Diffraction

CHAPTER 44

44–1 Introduction

(Diffraction, which is illustrated in Fig. 42–3, is the bending of light around an obstacle such as the edge of a slit) We can see the diffraction of light by looking through a crack between two fingers at a distant light source such as a tubular neon sign or by looking at a street light through a cloth umbrella. Usually diffraction effects are small and must be looked for carefully. Also, most sources of light have an extended area so that a diffraction pattern produced by one point of the source will overlap that produced by another. Finally, common sources of light are not monochromatic. The patterns for the various wavelengths overlap and again the effect is less apparent.

Diffraction was discovered by Francesco Maria Grimaldi (1618–1663), and the phenomenon was known both to Huygens (1629–1695) and to Newton (1642–1727). Newton did not see in it any justification for a wave theory for light. Huygens, although he believed in a wave theory, did not believe in diffraction! He imagined his secondary wavelets to be effective only at the point of tangency to their common envelope, thus denying the possibility of diffraction. In his words:

> And thus we see the reasons why light . . . proceeds only in straight lines in such a way that it does not illuminate any object except when the path from the source to the object is open along such a line.

Fresnel (1788–1827) correctly applied Huygens' principle (which is called the Huygens-Fresnel principle in Europe) to explain diffraction. In these early days the light waves were believed to be mechanical waves in an all-

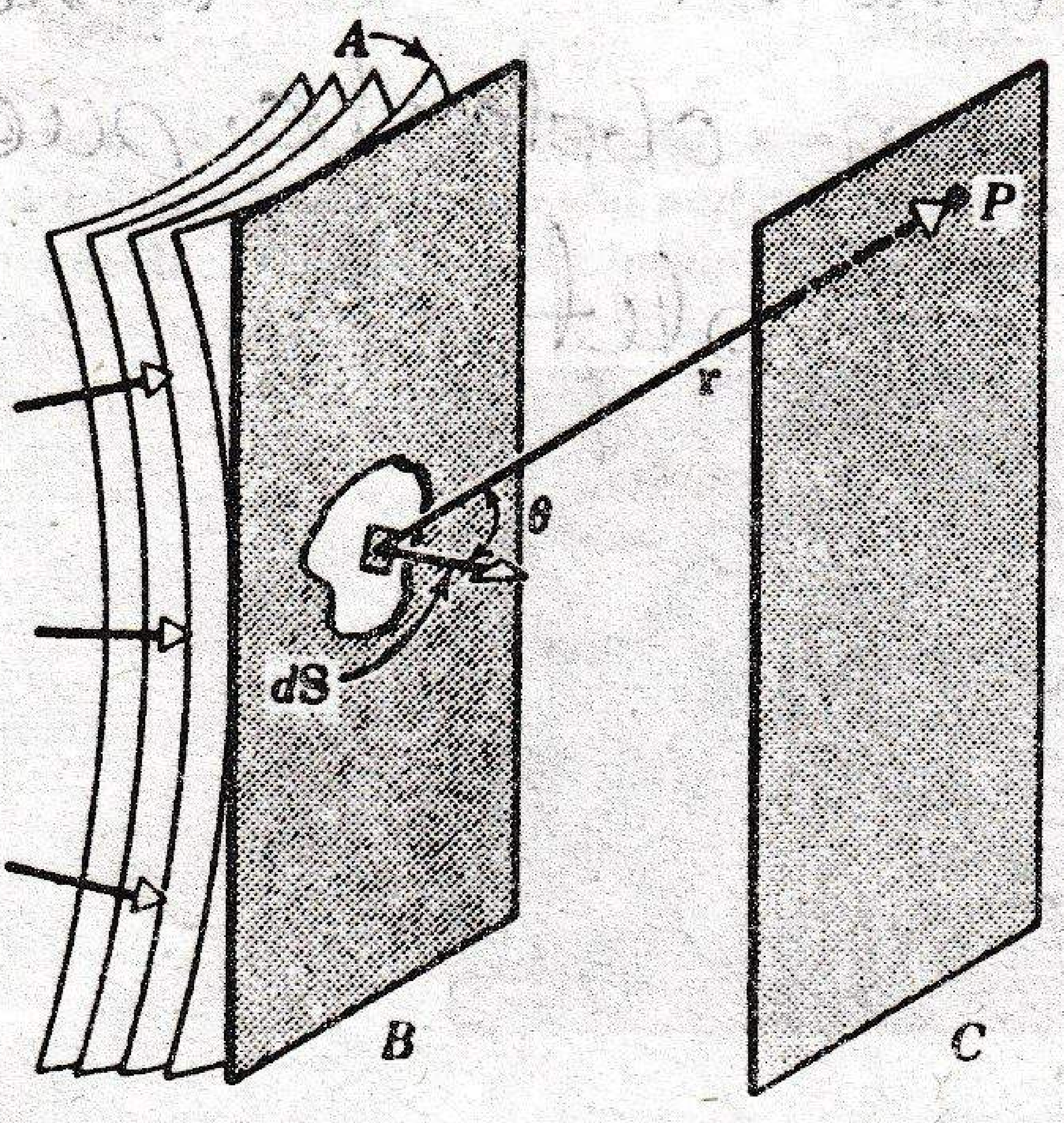

Fig. 44–1 Light is diffracted at the aperture in screen B and illuminates screen C. The intensity at P is found by dividing the wavefront at B into elementary radiators dS and combining their effects at P.

pervading ether. We have seen (Section 39–5) how Maxwell (1831–1879) showed that light waves were not mechanical in nature but electromagnetic. Einstein (1879–1955) rounded out our modern view of light waves by eliminating the need to postulate an ether (see Section 40–4).

Figure 44–1 shows a general diffraction situation. Surface A is a wavefront that falls on B, which is an opaque screen containing an aperture of arbitrary shape; C is a diffusing screen that receives the light that passes through this aperture. This pattern of light intensity on C can be calculated by subdividing the wavefront into elementary areas dS, each of which becomes a source of an expanding Huygens' wavelet. The light intensity at an arbitrary point P is found by superimposing the wave disturbances (that is, the **E** vectors) caused by the wavelets reaching P from all these elementary radiators.

The wave disturbances reaching P differ in amplitude and in phase because (*a*) the elementary radiators are at varying distances from P, (*b*) the light leaves the radiators at various angles to the normal to the wavefront (see p. 1020), and (*c*) some radiators are blocked by screen B; others are not. Diffraction calculations—simple in principle—may become difficult in practice. The calculation must be repeated for every point on screen C at which we wish to know the light intensity. We followed exactly this program in calculating the double-slit intensity pattern in Section 43–3. The calculation there was simple because we assumed only two elementary radiators, the two narrow slits.

Figure 44–2*a* shows the general case of *Fresnel diffraction*, in which the light source and the screen on which the diffraction pattern is displayed are a finite distance from the diffracting aperture; the wavefronts that fall on the diffracting aperture in this case and that leave it to illuminate any point P of the diffusing screen are not planes; the corresponding rays are not parallel.

A simplification results if source S and screen C are moved to a large distance from the diffraction aperture, as in Fig. 44-2*b*. This limiting case is called *Fraunhofer diffraction*. The wavefronts arriving at the diffracting aperture from the distant source S are planes, and the rays associated with these wavefronts are parallel to each other. Similarly, the wavefronts arriving at any point P on the distant screen C are planes, the corresponding rays

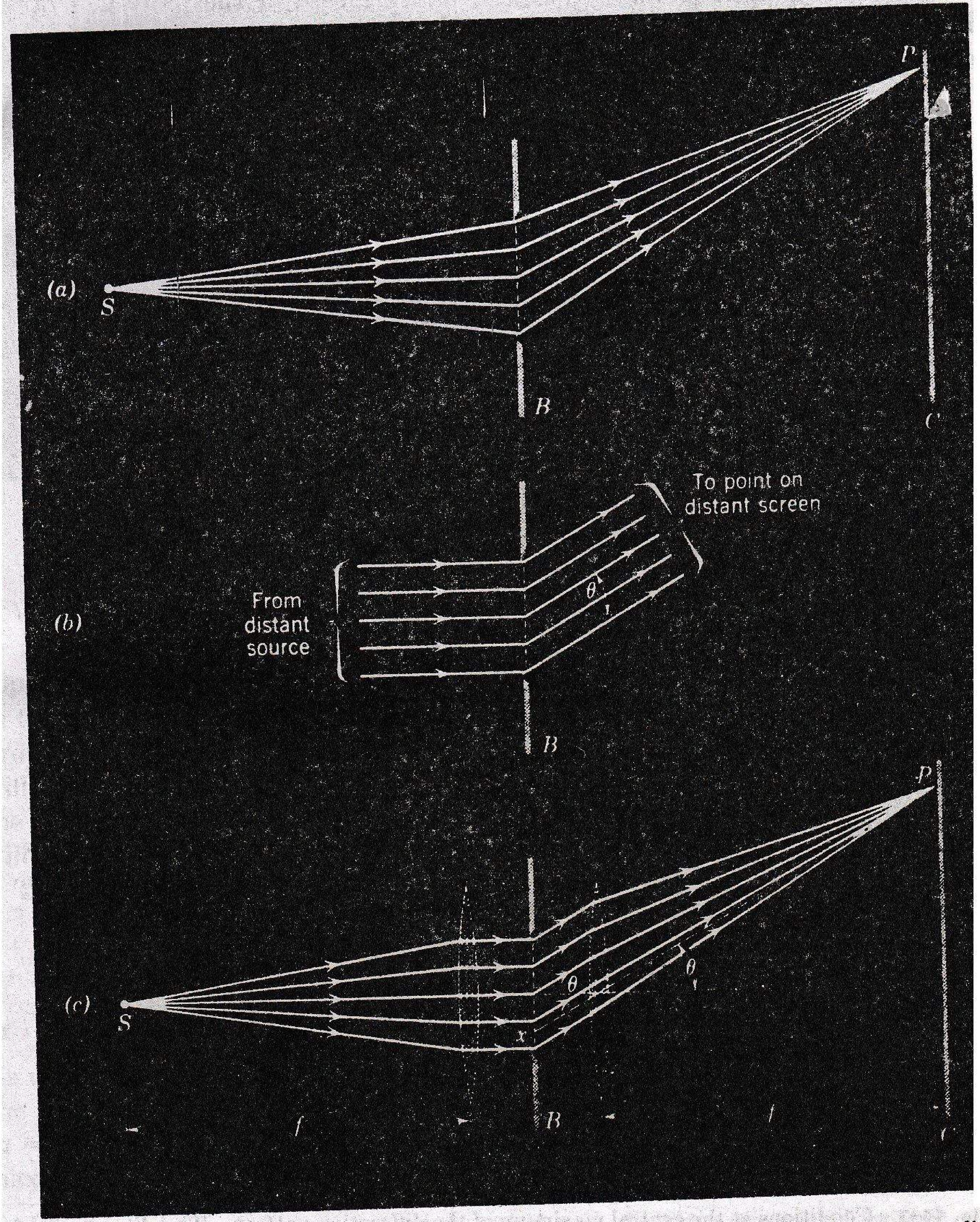

Fig. 44-2 (*a*) Fresnel diffraction. (*b*) Source S and screen C are moved to a large distance, resulting in Fraunhofer diffraction. (*c*) Fraunhofer diffraction conditions produced by lenses, leaving source S and screen C in their original positions.

also being parallel. Fraunhofer conditions can be established in the laboratory by using two converging lenses, as in Fig. 44-2*c*. The first of these converts the diverging wave from the source into a plane wave. The second lens causes plane waves leaving the diffracting aperture to converge to point *P*. All rays that illuminate *P* will leave the diffracting aperture parallel to the dashed line *Px* drawn from *P* through the center of this second (thin) lens. We assumed Fraunhofer conditions for Young's double-slit experiment in Section 43-1 (see Fig. 43-5).

Although Fraunhofer diffraction is a limiting case of the more general Fresnel diffraction, it is an important limiting case and is easier to handle mathematically. This book deals only with Fraunhofer diffraction.

44-2 Single Slit

Figure 44-3 shows a plane wave falling at normal incidence on a long narrow slit of width *a*. Let us focus our attention on the central point P_0 of screen *C*. The rays extending from the slit to P_0 all have the same optical path lengths, as we saw in Section 42-5. Since they are in phase at the plane of the slit, they will still be in phase at P_0, and the central point of the diffraction pattern that appears on screen *C* has a maximum intensity

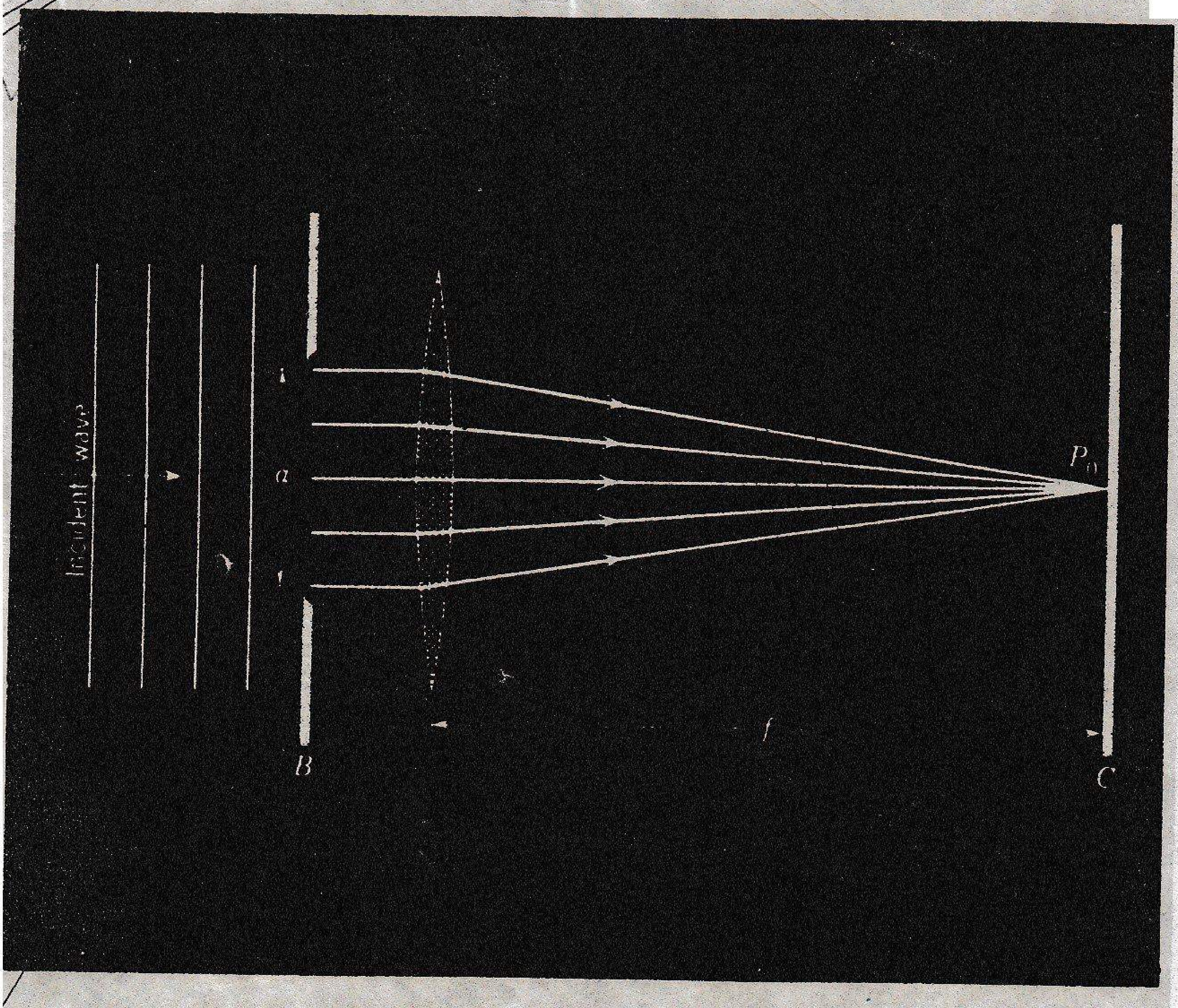

Fig. 44-3 Conditions at the central maximum of the diffraction pattern. The slit extends a distance above and below the figure, this distance being much greater than the slit width *a*.

