Electromagnetic Oscillations

CHAPTER 38

38-1 LC Oscillations

The *LC* system of Fig. 38-1 resembles a mass-spring system (see Fig. 8-4) in that, among other things, each system has a characteristic frequency of oscillation. To see this, we assume that initially the capacitor *C* in Fig. 38-1*a* carries a charge q_m and the current i in the inductor is zero. At this instant the energy stored in the capacitor is given by Eq. 30-25, or

$$U_E = \frac{1}{2}\frac{q_m{}^2}{C}. \tag{38-1}$$

The energy stored in the inductor, given by

$$U_B = \tfrac{1}{2}Li^2, \tag{38-2}$$

is zero because the current is zero. The capacitor now starts to discharge through the inductor, positive charge carriers moving counterclockwise, as shown in Fig. 38-1*b*. This means that a current i, given by dq/dt and pointing down in the inductor, is established.

As q decreases, the energy stored in the electric field in the capacitor also decreases. This energy is transferred to the magnetic field that appears around the inductor because of the current i that is building up there. Thus the electric field decreases, the magnetic field builds up, and energy is transferred from the former to the latter.

At a time corresponding to Fig. 38-1*c*, all the charge on the capacitor will have disappeared. The electric field in the capacitor will be zero, the energy stored there having been transferred entirely to the magnetic field of the

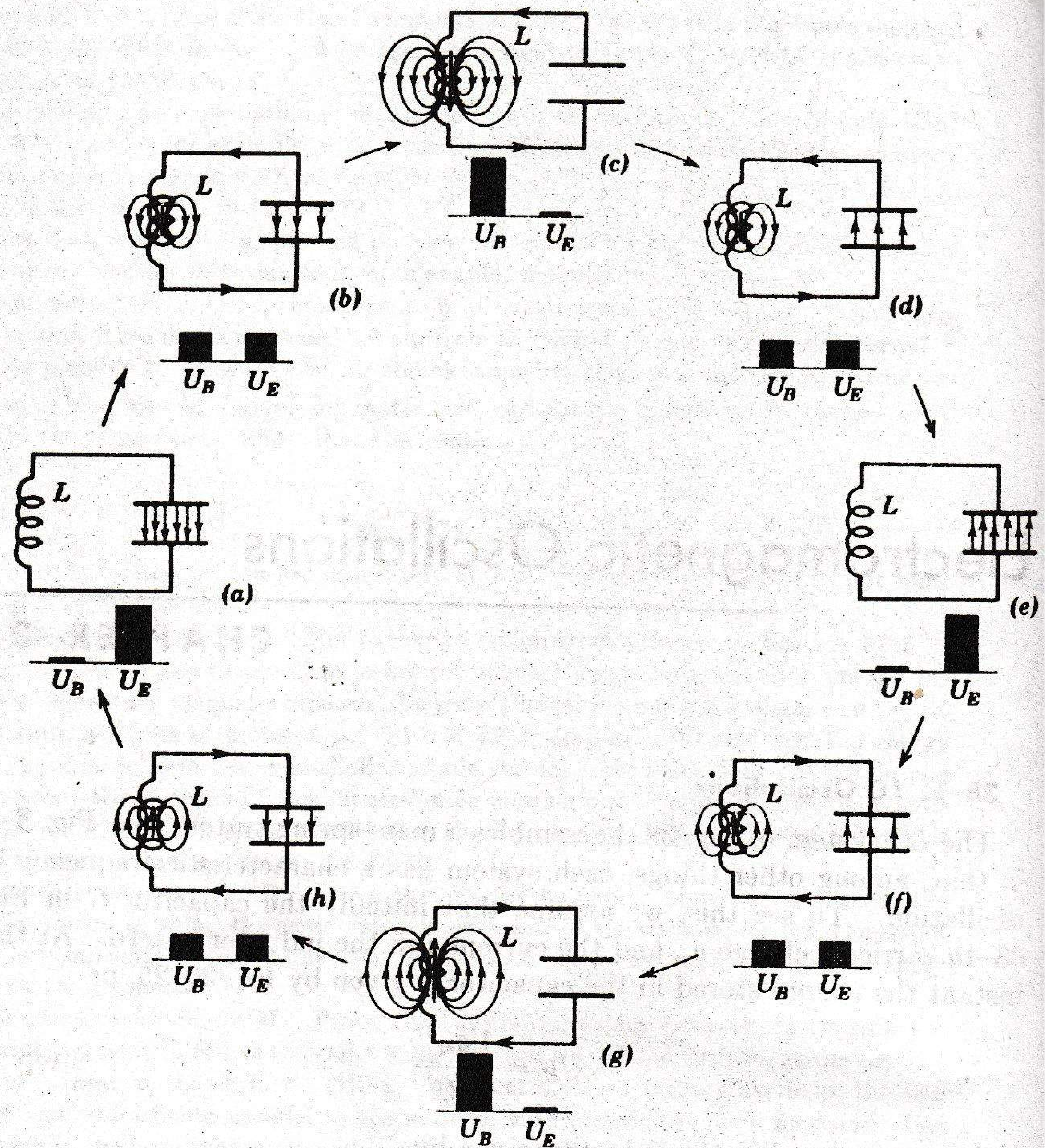

Fig. 38-1 Showing eight stages in a cycle of oscillation of an LC circuit. The bar graphs below each figure show the stored magnetic and electric potential energy. The vertical arrows on the inductor axis show the current. The student should compare this figure in detail with Fig. 8-4, to which it exactly corresponds.

inductor. According to Eq. 38-2, there must then be a current—and indeed one of maximum value—in the inductor. Note that even though q equals zero the current (which is dq/dt) is *not* zero at this time.

The large current in the inductor in Fig. 38-1*c* continues to transport positive charge from the top plate of the capacitor to the bottom plate, as shown in Fig. 38-1*d*; energy now flows from the inductor back to the capacitor as the electric field builds up again. Eventually, the energy will have been transferred completely back to the capacitor, as in Fig. 38-1*e*. The situation of Fig. 38-1*e* is like the initial situation, except that the capacitor is charged in the opposite direction.

The capacitor will start to discharge again, the current now being clockwise, as in Fig. 38–1*f*. Reasoning as before, we see that the circuit eventually returns to its initial situation and that the process continues at a definite frequency ν (measured, say, in cycles/sec) to which corresponds a definite *angular* frequency ω ($= 2\pi\nu$ and measured, say, in radians/sec). Once started, such *LC* oscillations (in the ideal case described, in which the circuit contains no resistance) continue indefinitely, energy being shuttled back and forth between the electric field in the capacitor and the magnetic field in the inductor. Any configuration in Fig. 38–1 can be set up as an initial condition. The oscillations will then continue from that point, proceeding clockwise around the figure. The student should compare these oscillations carefully with those of the mass-spring system described in Fig. 8–4.

To measure the charge q as a function of time, we can measure the variable potential difference $V_C(t)$ that exists across capacitor C. The relation

$$V_C = \left(\frac{1}{C}\right) q$$

shows that V_C is proportional to q. To measure the current we can insert a small resistance R in the circuit and measure the potential difference across it. This is proportional to i through the relation

$$V_R = (R)i.$$

We assume here that R is so small that its effect on the behavior of the circuit is negligible. Both q and i, or more correctly V_C and V_R, which are proportional to them, can be displayed on a cathode-ray oscilloscope. This instrument can plot automatically on its screen graphs proportional to $q(t)$ and $i(t)$, as in Fig. 38–2.

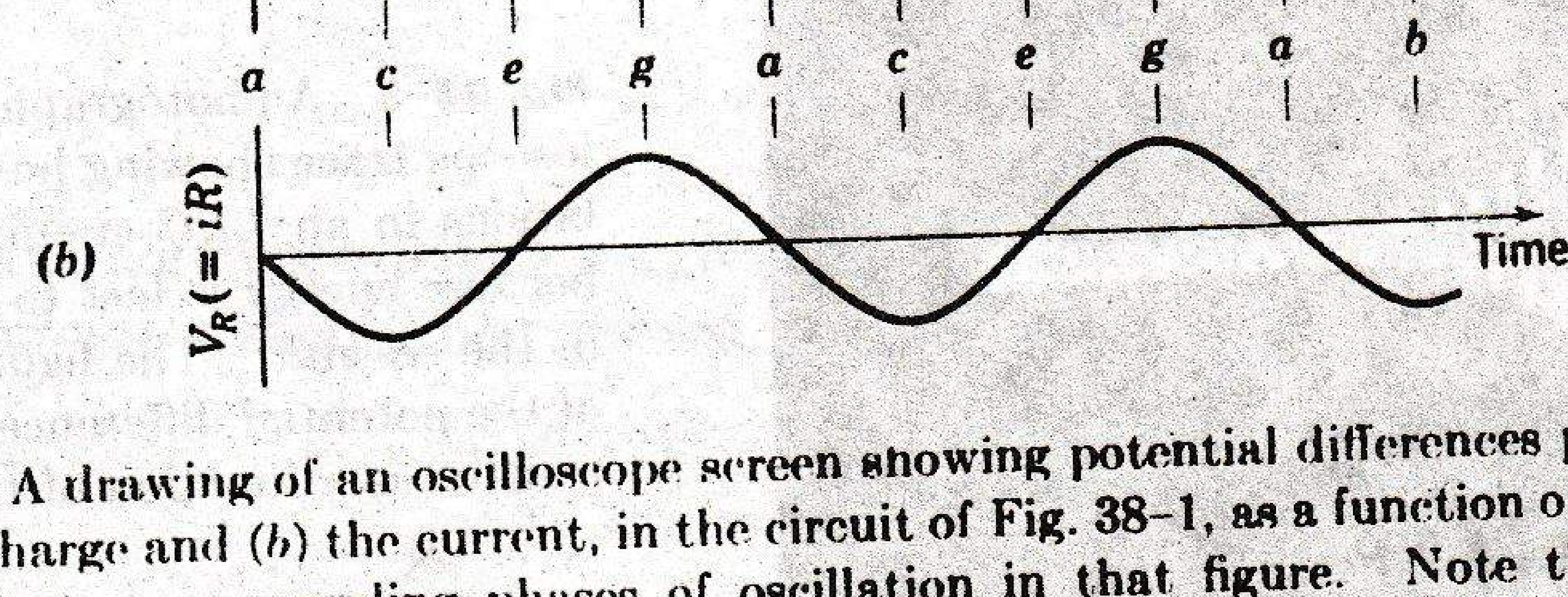

Fig. 38–2 A drawing of an oscilloscope screen showing potential differences proportional to (*a*) the charge and (*b*) the current, in the circuit of Fig. 38–1, as a function of time. The letters indicate corresponding phases of oscillation in that figure. Note that because $i = dq/dt$ the lower curve is proportional to the derivative of the upper.

▶ **Example 1.** A 1.0-μf capacitor is charged to 50 volts. The charging battery is then disconnected and a 10-mh coil is connected across the capacitor, so that LC oscillations occur. What is the maximum current in the coil? Assume that the circuit contains no resistance.

The maximum stored energy in the capacitor must equal the maximum stored energy in the inductor, from the conservation-of-energy principle. This leads, from Eqs. 38-1 and 38-2, to

$$\frac{1}{2}\frac{q_m^2}{C} = \tfrac{1}{2}Li_m^2,$$

where i_m is the *maximum* current and q_m is the *maximum* charge. Note that the maximum current and the maximum charge do not occur at the same time but one-fourth of a cycle apart; see Figs. 38-1 and 38-2. Solving for i_m and substituting CV_0 for q_m gives

$$i_m = V_0\sqrt{\frac{C}{L}} = (50 \text{ volts})\sqrt{\frac{1.0 \times 10^{-6} \text{ farad}}{10 \times 10^{-3} \text{ henry}}} = 0.50 \text{ amp.}$$ ◀

In an actual LC circuit the oscillations will not continue indefinitely because there is always some resistance present that will drain away energy by Joule heating. The oscillations, once started, will die away, as in Fig. 38-3. This figure should be compared to Fig. 15-19, which shows the decay of the mechanical oscillations of a mass-spring system caused by frictional damping.

It is possible to have sustained electromagnetic oscillations if arrangements are made to supply, automatically and periodically (once a cycle, say), enough energy from an outside source to compensate for that lost to Joule heat. We are reminded of a clock escapement, which is a device for feeding energy from a spring or a falling weight into an oscillating pendulum, thus compensating for frictional losses that would otherwise cause the oscillations to die away. Oscillators whose frequency ν may be varied between certain limits are commercially available as packaged units over a wide range of frequencies, extending from low audio-frequencies (lower than 10 cycles/sec) to microwave frequencies (higher than 10^{10} cycles/sec).

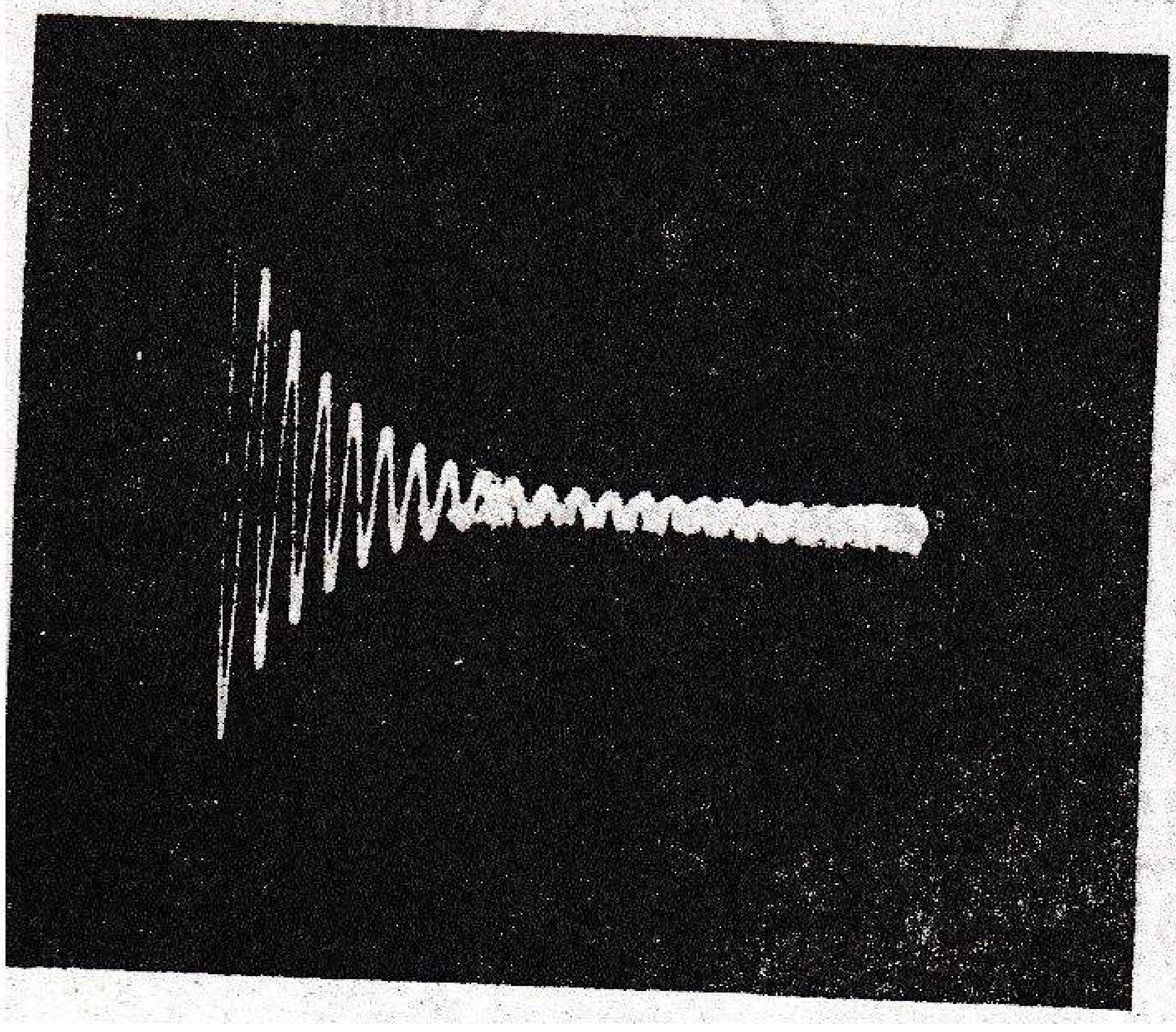

Fig. 38-3 A photograph of an oscilloscope trace showing how the oscillations in an LRC circuit die away because energy is lost to Joule heat in the resistor. The figure is a plot of the potential difference across the resistor as a function of time.

38-2 Analogy to Simple Harmonic Motion

Figure 8-4 shows that in a mass-spring system performing simple harmonic motion, as in an oscillating LC circuit, two kinds of energy occur. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving mass. These are given by the familiar formulas in the first column of Table 38-1. The table suggests that a capacitor is in

Table 38-1

SOME ENERGY FORMULAS

Mechanical		Electromagnetic	
spring	$U_P = \frac{1}{2}kx^2$	capacitor	$U_E = \frac{1}{2}\frac{q^2}{C}$
mass	$U_K = \frac{1}{2}mv^2$	inductor	$U_B = \frac{1}{2}Li^2$
	$v = dx/dt$		$i = dq/dt$

some formal way like a spring and an inductor is like a mass and that certain electromagnetic quantities "correspond" to certain mechanical ones, namely,

$$q \text{ corresponds to } x,$$
$$i \text{ corresponds to } v,$$
$$C \text{ corresponds to } 1/k,$$
$$L \text{ corresponds to } m.$$

Comparison of Fig. 38-1, which shows the oscillations of the LC circuit, with Fig. 8-4, which shows the oscillations in a mass-spring system, indicates how close the correspondence is. Note how v and i correspond in the two figures; also x and q. Note, too, how in each case the energy alternates between two forms, magnetic and electric for the LC system, and kinetic and potential for the mass-spring system.

In Section 15-3 we saw that the natural angular frequency of oscillation of an undamped mass-spring system is

$$\omega = 2\pi\nu = \sqrt{\frac{k}{m}}.$$

The method of correspondences suggests that to find the natural frequency for the LC circuit k should be replaced by $1/C$ and m by L, obtaining

$$\omega = 2\pi\nu = \sqrt{\frac{1}{LC}}. \tag{38-3}$$

This formula is indeed correct, as we show in the next section.

38-3 Electromagnetic Oscillations—Quantitative

We now derive an expression for the frequency of oscillation of an LC circuit, our derivation being based on the conservation-of-energy principle. The total energy U present at any instant in an oscillating LC circuit is given by

$$U = U_B + U_E = \tfrac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C},$$

which expresses the fact that at any arbitrary time the energy is stored partly in the magnetic field in the inductor and partly in the electric field in the capacitor. If we assume the circuit resistance to be zero, there is no energy transfer to Joule heat and U remains constant with time, even though i and q vary. In more formal language, dU/dt must be zero. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0. \qquad (38\text{–}4)$$

Now, q and i are not independent variables, being related by

$$i = \frac{dq}{dt}.$$

Differentiating yields $$\frac{di}{dt} = \frac{d^2q}{dt^2}.$$

Substituting these two expressions into Eq. 38–4 leads to

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0. \qquad (38\text{–}5)$$

This is the differential equation that describes the oscillations of a (resistanceless) LC circuit. To solve it, the student should note that Eq. 38–5 is mathematically of exactly the same form as Eq. 15–6,

$$m\frac{d^2x}{dt^2} + kx = 0, \qquad (15\text{–}3)$$

which is the differential equation for the mass-spring oscillations. Fundamentally, it is by comparing these two equations that the correspondences on p. 947 arise.

The solution of Eq. 15–6 proved to be

$$x = A\cos(\omega t + \phi), \qquad (15\text{–}5)$$

where A $(= x_m)$ is the amplitude of the motion and ϕ is an arbitrary *phase angle*. Since q corresponds to x, we can write the solution of Eq. 38–5 as

$$q = q_m\cos(\omega t + \phi), \qquad (38\text{–}6)$$

where ω is the still unknown angular frequency of the electromagnetic oscillations.

We can test whether Eq. 38-6 is indeed a solution of Eq. 38-5 by substituting it and its second derivative in that equation. To find the second derivative, we write

$$\frac{dq}{dt} = i = -\omega q_m \sin(\omega t + \phi) \tag{38-7}$$

and

$$\frac{d^2q}{dt^2} = -\omega^2 q_m \cos(\omega t + \phi).$$

Substituting q and d^2q/dt^2 into Eq. 38-5 yields

$$-L\omega^2 q_m \cos(\omega t + \phi) + \frac{1}{C} q_m \cos(\omega t + \phi) = 0.$$

Canceling $q_m \cos(\omega t + \phi)$ and rearranging leads to

$$\omega = \sqrt{\frac{1}{LC}}.$$

Thus, if ω is given the constant value $1/\sqrt{LC}$, Eq. 38-6 is indeed a solution of Eq. 38-5. This expression for ω agrees with Eq. 38-3, which was arrived at by the method of correspondences.

The phase angle ϕ in Eq. 38-6 is determined by the conditions that prevail at $t = 0$. If the initial condition is as represented by Fig. 38-1*a*, then we put $\phi = 0$ in order that Eq. 38-6 may predict $q = q_m$ at $t = 0$. What initial condition in Fig. 38-1 is implied if we select $\phi = 90°$?

▶ **Example 2.** (*a*) In an oscillating LC circuit, what value of charge, expressed in terms of the maximum charge, is present on the capacitor when the energy is shared equally between the electric and the magnetic field? (*b*) How much time is required for this condition to arise, assuming the capacitor to be fully charged initially? Assume that $L = 10$ mh and $C = 1.0\ \mu$f.

(*a*) The stored energy and the *maximum* stored energy in the capacitor are, respectively,

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_{E,m} = \frac{q_m^2}{2C}.$$

Substituting $U_E = \frac{1}{2} U_{E,m}$ yields

$$\frac{q^2}{2C} = \frac{1}{2}\frac{q_m^2}{2C} \quad \text{or} \quad q = \frac{1}{\sqrt{2}} q_m.$$

(*b*) To find the time, we write, assuming $\phi = 0$ in Eq. 38-6,

$$q = q_m \cos \omega t = \frac{1}{\sqrt{2}} q_m,$$

which leads to

$$\omega t = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4} \quad \text{or} \quad t = \frac{\pi}{4\omega}.$$

The angular frequency ω is found from Eq. 38-3, or

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(10 \times 10^{-3}\ \text{henry})(1.0 \times 10^{-6}\ \text{farad})}} = 1.0 \times 10^4\ \text{radians/sec}.$$

The time t is then

$$t = \frac{\pi}{4\omega} = \frac{\pi}{(4)(1.0 \times 10^4 \text{ radians/sec})} = 7.9 \times 10^{-5} \text{ sec.}$$

What is the frequency ν in cycles/sec?

The stored electric energy in the LC circuit, using Eq. 38–6, is

$$U_E = \frac{1}{2}\frac{q^2}{C} = \frac{q_m^{\,2}}{2C}\cos^2(\omega t + \phi), \tag{38-8}$$

and the magnetic energy, using Eq. 38–7, is

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2 q_m^{\,2}\sin^2(\omega t + \phi).$$

Substituting the expression for ω (Eq. 38–3) into this last equation yields

$$U_B = \frac{q_m^{\,2}}{2C}\sin^2(\omega t + \phi). \tag{38-9}$$

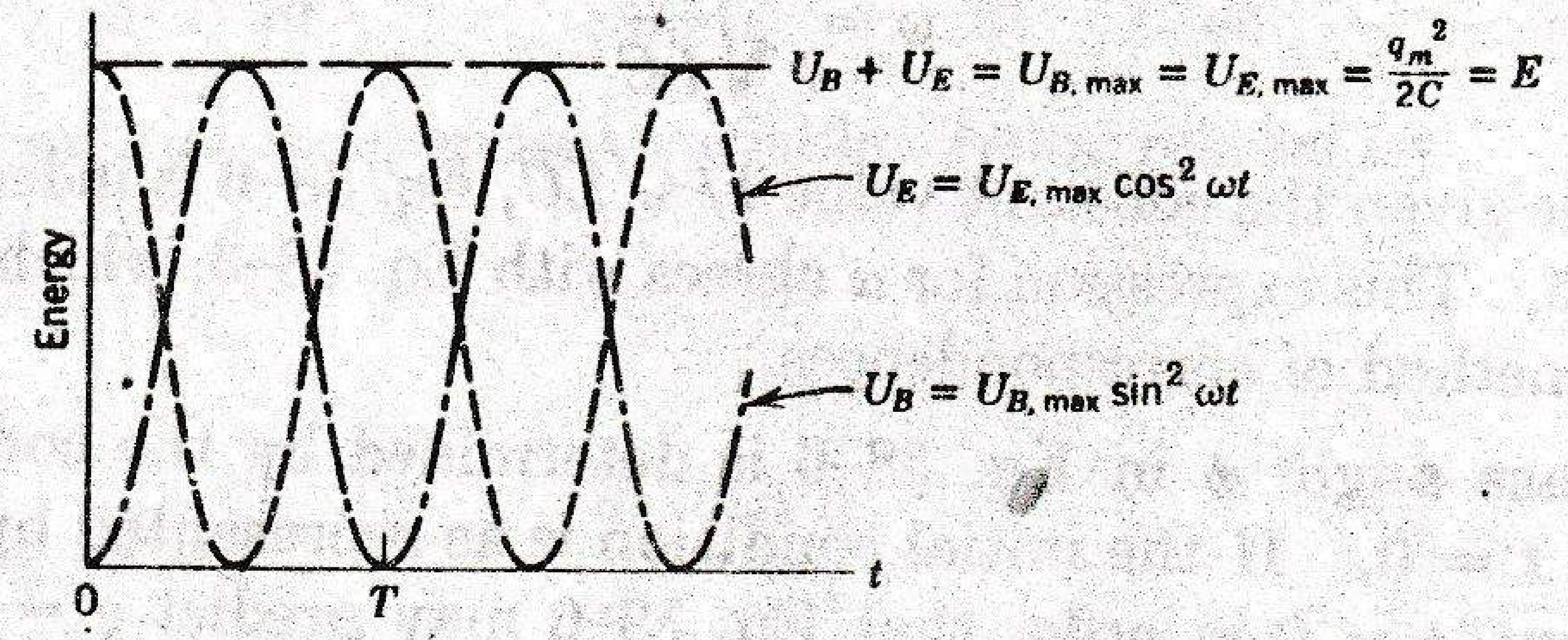

Fig. 38–4 The stored magnetic and electric energy in the circuit of Fig. 38–1. Note that their sum is a constant. The letters indicate corresponding phases of oscillation in Fig. 38–1.

Figure 38–4 shows plots of $U_E(t)$ and $U_B(t)$ for the case of $\phi = 0$. Note that (*a*) the maximum values of U_E and U_B are the same ($= q_m^{\,2}/2C$); (*b*) at any instant the sum of U_E and U_B is a constant ($= q_m^{\,2}/2C$); and (*c*) when U_E has its maximum value, U_B is zero and conversely. This analysis supports the qualitative analysis of Section 38–1. The student should compare this discussion with that given in Section 15–4 for the energy transfers in a mass-spring system.

▶ **Example 3.** *The LCR circuit.* (*a*) Derive an expression for the quantity $q(t)$ for a single-loop circuit containing a resistance R as well as an inductance L and a capacitance C. (*b*) After how long a time will the charge oscillations decay to half-amplitude if $L = 10$ mh, $C = 1.0\ \mu$f, and $R = 0.1$ ohm?

(*a*) If U is the total stored field energy, we have, as before,

$$U = \tfrac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}.$$

U is no longer constant but rather

$$\frac{dU}{dt} = -i^2R,$$

the minus sign signifying that the stored energy U *decreases* with time, being converted to Joule heat at the rate i^2R. Combining these two equations leads to

$$Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

Substituting dq/dt for i and d^2q/dt^2 for di/dt leads finally to

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0,$$

which is the differential equation that describes the damped LC oscillations. If we put $R = 0$, this equation reduces, as expected, to Eq. 38-5.

The student should compare this differential equation for damped LC oscillations with Eq. 15-37, or

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0, \tag{15-28}$$

which describes damped mass-spring oscillations. Once again the equations are mathematically identical, the resistance R corresponding to the mechanical damping constant b.

The solution of the LCR circuit follows at once, by correspondence, from the solution of Eq. 15-37. It is (see Eqs. 15-38 and 15-39) for R reasonably small, and for an initial condition in which the capacitor has a maximum charge

$$q = q_m e^{-Rt/2L}\cos\omega' t, \tag{38-10}$$

where

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}. \tag{38-11}$$

Note that Eq. 38-10, which can be described as a cosine function with an exponentially decreasing amplitude, is the equation of the decay curve of Fig. 38-3. Note, too (Eq. 38-11), that the presence of resistance reduces the oscillation frequency. These two equations reduce to familiar results as $R \to 0$.

(*b*) The oscillation amplitude will have decreased to half when the amplitude factor $e^{-Rt/2L}$ in Eq. 38-10 has the value one-half, or

$$\tfrac{1}{2} = e^{-Rt/2L},$$

which leads readily to

$$t = \frac{2L}{R}\ln 2 = \frac{(2)(10\times 10^{-3}\text{ henry})(0.69)}{0.10\text{ ohm}} = 0.14\text{ sec}.$$

The angular frequency, from Eq. 38-11, is

$$\omega' = \sqrt{\frac{1}{(10\times10^{-3}\text{ henry})(1.0\times10^{-6}\text{ farad})} - \left(\frac{0.10\text{ ohm}}{2\times10\times10^{-3}\text{ henry}}\right)^2}$$

$$= \sqrt{10^8\text{ radians/sec}^2 - 25\text{ radians/sec}^2} = 1.0\times10^4\text{ radians/sec}.$$

Note that the second term is rather small, so that in this case, as in many practical cases, the resistance has a negligible effect on the frequency. The student should show that 0.14 sec, the time at which the oscillations decrease to half-amplitude, corresponds to about 220 cycles of oscillation. The damping is much less severe than that illustrated in Fig. 38-3. ◀

38-4 Forced Oscillations and Resonance

Figure 38-5 shows an LCR circuit containing a sinusoidally varying emf $\mathcal{E}(t)$ given by

$$\mathcal{E} = \mathcal{E}_m \cos \omega'' t,$$

in which ω'' can be varied at will. The emf $\mathcal{E}(t)$ might be provided by a variable-frequency oscillator. What amplitudes of electromagnetic oscillations are set up in this circuit for various angular frequencies ω'' of the "driving force"?

The problem corresponds to that of forced oscillations in the damped mass-spring system of Section 15-10. The differential equation describing that motion is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_m \cos \omega'' t, \tag{15-31}$$

where ω'' is the angular frequency of the external periodic driving force applied to the system and F_m is its amplitude.

For the circuit of Fig. 38-5, the differential equation that follows from the correspondences of p. 947 and the additional reasonable correspondence of $\mathcal{E}$ to F, is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathcal{E}_m \cos \omega'' t. \tag{38-12}$$

This equation can also be derived by applying the energy conservation principle to the circuit of Fig. 38-5.

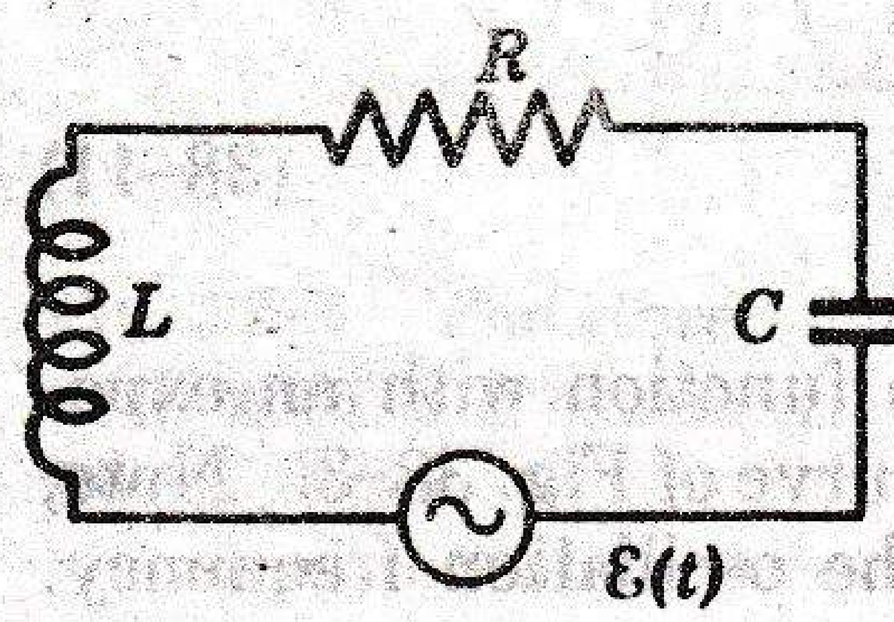

Fig. 38-5 An LCR circuit containing a sinusoidal emf of angular frequency ω''.

We can use the correspondences of p. 947 to write down the solution to Eq. 38-12. Starting from Eq. 15-4 and making the appropriate substitutions, we obtain

$$q = \frac{\mathcal{E}_m}{G} \sin(\omega'' t - \phi), \tag{38-13}$$

where

$$G = \sqrt{\left(\omega''^2 L - \frac{1}{C}\right)^2 + R^2\omega''^2},$$

and ϕ, the *phase angle* between the "driving force" and the "response," is given by

$$\phi = \cos^{-1}\frac{R\omega''}{G}.$$

As often as not, we are interested in the current $i(t)$ in the circuit, rather than the charge; the current corresponds to the velocity $v(t)$ of the moving mass in Section 15-10. We can find $i(t)$ by differentiating Eq. 38-13, or

$$i = \frac{dq}{dt} = \frac{\omega''\mathcal{E}_m}{G}\cos(\omega'' t - \alpha) = i_m \cos(\omega'' t - \alpha).$$

The amplitude i_m of the current oscillations is given, from these equations, by

$$i_m = \frac{\omega''\mathcal{E}_m}{G} = \frac{\mathcal{E}_m}{\sqrt{\left(\omega''L - \frac{1}{\omega''C}\right)^2 + R^2}}. \quad (38\text{-}14)$$

Inspection of Eq. 38-14 shows that the current (not the charge; see Question 5) will have its maximum amplitude when

$$\omega''L = \frac{1}{\omega''C},$$

which can be written as

$$\omega'' = \sqrt{\frac{1}{LC}}. \quad (38\text{-}15)$$

Comparison with Eq. 38-3 shows that the maximum amplitude of the current oscillations occurs when the frequency ω'' of the applied emf is exactly equal to the natural (undamped) frequency ω of the system.

