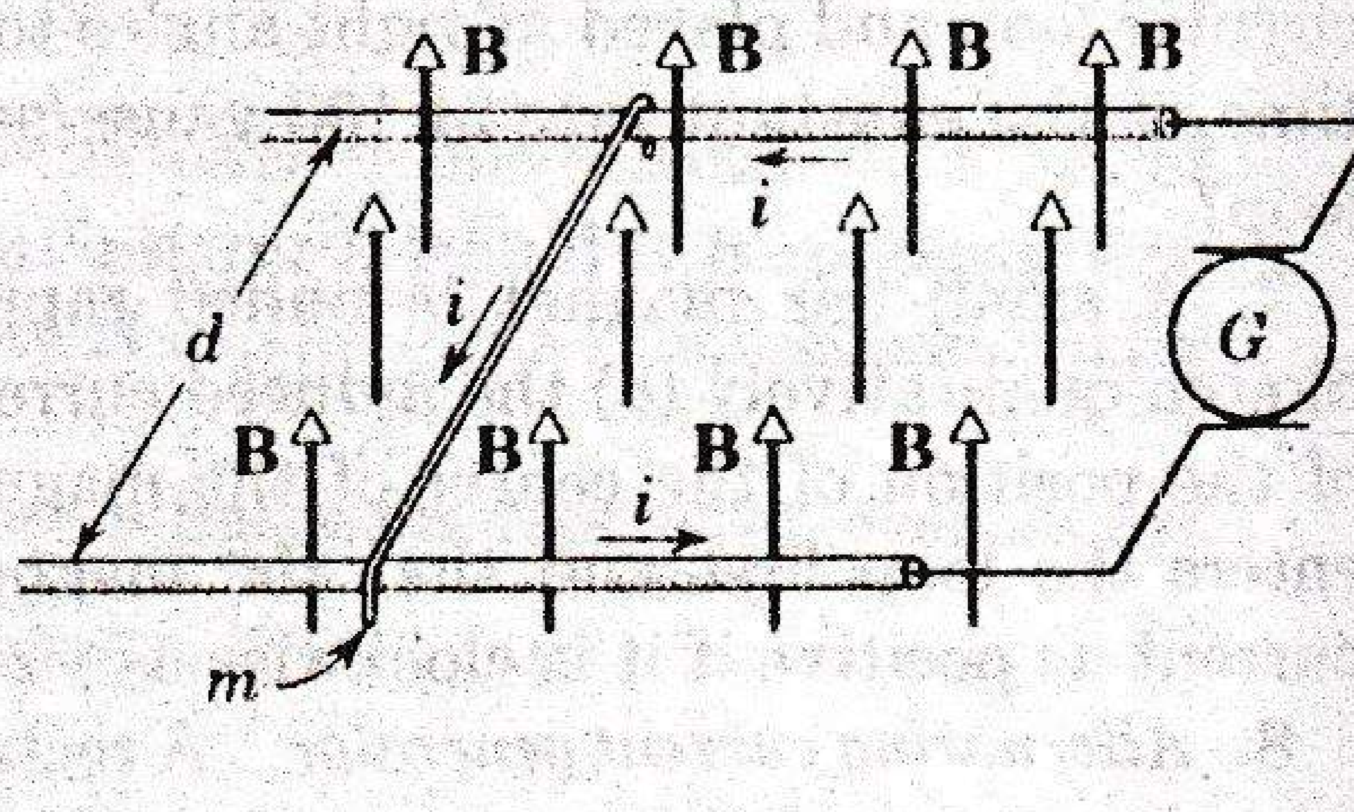


Fig. 35-31

12. A metal wire of mass m slides without friction on two rails spaced a distance d apart, as in Fig. 35-31. The track lies in a vertical uniform field of induction **B**. (*a*) A *constant current* i flows from generator G along one rail, across the wire, and back down the other rail. Find the velocity (speed and direction) of the wire as a function of time, assuming it to be at rest at $t = 0$. (*b*) The generator is replaced by a battery with *constant emf* $\mathcal{E}$. The velocity of the wire now approaches a constant final value. What is this terminal speed? (*c*) What is the current in part (*b*) when the terminal speed has been reached?

13. In Fig. 35-32 the magnetic flux through the loop perpendicular to the plane of the coil and directed into the paper is varying according to the relation

$$\Phi_B = 6t^2 + 7t + 1,$$

where Φ_B is in milliwebers (1 milliweber $= 10^{-3}$ weber) and t is in seconds. (*a*) What is the magnitude of the emf induced in the loop when $t = 2$ sec? (*b*) What is the direction of the current through R?

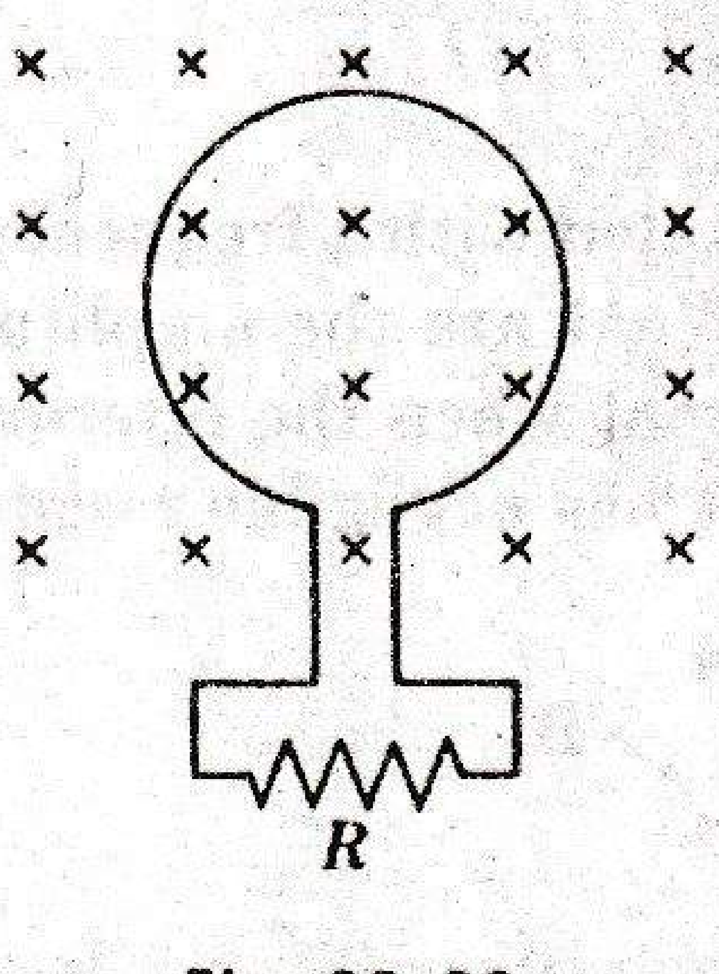


Fig. 35-32

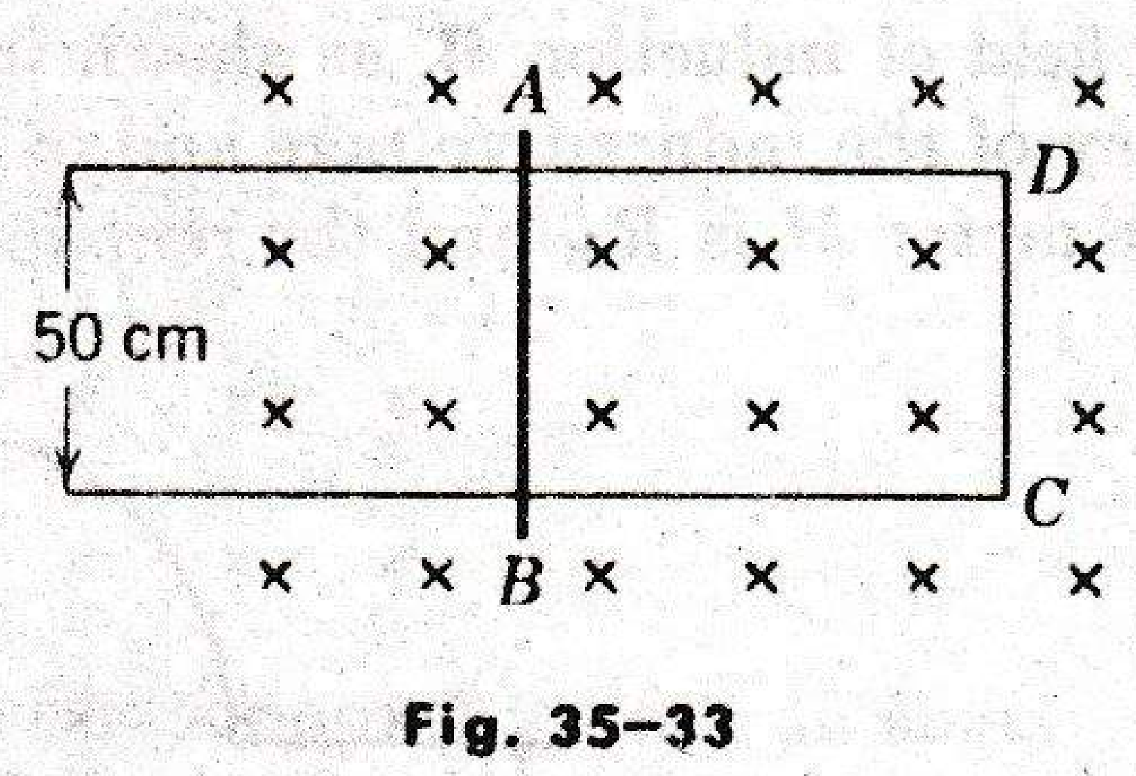


Fig. 35-33

14. In Fig. 35-33 a conducting rod AB makes contact with the metal rails AD and BC which are 50 cm apart in a uniform magnetic field of induction 1.0 weber/meter^2 perpendicular to the plane of the paper as shown. The total resistance of the circuit $ABCD$ is 0.4 ohm (assumed constant). (*a*) What is the magnitude and direction of the emf induced in the rod when it is moved to the left with a velocity of 8 meters/sec? (*b*) What force is required to keep the rod in motion? (*c*) Compare the rate at which mechanical work is done by the force **F** with the rate of development of heat in the circuit.

15. A square wire of length l, mass m, and resistance R slides without friction down parallel conducting rails of negligible resistance, as in Fig 35-34. The rails are connected

to each other at the bottom by a resistanceless rail parallel to the wire, so that the wire and rails form a closed rectangular conducting loop. The plane of the rails makes an angle θ with the horizontal, and a uniform vertical field of magnetic induction **B** exists throughout the region. (*a*) Show that the wire acquires a steady-state velocity of magnitude

$$v = \frac{mgR \sin\theta}{B^2 l^2 \cos^2\theta}.$$

(*b*) Prove that this result is consistent with the conservation-of-energy principle. (*c*) What change, if any, would there be if **B** were directed down instead of up?

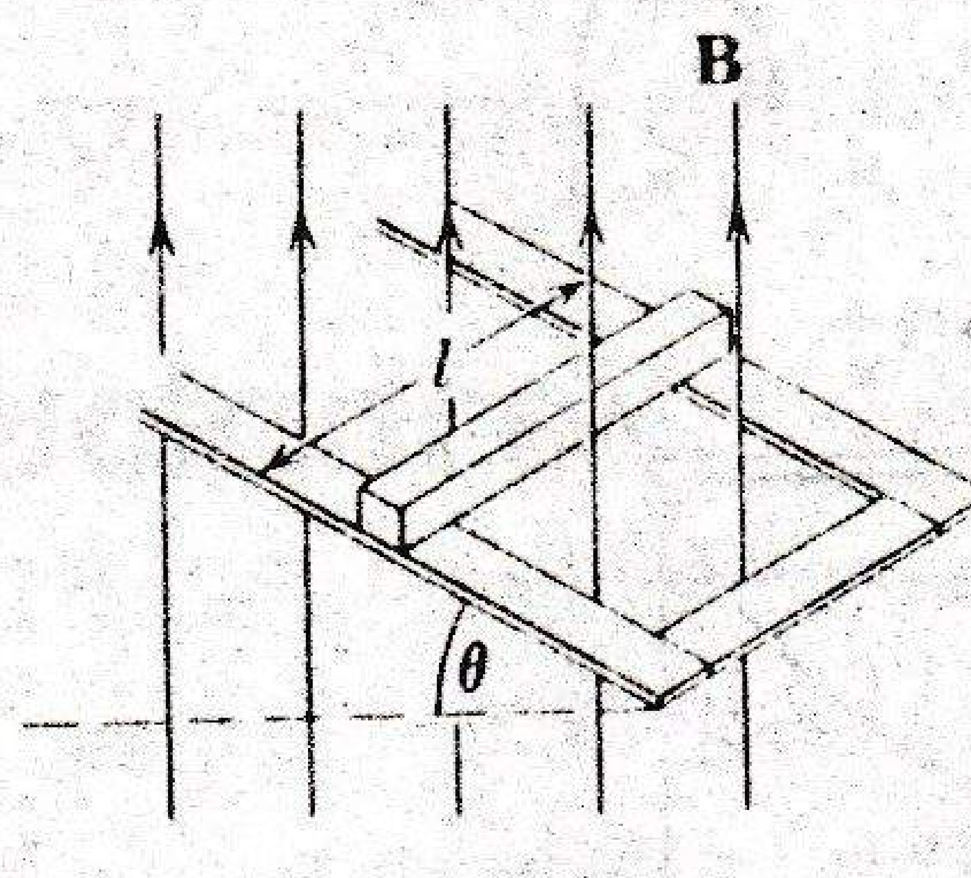


Fig. 35-34

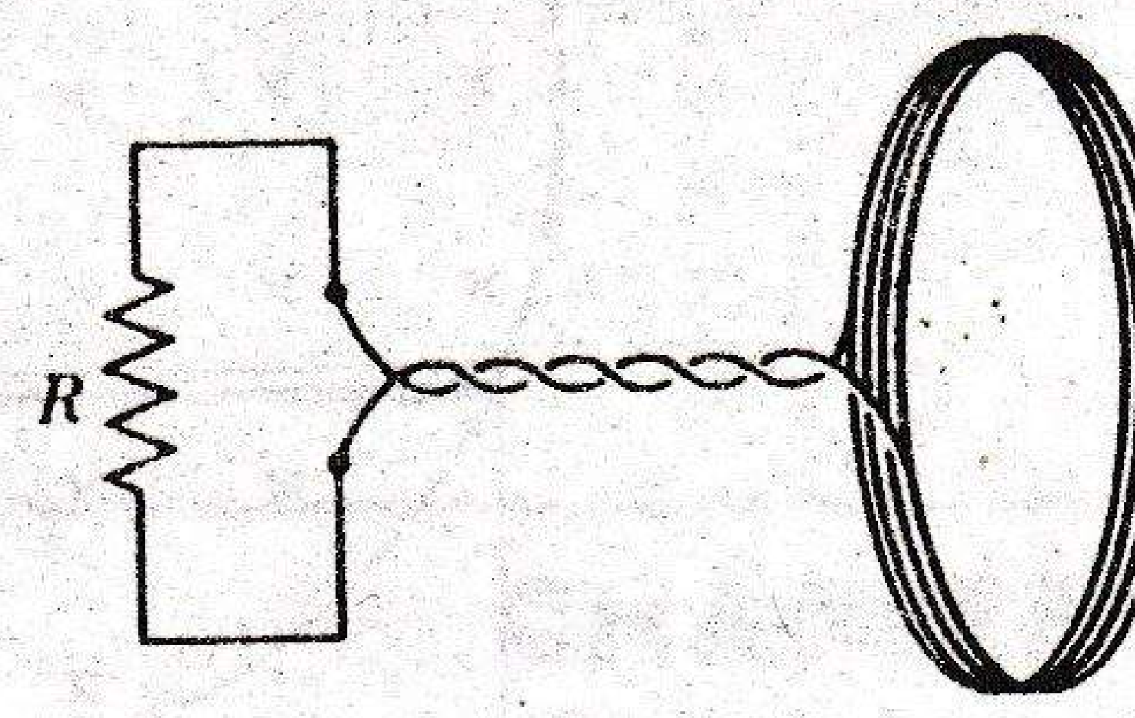


Fig. 35-35

16. Prove that if the flux of magnetic induction through the coil of N turns of Fig. 35-35 changes in any way from Φ_1 to Φ_2, then the charge q that flows through the circuit of total resistance R is given by

$$q = \frac{N(\Phi_2 - \Phi_1)}{R}.$$

17. Figure 35-36 shows a uniform field of induction **B** confined to a cylindrical volume of radius R. **B** is decreasing in magnitude at a constant rate of 100 gauss/sec. What is the instantaneous acceleration (direction and magnitude) experienced by an electron placed at a, at b, and at c? Assume r = 5.0 cm. (The necessary fringing of the field beyond R will not change your answer as long as there is axial symmetry about a perpendicular axis through b.)

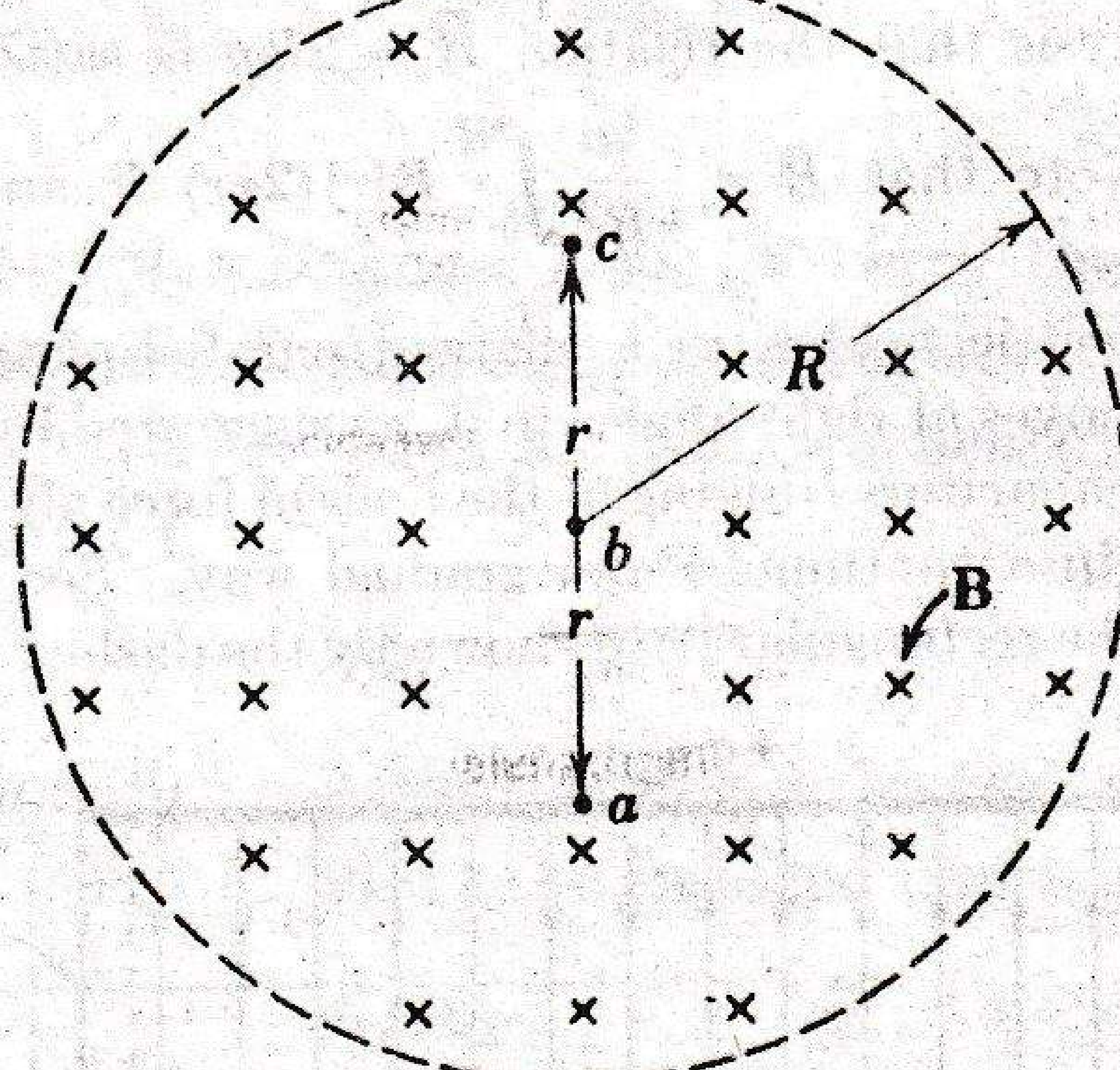


Fig. 35-36

18. A uniform magnetic field of induction **B** fills a cylindrical volume of radius R. A metal rod of length l is placed as shown in Fig. 35-37. If B is changing at the rate dB/dt,

show that the emf that is produced by the changing magnetic field and that acts between the ends of the rod is given by

$$\mathcal{E} = \frac{dB}{dt}\frac{l}{2}\sqrt{R^2 - \left(\frac{l}{2}\right)^2}.$$

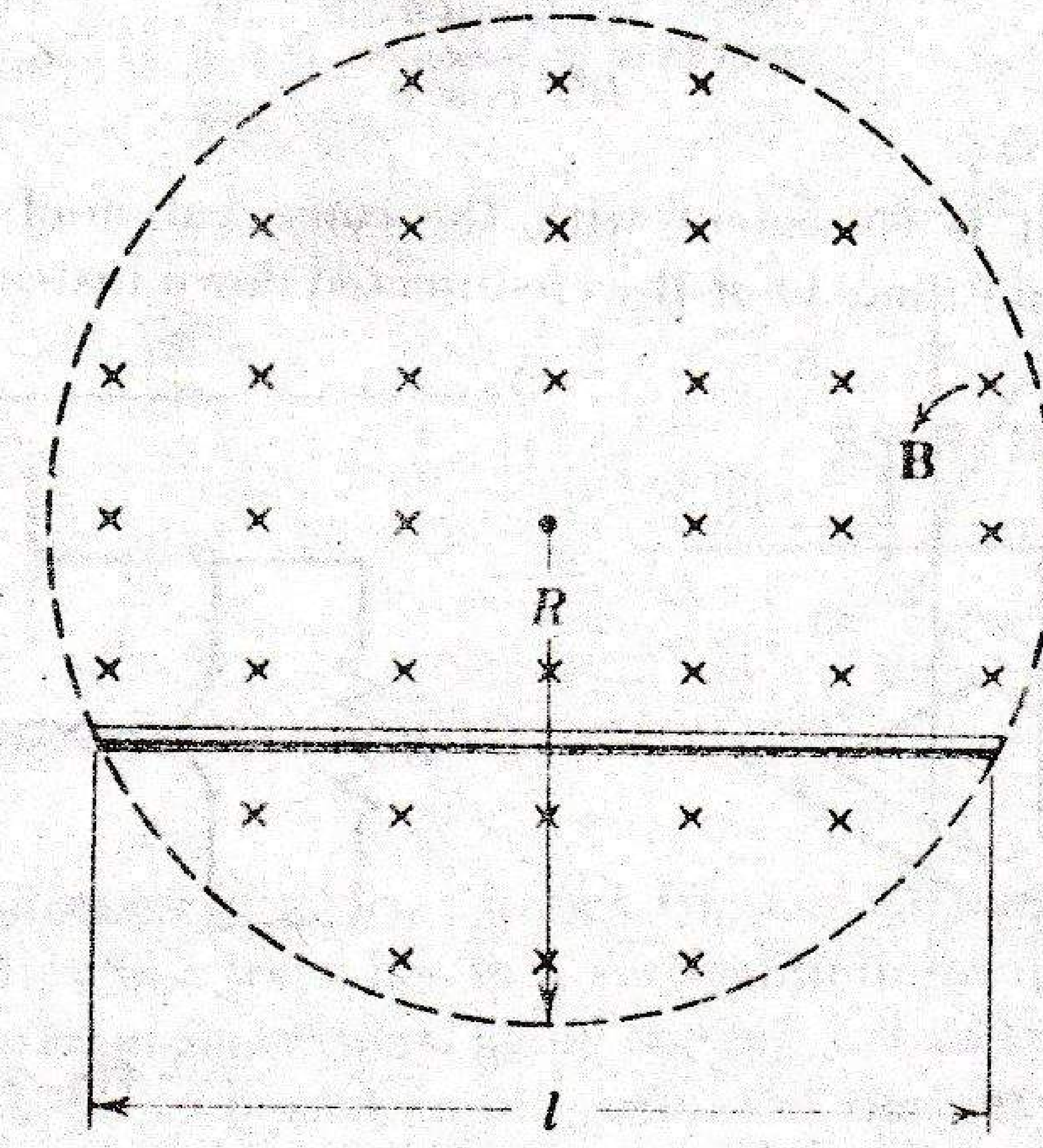


Fig. 35-37

19. Some measurements of the maximum magnetic induction as a function of radius for the General Electric Company betatron are as follows:

r, cm	B, gauss	r, cm	B, gauss
0	4000	81.2	4090
10.2	9500	83.7	4000
68.2	9500	88.9	3810
73.2	5280	91.4	3720
75.2	4510	93.5	3600
77.3	4280	95.5	3400

Show by graphical analysis that the relation $\bar{B} = 2B_R$ is satisfied at the orbit radius, $R = 84$ cm. (Hint: Note that $\bar{B} = \frac{1}{\pi R^2}\int_0^R B(r)(2\pi r)\,dr$ and evaluate the integral graphically.)

20. Prove that the electric field **E** in a charged parallel-plate capacitor cannot drop abruptly to zero as one moves at right angles to it, as suggested by the arrow in Fig. 35-38 (see point a). In actual capacitors fringing of the lines of force always occurs, which means that **E** approaches zero in a continuous and gradual way. See Problem 34-28. (Hint: Apply Faraday's law to the rectangular path shown by the dashed lines.)

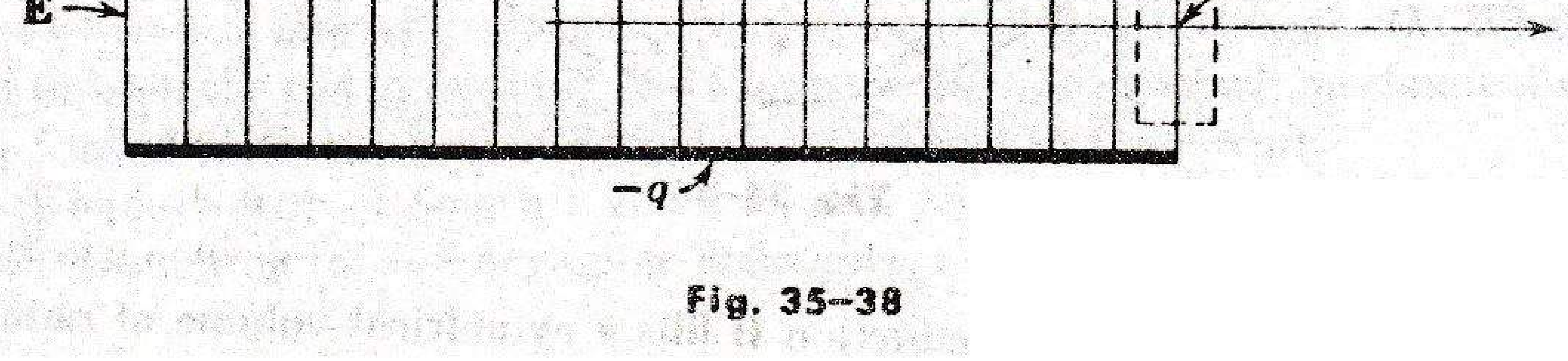


Fig. 35-38

Inductance

CHAPTER 36

36–1 Inductance

If two coils are near each other, a current i in one coil will set up a flux Φ_B through the second coil. If this flux is changed by changing the current, an induced emf will appear in the second coil according to Faraday's law. However, two coils are not needed to show an inductive effect. An induced emf appears in a coil if the current *in that same coil* is changed. This is called *self-induction* and the electromotive force produced is called a *self-induced emf*. It obeys Faraday's law of induction just as other induced emfs do.

Consider first a "close-packed" coil, a toroid, or the central section of a long solenoid. In all three cases the flux Φ_B set up in each turn by a current i is essentially the same for every turn. Faraday's law for such coils (Eq. 35–2)

$$\mathcal{E} = -\frac{d(N\Phi_B)}{dt} \tag{36–1}$$

shows that the number of *flux linkages* $N\Phi_B$ (N being the number of turns) is the important characteristic quantity for induction. For a given coil, provided no magnetic materials such as iron are nearby, this quantity is proportional to the current i, or

$$N\Phi_B = Li, \tag{36–2}$$

in which L, the proportionality constant, is called the *inductance* of the device.