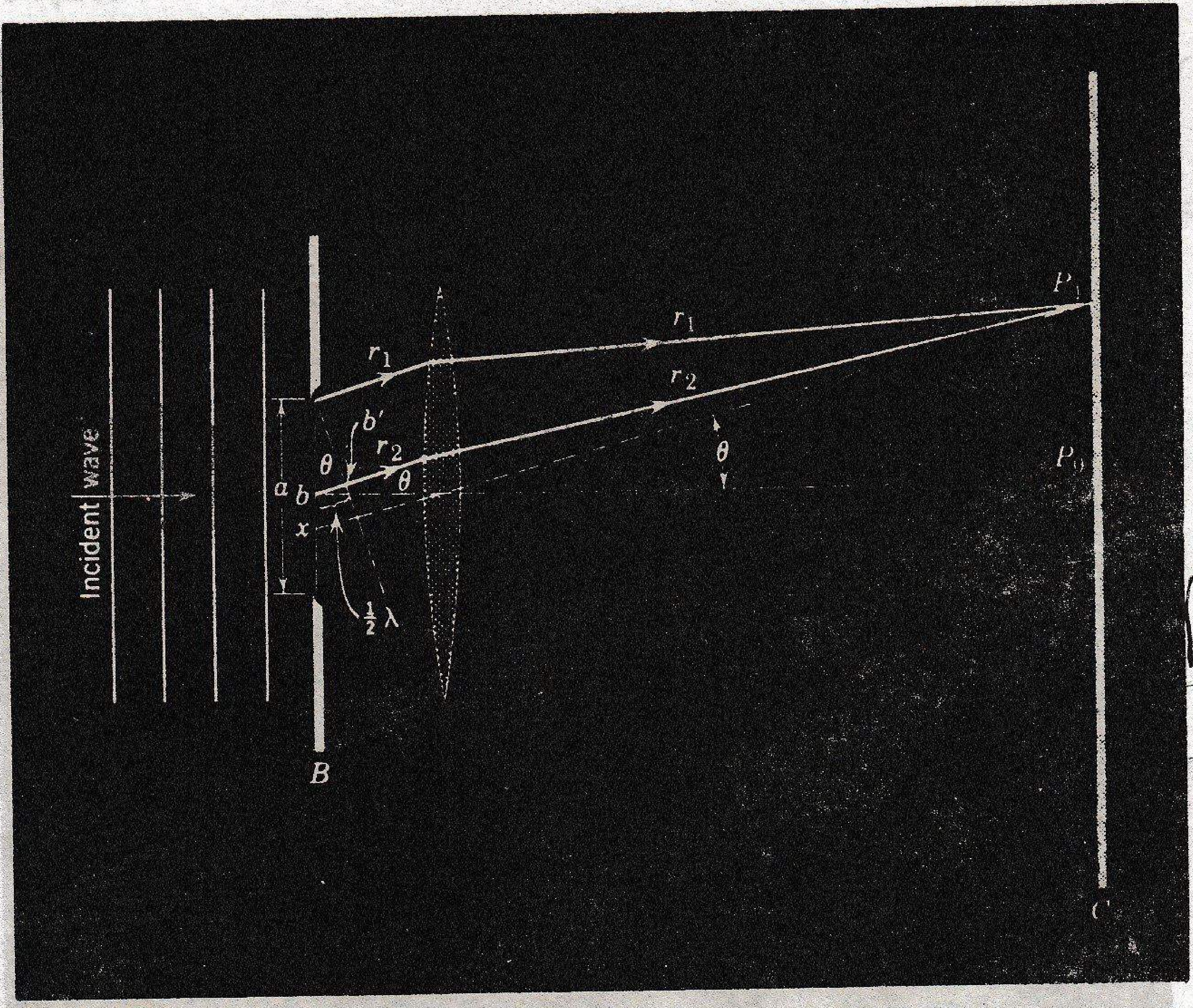

Fig. 44–4 Conditions at the first minimum of the diffraction pattern.

We now consider another point on the screen. Light rays which reach P_1 in Fig. 44–4 leave the slit at an angle θ as shown. Ray r_1 originates at the top of the slit and ray r_2 at its center. If θ is chosen so that the distance bb' in the figure is one-half a wavelength, r_1 and r_2 will be out of phase and will produce no effect at P_1.* In fact, every ray from the upper half of the slit will be canceled by a ray from the lower half, originating at a point $a/2$ below the first ray. The point P_1, the first minimum of the diffraction pattern, will have zero intensity (compare Fig. 42–3).

The condition shown in Fig. 44–4 is

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

or

$$a \sin\theta = \lambda. \tag{44-1}$$

As we stated earlier (see Fig. 42–1), the central maximum becomes wider as the slit is made narrower. If the slit width is as small as one wavelength $(a = \lambda)$, the first minimum occurs at $\theta = 90°$, which implies that the central maximum fills the entire forward hemisphere. We assumed a condition ap-

* Whatever phase relation exists between r_1 and r_2 at the plane represented by the sloping dashed line in Fig. 44–4 that passes through b' also exists at P_1, not being affected by the lens (see Section 42–5).

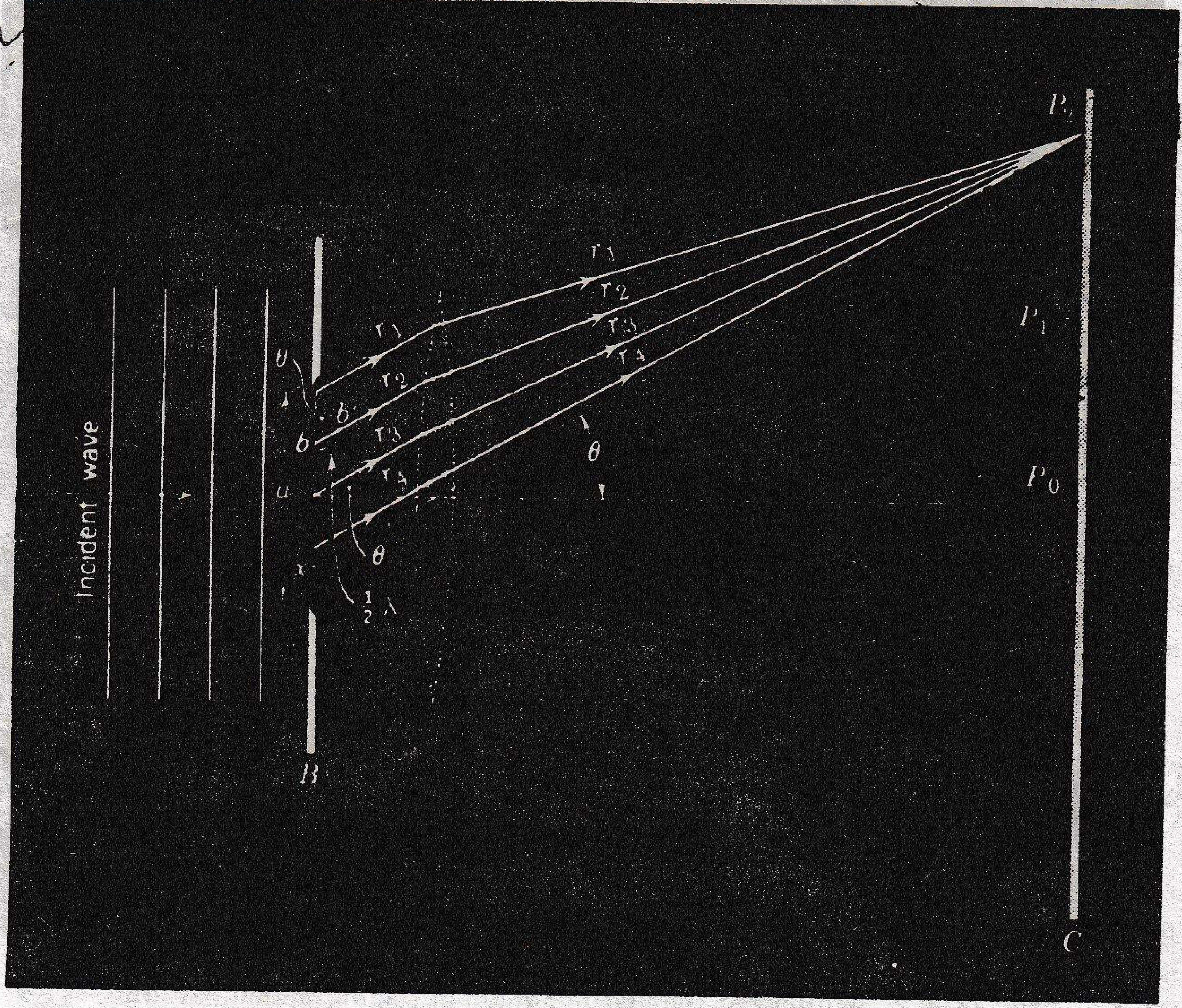

Fig. 44-5 Conditions at the second minimum of the diffraction pattern.

proaching this in our discussion of Young's double-slit interference experiment in Section 43-1.

In Fig. 44-5 the slit is divided into four equal zones, with a ray leaving the top of each zone. Let θ be chosen so that the distance bb' is one-half a wavelength. Rays r_1 and r_2 will then cancel at P_2. Rays r_3 and r_4 will also be half a wavelength out of phase and will also cancel. Consider four other rays, emerging from the slit a given distance below the four rays above. The two rays below r_1 and r_2 will cancel uniquely, as will the two rays below r_3 and r_4. We can proceed across the entire slit and conclude again that no light reaches P_2; we have located a second point of zero intensity.

The condition described (see Fig. 44-5) requires that

$$\frac{a}{4}\sin\theta = \frac{\lambda}{2},$$

or

$$a\sin\theta = 2\lambda.$$

By extension, the general formula for the minima in the diffraction pattern on screen C is

$$a\sin\theta = m\lambda \qquad m = 1, 2, 3, \ldots \text{ (minima).} \qquad (44\text{-}2)$$

There is a maximum approximately halfway between each adjacent pair of minima. The student should consider the simplification that has resulted in this analysis by examining Fraunhofer (Fig. 44-2c) rather than Fresnel conditions (Fig. 44-2a).

▶ **Example 1.** A slit of width a is illuminated by white light. For what value of a will the first minimum for red light ($\lambda = 6500$ A) fall at $\theta = 30°$?

At the first minimum we put $m = 1$ in Eq. 44-2. Doing so and solving for a yields

$$a = \frac{m\lambda}{\sin\theta} = \frac{(1)(6500 \text{ A})}{\sin 30°} = 13{,}000 \text{ A}.$$

Note that the slit width must be twice the wavelength in this case.

Example 2. In Example 1 what is the wavelength λ' of the light whose first diffraction maximum (not counting the central maximum) falls at $\theta = 30°$, thus coinciding with the first minimum for red light?

This maximum is about halfway between the first and second minima. It can be found without too much error by putting $m = 1.5$ in Eq. 44-2, or

$$a \sin\theta \cong 1.5\lambda'.$$

From Example 1, however,

$$a \sin\theta = \lambda.$$

Dividing gives

$$\lambda' = \frac{\lambda}{1.5} = \frac{6500 \text{ A}}{1.5} = 4300 \text{ A}.$$

Light of this color is violet. The second maximum for light of wavelength 4300 A will *always* coincide with the first minimum for light of wavelength 6500 A, no matter what the slit width. If the slit is relatively narrow, the angle θ at which this overlap occurs will be relatively large. ◀

44-3 Single Slit—Qualitative

Figure 44-6 shows a slit of width a divided into N parallel strips of width Δx. Each strip acts as a radiator of Huygens' wavelets and produces a characteristic wave disturbance at point P, whose position on the screen, for a particular arrangement of apparatus, can be described by the angle θ.

If the strips are narrow enough—which we assume—all points on a given strip have essentially the same optical path length to P, and therefore all the light from the strip will have the same phase when it arrives at P. The amplitudes ΔE_0 of the wave disturbances at P from the various strips may be taken as equal if θ in Fig. 44-6 is not too large.

We limit our considerations to points that lie in, or infinitely close to, the plane of Fig. 44-6. It can be shown that this procedure is valid for a slit whose length is much greater than its width a. We made this same assumption tacitly both earlier in this chapter and in Chapter 43; see Figs. 43-5 and 44-3, for example.

The wave disturbances from adjacent strips have a constant phase difference $\Delta\phi$ between them at P given by

$$\frac{\text{phase difference}}{2\pi} = \frac{\text{path difference}}{\lambda},$$

or

$$\Delta\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x \sin\theta), \tag{44-3}$$

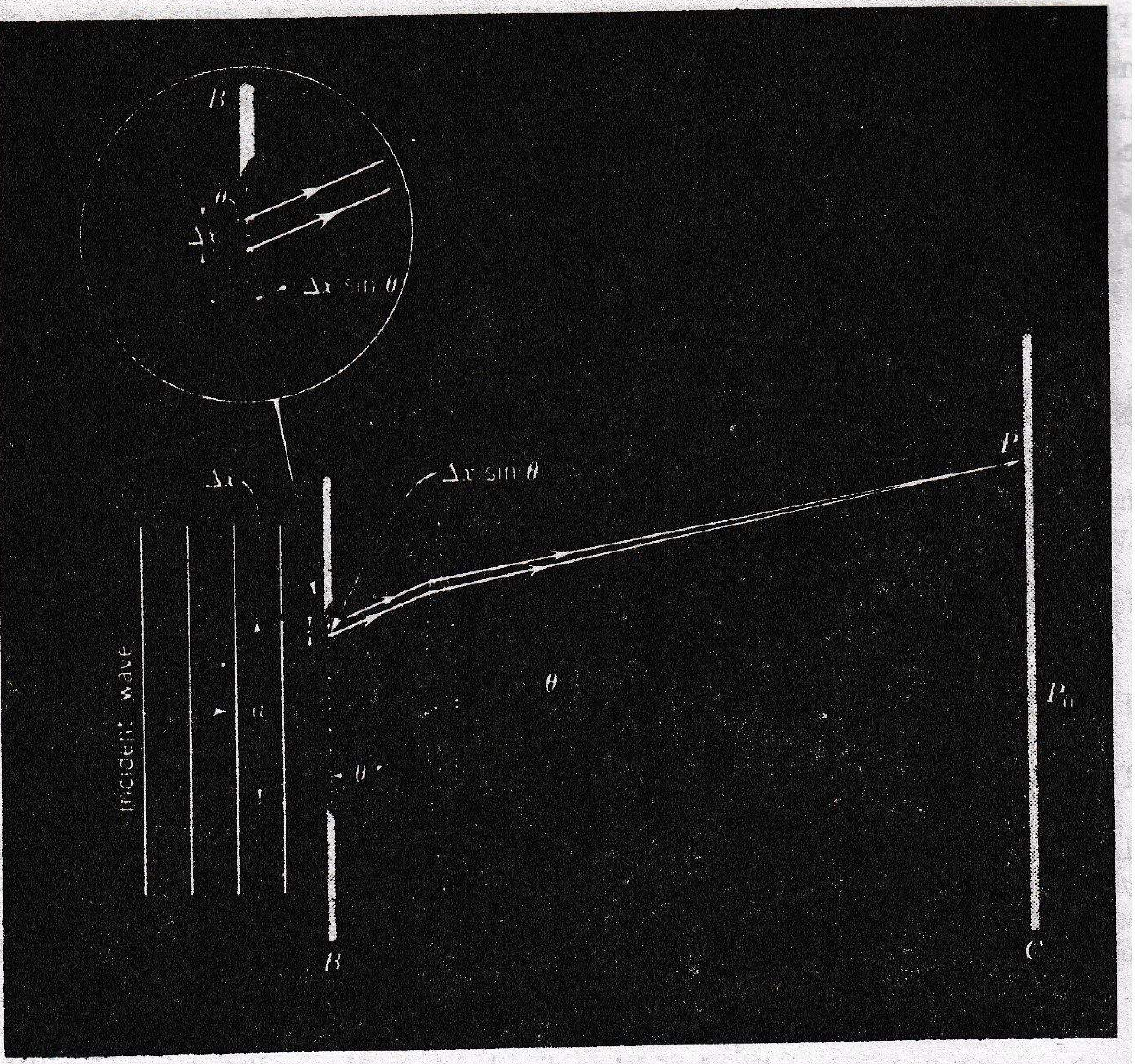

Fig. 44-6 A slit of width a is divided into N strips of width Δx. The insert shows conditions at the second strip more clearly. In the differential limit the slit is divided into an infinite number of strips (that is, $N \rightarrow \infty$) of differential width dx. For clarity in this and the following figure, we take $N = 18$.

where $\Delta x \sin \theta$ is, as the figure insert shows, the path difference for rays originating at the top edges of adjacent strips. Thus, at P, N vectors with the same amplitude ΔE_0, the same frequency, and the same phase difference $\Delta\phi$ between adjacent members combine to produce a resultant disturbance. We ask, for various values of $\Delta\phi$ [that is, for various points P on the screen, corresponding to various values of θ (see Eq. 44-3)], what is the amplitude E_θ of the resultant wave disturbance? We find the answer by representing the individual wave disturbances ΔE_0 by phasors and calculating the resultant phasor amplitude, as described in Section 43-4.

At the center of the diffraction pattern θ equals zero, and the phase shift between adjacent strips (see Eq. 44-3) is also zero. As Fig. 44-7a shows, the phasor arrows in this case are laid end to end and the amplitude of the resultant has its maximum value E_m. This corresponds to the center of the central maximum.

As we move to a value of θ other than zero, $\Delta\phi$ assumes a definite nonzero value (again see Eq. 44-3), and the array of arrows is now as shown in Fig. 44-7*b*. The resultant amplitude E_θ is less than before. Note that the length of the "arc" of small arrows is the same for both figures and indeed for all figures of this series. As θ increases further, a situation is reached (Fig. 44-7*c*) in which the chain of arrows curls around through 360°, the tip of the last arrow touching the foot of the first arrow. This corresponds to $E_\theta = 0$, that is, to the first minimum. For this condition the ray from the top of the slit (1 in Fig. 44-7*c*) is 180° out of phase with the ray from the center of the slit ($\frac{1}{2}N$ in Fig. 44-7*c*). These phase relations are consistent with Fig. 44-4, which also represents the first minimum.