At *resonance* ($\omega'' = \omega$) the amplitude of the current oscillations is determined entirely by the resistance; this follows by combining Eqs. 38-14 and 38-15, or

$$i_m = \frac{\mathcal{E}_m}{R} \quad \text{(at resonance)}.$$

Figure 38-6 shows i_m as a function of ω'' for an oscillating *LCR* circuit containing three different values of resistance. Note that the smaller the resistance, the sharper the resonance curve. The sharpness of such a curve is measured by its ***half-width***, which is the difference between two frequencies, each of which corresponds to a current amplitude of one-half the maximum current amplitude. The half-width of the curve for R = 10 ohms in Fig. 38-6 is shown in that figure by the arrow marked $\Delta\omega$.

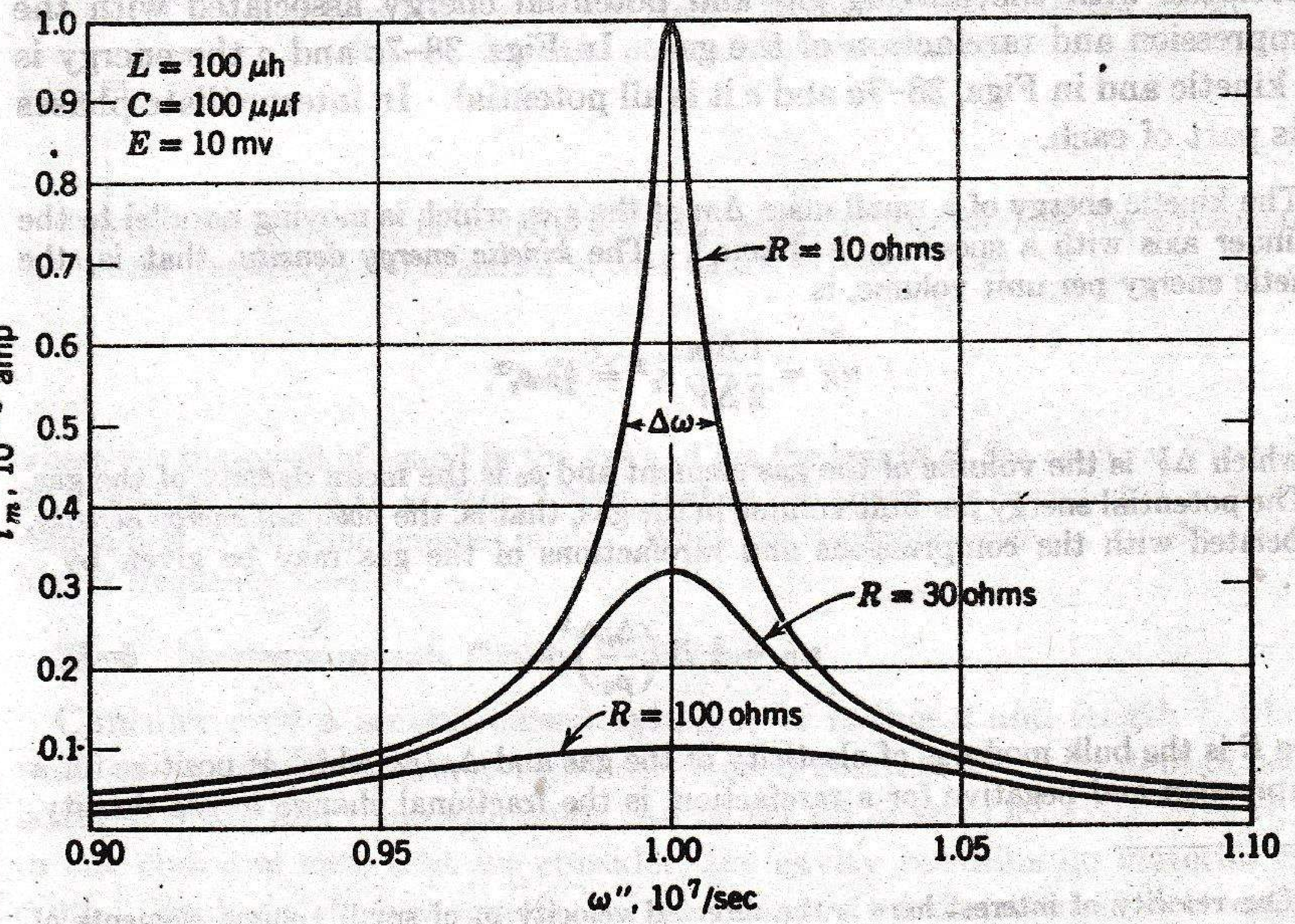

Fig. 38-6 The current amplitude as a function of frequency in the circuit of Fig. 38-5. The arrow marked $\Delta\omega$ on the curve for R = 10 ohms is the *half-width* of that curve.

38-5 Lumped and Distributed Elements

In the oscillating mass-spring system the two kinds of energy involved appear in separate parts of the system, the potential energy being stored in the spring and the kinetic energy in the moving mass. An *acoustic cavity resonator*, such as an organ pipe, is a mechanical oscillating system in which the two forms of energy are *not* separated in space. Kinetic energy, associated with moving air in the cavity, and potential energy, associated with compressions or rarefactions of the air, can both be present throughout the volume of the cavity. Such a cavity is an example of an oscillating system with *distributed* rather than *lumped* (as in the mass-spring system) elements.

A similar distinction exists in electromagnetic systems. The LC circuit of Fig. 38-1 is an example of lumped elements, in that the two kinds of energy are stored in rather different places; the circuit is described by giving the (lumped) system parameters L and C. Actually, in modern engineering practice and in physics research electromagnetic systems with *distributed elements* play a fundamental role.

Figure 38-7, a series of "snapshots" taken one-eighth of a cycle apart, shows the pressure and velocity variations in the fundamental mode of a particular acoustic resonator. There is a pressure node at the center and a pressure antinode at each end. There is a velocity * node at each end and a velocity antinode at the center. When the pressure variation is the greatest, the velocity is zero (Figs. 38-7*a* and *e*). When the pressure is uniform, the velocities have their maximum values (Figs. 38-7*c*, and *g*).

The energy in the acoustic resonator alternates between kinetic energy associated with the moving gas and potential energy associated with the compression and rarefaction of the gas. In Figs. 38-7*c* and *g* the energy is all kinetic and in Figs. 38-7*a* and *e* it is all potential. In intermediate phases it is part of each.

The kinetic energy of a small mass Δm of the gas, which is moving parallel to the cylinder axis with a speed v_g, is $\frac{1}{2}\Delta m v_g^2$. The *kinetic energy density*, that is, the kinetic energy per unit volume, is

$$u_K = \frac{1}{2}\frac{\Delta m}{\Delta V} v_g^2 = \tfrac{1}{2}\rho_0 v_g^2,$$

in which ΔV is the volume of the gas element and ρ_0 is the mean density of the gas.

The potential energy per unit volume in the gas, that is, the *potential energy density*, associated with the compressions and rarefactions of the gas may be given by

$$u_P = \tfrac{1}{2} B \left(\frac{\Delta\rho}{\rho_0}\right)^2.$$

Here B is the bulk modulus of elasticity of the gas and $\Delta\rho/\rho_0$, which is positive for a compression and negative for a rarefaction, is the fractional change in gas density.

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* The velocity of interest here is the directed velocity v_g of small volume elements of the gas which are, however, large enough to contain a great number of molecules. The thermal velocities of the molecules have no directional preference and are ignored.

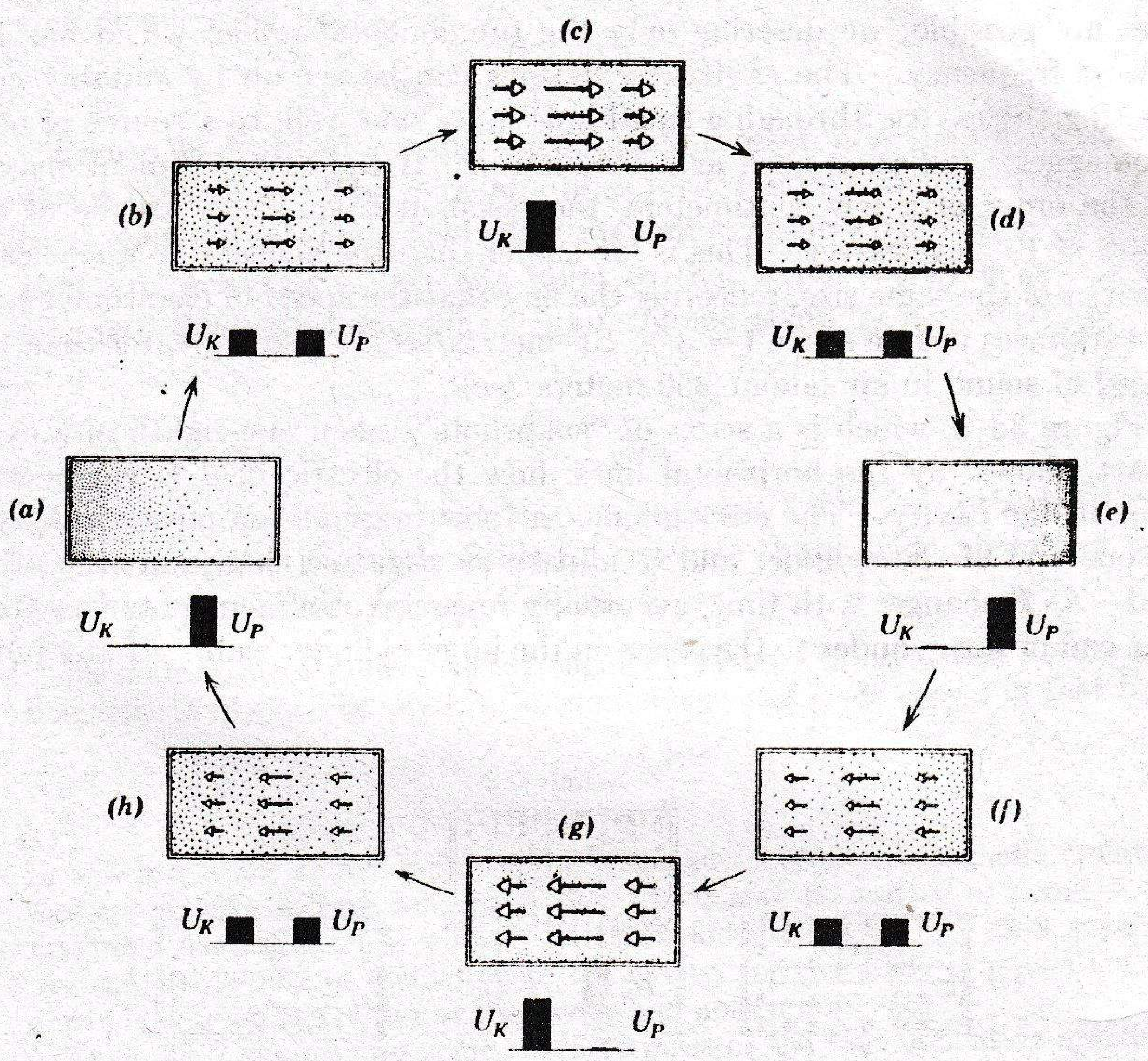

Fig. 38-7 Showing eight stages in a cycle of oscillation of a cylindrical acoustic resonant cavity. The bar graphs below each figure show the kinetic and potential energy. The arrows represent the directed velocities of small volume elements of the gas. Compare with Fig. 38-1.

The angular frequency of oscillation for the cavity of Fig. 38-7, in the fundamental (or lowest frequency) mode shown in that figure, is found from

$$\omega_1 = 2\pi\nu = \frac{2\pi v}{\lambda} = \frac{\pi v}{l},$$

where v is the speed of sound in the gas and l is the length of the cavity. From Eq. 20-1 we may write v as $\sqrt{B_0/\rho_0}$. Note that in the above we have put $\lambda = 2l$, corresponding to the fundamental mode. What are the frequencies ω_2, ω_3, etc., of the higher frequency modes?

38-6 Electromagnetic Cavity Oscillator

Consider now a second closed cylinder, of radius a and length l, whose walls are made of copper or some other good conductor. A system of oscillating electric and magnetic fields can be set up in such a cavity even if, as in the common case that we consider, the cavity contains no material medium. Such an *electromagnetic cavity resonator* is a distributed electromagnetic oscillator, in contrast to an LC circuit, which is a lumped system. As for the acoustic resonator, many modes of oscillation with discrete frequencies are possible; we describe only the fundamental mode, which has the lowest frequency. The cavity oscillations can be set up by suitably connecting the cavity, through a small hole in its side wall, to a source of electromagnetic radiation such as a magnetron. If the cavity dimensions are of the order of a few centimeters, the resonant frequencies will be of the order of 10^{10} cycles/sec. This is far higher than the acoustic frequencies in cavities of the same size, reflecting the fact that the speed of electromagnetic disturbances in free space ($= 3 \times 10^8$ meters/sec) is much greater than the speed of sound in air (about 350 meters/sec).

Figure 38–8, which is a series of "snapshots" taken one-eighth of a cycle apart, shows, by the horizontal lines, how the electric field **E** varies with time in the cavity. The electric lines of force originate on positive charges at one end of the cylinder and terminate on negative charges at the other end. As **E** changes with time, eventually reversing itself, currents flow from one end of the cylinder to the other on the inner cylinder wall. At any point

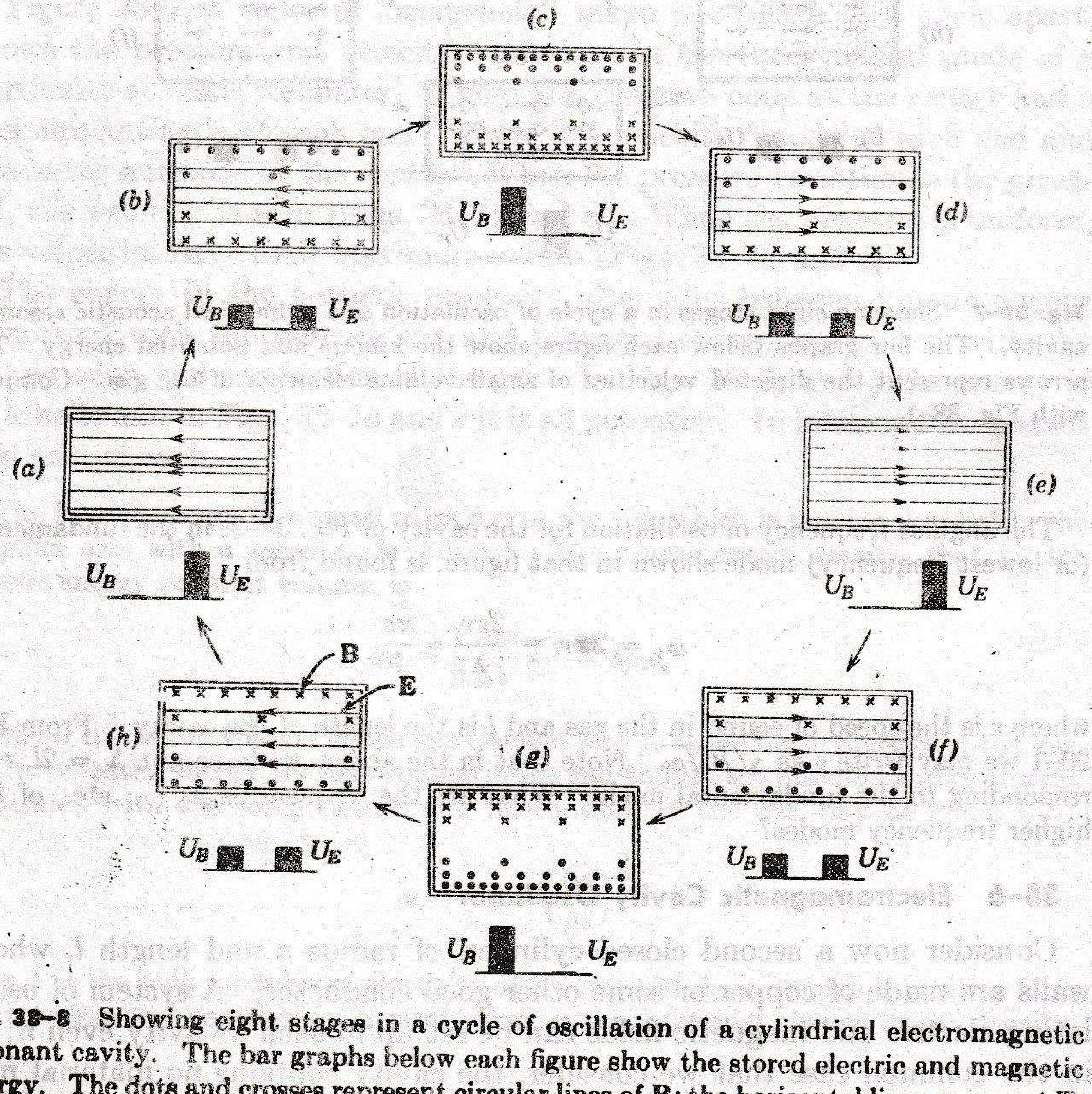

Fig. 38–8 Showing eight stages in a cycle of oscillation of a cylindrical electromagnetic resonant cavity. The bar graphs below each figure show the stored electric and magnetic energy. The dots and crosses represent circular lines of **B**; the horizontal lines represent **E**. Compare with Fig. 38–7.

in the cavity, energy is stored in the electric field in an amount per unit volume given by Eq. 30-27, or

$$u_E = \tfrac{1}{2}\epsilon_0 E^2. \quad (38\text{-}16)$$

Figure 38-8 also shows, by the dots and crosses, how the magnetic field **B** varies with time. The magnetic lines form circles about the cylinder axis. Note that the magnetic field has a maximum value when the electric field is zero, and conversely. At any point in the cavity energy is stored in the magnetic field in an amount per unit volume given by Eq. 36-19, or

$$u_B = \frac{1}{2\mu_0} B^2. \quad (38\text{-}17)$$

Thus, as in the *LC* circuit, energy is shuttled back and forth between the electric and the magnetic fields. However, these fields no longer occupy completely separate regions of space.

We state without proof that the angular frequency of oscillation for the electromagnetic cavity of Fig. 38-8 is, in the fundamental mode shown in that figure,

$$\omega_1 = \frac{1.19c}{a},$$

in which a is the cavity radius and c is the speed of electromagnetic radiations in free space. We will see in Section 39-5 that c may be written in terms of electromagnetic quantities as $1/\sqrt{\mu_0\epsilon_0}$. As the field patterns of Fig. 38-8 suggest, the resonant frequency of oscillation of the cavity, for the mode of oscillation shown, depends only on the radius of the cavity and not on its length.

Table 38-2 summarizes some characteristics of the four oscillating systems that we have discussed so far. For lumped systems it gives expressions for the two kinds

Table 38-2

FOUR OSCILLATING SYSTEMS

	Mechanical Systems	Electromagnetic Systems
Lumped systems	Mass + spring $U_K = \tfrac{1}{2}mv^2$ $U_P = \tfrac{1}{2}kx^2$ $\omega = \sqrt{\frac{k}{m}}$	*LC* circuit $U_B = \tfrac{1}{2}Li^2$ $U_E = \tfrac{1}{2}(1/C)q^2$ $\omega = \sqrt{\frac{(1/C)}{L}}$
Distributed systems	Acoustic cavity $u_K = \tfrac{1}{2}\rho_0 v_g^2$ $u_P = \tfrac{1}{2}B(\Delta\rho/\rho_0)^2$ $\omega_1 = \frac{3.14v}{l}$; $v = \sqrt{\frac{B}{\rho_0}}$	Electromagnetic cavity $u_B = \tfrac{1}{2}(1/\mu_0)B^2$ $u_E = \tfrac{1}{2}\epsilon_0 E^2$ $\omega_1 = \frac{1.19c}{a}$; $c = \sqrt{\frac{1}{\epsilon_0\mu_0}}$

of energy involved and for the (single) oscillation frequency. For the distributed systems it gives expressions for the two kinds of energy density involved and for the oscillation frequency in the fundamental mode. The student should study carefully all the correspondences, similarities, and differences that occur in the table.

The expression given above for the fundamental resonant angular frequency ω_1 of the cavity is derived by applying the basic equations of electromagnetism (Table 38-3) to the space inside the cavity and by invoking this boundary condition: **E** *must be zero inside the cavity wall and can have no tangential component anywhere on the cavity wall.* If this were not true, an infinite current would be set up in the assumed resistanceless wall by the tangential component of **E**. This boundary condition is similar in spirit to the requirement that a clamped string must have zero amplitude of oscillation at the points at which it is clamped or that a velocity node must exist at the end walls of an acoustic cavity resonator. Granted this boundary condition, on **E**, it can be shown that, assuming a perfectly conducting wall, (*a*) no time-varying magnetic field can exist inside the cavity wall and (*b*) no time-varying currents can exist inside the wall. A tangential magnetic field can exist *on the surface*, however; surface charges can exist, and surface currents can flow.

▶ **Example 4.** In the cavity of Fig. 38-8, what is the relationship between the "average" value of **E** throughout the cavity, measured at the instant corresponding to Fig. 38-8*a*, to the "average" value of **B**, measured at the instant corresponding to Fig. 38-8*c*?

At the first instant the energy is all electric and at the second it is all magnetic. The total energy U, found by integrating the energy density over the volume of the cavity, must be the same at these two instants, or

$$U = \int u_{E,m}\, dV = \int u_{B,m}\, dV,$$

where dV is a volume element in the cavity and $u_{E,m}$ and $u_{B,m}$ are the *maximum* values of u_E and of u_B at the site of this volume element; these maximum values occur one-fourth of a cycle apart, as Fig. 38-8 shows. Substituting Eqs. 38-16 and 38-17 leads to

$$\int \frac{\epsilon_0 E_m^2}{2}\, dV = \int \frac{B_m^2}{2\mu_0}\, dV$$

or

$$\mu_0\epsilon_0 \int E_m^2\, dV = \int B_m^2\, dV.$$

The quantity $\int E_m^2\, dV$ can be written as $\overline{E_m^2}V$, where V is the cavity volume and $\overline{E_m^2}$ is the average value of E_m^2 throughout the cavity. Treating B_m in the same way leads to

$$\mu_0\epsilon_0\overline{E_m^2} = \overline{B_m^2}$$

or, taking square roots,

$$\sqrt{\overline{B_m^2}} = \sqrt{\mu_0\epsilon_0}\,\sqrt{\overline{E_m^2}}.$$

We can represent $\sqrt{\overline{B_m^2}}$ by B_{rms}, a so-called "root-mean-square" value of B_m. In computing B_{rms}, note that the averaging is done throughout the volume of the cavity, at the instant corresponding to Fig. 38-8*c*. It is not a time average for a particular point in the cavity. Doing the same for **E** yields

$$B_{rms} = \sqrt{\mu_0\epsilon_0}\, E_{rms} = \sqrt{(4\pi \times 10^{-7}\ \text{weber/amp-m})(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)}\, E_{rms}$$
$$= (3.3 \times 10^{-9}\ \text{sec/meter})E_{rms}.$$

If E_{rms} equals 10^4 volts/meter, a reasonable value, then

$$B_{rms} = (3.3 \times 10^{-9}\ \text{sec/meter})(10^4\ \text{volts/meter})$$

$$= 3.3 \times 10^{-5}\ \text{weber/meter}^2 = 0.33\ \text{gauss}.$$

What is the total stored energy in the cavity under these conditions, assuming the cavity to be 10 cm long and 3.0 cm in diameter?

The student will recall that in Example 6, Chapter 36, we showed the energy density for a magnetic field of "ordinary" laboratory magnitude (say, 1 weber/meter^2) to be enormously greater than that for an electric field of "ordinary" magnitude (say, 10^5 volts/meter). This fact is consistent with the present example. ◀

38-7 Induced Magnetic Fields

To understand the electromagnetic cavity oscillations in terms of electromagnetic theory, we must complete our description of the basic equations of electromagnetism by introducing a new concept, namely, that *a changing electric field produces a magnetic field.* This concept, which is the symmetrical counterpart of Faraday's law of induction, is of fundamental significance. We will develop this concept by a symmetry argument and will let the agreement with experiment of our final conclusions speak for itself. This comparison with experiment, which is worked out largely in Chapters 39 and 40, forms one of the chief experimental bases of electromagnetic theory. A central achievement was the demonstration that the experimentally measured speed c of visible light in free space could be related to purely electromagnetic quantities by

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \tag{38-18}$$

This demonstration not only revealed optics as a branch of electromagnetism but led directly to the concept of the electromagnetic spectrum, which in turn resulted in the discovery of radio waves.

Figure 38-9*a* shows a uniform electric field **E** filling a cylindrical region of space. It might be produced by a circular parallel-plate capacitor, as suggested in Fig. 38-9*b*. We assume that E is increasing at a steady rate dE/dt, which means that charge must be supplied to the capacitor plates at a steady rate; to supply this charge requires a steady current i into the positive plate and an equal steady current i out of the negative plate.

If a sufficiently delicate experiment could be performed, it would be found that *a magnetic field is set up by this changing electric field.* Figure 38-9*a* shows **B** for four arbitrary points. Figure 38-9 suggests a beautiful example of the symmetry of nature. A changing *magnetic* field induces an *electric* field (Faraday's law); now we see that a changing *electric* field induces a *magnetic* field.

To describe this new effect quantitatively, we are guided by analogy with Faraday's law of induction,

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}, \tag{38-19}$$

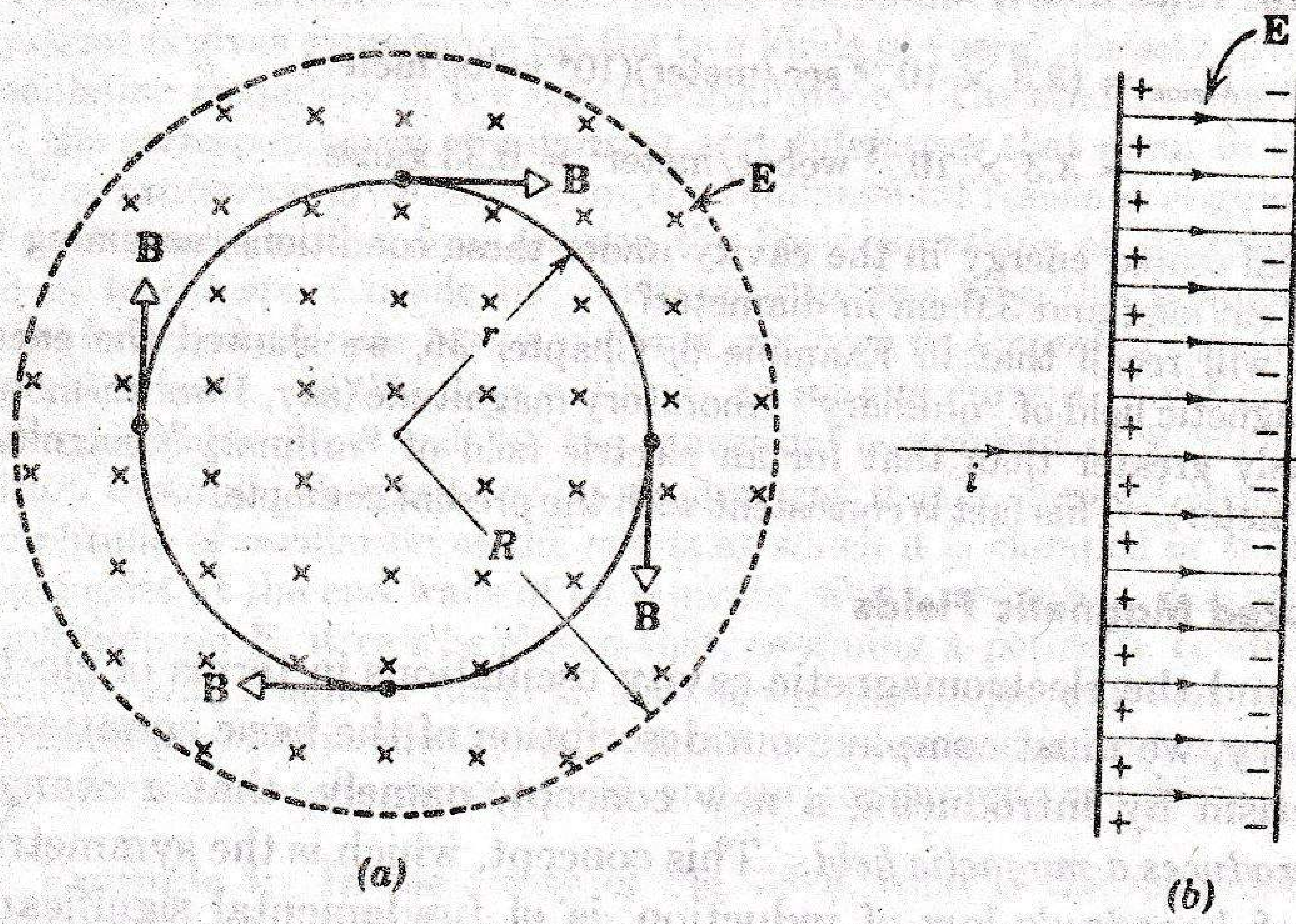

Fig. 38–9 (*a*) Showing the induced magnetic fields **B** at four points, produced by a changing electric field **E**. The electric field is increasing in magnitude. Compare Fig. 35–10. (*b*) Such a changing electric field may be produced by charging a parallel-plate capacitor as shown.

which asserts that an electric field (left term) is produced by a changing magnetic field (right term). For the symmetrical counterpart we might write *

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}. \tag{38-20}$$

Equation 38–20 asserts that a magnetic field (left term) can be produced by a changing electric field (right term). The student should compare carefully Fig. 35–10, which illustrates the production of an electric field by a changing magnetic field, with Fig. 38–9*a*. In each case the appropriate flux Φ_B or Φ_E is *increasing*. However, experiment shows that the lines of **E** in Fig. 35–10 are *counterclockwise*, whereas those of **B** in Fig. 38–9*a* are *clockwise*. This difference requires that the minus sign of Eq. 38–19 be omitted from Eq. 38–20.

In Section 34–1 we saw that a magnetic field can also be set up by a current in a wire. We described this quantitatively by Ampère's law:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i,$$

in which i is the conduction current passing through the loop around which the line integral is taken. Thus there are at least two ways of setting up

* Our system of units requires that we insert the constants ϵ_0 and μ_0 in Eq. 38–20. In some unit systems they would not appear.

a magnetic field: (*a*) by a changing electric field and (*b*) by a current. In general, both possibilities must be allowed for, or *

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i. \tag{38-21}$$

Maxwell is responsible for this important generalization of Ampère's law.

In Chapter 34 we assumed that no changing electric fields were present so that the term $d\Phi_E/dt$ in Eq. 38-21 is zero. In the discussion just given we assumed that there were no conduction currents in the space containing the electric field. Thus the term i in Eq. 38-21 is zero. We see now that each of these situations is a special case.

▶ **Example 5.** A parallel-plate capacitor with circular plates is being charged as in Fig. 38-9. (*a*) Derive an expression for the induced magnetic field at various radii r. Consider both $r < R$ and $r > R$.

From Eq. 38-20,

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt},$$

we can write, for $r \lesssim R$,

$$(B)(2\pi r) = \mu_0\epsilon_0 \frac{d}{dt}[(E)(\pi r^2)] = \mu_0\epsilon_0\pi r^2 \frac{dE}{dt}.$$

Solving for B yields $\quad B = \frac{1}{2}\mu_0\epsilon_0 r \frac{dE}{dt} \quad (r \lesssim R).$

For $r \gtrsim R$, Eq. 38-20 yields

$$(B)(2\pi r) = \mu_0\epsilon_0 \frac{d}{dt}[(E)(\pi R^2)] = \mu_0\epsilon_0\pi R^2 \left(\frac{dE}{dt}\right).$$

or $\quad B = \frac{\mu_0\epsilon_0 R^2}{2r}\frac{dE}{dt} \quad (r \gtrsim R).$

(*b*) Find B at $r = R$ for $dE/dt = 10^{12}$ volts/m-sec and for $R = 5.0$ m. At $r = R$ the two equations for B reduce to the same expression, or

$$\begin{aligned} B &= \tfrac{1}{2}\mu_0\epsilon_0 R \frac{dE}{dt} \\ &= (\tfrac{1}{2})(4\pi \times 10^{-7}\ \text{weber/amp-m})(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2) \\ &\quad (5.0 \times 10^{-2}\ \text{meter})(10^{12}\ \text{volts/m-sec}) \\ &= 2.8 \times 10^{-7}\ \text{weber/meter}^2 = 0.0028\ \text{gauss}. \end{aligned}$$

This shows that the induced magnetic fields in this example are so small that they can scarcely be measured with simple apparatus, in sharp contrast to induced *electric* fields (Faraday's law), which can be demonstrated easily. This experimental dif-

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* Actually, there is a third way of setting up a magnetic field, by the use of magnetized bodies. In Section 37-7 we saw that this could be accounted for by inserting a *magnetizing current* term i_M on the right side of Ampère's law. This law would then read, in its full generality,

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i + \mu_0 i_M.$$

In all that follows we assume that no magnetic materials are present so that $i_M = 0$.

ference is in part due to the fact that induced emfs can easily be multiplied by using a coil of many turns. No technique of comparable efficiency exists for magnetic fields. In experiments involving oscillations at very high frequencies dE/dt above can be very large, resulting in significantly larger values of the induced magnetic field. ◀

38–8 Displacement Current

Equation 38–21 shows that the term $\epsilon_0\, d\Phi_E/dt$ has the dimensions of a current. Even though no motion of charge is involved, there are advantages in giving this term the name *displacement* * *current*. Thus we can say that a magnetic field can be set up either by a conduction current i or by a displacement current i_d $(= \epsilon_0\, d\Phi_E/dt)$, and Eq. 38–21 can be rewritten as †

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0(i_d + i). \tag{38–22}$$

The concept of displacement current permits us to retain the notion that *current is continuous*, a principle established for steady conduction currents in Section 31–1. In Fig. 38–9*b*, for example, a current i enters the positive plate and leaves the negative plate. The *conduction* current is *not* continuous across the capacitor gap because no charge is transported across this gap. However, the displacement current i_d in the gap will prove to be exactly i, thus retaining the concept of the continuity of current.