From Faraday's law (see Eq. 36–1) the induced emf can be written as

$$\mathcal{E} = -\frac{d(N\Phi_B)}{dt} = -L\frac{di}{dt}. \tag{36–3a}$$

Written in the form

$$L = -\frac{\mathcal{E}}{di/dt}, \tag{36-3b}$$

this relation may be taken as the defining equation for inductance for coils of all shapes and sizes, whether or not they are close-packed and whether or not iron or other magnetic material is nearby. It is analogous to the defining relation for capacitance, namely

$$C = \frac{q}{V}.$$

If no iron or similar materials are nearby, L depends only on the geometry of the device. In an *inductor* (symbol ⌒ᴗᴗᴗᴗ⌒) the presence of a *magnetic field* is the significant feature, corresponding to the presence of an *electric* field in a *capacitor*.

The unit of inductance, from Eq. 36-3*b*, is the volt-sec/amp. A special name, the *henry*, has been given to this combination of units, or

$$1 \text{ henry} = 1 \text{ volt-sec/amp}.$$

The unit of inductance is named after Joseph Henry (1797–1878), an American physicist and a contemporary of Faraday. Henry independently discovered the law of induction at about the same time Faraday did. The units *millihenry* (1 mh = 10^{-3} henry) and *microhenry* (1 μh = 10^{-6} henry) are also commonly used.

The direction of a self-induced emf can be found from Lenz's law. Suppose that a steady current i, produced by a battery, exists in a coil. Let us suddenly reduce the (battery) emf in the circuit to zero. The current i will start to decrease at once; this *decrease* in current, in the language of Lenz's law, is the "change" which the self-induction must oppose. To oppose the falling current, the induced emf must point in the same direction as the current, as in Fig. 36-1*a*. When the current in a coil is increased, Lenz's law shows that the self-induced emf points in the *opposite* direction to that of the current, as in Fig. 36-1*b*. In each case the self-induced emf acts to op-

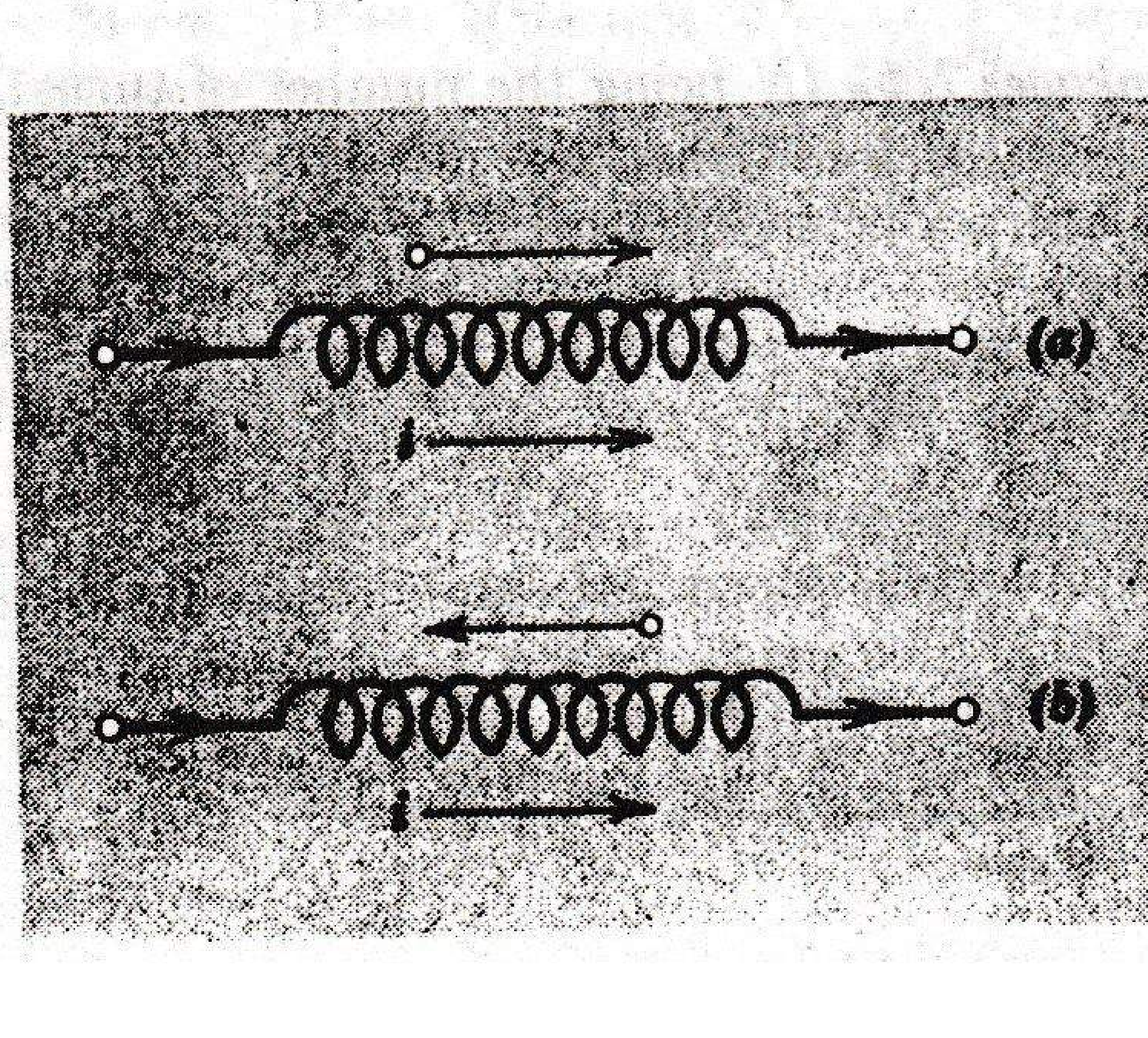


Fig. 36-1 In (a) the current i is *decreasing* and in (b) it is *increasing*. The self-induced emf $\mathcal{E}_L$ opposes the *change* in each case.

pose the *change* in the current. The minus sign in Eq. 36-3 shows that $\mathcal{E}$ and di/dt are opposite in sign, since L is always a positive quantity.

36-2 Calculation of Inductance

It has proved possible to make a direct calculation of capacitance in terms of geometrical factors for a few special cases, such as the parallel-plate capacitor. In the same way, it is possible to calculate the self-inductance L for a few special cases.

For a close-packed coil with no iron nearby, we have, from Eq. 36-2,

$$L = \frac{N\Phi_B}{i}. \tag{36-4}$$

Let us apply this equation to calculate L for a section of length l near the center of a long solenoid. The number of flux linkages in the length l of the solenoid is

$$N\Phi_B = (nl)(BA),$$

where n is the number of turns per unit length, B is the magnetic induction inside the solenoid, and A is the cross-sectional area. From Eq. 34-7, B is given by

$$B = \mu_0 ni.$$

Combining these equations gives

$$N\Phi_B = \mu_0 n^2 liA.$$

Finally, the inductance, from Eq. 36-4, is

$$L = \frac{N\Phi_B}{i} = \mu_0 n^2 lA. \tag{36-5}$$

The inductance of a length l of a solenoid is proportional to its volume (lA) and to the square of the number of turns per unit length. Note that it depends on geometrical factors only. The proportionality to n^2 is expected. If the number of turns per unit length is doubled, not only is the *total* number of turns N doubled but also the flux *through each turn* Φ_B is also doubled, an over-all factor of four for the flux linkages $N\Phi_B$, hence also a factor of four for the inductance (Eq. 36-4).

▶ **Example 1.** Derive an expression for the inductance of a toroid of rectangular cross section as shown in Fig. 36-2. Evaluate for $N = 10^3$, $a = 5.0$ cm, $b = 10$ cm, and $h = 1.0$ cm.

The lines of $\mathbf{B}$ for the toroid are concentric circles. Applying Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i,$$

to a circular path of radius r yields

$$(B)(2\pi r) = \mu_0 i_0 N,$$

where N is the number of turns and i_0 is the current in the toroid windings; recall that

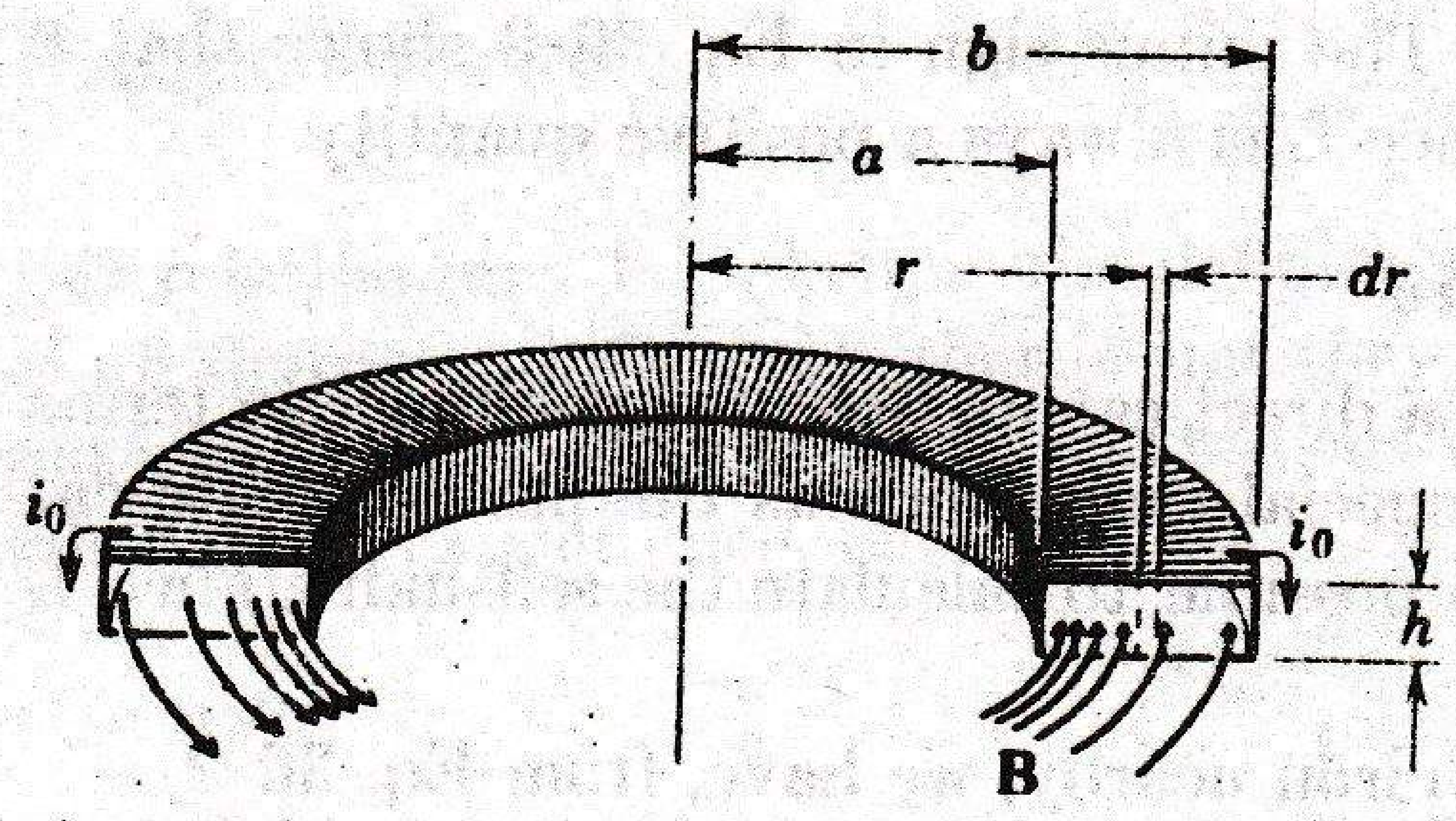


Fig. 36-2 Example 1. A cross section of a toroid, showing the current in the windings and the magnetic field.

i in Ampère's law is the *total* current that passes through the path of integration. Solving for B yields

$$B = \frac{\mu_0 i_0 N}{2\pi r}.$$

The flux Φ_B for the cross section of the toroid is

$$\Phi_B = \int \mathbf{B}\cdot d\mathbf{S} = \int_a^b (B)(h\,dr) = \int_a^b \frac{\mu_0 i_0 N}{2\pi r}\,h\,dr$$

$$= \frac{\mu_0 i_0 N h}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 i_0 N h}{2\pi}\ln\frac{b}{a},$$

where $h\,dr$ is the area of the elementary strip shown in the figure.

The inductance follows from Eq. 36-4, or

$$L = \frac{N\Phi_B}{i_0} = \frac{\mu_0 N^2 h}{2\pi}\ln\frac{b}{a}.$$

Substituting numerical values yields

$$L = \frac{(4\pi \times 10^{-7}\ \text{weber/amp-m})(10^3)^2(1.0 \times 10^{-2}\ \text{meter})}{2\pi}\ln\frac{10 \times 10^{-2}\ \text{meter}}{5 \times 10^{-2}\ \text{meter}}$$

$$= 1.4 \times 10^{-3}\ \text{weber/amp} = 1.4\ \text{mh}.$$ ◀

36-3 An *LR* Circuit

In Section 32-8 we saw that if an emf $\mathcal{E}$ is suddenly introduced, perhaps by using a battery, into a single loop circuit containing a resistor R and a capacitor C the charge does not build up immediately to its final equilibrium value ($= C\mathcal{E}$) but approaches it in an exponential fashion described by Eq. 32-15, or

$$q = C\mathcal{E}\,(1 - e^{-t/\tau_c}). \quad (36\text{-}6)$$

The delay in the rise of the charge is described by the *capacitative time constant* τ_c, defined from

$$\tau_c = RC. \quad (36\text{-}7)$$

If in this same circuit the battery emf $\mathcal{E}$ is suddenly removed, the charge does not immediately fall to zero but approaches zero in an exponential fashion, described by Eq. 32-18b, or

$$q = C\mathcal{E}\,e^{-t/\tau_c}. \quad (36\text{-}8)$$

The same time constant τ_c describes the fall of the charge as well as its rise.

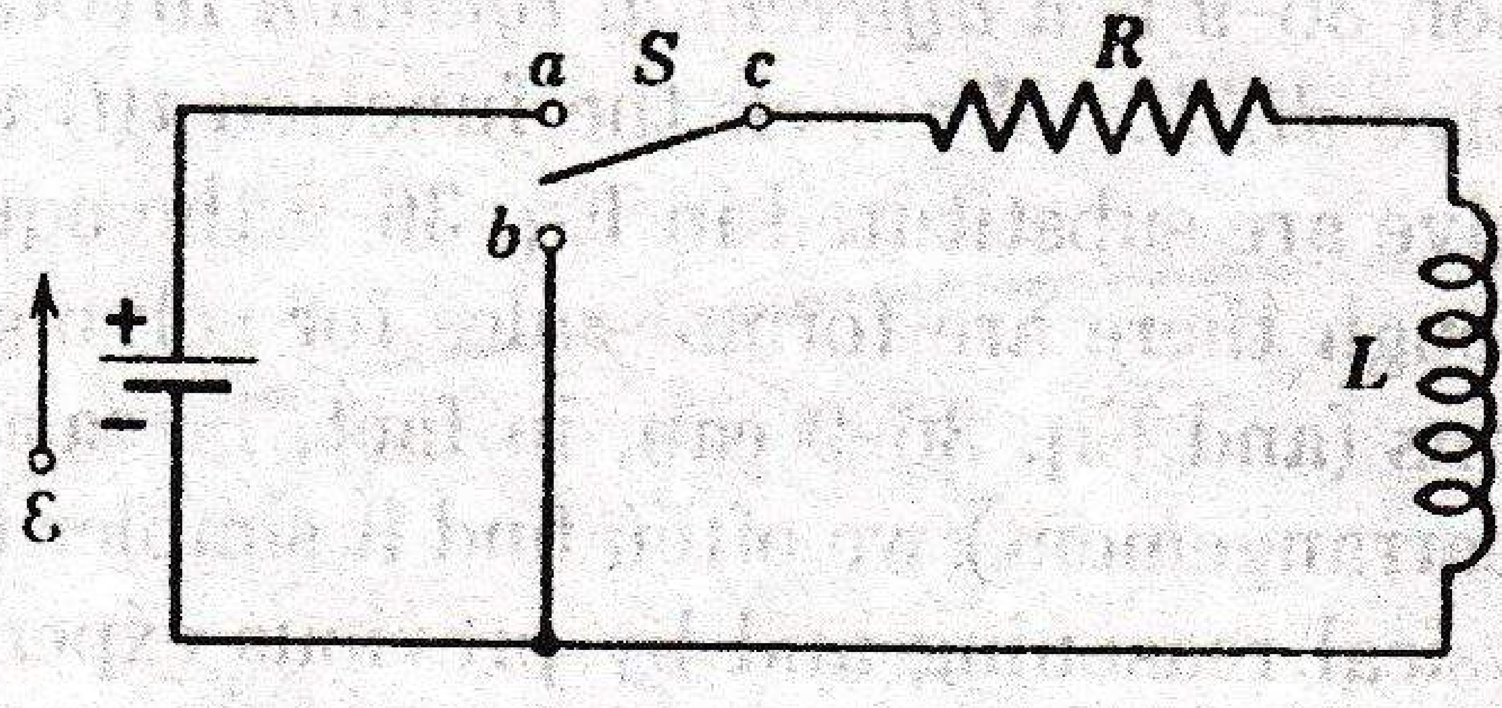


Fig. 36-3 An LR circuit.

An analogous delay in the rise or fall of the current occurs if an emf $\mathcal{E}$ is suddenly introduced into, or removed from, a single loop circuit containing a resistor R and an inductor L. When the switch S in Fig. 36-3 is closed on a, for example, the current in the resistor starts to rise. If the inductor were not present, the current would rise rapidly to a steady value $\mathcal{E}/R$. Because of the inductor, however, a self-induced emf $\mathcal{E}_L$ appears in the circuit and, from Lenz's law, this emf opposes the *rise* of current, which means that it opposes the battery emf $\mathcal{E}$ in polarity. Thus the resistor responds to the difference between two emfs, a constant one $\mathcal{E}$ due to the battery and a variable one $\mathcal{E}_L$ $(= -L\, di/dt)$ due to self-induction. As long as this second emf is present, the current in the resistor will be less than $\mathcal{E}/R$.

As time goes on, the rate at which the current increases becomes less rapid and the self-induced emf $\mathcal{E}_L$, which is proportional to di/dt, becomes smaller. Thus a time delay is introduced, and the current in the circuit approaches the value $\mathcal{E}/R$ asymptotically.

When the switch S in Fig. 36-3 is thrown to a, the circuit reduces to that of Fig. 36-4. Let us apply the loop theorem, starting at x in this figure and going clockwise around the loop. For the direction of current shown, x will be higher in potential than y, which means that we encounter a drop in potential of $-iR$ as we traverse the resistor. Point y is higher in potential than point z because, for an increasing current, the induced emf will oppose the *rise* of the current by pointing as shown. Thus as we traverse the inductor from y to z we encounter a drop in potential of $-L(di/dt)$. We encounter a rise in potential of $+\mathcal{E}$ in traversing the battery from z to x. The loop theorem thus gives

$$-iR - L\frac{di}{dt} + \mathcal{E} = 0$$

or

$$L\frac{di}{dt} + iR = \mathcal{E}. \tag{36-9}$$

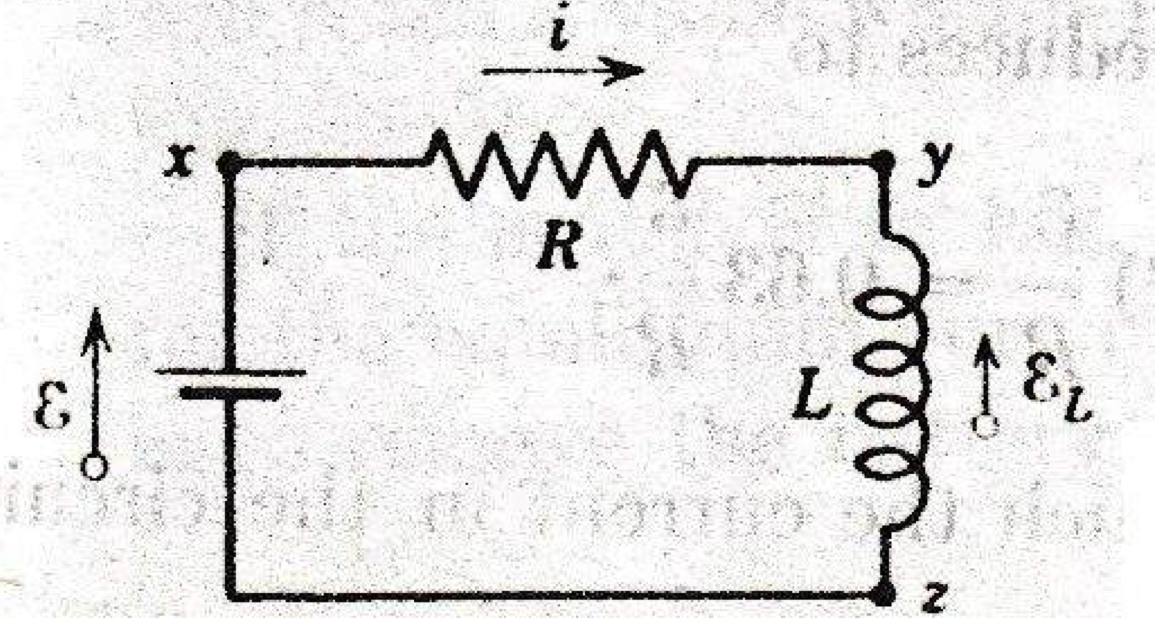


Fig. 36-4 The circuit of Fig. 36-3 just after switch S is closed on a.

Equation 36–9 is a *differential equation* involving the variable i and its first derivative di/dt. We seek the function $i(t)$ such that when it and its first derivative are substituted in Eq. 36–9 the equation is satisfied.

Although there are formal rules for solving various classes of differential equations (and Eq. 36–9 can, in fact, be easily solved by direct integration, after rearrangement) we often find it simpler to guess at the solution, guided by physical reasoning and by previous experience. Any proposed solution can be tested by substituting it in the differential equation and seeing whether this equation reduces to an identity.

The solution to Eq. 36–9 is, we assert,

$$i = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L}). \tag{36–10}$$

To test this solution by substitution, we find the derivative di/dt, which is

$$\frac{di}{dt} = \frac{\mathcal{E}}{L}e^{-Rt/L}. \tag{36–11}$$

Substituting i and di/dt into Eq. 36–9 leads to an identity, as the student can easily verify. Thus Eq. 36–10 is a solution of Eq. 36–9. Figure 36–5 shows how the potential difference V_R across the resistor ($= iR$; see Eq. 36–10) and V_L across the inductor ($= L\,di/dt$; see Eq. 36–11) vary with time for particular values of $\mathcal{E}$, L, and R. The student should compare this figure carefully with the corresponding figure for an RC circuit (Fig. 32–11).

We can rewrite Eq. 36–10 as

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}), \tag{36–12}$$

in which τ_L, the *inductive time constant*, is given by

$$\tau_L = L/R. \tag{36–13}$$

The student should note the correspondence between Eqs. 36–12 and 36–6.

To show that the quantity τ_L ($= L/R$) has the dimensions of time, we put

$$\frac{1 \text{ henry}}{\text{ohm}} = \frac{1 \text{ henry}}{\text{ohm}}\left(\frac{1 \text{ volt-sec}}{1 \text{ henry-amp}}\right)\left(\frac{1 \text{ ohm-amp}}{1 \text{ volt}}\right) = 1 \text{ sec.}$$

The first quantity in parenthesis is a conversion factor based on the defining equation for inductance [$L = -\mathcal{E}/(di/dt)$; Eq. 36–3*b*]. The second conversion factor is based on the relation $V = iR$.

The physical significance of the time constant follows from Eq. 36–12. If we put $t = \tau_L = L/R$ in this equation, it reduces to

$$i = \frac{\mathcal{E}}{R}(1 - e^{-1}) = (1 - 0.37)\frac{\mathcal{E}}{R} = 0.63\frac{\mathcal{E}}{R}.$$

Thus the time constant τ_L is that time at which the current in the circuit

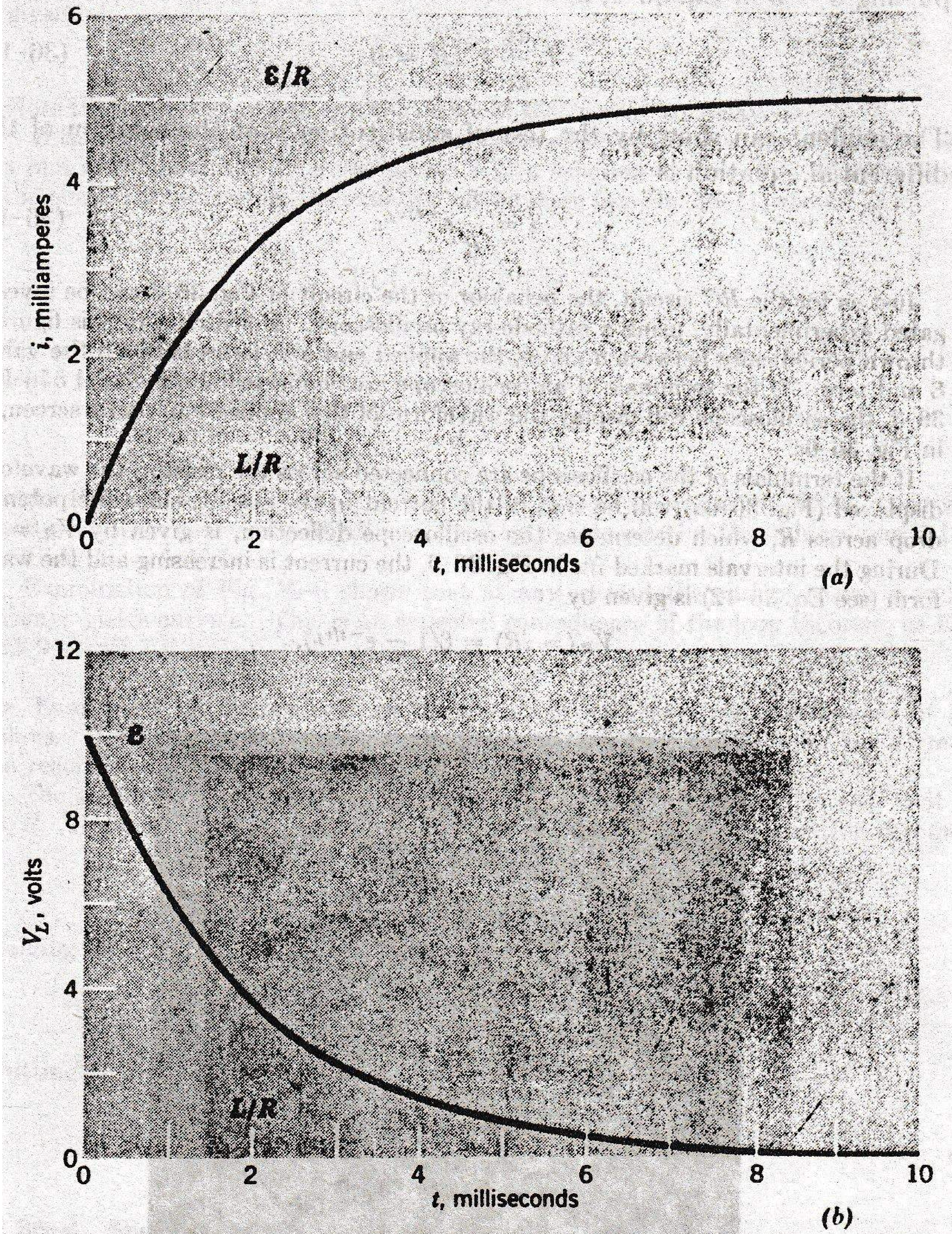


Fig. 36-5 If in Fig. 36-3 we assume that $R = 2000$ ohms, $L = 4$ henrys and $\mathcal{E} = 10$ volts, then (*a*) shows the variation of i with t during the current buildup after switch S is closed on a, and (*b*) the variations of V_L with t. The time constant is $L/R = 2.0 \times 10^{-3}$ sec.

will reach a value within $1/e$ (about 37%) of its final equilibrium value (see Fig. 36-5).