As θ increases further, the phase shift continues to increase, and the chain of arrows coils around through an angular distance greater than 360°, as in Fig. 44-7*d*, which corresponds to the first maximum beyond the central maximum. This maximum is much smaller than the central maximum. In making this comparison, recall that the arrows marked E_θ in Fig. 44-7 correspond to the *amplitudes* of the wave disturbance and not to the *in-*

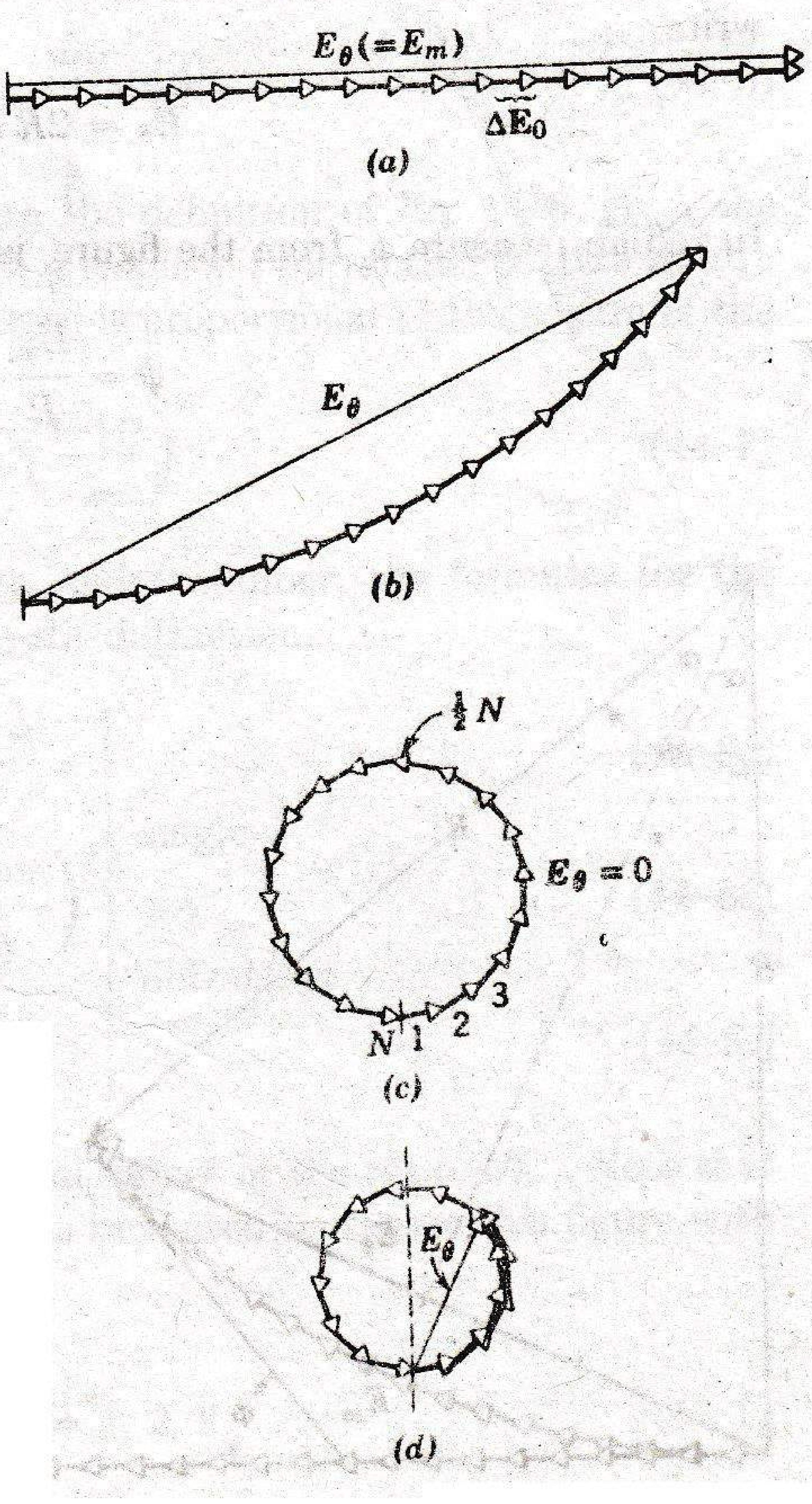

Fig. 44-7 Conditions at (*a*) the central maximum, (*b*) a direction slightly removed from the central maximum, (*c*) the first minimum, and (*d*) the first maximum beyond the central maximum for single-slit diffraction. This figure corresponds to $N = 18$ in Fig. 44-6.

tensity. The amplitudes must be squared to obtain the corresponding relative intensities (see Eq. 43-7).

44-4 Single Slit—Quantitative

The "arc" of small arrows in Fig. 44-8 shows the phasors representing, in amplitude and phase, the wave disturbances that reach an arbitrary point P on the screen of Fig. 44-6, corresponding to a particular angle θ. The resultant amplitude at P is E_θ. If we divide the slit of Fig. 44-6 into infinitesimal strips of width dx, the arc of arrows in Fig. 44-8 approaches the arc of a circle, its radius R being indicated in that figure. The length of the arc is E_m, the amplitude at the center of the diffraction pattern, for at the center of the pattern the wave disturbances are all in phase and this "arc" becomes a straight line as in Fig. 44-7*a*.

The angle ϕ in the lower part of Fig. 44-8 is revealed as the difference in phase between the infinitesimal vectors at the left and right ends of the arc E_m. This means that ϕ is the phase difference between rays from the top and the bottom of the slit of Fig. 44-6. From geometry we see that ϕ is also the angle between the two radii marked R in Fig. 44-8. From this figure we can write

$$E_\theta = 2R \sin \frac{\phi}{2}.$$

In radian measure ϕ, from the figure, is

$$\phi = \frac{E_m}{R}.$$

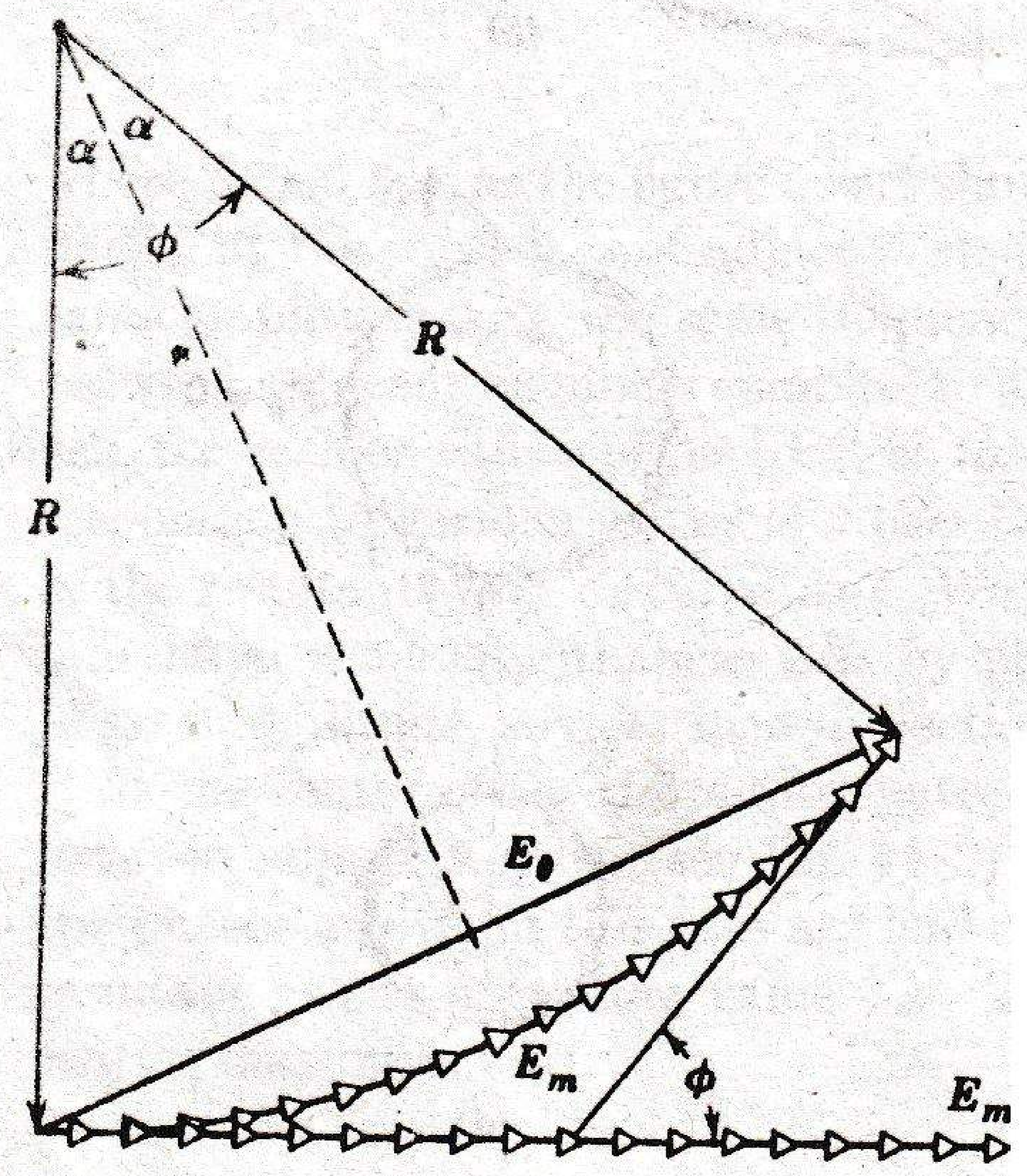

Fig. 44-8 A construction used to calculate the intensity in single-slit diffraction. The situation corresponds to that of Fig. 44-7.

Combining yields
$$E_\theta = \frac{E_m}{\phi/2} \sin\frac{\phi}{2},$$

or
$$E_\theta = E_m \frac{\sin\alpha}{\alpha}, \qquad (44\text{-}4)$$

in which
$$\alpha = \frac{\phi}{2}. \qquad (44\text{-}5)$$

From Fig. 44-6, recalling that ϕ is the phase difference between rays from the top and the bottom of the slit and that the path difference for these rays is $a \sin\theta$, we have

$$\frac{\text{phase difference}}{2\pi} = \frac{\text{path difference}}{\lambda},$$

or
$$\phi = \left(\frac{2\pi}{\lambda}\right)(a \sin\theta).$$

Combining with Eq. 44-5 yields

$$\alpha = \frac{\phi}{2} = \frac{\pi a}{\lambda} \sin\theta. \qquad (44\text{-}6)$$

Equation 44-4, taken together with the definition of Eq. 44-6, gives the amplitude of the wave disturbance for a single-slit diffraction pattern at any angle θ. The intensity I_θ for the pattern is proportional to the square of the amplitude, or

$$I_\theta = I_m \left(\frac{\sin\alpha}{\alpha}\right)^2. \qquad (44\text{-}7)$$

For convenience we display together, and renumber, the formulas for the amplitude and the intensity in single-slit diffraction.

[Eq. 44-4] $$E_\theta = E_m \frac{\sin\alpha}{\alpha} \qquad (44\text{-}8a)$$

[Eq. 44-7] $$I_\theta = I_m \left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad (44\text{-}8b)$$

[Eq. 44-6] $$\alpha\ (= \tfrac{1}{2}\phi) = \frac{\pi a}{\lambda} \sin\theta \qquad (44\text{-}8c)$$

single-slit diffraction

Figure 44-9 shows plots of I_θ for several values of the ratio a/λ. Note that the pattern becomes narrower as a/λ is increased; compare this figure with Figs. 42-1 and 42-3.

Minima occur in Eq. 44-8b when

$$\alpha = m\pi \qquad m = 1, 2, 3, \ldots. \qquad (44\text{-}9)$$

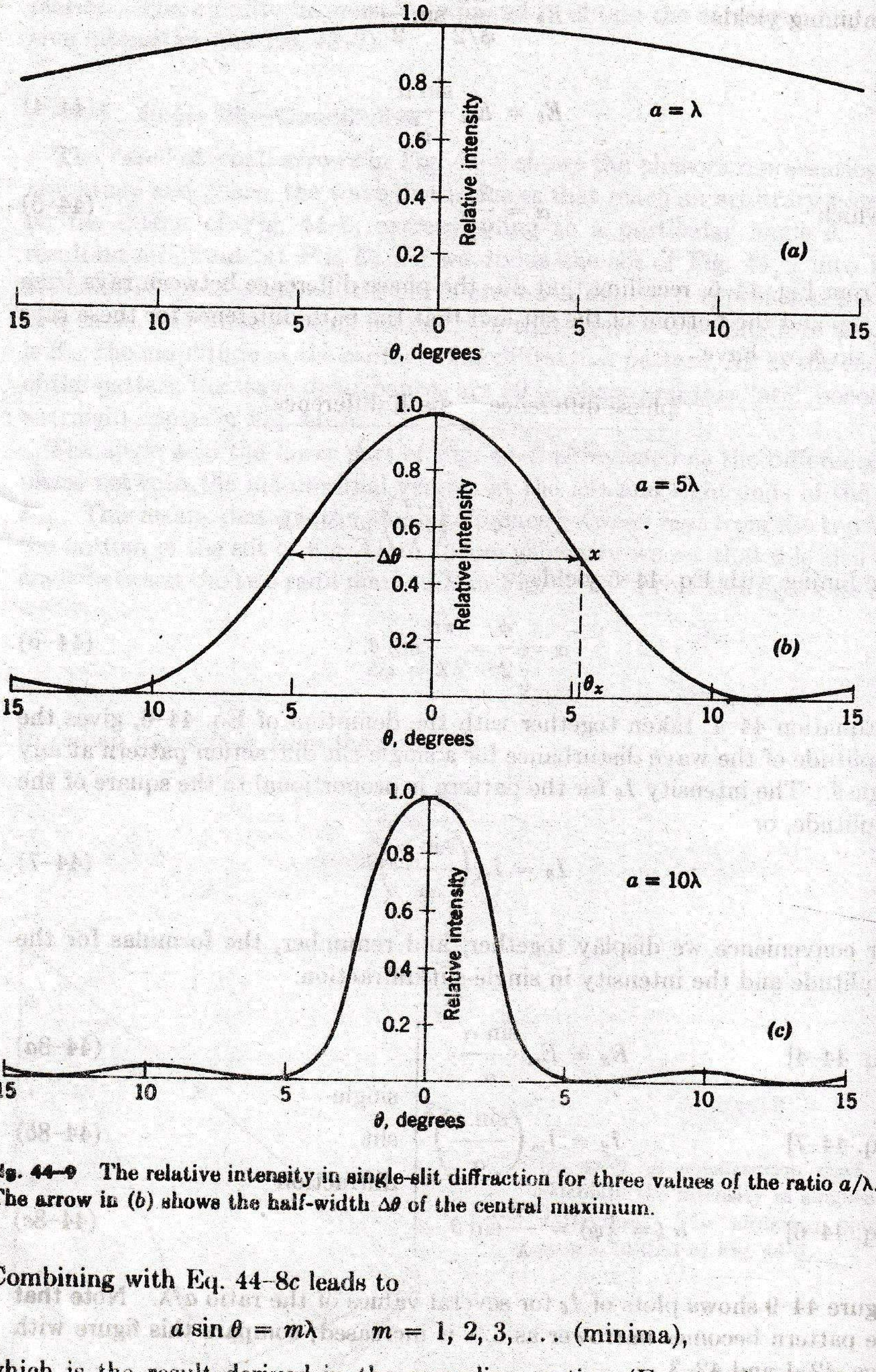

Fig. 44-9 The relative intensity in single-slit diffraction for three values of the ratio a/λ. The arrow in (*b*) shows the half-width $\Delta\theta$ of the central maximum.

Combining with Eq. 44-8*c* leads to

$$a \sin\theta = m\lambda \qquad m = 1, 2, 3, \ldots \text{ (minima)},$$

which is the result derived in the preceding section (Eq. 44-2). In that section, however, we derived *only* this result, obtaining no quantitative information about the intensity of the diffraction pattern at places in which it was not zero. Here (Eqs. 44-8) we have complete intensity information.

▶ **Example 3.** *Intensities of the secondary diffraction maxima.* Calculate, approximately, the relative intensities of the secondary maxima in the single-slit Fraunhofer diffraction pattern.

The secondary maxima lie approximately halfway between the minima and are found (compare Eq. 44-9) from

$$\alpha \cong (m + \tfrac{1}{2})\pi \qquad m = 1, 2, 3, \ldots.$$

Substituting into Eq. 44-8*b* yields

$$I_\theta = I_m \left[\frac{\sin (m + \frac{1}{2})\pi}{(m + \frac{1}{2})\pi}\right]^2,$$

which reduces to

$$\frac{I_\theta}{I_m} = \frac{1}{(m+\frac{1}{2})^2\pi^2}.$$

This yields, for $m = 1, 2, 3, \ldots$, $I_\theta/I_m = 0.045, 0.016, 0.0083$, etc. The successive maxima decrease rapidly in intensity.

Example 4. *Width of the central diffraction maximum.* Derive the ***half-width*** $\Delta\theta$ of the central maximum in a single-slit Fraunhofer diffraction (see Fig. 44-9*b*). The half-width is the angle between the two points in the pattern where the intensity is one-half that at the center of the pattern.

Point x in Fig. 44-9*b* is so chosen that $I_\theta = \frac{1}{2}I_m$, or, from Eq. 44-8*b*,

$$\frac{1}{2} = \left(\frac{\sin \alpha_x}{\alpha_x}\right)^2.$$

This equation cannot be solved analytically for α_x. It can be solved graphically, as accurately as one wishes, by plotting the quantity $(\sin \alpha_x/\alpha_x)^2$ as ordinate versus α_x as abscissa and noting the value of α_x at which the curve intersects the line "one-half" on the ordinate scale (see Problem 5). However, if only an approximate answer is desired, it is often quicker to use trial-and-error methods.