To calculate the displacement current, recall (see Eq. 30–5) that E in the gap is given by

$$E = \frac{q}{\epsilon_0 A}.$$

Differentiating gives $$\frac{dE}{dt} = \frac{1}{\epsilon_0 A}\frac{dq}{dt} = \frac{1}{\epsilon_0 A}\, i.$$

The displacement current i_d is by definition

$$i_d = \epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0 \frac{d(EA)}{dt} = \epsilon_0 A \frac{dE}{dt}.$$

Combining these two equations leads to

$$i_d = (\epsilon_0 A)\left(\frac{1}{\epsilon_0 A}\, i\right) = i,$$

which shows that the displacement current in the gap is identical with the conduction current in the lead wires.

---

* The word "displacement" was introduced for historical reasons that need not concern us here.

† We may write this more generally, taking the presence of magnetic materials into account, as

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0(i_d + i + i_M).$$

See the footnote on p. 961.

▶ **Example 6.** What is the displacement current for the capacitor of Example 5? From the definition of displacement current,

$$i_d = \epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0 \frac{d}{dt}[(E)(\pi R^2)] = \epsilon_0 \pi R^2 \frac{dE}{dt}$$

$$= (8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(\pi)(5.0 \times 10^{-2}\ \text{meter})^2(10^{12}\ \text{volts/m-sec})$$

$$= 0.070\ \text{amp}.$$

Even though this displacement current is reasonably large, it produces only a small magnetic field (see Example 5) because it is spread out over a large area. ◀

38-9 Maxwell's Equations

Equation 38-21 completes our presentation of the basic equations of electromagnetism, called *Maxwell's equations.* They are summarized in Table 38-3. All equations of physics that serve, as these do, to correlate experiments in a vast area and to predict new results have a certain beauty about them and can be appreciated, by those who understand them, on an aesthetic level. This is true for Newton's laws of motion, for the laws of thermodynamics, for the theory of relativity, and for the theories of quantum physics. As for Maxwell's equations, the German physicist Ludwig Boltzmann (quoting a line from Goethe) wrote "Was it a god who wrote these lines. . . ." In more recent times J. R. Pierce,* in a book chapter entitled "Maxwell's Wonderful Equations" says: "To anyone who is motivated by anything beyond the most narrowly practical, it is worth while to understand Maxwell's equations simply for the good of his soul." The scope of these equations is remarkable, including as it does the fundamental operating principles of all large-scale electromagnetic devices such as motors, cyclotrons, electronic computers, television, and microwave radar.

38-10 Maxwell's Equations and Cavity Oscillations

In this section we show how the oscillations of an electromagnetic cavity can be understood in terms of Maxwell's equations. A completely formal treatment, which is beyond our scope here, would start from these equations and would end with mathematical expressions for the variation of **B** and **E** with time and with position in the cavity for all modes of oscillation of the cavity. We confine ourselves to the fundamental mode only, illustrated in Fig. 38-8, for which we *postulated* the variations of **B** and **E** given in that figure; we will show that these postulated fields are completely consistent with Maxwell's equations.

Figure 38-10 presents two views of the cavity of Fig. 38-8*d*, in which both electric and magnetic fields are present. Study of Fig. 38-8 reveals that **B** is *decreasing* in magnitude and **E** is *increasing* at this phase of the cycle of oscillation. Let us apply Faraday's law,

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt},$$

* *Electrons, Waves and Messages*, Hanover House, 1956. This book is recommended as collateral reading in electromagnetism.

Table 38–3

THE BASIC EQUATIONS OF ELECTROMAGNETISM (MAXWELL'S EQUATIONS) *

Name	Equation	Describes	Crucial Experiment	Text Reference
Gauss's law for electricity	$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = q$	Charge and the electric field	1. Like charges repel and unlike charges attract, as the inverse square of their separation. 1′. A charge on an insulated conductor moves to its outer surface.	Chapter 28
Gauss's law for magnetism	$\oint \mathbf{B} \cdot d\mathbf{S} = 0$	The magnetic field	2. It is impossible to create an isolated magnetic pole.	Section 37–2
Ampère's law (as extended by Maxwell)	$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left(\epsilon_0 \frac{d\Phi_E}{dt} + i \right)$	The magnetic effect of a changing electric field or of a current	3. The speed of light can be calculated from purely electromagnetic measurements. 3′. A current in a wire sets up a magnetic field near the wire.	Section 39–5 Chapter 34
Faraday's law of induction	$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}$	The electrical effect of a changing magnetic field	4. A bar magnet, thrust through a closed loop of wire, will set up a current in the loop.	Chapter 35

* Written on the assumption that no dielectric or magnetic material is present.

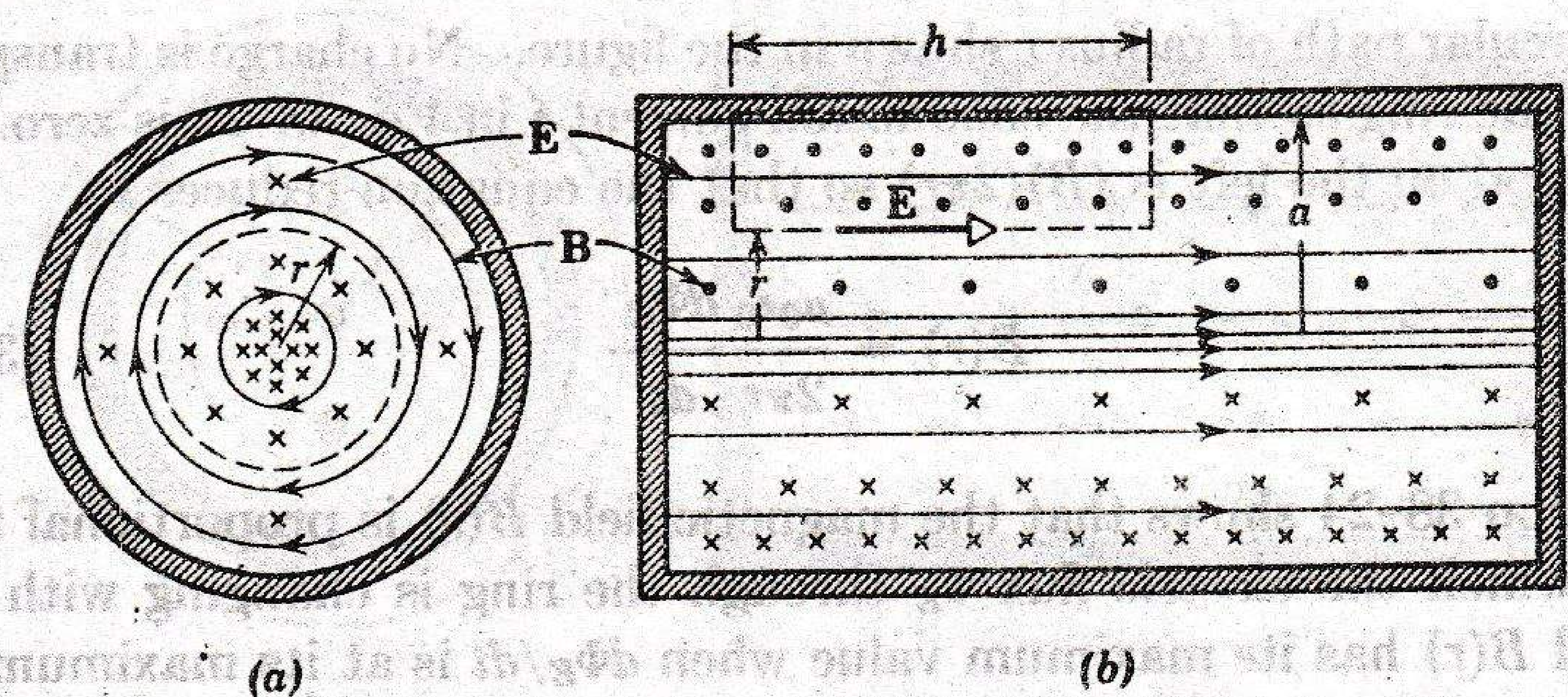


Fig. 38–10 Two cross sections of an electromagnetic resonant cavity at a phase of oscillation corresponding to Fig. 38–8*d*. (*a*) The dashed circle is a path suitable for applying Ampère's law. (*b*) The dashed rectangle is a path suitable for applying Faraday's law.

to the rectangle of dimensions h and $a - r$. There is a definite flux Φ_B through the rectangular path in question, and this flux is decreasing with time because **B** is decreasing. The line integral above is

$$\oint \mathbf{E} \cdot d\mathbf{l} = hE(r),$$

in which $E(r)$ is the value of E at a radius r from the center of the cavity. Note that **E** equals zero for the upper leg of the integration path (which lies in the cavity wall) and that **E** and $d\mathbf{l}$ are at right angles on the two side legs. Combining these equations yields

$$E(r) = -\frac{1}{h}\frac{d\Phi_B}{dt}. \qquad (38\text{–}23)$$

Equation 38–23 shows that $E(r)$ depends on the rate at which Φ_B through the path shown is changing with time and that it has its maximum value when $d\Phi_B/dt$ is a maximum. This occurs when **B** is zero, that is, when **B** is changing its direction; the student will recall that a sine or cosine is changing most rapidly, that is, it has the steepest tangent, at the instant it crosses the axis between positive and negative values. Thus the electric field pattern in the cavity will have its *maximum* value when the magnetic field is *zero* everywhere, consistent with Figs. 38–8*a* and *e* and with the concept of the interchange of energy between electric and magnetic forms. The student should demonstrate, by applying Lenz's law, that the electric field in Fig. 38–10*b* points to the right as shown, if the magnetic field is decreasing.

Figure 38–10*a* shows an end view of the cavity; the electric lines of force are entering the page at right angles to the page and the magnetic lines form clockwise circles. Let us apply Ampère's law in the form

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i, \qquad (38\text{–}21)$$

to the circular path of radius r shown in the figure. No charge is transported through the ring so that the conduction current i in Eq. 38–21 is zero. The line integral on the left is $(B)(2\pi r)$ so that the equation reduces to

$$B(r) = \frac{\mu_0 \epsilon_0}{2\pi r} \frac{d\Phi_E}{dt}. \tag{38-24}$$

Equation 38–24 shows that the magnetic field $B(r)$ is proportional to the rate at which the electric flux Φ_E through the ring is changing with time. The field $B(r)$ has its maximum value when $d\Phi_E/dt$ is at its maximum; this occurs when $\mathbf{E} = 0$, that is, when $\mathbf{E}$ is reversing its direction. Thus we see that $\mathbf{B}$ has its *maximum* value when $\mathbf{E}$ is *zero* for all points in the cavity. This is consistent with Figs. 38–8*c* and *g* and with the concept of the interchange of energy between electric and magnetic forms. A comparison with Fig. 38–9*a*, which, like Fig. 38–10*a*, corresponds to an increasing electric field, shows that the lines of $\mathbf{B}$ are indeed clockwise, as viewed along the direction of the electric field.

Comparison of Eqs. 38–23 and 38–24 suggests the complete interdependence of $\mathbf{B}$ and $\mathbf{E}$ in the cavity. As the magnetic field changes with time, it induces the electric field in a way described by Faraday's law. The electric field, which also changes with time, induces the magnetic field in a way described by Maxwell's extension of Ampère's law. The oscillations, once established, sustain each other and would continue indefinitely were it not for losses due to Joule heating in the cavity walls or leakage of energy from openings that might be present in the walls. In Chapter 39 we show that this interplay of $\mathbf{B}$ and $\mathbf{E}$ occurs not only in standing electromagnetic waves in cavities but also in traveling electromagnetic waves, such as radio waves or visible light.

Let us now analyze the currents—both conduction and displacement—that occur in the cavity and examine their connections to the electric and the magnetic fields. Figure 38–11 shows two views of the cavity, taken at an instant corresponding to that of Fig. 38–10. For simplicity the fields $\mathbf{E}$ and $\mathbf{B}$ are not shown; the arrows represent currents.

Since $\mathbf{E}$ is increasing at this instant, the positive charge on the left end cap must be increasing. Thus there must be conduction currents in the walls pointing from right to left in Fig. 38–11*b*. These currents are also shown by the dots (representing the tips of arrows) near the cavity walls in Fig. 38–11*a*.

Bearing in mind that $\epsilon_0\, d\Phi_E/dt$ is a displacement current, we can write Eq. 38–24 as

$$B(r) = \frac{\mu_0}{2\pi r} \epsilon_0 \frac{d\Phi_E}{dt} = \frac{\mu_0}{2\pi r} i_d.$$

This equation stresses that $\mathbf{B}$ in the cavity is associated with a displacement current; compare Eq. 34–4. Applying the right-hand rule in Fig. 38–10*a* shows that the displacement current i_d must be directed into the plane of the figure if it is to be associated with the clockwise lines of $\mathbf{B}$ that are present.

The displacement current is represented in Fig. 38–11*b* by the arrows that point to the right and in Fig. 38–11*a* by the crosses that represent arrows entering the page. Study of Fig. 38–11 shows that the current is continuous, being directed up the

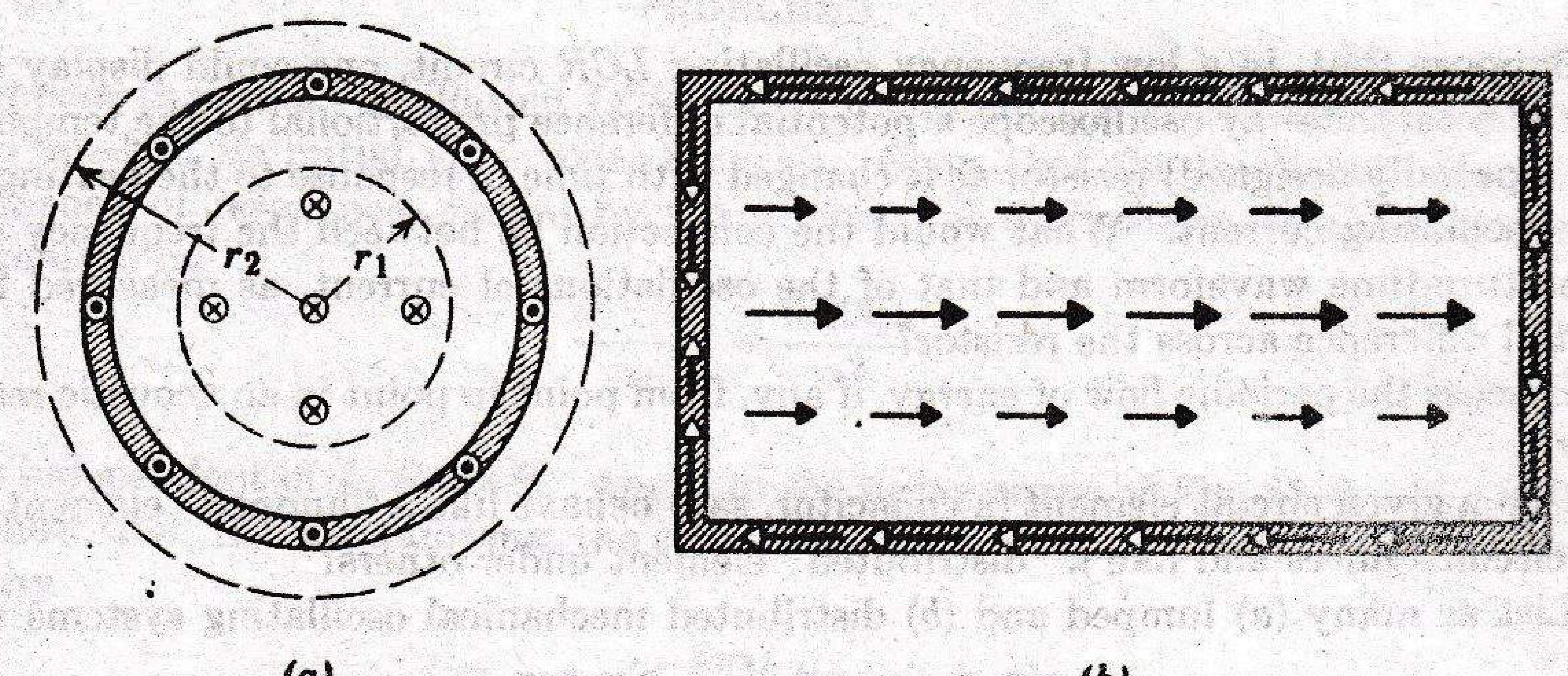


Fig. 38–11 The cavity of Fig. 38–10 showing (*a*) the conduction current coming up the walls and displacement current going down the cavity volume and (*b*) the displacement current (black arrowheads) in the volume of the cavity and the conduction currents (white arrowheads) at the walls. The arrows in each case represent *current densities*. Note the continuity of current (conduction + displacement), that is, it is possible to form closed current loops. (For a *truly* resistanceless cavity the conduction current would lie *entirely* on the surface, requiring a modification of our definition of current density; we leave this small complication for the student to consider.)

walls as a conduction current and then back down through the volume of the cavity as a displacement current. If we apply Ampère's law as extended by Maxwell,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(i_d + i), \qquad (38\text{–}25)$$

to the circular path of radius r_1 in Fig. 38–11*a*, we see that **B** at that path is due entirely to the displacement current, the conduction current *i within the path* being zero.

For the path of radius r_2, the *net* current enclosed is zero because the conduction current in the walls is exactly equal and opposite to the displacement current in the cavity volume. Since i equals i_d in magnitude, but is oppositely directed, it follows from Eq. 38–25 that B must be zero for all points outside the cavity, in agreement with observation.

QUESTIONS

1. Why doesn't the LC circuit of Fig. 38–1 simply stop oscillating when the capacitor has been completely discharged?

2. How might you start an LC circuit into oscillation with its initial condition being represented by Fig. 38–1*c*? Devise a switching scheme to bring this about.

3. In an oscillating LC circuit, assumed resistanceless, what determines (*a*) the frequency and (*b*) the amplitude of the oscillations?

4. Tabulate as many mechanical or electric systems as you can think of that possess a natural frequency, along with the formula for that frequency if given in the text.

5. Resonance in LC circuits, judged by Eq. 38–14 and Fig. 38–6, occurs when the frequency ω'' of the "driving force" is exactly equal to the (undamped) natural frequency of the LC circuit. In Section 15–10 we saw that resonance for mass-spring systems, judged by Eq. 15–41 and Fig. 15–19, occurs when ω'' is close to, *but not exactly equal to*, the (undamped) natural frequency of the mass-spring system. Is there a failure of the principle of correspondence here?

6. Suppose that, in a low frequency oscillating LCR circuit, one could display on the screen of a cathode-ray oscilloscope a potential difference proportional to the temperature of the (specially designed) resistor as it changed with time in response to the heating effect of the oscillating current. What would the connection be between the frequency of this temperature-time waveform and that of the oscillations of current, as measured by the potential difference across the resistor?

7. Discuss the periodic flow of energy, if any, from point to point in an acoustic resonant cavity.

8. Can a given circuit element (a capacitor, say) behave like a "lumped" element under some circumstances and like a "distributed" element under others?

9. List as many (*a*) lumped and (*b*) distributed mechanical oscillating systems as you can.

10. An air-filled acoustic resonant cavity and an electromagnetic resonant cavity of the same size have resonant frequencies that are in the ratio of 10^6 or so. Which has the higher frequency and why?

11. What constructional difficulties would you encounter if you tried to build an LC circuit of the type shown in Fig. 38–1 to oscillate (*a*) at 0.01 cps, or (*b*) at 10^{10} cycles/sec?

12. Could an electromagnetic cavity of the type shown in Fig. 38–8 be designed to operate at 60 cycles/sec? If so, give some details of its construction.

13. The "sharpness of tuning" of a copper electromagnetic cavity can be considerably increased by immersing it in liquid air. Explain. Be guided by Fig. 38–6, which shows "tuning curves" for an LC circuit.

14. Electromagnetic cavities are often silver plated on the inside. Explain.

15. Why is Faraday's law of induction more familiar than its symmetrical counterpart, Eq. 38–20?

16. Why is the quantity $\epsilon_0\, d\Phi_E/dt$ referred to as a (displacement) *current*?

17. In Fig. 38–1*c* a displacement current is needed to maintain continuity of current in the capacitor. How can one exist, since there is no charge on the capacitor?

18. Discuss the symmetries that appear between (*a*) the first two and (*b*) the second two of Maxwell's equations.

19. At what parts of the cycle will (*a*) the conduction current and (*b*) the displacement current in the cavity of Fig. 38–8 be zero?

20. Discuss the time variation during one complete cycle of the charges that appear at various points on the inner walls of the oscillating electromagnetic cavity of Fig. 38–8.

PROBLEMS

1. You are given a 10-mh inductor and two capacitors, of 5.0- and 2.0-μf capacitance. What resonant frequencies can be obtained by connecting these elements in various ways?

2. Given a 1.0-mh inductor, how would you make it oscillate at 1.0×10^6 cycles/sec?

3. An inductor is connected across a capacitor whose capacitance can be varied by turning a knob. We wish to make the frequency of the LC oscillations vary linearly with the angle of rotation of the knob, going from 2×10^5 cycles/sec to 4×10^5 cycles/sec as the knob turns through 180°. If $L = 1.0$ mh, plot C as a function of angle for the 180° rotation.

4. A 10-henry coil has a resistance of 180 ohms. What size of capacitor must be put in series with it if the combination is to "resonate" when connected to a 60-cycle/sec outlet?

5. Derive the differential equation for an LC circuit (Eq. 38–5) using the loop theorem.

6. Derive Eq. 38–12, the differential equation for forced oscillations in an LCR circuit, from the principle of conservation of energy.

7. Show that a damped LC circuit (see Example 3) loses half its energy to Joule heat in a time given approximately by $0.69\tau_L$, in which τ_L is the inductive time constant.

8. A circuit has $L = 10$ mh and $C = 1.0\ \mu$f. How much resistance must be inserted in the circuit to reduce the (undamped) resonant frequency by 0.01%?

9. Suppose that in an oscillating LCR circuit the amplitude of the charge oscillations drops to one-half its initial value after n cycles. Show that the fractional reduction in the frequency of resonance, caused by the presence of the resistor, is given to a close approximation by

$$\frac{\omega - \omega'}{\omega} = \frac{0.0061}{n^2},$$

which is independent of L, C, or R. Apply to the decay curve of Fig. 38–3.

10. Show that, for low damping, the current in a damped LC circuit is given approximately by

$$i = -q_m\omega' e^{-Rt/2L} \sin(\omega' t + \phi),$$

in which

$$\phi = \tan^{-1} \frac{R}{2L\omega'}.$$

Start from Eq. 38–10.

11. "Q" *for a circuit.* In the damped LC circuit of Example 3 show that the fraction of the energy lost per cycle of oscillation, $\Delta U/U$, is given to a close approximation by $2\pi R/\omega L$. The quantity $\omega L/R$ is often called the "Q" of the circuit (for "quality"). A "high-Q" circuit has low resistance, low fractional energy loss per cycle ($= 2\pi/Q$) and (see Fig. 38–6) a sharp resonance or "tuning" curve.

12. Show that the amplitude of the charge oscillations in an oscillating LCR circuit is given by

$$q_m = \frac{\mathcal{E}_m}{\sqrt{\left(\omega''^2 L - \frac{1}{C}\right)^2 + (\omega'' R)^2}}.$$

For what value of ω'' will q_m be a maximum?

13. An LCR circuit has $L = 1.0$ henry, $C = 20\ \mu$f, and $R = 20$ ohms. For what frequency ω'' of an applied emf will it resonate with maximum response? At what frequencies will the response be one-half its maximum value?

14. Show that the fractional half-width of the resonance curves of Fig. 38–6 is given, to a close approximation, by

$$\frac{\Delta\omega}{\omega} = \frac{\sqrt{3}R}{\omega L},$$

in which ω is the resonant frequency and $\Delta\omega$ is the width of the resonance peak at $i = \frac{1}{2}i_m$. Note (see Problem 11) that this expression may be written as $\sqrt{3}/Q$ which shows clearly that a "high-Q" circuit has a sharp resonance peak, that is, a small $\Delta\omega/\omega$.

15. A resistor-inductor-capacitor combination R_1, L_1, C_1 (connected in series) exhibits resonance at the same frequency as a second combination R_2, L_2, C_2. If the two combinations are now connected in series, at what frequency would the whole circuit resonate?

16. In Example 5 show that the *displacement current density* j_d is given, for $r < R$, by

$$j_d = \epsilon_0 \frac{dE}{dt}.$$

17. Prove that the displacement current in a parallel-plate capacitor can be written as

$$i_d = C\frac{dV}{dt}.$$

18. You are given a 1.0-μf capacitor. How would you establish an (instantaneous) displacement current of 1.0 amp in the space between its plates?

19. In Example 5 how does the displacement current through a concentric circular loop of radius r vary with r? Consider both $r < R$ and $r > R$.

20. In microscopic terms the principle of continuity of current may be expressed as

$$\oint (\mathbf{j} + \mathbf{j}_d) \cdot d\mathbf{S} = 0,$$

in which $\mathbf{j}$ is the conduction current density and $\mathbf{j}_d$ is the displacement current density. The integral is to be taken over any closed surface; the equation essentially says that whatever current flows into the enclosed volume must also flow out. (*a*) Apply this equation to the surface shown by the dashed lines in Fig. 38-12 shortly after switch S is closed. (*b*) Apply it to various surfaces that may be drawn in the cavity of Fig. 38-11, including some that cut the cavity walls.

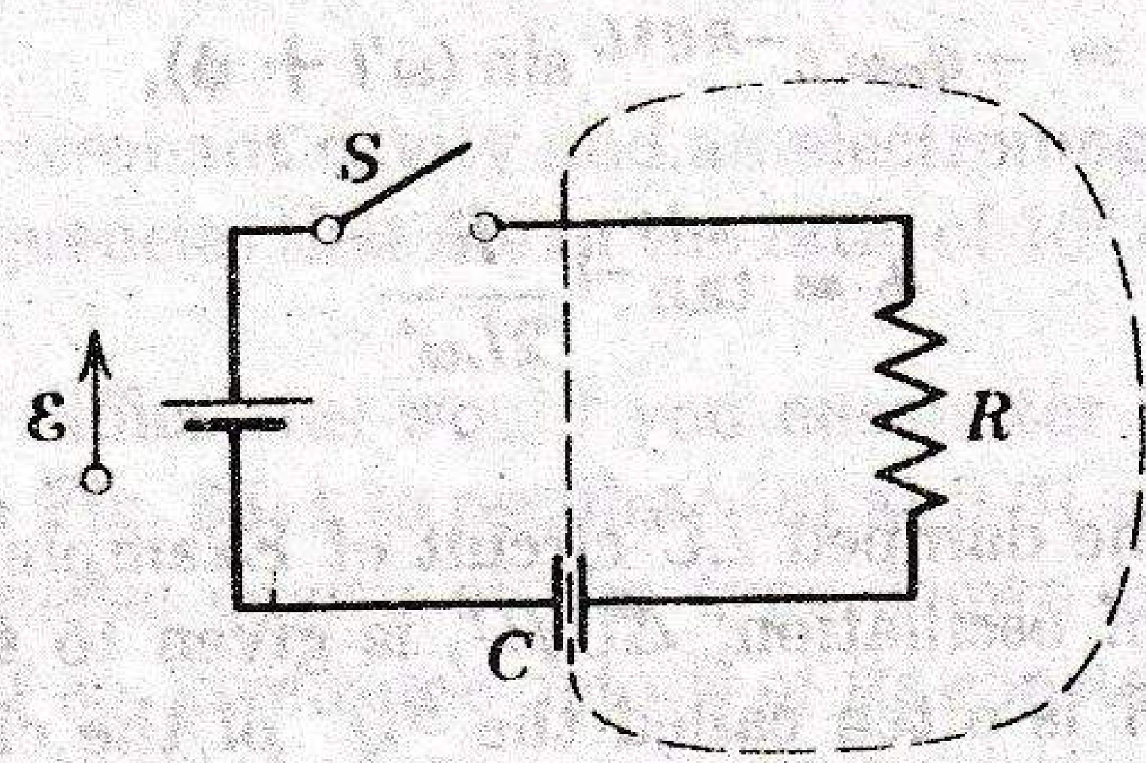


Fig. 38-12

21. A cylindrical electromagnetic cavity 5.0 cm in diameter and 7.0 cm long is oscillating in the mode shown in Fig. 38-8. (*a*) Assume that, for points on the axis of the cavity, $E_m = 10^4$ volts/meter. For such axial points what is the maximum rate $(dE/dt)_m$ at which E changes? (*b*) Assume that the average value of $(dE/dt)_m$, for all points over a cross section of a cavity, is about one-half the value found above for axial points. On this assumption, what is the maximum value of B at the cylindrical surface of the cavity?

22. Collect and tabulate expressions for the following four quantities, considering both $r < R$ and $r > R$. Copy down the derivations side by side and study them as interesting applications of Maxwell's equations to problems having cylindrical symmetry.

(*a*) $B(r)$ for a current i in a long wire of radius R (see Section 34-2).

(*b*) $E(r)$ for a long uniform cylinder of charge of radius R (see Section 28-6; also Problem 14, Chapter 28).

(*c*) $B(r)$ for a parallel-plate capacitor, with circular plates of radius R, in which E is changing at a constant rate (see Section 38-7).

(*d*) $E(r)$ for a cylindrical region of radius R in which a uniform magnetic field B is changing at a constant rate (see Section 35-5).

Electromagnetic Waves *

CHAPTER 39

39-1 Transmission Line

In Chapter 38 we studied electromagnetic energy confined, as a standing wave, to a restricted region of space, the interior of an electromagnetic resonant cavity. Such energy can also be transferred from place to place as a traveling wave. An arrangement of conductors for facilitating such transfers is called a *transmission line*. Figure 39-1 shows one type of line, a *coaxial cable*, its input end being connected to a switch S. For the time being we assume that the cable is infinitely long and that the cable elements have zero resistance.

When S is closed on b, the central and the outer conductors are at the same potential. If the switch is then thrown to a, a potential difference V suddenly appears between these elements. This potential difference does not appear instantaneously all along the line but is propagated with a finite

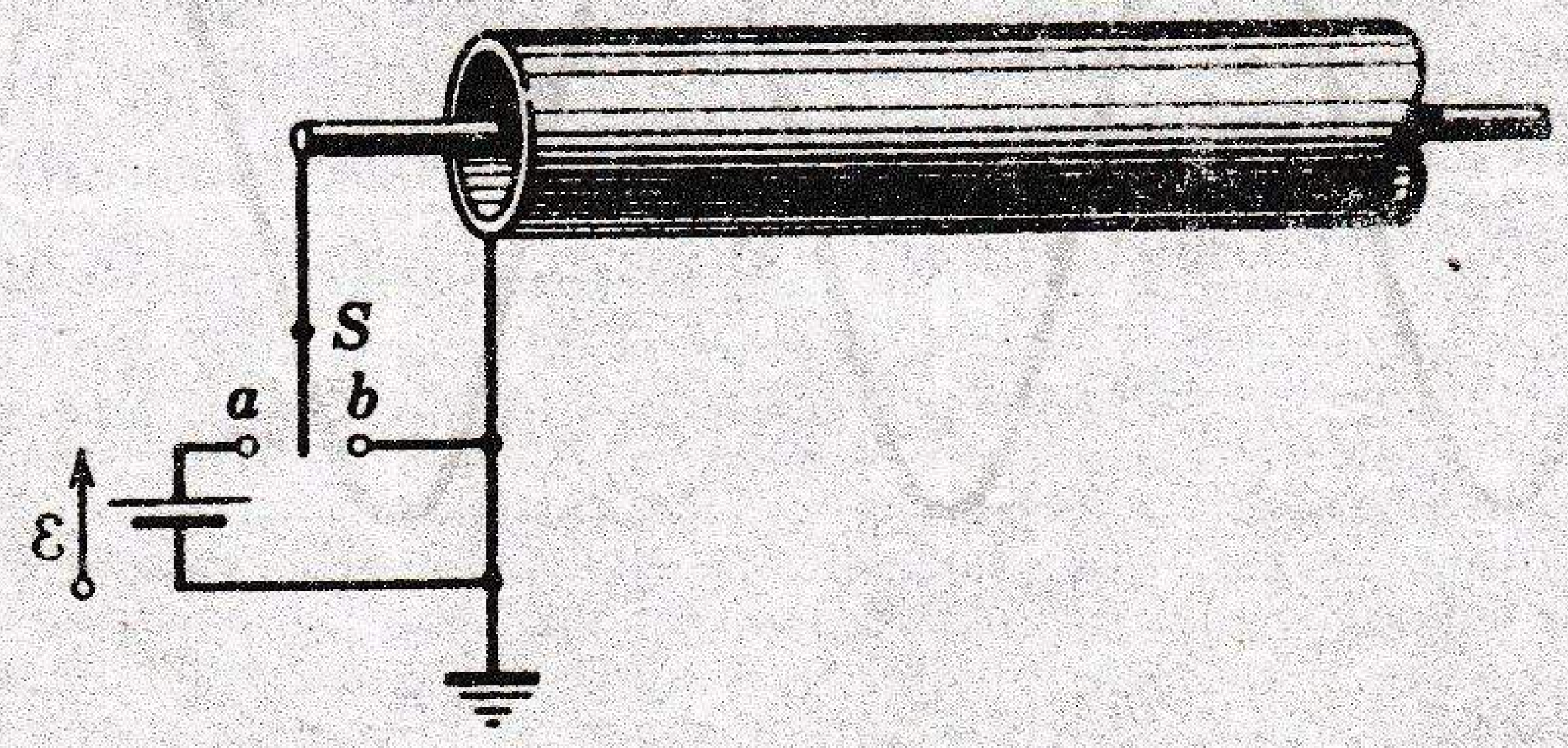


Fig. 39-1 An electromagnetic pulse can be sent along the coaxial cable by throwing switch S from b to a.

* This subject is treated in greater depth in Supplementary Topic V at the end of the book.

speed c that will turn out to be exactly that of light, assuming a resistanceless line. Figure 39–2a shows that the potential difference between the conductors at a distance l along the line suddenly rises, at a time given by $t = l/c$, from zero to a value determined by the battery emf. We can also consider the variation of V with position x along the line at a given time t_1 after closing the switch. Figure 39–2b shows such an instantaneous "snapshot." It, too, suggests a traveling "wavefront" moving along the line at speed c. At $t = t_1$ the signal has not yet reached points where $x > ct_1$.