If the switch S in Fig. 36-3, having been left in position a long enough for the equilibrium current $\mathcal{E}/R$ to be established, is thrown to b, the effect is to remove the battery from the circuit. The differential equation that governs the subsequent decay of the current in the circuit can be found by putting $\mathcal{E} = 0$ in Eq. 36–9, or

$$L\frac{di}{dt} + iR = 0. \tag{36-14}$$

The student can show by the test of substitution that the solution of this differential equation is

$$i = \frac{\mathcal{E}}{R}e^{-t/\tau_L}. \tag{36-15}$$

Just as for the *RC* circuit, the behavior of the circuit of Fig. 36–3 can be investigated experimentally, using a cathode-ray oscilloscope. If switch *S* in this figure is thrown periodically between *a* and *b*, the applied emf alternates between the values $\mathcal{E}$ and zero. If the terminals of an oscilloscope are connected across *b* and *c* in Fig. 36–3, the oscilloscope will display the waveform of this applied emf on its screen, as in Fig. 36–6*c*.

If the terminals of the oscilloscope are connected across the resistor, the waveform displayed (Fig. 36–6*a*) will be that of the current in the circuit, since the potential drop across *R*, which determines the oscilloscope deflection, is given by $V_R = iR$. During the intervals marked *inc* in Fig. 36–6, the current is increasing and the waveform (see Eq. 36–12) is given by

$$V_R(= iR) = \mathcal{E}(1 - e^{-t/\tau_L}).$$

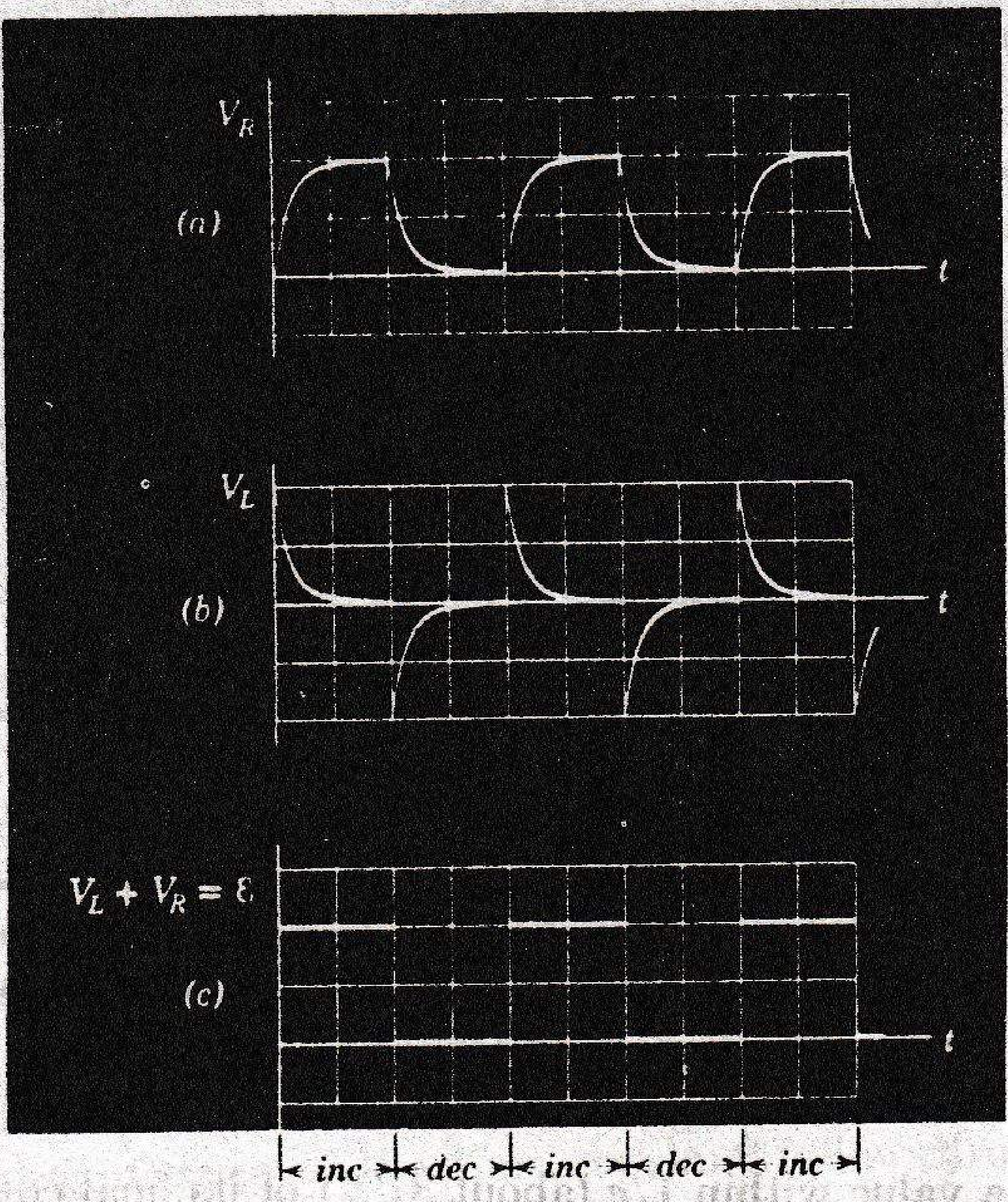


Fig. 36–6 Oscilloscope photograph showing the variation with time of (*a*) the potential drop V_R across the resistor, (*b*) the potential drop V_L across the inductor, and (*c*) the applied emf $\mathcal{E}$. During the intervals marked *inc* the current is increasing; during those marked *dec* it is decreasing. Compare with Fig. 32–13. (Courtesy E. K. Hege.)

During the intervals marked *dec*, the current is decreasing and V_R (see Eq. 36–15) is given by

$$V_R(= iR) = \mathcal{E}e^{-t/\tau_L}.$$

Note that both the growth and the decay of the current are delayed.

If the oscilloscope terminals are connected across the inductor, the screen will show a plot of the potential difference across it as a function of time (Fig. 36–6*b*). While the current is increasing, the equation of the trace (see Eq. 36–11) should be

$$V_L\left(= L\frac{di}{dt}\right) = \mathcal{E}e^{-t/\tau_L}.$$

When the current is decreasing, V_L is given in terms of the time derivative of Eq. 36–15 and is

$$V_L\left(= L\frac{di}{dt}\right) = -\mathcal{E}e^{-t/\tau_L}.$$

Note that V_L is opposite in sign when the current is increasing (di/dt positive) and when it is decreasing (di/dt negative), as is true also for the induced emf $\mathcal{E}_L[= -L(di/dt) = -V_L]$.

Examination of Fig. 36–6 shows that at any instant the sum of curves *a* and *b* always yields curve *c*. This is an expected consequence of the loop theorem, as Eq. 36–9 shows.

▶ **Example 2.** A solenoid has an inductance of 50 henrys and a resistance of 30 ohms. If it is connected to a 100-volt battery, how long will it take for the current to reach one-half its final equilibrium value?

The equilibrium value of the current is reached as $t \to \infty$; from Eq. 36–12 it is $\mathcal{E}/R$. If the current has half this value at a particular time t_0, this equation becomes

$$\frac{1}{2}\frac{\mathcal{E}}{R} = \frac{\mathcal{E}}{R}(1 - e^{-t_0/\tau_L}).$$

Solving for t_0 yields

$$t_0 = \tau_L \ln 2 = 0.69\frac{L}{R}.$$

Putting $\tau_L = L/R$ and using the values given, this reduces to

$$t_0 = 0.69\tau_L = 0.69\left(\frac{50 \text{ henrys}}{30 \text{ ohms}}\right) = 1.2 \text{ sec.}$$ ◀

36–4 Energy and the Magnetic Field

In Section 30–7 we saw that the electric field could be viewed as the site of stored energy, the energy per unit volume being given, in a vacuum, by

$$u_E = \tfrac{1}{2}\epsilon_0 E^2,$$

where E is the electric field strength at the point in question. Although this formula was derived for a parallel-plate capacitor, it holds for all kinds of electric field configurations.

Energy can also be stored in a magnetic field. For example, two parallel wires carrying currents in the same direction attract each other, and to pull them further apart work must be done. It is useful to think that this expended energy is stored in the magnetic field between and around the wires. The energy can be recovered from the field if the wires are allowed to move back to their original separation. In the electrostatic case the same argument was applied to the pulling apart of two unlike charges and in the gravitational case to the pulling apart of two masses.

To derive a quantitative expression for the storage of energy in the magnetic field, consider Fig. 36-4, which shows a source of emf $\mathcal{E}$ connected to a resistor R and an inductor L.

$$\mathcal{E} = iR + L\frac{di}{dt}, \qquad (36\text{-}9)$$

is the differential equation that describes the growth of current in this circuit. We stress that this equation follows immediately from the loop theorem and that the loop theorem in turn *is an expression of the principle of conservation of energy for single-loop circuits.* If we multiply each side of Eq. 36-9 by i, we obtain

$$\mathcal{E}i = i^2R + Li\frac{di}{dt}, \qquad (36\text{-}16)$$

which has the following physical interpretation in terms of work and energy:

1. If a charge dq passes through the seat of emf $\mathcal{E}$ in Fig. 36-4 in time dt, the seat does work on it in amount $\mathcal{E}\,dq$. The *rate* of doing work is $(\mathcal{E}\,dq)/dt$, or $\mathcal{E}i$. Thus the left term in Eq. 36-16 is the *rate at which the seat of emf delivers energy to the circuit.*

2. The second term in Eq. 36-16 is the *rate at which energy appears as Joule heat in the resistor.*

3. Energy that does not appear as Joule heat must, by our hypothesis, be stored in the magnetic field. Since Eq. 36-16 represents a statement of the conservation of energy for LR circuits, the last term must represent the *rate dU_B/dt at which energy is stored in the magnetic field,* or

$$\frac{dU_B}{dt} = Li\frac{di}{dt}. \qquad (36\text{-}17)$$

We can write this as $\qquad dU_B = Li\,di.$

Integrating yields

$$U_B = \int_0^{U_B} dU_B = \int_0^i Li\,di = \tfrac{1}{2}Li^2, \qquad (36\text{-}18)$$

which represents the total stored magnetic energy in an inductance L carrying a current i.

This relation can be compared with the expression for the energy associated with a capacitor C carrying a charge q, namely

$$U_E = \frac{1}{2}\frac{q^2}{C}.$$

Here the energy is stored in an electric field. In each case the expression for the stored energy was derived by setting it equal to the work that must be done to set up the field.

▶ **Example 3.** A coil has an inductance of 5.0 henrys and a resistance of 20 ohms. If a 100-volt emf is applied, what energy is stored in the magnetic field after the current has built up to its maximum value $\mathcal{E}/R$?

The maximum current is given by

$$i = \frac{\mathcal{E}}{R} = \frac{100 \text{ volts}}{20 \text{ ohms}} = 5.0 \text{ amp.}$$

The stored energy is given by Eq. 36-18:

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}(5.0 \text{ henrys})(5.0 \text{ amp})^2 = 63 \text{ joules.}$$

Note that the time constant for this coil ($= L/R$) is 0.25 sec. After how many time constants will *half* of this equilibrium energy be stored in the field?

Example 4. A 3.0-henry inductor is placed in series with a 10-ohm resistor, an emf of 3.0 volts being suddenly applied to the combination. At 0.30 sec (which is one inductive time constant) after the contact is made, (*a*) what is the rate at which energy is being delivered by the battery?

The current is given by Eq. 36-12, or

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}),$$

which at $t = 0.30$ sec ($= \tau_L$) has the value

$$i = \left(\frac{3.0 \text{ volts}}{10 \text{ ohms}}\right)(1 - e^{-1}) = 0.189 \text{ amp.}$$

The rate $P_\mathcal{E}$ at which energy is delivered by the battery is

$$\begin{aligned} P_\mathcal{E} &= \mathcal{E}i \\ &= (3.0 \text{ volts})(0.189 \text{ amp}) \\ &= 0.567 \text{ watt.} \end{aligned}$$

(*b*) At what rate does energy appear as Joule heat in the resistor? This is given by

$$\begin{aligned} P_J &= i^2R \\ &= (0.189 \text{ amp})^2(10 \text{ ohms}) \\ &= 0.357 \text{ watt.} \end{aligned}$$

(*c*) At what rate P_B is energy being stored in the magnetic field? This is given by the last term in Eq. 36-16, which requires that we know di/dt. Differentiating Eq. 36-12 yields

$$\begin{aligned} \frac{di}{dt} &= \left(\frac{\mathcal{E}}{R}\right)\left(\frac{R}{L}\right)e^{-t/\tau_L} \\ &= \frac{\mathcal{E}}{L}e^{-t/\tau_L}. \end{aligned}$$

At $t = \tau_L$ we have

$$\frac{di}{dt} = \left(\frac{3.0 \text{ volts}}{3.0 \text{ henrys}}\right)e^{-1} = 0.37 \text{ amp/sec.}$$

From Eq. 36–17, the desired rate is

$$P_B = \frac{dU_B}{dt} = Li\frac{di}{dt}$$

$$= (3.0 \text{ henrys})(0.189 \text{ amp})(0.37 \text{ amp/sec})$$

$$= 0.210 \text{ watt.}$$

Note that as required by the principle of conservation of energy (see Eq. 36–16)

$$P_\mathcal{E} = P_J + P_B,$$

or

$$0.567 \text{ watt} = 0.357 \text{ watt} + 0.210 \text{ watt}$$

$$= 0.567 \text{ watt.}$$

◀

36–5 Energy Density and the Magnetic Field

We now derive an expression for the *density* of energy u in a magnetic field. Consider a length l near the center of a very long solenoid; Al is the volume associated with this length. The stored energy must lie entirely within this volume because the magnetic field outside such a solenoid is essentially zero. Moreover, the stored energy must be uniformly distributed throughout the volume of the solenoid because the magnetic field is uniform everywhere inside.

Thus we can write

$$u_B = \frac{U_B}{Al}$$

or, since

$$U_B = \tfrac{1}{2}Li^2,$$

$$u_B = \frac{\tfrac{1}{2}Li^2}{Al}.$$

To express this in terms of the magnetic field, we can substitute for L in this equation, using the relation $L = \mu_0 n^2 lA$ (Eq. 36–5). Also we can solve Eq. 34–7 ($B = \mu_0 in$) for i and substitute in this equation. Doing so yields finally

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0}. \qquad (36\text{–}19)$$

This equation gives the energy density stored at any point (in a vacuum or in a nonmagnetic substance) where the magnetic induction is **B**. The equation is true for all magnetic field configurations, even though it was derived by considering a special case, the solenoid. Equation 36–19 is to be compared with Eq. 30–27,

$$u_E = \tfrac{1}{2}\epsilon_0 E^2, \qquad (\kappa = 1) \qquad (36\text{–}20)$$

which gives the energy density (in a vacuum) at any point in an electric field. Note that both u_B and u_E are proportional to the square of the appropriate field quantity, B or E.

The solenoid plays a role with relationship to magnetic fields similar to the role the parallel-plate capacitor plays with respect to electric fields. In each case we have a simple device that can be used for setting up a uniform field throughout a well-defined region of space and for deducing, in a simple way, some properties of these fields.

▶ **Example 5.** A long *coaxial cable* (Fig. 36-7) consists of two concentric cylinders with radii a and b. Its central conductor carries a steady current i, the outer conductor providing the return path. (*a*) Calculate the energy stored in the magnetic field for a length l of such a cable.

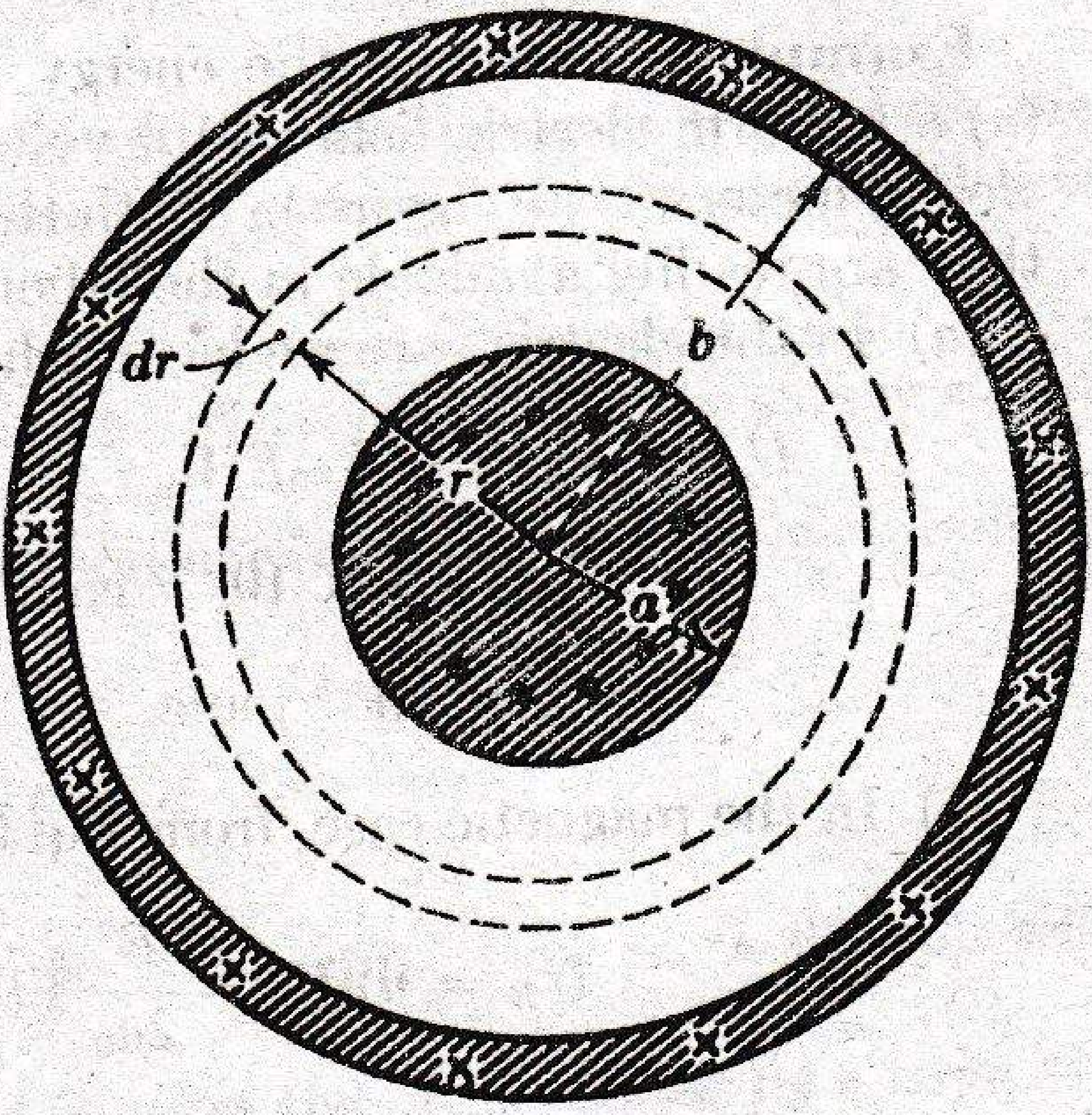


Fig. 36-7 Example 5. Cross section of a coaxial cable, showing steady currents in the central and outer conductors.

In the space between the two conductors Ampère's law

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i$$

leads to

$$(B)(2\pi r) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi r}.$$

Ampère's law shows further that the magnetic field is zero for points outside the outer conductor (why?). Magnetic fields exist *inside* each of the conductors; although their values can readily be found from Ampère's law, we choose to ignore them, on the assumption that the cable dimensions are chosen so that most of the stored magnetic energy is in the space between the conductors.

The energy density for points between the conductors, from Eq. 36-19, is

$$u = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2\pi r}\right)^2 = \frac{\mu_0 i^2}{8\pi^2 r^2}.$$

Consider a volume element dV consisting of a cylindrical shell whose radii are r and $r + dr$ and whose length is l. The energy dU contained in it is

$$dU = u\,dV = \frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi r l)(dr) = \frac{\mu_0 i^2 l}{4\pi}\frac{dr}{r}.$$

The total stored magnetic energy is found by integration, or

$$U = \int dU = \frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 i^2 l}{4\pi}\ln\frac{b}{a},$$

which is the desired expression.

(*b*) What is the inductance of a length l of coaxial cable?

The inductance L can be found from Eq. 36-18 ($U = \frac{1}{2}Li^2$), which leads to

$$L = \frac{2U}{i^2} = \frac{\mu_0 l}{2\pi} \ln \frac{b}{a}.$$

The student should also derive this expression directly from the definition of inductance, using the procedures of Example 1.

Example 6. Compare the energy required to set up, in a cube 10 cm on edge, (*a*) a uniform electric field of 10^5 volts/meter and (*b*) a uniform magnetic field of 1 weber/meter^2 (= 10^4 gauss). Both these fields would be judged reasonably large but they are readily available in the laboratory.

(*a*) In the electric case we have, where V_0 is the volume of the cube,

$$U_E = u_E V_0 = \tfrac{1}{2}\epsilon_0 E^2 V_0$$

$$= (0.5)(8.9 \times 10^{-12} \text{ coul}^2/\text{nt-m}^2)(10^5 \text{ volts/meter})^2(0.1 \text{ meter})^3$$

$$= 4.5 \times 10^{-5} \text{ joule}.$$

(*b*) In the magnetic case, from Eq. 36-19, we have

$$U_B = u_B V_0 = \frac{B^2}{2\mu_0} V_0 = \frac{(1 \text{ weber/meter}^2)^2(0.1 \text{ meter})^3}{(2)(4\pi \times 10^{-7} \text{ weber/amp-m})}$$

$$= 400 \text{ joules}.$$

In terms of fields normally available in the laboratory, much larger amounts of energy can be stored in a magnetic field than in an electric one, the ratio being about 10^7 in this example. Conversely, much more energy is required to set up a magnetic field of reasonable laboratory magnitude than is required to set up an electric field of similarly reasonable magnitude. ◀

QUESTIONS

1. Two coils are connected in series. Does their equivalent inductance depend on their geometrical relationship to each other?

2. Is the inductance per unit length for a solenoid near its center (*a*) the same as, (*b*) less than, or (*c*) greater than the inductance per unit length near its ends?

3. Two solenoids, A and B, have the same diameter and length and contain only one layer of windings, with adjacent turns touching, insulation thickness being negligible. Solenoid A contains many turns of fine wire and solenoid B contains fewer turns of heavier wire. (*a*) Which solenoid has the larger inductance? (*b*) Which solenoid has the larger inductive time constant?

4. If the flux passing through each turn of a coil is the same, the inductance of the coil may be computed from $L = N\Phi_B/i$ (Eq. 36-4). How might one compute L for a coil for which this assumption is not valid?

5. If a current in a source of emf is in the direction of the emf, the energy of the source decreases; if a current is in a direction opposite to the emf (as in charging a battery), the energy of the source increases. Do these statements apply to the inductor in Fig. 36-1*a* and 36-1*b*?

6. Show that the dimensions of the two expressions for L, $N\Phi_B/i$ (Eq. 36-4) and $\mathcal{E}/(di/dt)$ (Eq. 36-3*b*), are the same.

7. You are given N turns of wire connected in series. How should the turns be arranged to obtain the maximum self-inductance?

8. Does the time required for the current in a particular LR circuit to build up to any given fraction of its equilibrium value depend on the value of the applied emf?

9. A steady current is set up in a coil with a very large inductive time constant. When the current is interrupted with a switch, a heavy arc tends to appear at the switch blades. Explain. (Note: Interrupting currents in highly inductive circuits can be dangerous.)

10. In an LR circuit like that of Fig. 36–4 can the self-induced emf ever be larger than the battery emf?

11. In an LR circuit like that of Fig. 36–4 is the current in the resistance *always* the same as the current in the inductance?

12. In the circuit of Fig. 36–3 the self-induced emf is a maximum at the instant the switch is closed on a. How can this be since there is no current in the inductance at this instant?

13. Give some arguments to show that energy can be stored in a magnetic field.

14. The switch in Fig. 36–3 is thrown from a to b. What happens to the energy stored in the inductor?

15. In a toroid is the energy density larger near the inner radius or near the outer radius?

PROBLEMS

1. A 10-henry inductor carries a steady current of 2.0 amp. How can a 100-volt self-induced emf be made to appear in the inductor?

2. Two inductances L_1 and L_2 are connected in series and are separated by a large distance. (*a*) Show that the equivalent inductance L is $L_1 + L_2$. (*b*) Why must their separation be large?

3. Show that if two inductors with equal inductance L are connected in parallel the equivalent inductance of the combination is $\frac{1}{2}L$. The inductors are separated by a large distance.

4. Two parallel wires whose centers are a distance d apart carry equal currents in opposite directions. Show that, neglecting the flux within the wires themselves, the inductance of a length l of such a pair of wires is given by

$$L = \frac{\mu_0 l}{\pi} \ln \frac{d - a}{a},$$

where a is the wire radius. See Example 4, Chapter 34.

5. A long thin solenoid can be bent into a ring to form a toroid. Show that if the solenoid is long and thin enough the equation for the inductance of a toroid (see Example 1) reduces to that for a solenoid (Eq. 36–5).

6. A solenoid is wound with a single layer of #10 copper wire (diameter, 0.10 in.). It is 4.0 cm in diameter and 2.0 meters long. What is the inductance per unit length for the solenoid near its center? Assume that adjacent wires touch and that insulation thickness is negligible.

7. The inductance of a close-packed coil of 400 turns is 8 mh. What is the magnetic flux through the coil when the current is 5×10^{-3} amp?

8. A wooden toroidal core with a square cross section has an inner radius of 10 cm and an outer radius of 12 cm. It is wound with one layer of #18 wire (diameter, 0.040 in.; resistance, 160 ft/ohm). What are (*a*) the inductance and (*b*) the inductive time constant? Ignore the thickness of the insulation.

9. The current in an LR circuit builds up to one-third of its steady-state value in 5.0 sec. What is the inductive time constant?

10. How many "time constants" must we wait for the current in an LR circuit to build up to within 0.1 per cent of its equilibrium value?

11. The switch S in Fig. 36–3 is thrown from b to a. After one inductive time constant show that (*a*) the total energy transformed to Joule heat in the resistor is $0.168\mathcal{E}^2\tau_L/R$

and that (*b*) the energy stored in the magnetic field is $0.200\mathcal{E}^2\tau_L/R$. (*c*) Show that the equilibrium energy stored in the magnetic field is $0.500\mathcal{E}^2\tau_L/R$.

12. Show that the inductive time constant τ_L can also be defined as the time that would be required for the current in an *LR* circuit to reach its equilibrium value *if it continued to increase at its initial rate.*

13. A 50-volt potential difference is suddenly applied to a coil with $L = 50$ mh and $R = 180$ ohms. At what rate is the current increasing after 0.001 sec?

14. A coil with an inductance of 2.0 henrys and a resistance of 10 ohms is suddenly connected to a resistanceless battery with $\mathcal{E} = 100$ volts. At 0.1 sec after the connection is made, what are the rates at which (*a*) energy is being stored in the magnetic field, (*b*) Joule heat is appearing, and (*c*) energy is being delivered by the battery?

15. A coil with an inductance of 2.0 henrys and a resistance of 10 ohms is suddenly connected to a resistanceless battery with $\mathcal{E} = 100$ volts. (*a*) What is the equilibrium current? (*b*) How much energy is stored in the magnetic field when this current exists in the coil?

16. Prove that when switch *S* in Fig. 36–3 is thrown from *a* to *b* all the energy stored in the inductor appears as Joule heat in the resistor.

17. A circular loop of wire 5.0 cm in radius carries a current of 100 amp. What is the energy density at the center of the loop?

18. What is the magnetic energy density at the center of a circulating electron in the hydrogen atom (see Example 9, Chapter 34)?

19. A long wire carries a current of uniform density. Let i be the total current carried by the wire and show that the magnetic energy per unit length stored *within* the wire equals $\mu_0 i^2/16\pi$. Note that it does not depend on the wire diameter.