We know that α equals π at the first minimum; we guess that α_x is perhaps $\pi/2$ ($= 90° = 1.57$ radians). Trying this in Eq. 44-8*b* yields

$$\frac{I_\theta}{I_m} = \left[\frac{\sin (\pi/2)}{\pi/2}\right]^2 = 0.4.$$

This intensity ratio is *less* than 0.5, so that α_x must be *less* than 90°. After a few more trials we find easily enough that

$$\alpha_x = 1.40 \text{ radians} = 80°$$

does yield a ratio close to the correct value of 0.5.

We now use Eq. 44-8*c* to find the corresponding angle θ:

$$\alpha_x = \frac{\pi a}{\lambda} \sin \theta_x = 1.40,$$

or, noting that $a/\lambda = 5$ for Fig. 44-9*b*,

$$\sin \theta_x = \frac{1.40\lambda}{\pi a} = \frac{1.40}{5\pi} = 0.0892.$$

The half-width $\Delta\theta$ of the central maximum (see Fig. 44-9*b*) is given by

$$\Delta\theta = 2\theta_x = 2 \sin^{-1} 0.0892 = 2 \times 5.1° = 10.2°,$$

which is in agreement with the figure. ◀

44-5 Diffraction at a Circular Aperture

Diffraction will occur when a wavefront is partially blocked off by an opaque object such as a metal disk or an opaque screen containing an aperture. Here we consider diffraction at a circular aperture of diameter d, the aperture constituting the boundary of a circular lens.

Our previous treatment of lenses was based on geometrical optics, diffraction being specifically assumed not to occur. A rigorous analysis would be based from the beginning on wave optics, since geometrical optics is always an approximation, although often a good one. Diffraction phenomena would emerge in a natural way from such a wave-optical analysis.

Figure 44-10 shows the image of a distant point source of light (a star) formed on a photographic film placed in the focal plane of a converging lens. It is not a point, as the (approximate) geometrical optics treatment suggests, but a circular disk surrounded by several progressively fainter secondary rings. Comparison with Fig. 42-3c leaves little doubt that we are dealing with a diffraction phenomenon in which, however, the aperture is a circle rather than a long narrow slit. The ratio d/λ, where d is the diameter of the lens (or of a circular aperture placed in front of the lens), determines the scale of the diffraction pattern, just as the ratio a/λ does for a slit.

Analysis shows that the first minimum for the diffraction pattern of a circular aperture of diameter d, assuming Fraunhofer conditions, is given by

$$\sin\theta = 1.22\frac{\lambda}{d} \qquad (44\text{-}10)$$

This is to be compared with Eq. 44-1, or

$$\sin\theta = \frac{\lambda}{a},$$

which locates the first minimum for a long narrow slit of width a. The factor 1.22 emerges from the mathematical analysis when we integrate over the elementary radiators into which the circular aperture may be divided.

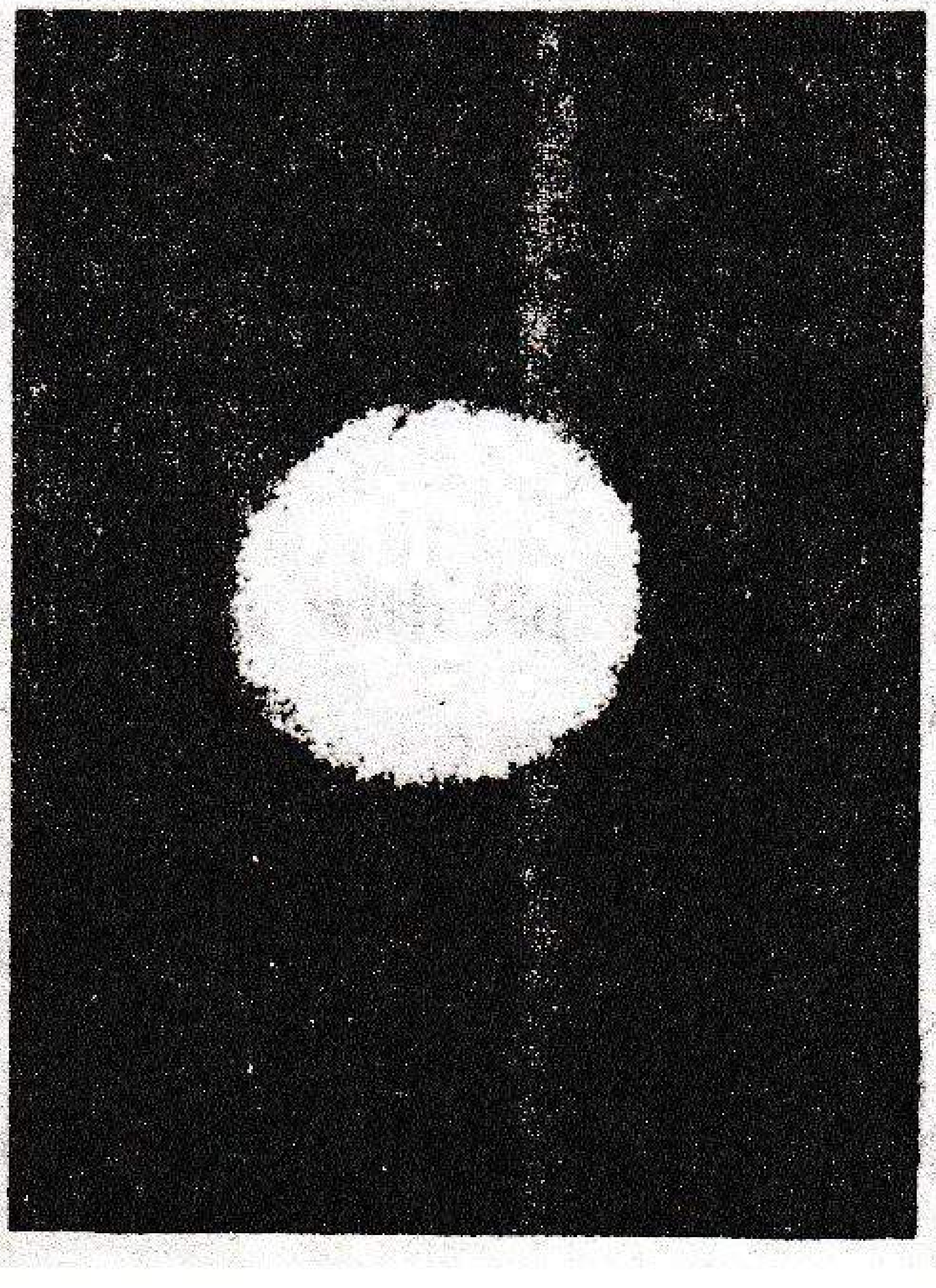

Fig. 44-10 The image of a star formed by a converging lens is a diffraction pattern. Note the central maximum, sometimes called the Airy disk (after Sir George Airy, who first solved the problem of diffraction at a circular aperture in 1835), and the circular secondary maximum. Other secondary maxima occur at larger radii but are too faint to be seen.

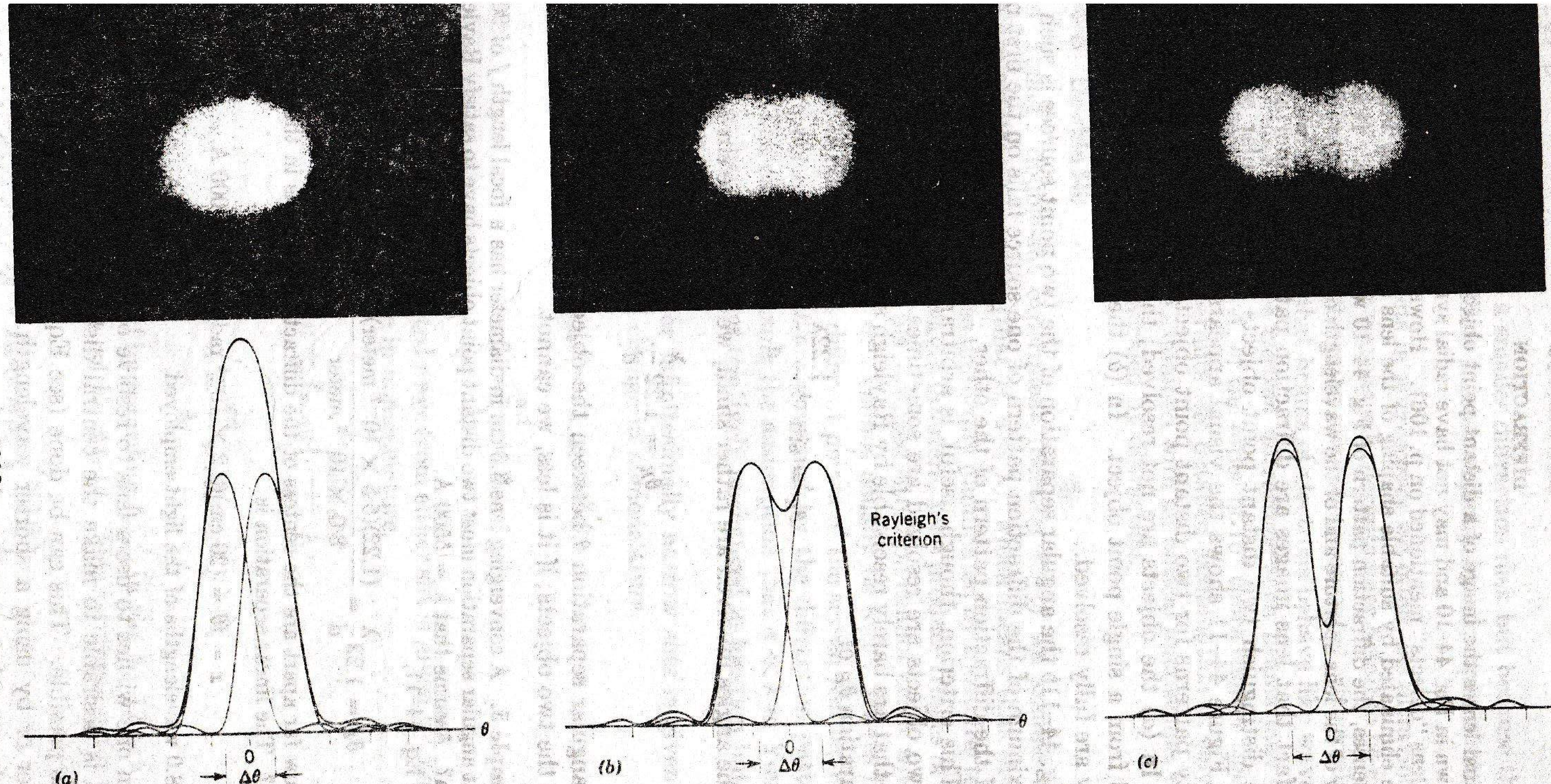

Fig. 44–11 The images of two distant point objects are formed by a converging lens whose diameter (= 10 cm) is 200,000 times the effective wavelength (= 5000 A). Sketches of the images as they appear in the focal plane of the lens are shown with the corresponding intensity plots below them. (*a*) The angular separation of the objects (see vertical ticks) is so small that the images are not resolved. (*b*) The objects are farther apart and the images meet Rayleigh's criterion for resolution. (*c*) The objects are still farther apart and the images are well resolved.

In actual lenses the image of a distant point object will be somewhat larger than that shown in Fig. 44–10 and may not have radial symmetry. This is caused by the various lens "defects" mentioned on p. 1061. However, even if all of these defects could be eliminated by suitable shaping of the lens surfaces or by introducing correcting lenses, the diffraction pattern of Fig. 44–10 would remain. It is an inherent property of the lens aperture and of the wavelength of light used.

The fact that lens images are diffraction patterns is important when we wish to distinguish two distant point objects whose angular separation is small. Figure 44–11 shows the visual appearances and the corresponding intensity patterns for two distant point objects with small angular separations. In (*a*) the objects are not resolved; that is, they cannot be distinguished from a single point object. In (*b*) they are barely resolved and in (*c*) they are fully resolved.

In Fig. 44–11*b* the angular separation of the two point sources is such that the maximum of the diffraction pattern of one source falls on the first minimum of the diffraction pattern of the other. This is called *Rayleigh's criterion.* This criterion, though useful, is arbitrary; other criteria for deciding when two objects are resolved are sometimes used. From Eq. 44–10, two objects that are barely resolvable by Rayleigh's criterion must have an angular separation θ_R of

$$\theta_R = \sin^{-1} \frac{1.22\lambda}{d}.$$

Since the angles involved are rather small, we can replace $\sin \theta_R$ by θ_R, or

$$\theta_R = 1.22 \frac{\lambda}{d}. \qquad (44\text{–}11)$$

If the angular separation θ between the objects is greater than θ_R, we can resolve the two objects; if it is less, we cannot.

▶ **Example 5.** A converging lens 3.0 cm in diameter has a focal length f of 20 cm. (*a*) What angular separation must two distant point objects have to satisfy Rayleigh's criterion? Assume that $\lambda = 5500$ A.

From Eq. 44–11,

$$\theta_R = 1.22 \frac{\lambda}{d} = \frac{(1.22)(5.5 \times 10^{-7} \text{ meter})}{3.0 \times 10^{-2} \text{ meter}} = 2.2 \times 10^{-5} \text{ radian.}$$

(*b*) How far apart are the centers of the diffraction patterns in the focal plane of the lens? The linear separation is

$$x = f\theta = (20 \text{ cm})(2.2 \times 10^{-5} \text{ radian}) = 44{,}000 \text{ A.}$$

This is 8.0 wavelengths of the light employed. ◀

When one wishes to use a lens to resolve objects of small angular separation, it is desirable to make the central disk of the diffraction pattern as small as possible. This can be done (see Eq. 44–11) by increasing the lens diameter or by using a shorter wavelength. One reason for constructing large telescopes is to produce *sharper* images so that celestial objects can be examined in finer detail. The images are also *brighter*, not only because the energy is concentrated into a smaller diffraction disk but because the larger lens collects more light. Thus fainter objects, for example, more distant stars, can be seen.

To reduce diffraction effects in *microscopes* we often use ultraviolet light, which, because of its shorter wavelength, permits finer detail to be examined than would be possible for the same microscope operated with visible light. We shall see in Chapter 48 that beams of electrons behave like waves under some circumstances. In the *electron microscope* such beams may have an effective wavelength of 0.04 A, of the order of 10^5 times shorter than visible light ($\lambda \cong 5000$ A). This permits the detailed examination of tiny objects like viruses. If a virus were examined with an optical microscope, its structure would be hopelessly concealed by diffraction.

44-6 Double Slit

In Young's double-slit experiment (Section 43-1) we assume that the slits are arbitrarily narrow (that is, $a \ll \lambda$), which means that the central part of the diffusing screen was uniformly illuminated by the diffracted waves from each slit. When such waves interfere, they produce fringes of uniform intensity, as in Fig. 43-9. This idealized situation cannot occur with actual slits because the condition $a \ll \lambda$ cannot usually be met. Waves from the two actual slits combining at different points of the screen will have intensities that are *not* uniform but are governed by the diffraction pattern of a single slit. The effect of relaxing the assumption that $a \ll \lambda$ in Young's experiment is to leave the fringes relatively unchanged in location but to alter their intensities.

The interference pattern for infinitesimally narrow slits is given by Eq. 43-11*b* and *c* or, with a small change in nomenclature,

$$I_{\theta,\text{int}} = I_{m,\text{int}} \cos^2 \beta, \tag{44-12}$$

where

$$\beta = \frac{\pi d}{\lambda} \sin \theta. \tag{44-13}$$

The intensity for the diffracted wave from either slit is given by Eqs. 44-8*b* and *c*, or, with a small change in nomenclature,

$$I_{\theta,\text{dif}} = I_{m,\text{dif}} \left(\frac{\sin \alpha}{\alpha}\right)^2, \tag{44-14}$$

where

$$\alpha = \frac{\pi a}{\lambda} \sin \theta. \tag{44-15}$$

The combined effect is found by regarding $I_{m,\text{int}}$ in Eq. 44-12 as a variable amplitude, given in fact by $I_{\theta,\text{dif}}$ of Eq. 44-14. This assumption, for the combined pattern, leads to

$$I_\theta = I_m (\cos \beta)^2 \left(\frac{\sin \alpha}{\alpha}\right)^2. \tag{44-16}$$

in which we have dropped all subscripts referring separately to interference and diffraction.