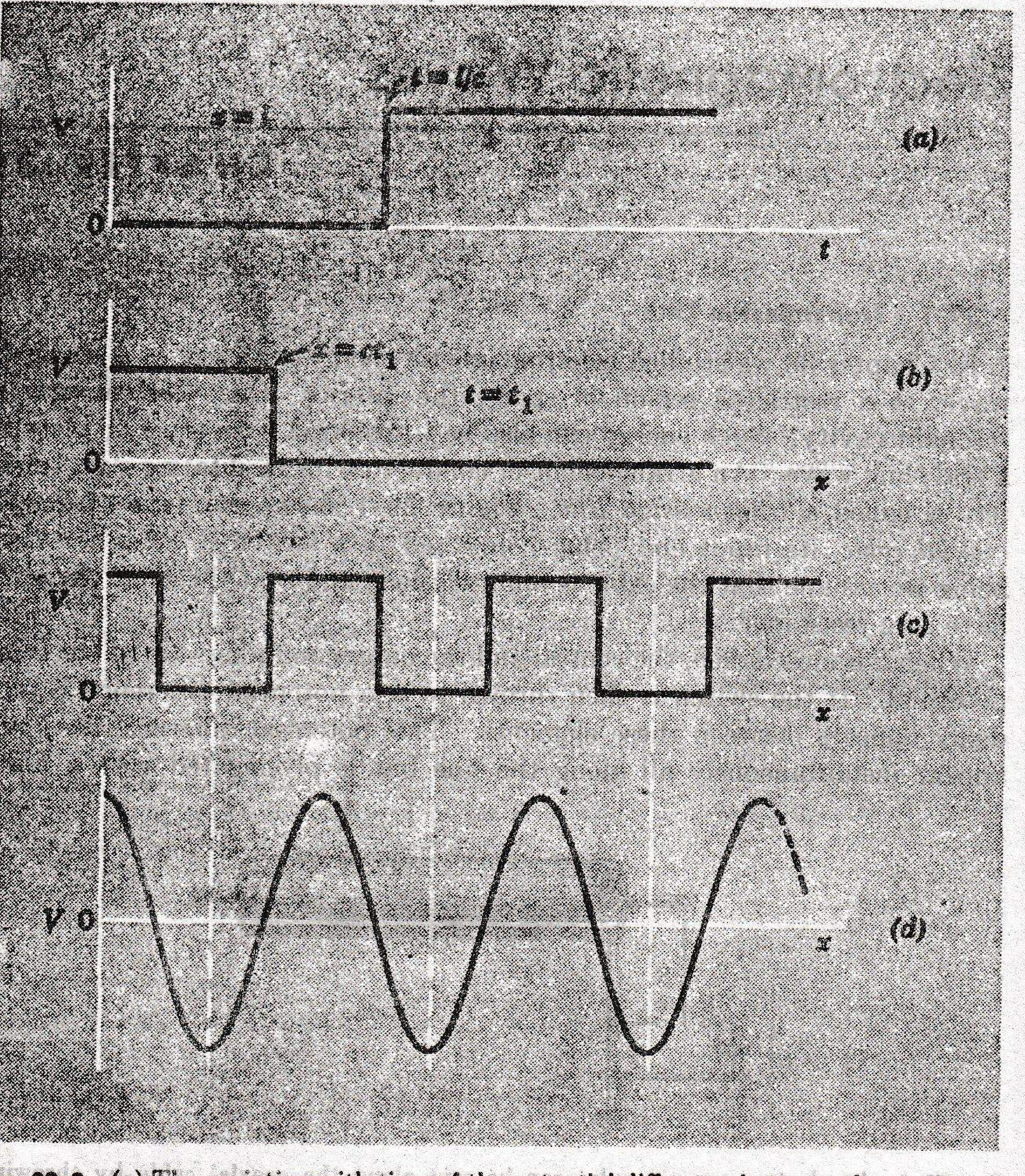


Fig. 39–2 (a) The variation with time of the potential difference between the conductors of a coaxial cable at a distance l from the input end. (b) An instantaneous "snapshot" of the pulse in the cable at a certain time t_1. (c) The waveform if switch S in Fig. 39–1 is periodically thrown between a and b. (d) The waveform if switch S is replaced by an electromagnetic oscillator with a sinusoidal output.

If switch S is periodically thrown from b to a and back again, a wave disturbance like that of Fig. 39-2c is propagated. This suggests that if the battery and switch arrangement is replaced by an electromagnetic oscillator with a sinusoidal output of frequency f a wave like that of Fig. 39-2d will be propagated.

A traveling wave in a resistanceless transmission line will exhibit a wavelength λ given by

$$\lambda = \frac{c}{\nu}.$$

If the oscillator frequency is 60 cycles/sec, the common commercial power frequency, the wavelength is 5×10^6 meters, which is about 3000 miles. At this low frequency traveling waves are not apparent in any line of normal length. By the time the polarity of the oscillator has changed appreciably, the energy fed into the line at the oscillator end has been delivered to the load.

Frequencies in the radio or the microwave range are much higher and the wavelengths correspondingly smaller. Commercial television frequencies as established by the Federal Communications Commission range from 54×10^6 to 980×10^6 cycles/sec. In terms of wavelength this is a range of 5.6 to 0.31 meter. At these wavelengths the patterns of potential difference in the transmission lines used to send television signals across the country can be described aptly as traveling waves. Microwaves, used in radar systems and for communication purposes, have even smaller wavelengths, in the range of about 20 cm to about 0.5 mm.

These considerations suggest another way of viewing the difference between lumped and distributed circuit elements. A system is "distributed" if the wavelength is about the same size as, or less than, the dimensions of the system. If the wavelength is much larger than the dimensions of the system, we are dealing with lumped components. A transmission line 50 meters long would be a lumped system for electromagnetic radiation at 60 cycles/sec ($\lambda = 5 \times 10^6$ meters) but a distributed system at 10^8 cycles/sec ($\lambda = 3$ meters). In a lumped system the circuit analysis is normally carried out in terms of lumped system parameters such as L, C, and R; in a distributed system the analysis is often carried out in terms of the fields that are set up and the charges and currents that are related to them.

▶ **Example 1.** A potential difference given by

$$V_0 = V_m \sin \omega t$$

is applied between the terminals of a long resistanceless transmission line; the frequency ν $(= \omega/2\pi)$ is 3×10^9 cycles/sec. Write an equation for $V(t)$ at a point which is 1.5 wavelengths down the line from the oscillator.

The general equation for a wave traveling in the x direction (see Eq. 19-10) is

$$V = V_m \sin (\omega t - kx),$$

where $k\ (= 2\pi/\lambda)$ is the wave number. At $x = 0$ this gives correctly the time variation of the input terminal potential difference. At $x = 1.5\lambda$ we have

$$V_P = V_m \sin\left[\omega t - \left(\frac{2\pi}{\lambda}\right)(1.5\lambda)\right] = V_m \sin(\omega t - 3\pi)$$
$$= -V_m \sin \omega t.$$

Thus V_P is always equal in magnitude to V_0 but is opposite in sign. What is the wavelength in this example? ◀

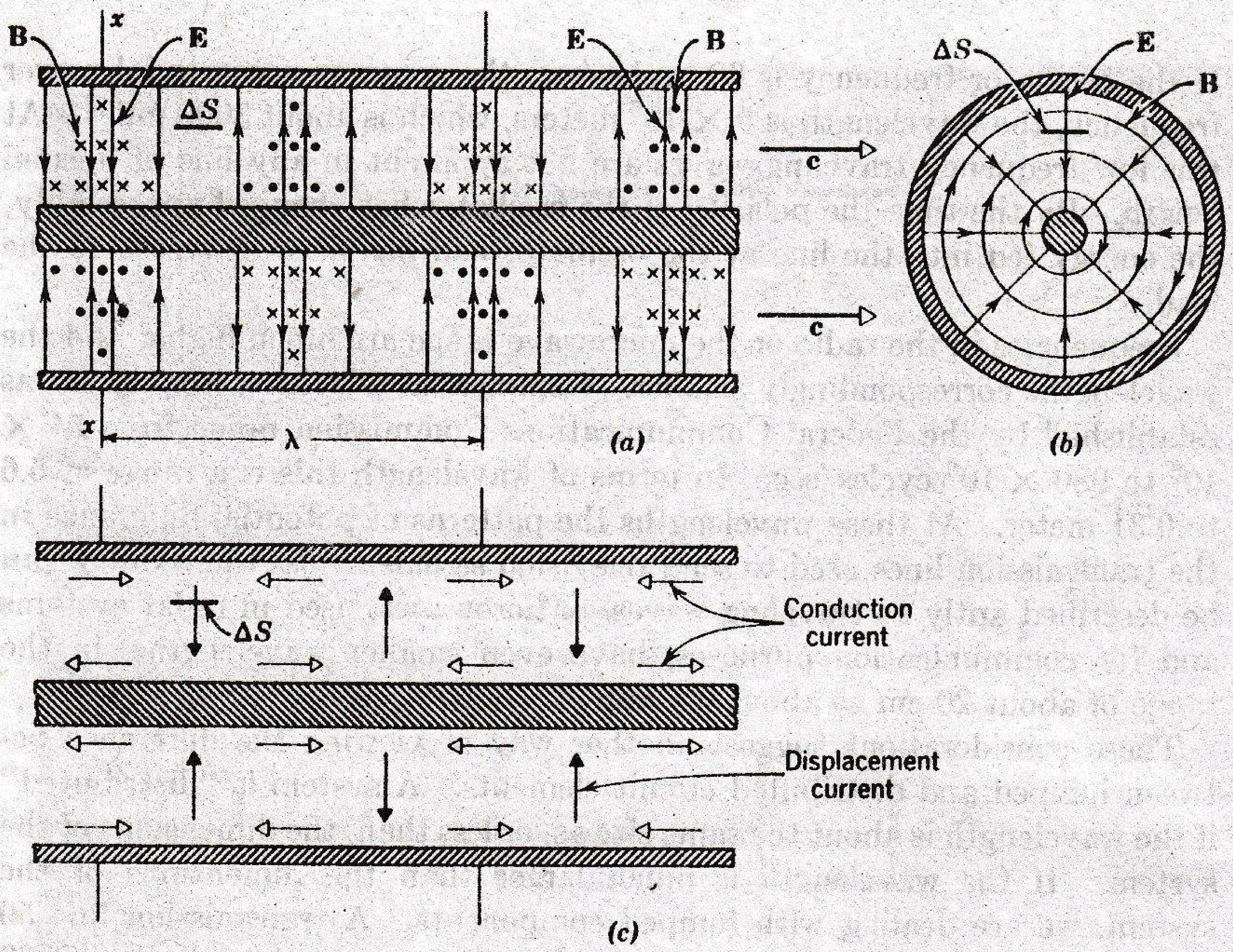


Fig. 39–3 (*a*) The electric and magnetic fields in a coaxial cable, showing a wave traveling to the right at speed *c*. (*b*) A cross-sectional view at a plane through *xx* in (*a*); the wave is emerging from the page. (*c*) Conduction current (open arrows) and displacement currents (filled arrows) associated with the wave in (*a*); the arrows in each case represent current density vectors.

39–2 Coaxial Cable—Fields and Currents

Figures 39–3*a* and *b* are "snapshots" of the electric and magnetic field configurations in a coaxial cable. The electric field is radial and the magnetic field forms concentric lines about the central conductor. The entire pattern moves along the line, assumed resistanceless, at speed *c*.

The field patterns in this figure obey the *boundary condition* required for a line that is assumed to be resistanceless, namely, that **E** for all points on either conducting surface has no tangential component (see p. 957). The field patterns can be deduced mathematically from Maxwell's equations by imposing this requirement. The configuration shown is the simplest of many different wave patterns that can travel along the line. The coaxial cable, unlike the electromagnetic cavity of Fig. 38-8, is not a resonant device. The angular frequency ω of waves that travel along it can be varied continuously, as is the case for all traveling waves, such as transverse waves in a long stretched cord.

Figure 39-3c shows the currents in the cable at the instant corresponding to Figs. 39-3a and b. The arrows parallel to the cable axis represent conduction currents in the central and the outer conductors. The vertical arrows with filled heads represent displacement currents that exist in the space between the conductors. Note that the conduction current and the displacement current arrows form closed loops, preserving the concept of the continuity of current.

▶ **Example 2.** Verify that the displacement current represented in Fig. 39-3c is consistent with the pattern of **B** and **E** shown in Fig. 39-3a.

Consider a small surface element ΔS shown edge-on in Fig. 39-3; it is shown as viewed from above in Fig. 39-4. This hypothetical element is stationary with respect to the cable while the field configuration moves through it at speed c. Figure 39-4a shows the electric lines of force in the vicinity of this element. It is clear from symmetry that, at the instant shown, the net flux Φ_E through this area is zero. However, even though Φ_E is zero in magnitude, it is, at this instant, *changing at its most rapid*

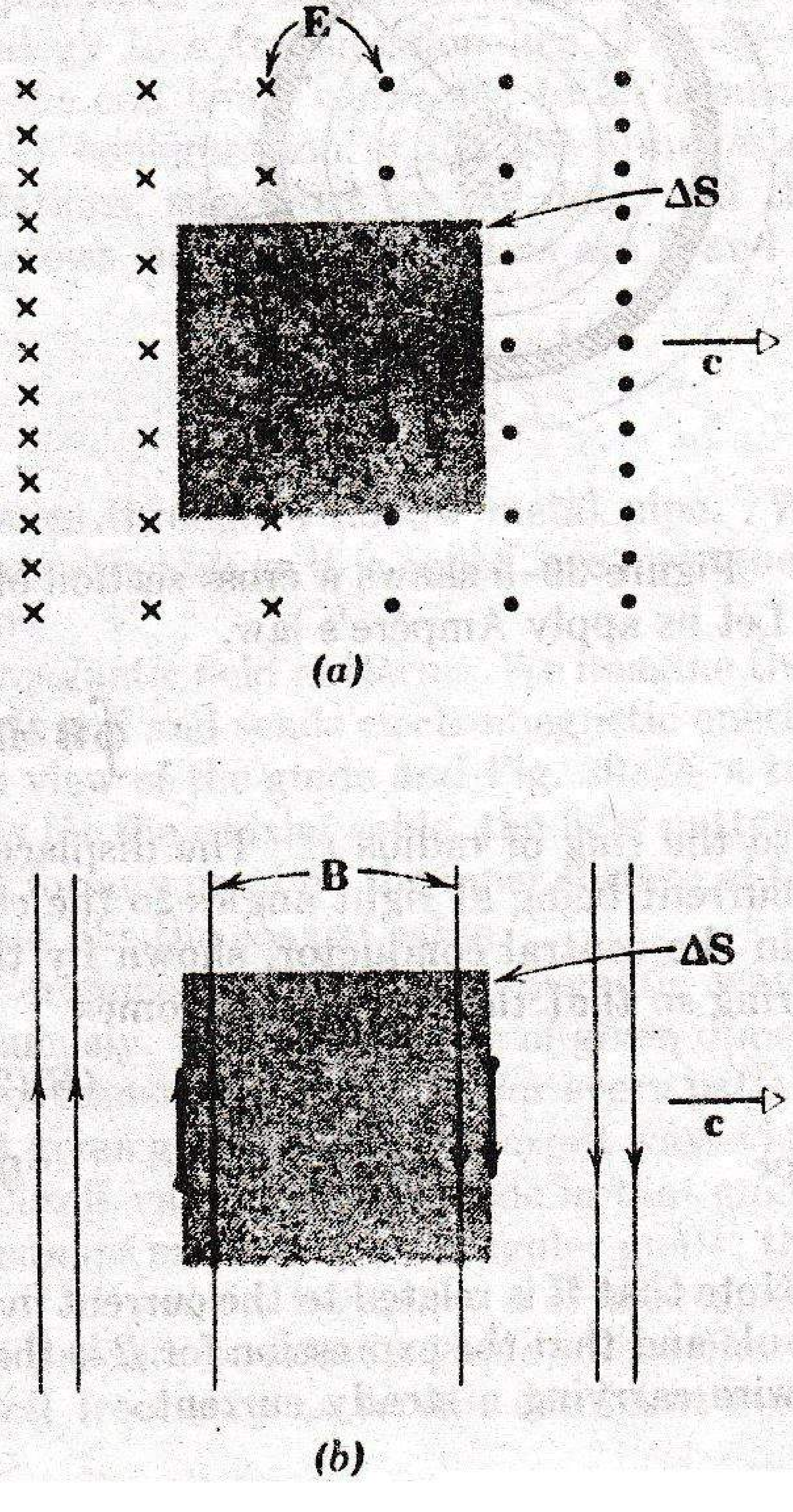


Fig. 39-4 The area element S in Fig. 39-3 enlarged and viewed from above, showing the adjacent (a) electric and (b) magnetic fields.

rate, since $\mathbf{E}$ at the element ΔS is at the very moment of reversing its direction as the wave moves through. Thus the displacement current, which is given by

$$i_d = \epsilon_0 \frac{d\Phi_E}{dt},$$

also has its maximum value.

Figure 39–4*b* shows the magnetic field in the vicinity of the element of area. Let us apply the generalized form of Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(i + i_d),$$

to the element. The conduction current i is zero since no charge is transported through ΔS. The displacement current i_d is not zero, having, in fact, its maximum value. Thus, since the right side of this equation ($= \mu_0 i_d$) does not vanish, the left side must not vanish. Study of Fig. 39–4*b* shows that $\oint \mathbf{B} \cdot d\mathbf{l}$ around the boundary of this square has, indeed, a nonzero value. Thus the field and displacement current configurations of Figs. 39–3*a*–*c* are consistent. We have not discussed the direction of i_d; that is, does it point into the plane of Fig. 39–4, as Fig. 39–3*c* asserts, or out of it? We leave this as a question for the student. He may be guided by considering the direction of the displacement current in Figs. 38–9*a* and *b*.

Example 3. Show that the conduction currents in Fig. 39–3*c* are appropriately related to the magnetic field pattern.

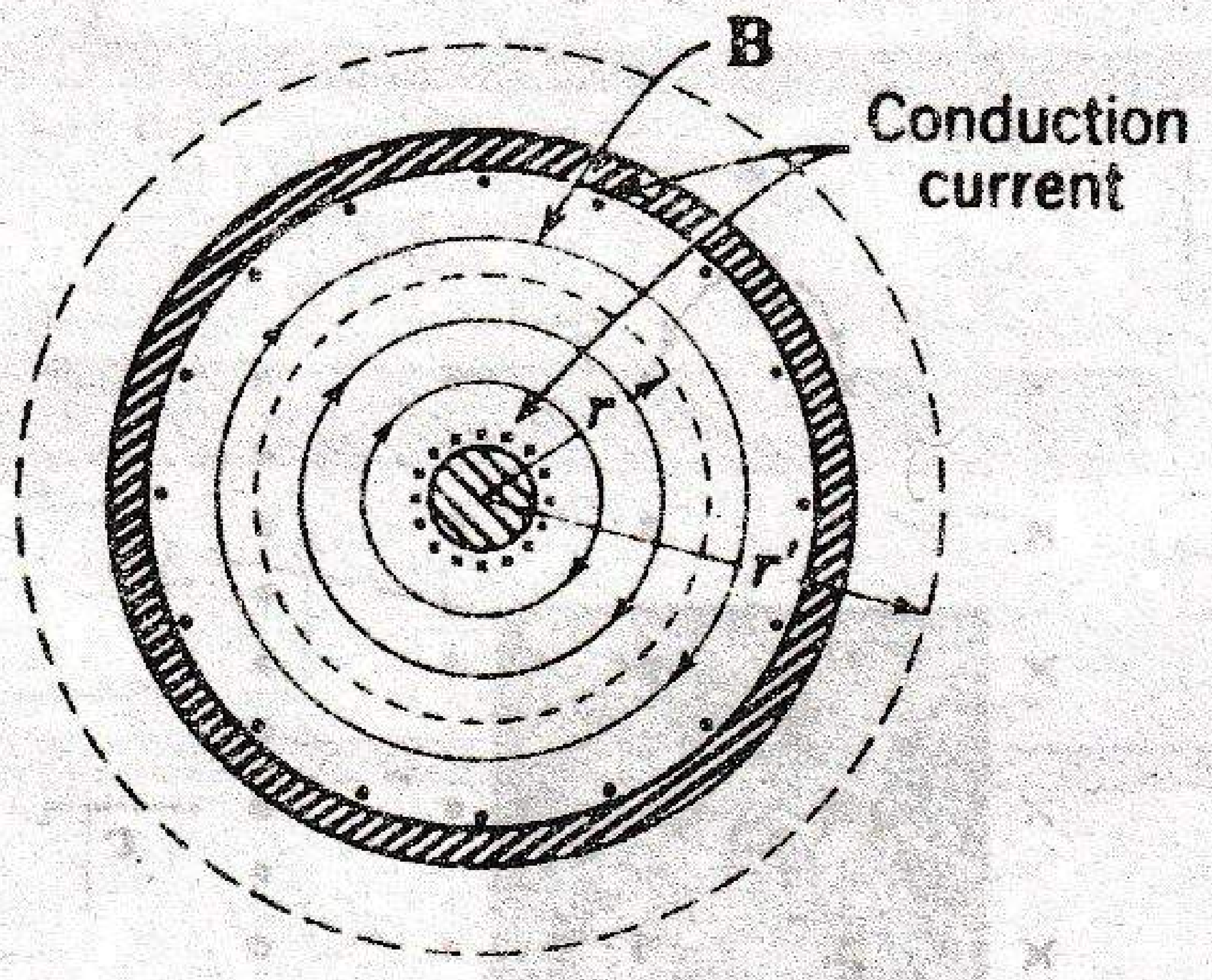


Fig. 39–5 Example 3. The coaxial cable of Fig. 39–3*b*, showing the conduction currents in the central and outer conductors. The wave is emerging from the page.

Figure 39–5 shows a cross section of the cable at a plane through *xx* in Fig. 39–3*a*. Let us apply Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(i + i_d),$$

to the ring of radius r. The displacement current through this ring is zero, this current being at right angles to the central conductor. The conduction current i in the central conductor, shown by the ×'s in the figure, does pass through this ring so that the equation becomes

$$(B)(2\pi r) = \mu_0 i,$$

or

$$B = \frac{\mu_0 i}{2\pi r}.$$

Note that $\mathbf{B}$ is related to the current in the central conductor by the usual right-hand rule and that the expression for B is that found earlier (Eq. 34–4) for a long straight wire carrying a steady current.

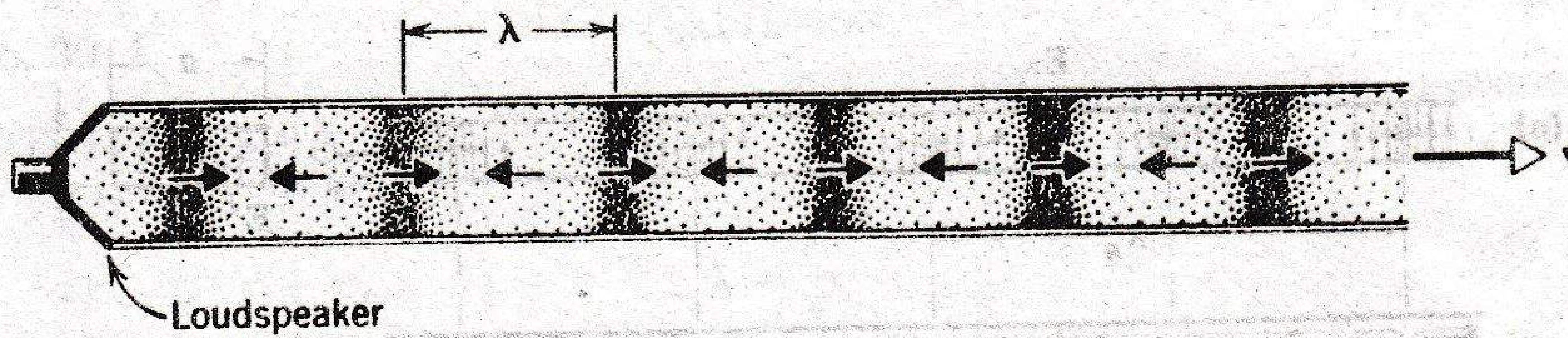


Fig. 39-6 An acoustic transmission line, showing a sound wave traveling to the right. The small arrows with filled heads show the directed drift velocities for small volume elements of the gas. Compare Fig. 39-3*a*.

If we apply Ampère's law to the large ring of radius r', we must put the *net* conduction current equal to zero because the current in the outer conductor, shown by the dots, is equal and opposite to that in the inner conductor. This means that **B** must be zero for points outside the cable, in agreement with experiment. ◀

It is interesting to compare the electromagnetic oscillations in a typical *traveling* wave, such as that of Fig. 39-3, to those in a cavity resonator, such as that of Fig. 38-8. The latter oscillations are an electromagnetic *standing* wave. In a traveling wave **E** and **B** are in phase, which means that at a given position along the transmission line they reach their maxima at the same time. However, Fig. 38-8 shows that at a given position in a cavity resonator **E** and **B** reach their maxima one-fourth of a cycle apart; they are 90° out of phase.

A complete analogy exists in mechanical systems. In the acoustic resonator of Fig. 38-7 the time variations of pressure and velocity for the standing acoustic wave are also 90° out of phase, in exact correspondence to the electromagnetic cavity oscillations of Fig. 38-8. The acoustic analogy to a transmission line (Fig. 39-6) would be an infinitely long gas-filled tube, one end being connected to an acoustic oscillator such as a loud-speaker. The entire configuration of Fig. 39-6 moves to the right with speed v. The pressure variations, suggested by the dots, and the instantaneous velocities, suggested by the arrows, are *in phase*, just as are **E** and **B** in the coaxial cable of Fig. 39-3.

39-3 Waveguide

It is possible to send electromagnetic waves through a hollow metal pipe. We assume that the inner walls of such a pipe, or *waveguide* as it is called, are resistanceless and that the cross section is rectangular.

Figure 39-7 shows a typical electric and magnetic field pattern. We imagine that a microwave oscillator is connected to the left end and sends electromagnetic energy down the guide. Figure 39-7*a* shows a side view of the guide and Fig. 39-7*b*, a top view; Fig. 39-7*c* shows the cross section. As for the coaxial cable, the field patterns are such that **E** has no tangential component for any point on the inner surface of the guide. The fields **E** and **B** are in phase, again like the coaxial cable.

As for all traveling waves, the angular frequency ω of electromagnetic waves traveling down a guide can be varied continuously. In a waveguide of given dimensions, however, there exists, for every mode of transmission, that is, for every pattern of **E** and **B**, a so-called *cutoff frequency* ω_0. A given guide will not transmit waves in a given mode if their frequency is below the cutoff value for that mode in that guide. The field patterns of Fig. 39-7 show the *dominant mode* for a rectangular guide; this the mode with the lowest cutoff frequency. Given the frequency ω of electromagnetic waves to be transmitted, it is common practice to select a guide whose dimensions are such that ω is larger than the cutoff frequency ω_0 for the dominant mode

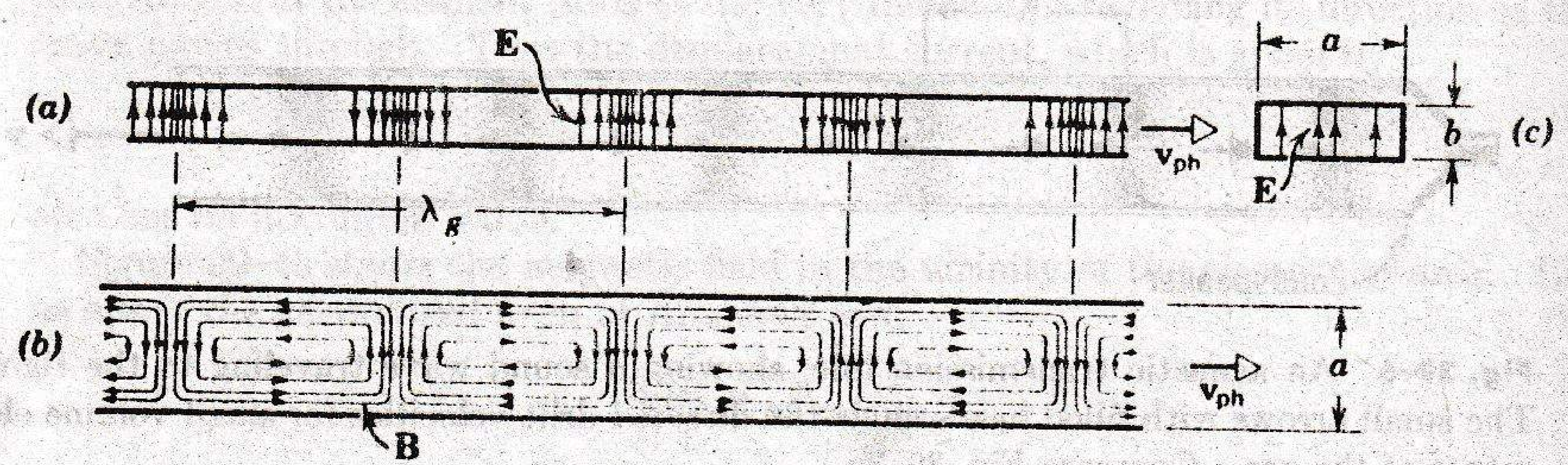


Fig. 39–7 A waveguide, showing (*a*) a side view of the lines of **E**, (*b*) a top view of the lines of **B**, and (*c*) a cross-sectional view of the lines of **E**. In (*c*) the wave is emerging from the page. For simplicity the lines of **B** are not shown in (*a*) and (*c*), nor are the lines of **E** shown in (*b*).

but smaller than the cutoff frequencies of all other modes. Under these conditions the dominant mode of propagation is the only one possible.

In a (resistanceless) coaxial cable the wave patterns travel at speed c. In the acoustic transmission line of Fig. 39–6 (assumed "resistanceless") the waves also travel at a speed v, which is the same as the propagation speed in an infinite medium. In a waveguide, however, the speed is *not* c. In waveguides we must distinguish between (*a*) the *phase speed* v_{ph}, which is the speed at which the wave patterns of Fig. 39–7 travel, and (*b*) the *group speed* v_{gr}, which is the speed at which electromagnetic energy or information-carrying "signals" travel along the guide. These speeds, which are identical for electromagnetic waves in a coaxial cable and for acoustic waves in a tube, are different for waves in a waveguide.

The phase speed is not directly measurable. The wave pattern is a repetitive structure, and there is no way to distinguish one wave maximum from another. The waves can be observed to enter one end of the guide and to leave at the other, but there is no way to identify a particular wave maximum so that its passage down the guide can be timed. We can put a "signal" on the wave by increasing the power level of the oscillator for a short time. This power pulse could be timed as it passes through the guide, but there is no guarantee that it travels at the same speed as the wave pattern and, indeed, it does not. The speed of such signals or markers is the speed at which *energy* is propagated, that is, the group speed.

From Maxwell's equations it can be shown that the phase speed and the group speed for the mode of Fig. 39–7 are

$$v_{ph} = \frac{c}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}} \tag{39-1}$$

and

$$v_{gr} = c\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}, \tag{39-2}$$

in which a is the width of the guide and λ the free-space wavelength. Note that as $a \rightarrow \infty$, which corresponds to free-space conditions, $v_{ph} = v_{gr} = c$.

The phase speed v_{ph} is *greater* than the velocity of light, the group speed v_{gr} being correspondingly less. In relativity theory we learn that no speed at which signals or energy travel can be faster than that of light. However, signals or energy cannot be transmitted down a guide at speeds exceeding c; they travel with speed v_{gr} which is always less than c so there is no conflict with the theory of relativity.

The wavelength λ in Eqs. 39-1 and 39-2 is the wavelength that would be measured for the oscillations in free space, that is,

$$\lambda = \frac{c}{\nu}, \tag{39-3}$$

where c is the speed in free space and ν is the frequency. For waves of a given frequency, the wavelength exhibited in a guide (λ_g) must differ from the free-space wavelength λ because the speed v_{ph} has changed. The so-called *guide wavelength* λ_g is given by

$$\lambda_g = \frac{v_{ph}}{\nu} = \frac{v_{ph}}{c/\lambda} = \lambda\,\frac{v_{ph}}{c}.$$

From Eq. 39-1 this yields

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}}. \tag{39-4}$$

Thus the guide wavelength, which is the wavelength exhibited by the field patterns in Fig. 39-7, is larger than the free-space wavelength.

▶ **Example 4.** What must be the width a of a rectangular guide such that the energy of electromagnetic radiation whose free-space wavelength is 3.0 cm travels down the guide (*a*) at 95% of the speed of light? (*b*) At 50% of the speed of light?

From Eq. 39-2 we have

$$v_{gr} = 0.95c = c\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}.$$

Solving for a yields a = 4.8 cm; repeating for $v_{gr} = 0.50c$ yields a = 1.7 cm.

This formula illustrates the cutoff phenomenon described above. If $\lambda = 2a$, then $v_{gr} = 0$ and energy cannot travel down the guide. For the radiation considered in this example λ = 3.0 cm, so that the guide must have a width a of *at least* ½ x 3.0 cm = 1.5 cm if it is to transmit this wave. The guide whose width we calculated in (*a*) above can transmit radiations whose free-space wavelength is 2 x 4.8 cm = 9.6 cm *or less.* ◀

39-4 Radiation

The acoustic transmission line of Fig. 39-6 cannot be infinitely long. Its far end may be sealed by a solid cap or left open, or it may have a flange, a horn, or some similar device mounted on it. If the far end is not sealed, energy will escape into the medium beyond. This is called *acoustic radiation.* In general, some energy will also be reflected back down the transmission line. If acoustic radiation is desirable, the designer's task is to fashion a termination (that is, an "acoustic antenna") for the transmission line such that the smallest possible fraction of the incident energy will be reflected back down the line. Such a termination might take the form of a flared horn. Acoustic radiation, of course, requires a medium such as air in order to be propagated.