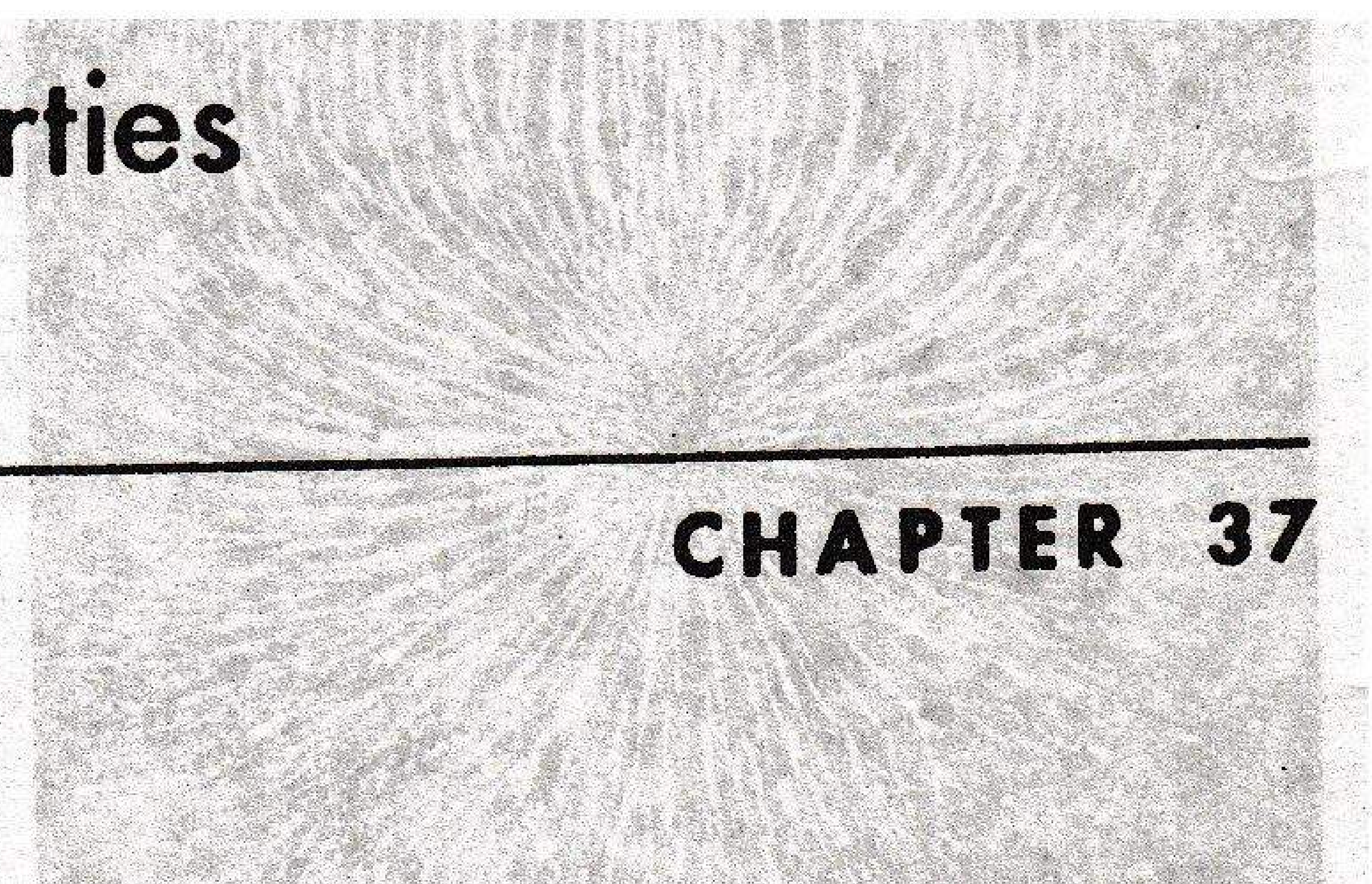
20. Show that the self-inductance for a length l of a long wire associated with the flux *inside* the wire only is $\mu_0 l/8\pi$, independent of the wire diameter.

21. The coaxial cable of Example 5 has $a = 1.0$ mm, $b = 4.0$ mm, and $c = 5.0$ mm (c is the radius of the outer surface of the outer conductor). It carries a current of 10 amp in the inner conductor and an equal but oppositely directed return current in the outer conductor. Calculate and compare the stored magnetic energy per meter of cable length (*a*) within the central conductor, (*b*) in the space between the conductors, and (*c*) within the outer conductor.

22. A length of #10 copper wire carries a current of 10 amp. Calculate (*a*) the magnetic energy density and (*b*) the electric energy density at the surface of the wire. The wire diameter is 0.10 in. and its resistance per unit length is 1.0 ohm/1000 ft.

23. What must be the strength of a uniform electric field if it is to have the same energy density as that possessed by a 5000-gauss magnetic field?

Magnetic Properties of Matter



CHAPTER 37

37–1 Poles and Dipoles

In electricity the *isolated charge q* is the simplest structure that can exist. If two such charges of opposite sign are placed near each other, they form an *electric dipole*, characterized by an electric dipole moment **p**. In magnetism isolated magnetic "poles," which would correspond to isolated electric charges, apparently do not exist. The simplest magnetic structure is the *magnetic dipole*, characterized by a magnetic dipole moment $\boldsymbol{\mu}$. Table 34–1 summarizes some characteristics of electric and magnetic dipoles.

A current loop, a bar magnet, and a solenoid of finite length are examples of magnetic dipoles. Their magnetic dipole moments can be measured by placing the dipole in an external magnetic field **B**, measuring the torque $\boldsymbol{\tau}$ that acts on it, and computing $\boldsymbol{\mu}$ from Eq. 33–11, or

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}. \tag{37–1}$$

Alternatively, we can measure **B** due to the dipole at a point along its axis a (large) distance r from its center and compute μ from the expression in Table 34–1, or

$$B = \frac{\mu_0}{2\pi}\frac{\mu}{r^3}. \tag{37–2}$$

Figure 37–1, which shows iron filings sprinkled on a sheet of paper under which there is a bar magnet, suggests that this dipole might be viewed as two "poles" separated by a distance d. However, all attempts to isolate these poles fail. If the magnet is broken, as in Fig. 37–2, the fragments prove to be dipoles and not isolated poles. If we break up a magnet into the electrons

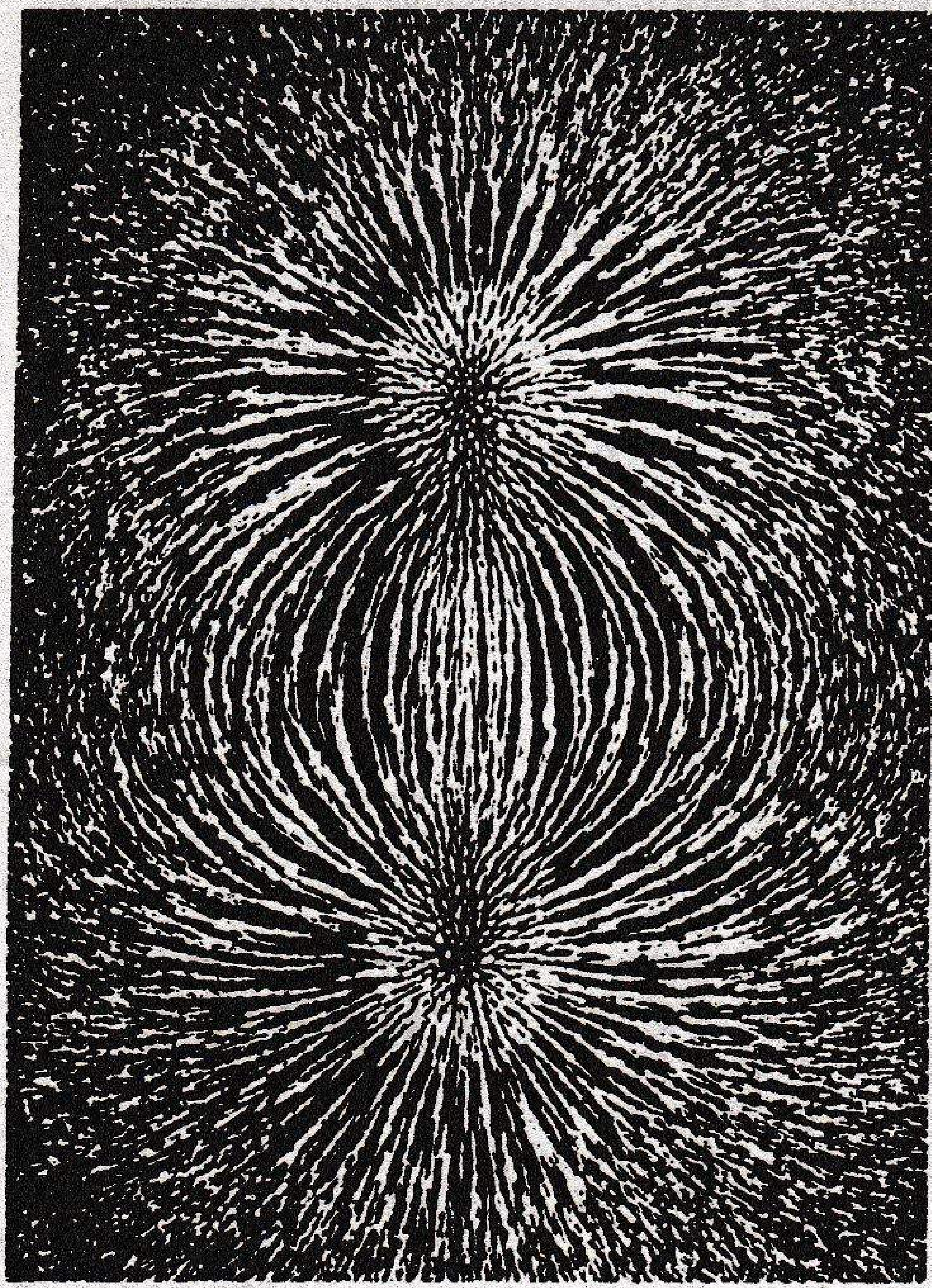


Fig. 37–1 A bar magnet is a magnetic dipole. The iron filings suggest the pattern of lines of force in Fig. 37–4*a*. (Courtesy Physical Science Study Committee.)

and nuclei that make up its atoms, it will be found that even these elementary particles are magnetic dipoles. Figure 37–3 contrasts the electric and the magnetic characters of the free electron.

All electrons have a characteristic *"spin" angular momentum* about a certain axis, which has the value of

$$L_s = 0.52723 \times 10^{-34} \text{ joule-sec.}$$

This is suggested by the vector $\mathbf{L}_s$ in Fig. 37–3*b*. Such a spinning charge can be viewed classically as being made up of infinitesimal current loops. Each such loop is a tiny magnetic dipole, its moment being given by (Eq. 33–10)

$$\mu = NiA, \tag{37–3}$$

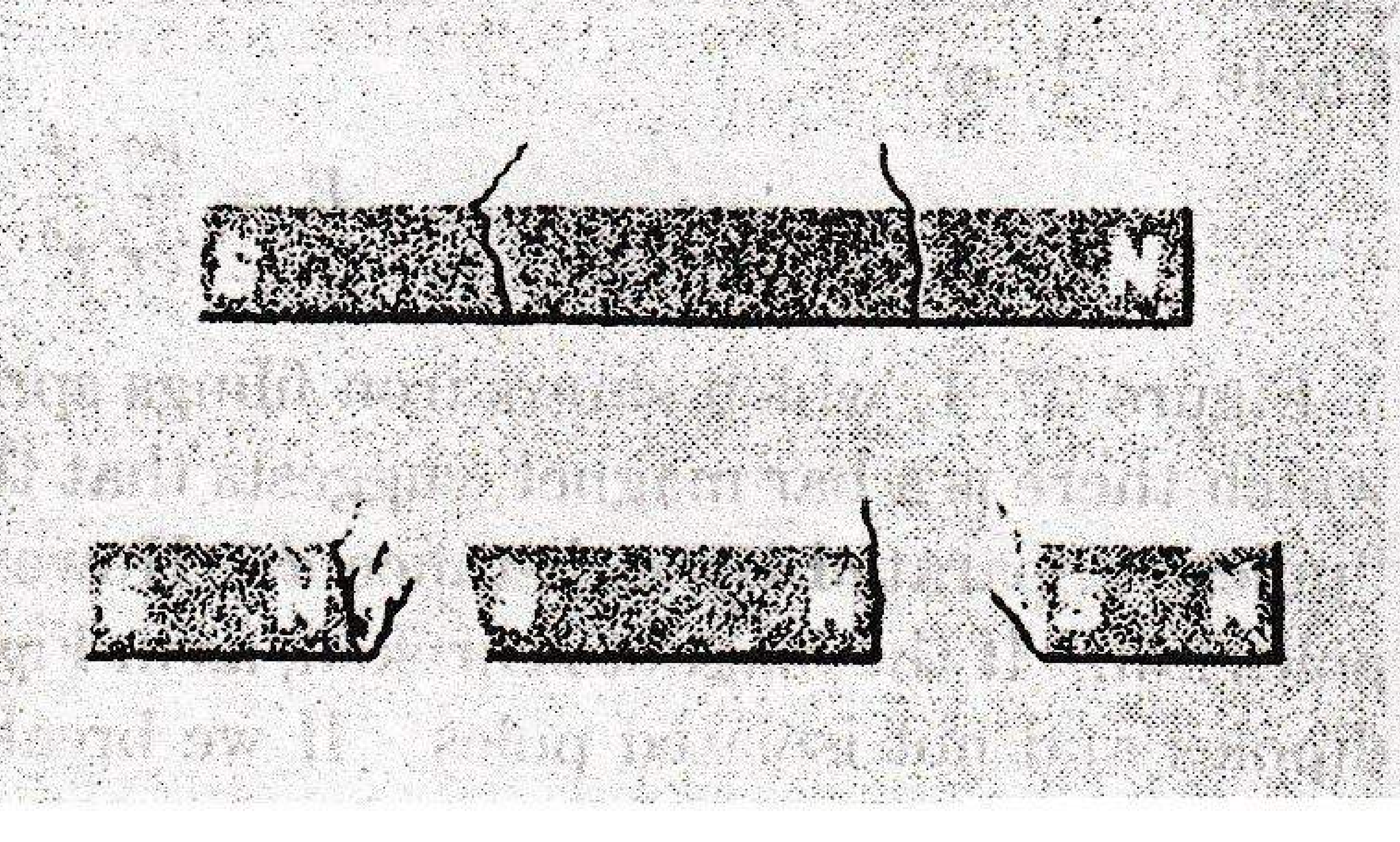


Fig. 37–2 If a bar magnet is broken, each fragment becomes a small dipole.

where i is the equivalent current in each infinitesimal loop and A is the loop area. The number of turns, N, is unity for each loop. The magnetic dipole moment of the spinning charge can be found by integrating over the moments of the infinitesimal current loops that make it up; see Problem 2.

Although this model of the spinning electron is too mechanistic and is not in accord with modern quantum physics, it remains true that the magnetic dipole moments of elementary particles are closely connected with their intrinsic angular momenta, or spins. Those particles and nuclei whose spin angular momentum is zero (the α-particle, the pion, the O^{16} nucleus, etc.) have no magnetic dipole moment. The "intrinsic" or "spin" magnetic moment of the electron must be distinguished from any additional magnetic moment it may have because of its *orbital* motion in an atom; see Example 2.

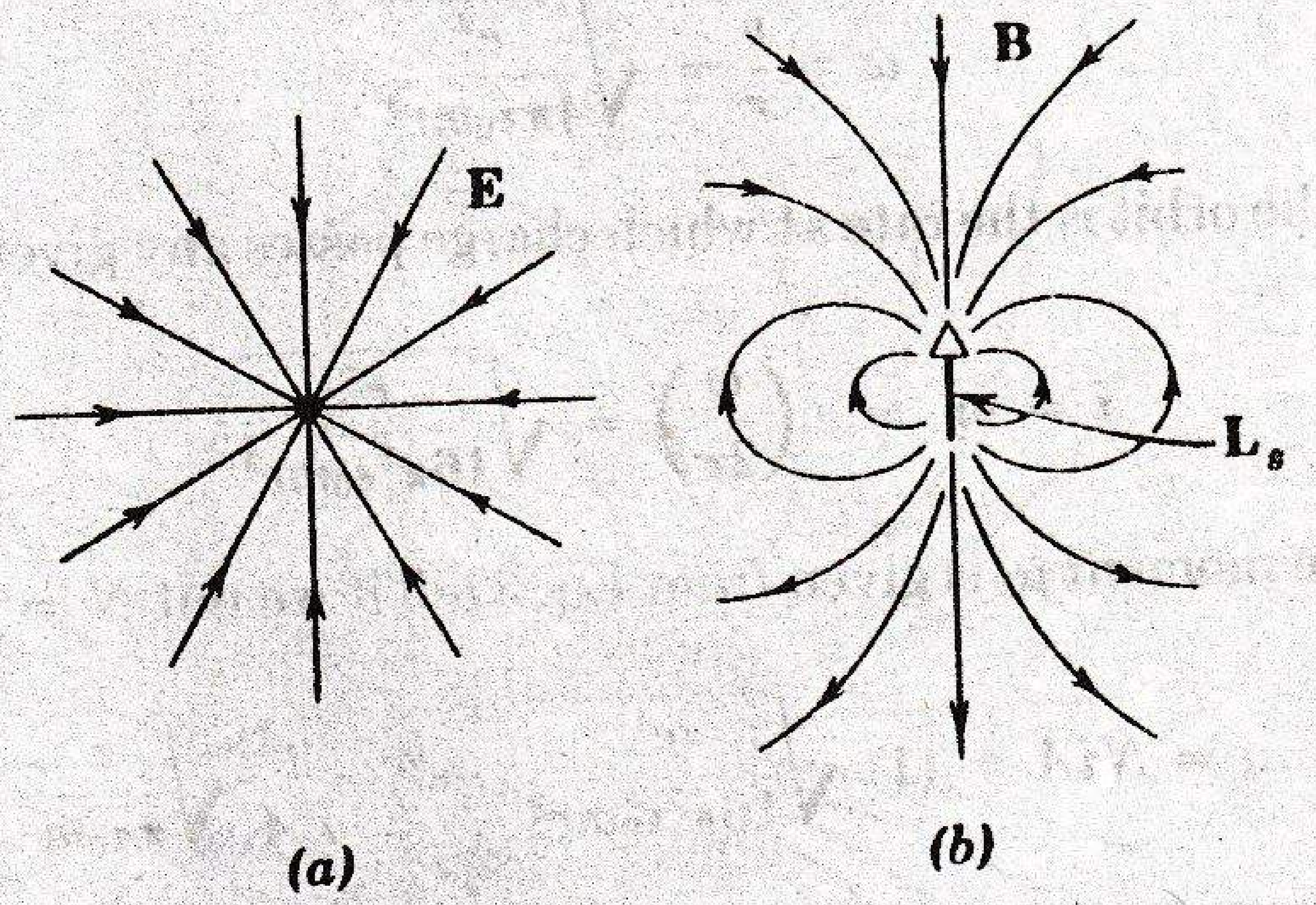


Fig. 37-3 (*a*) The lines of **E** and (*b*) the lines of **B** for an electron. The magnetic dipole moment of the electron, μ_l, is directed opposite to the spin angular momentum vector, $\mathbf{L}_s$.

▶ **Example 1.** Devise a method for measuring μ for a bar magnet.

(*a*) Place the magnet in a uniform external magnetic field **B**, with $\boldsymbol{\mu}$ making an angle θ with **B**. The magnitude of the torque acting on the magnet (see Eq. 37-1) is given by

$$\tau = \mu B \sin \theta.$$

Clearly μ can be learned if τ, B, and θ are measured.

(*b*) A second technique is to suspend the magnet from its center of mass and to allow it to oscillate about its stable equilibrium position in the external field **B**. For *small* oscillations, $\sin \theta$ can be replaced by θ and the equation just given becomes

$$\tau = -(\mu B)\theta = -\kappa\theta,$$

where κ is a constant. The minus sign has been inserted to show that τ is a *restoring torque*. Since τ is proportional to θ, the condition for simple angular harmonic motion is met. The frequency ν is given by the reciprocal of Eq. 15-24, or

$$\nu = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} = \frac{1}{2\pi}\sqrt{\frac{\mu B}{I}}.$$

With this equation μ can be found from the measured quantities ν, B, and I.

Example 2. An electron in an atom circulating in an assumed circular orbit of radius r behaves like a tiny current loop and has an *orbital magnetic dipole moment* * usually represented by μ_l. Derive a connection between μ_l and the *orbital angular momentum* L_l.

Newton's second law ($F = ma$) yields, if we substitute Coulomb's law for F,

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = ma = \frac{mv^2}{r}$$

or

$$v = \sqrt{\frac{e^2}{4\pi\epsilon_0 mr}}. \tag{37-4}$$

The angular velocity ω is given by

$$\omega = \frac{v}{r} = \sqrt{\frac{e^2}{4\pi\epsilon_0 mr^3}}.$$

The current for the orbit is the rate at which charge passes any given point, or

$$i = e\nu = e\left(\frac{\omega}{2\pi}\right) = \sqrt{\frac{e^4}{16\pi^3\epsilon_0 mr^3}}.$$

The orbital dipole moment μ_l is given from Eq. 37-3 if we put $N = 1$ and $A = \pi r^2$, or

$$\mu_l = NiA = (1)\sqrt{\frac{e^4}{16\pi^3\epsilon_0 mr^3}}\,(\pi r^2) = \frac{e^2}{4}\sqrt{\frac{r}{\pi\epsilon_0 m}}. \tag{37-5}$$

The orbital angular momentum L_l is

$$L_l = (mv)r.$$

Combining with Eq. 37-4 leads to

$$L_l = \sqrt{\frac{e^2mr}{4\pi\epsilon_0}}.$$

Finally, eliminating r between this equation and Eq. 37-5 yields

$$\mu_l = L_l\left(\frac{e}{2m}\right),$$

which shows that the orbital magnetic moment of an electron is proportional to its orbital angular momentum.

For $r = 5.1 \times 10^{-11}$ meter, which corresponds to hydrogen in its normal state, we have from Eq. 37-5

$$\mu_l = \frac{e^2}{4}\sqrt{\frac{r}{\pi\epsilon_0 m}}$$

$$= \frac{(1.6 \times 10^{-19}\ \text{coul})^2}{4}\sqrt{\frac{5.1 \times 10^{-11}\ \text{meter}}{(\pi)(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(9.1 \times 10^{-31}\ \text{kg})}}$$

$$= 9.1 \times 10^{-24}\ \text{amp-m}^2.$$ ◀

* This must not be confused with the magnetic dipole moment μ_s of the electron spin, which is also present.

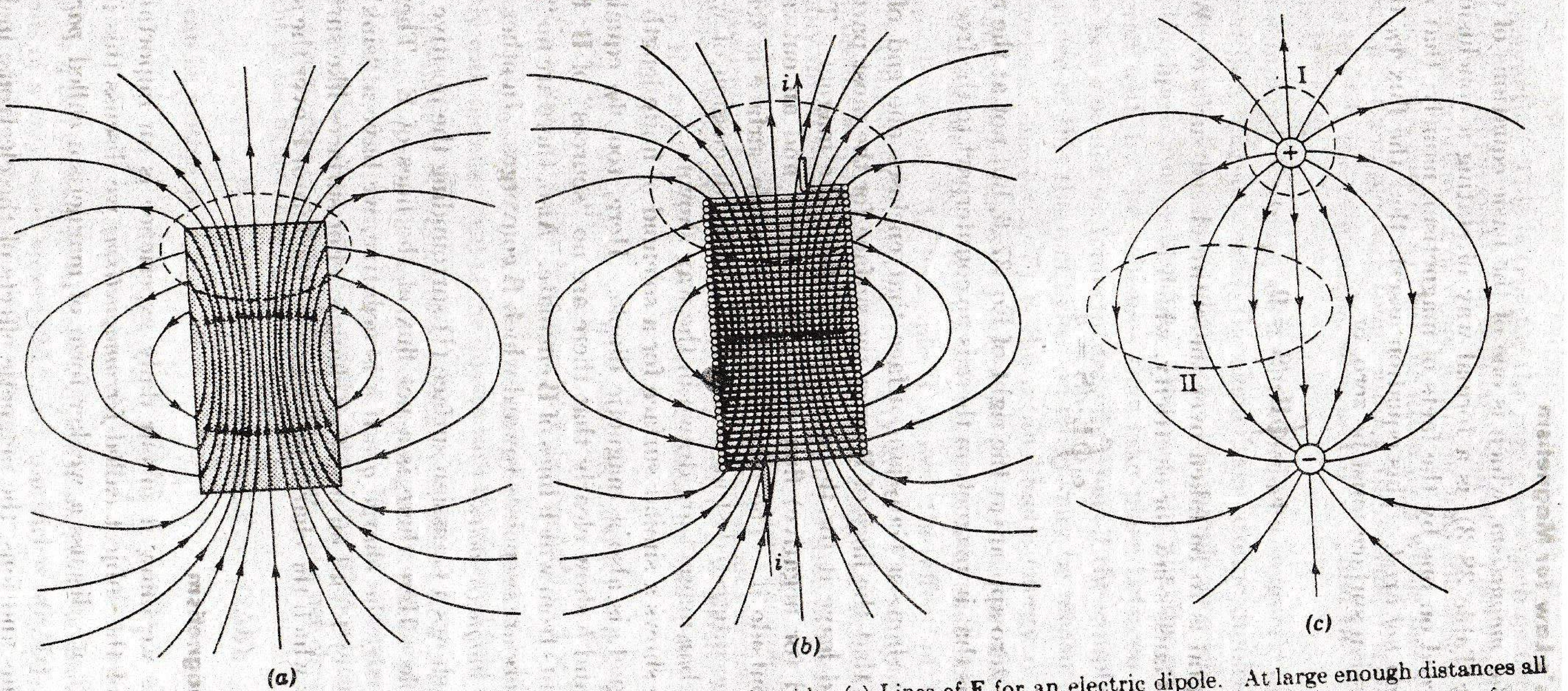


Fig. 37–4 Lines of **B** (*a*) for a bar magnet and (*b*) for a short solenoid. (*c*) Lines of **E** for an electric dipole. At large enough distances all three fields vary like those for a dipole. The four dashed curves are intersections with the plane of the figure of closed Gaussian surfaces. Note that Φ_B equals zero for (*a*) or (*b*). Φ_E equals zero for surfaces like II in (*c*), which do not contain any charge, but Φ_E is not zero for surfaces like I.

37–2 Gauss's Law for Magnetism

Gauss's law for magnetism, which is one of the basic equations of electromagnetism (see Table 38–3), is a formal way of stating a conclusion that seems to be forced on one by the facts of magnetism, namely, that *isolated magnetic poles do not exist*. This equation asserts that the flux Φ_B through any *closed* Gaussian surface must be zero, or

$$\Phi_B = \oint \mathbf{B} \cdot d\mathbf{S} = 0, \tag{37–6}$$

where the integral is to be taken over the entire closed surface. We contrast this with Gauss's law for electricity, which is

$$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = q. \tag{37–7}$$

The fact that a zero appears at the right of Eq. 37–6, but not at the right of Eq. 37–7, means that in magnetism there is no counterpart to the free charge q in electricity.

Figure 37–4*a* shows a Gaussian surface that encloses one end of a bar magnet. Note that the lines of **B** enter the surface for the most part inside the magnet and leave it for the most part outside the magnet. There is thus an inward (or negative) flux inside the magnet and an outward (or positive) flux outside it. The total flux for the whole surface is zero. In Section 37–7 the interested student will find a fuller discussion of why the lines of **B** for a bar magnet are drawn in the way shown.

Figure 37–4*b* shows a similar surface for a solenoid of finite length which, like a bar magnet, is also a magnetic dipole. Here, too, Φ_B equals zero. Figures 37–4*a* and *b* show clearly that there are no "sources" of **B**; that is, there are no points from which lines of **B** emanate. Also, there are no "sinks" of **B**; that is, there are no points toward which **B** converges. In other words, *there are no free magnetic poles.*

Figure 37–4*c* shows a Gaussian surface (I) surrounding the positive end of an electric dipole. Here there *is* a net flux of the lines of **E**. There is a "source" of **E**; it is the charge q. If q is negative, we have a "sink" of **E** because the lines of **E** end on negative charges. For surfaces like surface II in Fig. 37–4*c* for which the charge inside is zero, the flux of **E** over the surface is also zero.

37–3 Paramagnetism

Magnetism as we know it in our daily experience is an important but special branch of the subject called *ferromagnetism;* we discuss this in Section 37–5. Here we discuss a weaker form of magnetism called *paramagnetism*.

For most atoms and ions, the magnetic effects of the electrons, including both their spins and orbital motions, exactly cancel so that the atom or ion is not magnetic. This is true for the rare gases such as neon and for ions * such as Cu^+, which makes up ordinary copper. These materials do not exhibit paramagnetism. For other atoms or ions the magnetic effects of the electrons do not cancel, so that the atom as a whole has a magnetic dipole moment $\boldsymbol{\mu}$. Examples are found among the so-called transition elements, such as Mn^{++}, the rare earths, such as Gd^{+++}, and the actinide elements, such as U^{++++}.