Let us express this result in words. At any point on the screen the available light intensity from each slit, considered separately, is given by the diffraction pattern of that slit (Eq. 44-14). The diffraction patterns for the two slits, again considered separately, coincide because parallel rays in

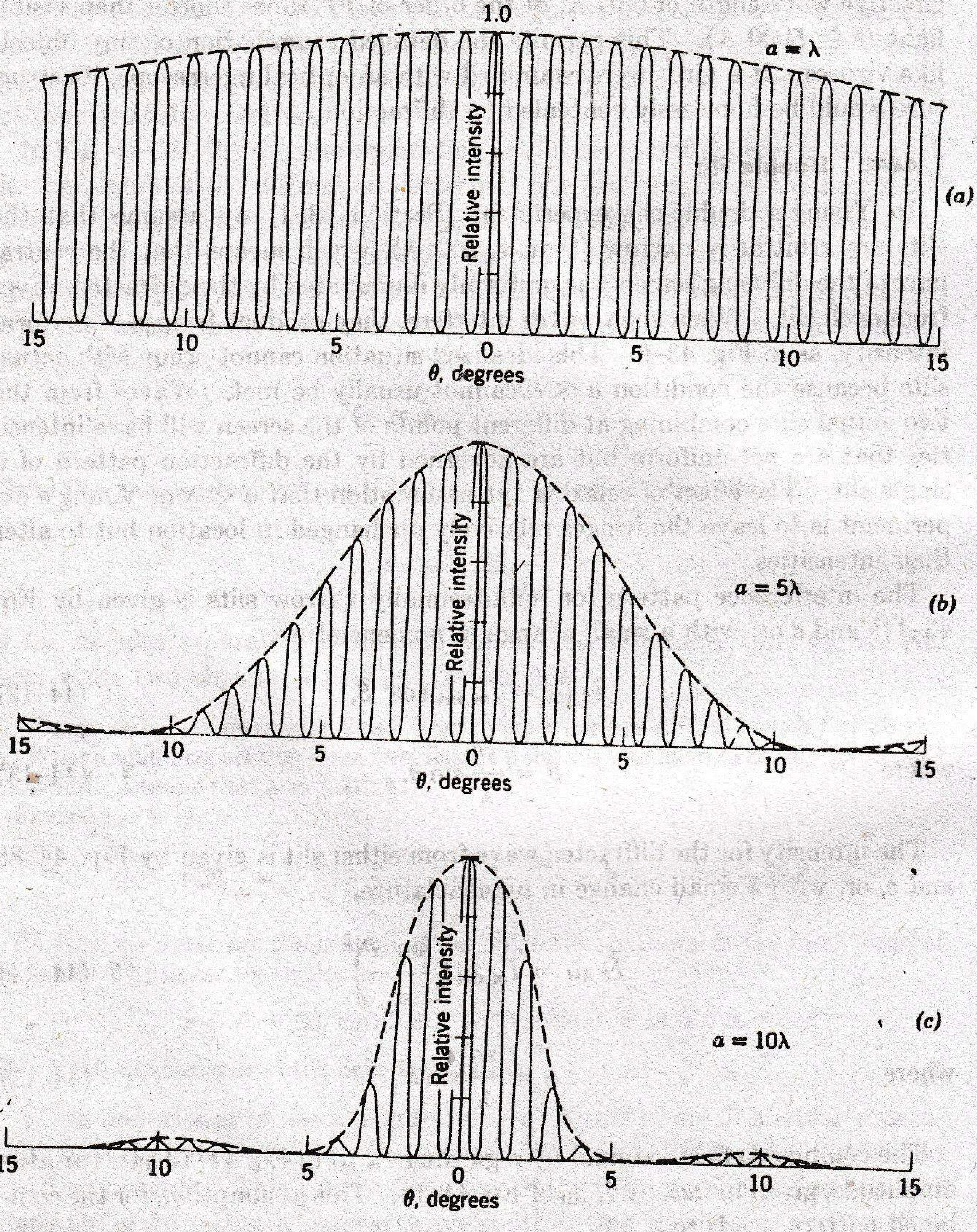

Fig. 44-12 Interference fringes for a double slit with slit separation $d = 50\lambda$. Three different slit widths, described by $a/\lambda = 1$, 5, and 10 are shown.

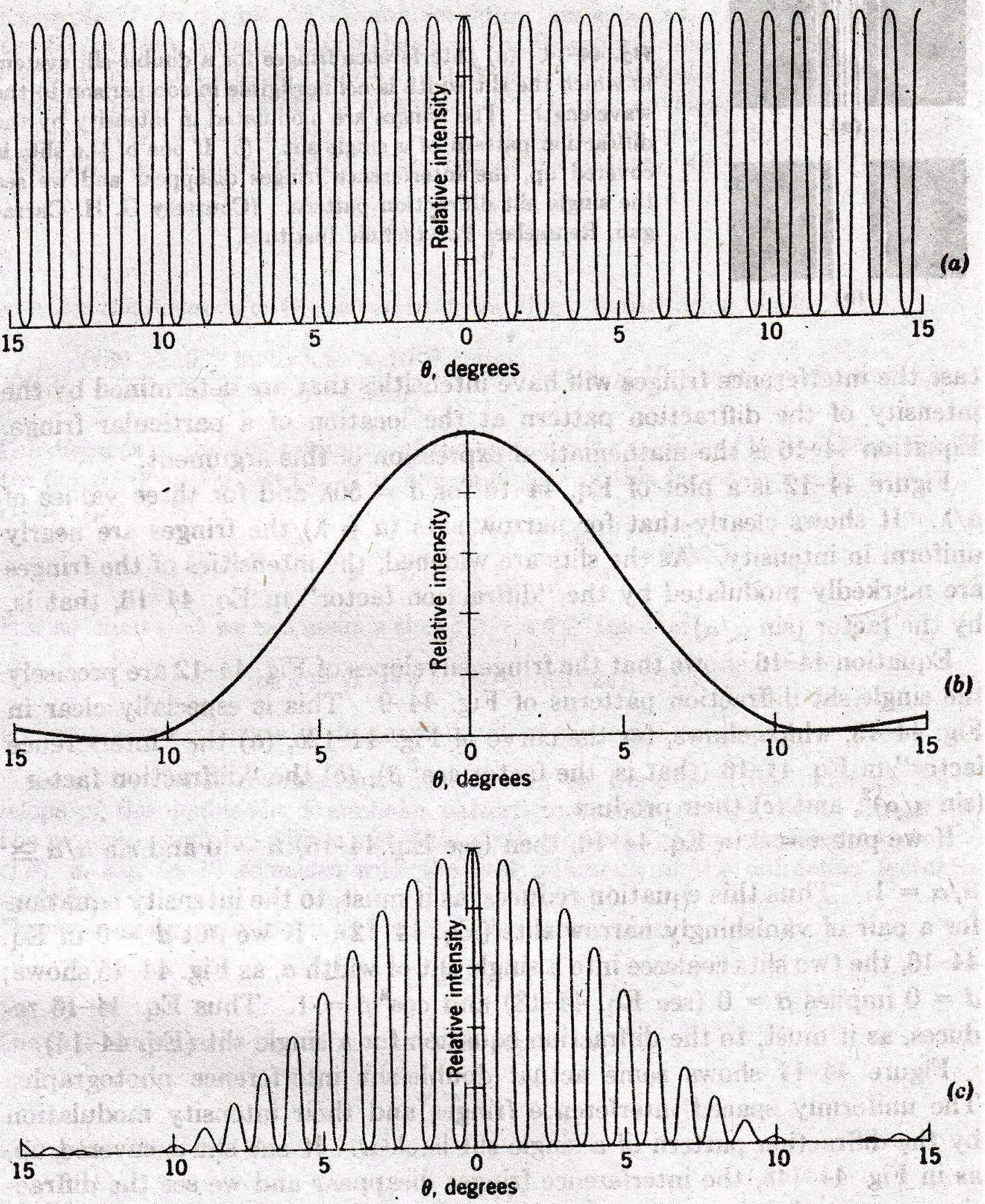

Fig. 44–13 (*a*) The "interference factor" and (*b*) the "diffraction factor" in Eq. 44–16 and (*c*) their product; compare Fig. 44–12*b*.

Fraunhofer diffraction are focused at the same spot (see Fig. 44–5). Because the two diffracted waves are coherent, they will interfere.

The effect of interference is to redistribute the available energy over the screen, producing a set of fringes. In Section 43–1, where we assumed $a \ll \lambda$, the available energy was virtually the same at all points on the screen so that the interference fringes had virtually the same intensities (see Fig. 43–9). If we relax the assumption $a \ll \lambda$, the available energy is *not* uniform over the screen but is given by the diffraction pattern of a slit of width a. In this

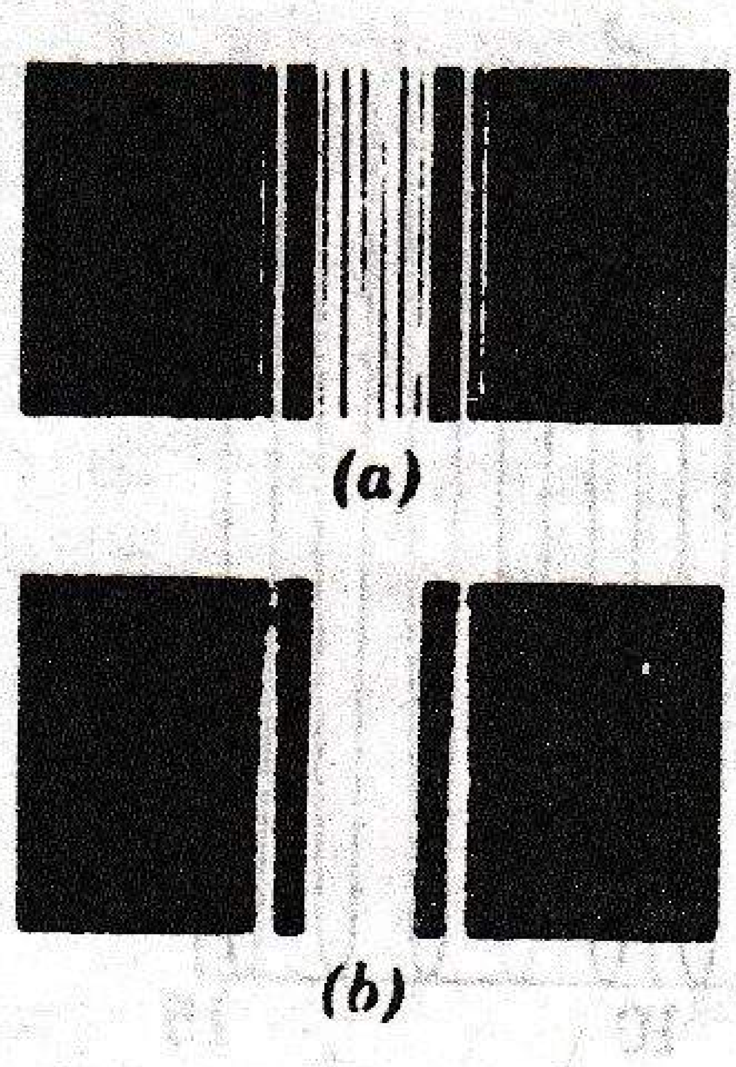

Fig. 44–14 (*a*) Interference fringes for a double-slit system in which the slit width is *not* negligible in comparison to the wavelength. The fringes are modulated in intensity by the diffraction pattern of a single slit. (*b*) If one of the slits is covered up, the interference fringes disappear and we see the single slit diffraction pattern. (Courtesy G. H. Carragan, Rensselaer Polytechnic Institute.)

case the interference fringes will have intensities that are determined by the intensity of the diffraction pattern at the location of a particular fringe. Equation 44–16 is the mathematical expression of this argument.

Figure 44–12 is a plot of Eq. 44–16 for $d = 50\lambda$ and for three values of a/λ. It shows clearly that for narrow slits ($a = \lambda$) the fringes are nearly uniform in intensity. As the slits are widened, the intensities of the fringes are markedly modulated by the "diffraction factor" in Eq. 44–16, that is, by the factor $(\sin \alpha/\alpha)^2$.

Equation 44–16 shows that the fringe envelopes of Fig. 44–12 are precisely the single-slit diffraction patterns of Fig. 44–9. This is especially clear in Fig. 44–13, which shows, for the curve of Fig. 44–12*b*, (*a*) the "interference factor" in Eq. 44–16 (that is, the factor $\cos^2 \beta$), (*b*) the "diffraction factor" $(\sin \alpha/\alpha)^2$, and (*c*) their product.

If we put $a = 0$ in Eq. 44–16, then (see Eq. 44–15) $\alpha = 0$ and $\sin \alpha/\alpha \underset{\alpha \to 0}{\cong} \alpha/\alpha = 1$. Thus this equation reduces, as it must, to the intensity equation for a pair of vanishingly narrow slits (Eq. 44–12). If we put $d = 0$ in Eq. 44–16, the two slits coalesce into a single slit of width a, as Fig. 44–15 shows; $d = 0$ implies $\beta = 0$ (see Eq. 44–13) and $\cos^2 \beta = 1$. Thus Eq. 44–16 reduces, as it must, to the diffraction equation for a single slit (Eq. 44–14).

Figure 44–14 shows some actual double-slit interference photographs. The uniformly spaced interference fringes and their intensity modulation by the diffraction pattern of a single slit is clear. If one slit is covered up, as in Fig. 44–14*b*, the interference fringes disappear and we see the diffraction pattern of a single slit.

▶ **Example 6.** Starting from the curve of Fig. 44–12*b*, what is the effect of (*a*) increasing the slit width, (*b*) increasing the slit separation, and (*c*) increasing the wavelength?

(*a*) If we increase the slit width a, the envelope of the fringe pattern changes so that its central peak is sharper (compare Fig. 44–12*c*). The fringe spacing, which depends on d/λ, does not change.

(*b*) If we increase d, the fringes become closer together, the envelope of the pattern remaining unchanged.

(*c*) If we increase λ, the envelope becomes broader and the fringes move further apart. Increasing λ is equivalent to decreasing both of the ratios a/λ and d/λ. The general relationship of the envelope to the fringes, which depends only on d/a, does not change with wavelength.

Example 7. In double-slit Fraunhofer diffraction what is the fringe spacing on a screen 50 cm away from the slits if they are illuminated with blue light ($\lambda = 4800$ A), if $d = 0.10$ mm, and if the slit width $a = 0.02$ mm? What is the linear distance from the central maximum to the first minimum of the fringe envelope?

The intensity pattern is given by Eq. 44-16, the fringe spacing being determined by the interference factor $\cos^2 \beta$. From Example 2, Chapter 43, we have

$$\Delta y = \frac{\lambda D}{d},$$

where D is the distance of the screen from the slits. Substituting yields

$$\Delta y = \frac{(480 \times 10^{-9}\text{ meter})(50 \times 10^{-2}\text{ meter})}{0.10 \times 10^{-3}\text{ meter}} = 2.4 \times 10^{-3}\text{ meter} = 2.4\text{ mm.}$$

The distance to the first minimum of the envelope is determined by the diffraction factor $(\sin \alpha/\alpha)^2$ in Eq. 44-16. The first minimum in this factor occurs for $\alpha = \pi$. From Eq. 44-15,

$$\sin \theta = \frac{\alpha\lambda}{\pi a} = \frac{\lambda}{a} = \frac{480 \times 10^{-9}\text{ meter}}{0.02 \times 10^{-3}\text{ meter}} = 0.024.$$

This is so small that we can assume that $\theta \cong \sin \theta \cong \tan \theta$, or

$$y = D \tan \theta \cong D \sin \theta = (50\text{ cm})(0.024) = 1.2\text{ cm.}$$

There are about ten fringes in the central peak of the fringe envelope.

Example 8. What requirements must be met for the central maximum of the envelope of the double-slit Fraunhofer pattern to contain exactly eleven fringes?

The required condition will be met if the sixth minimum of the interference factor ($\cos^2 \beta$) in Eq. 44-16 coincides with the first minimum of the diffraction factor $(\sin \alpha/\alpha)^2$.

The sixth minimum of the interference factor occurs when

$$\beta = \tfrac{11}{2}\pi$$

in Eq. 44-12.

The first minimum in the diffraction term occurs for

$$\alpha = \pi.$$

Dividing (see Eqs. 44-13 and 44-15) yields

$$\frac{\beta}{\alpha} = \frac{d}{a} = \frac{11}{2}.$$

This condition depends only on the slit geometry and not on the wavelength. For long waves the pattern will be broader than for short waves, but there will always be eleven fringes in the central peak of the envelope. ◀

The double-slit problem as illustrated in Fig. 44-12 combines interference and diffraction in an intimate way. At root both are superposition effects and depend on adding wave disturbances at a given point, taking phase differences properly into account. If the waves to be combined originate from a *finite* (and usually small) number of elementary coherent radiators, as in Young's double-slit experiment, we call the effect *interference*. If the waves to be combined originate by subdividing a wave into *infinitesimal* coherent radiators, as in our treatment of a single slit (Fig. 44–6), we call the effect *diffraction*. This distinction between interference and diffraction is convenient and useful. However, it should not cause us to lose sight of the fact that both are superposition effects and that often both are present simultaneously, as in Young's experiment.

QUESTIONS

1. Why is the diffraction of sound waves more evident in daily experience than that of light waves?

2. Why do radio waves diffract around buildings, although light waves do not?

3. A loud-speaker horn has a rectangular aperture 4 ft high and 1 ft wide. Will the pattern of sound intensity be broader in the horizontal plane or in the vertical?

4. A radar antenna is designed to give accurate measurements of the height of an aircraft but only reasonably good measurements of its direction in a horizontal plane. Must the height-to-width ratio of the radar reflector be less than, equal to, or greater than unity?

5. A person holds a single narrow vertical slit in front of the pupil of his eye and looks at a distant light source in the form of a long heated filament. Is the diffraction pattern that he sees a Fresnel or a Fraunhofer pattern?

6. In a single-slit Fraunhofer diffraction, what is the effect of increasing (*a*) the wavelength and (*b*) the slit width?

7. Sunlight falls on a single slit of width 10^4 A. Describe qualitatively what the resulting diffraction pattern looks like.

8. In Fig. 44–5 rays r_1 and r_3 are in phase; so are r_2 and r_4. Why isn't there a *maximum* intensity at P_2 rather than a minimum?

9. Describe what happens to a Fraunhofer single-slit diffraction pattern if the whole apparatus is immersed in water.

10. Distinguish clearly between θ, α, and ϕ in Eq. 44–8*c*.

11. Do diffraction effects occur for virtual images as well as for real images? Explain.

12. Do diffraction effects occur for images formed by (*a*) plane mirrors and (*b*) spherical mirrors? Explain.

13. If we were to redo our analysis of the properties of lenses in Section 42–5 by the methods of geometrical optics but *without* restricting our considerations to paraxial rays and to "thin" lenses, would diffraction phenomena, such as that of Fig. 44–10, emerge from the analysis? Discuss.