An electromagnetic transmission line such as a coaxial cable or a waveguide can also be terminated in many ways, and energy can escape from the end of the line into the space beyond. In contrast to sound waves, a physical

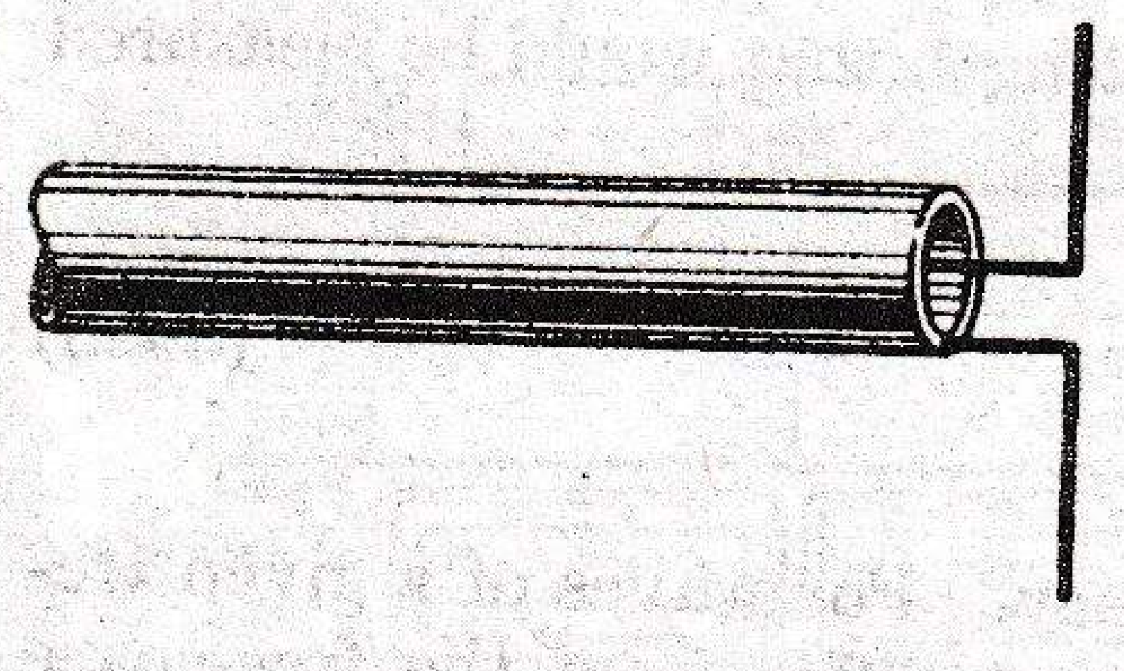


Fig. 39-8 An electric dipole antenna on the end of a coaxial cable.

medium is not required. Thus electromagnetic energy can be radiated from the end of the transmission line, to form a traveling electromagnetic wave in free space.

Figure 39-8 shows an effective termination for a coaxial cable; it consists of two wires arranged as shown and is called an *electric dipole antenna*. The potential difference between the two conductors alternates sinusoidally as the wave reaches them, the effect being that of an electric dipole whose dipole moment **p** varies with time.

Figure 39-9 shows such a dipole, represented by two equal and opposite charges. Its dipole moment, represented by the arrows marked **p** in the

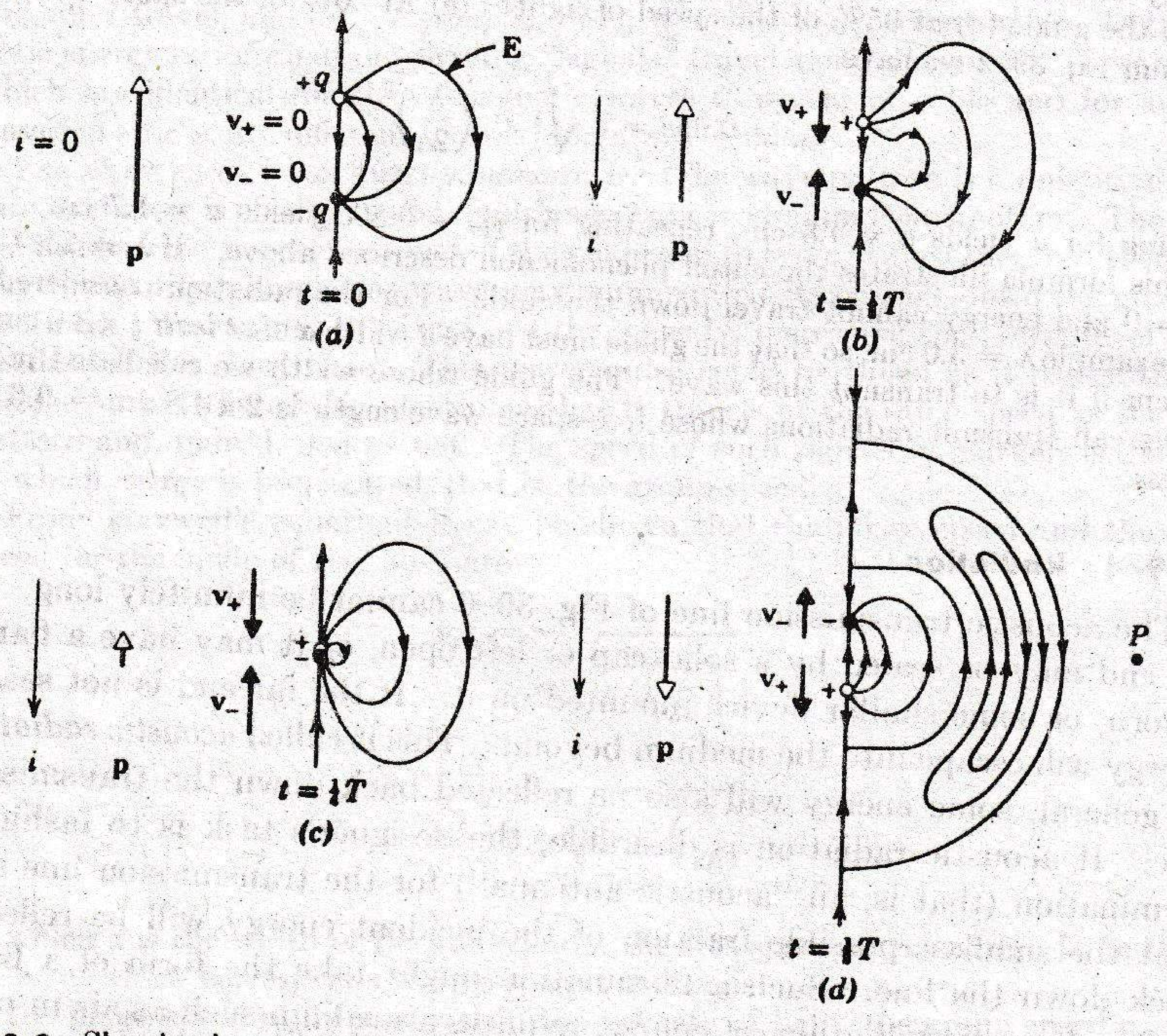


Fig. 39-9 Showing how radiation escapes from an oscillating electric dipole. The velocities of the charges, the electric dipole moment, the equivalent current, and the lines of E are shown at four stages of oscillation, one-eighth of a period apart. The lines of B are not shown. (Adapted from Attwood, *Electric and Magnetic Fields*, John Wiley and Sons, third edition, 1949.)

figure, oscillates sinusoidally as the charges oscillate; in Fig. 39-9*c* it is on the verge of reversing its direction. The four views, one-eighth of a cycle apart, show how the electric lines of force break away from the dipole and form closed loops that travel through free space with speed *c*. The oscillating charges constitute a current, which is represented by the arrows marked *i* in Fig. 39-9. These oscillating currents generate a field of **B** which, for simplicity, is not represented in the figure. The lines of **B** as well as those of **E** also form closed loops that move away from the dipole with speed *c*. These traveling electric and magnetic fields, which, as we shall see, are strongly interdependent, constitute *electromagnetic radiation.*

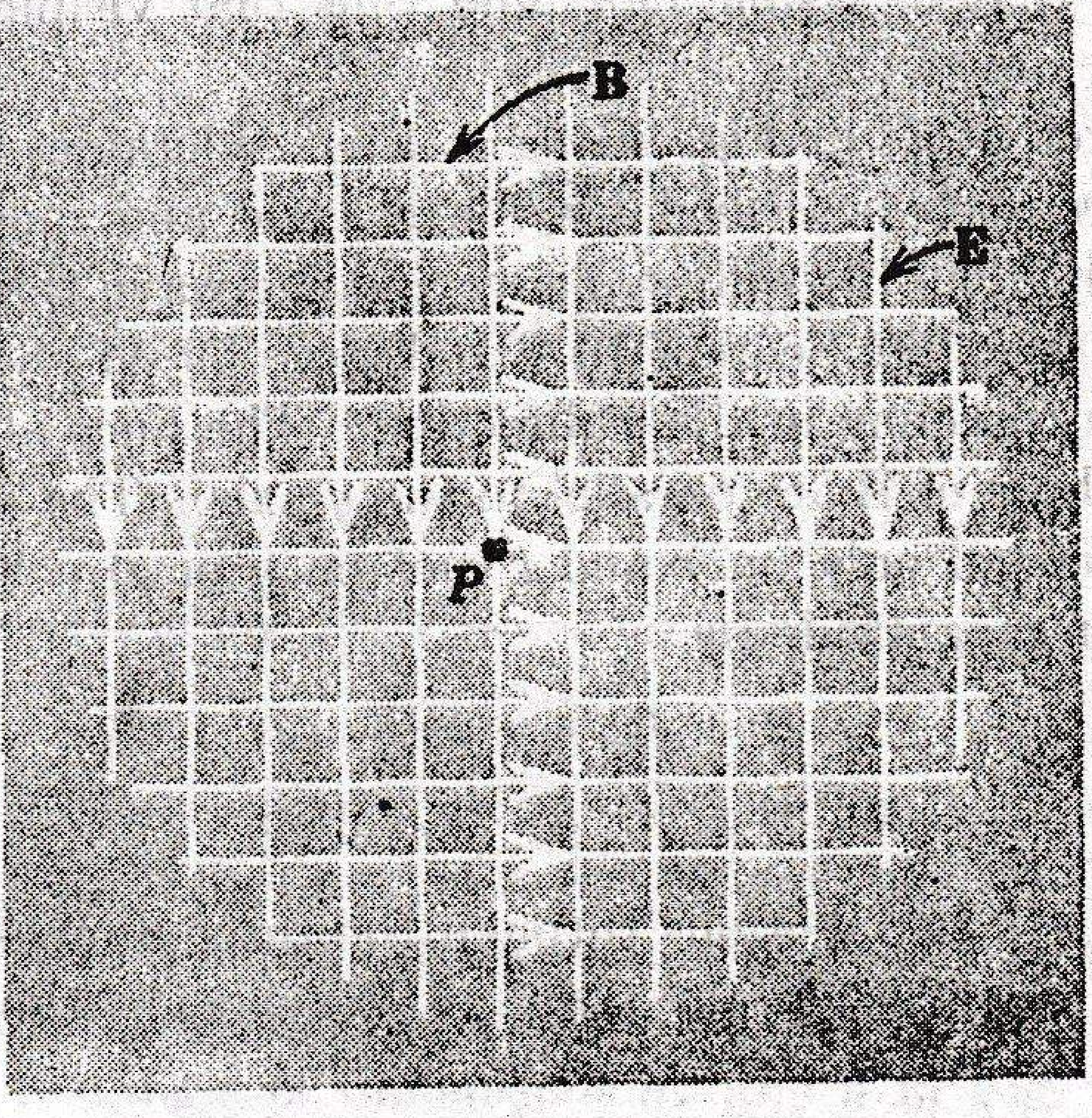


Fig. 39-10 An instantaneous view of an electromagnetic wave as seen by an observer at *P* in Fig. 39-9*d*.

Figure 39-10 shows a part of the wavefront as it would appear to an observer at point *P* in Fig. 39-9*d*. The wave is moving directly out of the page. One-half a period later the observer at *P* will see a field pattern like that of Fig. 39-10, except that the directions of both the electric and the magnetic fields will be reversed. The speed *c* of the wave in free space is given by $c = \nu\lambda$, which can be written as

$$c = \frac{\omega}{k}, \tag{39-5}$$

where ω, the angular frequency, and k, the wave number, are related to the frequency ν and the wavelength λ by

$$\omega = 2\pi\nu \quad \text{and} \quad k = 2\pi/\lambda.$$

39-5 Traveling Waves and Maxwell's Equations

In earlier sections we have postulated the existence of certain magnetic and electric field distributions, in resonant cavities, coaxial cables, and wave-guides, and we have shown that these postulated distributions are consistent with Maxwell's equations, as are the distributions of conduction and displacement currents associated with the fields. The student who pursues his studies of electromagnetism will learn how to derive mathematical expressions for **E** and **B** by subjecting Maxwell's equations to the boundary conditions appropriate to the problem at hand. In this section we continue our program by showing that the postulated patterns of **E** and **B** for a traveling electromagnetic wave are completely consistent with Maxwell's equations. In doing so, we will be able to show that the speed of such waves in free space is that of visible light and thus that visible light is itself an electromagnetic wave.

If the observer at P in Fig. 39–9*d* is a considerable distance from the source, the *wavefronts* described by the electric and magnetic fields that reach him (see Fig. 39–10) will be planes and the wave that moves past him will be a *plane wave* (see Section 19–2). Figure 39–11 shows a "snapshot" of a plane wave traveling in the x direction. The lines of **E** are parallel to the z axis and those of **B** are parallel to the y axis. The values of **B** and **E** for this wave depend only on x and t (not on y or z). We postulate that they are given in magnitude by

$$B = B_m \sin (kx - \omega t) \tag{39-6}$$

and

$$E = E_m \sin (kx - \omega t). \tag{39-7}$$

Figure 39–12 shows two sections through the three-dimensional diagram of Fig. 39–11. In Fig. 39–12*a* the plane of the page is the xz plane and in

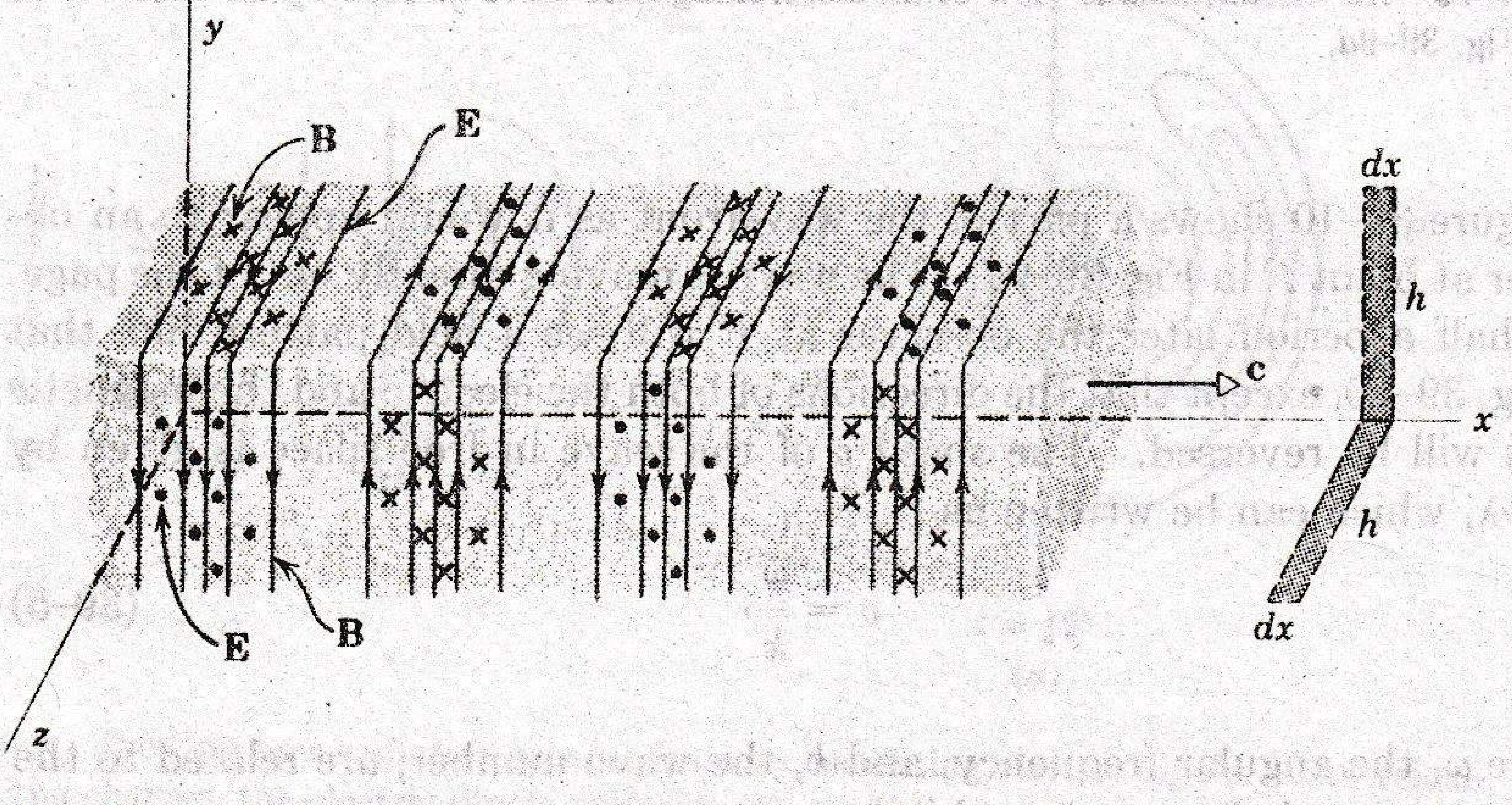


Fig. 39–11 A plane electromagnetic wave traveling to the right at speed c. Lines of **B** are parallel to the y axis; those of **E** are parallel to the z axis. The shaded rectangles on the right refer to Fig. 39–12.

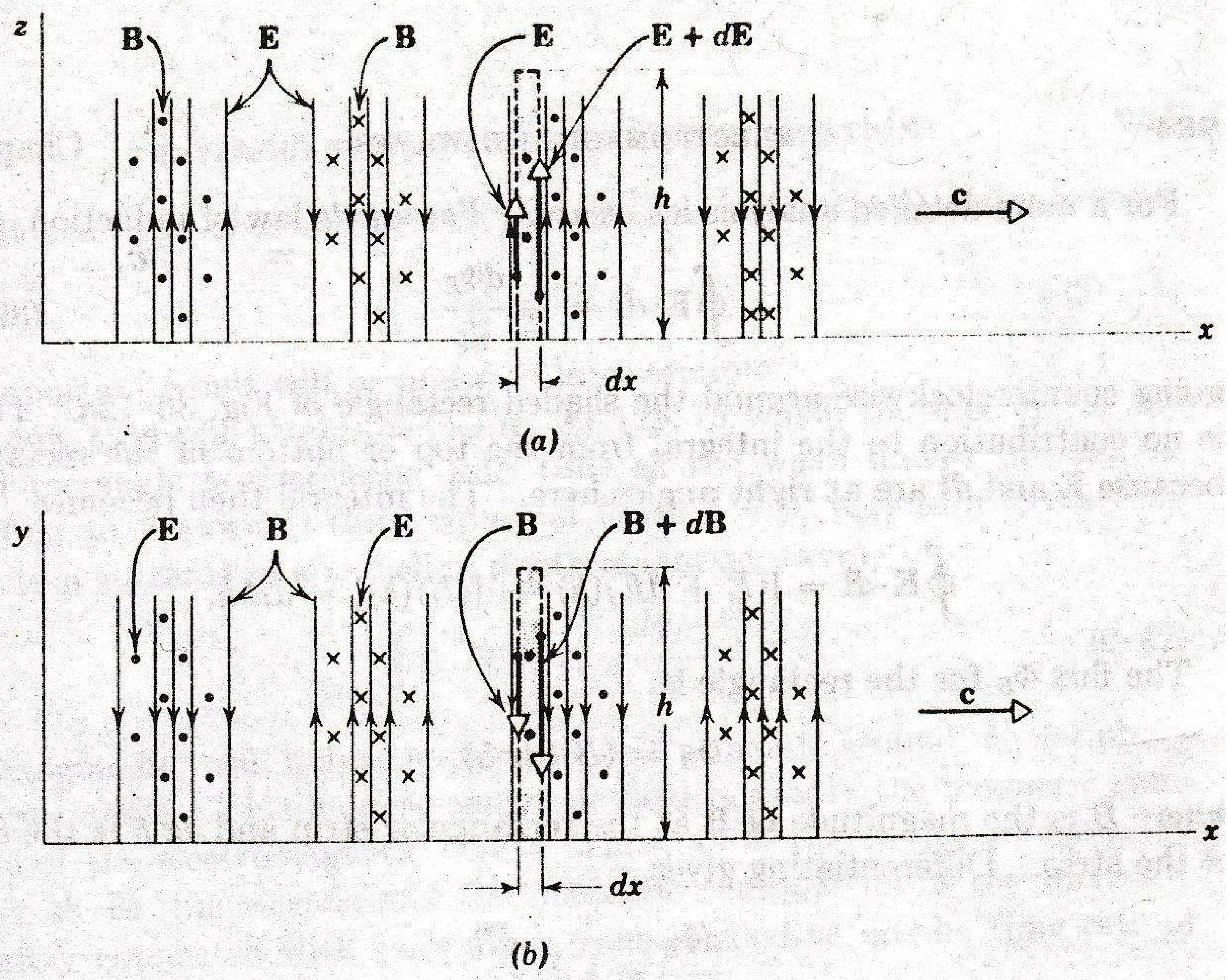


Fig. 39-12 The wave of Fig. 39-11 viewed (*a*) in the *xz* plane and (*b*) in the *xy* plane.

Fig. 39-12*b* it is the *xy* plane. Note that, as for the traveling waves in a coaxial cable (Fig. 39-3*a*) and in a waveguide (Fig. 39-7), **E** and **B** are in phase, that is, at any point through which the wave is moving they reach their maximum values at the same time.

The shaded rectangle of dimensions dx and h in Fig. 39-12*a* is fixed in space. As the wave passes over it, the magnetic flux Φ_B through the rectangle will change, which will give rise to induced electric fields around the rectangle, according to Faraday's law of induction. These induced electric fields are, in fact, simply the electric component of the traveling wave.

Let us apply Lenz's law to this induction process. The flux Φ_B for the shaded rectangle of Fig. 39-12*a* is *decreasing* with time because the wave is moving through the rectangle to the right and a region of weaker magnetic fields is moving into the rectangle. The induced field will act to oppose this change, which means that if we imagine that the boundary of the rectangle is a conducting loop a *counterclockwise* induced current would appear in it. This current would produce a field of **B** that, within the rectangle, would point out of the page, thus opposing the decrease in Φ_B. There is, of course, no conducting loop, but the net induced electric field **E** does indeed act counterclockwise around the rectangle because $E + dE$, the magnitude of **E** at the right edge of the rectangle, is greater than E, the magnitude of **E** at the left edge. Thus the electric field configuration is entirely consistent with the concept that it is induced by the changing magnetic field.

For a more detailed analysis let us apply Faraday's law of induction, or

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}, \tag{39-8}$$

going counterclockwise around the shaded rectangle of Fig. 39–12*a*. There is no contribution to the integral from the top or bottom of the rectangle because $\mathbf{E}$ and $d\mathbf{l}$ are at right angles here. The integral then becomes

$$\oint \mathbf{E}\cdot d\mathbf{l} = [(E + dE)(h)] - [(E)(h)] = dE\, h.$$

The flux Φ_B for the rectangle is

$$\Phi_B = (B)(dx\, h),$$

where B is the magnitude of $\mathbf{B}$ at the rectangular strip and $dx\, h$ is the area of the strip. Differentiating gives

$$\frac{d\Phi_B}{dt} = h\, dx \frac{dB}{dt}.$$

From Eq. 39–8 we have then

$$dE\, h = -h\, dx \frac{dB}{dt},$$

or

$$\frac{dE}{dx} = -\frac{dB}{dt}. \tag{39-9}$$

Actually, both B and E are functions of x and t; see Eqs. 39–6 and 39–7. In evaluating dE/dx, it is assumed that t is constant because Fig. 39–12*a* is an "instantaneous snapshot." Also, in evaluating dB/dt it is assumed that x is constant since what is required is the time rate of change of B at a particular place, the strip in Fig. 39–12*a*. The derivatives under these circumstances are called *partial derivatives*, and a somewhat different notation is used for them; see the footnote on p. 725. In this notation Eq. 39–9 becomes

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}. \tag{39-10}$$

The minus sign in this equation is appropriate and necessary, for although E is increasing with x at the site of the shaded rectangle in Fig. 39–12*a*, B is decreasing with t. Since $E(x,t)$ and $B(x,t)$ are known (see Eqs. 39–6 and 39–7), Eq. 39–10 reduces to

$$kE_m \cos(kx - \omega t) = \omega B_m \cos(kx - \omega t),$$

or (see Eq. 39–5)

$$\frac{\omega}{k} = \frac{E_m}{B_m} = c. \tag{39-11a}$$

Thus the speed of the wave c is the ratio of the amplitudes of the electric and the magnetic components of the wave. From Eqs. 39–6 and 39–7 we see that the ratio of amplitudes is the same as the ratio of instantaneous values, or

$$E = cB. \tag{39-11b}$$

This important result will be useful in later sections.

We now turn our attention to Fig. 39-12*b*, in which the flux Φ_E for the shaded rectangle is decreasing with time as the wave moves through it. According to Maxwell's third equation (with $i = 0$, because there are no conduction currents in a traveling electromagnetic wave),

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt}, \tag{39-12}$$

this changing flux will induce a magnetic field at points around the periphery of the rectangle. This induced magnetic field is simply the magnetic component of the electromagnetic wave. Thus, as in the cavity resonator of Section 38-10, the electric and the magnetic components of the wave are intimately connected with each other, each depending on the time rate of change of the other.

Comparison of the shaded rectangles in Fig. 39-12 shows that for each the appropriate flux, Φ_B or Φ_E, is *decreasing* with time. However, if we proceed counterclockwise around the upper and lower shaded rectangles, we see that $\oint \mathbf{E}\cdot d\mathbf{l}$ is *positive*, whereas $\oint \mathbf{B}\cdot d\mathbf{l}$ is *negative*. This is as it should be. If the student will compare Figs. 35-10 and 38-9*a*, he will be reminded that although the fluxes Φ_B and Φ_E in those figures are changing with time in the same way (both are increasing) the lines of the induced fields, **E** and **B**, respectively, circulate in opposite directions.

The integral in Eq. 39-12, evaluated by proceeding counterclockwise around the shaded rectangle of Fig. 39-12*b*, is

$$\oint \mathbf{B}\cdot d\mathbf{l} = [-(B + dB)(h)] + [(B)(h)] = -h\, dB,$$

where B is the magnitude of **B** at the left edge of the strip and $B + dB$ is its magnitude at the right edge.

The flux Φ_E through the rectangle of Fig. 39-12*b* is

$$\Phi_E = (E)(h\, dx).$$

Differentiating gives
$$\frac{d\Phi_E}{dt} = h\, dx \frac{dE}{dt}.$$

Equation 39-12 can thus be written

$$-h\, dB = \mu_0\epsilon_0\left(h\, dx \frac{dE}{dt}\right)$$

or, substituting partial derivatives,

$$-\frac{\partial B}{\partial x} = \mu_0\epsilon_0 \frac{\partial E}{\partial t}. \tag{39-13}$$

Again, the minus sign in this equation is appropriate and necessary, for, although B is increasing with x at the site of the shaded rectangle in Fig. 39-12b, E is decreasing with t.

Combining this equation with Eqs. 39-6 and 39-7 yields

$$-kB_m \cos(kx - \omega t) = -\mu_0\epsilon_0\omega E_m \cos(kx - \omega t),$$

or (see Eq. 39-5)

$$\frac{E_m}{B_m} = \frac{k}{\mu_0\epsilon_0\omega} = \frac{1}{\mu_0\epsilon_0 c}. \tag{39-14}$$

Eliminating E_m/B_m between Eqs. 39-11a and 39-14 yields

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}. \tag{39-15}$$

Substituting numerical values yields

$$c = \frac{1}{\sqrt{(4\pi \times 10^{-7}\ \text{weber/amp-m})(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)}}$$
$$= 3.0 \times 10^8\ \text{meters/sec}, \tag{39-16}$$

which is the speed of light in free space! This emergence of the speed of light from purely electromagnetic considerations is the crowning achievement of Maxwell's electromagnetic theory. Maxwell made this prediction before radio waves were known and before it was realized that light was electromagnetic in nature. His prediction led to the concept of the electromagnetic spectrum, which we discuss in Chapter 40, and to the discovery of radio waves by Heinrich Hertz in 1890. It made it possible to discuss optics as a branch of electromagnetism and to derive its fundamental laws from Maxwell's equations.

A conclusion as fundamental as Eq. 39-15 must be subject to rigorous experimental verification. Of the three quantities in that equation, one, μ_0, has an assigned value, namely, $4\pi \times 10^{-7}$ weber/amp-m. The speed of light c is one of the most precisely measured physical constants, having the presently accepted value of 2.997924×10^8 meters/sec. The remaining quantity, ϵ_0, can be measured by making measurements on an accurately constructed parallel-plate capacitor, as described in Section 30-2. The best measured value, by Rosa and Dorsey of the National Bureau of Standards (U.S.A.) in 1906, is 8.84025×10^{-12} coul^2/nt-m^2. To this accuracy Eq. 39-15 is completely verified. Our confidence in electromagnetic theory, bolstered by numerous successful predictions and agreements with experiment, is now such that we reverse the foregoing procedure and calculate our presently accepted value of ϵ_0 (see Appendix A) from the measured speed of light, using Eq. 39-15.

39-6 The Poynting Vector

One of the important characteristics of an electromagnetic wave is that it can transport energy from point to point. As we show below, the rate of energy flow per unit area in a plane electromagnetic wave can be described by a vector **S**, called the *Poynting vector* after John Henry Poynting (1852–1914), who first pointed out its properties. We define **S** from

$$\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B}. \tag{39-17}$$

In the mks system **S** is expressed in watts/meter^2; the direction of **S** gives the direction in which the energy moves. The vectors **E** and **B** refer to their instantaneous values at the point in question. If Eq. 39-17 is applied to the traveling plane electromagnetic wave of Fig. 39-11, it is clear that $\mathbf{E}\times\mathbf{B}$, hence **S**, point in the direction of propagation. Note, too, that **S** points parallel to the axis for all points in the coaxial cable of Fig. 39-3.

We get meaningful results if we extend the Poynting vector concept to other electromagnetic situations involving either traveling or standing electromagnetic waves, as we will see in Examples 5 and 6. If we extend it to circuit situations involving steady or almost steady currents and lumped circuit elements, we are led to some interesting conclusions, which we explore in Problems 11, 17, and 18.

▶ **Example 5.** Analyze energy flow in the cavity of Fig. 38-8, using the Poynting vector.

Study of Fig. 39-13 shows that when the energy is all electric (Figs. 39-13*a* and *e*) it is concentrated along the axis, because this is the region in which **E** has its maximum

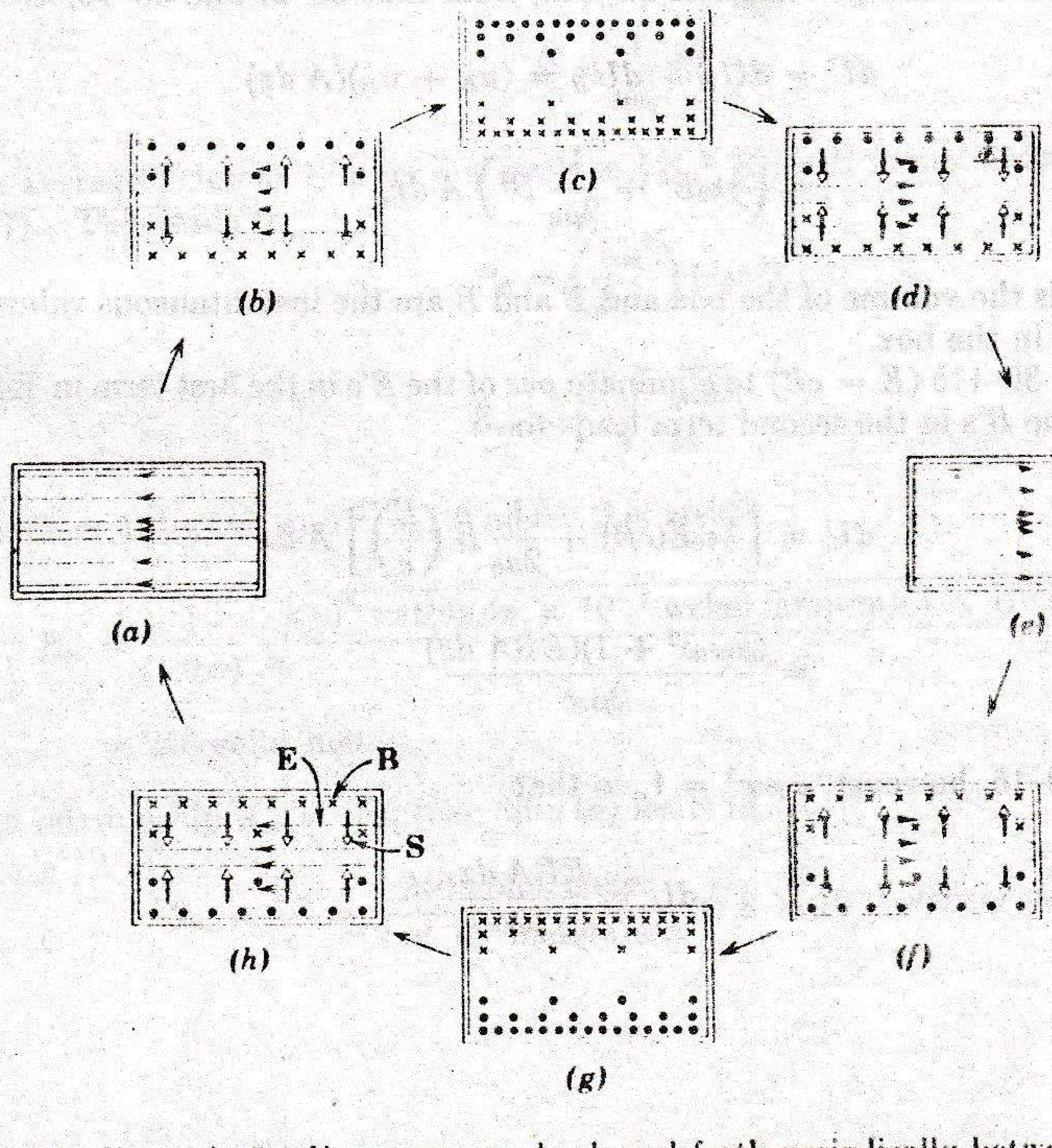


Fig. 39-13 Example 5. Energy surges back and forth periodically between the central region of the cavity and the region near the walls, as indicated by the Poynting vector **S**.

value. When the energy is all magnetic (Figs. 39–13*c* and *g*), it is concentrated near the walls. Thus the energy surges back and forth periodically between the central region and the region near the walls. The figure shows, by the open arrows, the direction of **S** at various points in the cavity and at various times in the cycle. Note that **S** equals zero for Figs. 39–13*a*, *c*, *e*, and *g*, which is appropriate because at these instants of time the field configurations are momentarily stationary and energy is not flowing. A pendulum bob at the end of its swing and at the bottom of its trajectory forms a mechanical analogy. The student should verify from Eq. 39–17 that these arrows point in the correct directions. ◀

Figure 39–14 can be used to derive the Poynting relation for the special case of a traveling plane electromagnetic wave. It shows a cross section of a traveling plane wave, along with a thin "box" of thickness dx and area A. The box, a mathematical construction, is fixed with respect to the axes while the wave moves through it.