If a sample of N atoms, each of which has a magnetic dipole moment $\boldsymbol{\mu}$, is placed in a magnetic field, the elementary atomic dipoles tend to line up with the field. For perfect alignment, the sample as a whole would have a magnetic dipole moment of $N\boldsymbol{\mu}$. However, the aligning process is seriously interfered with by the collisions that take place between the atoms if the sample is a gas and by temperature vibrations if the sample is a solid. The importance of this thermal agitation effect may be measured by comparing two energies: one ($= \frac{3}{2}kT$) is the mean kinetic energy of translation of a gas atom at temperature T; the other ($= 2\mu B$) is the difference in energy between an atom lined up with the magnetic field and one pointing in the opposite direction. As Example 3 shows, the effect of the collisions at ordinary temperatures and fields is very great. The sample acquires a magnetic moment when placed in an external magnetic field, but this moment is usually much smaller than the maximum possible moment $N\mu$.

▶ **Example 3.** A paramagnetic gas, whose atoms (see Example 2) have a magnetic dipole moment of about 10^{-23} amp-m², is placed in an external magnetic field of magnitude 1 weber/meter². At room temperature ($T = 300°K$) calculate and compare U_T, the mean kinetic energy of translation ($= \frac{3}{2}kT$), and U_B, the magnetic energy ($= 2\mu B$):

$$U_T = \tfrac{3}{2}kT = (\tfrac{3}{2})(1.38 \times 10^{-23}\ \text{joule/°K})(300°\text{K}) = 6 \times 10^{-21}\ \text{joule},$$

$$U_B = 2\mu B = (2)(10^{-23}\ \text{amp-m}^2)(1\ \text{weber/meter}^2) = 2 \times 10^{-23}\ \text{joule}.$$

Because U_T equals 300 U_B, we see that energy exchanges in collisions can interfere seriously with the alignment of the dipoles with the external field. ◀

If a specimen of a paramagnetic substance is placed in a nonuniform magnetic field, such as that near the pole of a strong magnet, it will be attracted toward the region of higher field, that is, toward the pole. We can understand this by drawing an analogy with the corresponding electric case of Fig. 37-5, which shows a dielectric specimen (a sphere) in a nonuniform electric field. The net electric force points to the right in the figure and is

$$F_e = q(E + \Delta E) - q(E - \Delta E) = q(2\Delta E),$$

which can be written as

$$F_e = \frac{(q\,\Delta x)}{\Delta x}\,2\Delta E = p\left(\frac{2\Delta E}{\Delta x}\right) \cong p\left(\frac{dE}{dx}\right)_{\max}$$

Here p ($= q\,\Delta x$) is the induced electric dipole moment of the sphere. In the differential limit of a very small sphere $(2\Delta E/\Delta x)$ approaches $(dE/dx)_{\max}$, the gradient of the electric field at the center of the sphere.

* Cu^+ indicates a copper atom from which one electron has been removed; Al^{+++} indicates an aluminum atom from which three electrons have been removed, etc.

In the corresponding magnetic case we have, by analogy,

$$F_m = \mu \left(\frac{dB}{dx}\right)_{\max} \tag{37-8}$$

Thus, by measuring the magnetic force F_m that acts on a small paramagnetic specimen when it is placed in a nonuniform magnetic field whose field gradient $(dB/dx)_{\max}$ is known, we can learn its magnetic dipole moment μ. The *magnetization* $\mathbf{M}$ is defined as the magnetic moment per unit volume, or

$$\mathbf{M} = \frac{\boldsymbol{\mu}}{V},$$

where V is the volume of the specimen. It is a vector because $\boldsymbol{\mu}$, the dipole moment of the specimen, is a vector.

In 1895 Pierre Curie (1859–1906) discovered experimentally that the magnetization M of a paramagnetic specimen is directly proportional to B, the effective value of magnetic induction in which the specimen is placed, and inversely proportional to the temperature, or

$$M = C\frac{B}{T}, \tag{37-9}$$

in which C is a constant. This equation is known as *Curie's law*. The law is physically reasonable in that increasing B tends to align the elementary dipoles in the specimen, that is, to increase M, whereas increasing T tends to interfere with this alignment, that is, to decrease M. Curie's law is well verified experimentally, provided that the ratio B/T does not become too large.

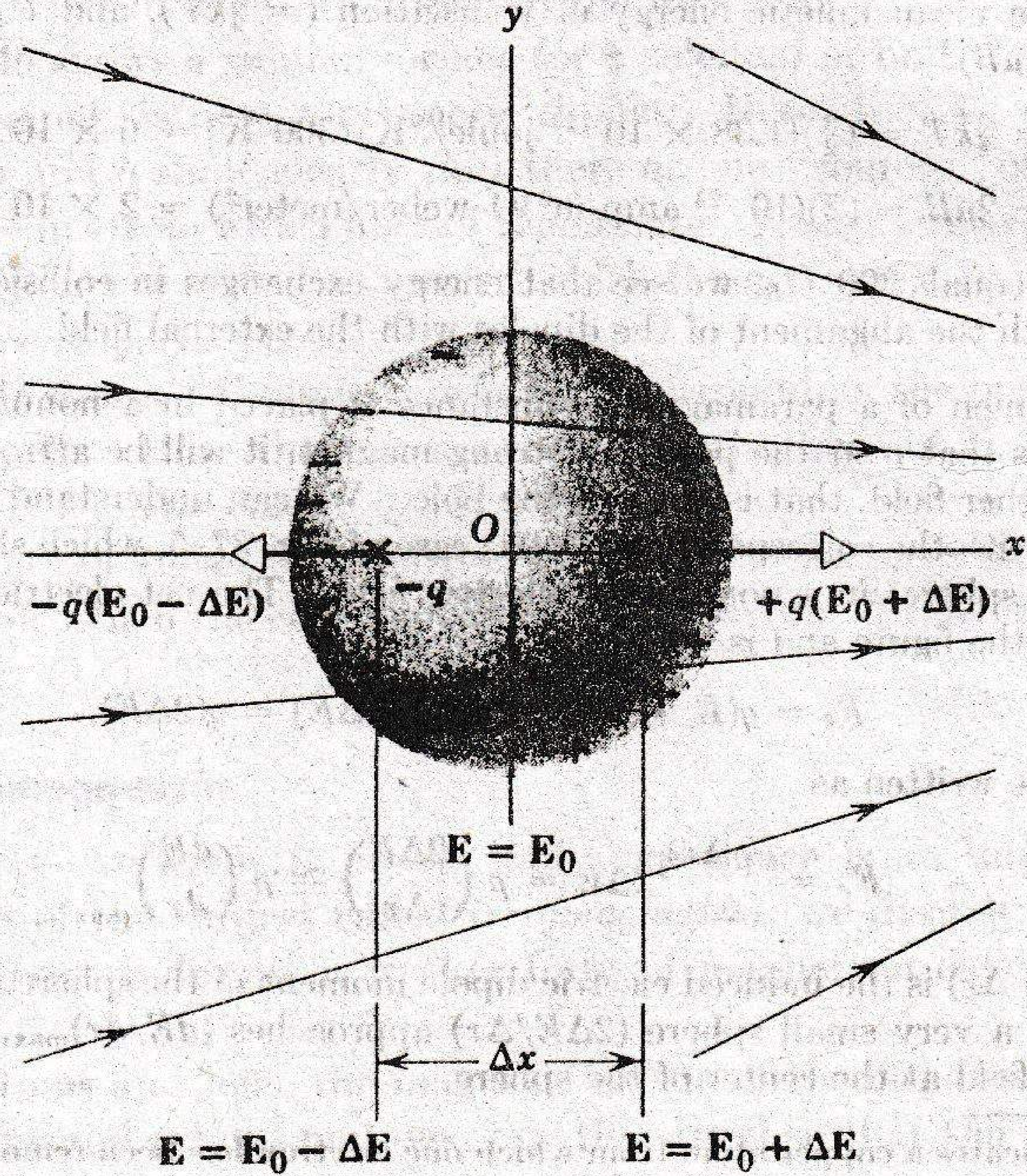


Fig. 37–5 A dielectric sphere in a nonuniform electric field.

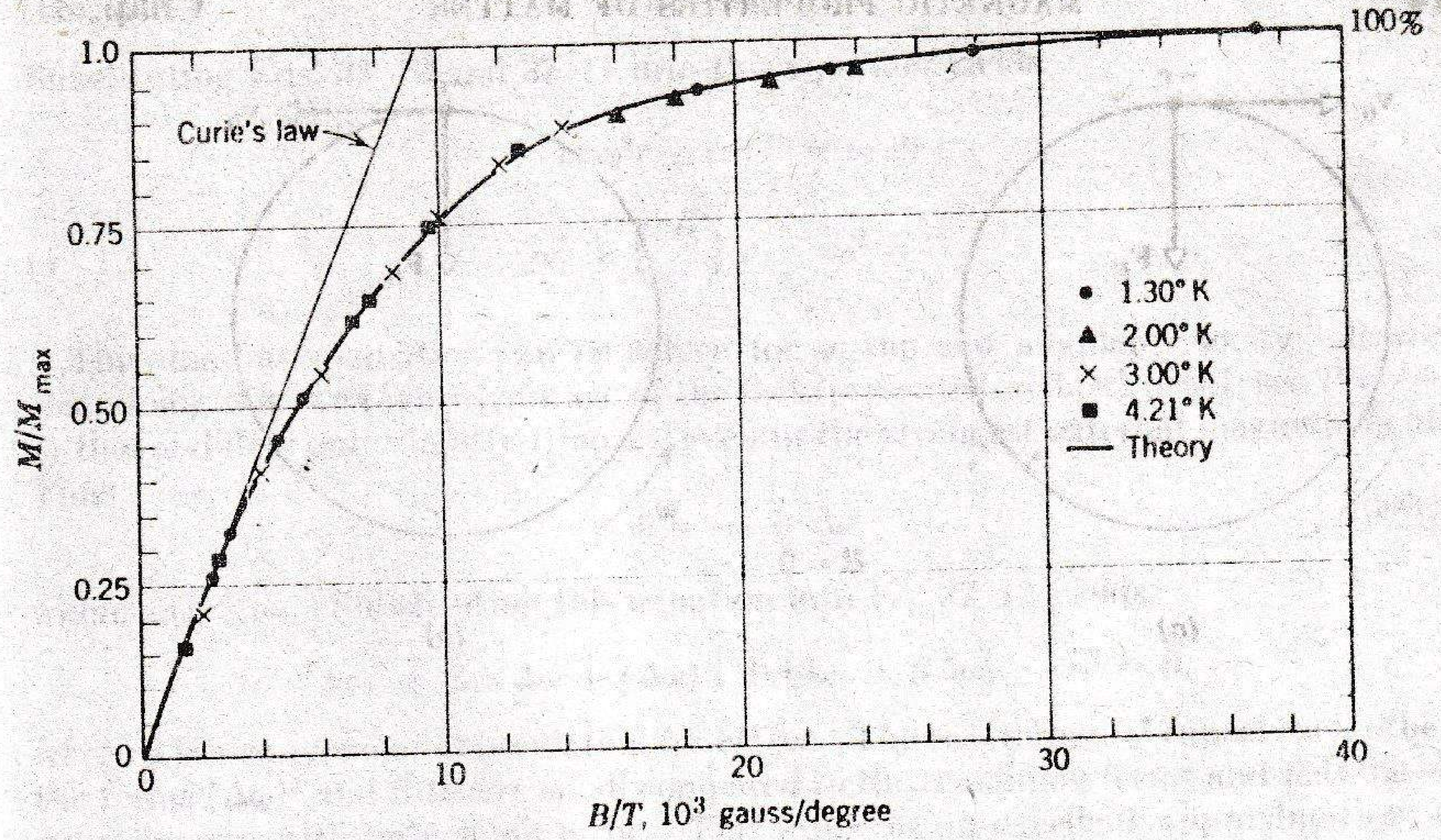


Fig. 37-6 The ratio M/M_{max} for a paramagnetic salt (chromium potassium alum) in various magnetic fields and at various temperatures. The curve through the experimental points is a theoretical curve calculated from modern quantum physics. (From measurements by W. E. Henry.)

M cannot increase without limit, as Curie's law implies, but must approach a value M_{max} ($= \mu N/V$) corresponding to the complete alignment of the N dipoles contained in the volume V of the specimen. Figure 37-6 shows this saturation effect for a sample of $CrK(SO_4)_2 \cdot 12H_2O$. The chromium ions are responsible for all the paramagnetism of this salt, all the other elements being paramagnetically inert. To achieve 99.5% saturation, it is necessary to use applied magnetic fields as high as 50,000 gauss and temperatures as low as 1.3°K. Note that for more readily achievable conditions, such as $B = 10{,}000$ gauss and $T = 10°$K, the abscissa in Fig. 37-6 is only 1.0 so that Curie's law would appear to be well obeyed for this and for all lower values of B/T. The curve that passes through the experimental points in this figure is calculated from a theory based on modern quantum physics; it is in excellent agreement with experiment.

37-4 Diamagnetism

In 1846 Michael Faraday discovered that a specimen of bismuth brought near to the pole of a strong magnet is *repelled*. He called such substances *diamagnetic*. Diamagnetism, present in all substances, is such a feeble effect that its presence is masked in substances made of atoms that have a net magnetic dipole moment, that is, in paramagnetic or ferromagnetic substances.

Figures 37-7*a* and *b* show an electron circulating in a diamagnetic atom at angular frequency ω_0 in an assumed circular orbit of radius r. Each electron is moving under the action of a centripetal force $\mathbf{F}_E$ of electrostatic origin where, from Newton's second law,

$$F_E = ma = m\omega_0^2 r. \tag{37-10}$$

Each rotating electron has an orbital magnetic moment, but for the atom as a whole the orbits are randomly oriented so that there is no *net* magnetic effect. In Fig. 37-7*a*, for example, the magnetic dipole moment $\boldsymbol{\mu}_l$ points into the page; in Fig.

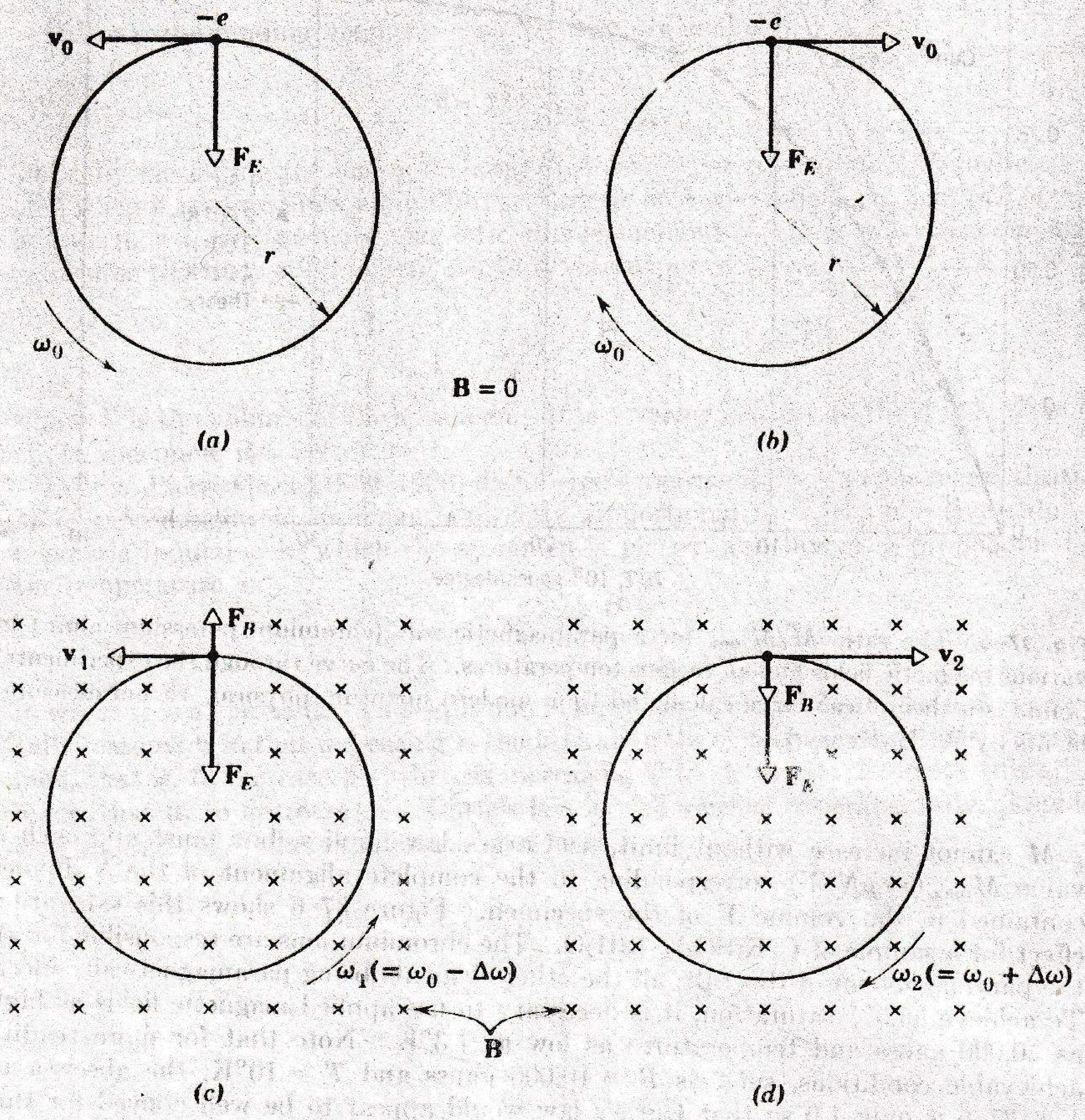


Fig. 37–7 (*a*) An electron circulating in an atom. (*b*) An electron circulating in the opposite direction. (*c*) A magnetic field is introduced, *decreasing* the linear speed of the electron in (*a*), that is, $v_1 < v_0$. (*d*) The magnetic field *increases* the linear speed of the electron in (*b*), that is, $v_2 > v_0$.

37–7*b* it points out and the net effect for the two orbits shown is cancellation. This cancellation is also shown at the left in Fig. 37–8.

If an external field **B** is applied as in Fig. 37–7*c* and *d*, an *additional* force, given by $-e(\mathbf{v} \times \mathbf{B})$, acts on the electron. This magnetic force acts always at right angles to the direction of motion; its magnitude is

$$F_B = evB = e(\omega r)B. \tag{37–11}$$

The student should show that in Fig. 37–7*c* $\mathbf{F}_B$ and $\mathbf{F}_E$ point in opposite directions and that in Fig. 37–7*d* they point in the same direction. Note that since the centripetal force changes when the magnetic field is applied (the radius can be shown to remain constant), the angular velocity must also change; thus ω in Eq. 37–11 differs from ω_0 in Eq. 37–10.

Applying Newton's second law to Figs. 37–7*c* and *d*, and allowing for both directions of circulation, yields for the *resultant* forces on the electrons

$$F_E \pm F_B = ma = m\omega^2 r.$$

Substituting Eqs. 37-10 and 37-11 into this equation yields

$$m\omega_0^2 r \pm e\omega r B = m\omega^2 r$$

or

$$\omega^2 \mp \left(\frac{eB}{m}\right)\omega - \omega_0^2 = 0. \tag{37-12}$$

This quadratic equation can be solved for ω, the new angular velocity. Rather than doing this, we take advantage of the fact (presented without proof; see Problem 7) that ω differs only slightly from ω_0, even in the strongest external magnetic fields. Thus

$$\omega = \omega_0 + \Delta\omega \tag{37-13}$$

where $\Delta\omega \ll \omega_0$. Substituting this equation into Eq. 37-12 yields

$$[\omega_0^2 + 2\omega_0\,\Delta\omega + (\Delta\omega)^2] \pm [\beta\omega_0 + \beta\,\Delta\omega] - \omega_0^2 = 0,$$

where β is a convenient abbreviation for eB/m. The two terms ω_0^2 cancel each other; the terms $(\Delta\omega)^2$ and $\beta\Delta\omega$ are small compared to the remaining terms and may be set equal to zero with only small error. This leads, as an excellent approximation, to

$$\Delta\omega \cong \mp \tfrac{1}{2}\beta = \mp \frac{eB}{2m} \tag{37-14}$$

or, from Eq. 37-13,

$$\omega = \omega_0 \mp \frac{eB}{2m}.$$

Thus the effect of applying a magnetic field is to increase or decrease (depending on the direction of circulation) the angular velocity. This, in turn, increases or decreases the orbital magnetic moment of the circulating electron (see Example 2).

In Fig. 37-7c the angular velocity is reduced (because the centripetal force is reduced) so that the magnitude of the magnetic moment is reduced. In Fig. 37-7d, however, the angular velocity is increased so that the magnitude of $\boldsymbol{\mu}_l$ is increased. These effects are shown on the right in Fig. 37-8, where it will be noted that the two magnetic moments *no longer cancel*.

We see that if a magnetic field **B** is applied to a diamagnetic substance (zero net magnetic moment in absence of applied field), a magnetic moment will be *induced* whose direction (out of the plane of Fig. 37-7) is *opposite* to **B**; see also Fig. 37-8. This is precisely the reverse of paramagnetism, in which the (*permanent*) magnetic dipoles tend to point in the *same* direction as the applied field.

We can now understand why a diamagnetic specimen is repelled when brought near to the pole of a strong magnet. If the pole is a north pole, there exists a non-uniform magnetic field of induction with **B** pointing away from the pole. If a sphere

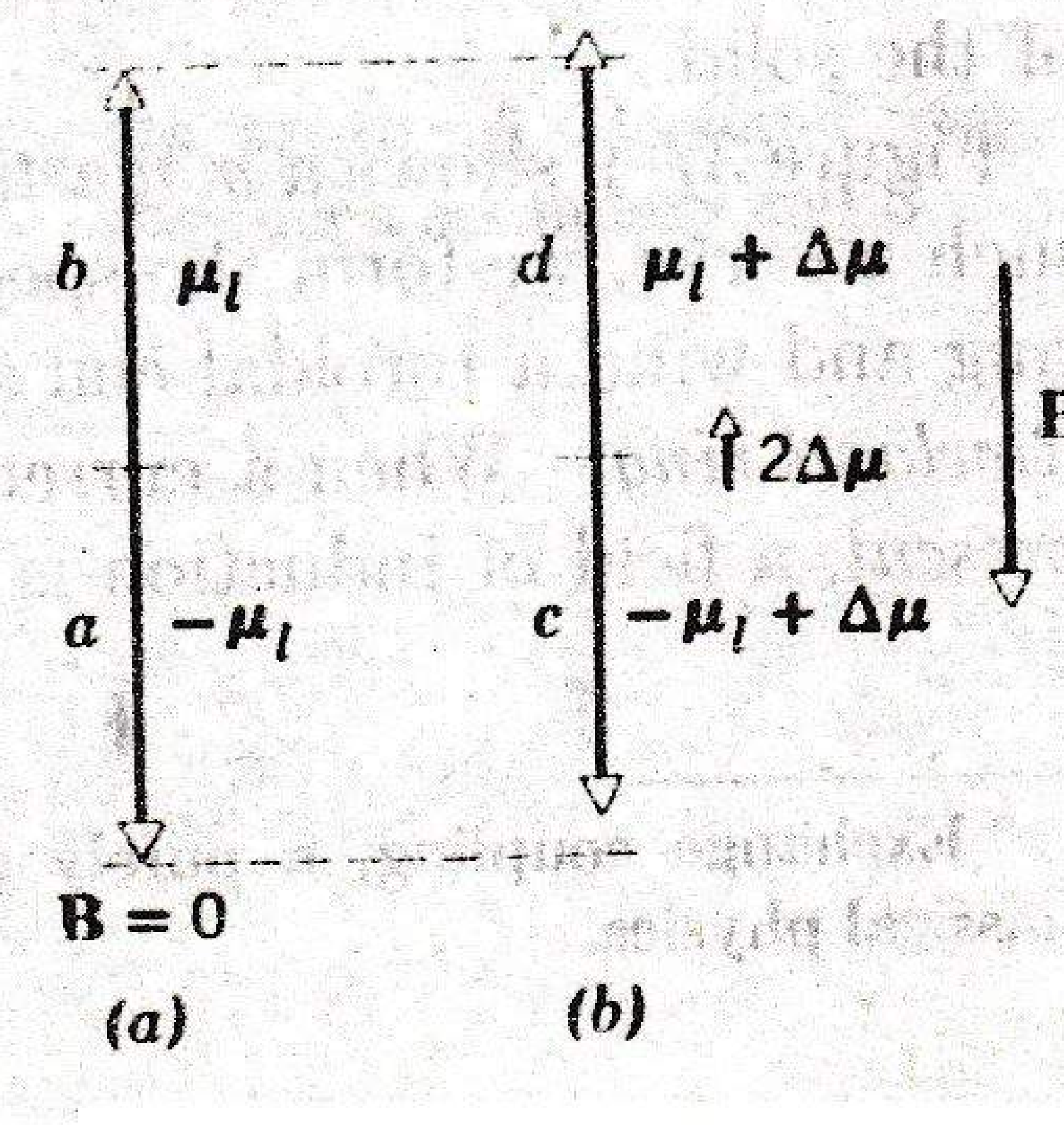


Fig. 37-8 The magnetic moments of the two oppositely circulating electrons in an atom cancel when there is no external magnetic field, as in (*a*), but do *not* cancel when a field is applied, as in (*b*). Note that the resultant moment in (*b*) points in the *opposite* direction to **B**. Compare carefully with Fig. 37-7.

made of a diamagnetic material (bismuth, say) is brought near to this pole, the magnetization **M** that is induced in it points toward the pole, that is, *opposite to* **B**. Thus the side of the sphere closest to the magnet behaves like a north pole and is *repelled* by the nearby north pole of the magnet. For a paramagnetic sphere, the vector **M** points *along the direction of* **B** and the side of the sphere nearest to the magnet is a south pole, which is *attracted* to the north pole of the magnet.

▶ **Example 4.** Calculate the *change* in magnetic moment for a circulating electron, as described in Example 2, if a magnetic field of induction **B** of 2.0 webers/meter² acts at right angles to the plane of the orbit.

We obtain μ from Eq. 37-3, or

$$\mu = NiA = (1)(ev)(\pi r^2) = (1)\left(\frac{e\omega}{2\pi}\right)(\pi r^2) = \tfrac{1}{2}er^2\omega.$$

The *change* in μ is $$\Delta\mu = \tfrac{1}{2}er^2\,\Delta\omega$$

or, from Eq. 37-14, $$\Delta\mu = \pm\,\tfrac{1}{2}er^2\left(\frac{eB}{2m}\right) = \pm\,\frac{e^2Br^2}{4m}.$$

Substituting numbers yields

$$\Delta\mu = \pm\,\frac{(1.6\times 10^{-19}\text{ coul})^2(2.0\text{ webers/meter}^2)(5.1\times 10^{-11}\text{ meter})^2}{(4)(9.1\times 10^{-31}\text{ kg})}$$

$$= \pm 3.7\times 10^{-29}\text{ amp-m}^2.$$

In Example 2 the moment μ_l was 9.1×10^{-24} amp-m², so that the change induced by even a strong external magnetic field is rather small, the ratio $\Delta\mu/\mu_l$ being about 4×10^{-6}. ◀

37-5 Ferromagnetism

For five elements (Fe, Co, Ni, Gd, and Dy) and for a variety of alloys of these and other elements a special effect occurs which permits a specimen to achieve a high degree of magnetic alignment in spite of the randomizing tendency of the thermal motions of the atoms. In such materials, described as *ferromagnetic*, a special form of interaction called *exchange coupling* occurs between adjacent atoms, coupling their magnetic moments together in rigid parallelism.* Modern quantum physics successfully predicts that this will occur only for the five elements listed. If the temperature is raised above a certain critical value, called the *Curie temperature*, the exchange coupling suddenly disappears and the materials become simply paramagnetic. For iron the Curie temperature is 1043°K. Ferromagnetism is evidently a property not only of the individual atom or ion but also of the interaction of each atom or ion with its neighbors in the crystal lattice (see Fig. 21-5) of the solid.

Figure 37-9 shows a *magnetization curve* for a specimen of iron. To obtain such a curve, we form the specimen, assumed initially unmagnetized, into a ring and wind a toroidal coil around it as in Fig. 37-10, to form a so-called *Rowland ring*. When a current i is set up in the coil, *if the iron core is not present*, a field of induction is set up within the toroid given by (Eq. 34-4)

$$B_0 = \mu_0 ni, \tag{37-15}$$

* Exchange coupling, a purely quantum effect, cannot be "explained" in terms of classical physics.