14. Distinguish carefully between interference and diffraction in Young's double-slit experiment.

15. In what way are interference and diffraction similar? In what way are they different?

16. In double-slit interference patterns such as that of Fig. 44–14*a* we said that the interference fringes were modulated in intensity by the diffraction pattern of a single slit. Could we reverse this statement and say that the diffraction pattern of a single slit is intensity-modulated by the interference fringes? Discuss.

PROBLEMS

1. In a single-slit diffraction pattern the distance between the first minimum on the right and the first minimum on the left is 5.2 mm. The screen on which the pattern is displayed is 80 cm from the slit and the wavelength is 5460 A. Calculate the slit width.

2. A plane wave ($\lambda = 5900$ A) falls on a slit with $a = 0.40$ mm. A converging lens ($f = +70$ cm) is placed behind the slit and focuses the light on a screen. What is the linear distance on the screen from the center of the pattern to (*a*) the first minimum and (*b*) the second minimum?

3. A single slit is illuminated by light whose wavelengths are λ_a and λ_b, so chosen that the first diffraction minimum of λ_a coincides with the second minimum of λ_b. (*a*) What relationship exists between the two wavelengths? (*b*) Do any other minima in the two patterns coincide?

4. (*a*) Show that the values of α at which intensity maxima for single-slit diffraction occur can be found exactly by differentiating Eq. 44–8*b* with respect to α and equating to zero, obtaining the condition

$$\tan \alpha = \alpha.$$

(*b*) Find the values of α satisfying this relation by plotting graphically the curve $y = \tan \alpha$ and the straight line $y = \alpha$ and finding their intersections. (*c*) Find the (nonintegral) values of m corresponding to successive maxima in the single-slit pattern. Note that the secondary maxima do not lie exactly halfway between minima.

5. In Example 4 solve the transcendental equation

$$\frac{1}{2} = \left(\frac{\sin \alpha_x}{\alpha_x}\right)^2$$

graphically for α_x, to an accuracy of three significant figures.

6. (*a*) In Fig. 44–7*d*, why is E_θ, which represents the first maximum beyond the central maximum, not vertical? (*b*) Calculate the angle it makes with the vertical, assuming the slit to be divided into infinitesimal strips of width dx.

7. What is the half-width of a diffracted beam for a slit whose width is (*a*) 1, (*b*) 5, and (*c*) 10 wavelengths?

8. (*a*) A circular diaphragm 0.60 meter in diameter oscillates at a frequency of 25,000 cycles/sec in an underwater source of sound for submarine detection. Far from the source the sound intensity is distributed as a Fraunhofer diffraction pattern for a circular hole whose diameter equals that of the diaphragm. Take the speed of sound in water to be 1450 meters/sec and find the angle between the normal to the diaphragm and the direction of the first minimum. (*b*) Repeat for a source having an (audible) frequency of 1000 cycles/sec.

9. The two headlights of an approaching automobile are 4 ft apart. At what maximum distance will the eye resolve them? Assume a pupil diameter of 5.0 mm and $\lambda = 5500$ A. Assume also that this distance is determined only by diffraction effects at the circular pupil aperture.

10. The wall of a large room is covered with acoustic tile in which small holes are drilled 5.0 mm from center to center. How far can a person be from such a tile and still distinguish the individual holes, assuming ideal conditions? Assume the diameter of the pupil to be 4.0 mm and λ to be 5500 A.

11. (*a*) How small is the angular separation of two stars if their images are barely resolved by the Thaw refracting telescope at the Allegheny Observatory in Pittsburgh? The lens diameter is 30 in. and its focal length is 46 ft. Assume $\lambda = 5000$ A. (*b*) Find the distance between these barely resolved stars if each of them is 10 light years distant from the earth. (*c*) For the image of a single star in this telescope, find the diameter of the first dark ring in the diffraction pattern, as measured on a photographic plate placed at the

focal plane. Assume that the star image structure is associated entirely with diffraction at the lens aperture and not with (small) lens "errors."

12. Find the separation of two points on the moon's surface that can just be resolved by the 200-in. telescope at Mount Palomar, assuming that this distance is determined by diffraction effects. The distance from the earth to the moon is 240,000 miles.

13. Construct qualitative vector diagrams like those of Fig. 44-7 for the double-slit interference pattern. For simplicity, consider $d = 2a$ (see Fig. 44-15). Can you interpret the main features of the intensity pattern this way?

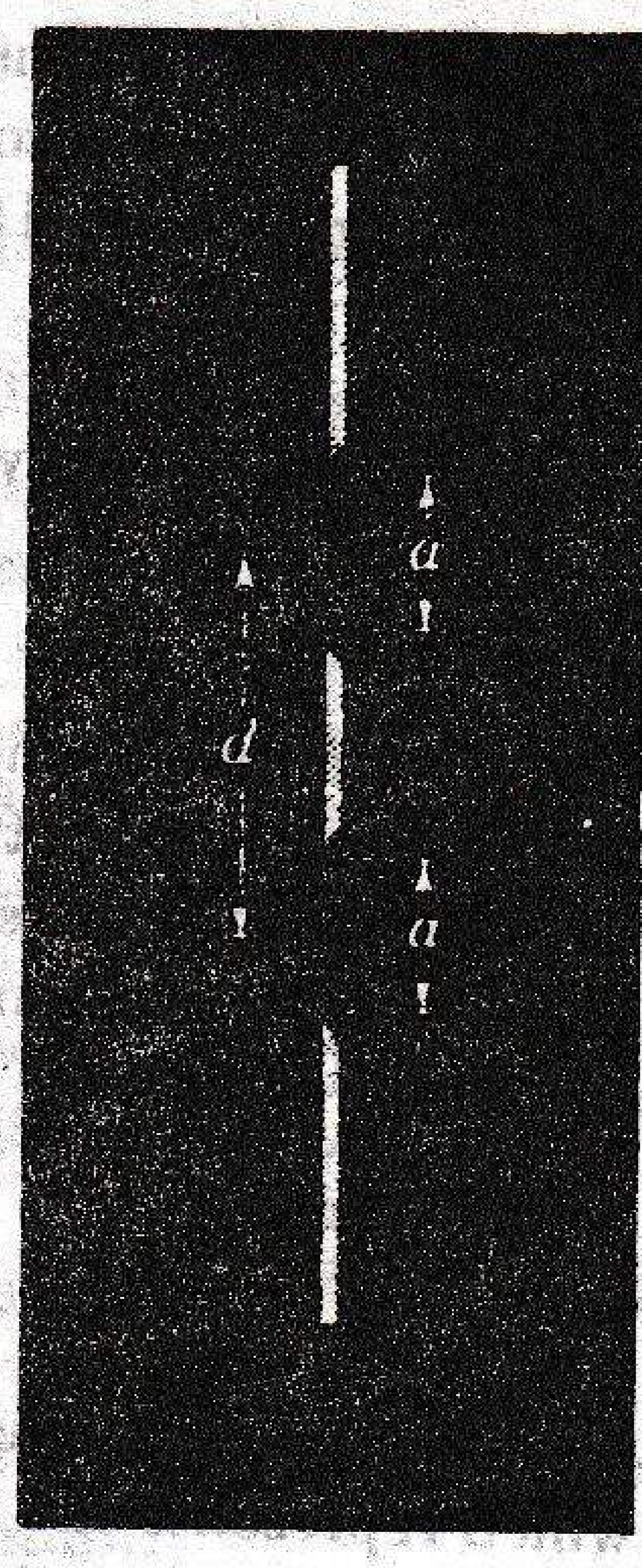

Fig. 44-15

14. Suppose that, as in Example 8, the envelope of the central peak contains eleven fringes. How many fringes lie between the first and second minima of the envelope?

15. For $d = 2a$ in Fig. 44-15, how many interference fringes lie in the central diffraction envelope?

16. If we put $d = a$ in Fig. 44-15, the two slits coalesce into a single slit of width $2a$. Show that Eq. 44-16 reduces to the diffraction pattern for such a slit.

17. (*a*) Design a double-slit system in which the fourth fringe, not counting the central maximum, is missing. (*b*) What other fringes, if any, are also missing?

Gratings and Spectra

CHAPTER 45

45-1 Introduction

In connection with Young's experiment (Sections 43-1 and 43-3) we discussed the interference of two coherent waves formed by diffraction at two elementary radiators (pinholes or slits). In our first treatment we assumed that the slit width was much less than the wavelength, so that light diffracted from each slit illuminated the observation screen essentially uniformly. Later, in Section 44-6, we took the slit width into account and showed that the intensity pattern of the interference fringes is modulated by a "diffraction factor" $(\sin \alpha/\alpha)^2$ (see Eq. 44-16).

Here we extend our treatment to cases in which the number N of radiators or diffracting centers is larger—and usually much larger—than two. We consider two situations:

1. An array of N parallel equidistant slits, called a *diffraction grating.*
2. A three-dimensional array of periodically arranged radiators—the atoms in a crystalline solid such as NaCl. In this case the average spacing between the elementary radiators is so small that interference effects must be sought at wavelengths much smaller than those of visible light. We speak of *X-ray diffraction.*

In each case we distinguish carefully between the diffracting properties of a single radiator (slit or atom) and the interference of the waves diffracted, coherently, from the assembly of radiators.

45-2 Multiple Slits

A logical extension of Young's double-slit interference experiment is to increase the number of slits from two to a larger number N. An arrangement like that of Fig. 45-1, usually involving many more slits, is called a *diffrac-*

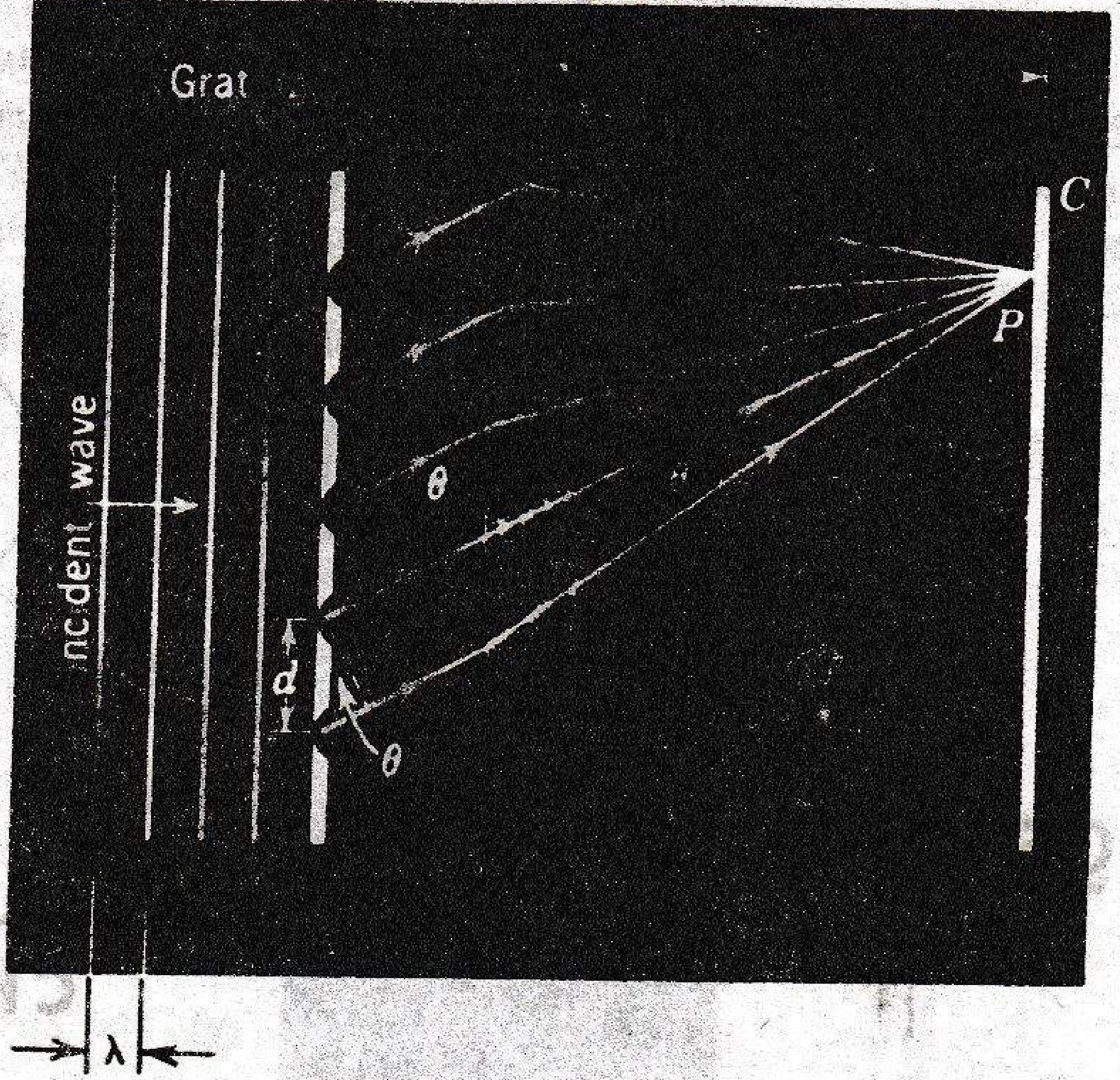

Fig. 45–1 An idealized diffraction grating containing five slits. The slit width a is shown for convenience to be considerably smaller than λ, although this condition is not realized in practice. The figure is distorted in that f is much greater than d in practice.

tion grating. As for a double slit, the intensity pattern that results when monochromatic light of wavelength λ falls on a grating consists of a series of interference fringes. The *angular separations* of these fringes are determined by the ratio λ/d, where d is the spacing between the centers of adjacent slits. The relative *intensities* of these fringes are determined by the diffraction pattern of a single grating slit, which depends on the ratio λ/a, where a is the slit width.

Figure 45-2, which compares the intensity patterns for $N = 2$ and $N = 5$, shows clearly that the "interference" fringes are modulated in intensity by a "diffraction" envelope, as in Fig. 44–14. Figure 45–3 presents a theoretical calculation of the intensity patterns for a few fringes near the centers of the patterns of Fig. 45–2. These two figures show that increasing N (*a*) does not change the spacing between the (principal) interference fringe maxima,

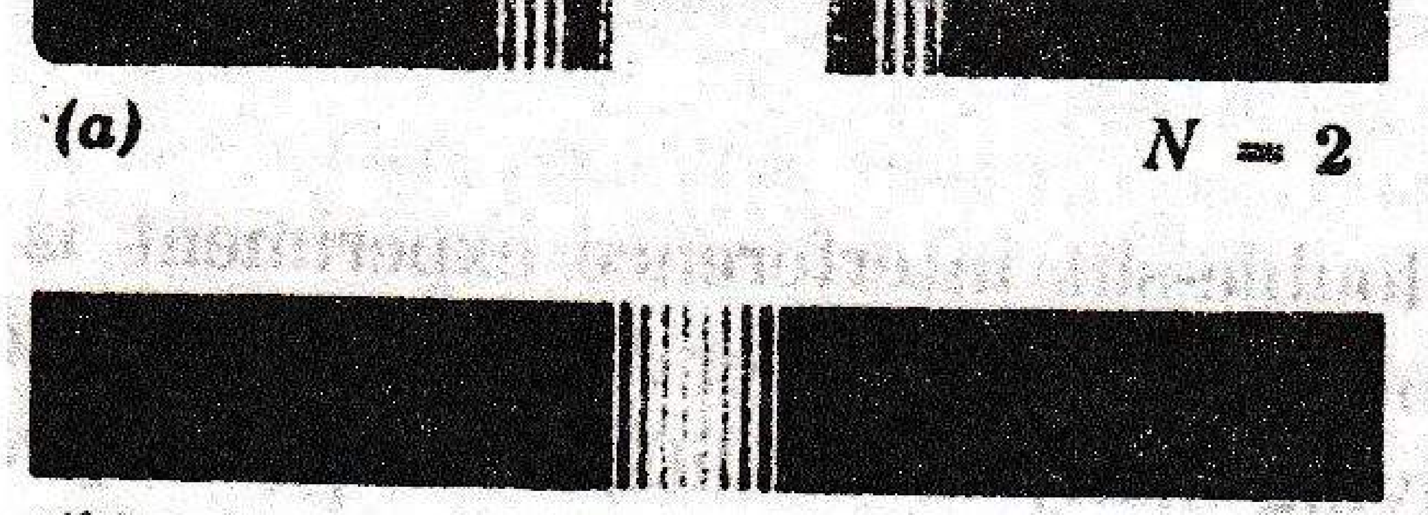

Fig. 45–2 Intensity patterns for "gratings" with (*a*) $N = 2$ and (*b*) $N = 5$ for the same value of d and λ. Note how the intensities of the fringes are modulated by a diffraction envelope as in Fig. 44–14; thus the assumption $a \ll \lambda$ is not realized in these actual "gratings." For $N = 5$ three very faint secondary maxima, not visible in this photograph, appear between each pair of adjacent primary maxima.