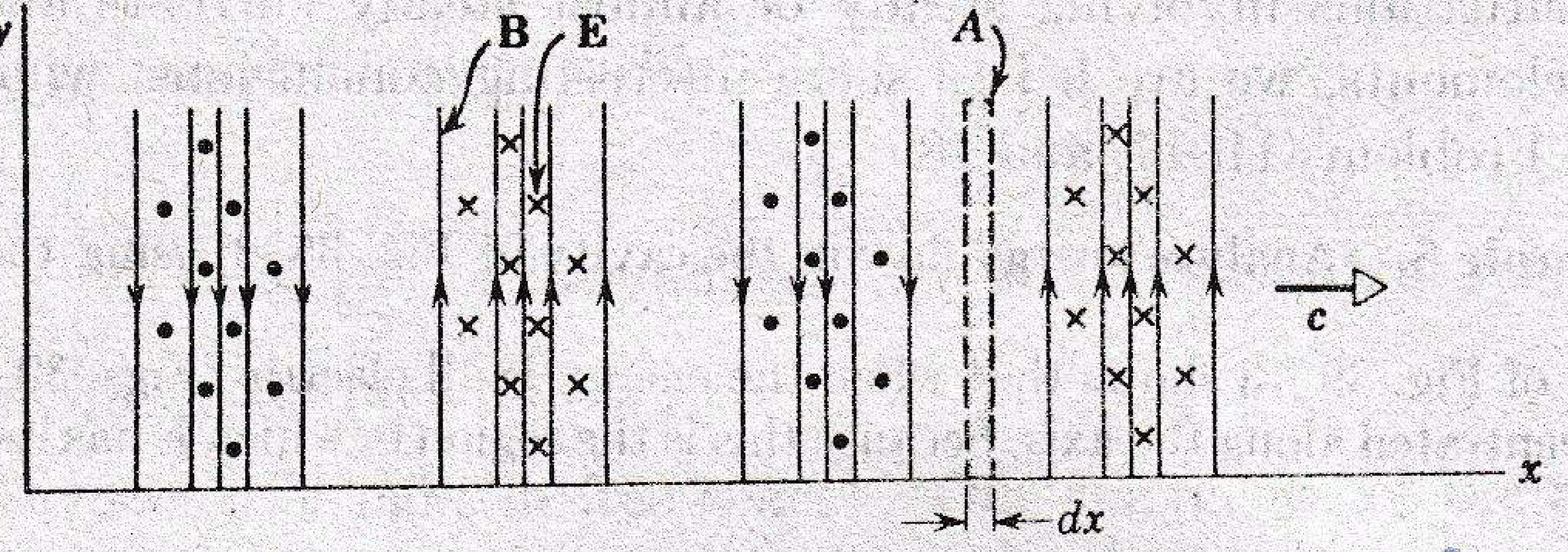


Fig. 39–14 A plane wave is traveling to the right at speed c; compare Fig. 39–12*b*. The dashed rectangle in this figure represents a three-dimensional box, of area A and thickness dx, that extends at right angles to the plane of the figure.

At any instant the energy stored in the box, from Eqs. 30–27 and 36–19, is

$$dU = dU_E + dU_B = (u_E + u_B)(A\,dx)$$

$$= \left(\tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) A\,dx, \tag{39–18}$$

where $A\,dx$ is the volume of the box and E and B are the instantaneous values of the field vectors in the box.

Using Eq. 39–11*b* ($E = cB$) to eliminate *one* of the E's in the first term in Eq. 39–18 and *one* of the B's in the second term leads to

$$dU = \left[\tfrac{1}{2}\epsilon_0 E(cB) + \frac{1}{2\mu_0} B\left(\frac{E}{c}\right)\right] A\,dx$$

$$= \frac{(\mu_0\epsilon_0 c^2 + 1)(EBA\,dx)}{2\mu_0 c}$$

From Eq. 39–15, however, $\mu_0\epsilon_0 c^2 = 1$, so that

$$dU = \frac{EBA\,dx}{\mu_0 c}.$$

This energy dU will pass through the right face of the box in a time dt equal to dx/c. Thus the energy per unit area per unit time, which is S, is given by

$$S = \frac{dU}{dt\,A} = \frac{EBA\,dx}{\mu_0 c\,(dx/c)\,A} = \frac{1}{\mu_0}EB.$$

This is exactly the prediction of the more general relation Eq. 39-17 for a traveling plane wave.

This relation refers to values of S, E, and B at any instant of time. We are usually more interested in the *average* value of S, taken over one or more cycles of the wave. An observer making intensity measurements on a wave moving past him would measure this average value $\bar{S}$. We can easily show (see Example 6) that $\bar{S}$ is related to the *maximum* values of E and B by

$$\bar{S} = \frac{1}{2\mu_0}E_m B_m.$$

▶ **Example 6.** An observer is at a great distance r from a point light source whose power output is P_0. Calculate the magnitudes of the electric and the magnetic fields. Assume that the source is monochromatic, that it radiates uniformly in all directions, and that at distant points it behaves like the traveling plane wave of Fig. 39-11.

The power that passes through a sphere of radius r is $(\bar{S})(4\pi r^2)$, where $\bar{S}$ is the *average* value of the Poynting vector at the surface of the sphere. This power must equal P_0, or

$$P_0 = \bar{S}4\pi r^2.$$

From the definition of **S** (Eq. 39-17), we have

$$\bar{S} = \overline{\left(\frac{1}{\mu_0}EB\right)}.$$

Using the relation $E = cB$ (Eq. 39-11*b*) to eliminate B leads to

$$\bar{S} = \frac{1}{\mu_0 c}\overline{E^2}.$$

The average value of E^2 over one cycle is $\frac{1}{2}E_m{}^2$, since E varies sinusoidally (see Eq. 39-7). This leads to

$$P_0 = \left(\frac{E_m{}^2}{2\mu_0 c}\right)(4\pi r^2),$$

or

$$E_m = \frac{1}{r}\sqrt{\frac{P_0\mu_0 c}{2\pi}}.$$

For $P_0 = 10^3$ watts and $r = 1.0$ meter this yields

$$E_m = \frac{1}{(1.0m)}\sqrt{\frac{(10^3 \text{ watts})(4\pi \times 10^{-7} \text{ weber/amp-m})(3 \times 10^8 \text{ meters/sec})}{2\pi}}$$

$$= 240 \text{ volts/meter}.$$

The relationship $E_m = cB_m$ (Eq. 39-11*a*) leads to

$$B_m = \frac{E_m}{c} = \frac{240 \text{ volts/meter}}{3 \times 10^8 \text{ meters/sec}} = 8 \times 10^{-7} \text{ weber/meter}^2.$$

Note that E_m is appreciable as judged by ordinary laboratory standards but that B_m (= 0.008 gauss) is quite small. ◀

QUESTIONS

1. In the coaxial cable of Fig. 39-1, what are the directions of the conduction current (*a*) in the central conductor and (*b*) in the outer conductor, shortly after the switch is thrown to position *a*? Consider points that have been reached by the wavefront of Fig. 39-2*a* and *b* and those that have not.

2. Compare a coaxial cable and a waveguide, used as a transmission line. Point out both similarities and differences.

3. What is the relation between the wavelength in the cable and that in free space for a coaxial cable?

4. Can traveling waves with a continuous range of wavelengths be sent down (*a*) a coaxial cable and (*b*) a waveguide? Can standing waves with a continuous range of wavelengths be set up in a resonant cavity? Develop mechanical or acoustical analogies to support your answers.

5. If a certain wavelength is larger than the cutoff wavelength for a guide in its dominant mode, can energy be sent down it in any other mode?

6. Explain why the term $\epsilon_0\, d\Phi_E/dt$ is needed in Ampère's equation to understand the propagation of electromagnetic waves.

7. In the equation $c = 1/\sqrt{\mu_0\epsilon_0}$ (Eq. 39-15), how can c always have the same value if μ_0 is arbitrarily assigned and ϵ_0 is measured?

8. Is it conceivable that electromagnetic theory might some day be able to predict the value of c (3×10^8 meters/sec), not in terms of μ_0 and ϵ_0, but directly and numerically without recourse to any measurements?

9. What is the direction of the displacement current in Fig. 39-4? Give an argument to support your answer.

10. In a coaxial cable is the energy transported in the conductors, through the agency of the currents, or in the space between them, through the agency of the fields?

PROBLEMS

1. Using Gauss's law, sketch the instantaneous charges that appear on the conductors of the coaxial cable of Fig. 39-3 and show that this pattern of charges is appropriately related to the conduction currents shown in Fig. 39-3*c*.

2. For a rectangular guide of width 3.0 cm, plot the phase speed, the group speed, and the guide wavelength as a function of the free-space wavelength. Assume the dominant mode.

3. For a rectangular guide of width 3.0 cm, what must the free-space wavelength of radiation be if it is to require 1.0 μsec (= 10^{-6} sec) for energy to traverse a 100 meter length of guide? What is the phase speed under these circumstances?

4. Under what conditions will the guide wavelength in the guide of Fig. 39-7 be double the free-space wavelength?

5. How does the displacement current vary with space and time in a traveling plane electromagnetic wave?

6. Prove that for any point in an electromagnetic wave such as that of Fig. 39-11 the density of energy stored in the electric field equals that stored in the magnetic field.

7. A resonant cavity is constructed by closing each end of the coaxial cable of Fig. 39-3 with a metal cap. The cavity contains three half-waves. Describe the patterns of **E** and **B**

that occur, assuming the same mode of oscillation as that shown in Fig. 39–3. (Hint: Remember that **E** can have no tangential component at a conducting surface and that **B** and **E** must be 90° out of phase.)

8. If a coaxial cable has resistance, energy must flow from the fields into the conducting surfaces to provide the Joule heating. How must the electric lines of force of Fig. 39–3*a* be modified in this case? (Hint: The Poynting vector near the surface must have a component pointing toward the surface.)

9. What guide wavelength does 10-cm radiation (free-space wavelength) exhibit in a rectangular guide whose width is 6.0 cm? Assume the dominant mode. What is the cutoff wavelength for this guide?

10. Sketch five more figures to complete the sequence of Fig. 39–9, showing radiation from an oscillating dipole. Include an indication of the lines of **B** in your drawings.

11. Figure 39–15 shows a long resistanceless transmission line, delivering power from a battery to a resistive load. A steady current i exists as shown. (*a*) Sketch qualitatively the electric and magnetic fields around the line, and (*b*) show that, according to the Poynting vector point of view, energy travels from the battery to the resistor through the space around the line and not through the line itself. (Hint: Each conductor in the line is an equipotential surface, since the line has been assumed to have no resistance.)

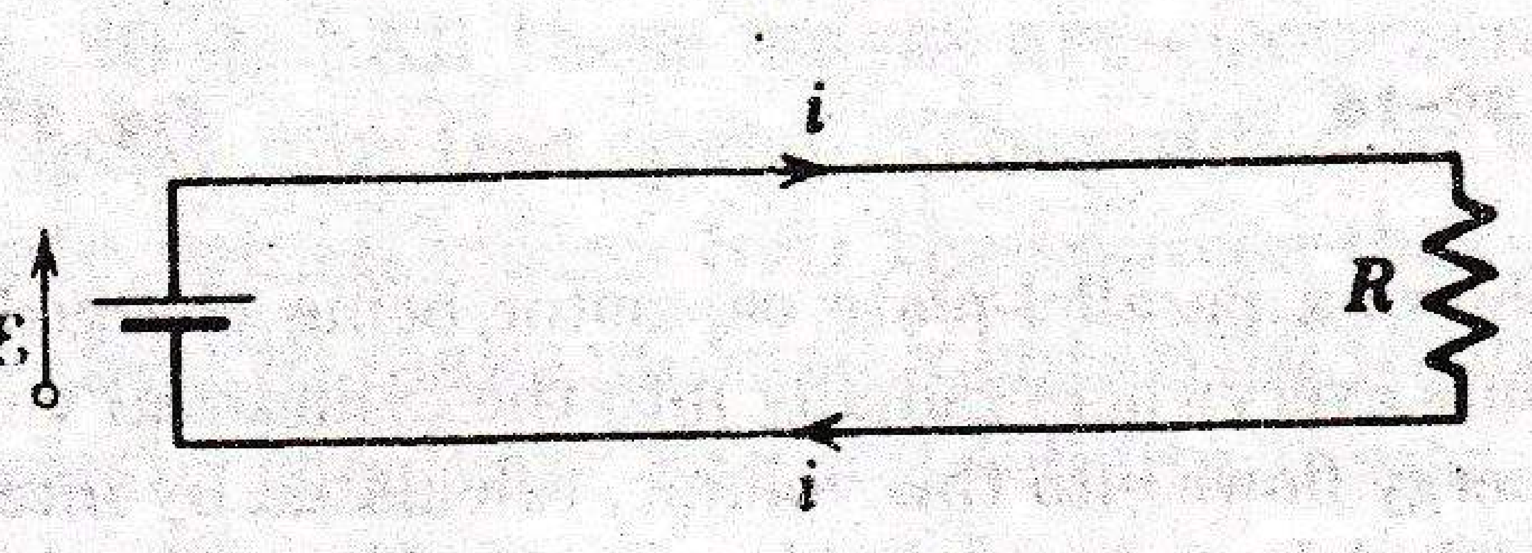


Fig. 39–15

12. Analyze the flow of energy in the waveguide of Fig. 39–7, using the Poynting vector.

13. A cube of edge a has its edges parallel to the x-, y-, and z-axes of a rectangular coordinate system. A uniform electric field **E** is parallel to the y-axis and a uniform magnetic field **B** is parallel to the x-axis. Calculate (*a*) the rate at which, according to the Poynting vector point of view, energy may be said to pass through each face of the cube and (*b*) the net rate at which the energy stored in the cube may be said to change.

14. A #10 copper wire (diameter, 0.10 in.; resistance per 1000 ft, 1.00 ohm) carries a current of 25 amp. Calculate **E**, **B**, and **S** for a point on the surface of the wire.

15. Sunlight strikes the earth, outside its atmosphere, with an intensity of 2.0 cal/cm^2-min. Calculate E_m and B_m for sunlight, assuming it to be a wave like that of Fig. 39–10.

16. A plane radio wave has $E_m \cong 10^{-4}$ volt/meter. Calculate (*a*) B_m and (*b*) the intensity of the wave, as measured by $\bar{S}$.

17. Figure 39–16 shows a cylindrical resistor of length l, radius a, and resistivity ρ, carrying a current i. (*a*) Show that the Poynting vector **S** at the surface of the resistor is everywhere directed normal to the surface, as shown. (*b*) Show that the rate P at which energy flows into the resistor through its cylindrical surface, calculated by integrating the Poynting vector over this surface, is equal to the rate at which Joule heat is produced; that is,

$$\int \mathbf{S}\cdot d\mathbf{A} = i^2R,$$

where $d\mathbf{A}$ is an element of area of the cylindrical surface. This shows that, according to the Poynting vector point of view, the energy that appears in a resistor as Joule heat does not enter it through the connecting wires but through the space around the wires and the resistor. (Hint: **E** is parallel to the axis of the cylinder, in the direction of the current; **B** forms concentric circles around the cylinder, in a direction given by the right-hand rule.)

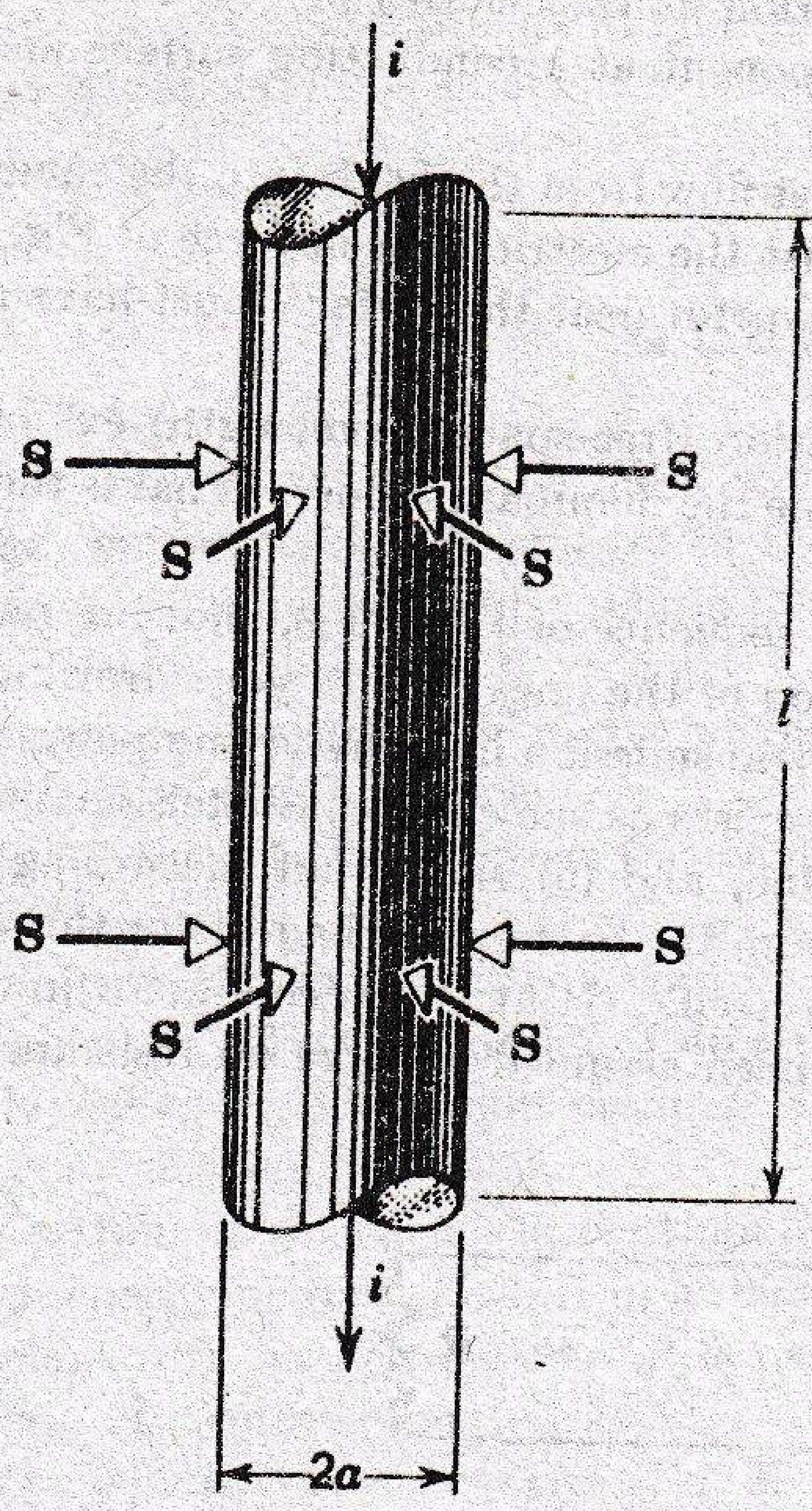


Fig. 39–16

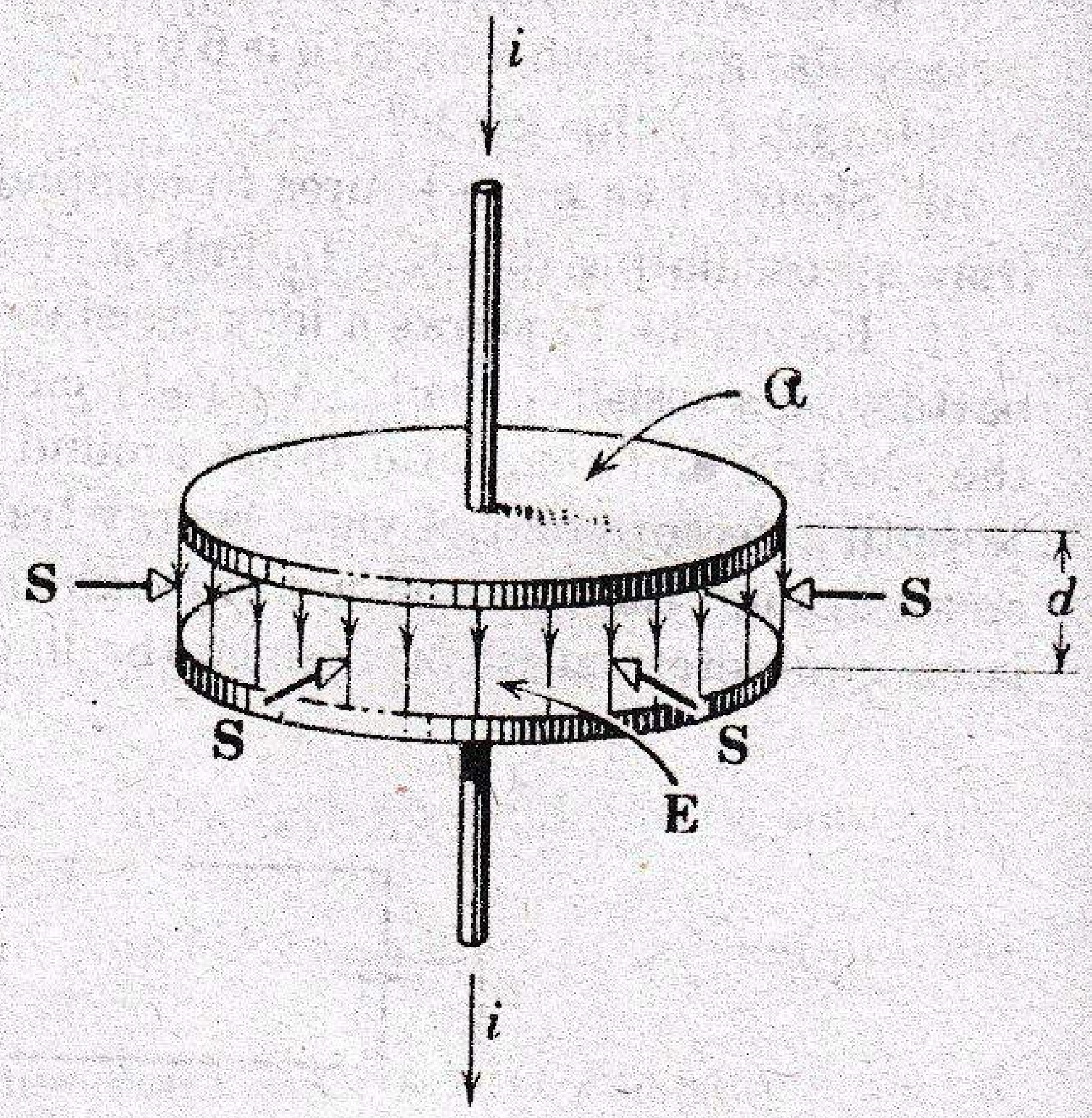


Fig. 39–17

18. Figure 39–17 shows a parallel-plate capacitor being charged. (*a*) Show that the Poynting vector **S** points everywhere radially into the cylindrical volume. (*b*) Show that the rate *P* at which energy flows into this volume, calculated by integrating the Poynting vector over the cylindrical boundary of this volume, is equal to the rate at which the stored electrostatic energy increases; that is, that

$$\int \mathbf{S}\cdot d\mathbf{A} = \mathcal{A}d\,\frac{d}{dt}\left(\tfrac{1}{2}\epsilon_0 E^2\right),$$

where $\mathcal{A}d$ is the volume of the capacitor and $\frac{1}{2}\epsilon_0 E^2$ is the energy density for all points within that volume. This analysis shows that, according to the Poynting vector point of view, the energy stored in a capacitor does not enter it through the wires but through the space around the wires and the plates. (Hint: To find **S** we must first find **B**, which is the magnetic field set up by the displacement current during the charging process; see Fig. 38–9. Ignore fringing of the lines of **E**.)

Nature and Propagation of Light

CHAPTER 40

40–1 Light and the Electromagnetic Spectrum

Light was shown by Maxwell to be a component of the *electromagnetic spectrum* of Fig. 40–1. All these waves are electromagnetic in nature and have the same speed c in free space. They differ in wavelength (and thus in frequency) only, which means that the sources that give rise to them and the instruments used to make measurements with them are rather different.* The electromagnetic spectrum has no definite upper or lower limit. The labeled regions in Fig. 40–1 represent frequency intervals within which a common body of experimental technique, such as common sources and common detectors, exists. All such regions overlap. For example, we can produce radiation of wavelength 10^{-3} meter either by microwave techniques (microwave oscillators) or by infrared techniques (incandescent sources).

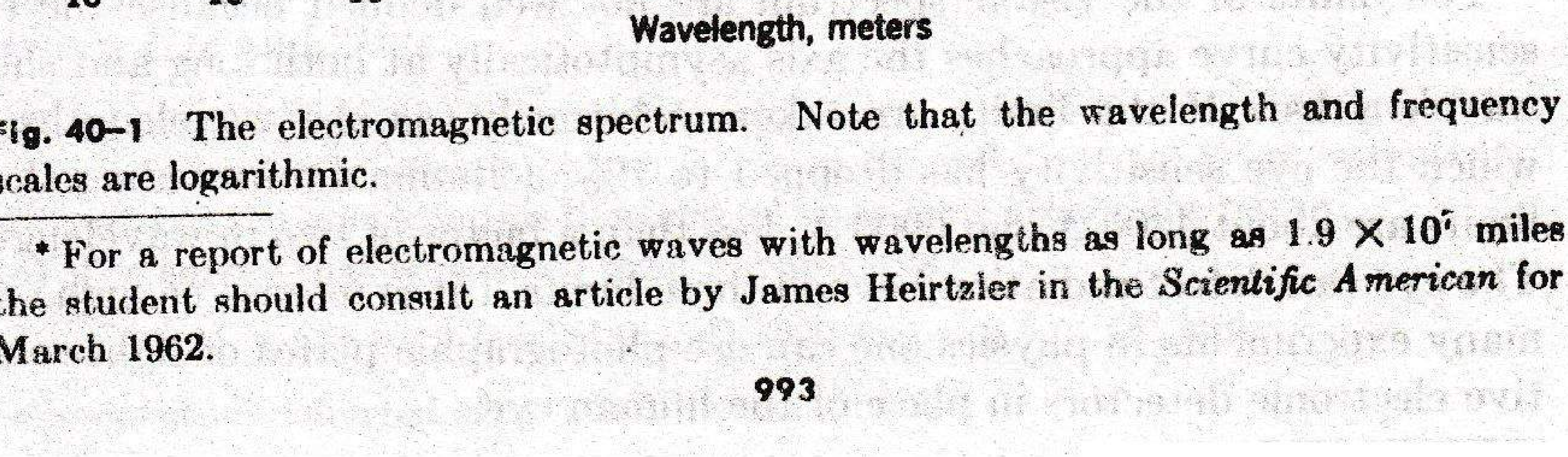


Fig. 40–1 The electromagnetic spectrum. Note that the wavelength and frequency scales are logarithmic.

* For a report of electromagnetic waves with wavelengths as long as 1.9×10^7 miles the student should consult an article by James Heirtzler in the *Scientific American* for March 1962.

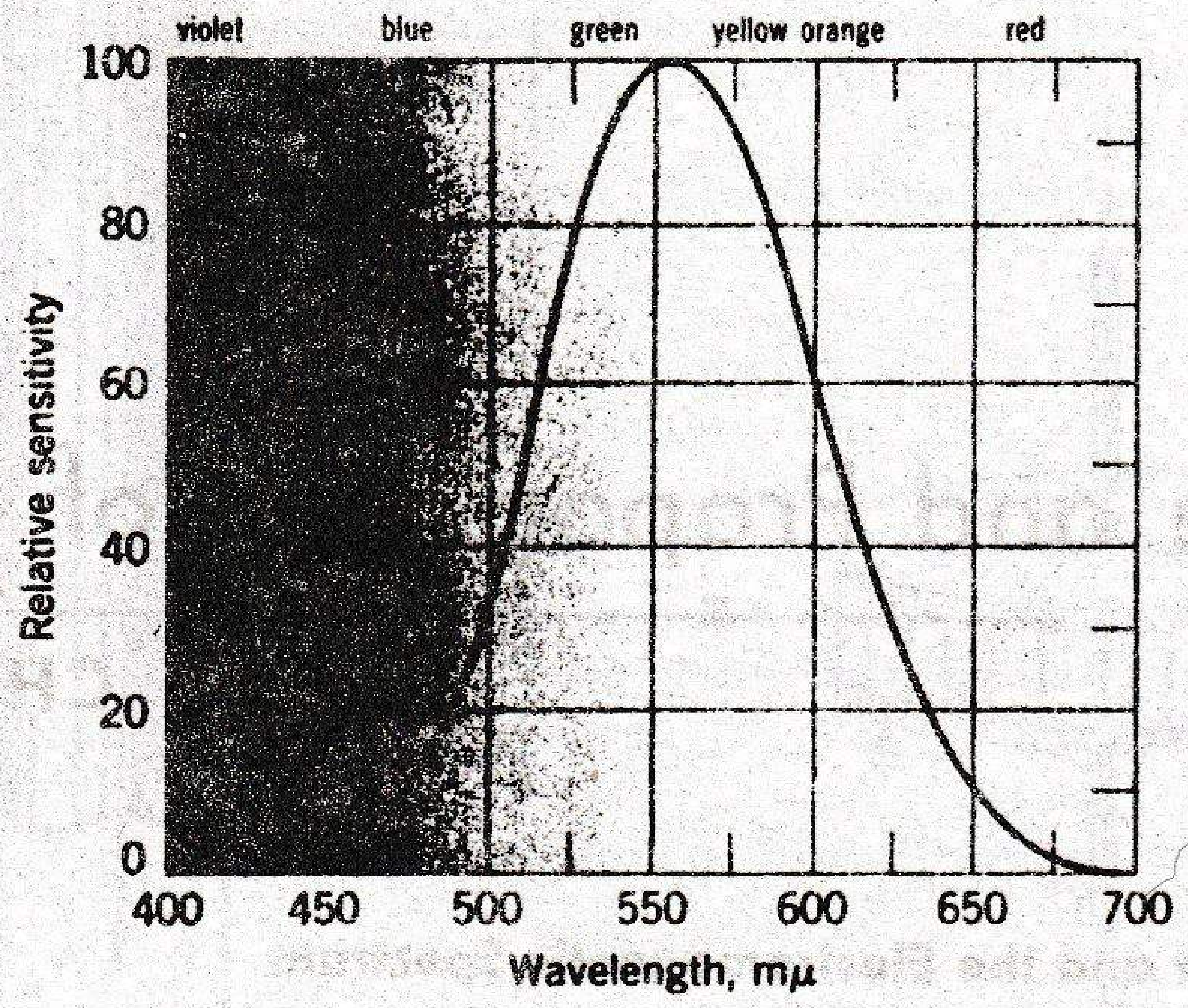


Fig. 40–2 The relative eye sensitivity of an assumed *standard observer* at different wavelengths for normal levels of illumination. The shaded areas represent the (continuously graded) color sensations for normal vision.

"Light" is defined here as radiation that can affect the eye. Figure 40–2, which shows the relative eye sensitivity of an assumed *standard observer* to radiations of various wavelengths, shows that the center of the visible region is about 5.55×10^{-7} meter. Light of this wavelength produces the sensation of yellow-green.*

In optics we often use the micron (abbr. μ) the millimicron (abbr. $m\mu$), and the Angstrom (abbr. A) as units of wavelength. They are defined from

$$1\ \mu = 10^{-6}\ \text{meter}$$

$$1\ m\mu = 10^{-9}\ \text{meter}$$

$$1\ \text{A} = 10^{-10}\ \text{meter}.$$

Thus the center of the visible region can be expressed as $0.555\ \mu$, $555\ m\mu$, or 5550 A.

The limits of the visible spectrum are not well defined because the eye sensitivity curve approaches the axis asymptotically at both long and short wavelengths. If the limits are taken, arbitrarily, as the wavelengths at which the eye sensitivity has dropped to 1% of its maximum value, these limits are about 4300 A and 6900 A, less than a factor of two in wavelength. The eye can detect radiation beyond these limits if it is intense enough. In many experiments in physics one can use photographic plates or light-sensitive electronic detectors in place of the human eye.

* See "Experiments in Color Vision" by Edwin H. Land, *Scientific American*, May 1959, and especially "Color and Perception: the Work of Edwin Land in the Light of Current Concepts" by M. H. Wilson and R. W. Brocklebank, *Contemporary Physics*, December 1961, for a fascinating discussion of the problems of perception and the distinction between color as a characteristic of light and color as a perceived property of objects.

40-2 Energy and Momentum

Energy is carried by electromagnetic waves from the sun to the earth or from an open fire to a hand placed nearby. The transport of energy by such a wave in free space was described in Section 39-6 by the Poynting vector **S**, or

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}, \tag{40-1}$$

where **E** and **B** are the instantaneous values of the electric and magnetic field vectors.

Less familiar is the fact that electromagnetic waves may also transport linear momentum. In other words, it is possible to exert a pressure (a *radiation pressure* *) on an object by shining a light on it. Such forces must be small in relation to forces of our daily experience because we do not ordinarily notice them. The first measurement of radiation pressure was made in 1901–1903 by Nichols and Hull in this country and by Lebedev in Russia, about thirty years after the existence of such effects had been predicted theoretically by Maxwell.

Let a parallel beam of light fall on an object for a time t, the incident light being *entirely absorbed* by the object. If energy U is absorbed during this time, the momentum p delivered to the object is given, according to Maxwell's prediction, by

$$p = \frac{U}{c} \quad \text{(total absorption)}, \tag{40-2a}$$

where c is the speed of light. The direction of **p** is the direction of the incident beam. If the light energy U is *entirely reflected*, the momentum delivered will be twice that given above, or

$$p = \frac{2U}{c} \quad \text{(total reflection)}. \tag{40-2b}$$

In the same way, twice as much momentum is delivered to an object when a perfectly elastic tennis ball is bounced from it as when it is struck by a perfectly inelastic ball of the same mass and speed. If the light energy U is partly reflected and partly absorbed, the delivered momentum will lie between U/c and $2U/c$.