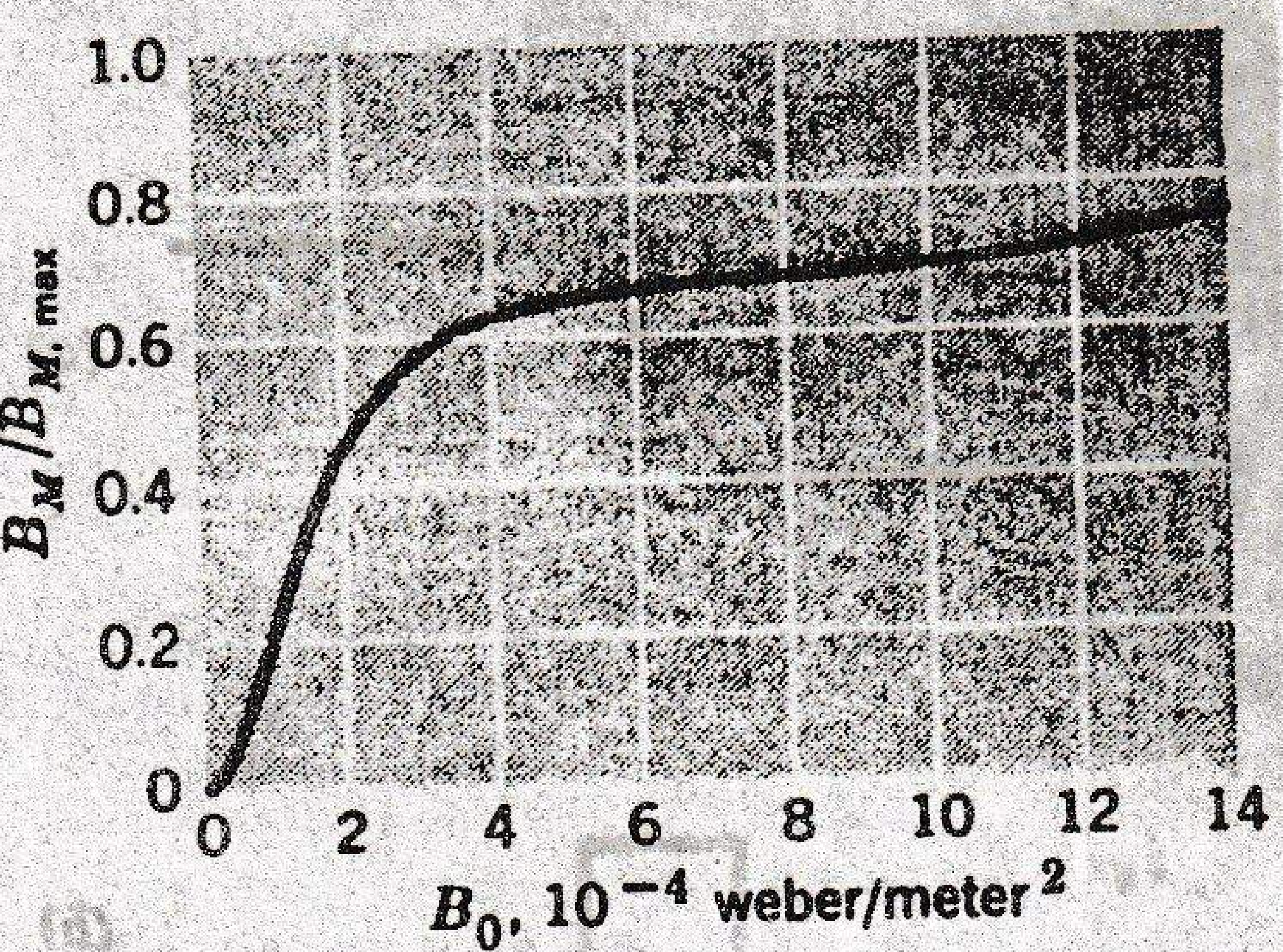


Fig. 37-9 A magnetization curve for iron.

where n is the number of turns per unit length for the toroid. Although this formula was derived for a long solenoid, it can be applied to a toroid if $d \ll r$ in Fig. 37-10. Because of the iron core, the actual value of **B** in the toroidal space will exceed $\mathbf{B}_0$, by a large factor in many cases, since the elementary atomic dipoles in the core line up with the applied field $\mathbf{B}_0$, thereby setting up their own field of induction. Thus we can write

$$B = B_0 + B_M \tag{37-16}$$

where B_M is the magnetic induction due to the specimen; it is proportional to the magnetization M of the specimen. Often $B_M \gg B_0$.

The field B_0 is proportional to the current in the toroid and can be calculated readily, using Eq. 37-15; B can be measured in a way that is described below. An experimental value for B_M can be derived from Eq.

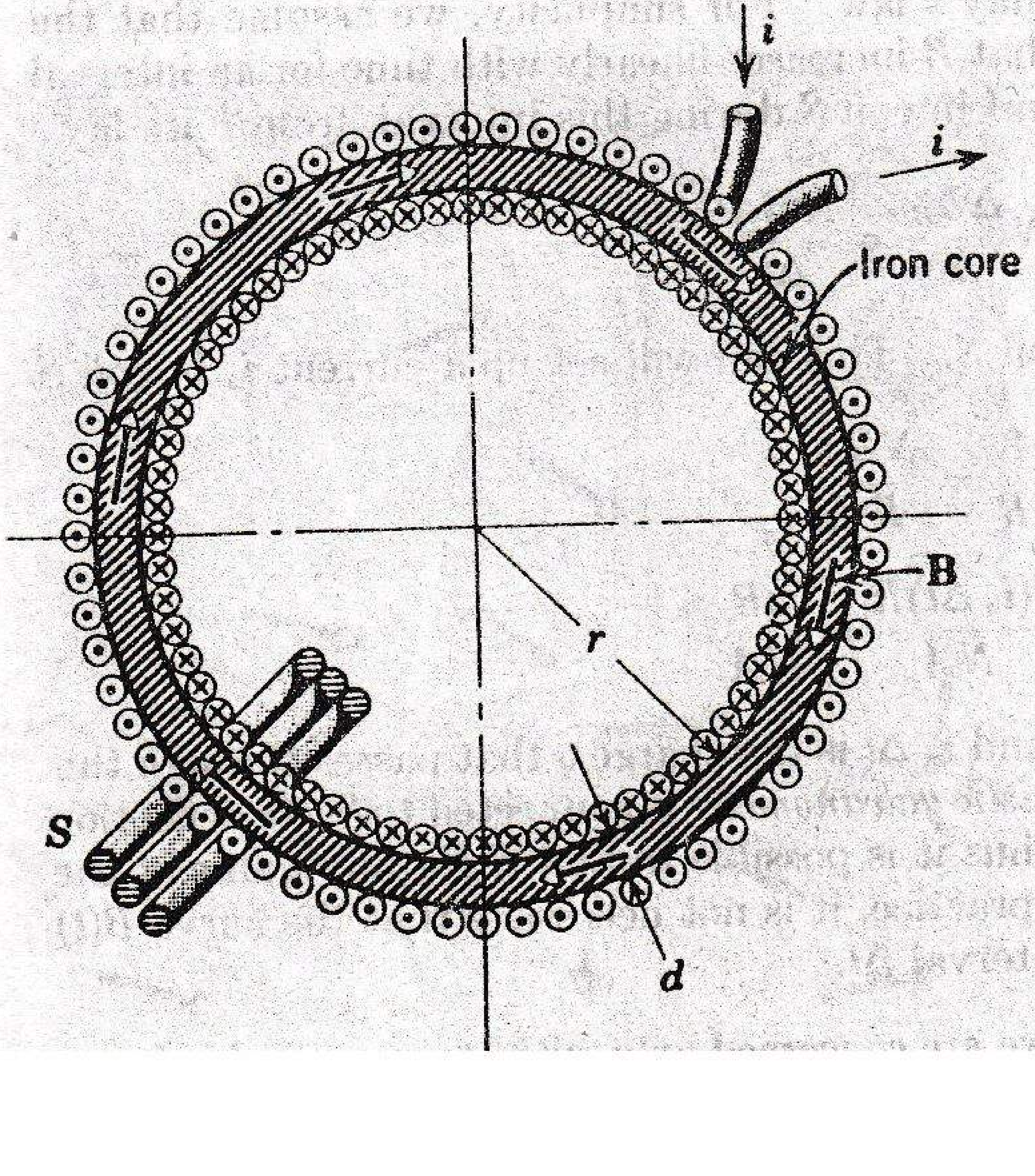


Fig. 37-10 A Rowland ring, showing a secondary coil S.

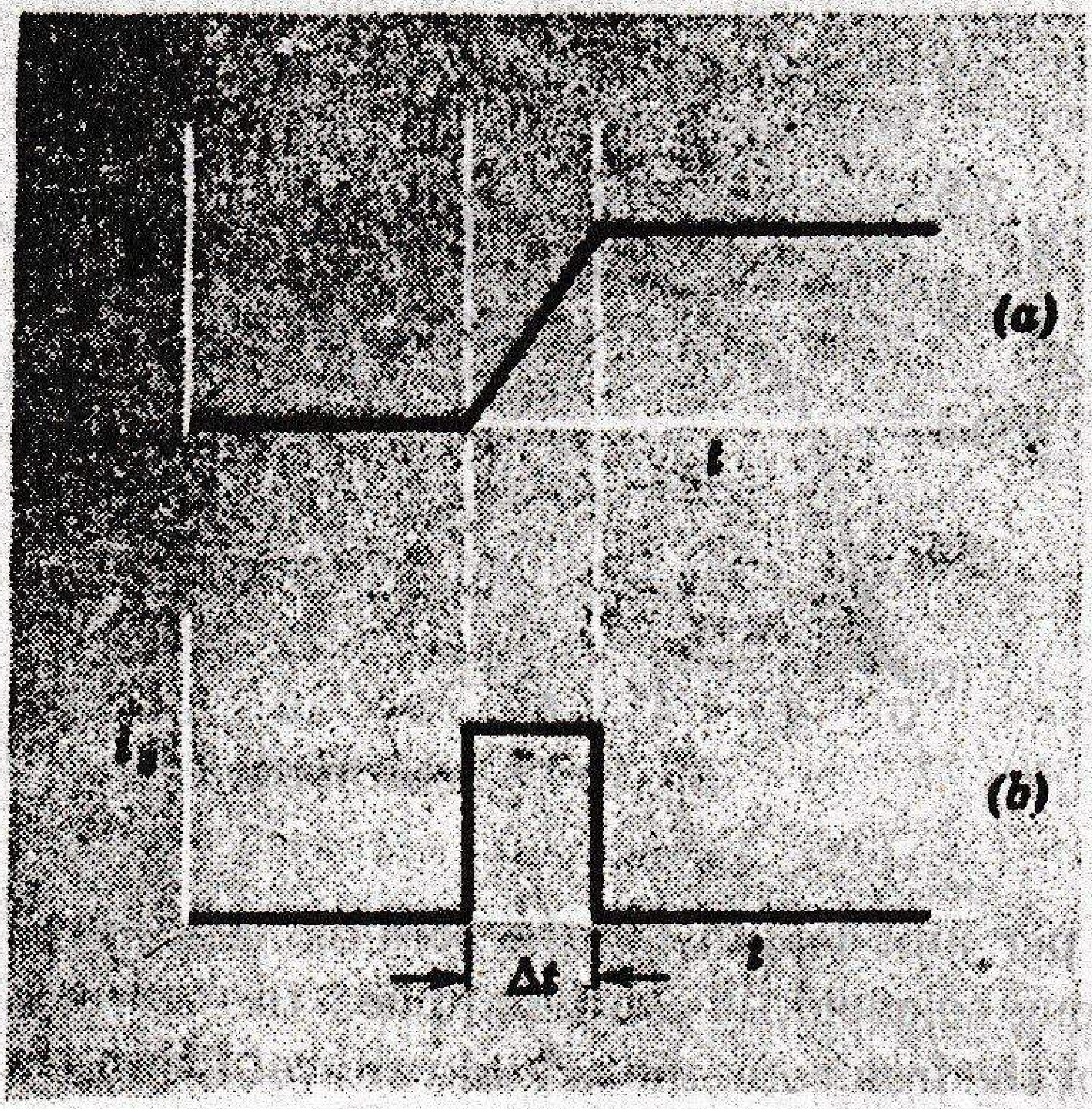


Fig. 37–11 (*a*) B for a Rowland ring as the current in the windings is increased from zero during an interval Δt. (*b*) The corresponding induced current in the secondary coil. Both curves are idealized; in practice, the sharp corners would be rounded off.

37–16. It has a maximum value $B_{M,\max}$ corresponding to complete alignment of the atomic dipoles in the iron. Thus we can plot, as in Fig. 37–9, the fractional degree of alignment ($= B_M/B_{M,\max}$) as a function of B_0. For this specimen a value of 96.5% saturation is achieved at $B_0 = 0.13$ weber/meter^2 ($= 1300$ gauss; this point is about 16 ft to the right of the origin in the figure); increasing B_0 to 1.0 weber/meter^2 ($= 10{,}000$ gauss; about 120 ft to the right in Fig. 37–9) increases the fractional saturation only to 97.7%.

To measure B in the system of Fig. 37–10, let the current in the toroid windings be increased from zero to i. The flux through the secondary coil S will change by BA, where A is the area of the toroid. While the flux is changing, an induced emf will appear in coil S, according to Faraday's law. For simplicity, we assume that the current in the toroid is so adjusted that B increases linearly with time for an interval Δt, as shown in Fig. 37–11*a*. The emf in coil S during this interval, from Faraday's law,* will then be

$$\mathcal{E} = N\frac{\Delta\Phi_B}{\Delta t} = \frac{NBA}{\Delta t},$$

where N is the number of turns in coil S. This emf will set up a current i_s in coil S given by

$$i_s = \frac{\mathcal{E}}{R} = \frac{NBA}{R\,\Delta t}$$

or

$$B = \frac{(i_s\,\Delta t)R}{NA} = \frac{qR}{NA},$$

in which R is the resistance of coil S and $i_s\,\Delta t$ is the charge q that passes through this coil during time Δt. If a so-called *ballistic galvanometer* is connected to S, its deflection will be a measure of the charge q. Thus it is possible to find B for any value of the current i in the toroid windings. In practice, it is not necessary that the curve $B(t)$ in Fig. 37–11*a* be linear during the interval Δt.

* We ignore the minus sign because we are concerned only with the magnitude of $\mathcal{E}$.

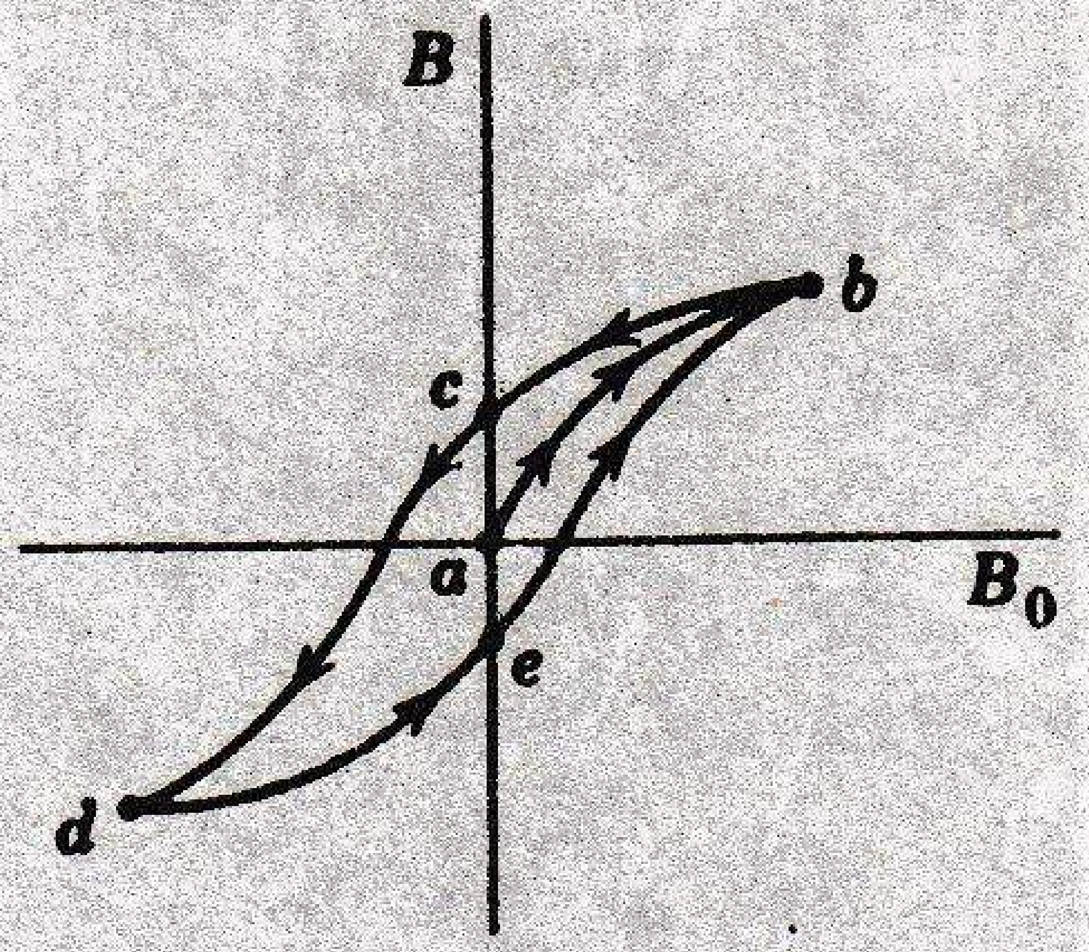


Fig. 37-12 A magnetization curve (*ab*) for a specimen of iron and an associated hysteresis loop (*ebcde*).

Magnetization curves for ferromagnetic materials *do not retrace themselves* as we increase and then decrease the toroid current. Figure 37-12 shows the following operations with a Rowland ring: (1) starting with the iron unmagnetized (point *a*), increase the toroid current until $B_0\ (= \mu_0 ni)$ has the value corresponding to point *b*; (2) reduce the current in the toroid winding back to zero (point *c*); (3) reverse the toroid current and increase it in magnitude until point *d* is reached; (4) reduce the current to zero again (point *e*); (5) reverse the current once more until point *b* is reached again. The lack of retraceability shown in Fig. 37-12 is called ***hysteresis.*** Note that at points *c* and *e* the iron core is magnetized, even though there is no current in the toroid windings; this is the familiar phenomenon of *permanent magnetism.*

The magnetization curve for paramagnetism (Fig. 37-6) is explained in terms of the mutually opposing tendencies of alignment with the external field and of randomization because of the temperature motions. In ferromagnetism, however, we have assumed that adjacent atomic dipoles are locked in rigid parallelism. Why, then, does the magnetic moment of the specimen not reach its saturation value for very low—even zero—values of B_0? The modern interpretation is to assume the existence within the specimen of *domains*, that is, of local regions within which there is essentially perfect alignment. The domains themselves, however, as Fig. 37-13 suggests, are not parallel at moderately low values of $\mathbf{B}_0$.

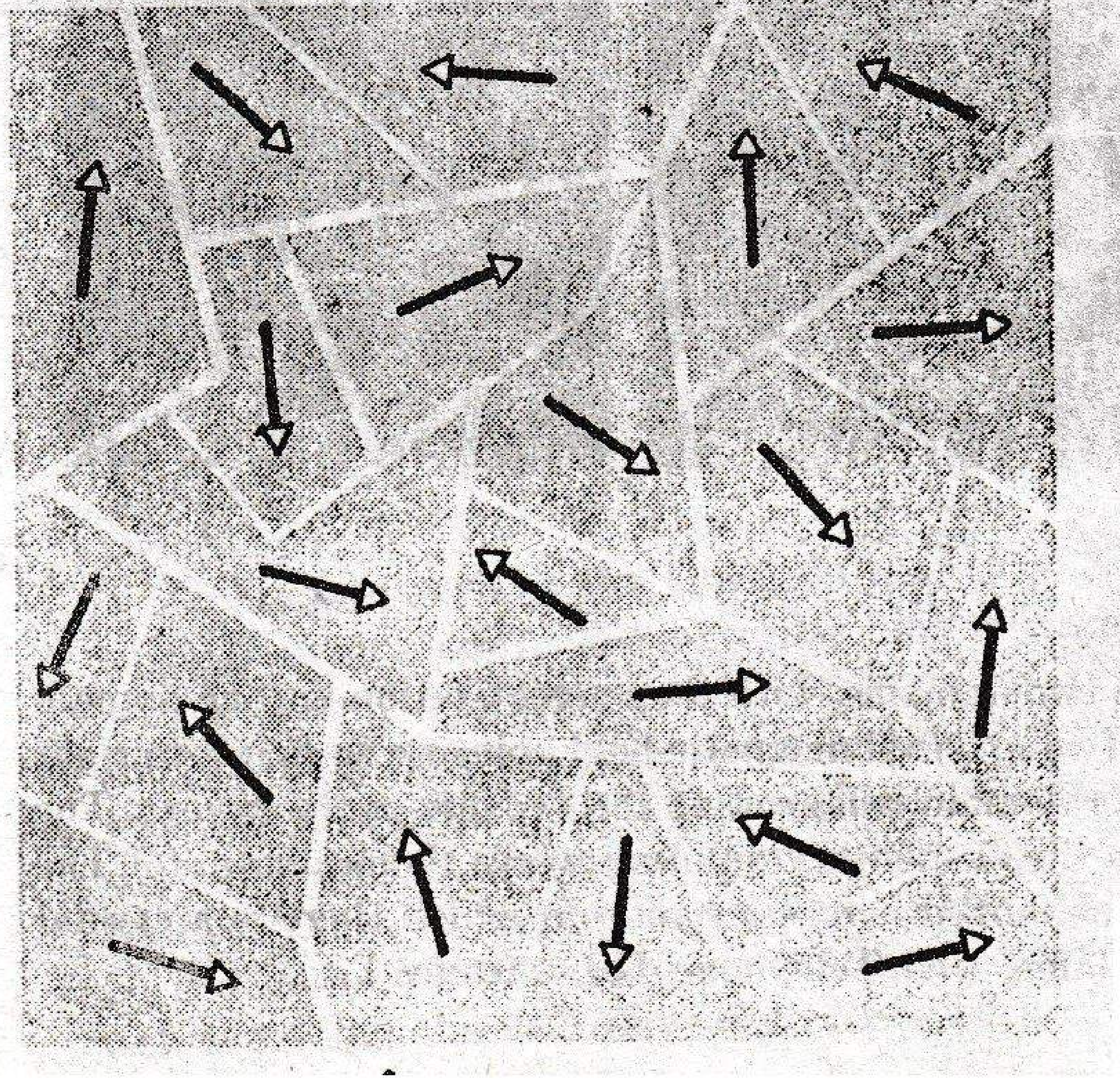


Fig. 37-13 The separate magnetic domains in an unmagnetized polycrystalline ferromagnetic sample are oriented to produce little external effect. Each domain, however, is made up of completely aligned atomic dipoles, as suggested by the arrows. The heavy boundaries define the crystals that make up the solid; the light boundaries define the domains within the crystals.

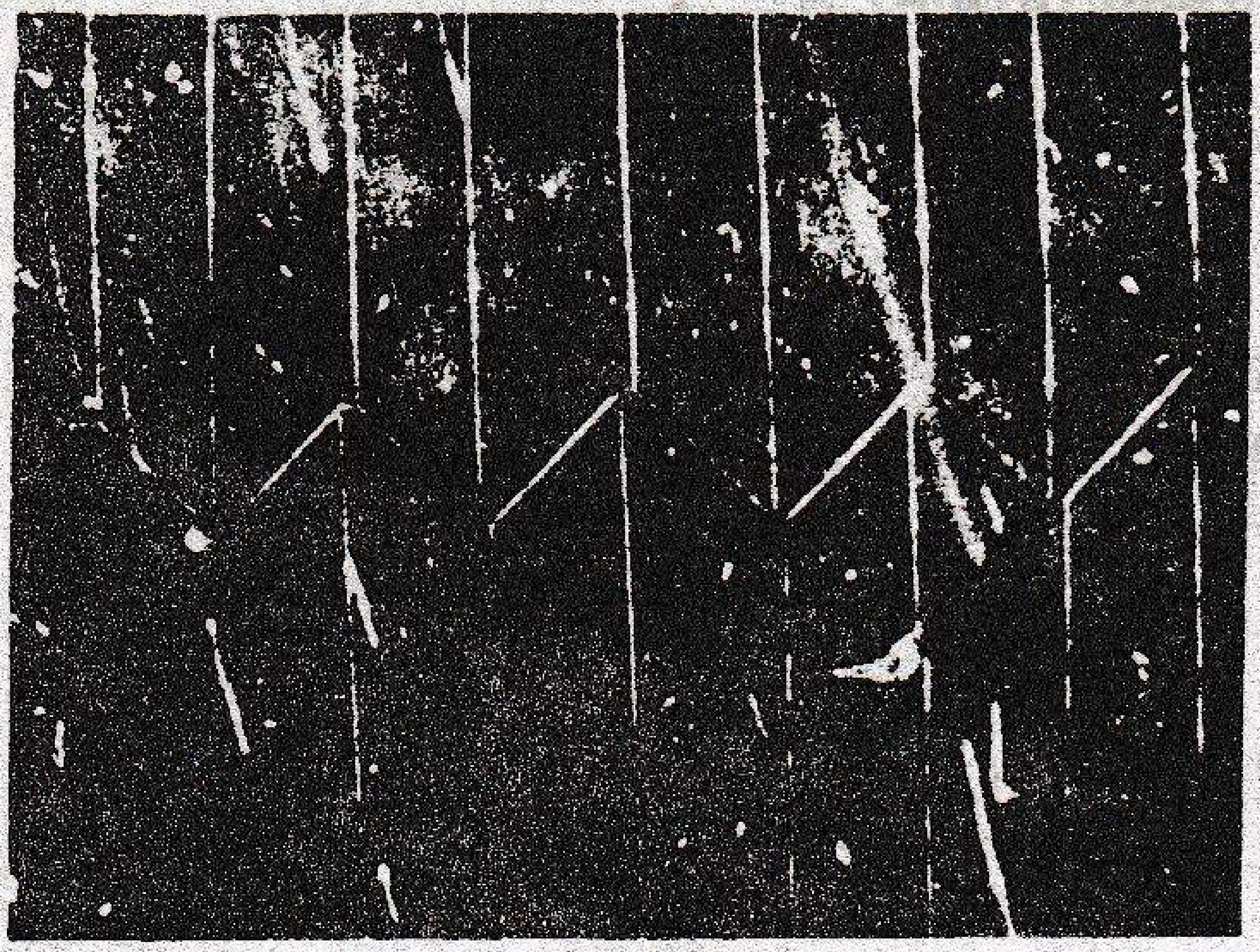


Fig. 37–14 Domain patterns for a single crystal of iron containing 3.8% silicon. The white lines show the boundaries between the domains. These boundaries are regular rather than irregular, as in Fig. 37–13, because the specimen is a single crystal. In Fig. 37–13 the specimen is made up of many crystallites or grains. (Courtesy H. J. Williams, Bell Telephone Laboratories.)

Figure 37–14 shows some domain photographs, taken by sprinkling a colloidal suspension of finely powdered iron oxide on a properly etched single crystal of iron. The domain boundaries, which are thin regions in which the alignment of the elementary dipoles changes from a certain direction in one domain to an entirely different direction in the other, are the sites of intense but highly localized and nonuniform magnetic fields. The suspended colloidal particles are attracted to these regions. Although the atomic dipoles in the individual domains are completely aligned, the specimen as a whole may have a very small resultant magnetic moment. This is the state of affairs in an unmagnetized iron nail.

As we magnetize a piece of iron by placing it in an external magnetic field, two effects take place. One is a growth in size of the domains that are favorably oriented at the expense of those that are not, as in Fig. 37–15. Second, the direction of orientation of the dipoles within a domain may swing around as a unit, becoming closer to the field direction. Hysteresis comes about because the domain boundaries do not move completely back to their original positions when the external field B_0 is removed.