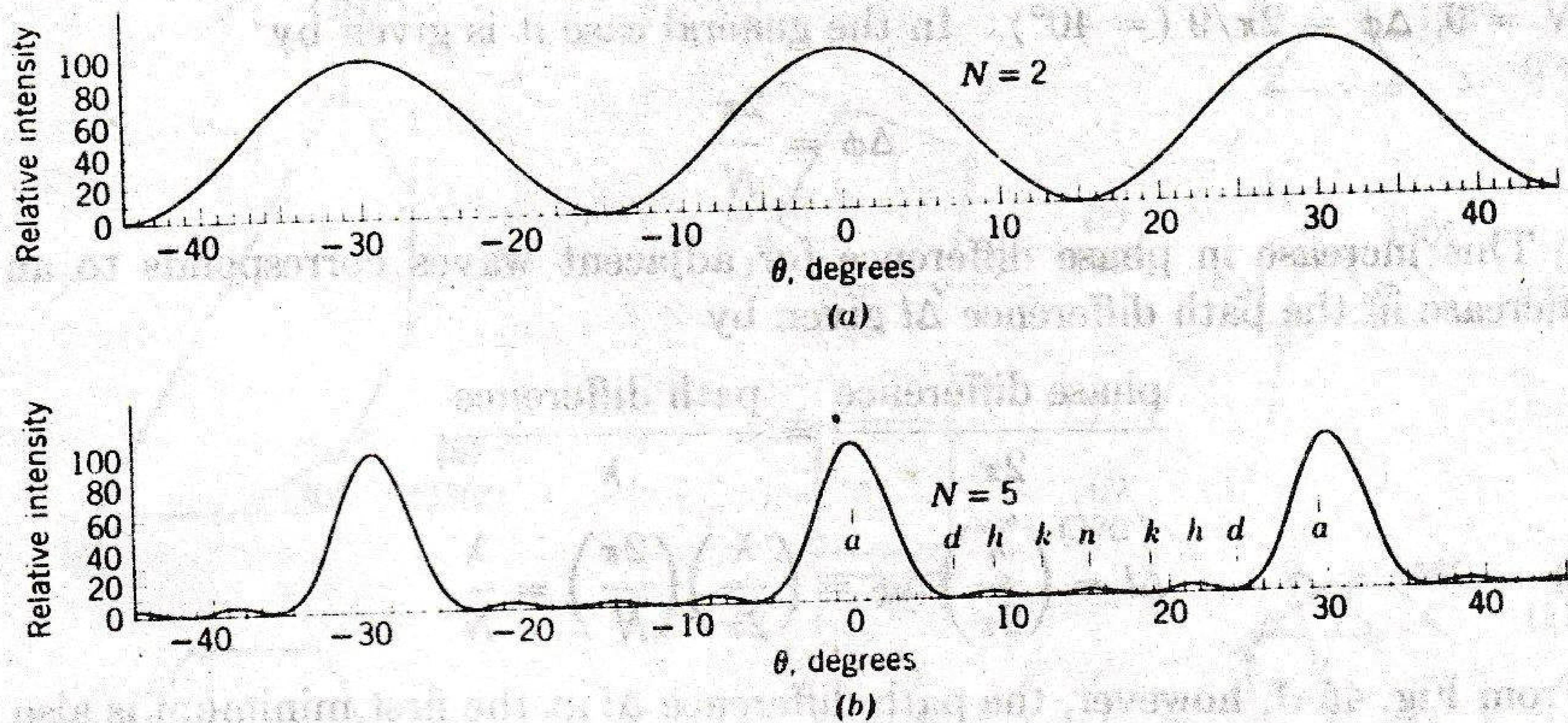

Fig. 45-3 Calculated intensity patterns for (*a*) a two-slit and (*b*) a five-slit grating for the same value of d and λ. This figure shows the sharpening of the principal maxima and the appearance of faint secondary maxima for $N > 2$. The letters on the five-slit pattern refer to Fig. 45-5. The figure assumes slits with $a \ll \lambda$ so that the principal maxima are of uniform intensity.

provided d and λ remain unchanged, (*b*) sharpens the (principal) maxima, and (*c*) introduces small secondary maxima between the principal maxima. Three such secondaries are present (but not readily visible) between each pair of adjacent principal maxima in Fig. 45-2*b*.

A principal maximum in Fig. 45-1 will occur when the path difference between rays from adjacent slits ($= d \sin \theta$) is given by

$$d \sin \theta = m\lambda \qquad m = 0, 1, 2, \ldots \qquad \text{(principal maxima)}, \qquad (45\text{-}1)$$

where m is called the *order number*. This equation is identical with Eq. 43-1, which locates the intensity maxima for a double slit. The *locations* of the (principal) maxima are thus determined only by the ratio λ/d and are independent of N. As for the double slit, the ratio a/λ determines the relative *intensities* of the principal maxima but does not alter their locations appreciably.

The sharpening of the principal maxima as N is increased can be understood by a graphical argument, using phasors. Figures 45-4*a* and *b* show conditions at any of the principal maxima for a two-slit and a nine-slit grating. The small arrows represent the amplitudes of the wave disturbances arriving at the screen at the position of each principal maximum. For simplicity we consider the central principal maximum only, for which $m = 0$, and thus $\theta = 0$, in Eq. 45-1.

Consider the angle $\Delta\theta_0$ corresponding to the position of zero intensity that lies on either side of the central principal maximum. Figures 45-4*c* and *d* show the phasors at this point. The phase difference between waves from adjacent slits, which is zero at the central principal maximum, must increase by an amount $\Delta\phi$ chosen so that the array of phasors just closes on itself, yielding zero resultant intensity. For $N = 2$, $\Delta\phi = 2\pi/2$ ($= 180°$); for $N = 9$, $\Delta\phi = 2\pi/9$ ($= 40°$). In the general case it is given by

$$\Delta\phi = \frac{2\pi}{N}.$$

This increase in phase difference for adjacent waves corresponds to an increase in the path difference Δl given by

$$\frac{\text{phase difference}}{2\pi} = \frac{\text{path difference}}{\lambda},$$

or

$$\Delta l = \left(\frac{\lambda}{2\pi}\right)\Delta\phi = \left(\frac{\lambda}{2\pi}\right)\left(\frac{2\pi}{N}\right) = \frac{\lambda}{N}.$$

From Fig. 45-1, however, the path difference Δl at the first minimum is also given by $d \sin \Delta\theta_0$, so that we can write

$$d \sin \Delta\theta_0 = \frac{\lambda}{N},$$

or

$$\sin \Delta\theta_0 = \frac{\lambda}{Nd}.$$

Since $N \gg 1$ for actual gratings, $\sin \Delta\theta_0$ will ordinarily be quite small (that is, the lines will be sharp), and we may replace it by $\Delta\theta_0$ to good approximation, or

$$\Delta\theta_0 = \frac{\lambda}{Nd} \quad \text{(central principal maximum).} \qquad (45\text{–}2)$$

This equation shows specifically that if we increase N for a given λ and d,

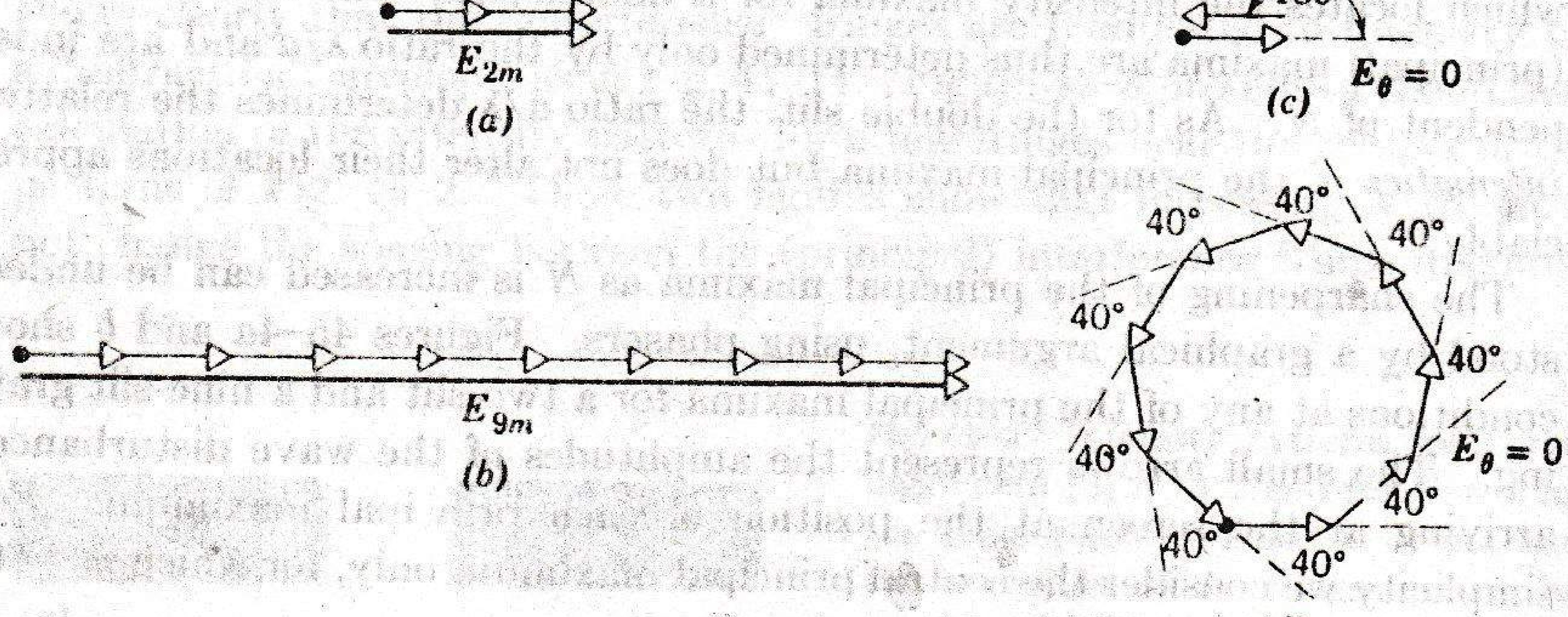

Fig. 45-4 Drawings (*a*) and (*b*) show conditions at the central principal maximum for a two-slit and a nine-slit grating, respectively. Drawings (*c*) and (*d*) show conditions at the minimum of zero intensity that lies on either side of this central principal maximum. In going from (*a*) to (*c*) the phase shift between waves from adjacent slits changes by 180° ($\Delta\phi = 2\pi/2$); in going from (*b*) to (*d*) it changes by 40° ($\Delta\phi = 2\pi/9$).

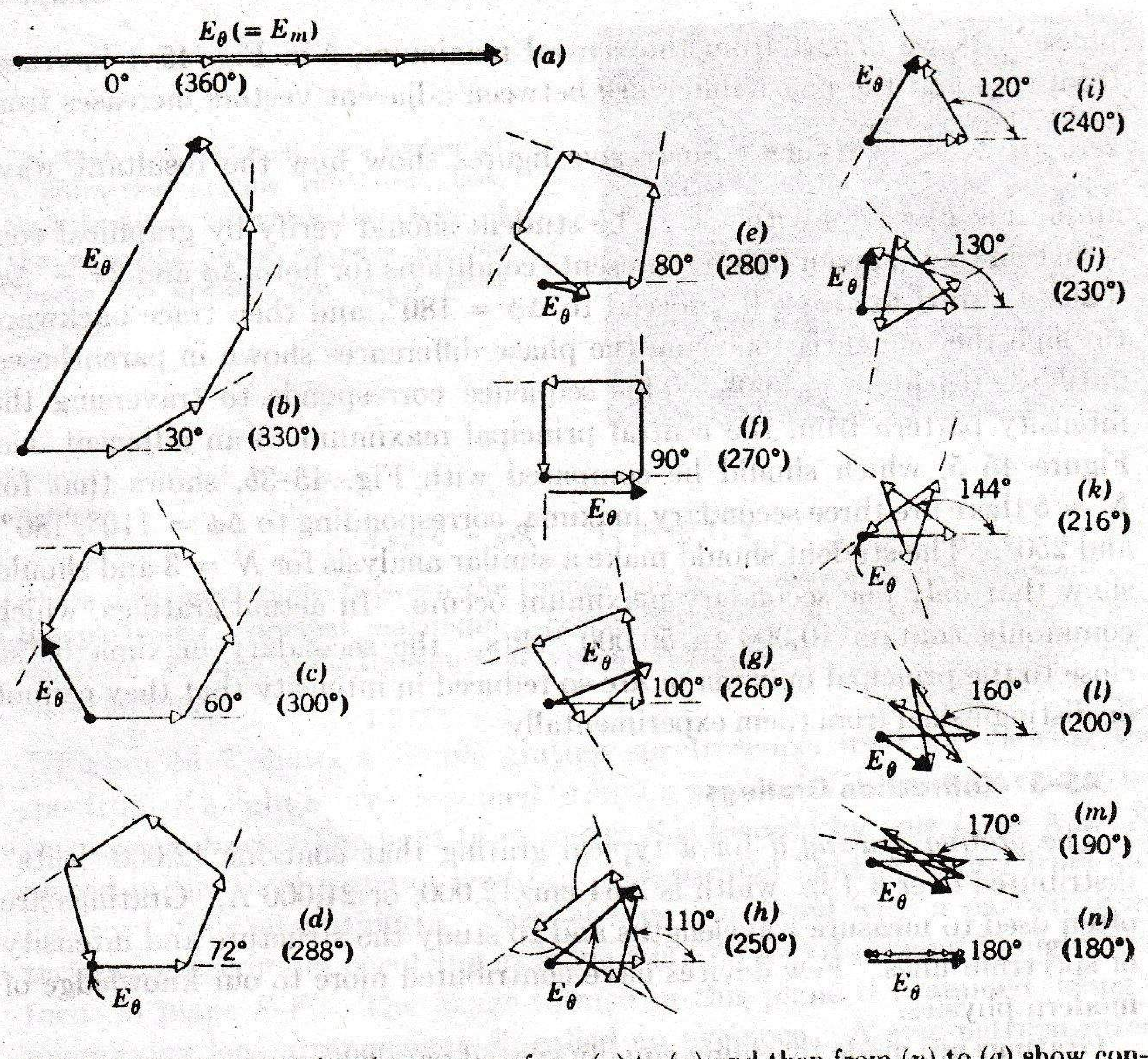

Fig. 45-5 The figures taken in sequence from (a) to (n) and then from (n) to (a) show conditions as the intensity pattern of a five-slit grating is traversed from the central principal maximum to an adjacent principal maximum. Phase differences between waves from adjacent slits are shown directly or, when going from (n) to (a), in parentheses. Principal maxima occur at (a), secondary maxima at, or near, (h) and (n), and points of zero intensity at (d) and (k). Compare Fig. 45-3b.

$\Delta\theta_0$ will decrease, which means that the central principal maximum becomes sharper.

We state without proof,* and for later use, that for principal maxima other than the central one (that is, for $m \neq 0$) the angular distance between the position θ_m of the principal maximum of order m and the minimum that lies on either side is given by

$$\Delta\theta_m = \frac{\lambda}{Nd \cos \theta_m} \quad \text{(any principal maximum).} \qquad (45\text{-}3)$$

For the central principal maximum we have $m = 0$, $\theta_m = 0$, and $\Delta\theta_m = \Delta\theta_0$, so that Eq. 45-3 reduces, as it must, to Eq. 45-2.

The origin of the secondary maxima that appear for $N > 2$ can also be understood using the phasor method. Figure 45-5a shows conditions at the central principal maximum for a five-slit grating. The vectors are in

* See Problem 15.

phase. As we depart from the central maximum, θ in Fig. 45-1 increases from zero and the phase difference between adjacent vectors increases from zero to $\Delta\phi = \frac{2\pi}{\lambda} d \sin\theta$. Successive figures show how the resultant wave amplitude E_θ varies with $\Delta\phi$. The student should verify by graphical construction that a given figure represents conditions for both $\Delta\phi$ and $2\pi - \Delta\phi$. Thus we start at $\Delta\phi = 0$, proceed to $\Delta\phi = 180°$, and then trace backward through the sequence, following the phase differences shown in parentheses, until we reach $\Delta\phi = 360°$. This sequence corresponds to traversing the intensity pattern from the central principal maximum to an adjacent one. Figure 45-5, which should be compared with Fig. 45-3b, shows that for $N = 5$ there are three secondary maxima, corresponding to $\Delta\phi = 110°$, $180°$, and $250°$. The student should make a similar analysis for $N = 3$ and should show that only one secondary maximum occurs. In actual gratings, which commonly contain 10,000 to 50,000 "slits," the secondary maxima lie so close to the principal maxima or are so reduced in intensity that they cannot be distinguished from them experimentally.

45-3 Diffraction Gratings

The *grating spacing* d for a typical grating that contains 12,000 "slits" distributed over a 1-in. width is 2.54 cm/12,000, or 21,000 A. Gratings are often used to measure wavelengths and to study the structure and intensity of spectrum lines. Few devices have contributed more to our knowledge of modern physics.

Gratings are made by ruling equally spaced parallel grooves on a glass or a metal * plate, using a diamond cutting point whose motion is automatically controlled by an elaborate ruling engine. Once such a master grating has been prepared, replicas can be formed by pouring a collodion solution on the grating, allowing it to harden, and stripping it off. The stripped collodion, fastened to a flat piece of glass or other backing, forms a good grating.

Figure 45-6 shows a cross section of a common type of grating ruled on glass. In the rudimentary grating of Fig. 45-1 open slits were separated by opaque strips; the *amplitude* of the wave disturbance varied in a periodic way as the grating was crossed, dropping to zero on the opaque strips. The grating of Fig. 45-6 is transparent everywhere, so that there is little periodic change in amplitude as the grating is crossed. The effect of the rulings is to change the *optical thickness* of the grating in a periodic way, rays traversing the grating between the rulings (b in Fig. 45-6) containing more wavelengths than rays traversing the grating in the center of the rulings (a in Fig. 45-6). This results in a periodic change of *phase* as one crosses the grating at right angles to the rulings. Reflection gratings also depend for their operation on a periodic change in phase of the reflected wave as one crosses the grating, the change in amplitude under these conditions being negligible. The principal maxima for *phase gratings*, assuming that the incident light falls on the grating at right angles, can be given by the same formula derived earlier for idealized amplitude or slit gratings,

* Gratings ruled on metal are called *reflection gratings* because the interference effects are viewed in reflected rather than in transmitted light. Many research gratings are the reflection type; commonly they are ruled on the surface of a concave mirror, which eliminates the need for lenses.