▶ **Example 1.** A parallel beam of light with an energy flux S of 10 watts/cm² falls for 1 hr on a perfectly reflecting plane mirror of 1.0-cm² area. (*a*) What momentum is delivered to the mirror in this time and (*b*) what force acts on the mirror?

(*a*) The energy that is reflected from the mirror is

$$U = (10 \text{ watts/cm}^2)(1.0 \text{ cm}^2)(3600 \text{ sec}) = 3.6 \times 10^4 \text{ joules}.$$

The momentum delivered after 1 hr's illumination is

$$p = \frac{2U}{c} = \frac{(2)(3.6 \times 10^4 \text{ joules})}{3 \times 10^8 \text{ meters/sec}} = 2.4 \times 10^{-4} \text{ kg-m/sec}.$$

* See "Radiation Pressure," G. E. Henry, *Scientific American*, p. 99, June 1957.

(*b*) From Newton's second law, the average force on the mirror is equal to the average rate at which momentum is delivered to the mirror, or

$$F = \frac{p}{t} = \frac{2.4 \times 10^{-4}\text{ kg-m/sec}}{3600\text{ sec}} = 6.7 \times 10^{-8}\text{ nt}.$$

This is a small force. ◀

Nichols and Hull, in 1903, measured radiation pressures and verified Eq. 40–2, using a torsion balance technique. They allowed light to fall on mirror M in Fig. 40–3; the radiation pressure caused the balance arm to turn through a measured angle θ, twisting the torsion fiber F. Assuming a suitable calibration for their torsion fiber, the experimenters could arrive at a numerical value for this pressure. Nichols and Hull measured the intensity of their light beam by allowing it to fall on a blackened metal disk of known absorptivity and by measuring the temperature rise of this disk. In a particular run these experimenters measured a radiation pressure of 7.01×10^{-6} nt/meter^2; for their light beam, the value predicted, using Eq. 40–2, was 7.05×10^{-6} nt/meter^2, in excellent agreement. Assuming a mirror area of 1 cm^2, this represents a force on the mirror of only 7×10^{-10} nt, about 100 times smaller than the force calculated in Example 1.

The success of the experiment of Nichols and Hull was the result in large part of the care they took to eliminate spurious deflecting effects caused by changes in the speed distribution of the molecules in the gas surrounding the mirror. These changes were brought about by the small rise in the temperature of the mirror as it absorbed light energy from the incident beam. This "radiometer effect" is responsible for the spinning action of the familiar toy radiometers when placed in a beam of sunlight. In a perfect vacuum such effects would not occur, but in the best vacuums available in 1903 radiometer effects were present and had to be taken specifically into account in the design of the experiment.

To demonstrate the transport of momentum from Maxwell's equations in a particular case, let a plane electromagnetic wave traveling in the z direction fall on a

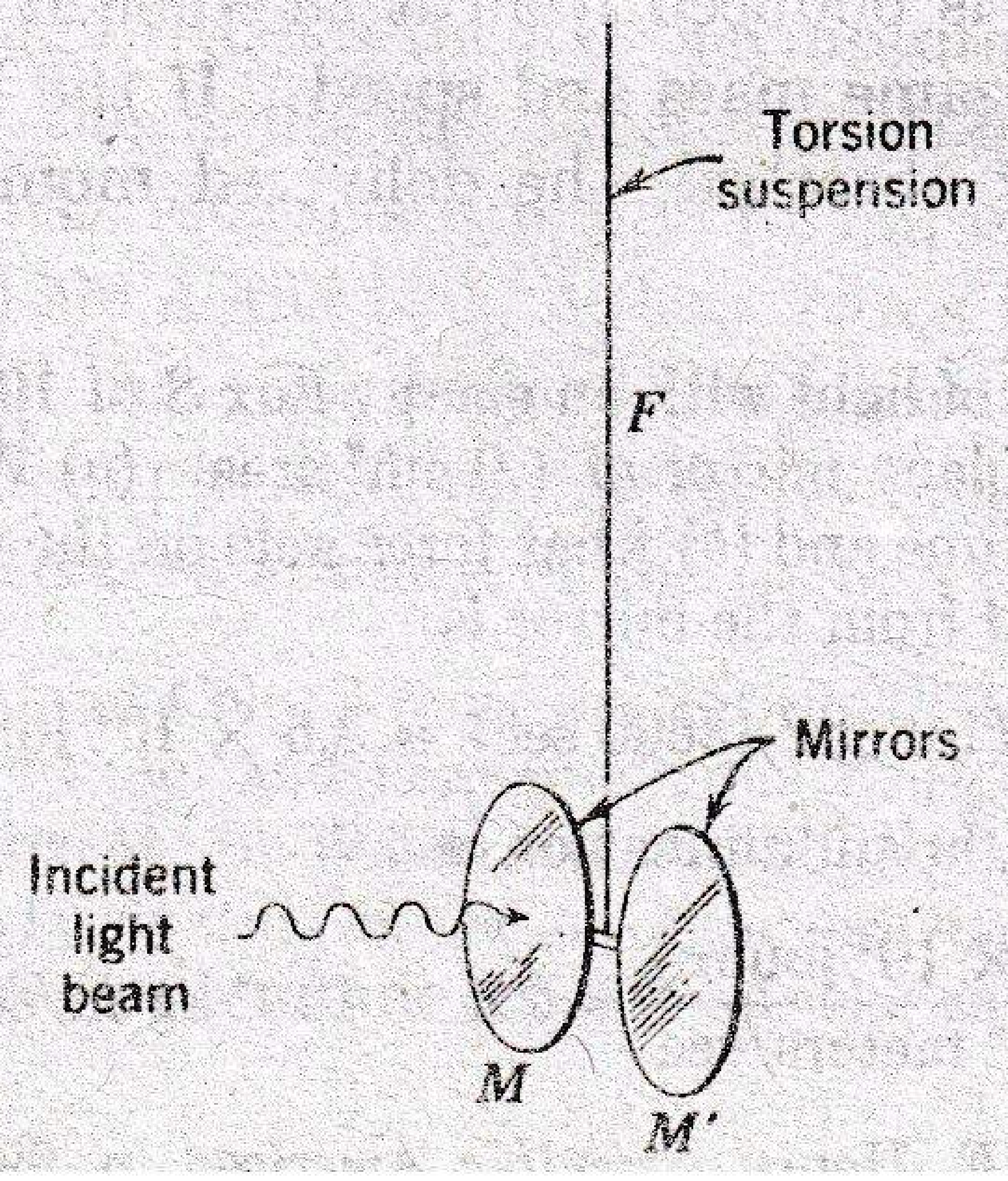


Fig. 40–3

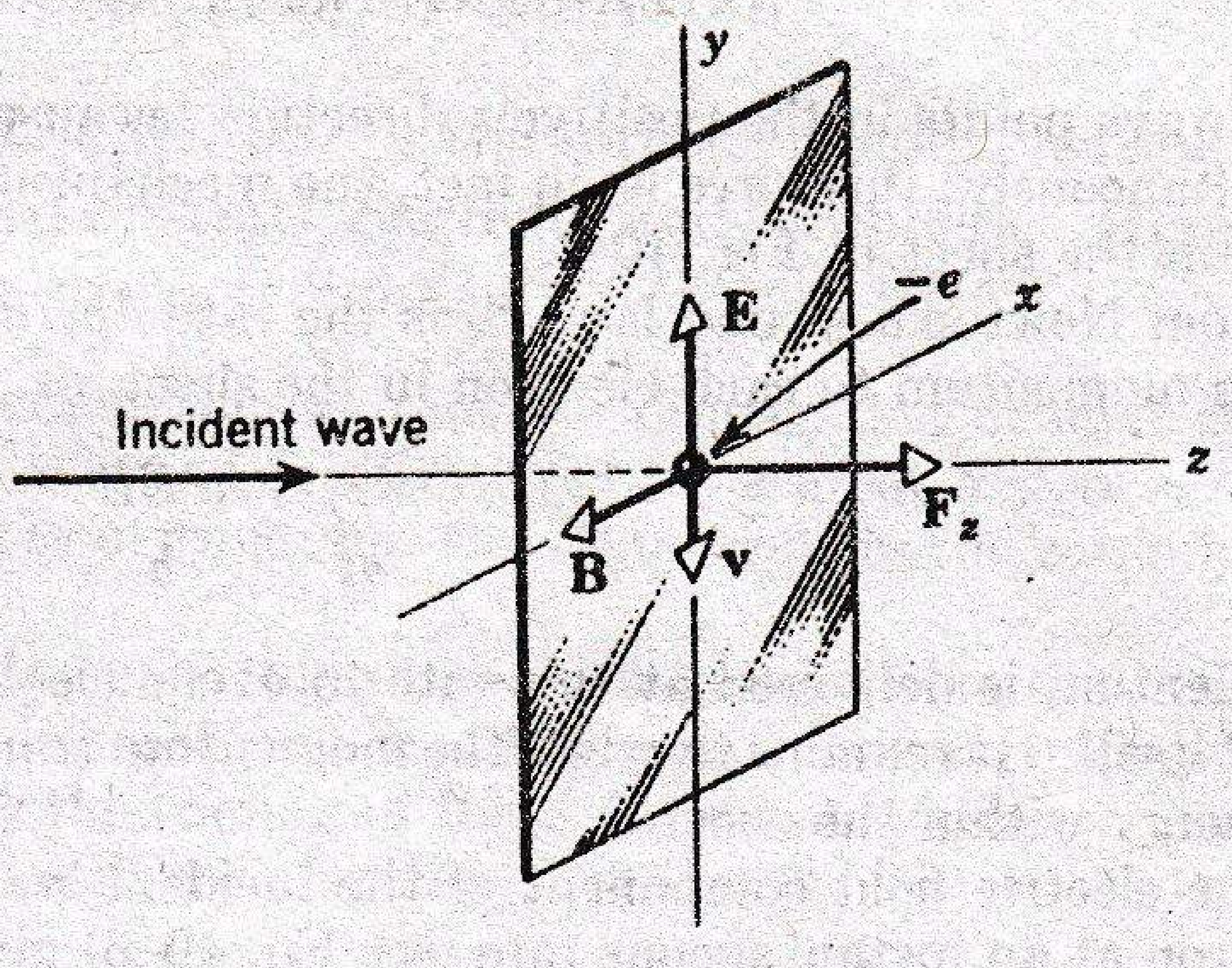


Fig. 40-4 An incident plane light wave falls on an electron in a thin resistive sheet. Instantaneous values of **E**, **B**, the electron velocity **v**, and the radiation force $\mathbf{F}_z$ are shown.

large thin sheet of a material of high resistivity as in Fig. 40-4. A small part of the incident energy will be absorbed within the sheet, but most of it will be transmitted if the sheet is thin enough.*

The incident wave **E** and **B** vary with time at the sheet as

$$\mathbf{E} = \mathbf{E}_m \sin \omega t \tag{40-3}$$

and

$$\mathbf{B} = \mathbf{B}_m \sin \omega t \tag{40-4}$$

where **E** is parallel to the $\pm y$ axis and **B** is parallel to the $\pm x$ axis.

In Section 31-4 we saw that the effect of a (constant) electric force ($= -eE$) on a conduction electron in a metal was to make it move with a (constant) drift speed v_d. The electron behaves as if it is immersed in a viscous fluid, the electric force acting on it being counterbalanced by a "viscous" force, which may be taken as proportional to the electron speed. Thus for a constant field E, after equilibrium is established,

$$eE = bv_d, \tag{40-5}$$

where b is a resistive damping coefficient. The electron equilibrium speed, dropping the subscript d, is thus

$$v = \frac{eE}{b}. \tag{40-6}$$

If the applied electric field varies with time and if the variation is slow enough, the electron speed can continually readjust itself to the changing value of E so that its speed continues to be given essentially by its equilibrium value (Eq. 40-6) at all times. These readjustments are more rapidly made the more viscous the medium, just as a stone falling in air reaches a constant equilibrium rate of descent only relatively slowly but one falling in a viscous oil does so quite rapidly. We assume that the sheet in Fig. 40-4 is so viscous, that is, that its resistivity is so high, that Eq. 40-6 remains valid even for the rapid oscillations of E in the incident light beam.

As the electron vibrates parallel to the y axis, it experiences a *second* force due to the *magnetic* component of the wave. This force $\mathbf{F}_z$ ($= -e\mathbf{v} \times \mathbf{B}$) points in the z direction, being at right angles to the plane formed by **v** and **B**, that is, the xy plane. The instantaneous magnitude of $\mathbf{F}_z$ is given by

$$F_z = evB = \frac{e^2EB}{b}. \tag{40-7}$$

* Some of the incident energy will also be reflected, but the reflected wave is of such low intensity that it can be ignored in the derivation that follows; see *Optics* by B. Rossi, Addison-Wesley Publishing Company, p. 411, 1957, from which this derivation is adapted.

$\mathbf{F}_z$ always points in the positive z direction because $\mathbf{v}$ and $\mathbf{B}$ reverse their directions simultaneously; this force is, in fact, the mechanism by which the radiation pressure acts on the sheet of Fig. 40-4.

From Newton's second law, F_z is the rate dp_e/dt at which the incident wave delivers momentum to each electron in the sheet, or

$$\frac{dp_e}{dt} = \frac{e^2EB}{b}. \tag{40-8}$$

Momentum is delivered at this rate to every electron in the sheet and thus to the sheet itself. It remains to relate the momentum transfer to the sheet to the absorption of energy within the sheet.

The electric field component of the incident wave does work on each oscillating electron at an instantaneous rate (see Eq. 40-6) given by

$$\frac{dU_e}{dt} = F_E v = (eE)\left(\frac{eE}{b}\right) = \frac{e^2E^2}{b}.$$

Note that the magnetic force $\mathbf{F}_z$, always being at right angles to the velocity $\mathbf{v}$, does no work on the oscillating electron. Equation 39-11b shows that for a plane wave in free space B and E are related by

$$E = Bc.$$

Substituting above for *one* of the E's leads to

$$\frac{dU_e}{dt} = \frac{e^2EBc}{b}. \tag{40-9}$$

This equation represents the rate, per electron, at which energy is absorbed from the incident wave.

Comparing Eqs. 40-8 and 40-9 shows that

$$\frac{dp_e}{dt} = \frac{1}{c}\frac{dU_e}{dt}.$$

Integrating yields
$$\int_0^t \frac{dp_e}{dt}\,dt = \frac{1}{c}\int_0^t \frac{dU_e}{dt}\,dt,$$

or
$$p_e = \frac{U_e}{c}, \tag{40-10}$$

where p_e is the momentum delivered to a single electron in any given time t and U_e is the energy absorbed by that electron in the same time interval. Multiplying each side by the number of free electrons in the sheet leads to Eq. 40-2*a*.

Although we derived this relation (Eq. 40-10) for a particular kind of absorber, no characteristics of the absorber—for example, the resistive damping coefficient b—remain in the final expression. This is as it should be because Eq. 40-10 is a general property of radiation absorbed by *any* material.

40-3 The Speed of Light *

Light travels so fast that there is nothing in our daily experience to suggest that its speed is not infinite. It calls for considerable insight even to ask "How fast does light travel?" Galileo asked himself this question and tried to answer it experimentally. His chief work, *Two New Sciences*, published

* See "The Speed of Light," J. H. Rush, *Scientific American*, p. 67, August 1955.

in the Netherlands in 1638, is written in the form of a conversation among three fictitious persons called Salviati, Sagredo, and Simplicio. Here is part of what they say about the speed of light.

Simplicio: Everyday experience shows that the propagation of light is instantaneous; for when we see a piece of artillery fired, at a great distance, the flash reaches our eyes without lapse of time; but the sound reaches the ear only after a noticeable interval.

Sagredo: Well, Simplicio, the only thing I am able to infer from this familiar bit of experience is that sound, in reaching our ear, travels more slowly than light; it does not inform me whether the coming of the light is instantaneous or whether, although extremely rapid, it still occupies time. . . .

Sagredo, who evidently is Galileo himself, then describes a possible method for measuring the speed of light. He and an assistant stand facing each other some distance apart, at night. Each carries a lantern which can be covered or uncovered at will. Galileo started the experiment by uncovering his lantern. When the light reached the assistant he uncovered his own lantern, whose light was then seen by Galileo. Galileo tried to measure the time between the instant at which he uncovered his own lantern and the instant at which the light from his assistant's lantern reached him. For a 1-mile separation we now know that the round trip travel time would be only 11×10^{-6} sec. This is much less than human reaction times, so the method fails.

To measure a large velocity directly, we must either measure a small time interval or use a long base line. This situation suggests that astronomy, which deals with great distances, might be able to provide an experimental value for the speed of light; this proved to be true. Although it would be desirable to time the light from the sun as it travels to the earth, there is no way of knowing when the light that reaches us at any instant left the sun; we must use subtler astronomical methods.

Note, however, that microwave pulses are quite regularly reflected from the moon; this gives a 7.68×10^8-meter base line (there and back) for timing purposes. The speed of light (and of microwaves) is so well known now from other experiments that these measurements are used to measure the lunar distance accurately. Microwave signals have also been reflected from Venus.

In 1675 Ole Roemer, a Danish astronomer working in Paris, made some observations of the moons of Jupiter (see Problem 9) from which a speed of light of 2×10^8 meters/sec may be deduced. About fifty years later James Bradley, an English astronomer, made some astronomical observations of an entirely different kind from which a value of 3.0×10^8 meters/sec may be deduced.

In 1849 Hippolyte Louis Fizeau (1819–1896), a French physicist, first measured the speed of light by a nonastronomical method, obtaining a value of 3.13×10^8 meters/sec. Figure 40–5 shows Fizeau's apparatus. Let us first ignore the toothed wheel. Light from source S is made to converge by lens L_1, is reflected from mirror M_1, and forms in space at F an image of the source. Mirror M_1 is a so-called "half-silvered mirror"; its reflecting coating is so thin that only half the light that falls on it is reflected, the other half being transmitted.

Light from the image at F enters lens L_2 and emerges as a parallel beam; after passing through lens L_3 it is reflected back along its original direction by mirror M_2. In Fizeau's experiment the distance l between M_2 and F was

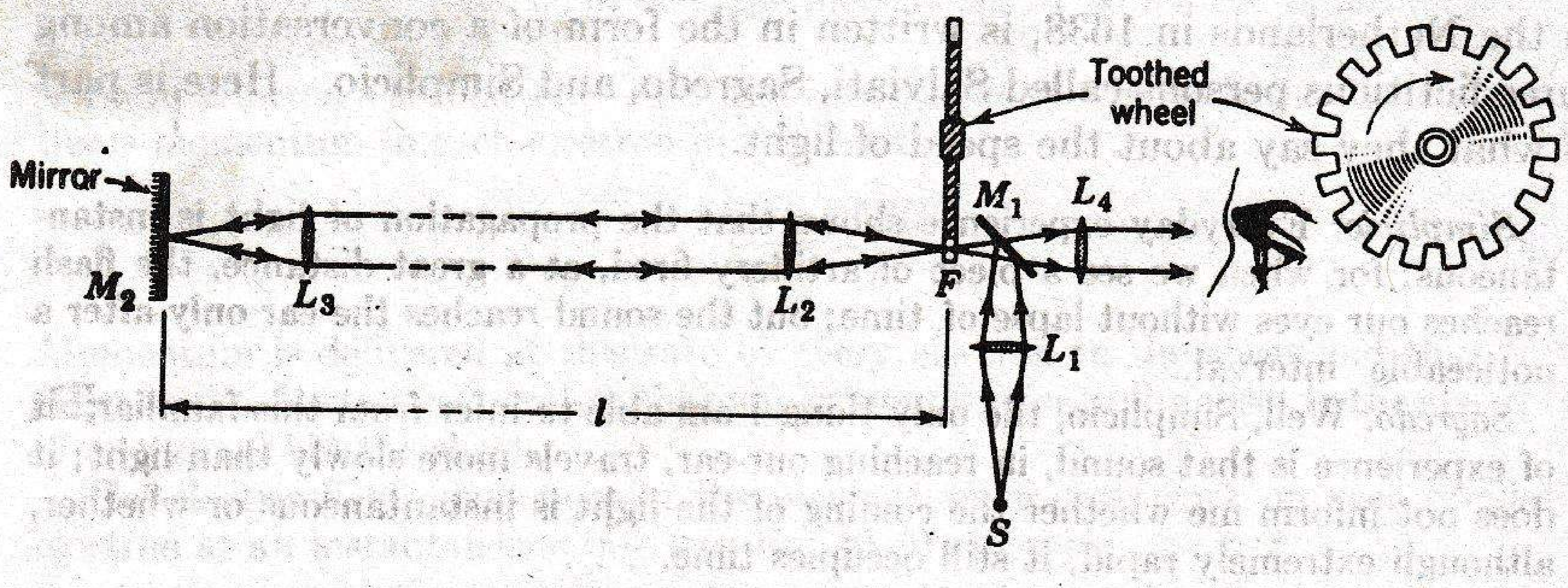


Fig. 40–5 Fizeau's apparatus for measuring the speed of light.

8630 meters or 5.36 miles. When the light strikes mirror M_1 again, some will be transmitted, entering the eye of the observer through lens L_4.

The observer will see an image of the source formed by light that has traveled a distance $2l$ between the wheel and mirror M_2 and back again. To time the light beam a marker of some sort must be put on it. This is done by "chopping" it with a rapidly rotating toothed wheel. Suppose that during the round-trip travel time of $2l/c$ the wheel has turned just enough so that, when the light from a given "burst" returns to the wheel, point F is covered by a tooth. The light will hit the face of the tooth that is toward M_2 and will not reach the observer's eye.

If the speed of the wheel is exactly right, the observer will not see any of he bursts because each will be screened by a tooth. The observer measures : by increasing the angular speed ω of the wheel from zero until the image of source S disappears. Let θ be the angular distance from the center of a gap to the center of a tooth. The time needed for the wheel to rotate a distance θ is the round-trip travel time $2l/c$. In equation form,

$$\frac{\theta}{\omega} = \frac{2l}{c} \quad \text{or} \quad c = \frac{2\omega l}{\theta}. \tag{40-11}$$

This "chopped beam" technique, suitably modified, is used today to measure the speeds of neutrons and other particles.

▶ **Example 2.** The wheel used by Fizeau had 720 teeth. What is the smallest angular speed at which the image of the source will vanish?

The angle θ is 1/1440 rev; solving Eq. 40–11 for ω gives

$$\omega = \frac{c\theta}{2l} = \frac{(3.00 \times 10^8 \text{ meters/sec})(1/1440 \text{ rev})}{(2)(8630 \text{ meters})} = 12.1 \text{ rev/sec.}$$ ◀

The French physicist Foucault (1819–1868) greatly improved Fizeau's method by substituting a rotating mirror for the toothed wheel. The American physicist Albert A. Michelson (1852–1931) conducted an extensive series of measurements of c, extending over a fifty-year period, using this technique.

Table 40-1

THE SPEED OF ELECTROMAGNETIC RADIATION IN FREE SPACE
(Some selected measurements)

Date	Experimenter	Country	Method	Speed, km/sec	Uncertainty, km/sec
1600(?)	Galileo	Italy	Lanterns and shutters	"If not instantaneous, it is extraordinarily rapid"	
1675	Roemer	France	Astronomical	200,000	
1729	Bradley	England	Astronomical	304,000	
1849	Fizeau	France	Toothed wheel	313,300	
1862	Foucault	France	Rotating mirror	298,000	500
1876	Cornu	France	Toothed wheel	299,990	200
1880	Michelson	U.S.A.	Rotating mirror	299,910	50
1883	Newcomb	England	Rotating mirror	299,860	30
1883	Michelson	U.S.A.	Rotating mirror	299,853	60
1906	Rosa and Dorsey	U.S.A.	Electromagnetic theory	299,781	10
1923	Mercier	France	Standing waves on wires	299,782	15
1926	Michelson	U.S.A.	Rotating mirror	299,796	4
1928	Karolus and Mittelstaedt	Germany	Kerr cell	299,778	10
1932	Michelson, Pease, and Pearson	U.S.A.	Rotating mirror	299,774	11
1940	Huettel	Germany	Kerr cell	299,768	10
1941	Anderson	U.S.A.	Kerr cell	299,776	14
1950	Bergstrand	Sweden	Geodimeter	299,792.7	0.25
1950	Essen	England	Microwave cavity	299,792.5	3
1950	Houston	Scotland	Vibrating crystal	299,775	9
1950	Bol and Hansen	U.S.A.	Microwave cavity	299,789.3	0.4
1951	Aslakson	U.S.A.	Shoran radar	299,794.2	1.9
1952	Rank, Ruth, and Ven der Sluis	U.S.A.	Molecular spectra	299,776	7
1952	Froome	England	Microwave interferometer	299,792.6	0.7
1954	Florman	U.S.A.	Radio interferometer	299,795.1	3.1
1954	Rank, Shearer, and Wiggins	U.S.A.	Molecular spectra	299,789.8	3.0
1956	Edge	Sweden	Geodimeter	299,792.9	0.2

We must view the speed of light within the larger framework of the speed of electromagnetic radiation in general. It is a significant experimental confirmation of Maxwell's theory of electromagnetism that the speed in free space of waves in all parts of the electromagnetic spectrum has the same value c. Table 40-1 shows some selected measurements that have been made of the speed of electromagnetic radiation since Galileo's day. It stands as a monument to man's persistence and ingenuity. Note in the last column how the uncertainty in the measurement has improved through the years. Note also the international character of the effort and the variety of methods.

The task of arriving at a single "best" value of c from the many listed in the table is difficult, for it involves a careful study of each reported measurement and a selection from among them, based on the reported uncertainty and the selector's judgment of the probable presence or absence of hidden error. In the final averaging measurements with small uncertainties will be given more weight than those with large ones. By careful analysis of such measurements the "best" value, as of 1964, is

$$c = 2.997925 \times 10^8 \text{ meters/sec.}$$

The uncertainty of measurement is less than 0.000003×10^8 meters/sec or 0.0001%.

Since about 1940, nearly all precise measurements of c have been made in the microwave or the short radio wave region of the electromagnetic spectrum. We describe here the "microwave cavity method" used by Essen in England and by Bol and Hansen in the U.S.A. It employs standing electromagnetic waves confined to a cavity rather than traveling waves in free space.

It is possible to convert a section of waveguide such as that of Fig. 39-7 into a resonant cavity by closing it with two metal caps; see Fig. 40-6. The pattern of oscillations in the cavity is closely related to that in the guide and exhibits the same "guide wavelength" λ_g. The guide wavelength is related to the cavity length l by

$$\lambda_g = \frac{2l}{n} \quad n = 1, 2, 3, \cdots, \tag{40-12}$$

which is the same relationship used for acoustic waves in closed pipes; n (= 3 for Fig. 40-6) gives the number of half-waves contained in the cavity.

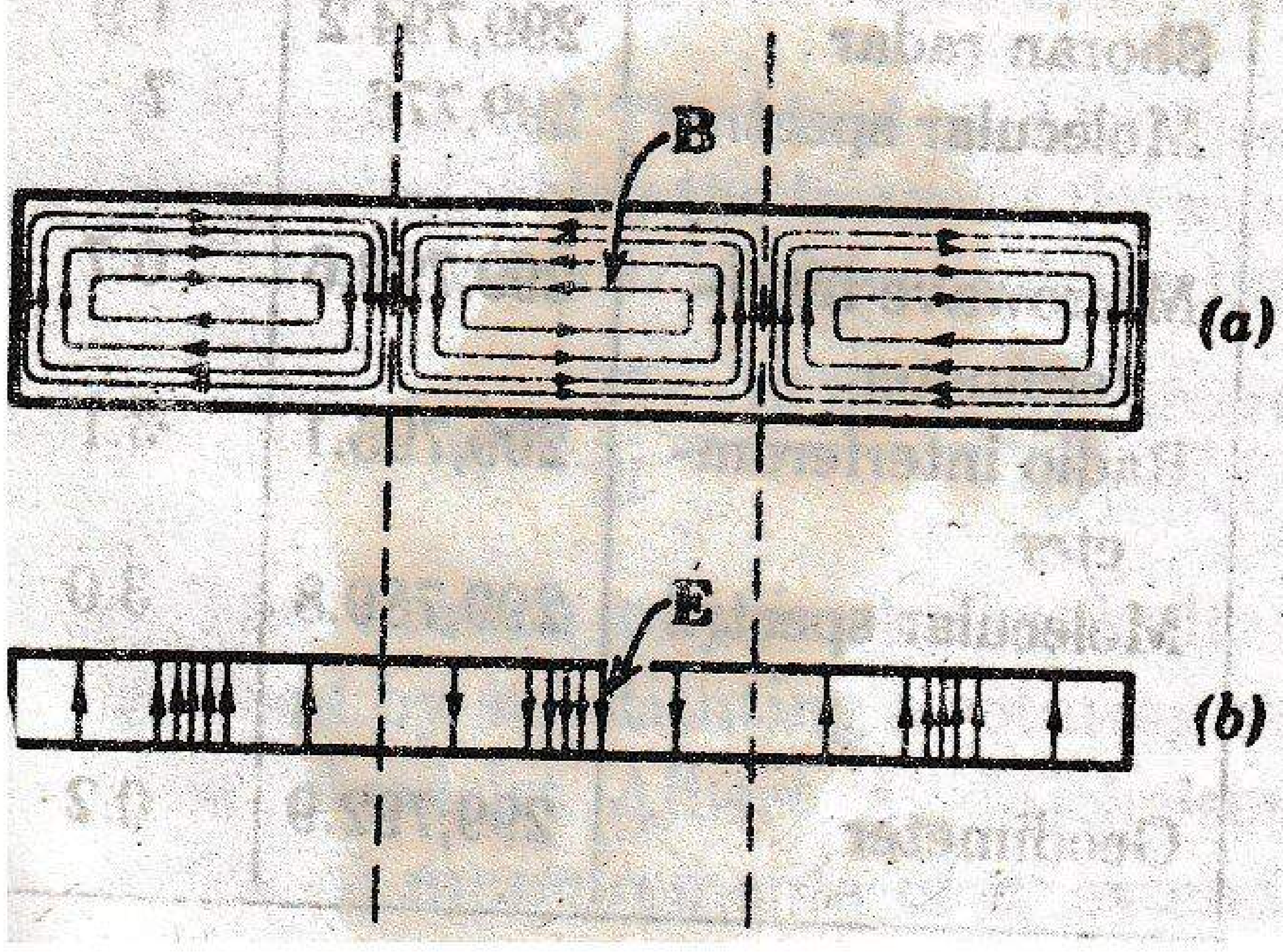


Fig. 40-6 A resonant cavity constructed from a section of waveguide; compare Fig. 39-7. For simplicity the lines of E are not shown in (a) and those of B are not shown in (b).

The procedure is to measure λ_g for such a cavity, which has been tuned to resonance, and then, using Eq. 39-4,

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}}, \tag{40-13}$$

calculate the free-space wavelength λ. From the measured resonant frequency, the speed c can be found from $c = \lambda\nu$.

▶ **Example 3.** Essen of the National Physical Laboratory in England made a resonant cavity measurement of the speed of electromagnetic waves. His cavity was made of a circular waveguide rather than a rectangular one; it can be shown that, for the oscillation pattern used by him, the geometrical factor $2a$ in Eq. 40-13 must be replaced by $1.64062R$, where R is the guide radius. The cavity radius was 3.25876 cm; the cavity length was 15.64574 cm and it proved to resonate at 9.498300×10^9 cycles/sec. At resonance it was determined that there were eight half-waves in the cavity. What value of c results?

From Eq. 40-12, computing only an approximate result,

$$\lambda_g = \frac{2l}{n} = \frac{(2)(15.6 \text{ cm})}{8} = 3.90 \text{ cm}.$$

Substituting into Eq. 40-13, suitably modified for a circular waveguide, yields

$$3.90 \text{ cm} = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{(1.64)(3.26 \text{ cm})}\right)^2}}.$$

Solving this equation for λ yields $\lambda = 3.15$ cm. Finally, we have

$$c = \lambda\nu = (3.15 \text{ cm})(9.50 \times 10^9 \text{ cycles/sec}) = 2.99 \times 10^8 \text{ meters/sec}.$$

For practical reasons Essen analyzed his data in a more roundabout way than that given. His final result, based on many measurements under different conditions and carried to much greater accuracy than that illustrated in the above example, was 2.997925×10^8 meters/sec with an uncertainty of 0.000030×10^8 meters/sec. ◀

40-4 Moving Sources and Observers

When we say that the speed of sound in dry air at 0°C is 331.7 meters/sec, we imply a reference frame fixed with respect to the air mass. When we say that the speed of light in free space is 2.997925×10^8 meters/sec, what reference frame is implied? It cannot be the medium through which the light wave travels because, in contrast to sound, no medium is required.

The concept of a wave requiring no medium was abhorrent to the physicists of the nineteenth century, influenced as they then were by a false analogy between light waves and sound waves or other purely mechanical disturbances. These physicists postulated the existence of an *ether*, which was a tenuous substance that filled all space and served as a medium of transmission for light. The ether was required to have a vanishingly small density to account for the fact that it could not be observed by any known means in an evacuated space.