Two other types of magnetism, closely related to ferromagnetism, are *antiferromagnetism* and *ferrimagnetism* (note spelling). In antiferromagnetic substances, of

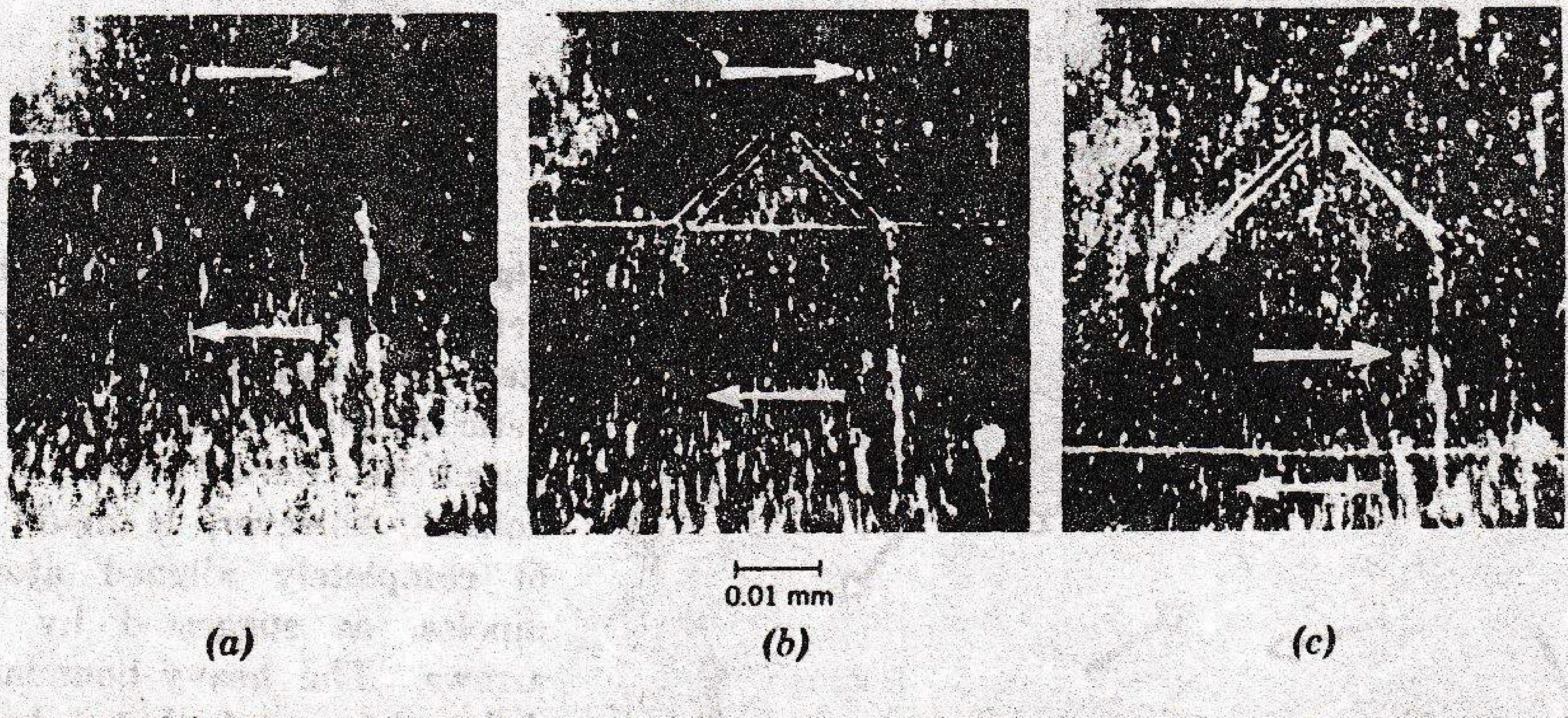


Fig. 37–15 (*a*) A boundary between two domains, with the magnetization in each domain as shown by the white arrows. (*b*) If an external magnetic field pointing from left to right is imposed on the specimen, the upper domain will grow at the expense of the lower. The domain boundary will move down as the elementary dipoles reverse themselves. (*c*) The process continues. The boundary has moved across a region in which there is a crystal imperfection. (Courtesy H. J. Williams, Bell Telephone Laboratories.)

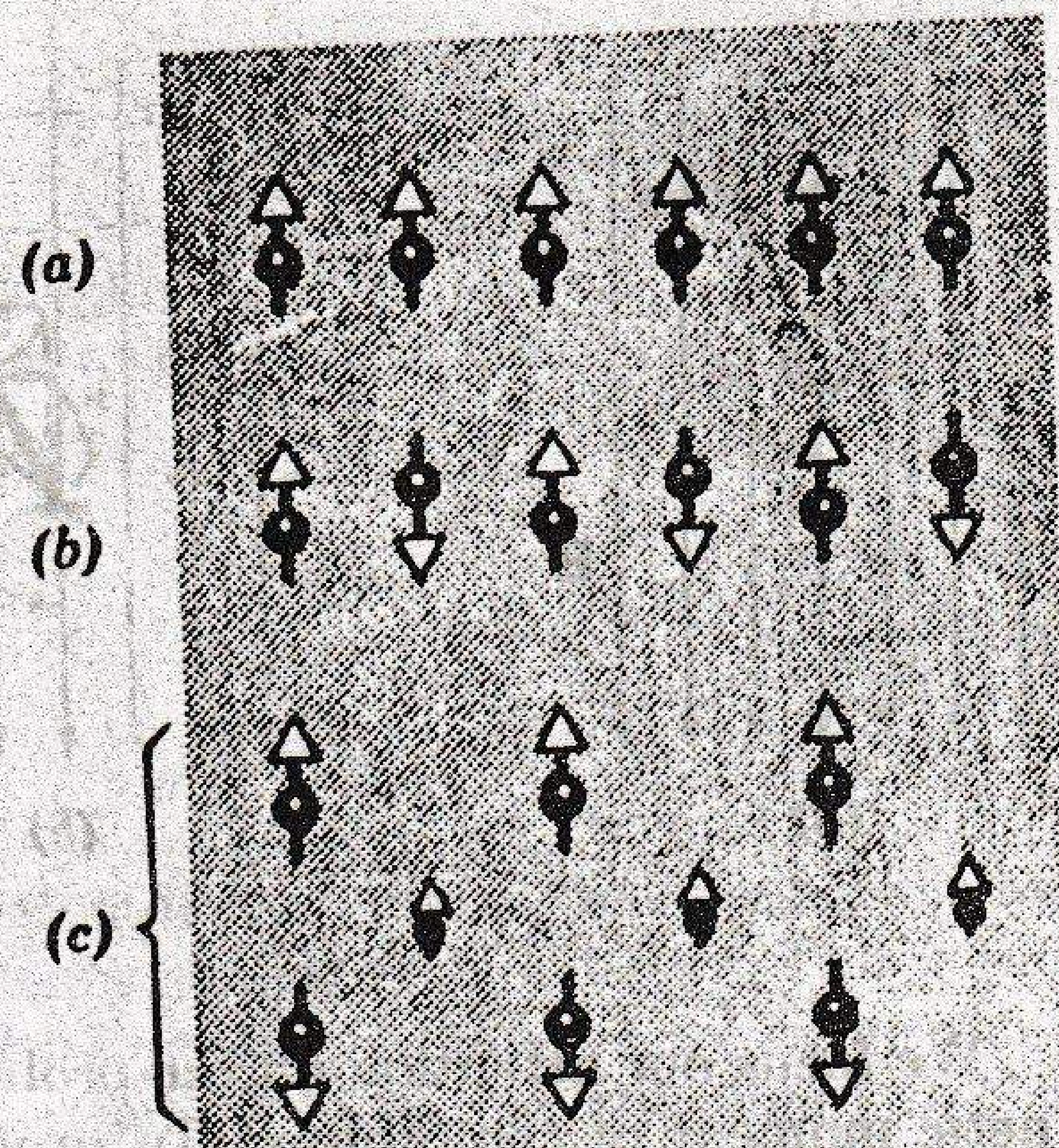


Fig. 37–16 Showing how elementary magnetic dipoles are oriented by the interatomic exchange coupling in (*a*) ferromagnetism, (*b*) antiferromagnetism, and (*c*) ferrimagnetism.

which MnO_2 is an example, the exchange coupling to which we referred on page 834 serves to lock adjacent ions into rigid *antiparallelism* (see Fig. 37–16*b*). Such materials exhibit very little gross external magnetism. However, if they are heated above a certain temperature, called the *Néel temperature*, the exchange coupling ceases to act and the material becomes paramagnetic. In ferrimagnetic substances, of which iron ferrite is an example, two different kinds of magnetic ions are present. In iron ferrite the two ions are Fe^{++} and Fe^{+++}. The exchange coupling locks the ions into a pattern like that of Fig. 37–16*c*, in which the external effects are intermediate between ferromagnetism and antiferromagnetism. Here, too, the exchange coupling disappears if the material is heated above a certain characteristic temperature.

37–6 Nuclear Magnetism

Many nuclei have magnetic dipoles, and the possibility arises that a specimen of matter may exhibit gross external magnetic effects associated with its nuclei. However, nuclear magnetic moments are several orders of magnitude smaller than those associated with the electronic motions in an atom or ion. The magnetic moment of an electron associated with its spin, for example, exceeds that of the proton (the nucleus of hydrogen) by a factor of 660.

Gross external effects for nuclear magnetism are smaller than the corresponding (electronic) paramagnetic effects by the *square* of ratios of this order of magnitude, because (*a*), *all else being equal*, the external magnetism is reduced by such a ratio, but (*b*) the very fact that the magnetic dipole moment of the nucleus is smaller means that (see Example 3) the thermal vibrations are proportionally (to a good approximation) more effective in reducing the degree of alignment of the elementary dipoles in an external magnetic field; thus all else is *not* equal and the ratio enters twice.

Techniques such as the Rowland ring (Fig. 37–10) are far too insensitive to detect nuclear magnetism. We describe here a *nuclear resonance technique* by means of which nuclear magnetism can readily reveal itself. This method is also vastly useful for studying paramagnetism, ferromagnetism, antiferromagnetism, and ferrimagnetism, in all of which cases the magnetic effects are associated not with the nuclei but with the atomic electrons. The nuclear-resonance technique was developed in 1946 by E. M. Purcell and his co-workers at Harvard. Simultaneously and independently, F. Bloch and his co-workers at Stanford discovered a very similar method. For these achievements the two physicists received a Nobel prize.

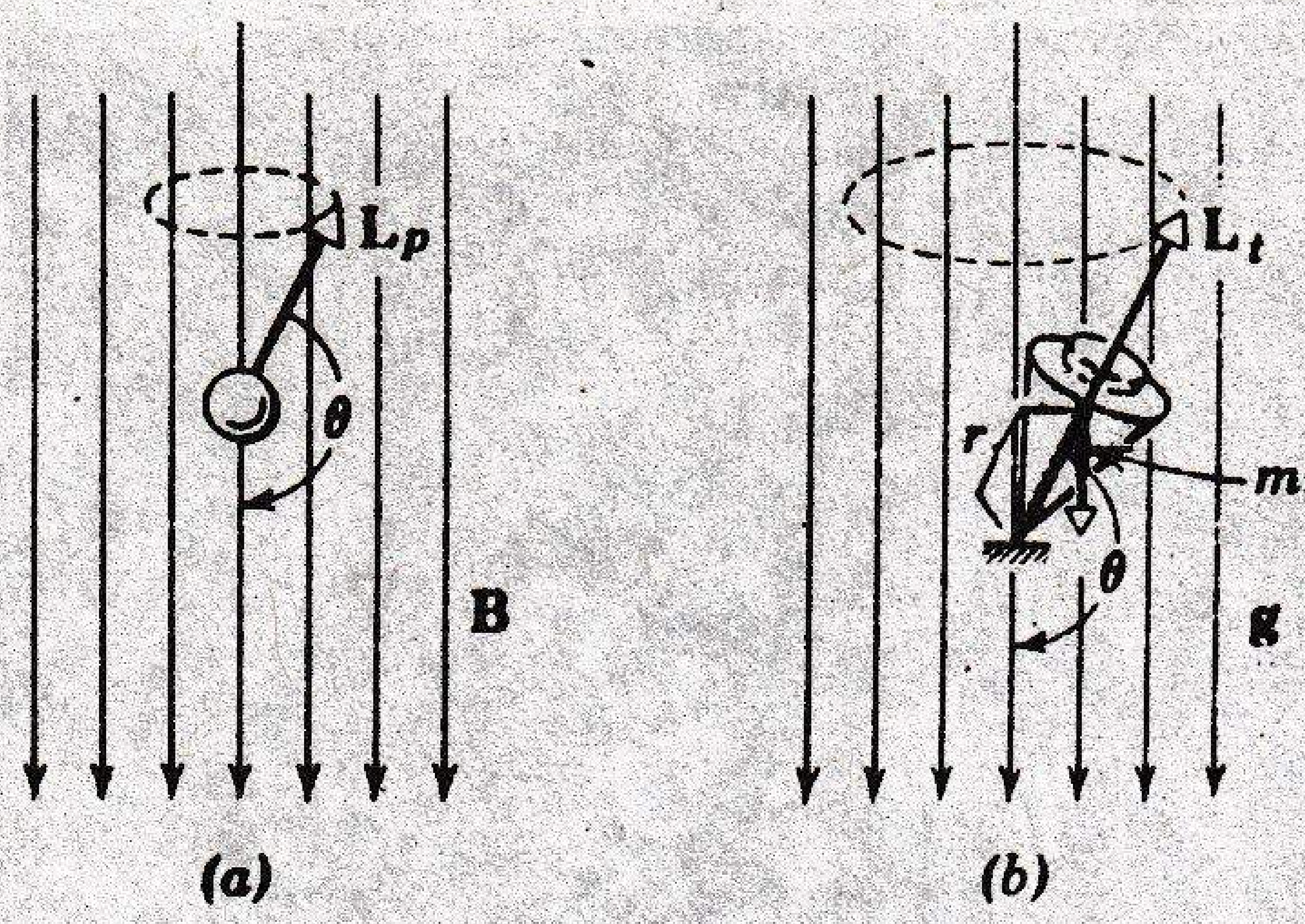


Fig. 37–17 (*a*) A spinning proton precessing in an external magnetic field and (*b*) a spinning top precessing in an external gravitational field. $\mathbf{L}_p$ and $\mathbf{L}_t$ are the two angular momentum vectors.

We focus our attention on the problem of measuring the magnitude μ of the magnetic moment of the proton. In principle, this can be done by placing a specimen containing protons in an external field of magnetic induction **B** and by measuring the energy ($= 2\mu B$) required to turn the protons end for end. A rigorously correct description of the procedures cannot be given without using quantum physics. The description given, although based entirely on classical physics, nevertheless leads to the correct conclusions.

Figure 37–17*a* shows a spinning proton with its axis making an angle θ with a uniform external magnetic field **B**. Figure 37–17*b* shows a spinning top with its axis making an angle θ with a uniform external gravitational field **g**. In each case there is a torque that tends to align the axis of the spinning object with the field. For the proton (Eq. 33–11) it is given by

$$\tau_p = \mu B \sin\theta. \tag{37–17a}$$

For the top it is given by

$$\tau_t = mgr \sin\theta, \tag{37–17b}$$

where r locates the center of mass of the top and m is its mass.

In Section 13–2, we saw that the spinning top processes about a vertical axis with an angular frequency given by

$$\omega_t = \frac{mgr}{L_t}, \tag{37–18a}$$

in which L_t is the spin angular momentum of the top.

The proton, which has a quantized spin angular momentum L_p, will also precess about the direction of the (magnetic) field because of the action of the (magnetic) torque. The student should derive the expression for the frequency of precession, being guided by the derivation of Section 13–2, but using the magnetic torque (Eq. 37–17*a*) instead of the gravitational torque (Eq. 37–17*b*). The relation is

$$\omega_p = \frac{\mu B}{L_p}. \tag{37–18b}$$

▶ **Example 5.** What is the precession frequency of a proton in a magnetic field of 0.5 weber/meter^2?

The quantities μ and L_p in Eq. 37–18*b* are 1.4×10^{-26} amp-m^2 and 0.53×10^{-34} joule-sec. This equation then yields

$$\nu_p = \frac{\omega_p}{2\pi} = \frac{\mu B}{2\pi L_p} = \frac{(1.4 \times 10^{-26}\text{ amp-m}^2)(0.50\text{ weber/meter}^2)}{(2\pi)(0.53 \times 10^{-34}\text{ joule-sec})} = 2.1 \times 10^7 \text{ cps}.$$

This frequency (= 21 mc/sec) is in the radio-frequency range. ◀

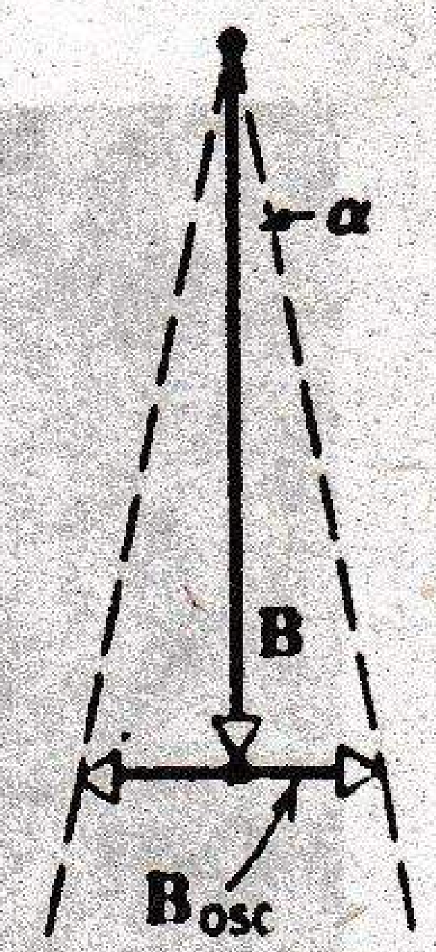


Fig. 37-18 In the nuclear magnetic resonance method a small oscillating magnetic field $\mathbf{B}_{osc}$ is placed at right angles to a steady field **B**.

It is possible to change the energy of any system in periodic motion if we allow an external influence to act on it at the same frequency as that of its motion. This is the familiar *resonance* condition. As an "external influence" for the precessing proton, we use a small alternating magnetic field $\mathbf{B}_{osc}$ arranged to be at right angles to the steady field **B**. This oscillating field combines vectorially with the steady field so that the *resultant* field rocks back and forth between the limits shown by the dashed lines in Fig. 37-18. Typical values for B and for the amplitude of $\mathbf{B}_{osc}$ are 5000 gauss and 1 gauss, respectively, so that the rocking angle α in the figure is quite small. If the angular frequency ω_0 of the oscillating field is chosen equal to the angular precession frequency ω_p of the proton, it turns out that the precessing proton can absorb energy. An increase in energy means an increase in θ in Fig. 37-17*a*.

The resonance condition

$$\omega_0 = \frac{\mu B}{L_p} \tag{37-19}$$

can be used to measure μ. We place the spinning proton in a known field **B**, apply a "perturbing field" at right angles to it, and vary the angular frequency ω_0 of this perturbing field until resonance occurs. It is possible to tell when Eq. 37-19 is satisfied because, at resonance, many spinning protons will tend to turn end for end in the field, absorbing energy which can be detected by appropriate electronic techniques.

Figure 37-19 is a schematic diagram of an experimental arrangement. The protons, present as hydrogen nuclei in a small vial V of water, are immersed in a strong steady magnetic field caused by the electromagnet whose pole faces N and S are shown. A rapidly alternating current in the small coil C provides the (horizontal) weak, perturbing magnetic field $\mathbf{B}_{osc}$. This current is provided by a radio-frequency oscillator

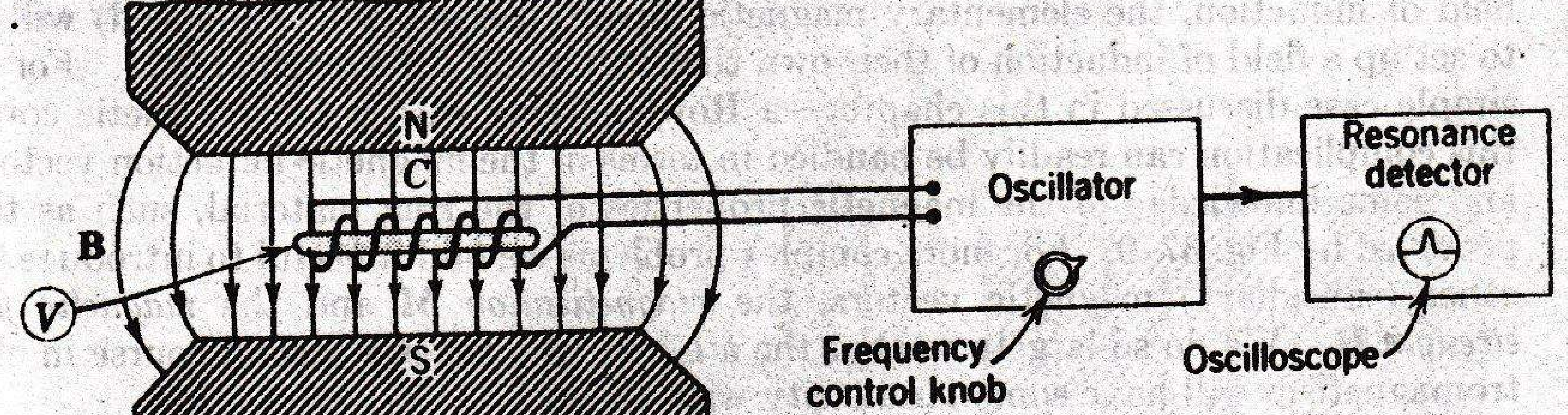


Fig. 37-19 An arrangement to observe nuclear resonance. The oscillating field is horizontal within the coil.

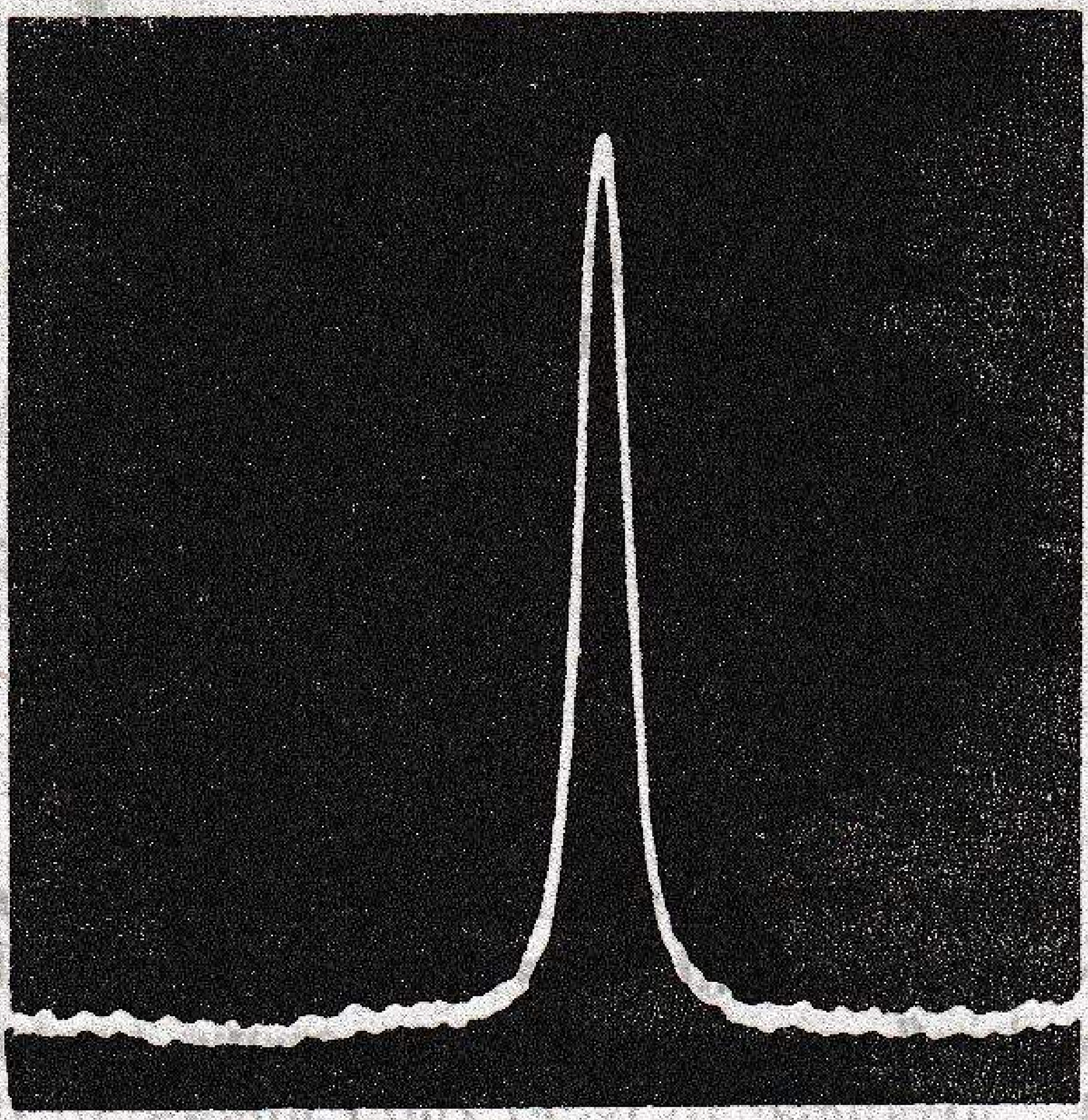


Fig. 37-20 An oscilloscope photograph of a proton resonance peak showing energy absorbed from the oscillator versus oscillator frequency. (From Bloembergen et al., *Phys. Rev.*, 73, 679.)

whose angular frequency ω_0 can be varied; an electronic "resonance detector," also connected to the oscillator, serves to indicate when energy is being drained from the oscillator and used to "flip the protons." In principle, the oscillator angular frequency ω_0 is varied until the resonance detector shows that Eq. 37-19 is satisfied (see Fig. 37-20). The magnetic moment μ can then be determined by measuring B and ω_0. Surprisingly enough, magnetic moments can be measured in this and similar ways to a much greater accuracy than we can measure μ for a bar magnet. For the proton we have

$$\mu_p = 1.41044 \times 10^{-26} \text{ amp-m}^2$$

37-7 Three Magnetic Vectors

In Chapter 30 we saw that if a dielectric is placed in an electric field polarization charges will appear on its surface. These surface charges, which find their origin in the elementary electric dipoles (permanent or induced) that make up the dielectric, set up a field of their own that modifies the original field. For the simple case discussed in Chapter 30—a dielectric slab in a parallel-plate capacitor—this complication can readily be handled in terms of the electric field strength vector **E** and some knowledge of the electric properties of the slab material, such as the dielectric constant. For more complex problems we asserted that it was useful to introduce two other (subsidiary) electric vectors, the *electric polarization* **P** and the *electric displacement* **D**. Table 30-2 shows some of the characteristics of these three vectors.

In magnetism we find a similar situation. If magnetic materials are placed in a field of induction, the elementary magnetic dipoles (permanent or induced) will act to set up a field of induction of their own that will modify the original field. For the simple case discussed in this chapter—a Rowland ring with a ferromagnetic core—this complication can readily be handled in terms of the magnetic induction vector **B** and some knowledge of the magnetic properties of the ring material, such as that provided by Fig. 37-9. For more complex problems we find it useful to introduce two other (subsidiary) magnetic vectors, the *magnetization* **M** and the *magnetic field strength* **H**. We do so largely so that the student who takes a second course in electromagnetism will have some familiarity with them.

Consider a Rowland ring carrying a current i_0 in its windings and designed so that its core, assumed to be iron, can be removed. The magnetic induction B, measured by the methods of Section 37-5, will be much greater when the core is in place than when it is not, assuming that the current in the windings remains unchanged.

Physically we can understand the large value of B in the iron core in terms of the alignment of the elementary dipoles in the iron. A hypothetical slice out of the iron core, as in Fig. 37-21*b*, has a magnetic moment $d\boldsymbol{\mu}$ equal to the vector sum of all of the elementary dipoles contained in it. We define our first subsidiary vector, the *magnetization* **M**, as the magnetic moment per unit volume of the core material. For the slice of Fig. 37-21*b* we have

$$d\boldsymbol{\mu} = \mathbf{M}(A\,dl),$$

where $(A\,dl)$ is the volume of the slice, A being the cross section of the core.

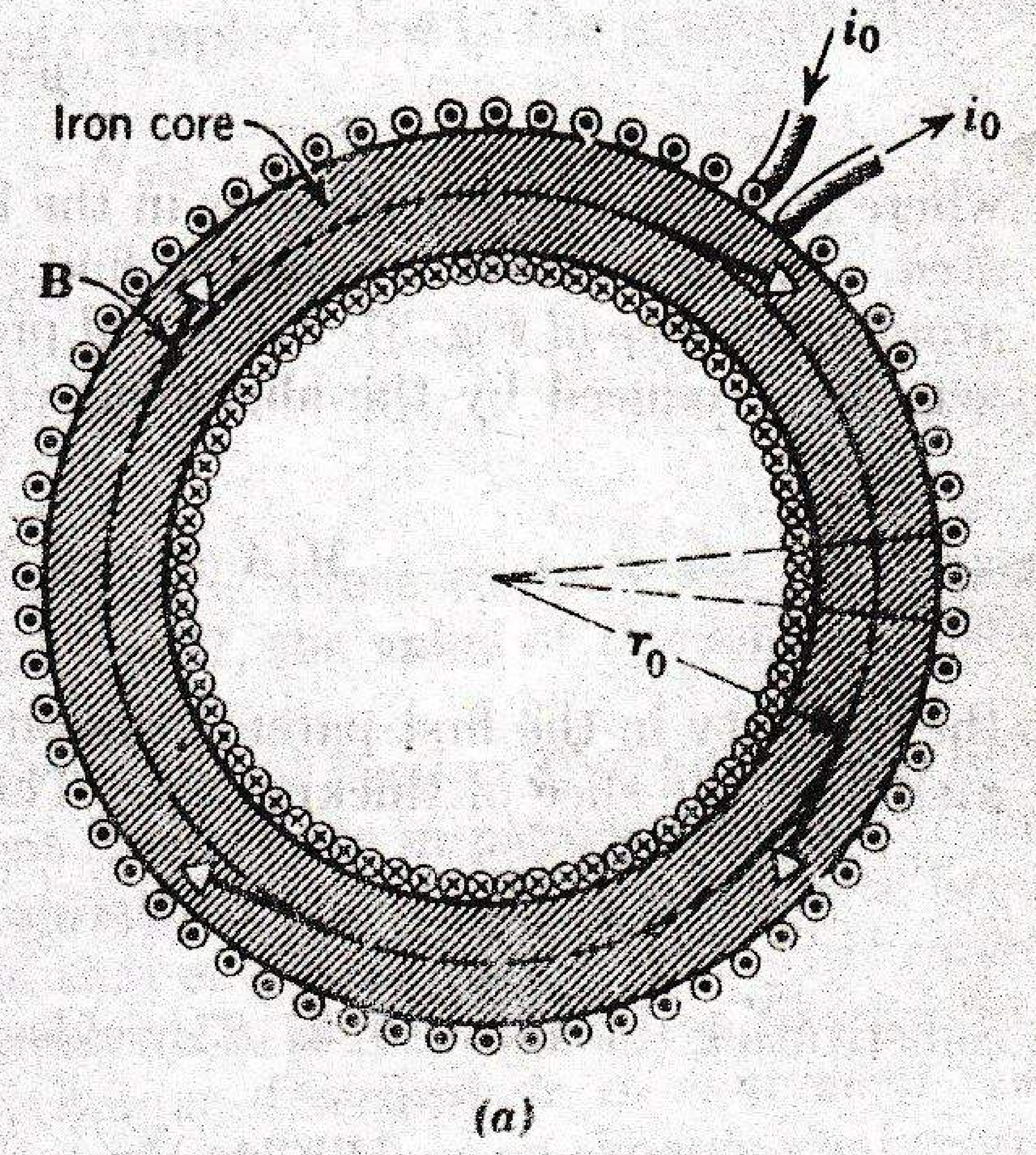
When we discussed Ampère's law in Chapter 34, we assumed that no magnetic materials were present. If we apply this law, namely

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i, \tag{37-20}$$

to the circular path of integration shown in Fig. 37-21*a*, we obtain

$$(B)(2\pi r_0) = \mu_0(N_0 i_0), \tag{37-21}$$

in which r_0 is the mean radius of the core, N_0 is the number of turns, and i_0 is the current in each turn. We see at once that Ampère's law, in the form of Eq. 37-20,



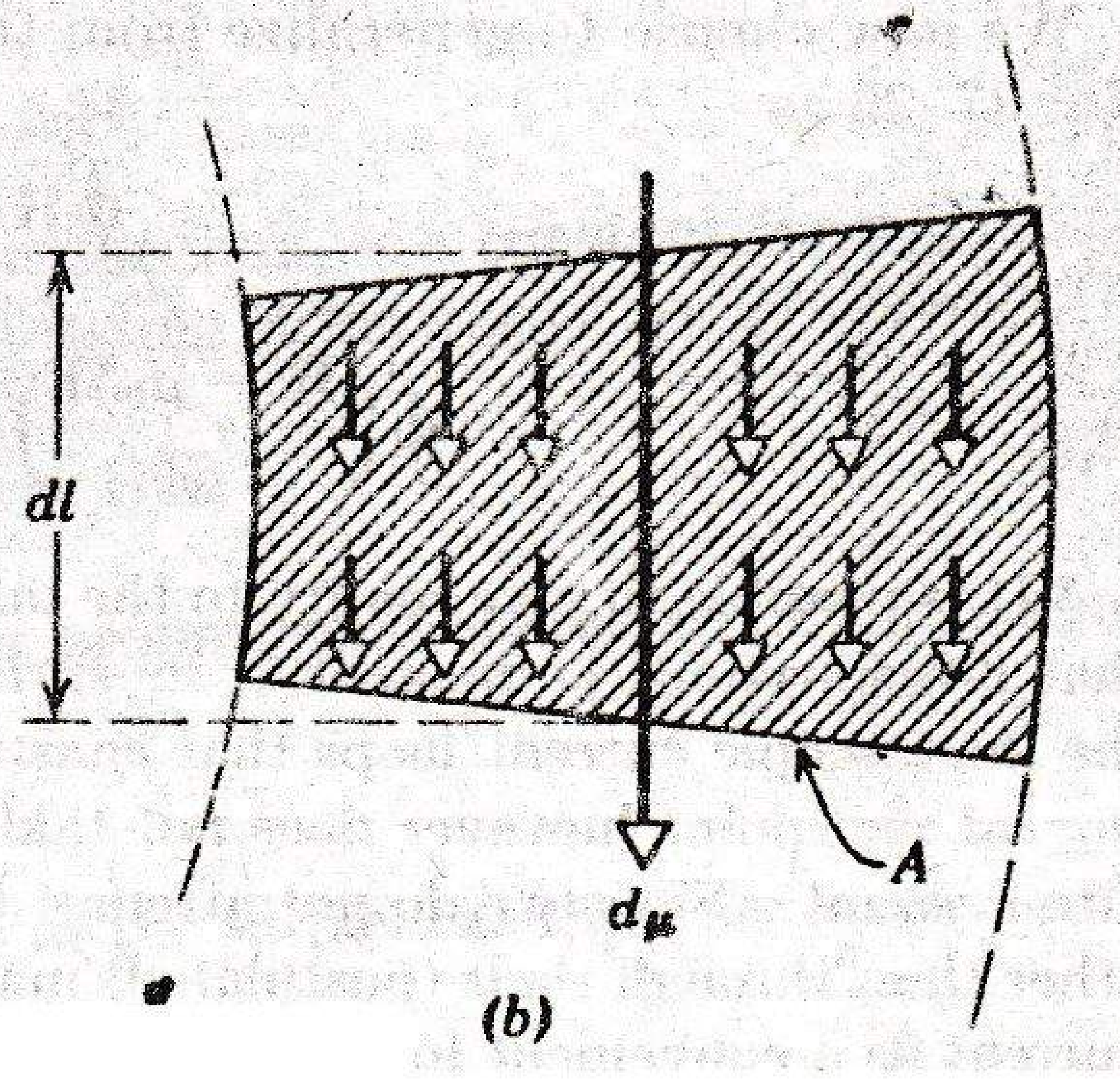


Fig. 37-21 (*a*) A Rowland ring with an iron core. (*b*) A slice of the core, showing its magnetic moment $d\boldsymbol{\mu}$ caused by the alignment of the elementary magnetic dipoles in the iron.

is not valid when magnetic materials are present. Equation 37-21 predicts that, since the right side is the same whether or not the core is in place, the induction B should also be the same, a prediction not in accord with experiment.

We can increase B in the absence of the iron core to the value that it has when the core is in place if we increase the current in the windings by an amount $i_{M,0}$. The magnetization of the iron core is thus *equivalent in its effect on* **B** to such a hypothetical current increase. We choose to modify Ampère's law by arbitrarily inserting a *magnetizing current* term i_M on the right, obtaining

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(i + i_M). \tag{37-22}$$

If we give i_M a suitable value when the iron core is in place, it is clear that Ampère's law, in this new form, can remain valid. It remains to relate this (largely hypothetical) magnetizing current to something more physical, the magnetization **M**.*

Applying Eq. 37-22 to the iron ring of Fig. 37-21*a* yields

$$(B)(2\pi r_0) = \mu_0(N_0 i_0) + \mu_0(N_0 i_{M,0}). \tag{37-23}$$

We can relate $i_{M,0}$ to the magnetization **M** if we recall (Eq. 33-10) that the magnetic moment of a magnetic dipole in the form of a current loop is given by

$$\mu = NiA,$$

where N is the number of turns in the loop, i is the loop current, and A is the loop area. Let us use this equation to find what increase $i_{M,0}$ in current in the windings around the slice of Fig. 37-21*b* would produce a magnetic moment equivalent to that actually produced by the alignment of elementary dipoles in the slice. We have

$$M(A\,dl) = \left(N_0 \frac{dl}{2\pi r_0}\right)(i_{M,0})(A),$$

the quantity in the first parentheses on the right being the number of turns associated with the slice of thickness dl. This reduces to

$$N_0 i_{M,0} = M(2\pi r_0). \tag{37-24}$$

Substituting this into Eq. 37-23 yields

$$(B)(2\pi r_0) = \mu_0(N_0 i_0) + \mu_0(M)(2\pi r_0). \tag{37-25}$$

We now choose to generalize from the special case of the Rowland ring by writing Eq. 37-25 as

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i + \mu_0 \oint \mathbf{M} \cdot d\mathbf{l}$$

or

$$\oint \left(\frac{\mathbf{B} - \mu_0 \mathbf{M}}{\mu_0}\right) \cdot d\mathbf{l} = i.$$

* It is possible to give reality to the magnetizing current by viewing it as a real current that flows around the magnet at its surface, being the resultant macroscopic effect of all the microscopic current loops that constitute the atomic electron orbits. This *Amperian current* viewpoint however does not take the magnetization due to electron spin readily into account. Since we do not attempt to measure magnetizing currents experimentally, other than through their (postulated) magnetic effects, we prefer to view the magnetizing current as a convenient formalism.

The quantity $(\mathbf{B} - \mu_0\mathbf{M})/\mu_0$ occurs so often in magnetic situations that we give it a special name, the *magnetic field strength* **H**, or

$$\mathbf{H} = \frac{\mathbf{B} - \mu_0\mathbf{M}}{\mu_0}$$

which we write as

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}. \tag{37-26}$$

Ampère's law can now be written in the simple form

$$\oint \mathbf{H}\cdot d\mathbf{l} = i, \tag{37-27}$$

which holds in the presence of magnetic materials and in which i is the *true current only*, that is, it does not include the magnetizing current. This reminds us that the electric displacement vector **D** permitted us to write Gauss's law for the case in which dielectric materials are present, in a form involving free charges only, that is, not polarization charges; see Table 30-2.

We state without proof (see Problems 12 and 13) that at a boundary between two media (1) the component of **H** tangential to the surface has the same value on each side of the surface * and (2) the component of **B** perpendicular to the surface has the same value on each side of the surface. These *boundary conditions* are of great value in solving complex problems.

To find H in our Rowland ring, let us apply Ampère's law in the generalized form of Eq. 37-27. We have

$$(H)(2\pi r_0) = N_0 i_0,$$

where i_0 is the (true) current in the windings. This gives

$$H = \left(\frac{N_0}{2\pi r_0}\right) i_0 = n i_0, \tag{37-28}$$

in which n is the number of turns per unit length. Since we have not introduced any information describing the core into Eq. 37-27, the value of H computed from Eq. 37-28 is independent of the core material.

B can be measured experimentally by the method of Section 37-5 and M can then be calculated from Eq. 37-26. The student should note in passing (see Eq. 37-15) that the abscissa B_0 in Fig. 37-9 is proportional to H $(= \mu_0 H)$, the ordinate being proportional to B. Curves such as this and that of Fig. 37-12 are called *B-H curves*.

Let us assume that we have made measurements of **H**, **B**, and **M** for a wide variety of magnetic materials, using either the technique described or an equivalent one. For *paramagnetic* and *diamagnetic* materials we would find, as an experimental result, that **B** is directly proportional to **H**, or

$$\mathbf{B} = \kappa_m \mu_0 \mathbf{H}, \tag{37-29}$$

in which κ_m, the *permeability* of the magnetic medium, is a constant for a given temperature and density of the material. Eliminating **B** between Eqs. 37-29 and 37-26 allows us to write

$$\mathbf{M} = (\kappa_m - 1)\mathbf{H}, \tag{37-30}$$

which is another expression of the linear or proportional character of paramagnetic and diamagnetic materials.

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* Assuming that there are no true currents at the surface, as there are in the Rowland ring of Fig. 37-21*a*, for example.

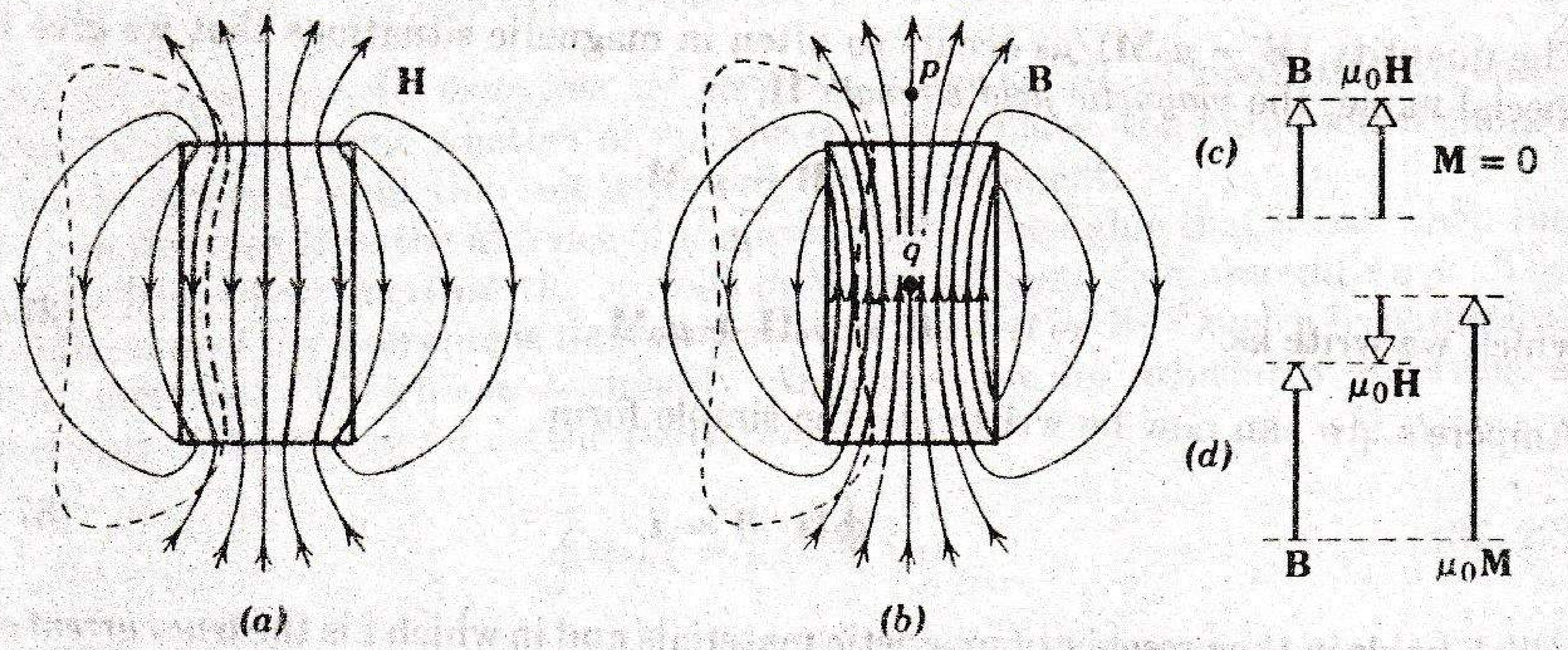

Fig. 37–22 (*a*) The lines of **H** and (*b*) the lines of **B** for a permanent magnet. Note that the lines of **H** change direction at the boundary. The closed dashed curves are paths of integration around which Ampère's law may be applied. The relation $\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}$ is shown to be satisfied for (*c*) a particular outside point *p* and (*d*) a particular inside point *q*.

For a vacuum, in which there are no magnetic dipoles present to be aligned, the magnetization **M** must be zero. Putting $\mathbf{M} = 0$ in Eq. 37-26 leads to

$$\mathbf{B} = \mu_0\mathbf{H} \quad \text{(a vacuum).} \tag{37-31}$$

Comparison with Eq. 37-29 shows that a vacuum must be described by $\kappa_m = 1$. Equation 37-30 verifies that the magnetization vanishes if we put κ_m equal to unity. For paramagnetic materials κ_m is slightly greater than unity. For diamagnetic materials it is slightly less than unity; Eq. 37-30 shows that this requires **M** and **H** to be oppositely directed, a fact discussed at length in Section 37-4.

In ferromagnetic materials the relationship between **B** and **H** is far from linear, as Figs. 37-9 and 37-12 show. Experimentally, κ_m proves to be a function not only of the value of H but also, because of hysteresis, of the magnetic and thermal history of the specimen.*

An interesting special case of ferromagnetism is the permanent magnet, for which **H**, **M**, and **B** all have nonvanishing values inside the magnet even though there is no true current. Figure 37-22 shows typical lines of **B** and **H** associated with such a magnet. The lines of **B** may be drawn as continuous loops, the boundary condition (2), mentioned above, being satisfied where the lines enter and leave the magnet. Equation 37-22 shows that the vector **B** is associated with the *total* current, both true and magnetizing. In Fig. 37-22*a* $\oint \mathbf{B}\cdot d\mathbf{l}$ around any loop such as that shown by the dashed curve is not zero and must be associated with a hypothetical magnetizing current i_M imagined to circulate around the magnet at its surface; actual or true currents (i) do not exist in this problem. Figure 37-22*b* shows that **H** reverses direction at the boundary. Since **H** (see Eq. 37-27) is associated with true currents only, we must have $\oint \mathbf{H}\cdot d\mathbf{l}$ around any loop such as that shown by the dashed lines.

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* In the dielectric case there are waxy materials, called *ferroelectrics*, for which the relationship between **D** and **E** is nonlinear, which exhibit hysteresis, and from which quasi-permanent electric dipoles (*electrets*) can be constructed. However, most commonly useful dielectric materials are linear whereas most commonly useful magnetic materials are nonlinear.

The reversal of **H** at the boundary makes this possible. Note that **M** and **H** point in opposite directions within the magnet. Table 37-1 summarizes the properties of the three vectors **B**, **H**, and **M**.

Table 37-1

THREE MAGNETIC VECTORS

Name	Symbol	Associated with	Boundary Condition
Magnetic induction	**B**	All currents	Normal component continuous
Magnetic field strength	**H**	True currents only	Tangential component continuous †
Magnetization (magnetic dipole moment per unit volume)	**M**	Magnetization currents only	Vanishes in a vacuum

Defining equations for **B**	$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ or $= i\mathbf{l} \times \mathbf{B}$	Eq. 33-3a Eq. 33-6a
General relation among the three vectors	$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}$	Eq. 37-26
Ampère's law when magnetic materials are present	$\oint \mathbf{H}\cdot d\mathbf{l} = i$ (i = true current only)	Eq. 37-27
Empirical relations for certain magnetic materials *	$\mathbf{B} = \kappa_m\mu_0\mathbf{H}$ $\mathbf{M} = (\kappa_m - 1)\mathbf{H}$	Eq. 37-29 Eq. 37-30

* For paramagnetic and diamagnetic materials only, if κ_m is to be independent of **H**.
† Assuming no true currents exist at the boundary.

▶ **Example 6.** In the Rowland ring the (true) current i_0 in the windings is 2.0 amp and the number of turns per unit length (n) in the toroid is 10 turns/cm. B, measured by the technique of Section 37-5, is 1.0 weber/meter². Calculate (*a*) H, (*b*) M, and (*c*) the magnetizing current $i_{M,0}$ both when the core is in place and when it is removed. (*d*) For these particular operating conditions, what is κ_m?

(*a*) H is independent of the core material and may be found from Eq. 37-28:

$$H = ni$$
$$= (10 \text{ turns/cm})(2.0 \text{ amp})$$
$$= 2.0 \times 10^3 \text{ amp/meter}.$$

(*b*) M is zero when the core is removed. With the core in place, we may solve Eq. 37-26 for **M**, obtaining for the magnitude of **M**

$$M = \frac{B - \mu_0 H}{\mu_0}$$

$$= \frac{(1.0 \text{ weber/meter}^2) - (4\pi \times 10^{-7} \text{ weber/amp-m})(2.0 \times 10^3 \text{ amp/meter})}{(4\pi \times 10^{-7} \text{ weber/amp-m})}$$

$$= 7.9 \times 10^5 \text{ amp/meter}.$$

(*c*) The effective magnetizing current follows from Eq. 37-24:

$$i_{M,0} = M\left(\frac{2\pi r_0}{N_0}\right) = \frac{M}{n}$$

$$= \frac{7.9 \times 10^5 \text{ amp/meter}}{2.0 \times 10^3 \text{ turns/meter}}$$

$$= 390 \text{ amp}.$$

An *additional* current of this amount in the windings would produce the same value of B in the absence of a core as that obtained, by alignment of the elementary dipoles, with the core in place.

(*d*) The permeability can be found from Eq. 37-29, or

$$\kappa_m = \frac{B}{\mu_0 H}$$

$$= \frac{1.0 \text{ weber/meter}^2}{(4\pi \times 10^{-7} \text{ weber/amp-m})(2.0 \times 10^3 \text{ amp/meter})}$$

$$= 397.$$

We emphasize that this value of κ_m holds only for the special conditions of this experiment. ◀

QUESTIONS

1. Two iron bars are identical in appearance. One is a magnet and one is not. How can you tell them apart? You are not permitted to suspend either bar as a compass needle or to use any other apparatus.

2. How could you reverse the magnetism of a compass needle?

3. Two iron bars always attract, no matter the combination in which their ends are brought near each other. Can you conclude that one of the bars must be unmagnetized?

4. If we sprinkle iron filings on a particular bar magnet, they cling both to the ends *and to the middle*. Sketch roughly the lines of **B**, both outside and inside the magnet.

5. The earth is a huge magnetic dipole. (*a*) Is the magnetic pole in the Northern Hemisphere a north or a south magnetic pole? (*b*) In the Northern Hemisphere do the magnetic lines of force associated with the earth's magnetic field point toward the earth's surface or away from it?

6. Cosmic rays are charged particles that strike our atmosphere from some external source. We find that more low-energy cosmic rays reach the earth at the north and south magnetic poles than at the (magnetic) equator. Why is this so?

7. How might the magnetic dipole moment of the earth be measured?

8. Give three reasons for believing that the flux Φ_B of the earth's magnetic field is greater through the boundaries of Alaska than through those of Texas.

9. The neutron, which has no charge, has a magnetic dipole moment. Is this possible on the basis of classical electromagnetism, or does this evidence alone indicate that classical electromagnetism has broken down?

10. Is the magnetization at saturation for a paramagnetic substance very much different from that for a saturated ferromagnetic substance of about the same size?

11. Explain why a magnet attracts an unmagnetized iron object such as a nail.

12. Does any net force or torque act on (*a*) an unmagnetized iron bar or (*b*) a permanent bar magnet when placed in a uniform magnetic field?

13. A nail is placed at rest on a smooth table top near a strong magnet. It is released and attracted to the magnet. What is the source of the kinetic energy it has just before it strikes the magnet?

14. The magnetization induced in a given diamagnetic sphere by a given external magnetic field does not vary with temperature, in sharp contrast to the situation in paramagnetism. Is this understandable in terms of the description that we have given of the origin of diamagnetism?

15. Compare the magnetization curves for a paramagnetic substance (Fig. 37-6) and for a ferromagnetic substance (Fig. 37-9). What would a similar curve for a diamagnetic substance look like? Do you think that it would show saturation effects in strong applied fields (say 10 weber/meter²)?

16. Distinguish between the precession frequency and the cyclotron frequency of a proton in a magnetic field.

17. In our discussion of nuclear magnetism we said that energy absorption occurs because the dipoles are turned end for end. However, a given dipole might initially be lined up either with the field or against it. In the first case there would be an *absorption* of energy, but in the second case there would be a *release* of energy, each amount being $2\mu B$. Why do we observe a *net* absorption? These two events would seem to cancel.

18. Discuss similarities and differences in Tables 30-2 and 37-1.

PROBLEMS

1. The earth has a magnetic dipole moment of 6.4×10^{21} amp-m². (*a*) What current would have to be set up in a single turn of wire going around the earth at its magnetic equator if we wished to set up such a dipole? (*b*) Could such an arrangement be used to cancel out the earth's magnetism at points in space well above the earth's surface? (*c*) On the earth's surface?

2. Assume that the electron is a small sphere of radius R, its charge and mass being spread uniformly throughout its volume. Such an electron has a "spin" angular momentum L of 0.53×10^{-34} joule-sec and a magnetic moment μ of 9.3×10^{-24} amp-m². Show that $e/m = 2\mu/L$. Is this prediction in agreement with experiment? (Hint: The spherical electron must be divided into infinitesimal current loops and an expression for the magnetic moment found by integration. This model of the electron is too mechanistic to be in the spirit of the modern quantum view of this particle.)

3. Calculate (*a*) the electric field strength and (*b*) the magnetic induction at a point 1.0 A (one angstrom unit) away from a proton, measured along its axis of spin. The magnetic moment of the proton is 1.4×10^{-26} amp-m².

4. A Rowland ring is formed of ferromagnetic material. It is circular in cross section, with an inner radius of 5.0 cm and an outer radius of 6.0 cm and is wound with 400 turns of wire. (*a*) What current must be set up in the windings to attain $B_0 = 2 \times 10^{-4}$ weber/meter² in Fig. 37-9? (*b*) A secondary coil wound around the toroid has 50 turns and has a resistance of 8.0 ohms. If, for this value of B_0, we have $B_M = 800B_0$, how much charge moves through the secondary coil when the current in the toroid windings is turned on?

5. The dipole moment associated with an atom of iron in an iron bar is 1.8×10^{-23} amp-m². Assume that all the atoms in the bar, which is 5 cm long and has a cross-sec-

tional area of 1 cm^2, have their dipole moments aligned. (*a*) What is the dipole moment of the bar? (*b*) What torque must be exerted to hold this magnet at right angles to an external field of 15,000 gauss?

6. Can you give an explanation of diamagnetism based on Faraday's law of induction? In Figs. 37–7*a* and *b*, for example, what inductive effects can be expected as the magnetic field is built up from zero to the final value **B**?

7. Prove that $\Delta\omega \ll \omega_0$ in Eq. 37–13.

8. Show that, classically, a spinning positive charge will have a spin magnetic moment that points in the same direction as its spin angular momentum.

9. Assume that the nuclei (protons) in 1 gm of water *could all be aligned*. What magnetic induction B would be produced 5.0 cm from the sample, along its alignment axis?

10. It is possible to measure e/m for the electron by measuring (*a*) the cyclotron frequency ν_c of electrons in a given magnetic field and (*b*) the precession frequency ν_p of protons in the same field. Show that the relation is

$$\frac{e}{m} = \frac{\nu_c}{\nu_p}\frac{\mu_s}{L_s}.$$

Since μ_s and L_s for the proton are accurately known, this experiment gives us our most precise value of e/m today.

11. *Dipole-dipole interaction.* The exchange coupling mentioned in Section 37–5 as being responsible for ferromagnetism is *not* the mutual magnetic interaction energy between two elementary magnetic dipoles. To show this (*a*) compute B a distance a (= 1.0 A) away from a dipole of moment μ (= 1.8×10^{-23} amp-m^2); (*b*) compute the energy (= $2\mu B$) required to turn a second similar dipole end for end in this field. What do you conclude about the strength of this dipole-dipole interaction? Compare with the results of Example 3. (Note: for the same distance, the field in the median plane of a dipole is only half as large as on the axis; see Eq. 37–2.)

12. *Boundary condition for* **B**. Prove that at the boundary between two media the normal component of **B** has the same value on each side of the surface. (Hint: Construct a closed Gaussian surface shaped like a flat pillbox with one face in each medium and apply Gauss's law for magnetism.)

13. *Boundary condition for* **H**. Prove that at the boundary between two media the tangential component of **H** has the same value on each side of the surface, assuming that there is no current at the surface. (Hint: Construct a closed rectangular loop, the two opposite longer sides being parallel to the surface, with one side in each medium. Use Ampère's law in the form that applies when magnetic materials are present.)

14. The magnetic energy density can be shown to be given in its most general form as

$$u_B = \tfrac{1}{2}\mathbf{B}\cdot\mathbf{H}.$$

Does this reduce to a familiar result for a vacuum?