The ether concept, although it proved useful for many years, did not survive the test of experiment. In particular, careful attempts to measure

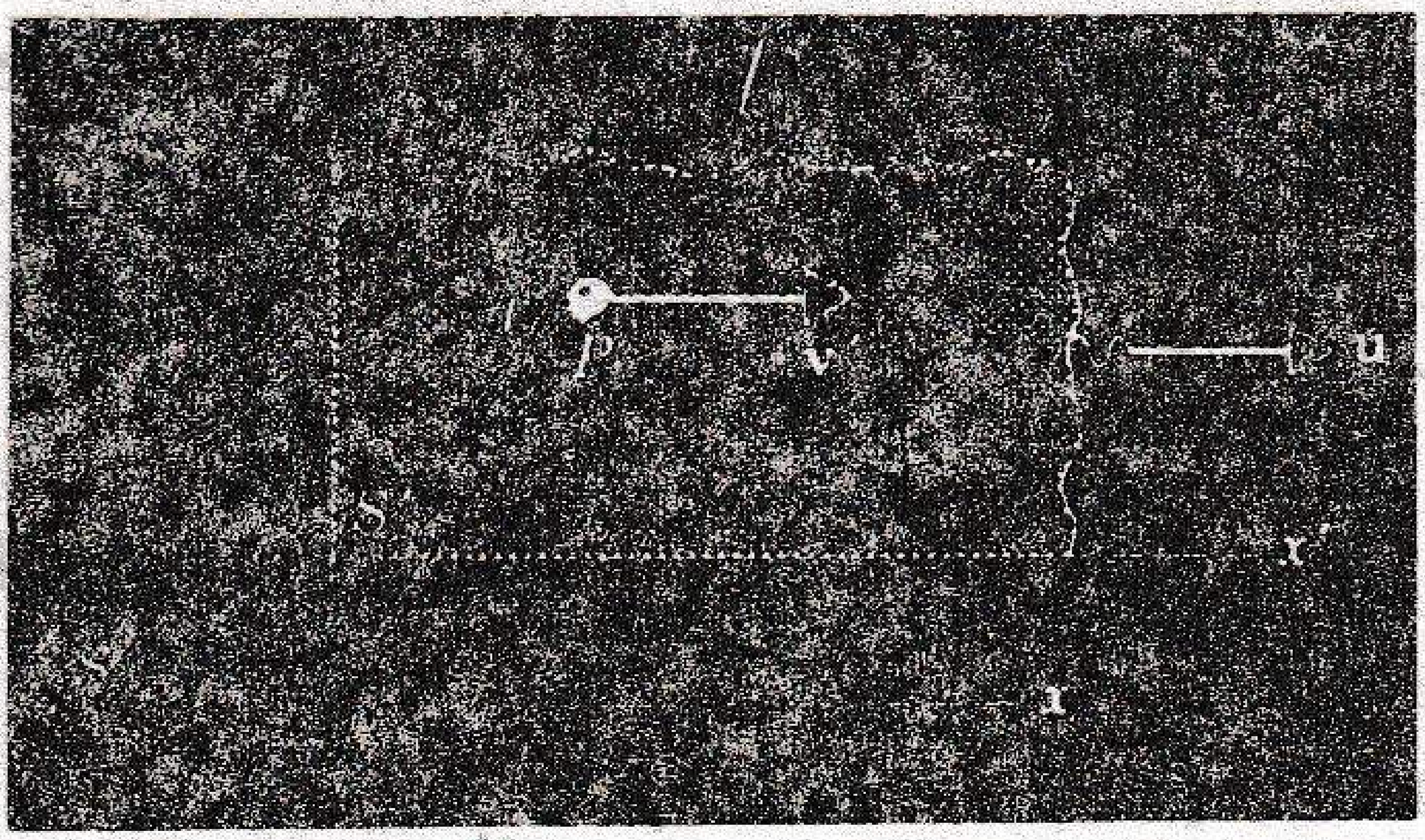


Fig. 40–7 Observers S and S', who are in relative motion, each observe a light pulse P. The pulse is emitted from a source, not shown, that is at rest in the S' frame of reference.

the speed of the earth through the ether always gave the result of zero.* Physicists were not willing to believe that the earth was permanently at rest in the ether and that all other bodies in the universe were in motion through it. Other hypotheses about the nature of the propagation of light also proved unsatisfactory for one reason or another.

Einstein in 1905 resolved the difficulty of understanding the propagation of light by making a bold postulate: If a number of observers are moving (at uniform velocity) with respect to each other and to a source of light and if each observer measures the speed of the light emerging from the source, *they will all obtain the same value.* This is the fundamental assumption of Einstein's theory of relativity. It does away with the need for an ether by asserting that the speed of light is the same in *all* reference frames; none is singled out as fundamental. The theory of relativity, derived from this postulate, has been subject to many experimental tests, from which agreement with the predictions of theory has always emerged. These agreements, extending over half a century, lend strong support to Einstein's basic postulate about light propagation.

Figure 40–7 focuses specifically on the fundamental problem of light propagation. A source of light, at rest in reference frame S', emits a light pulse P whose speed v' is measured by an observer at rest in this same frame. From the point of view of an observer in reference frame S, frame S' and its associated observer are moving in the positive x direction at speed u. Question: What speed v would observer S measure for the light pulse P? Einstein's hypothesis asserts that *each* observer would measure the same speed c, or that

$$v = v' = c.$$

This hypothesis contradicts the classical law of addition of velocities (see Section 4–6), which asserts that

$$v = v' + u. \tag{40-14}$$

This law, which is so familiar that it seems (incorrectly) to be intuitively true, is in fact based on observations of gross moving objects in the world about us. Even the fastest of these—an earth satellite, say—is moving at

* See Section 43–7, which describes the crucial experiment of Michelson and Morley.

a speed that is quite small compared to that of light. The body of experimental evidence that underlies Eq. 40-14 thus represents a severely restricted area of experience, namely, experiences in which $v' \ll c$ and $u \ll c$. If we assume that Eq. 40-14 holds for all particles regardless of speed, we are making a gross extrapolation. Einstein's theory of relativity predicts that this extrapolation is indeed not valid and that Eq. 40-14 is a limiting case of a more general relationship that holds for light pulses and for material particles, whatever their speed, or

$$v = \frac{v' + u}{1 + v'u/c^2}. \tag{40-15}$$

Equation 40-15 is quite indistinguishable from Eq. 40-14 at low speeds, that is, when $v' \ll c$ and $u \ll c$; see Example 4.

If we apply Eq. 40-15 to the case in which the moving object is a light pulse, and if we put $v' = c$, we obtain

$$v = \frac{c + u}{1 + cu/c^2} = c.$$

This is consistent, as it must be, with the fundamental assumption on which the derivation of Eq. 40-15 is based; it shows that *both* observers measure the same speed c for light. Equation 40-14 predicts (incorrectly) that the speed measured by S will be $c + u$. Figure 40-8 shows that the (correct) Eq. 40-15 and the (approximate) Eq. 40-14 cannot be distinguished from each other at speeds that are small compared to the speed of light.

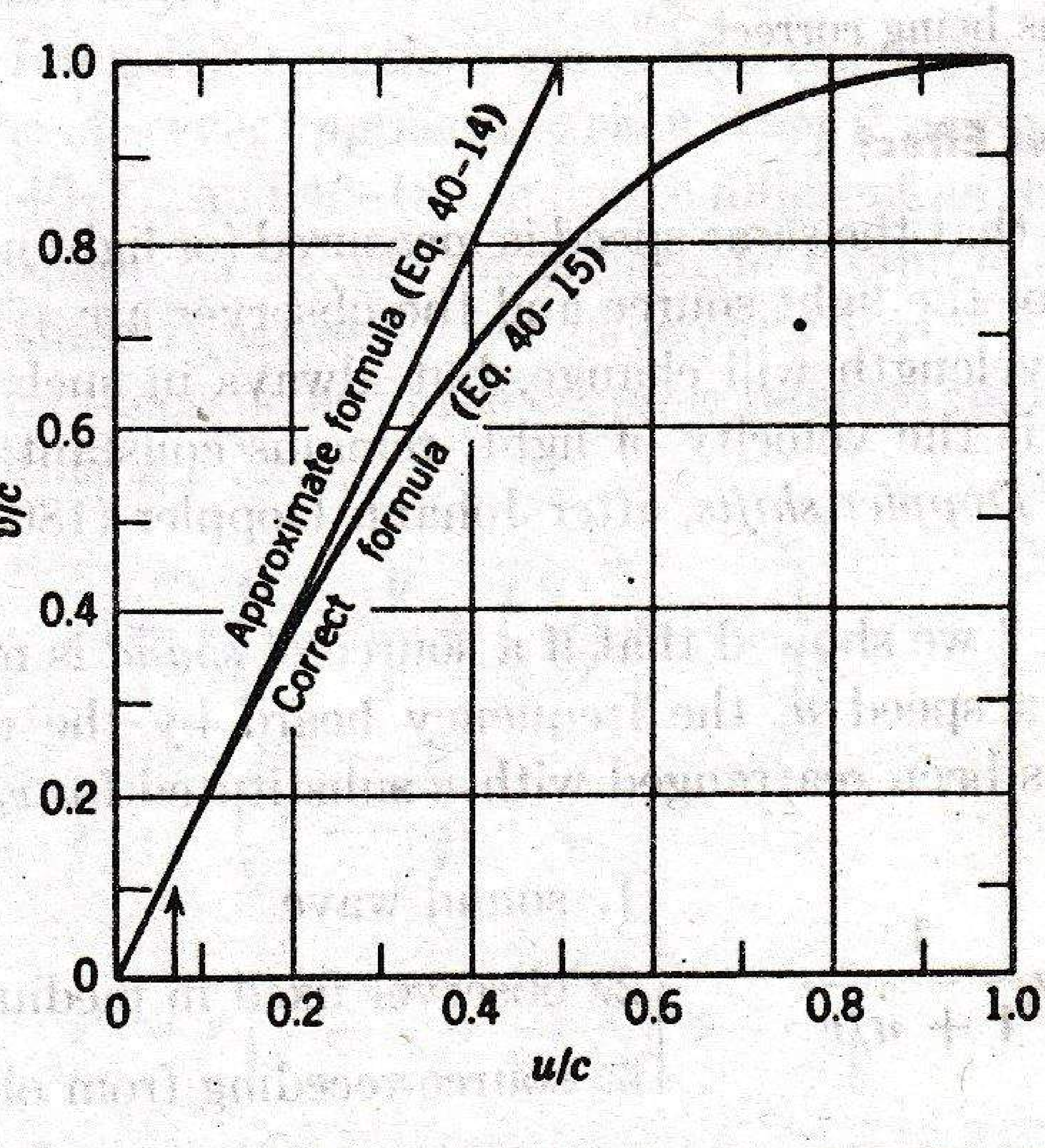


Fig. 40-8 The speed of a particle P, as seen by observer S in Fig. 40-7, for the special case of $v' = u$. All speeds are expressed as a ratio to c, the speed of light. The vertical arrow corresponds to 5×10^7 miles/hr.

▶ **Example 4.** Suppose that $v' = u = 25{,}000$ miles/hr. What per cent error is made in using Eq. 40–14 rather than Eq. 40–15 to calculate v? The speed of light is 6.7×10^8 miles/hr.

Equation 40–14 gives

$$v = v' + u = 25{,}000 \text{ miles/hr} + 25{,}000 \text{ miles/hr} = 50{,}000 \text{ miles/hr}.$$

Equation 40–15 gives

$$v = \frac{v' + u}{1 + v'u/c^2}$$

$$= \frac{25{,}000 \text{ miles/hr} + 25{,}000 \text{ miles/hr}}{1 + \dfrac{(25{,}000 \text{ miles/hr})^2}{(6.7 \times 10^8 \text{ miles/hr})^2}}$$

$$= \frac{50{,}000 \text{ miles/hr}}{1.0000000014}.$$

Even at 25,000 miles/hr the error in Eq. 40–14 is immeasurably small.

Example 5. Two electrons are ejected in opposite directions from radioactive atoms in a sample of radioactive material. Let each electron have a speed, as measured by a laboratory observer, of $0.6c$ (this corresponds to a kinetic energy of 130 kev). What is the speed of one electron as seen from the other?

Equation 40–14 gives

$$v = v' + u = 0.6c + 0.6c = 1.2c.$$

Equation 40–15 gives

$$v = \frac{v' + u}{1 + v'u/c^2} = \frac{0.6c + 0.6c}{1 + (0.6c)^2/c^2} = 0.88c.$$

This example shows that for speeds that are comparable to c, Eqs. 40–14 and 40–15 yield rather different results. A wealth of indirect experimental evidence points to the latter result as being correct. ◀

40–5 Doppler Effect

We have seen that the same speed is measured for light no matter what the relative speeds of the light source and the observer are. The measured frequency and wavelength will change, but always in such a way that their product, which is the velocity of light, remains constant. Such frequency shifts are called *Doppler shifts*, after Johann Doppler (1803–1853), who first predicted them.

In Section 20–7 we showed that if a source of *sound* is moving away from an observer at a speed u, the frequency heard by the observer (see Eq. 20–10, which has been rearranged with u substituted for v_s) is

$$\nu' = \nu \frac{1}{1 + u/v}. \quad \begin{cases} 1. \text{ sound wave} \\ 2. \text{ observer fixed in medium} \\ 3. \text{ source receding from observer} \end{cases} \quad (40\text{–}16)$$

In this equation ν is the frequency heard when the source is at rest and v is the speed of sound.

If the source is at rest in the transmitting medium but the observer is moving away from the source at speed u, the observed frequency (see Eq. 20-9b, in which u has been substituted for v_0) is

$$\nu' = \nu\left(1 - \frac{u}{v}\right). \quad \begin{cases} 1. \text{ sound wave} \\ 2. \text{ source fixed in medium} \\ 3. \text{ observer receding from source} \end{cases} \qquad (40\text{-}17)$$

Even if the relative separation speeds u of the source and the observer are the same, the frequencies predicted by Eqs. 40-16 and 40-17 are different. This is not surprising, because a sound source moving through a medium in which the observer is at rest is physically different from an observer moving through that medium with the source at rest, as comparison of Figs. 20-10 and 20-11 shows.

We might be tempted to apply Eqs. 40-16 and 40-17 to light, substituting c, the speed of light, for v, the speed of sound. For light, as contrasted with sound, however, it has proved impossible to identify a medium of transmission relative to which the source and the observer are moving. This means that "source receding from observer" and "observer receding from source" are physically identical situations and must exhibit *exactly the same* Doppler frequency. As applied to light, either Eq. 40-16 or Eq. 40-17 or both must be incorrect. The Doppler frequency predicted by the theory of relativity is, in fact,

$$\nu' = \nu\,\frac{1 - u/c}{\sqrt{1 - (u/c)^2}}. \quad \begin{cases} 1. \text{ light wave} \\ 2. \text{ source and observer separating} \end{cases} \qquad (40\text{-}18)$$

In all three of the foregoing equations we obtain the appropriate relations for the source and the observer *approaching* each other if we replace u by $-u$.

Equations 40-16, 40-17, and 40-18 are not so different as they seem if the ratio u/c is small enough. This was made clear in Example 3, Chapter 20, for the first two of these equations. Let us expand Eqs. 40-16, 40-17 and 40-18 by the binomial theorem, as in the example referred to. The equations then become, substituting c for v,

$$\nu' = \nu\left[1 - \frac{u}{c} + \left(\frac{u}{c}\right)^2 + \cdots\right]. \qquad (40\text{-}16a)$$

$$\nu' = \nu\left(1 - \frac{u}{c}\right). \qquad (40\text{-}17a)$$

and

$$\nu' = \nu\left[1 - \frac{u}{c} + \frac{1}{2}\left(\frac{u}{c}\right)^2 + \cdots\right]. \qquad (40\text{-}18a)$$

The ratio u/c for all available monochromatic light sources, even those of atomic dimensions, is small. This means that successive terms in these equations become small rapidly and, depending on the accuracy required, only a limited number of terms need be retained.

Under nearly all circumstances the differences among these three equations are not important. Nevertheless, it is of extreme interest to carry out at least one experiment precisely enough to serve as a test of Eq. 40–18*a* and thus, in part, of the theory of relativity.

H. E. Ives and G. R. Stilwell carried out such a precision experiment in 1938. They sent a beam of hydrogen atoms, generated in a gas discharge, down a tube at speed u, as in Fig. 40–9*a*. They could observe light emitted by these atoms in a direction opposite to $\mathbf{u}$ (atom 1, for example) using a mirror, and also in a direction parallel to $\mathbf{u}$ (atom 2, for example). With a precision spectrograph, they could photograph a particular characteristic spectrum line in this light, obtaining, on a frequency scale, the lines marked ν_1' and ν_2' in Fig. 40–9*b*. It is also possible to photograph, on the same photographic plate, a line corresponding to light emitted from *resting* atoms; such a line appears as ν in Fig. 40–9*b*. A fundamental measured quantity in this experiment is $\Delta\nu/\nu$, defined from

$$\frac{\Delta\nu}{\nu} = \frac{\Delta\nu_2 - \Delta\nu_1}{\nu}, \tag{40–19}$$

(see Fig. 40–9*b*). It measures the extent to which the frequency of the light from resting atoms fails to lie halfway between the frequencies ν_1' and ν_2'. Table 40–2 shows that the measured results agree with the formula predicted by the theory of relativity (Eq. 40–18*a*) and not with the classical formula borrowed from the theory of sound propagation in a material medium (Eq. 40–16*a*).

Ives and Stilwell did not present their experimental results as evidence for the support of Einstein's theory of relativity but rather gave them an alternative theoretical explanation. Modern observers, looking not only at their excellent experiment but at the whole range of experimental evidence, now give the Ives-Stilwell experiment the interpretation we have described for it above.

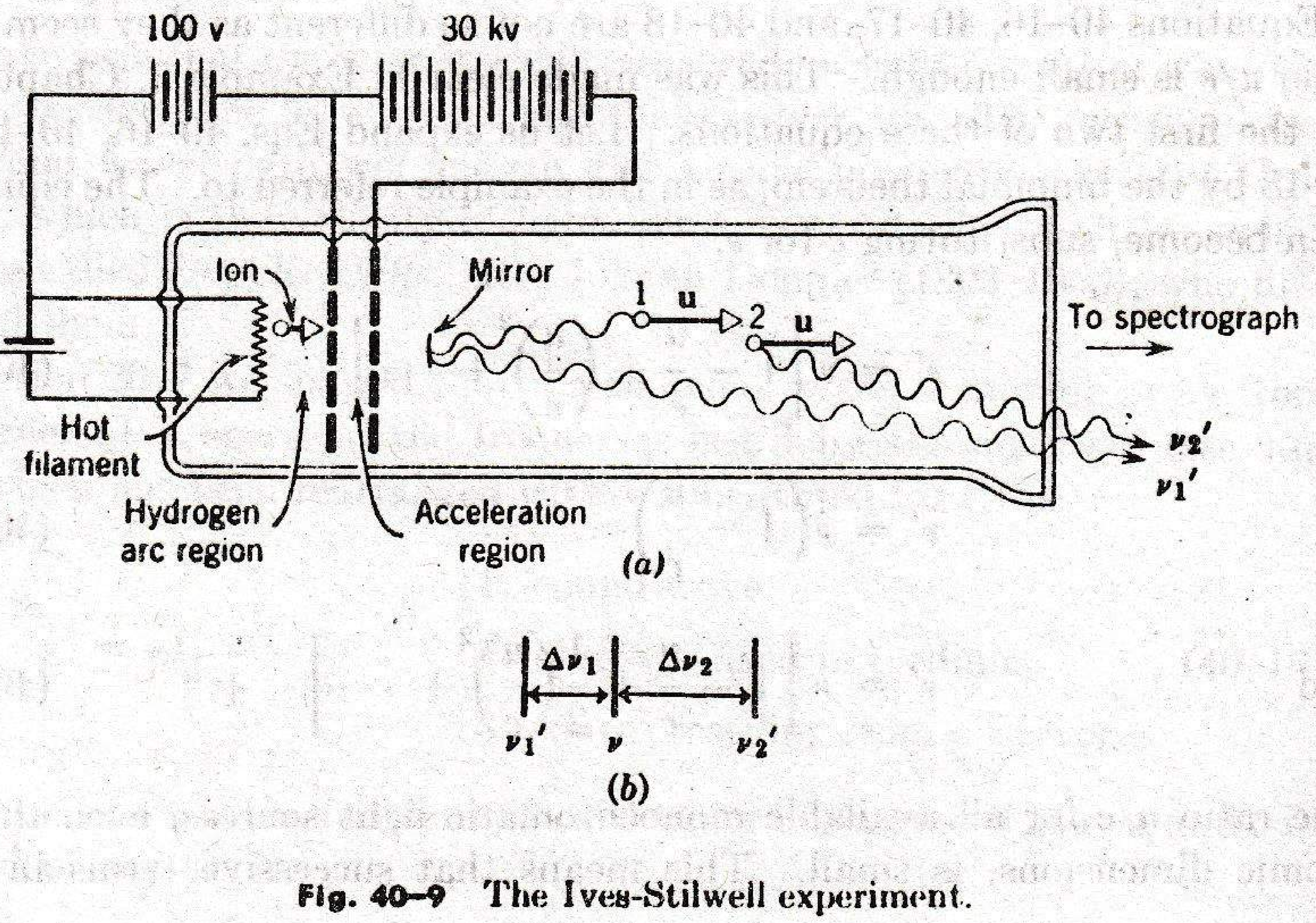


Fig. 40–9 The Ives-Stilwell experiment.

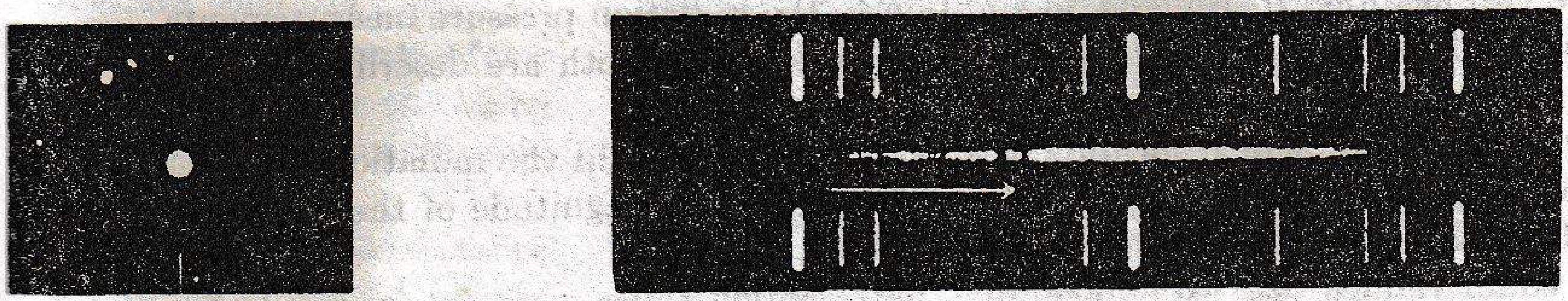
Table 40–2

THE IVES-STILWELL EXPERIMENT *

$\frac{\Delta\nu}{\nu}$, 10^{-5}	Speed of moving atoms ($= u$), 10^6 meters/sec			
	0.865	1.01	1.15	1.33
Theoretical value according to classical theory (Eq. 40–16a)	1.67	2.26	2.90	3.94
Theoretical value according to the theory of relativity (Eq. 40–18a)	0.835	1.13	1.45	1.97
Experimental value	0.762	1.1	1.42	1.9

* See Eq. 40–19; the table shows only part of the data taken by Ives and Stilwell.

The Doppler effect for light finds many applications in astronomy, where it is used to determine the speeds at which luminous heavenly bodies are moving toward us or receding from us. Such Doppler shifts measure only the radial or line-of-sight components of the relative velocity. All galaxies * for which such measurements have been made (Fig. 40–10) appear to be receding from us, the recession velocity being greater for the more distant galaxies; these observations are the basis of the expanding-universe concept.



(a) (b)

Fig. 40–10 (a) The central spot is a nebula in the constellation Corona Borealis; it is 130,000,000 light years distant. (b) The central streak shows the distribution in wavelength of the light emitted from this nebula. The two vertical dark bands show the presence of calcium. The horizontal arrow shows that these calcium lines occur at longer wavelengths than those for terrestrial light sources containing calcium, the length of the arrow representing the wavelength shift. Measurement of this shift shows that the galaxy is receding from us at 13,400 miles/sec. The lines above and below the central streak represent light from a terrestrial source, used to establish a wavelength scale. (Courtesy Mount Wilson and Mount Palomar Observatories.)

* See "The Red-Shift," Allen R. Sandage, *Scientific American*, p. 171, September 1956.

▶ **Example 6.** Certain characteristic wavelengths in the light from a galaxy in the constellation Virgo are observed to be increased in wavelength, as compared with terrestrial sources, by about 0.4%. What is the radial speed of this galaxy with respect to the earth? Is it approaching or receding?

If λ is the wavelength for a terrestrial source, then

$$\lambda' = 1.004\lambda.$$

Since we must have $\lambda'\nu' = \lambda\nu = c$, we can write this as

$$\nu' = 0.996\nu.$$

This frequency shift is so small that, in calculating the source velocity, it makes no practical difference whether we use Eq. 40–16, 40–17, or 40–18. Using Eq. 40–17 we obtain

$$\nu' = 0.996\nu = \nu\left(1 - \frac{u}{c}\right).$$

Solving yields $u/c = 0.004$, or $u = (0.004)(3 \times 10^8 \text{ meters/sec}) = 1.2 \times 10^6$ meters/sec or 2.7×10^6 miles/hr. The galaxy is *receding;* had u turned out to be negative, the galaxy would have been moving toward us. ◀

QUESTIONS

1. How might an eye-sensitivity curve like that of Fig. 40–2 be measured?
2. Why are danger signals in red, when the eye is most sensitive to yellow-green?
3. Comment on this definition of the limits of the spectrum of visible light given by a physiologist: "The limits of the visible spectrum occur when the eye is no better adapted than any other organ of the body to serve as a detector."
4. How can an object absorb light energy without absorbing momentum?
5. A searchlight sends out a parallel beam of light. Does the searchlight experience any force associated with the emission of light?
6. Name two historic experiments, in addition to the radiation pressure measurements of Nichols and Hull, in which a torsion balance was used. Both are described in this book, one in Part 1 and one in Part 2.
7. Show that for complete absorption of a parallel beam of light the radiation pressure on the absorbing object is given by $p = S/c$, where S is the magnitude of the Poynting vector and c is the speed of light in free space.
8. How could Galileo test experimentally that reaction times were an overwhelming source of error in his attempt to measure the speed of light, described on p. 999?
9. It has been suggested that the velocity of light may change slightly in value as time goes on. Can you find any evidence for this in Table 40–1?
10. A friend asserts that Einstein's postulate (that the speed of light is not affected by the uniform motion of the source or the observer) must be discarded because it violates "common sense." How would you answer him?
11. In a vacuum, does the speed of light depend on (*a*) the wavelength, (*b*) the frequency, (*c*) the intensity, (*d*) the speed of the source, or (*e*) the speed of the observer?
12. Can a galaxy be so distant that its recession speed equals c? If so, how can we see the galaxy? That is, will its light ever reach us?

PROBLEMS

1. (*a*) At what wavelengths does the eye sensitivity have half its maximum value? (*b*) What are the frequency and the period of the light for which the eye is most sensitive?

2. It has been proposed that a spaceship might propel itself in the solar system by radiation pressure, using a large sail made of aluminum foil. How large must the sail be if the radiation force is to be equal in magnitude to the sun's gravitational attraction? Assume that the mass of the *ship + sail* is 100 slugs, that the sail is perfectly reflecting, and that the sail is oriented at right angles to the sun's rays. The sun's mass is 1.97×10^{30} kg.

3. Radiation from the sun striking the earth has an intensity of 1400 watts/meter^2. Assuming that the earth behaves like a flat disk at right angles to the sun's rays and that all the incident energy is absorbed, calculate the force on the earth due to radiation pressure. Compare it with the force due to the sun's gravitational attraction.

4. Prove, for a plane wave at normal incidence on a plane surface, that the radiation pressure on the surface is equal to the energy density in the beam outside the surface. This relation holds no matter what fraction of the incident energy is reflected.

5. Prove, for a stream of bullets striking a plane surface at right angles, that the "pressure" is *twice* the (kinetic) energy density in the stream above the surface; assume that the bullets are completely "absorbed" by the surface. Contrast this with the behavior of light (Problem 4).

6. A small spaceship whose mass, with occupant, is 100 slugs is drifting in outer space, where no gravitational field exists. If it shines a searchlight, which radiates 10^4 watts, into space, what speed would the ship attain in one day because of the reaction force associated with the momentum carried away by the light beam?

7. What is the radiation pressure 1.0 meter away from a 500-watt light bulb? Assume that the surface on which the pressure is exerted faces the bulb and is perfectly absorbing and that the bulb radiates uniformly in all directions.

8. The uncertainty of the distance to the moon, as measured by the reflection of radar waves from it, is about 0.5 mile. Assuming that this uncertainty is associated only with the measurement of the elapsed time, what uncertainty in this time is implied?

9. Roemer's method for measuring the speed of light consisted in observing the apparent times of revolution of one of the moons of Jupiter. The true period of revolution is 42.5 hr. (*a*) Taking into account the finite speed of light, how would you expect the apparent time of revolution to alter as the earth moves in its orbit from point x to point y in Fig. 40–11? (*b*) What observations would be needed to compute the speed of light? Neglect the motion of Jupiter in its orbit. Figure 40–11 is not drawn to scale.

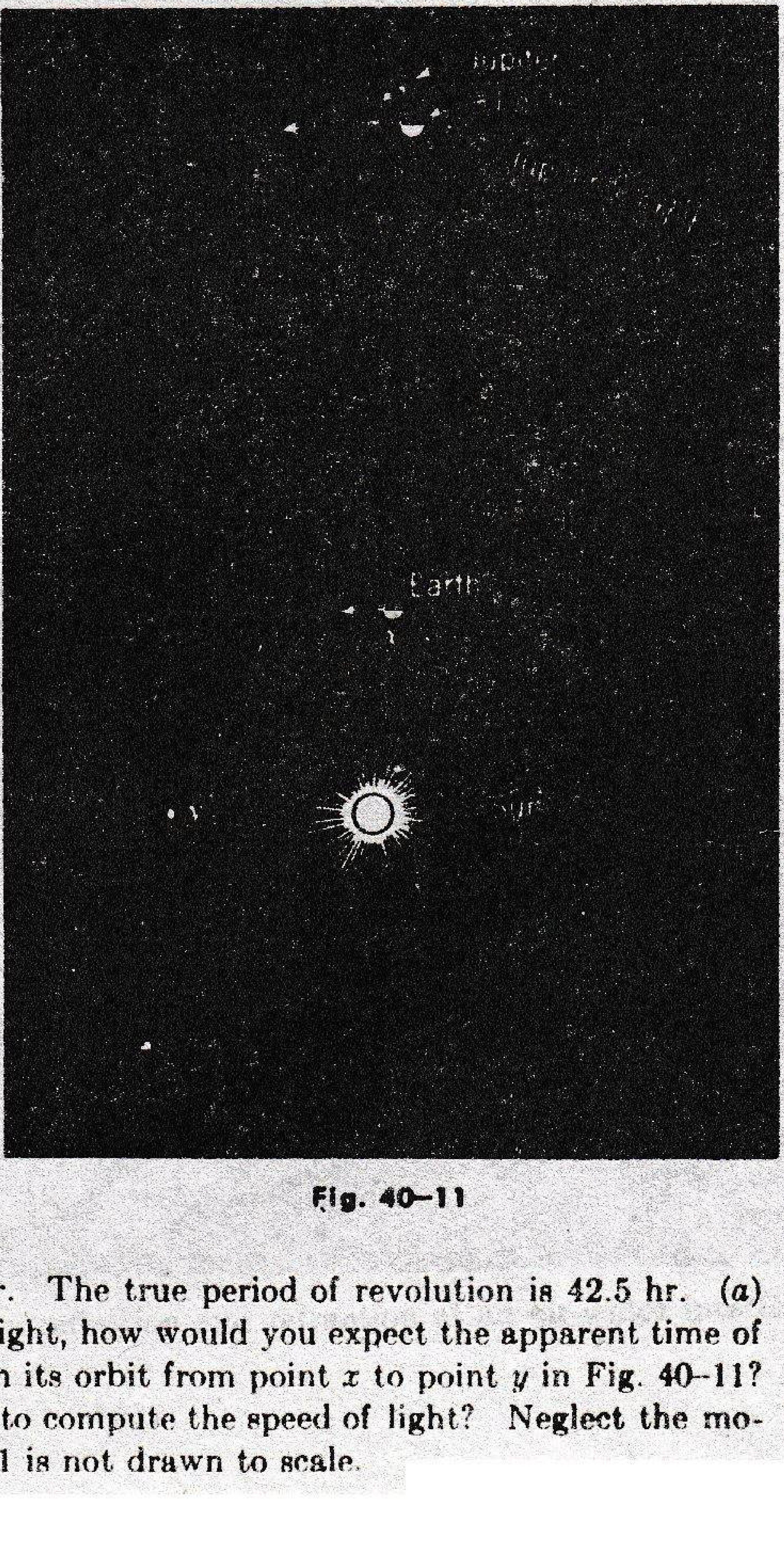


Fig. 40–11

10. Suppose that light is timed over a 1-mile base line and its speed is measured to the accuracy quoted on p. 1002. How large an error in the length of the base line could be tolerated, assuming other sources of error to be negligible?

11. For what value of u/c does Eq. 40–17 differ from Eq. 40–18 by 1%?

12. The "red shift" of radiation from a distant nebula consists of the light (H_γ), known to have a wavelength of 4340×10^{-8} cm when observed in the laboratory, appearing to have a wavelength of 6562×10^{-8} cm. What is the speed of the nebula in the line of sight relative to the earth? (*b*) Is it approaching or receding?

13. The difference in wavelength between an incident microwave beam and one reflected from an approaching or receding car is used to determine automobile speeds on the highway. (*a*) Show that if v is the speed of the car and ν the frequency of the incident beam, the change of frequency is approximately $2v\nu/c$, where c is the speed of the electromagnetic radiation. (*b*) For microwaves of frequency 2450 megacycles/sec, what is the change of frequency per mile/hr of speed?

14. Show that, for slow speeds, the Doppler shift can be written in the approximate form

$$\frac{\Delta\lambda}{\lambda} = \frac{u}{c},$$

where $\Delta\lambda$ is the change in wavelength.

15. The period of rotation of the sun at its equator is 24.7 days; its radius is 7.0×10^8 meters. What Doppler wavelength shifts are expected for characteristic wavelengths in the vicinity of 5500 A emitted from the edge of the sun's disk?

16. An earth satellite, transmitting on a frequency of 40×10^6 cycles/sec (exactly), passes directly over a radio receiving station at an altitude of 250 miles and at a speed of 18,000 miles/hr. Plot the change in frequency, attributable to the Doppler effect, as a function of time, counting $t = 0$ as the instant the satellite is over the station. (Hint: The speed u in the Doppler formula is not the actual velocity of the satellite but its component in the direction of the station. Use the nonrelativistic formula (Eq. 40–16*a*) and neglect the curvature of the earth and of the satellite orbit.)

17. A rocketship is receding from the earth at a speed of $0.2c$. A light in the rocketship appears blue to passengers on the ship. What color would it appear to be to an observer on the earth? See Fig. 40–2.

18. In the experiment of Ives and Stilwell the speed u of the hydrogen atoms in a particular run was 8.61×10^5 meters/sec. Calculate $\Delta\nu_1$, $\Delta\nu_2$, and $\Delta\nu/\nu$, on the assumptions that (*a*) Eq. 40–18*a* is correct and (*b*) that Eq. 40–16*a* is correct; compare your results with those given in Table 40–2 for this speed. Retain the first three terms only in Eqs. 40–18*a* and 40–16*a*.