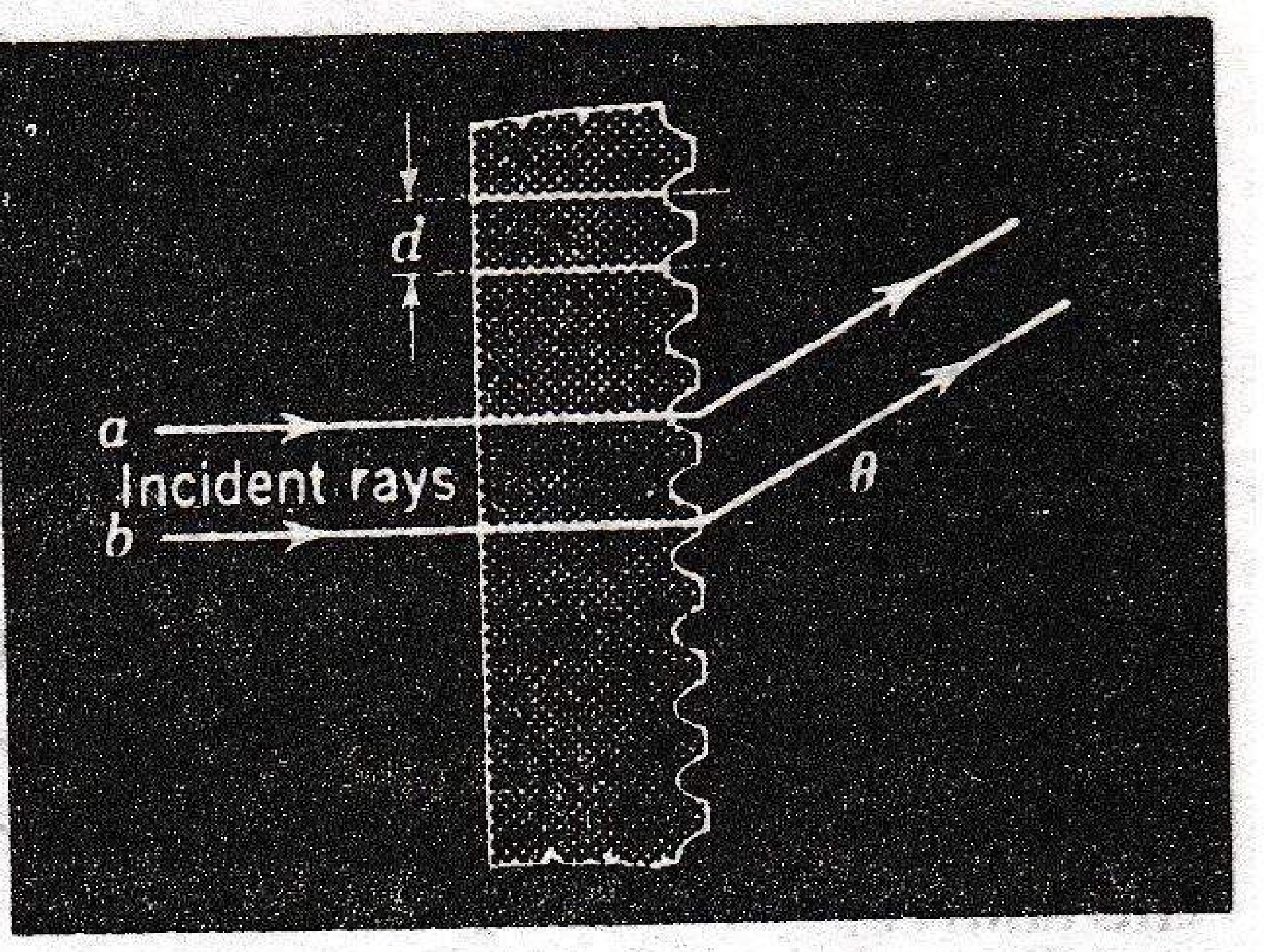

Fig. 45-6 An enlarged cross section of a diffraction grating ruled on glass. Such gratings, in which the phase of the emerging wave changes as one crosses the grating, are called *phase gratings*.

namely

$$d \sin \theta = m\lambda \qquad m = 0, 1, 2 \ldots,$$

where d is the distance between the rulings and the integer m is called the *order* of the particular principal maximum. Essentially all gratings used in the visible spectrum, whether of the transmission type, as in Fig. 45-6, or the reflection type, are phase gratings.

Figure 45-7 shows a simple grating spectroscope, used for viewing the spectrum of a light source, assumed to emit a number of discrete wavelengths, or *spectrum lines*. The light from source S is focused by lens L_1 on a slit S_1 placed in the focal plane of lens L_2. The parallel light emerging from collimator C falls on grating G. Parallel rays associated with a particular interference maximum occurring at angle θ fall on lens L_3, being brought to a focus in plane F-F'. The image formed in this plane is examined, using a magnifying lens arrangement E, called an eyepiece. A symmetrical inter-

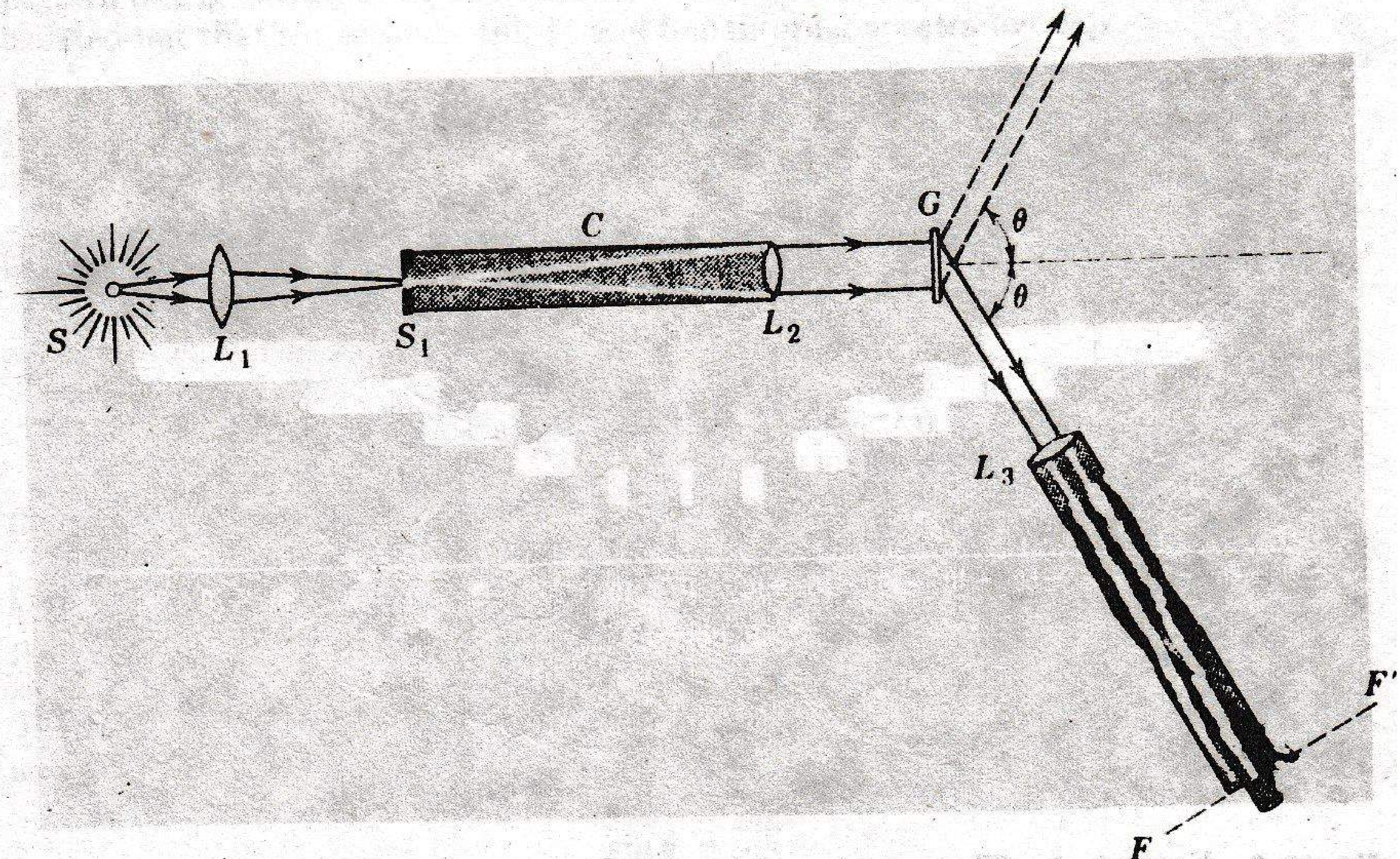

Fig. 45-7 A simple type of grating spectroscope used to analyze the wavelengths of the light emitted by source S.

ference pattern is formed on the other side of the central position, as shown by the dotted lines. The entire spectrum can be viewed by rotating telescope T' through various angles. Instruments used for scientific research or in industry are more complex than the simple arrangement of Fig. 45–7. They invariably employ photographic or photoelectric recording and are called *spectrographs*. Figure 47–12 shows a small portion of the spectrum of iron, produced by examining the light produced in an arc struck between iron electrodes, using a research type spectrograph with photographic recording. Each line in the figure represents a different wavelength that is emitted from the source.

Grating instruments can be used to make absolute measurements of wavelength, since the grating spacing d in Eq. 45–1 can be measured accurately with a traveling microscope. Several spectra are normally produced in such instruments, corresponding to $m = \pm 1, \pm 2$, etc., in Eq. 45–1 (see Fig. 45–8). This may cause some confusion if the spectra overlap. Further, this multiplicity of spectra reduces the recorded intensity of any given spectrum line because the available energy is divided among a number of spectra.

This disadvantage of the grating instrument can be overcome by shaping the profile of the grating grooves so that a large fraction of the light is thrown into a particular order on a particular side (for a given wavelength). This technique, called *blazing*, so alters the diffracting properties of the individual grooves (by controlling their profiles) that the light of wavelength λ diffracted by a single groove has a sharp peak of maximum intensity at a selected angle θ ($\neq 0$).

Light can also be analyzed into its component wavelengths if the grating in Fig. 45–7 is replaced by a prism. In a *prism spectrograph* each wavelength

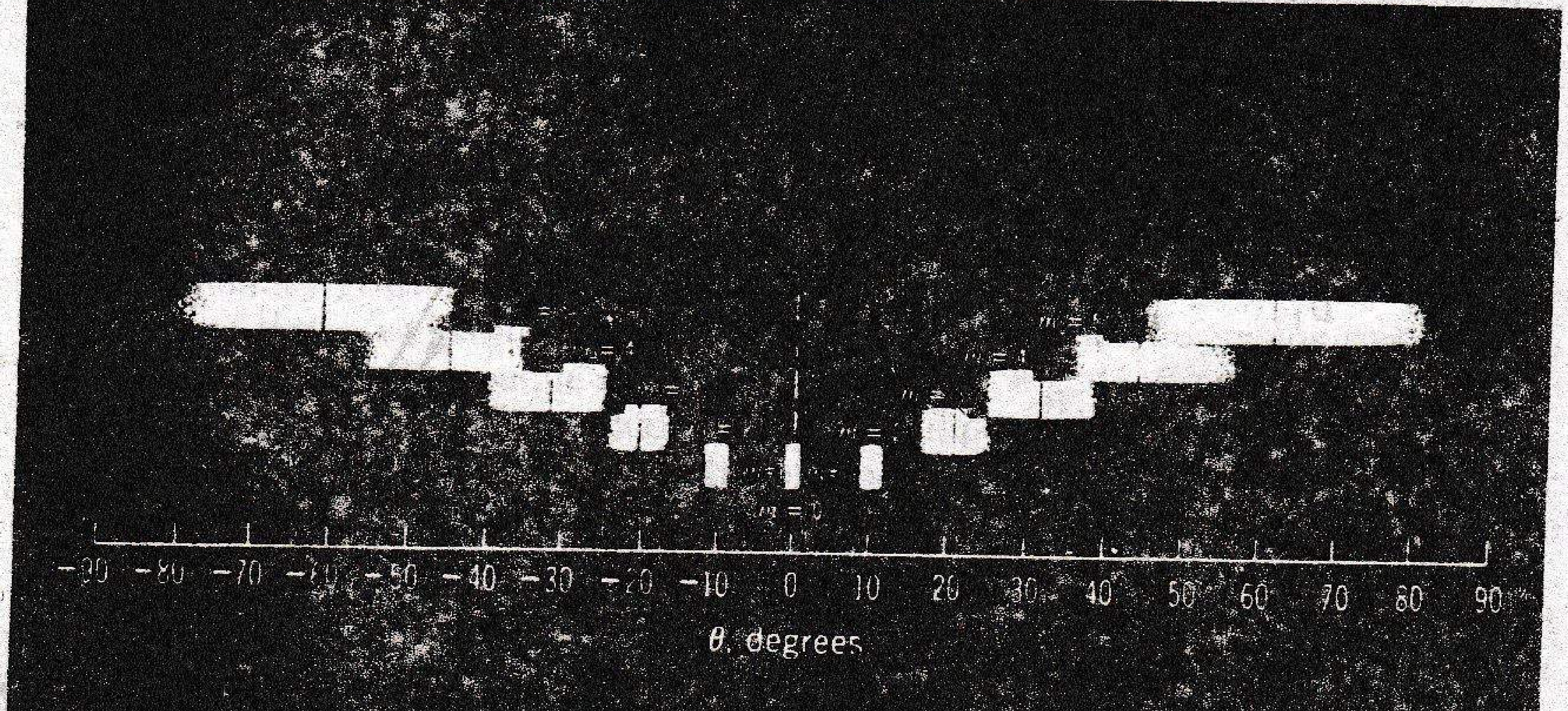

Fig. 45–8 Example 1. The spectrum of white light as viewed in a grating instrument like that of Fig. 45–7. The different orders, identified by the order number m, are shown separated vertically for clarity. As actually viewed, they would not be so displaced. The central line in each order corresponds to λ = 5500 A.

in the incident beam is deflected through a definite angle θ, determined by the index of refraction of the prism material for that wavelength. Curves such as Fig. 41-2, which gives the index of refraction of fused quartz as a function of wavelength, show that the shorter the wavelength, the larger the angle of deflection θ. Such curves vary from substance to substance and must be found by measurement. Prism instruments are not adequate for accurate *absolute* measurements of wavelength because the index of refraction of the prism material at the wavelength in question is usually not known precisely enough. Both prism and grating instruments make accurate *comparisons* of wa elength, using a suitable comparison spectrum such as that shown in Fig. 47-12, in which careful absolute determinations have been made of the wavelengths of the spectrum lines. The prism instrument has an advantage over a grating instrument (unblazed) in that its light energy is concentrated into a single spectrum so that brighter lines may be produced.

▶ **Example 1.** A grating with 8000 rulings/in. is illuminated with white light at perpendicular incidence. Describe the diffraction pattern. Assume that the wavelength of the light extends from 4000 to 7000 A.

The grating spacing d is 2.54 cm/8000, or 31,700 A. The central or zero-order maximum corresponds to $m = 0$ in Eq. 45-1. All wavelengths present in the incident light are superimposed at $\theta = 0$, as Fig. 45-8 shows.

The first-order diffraction pattern corresponds to $m = 1$ in Eq. 45-1. The 4000-A line occurs at an angle given by

$$\theta = \sin^{-1}\frac{m\lambda}{d} = \sin^{-1}\frac{(1)(4000\text{ A})}{31{,}700\text{ A}} = \sin^{-1} 0.126 = 7.3°.$$

In the same way the angle for the 7000-A line is found to be 12.8°, and the entire pattern of Fig. 45-8 can be calculated. Note that the *first-order spectrum* ($m = 1$) is isolated but that the second-, third-, and fourth-order spectra overlap.

Example 2. A diffraction grating has 10^4 rulings uniformly spaced over 1 in. It is illuminated at normal incidence by yellow light from a sodium vapor lamp. This light contains two closely spaced lines (the well-known *sodium doublet*) of wavelengths 5890.0 and 5895.9 A. (*a*) At what angle will the first-order maximum occur for the first of these wavelengths?

The grating spacing d is 10^{-4} in., or 25,400 A. The first-order maximum corresponds to $m = 1$ in Eq. 45-1. We thus have

$$\theta = \sin^{-1}\frac{m\lambda}{d} = \sin^{-1}\frac{(1)(5890\text{ A})}{25{,}400\text{ A}} = \sin^{-1} 0.232 = 13.3°.$$

(*b*) What is the angular separation between the first-order maxima for these lines?

The straightforward way to find this separation is to repeat this calculation for $\lambda = 5895.9$ A and to subtract the two angles. A difficulty, which can best be appreciated by carrying out the calculation, is that we must carry a large number of significant figures to obtain a meaningful value for the difference between the angles To calculate the difference in angular positions *directly*, let us write down Eq. 45-1 solved for $\sin\theta$, and differentiate it, treating θ and λ as variables:

$$\sin\theta = \frac{m\lambda}{d}$$

$$\cos\theta\, d\theta = \frac{m}{d}\, d\lambda.$$

If the wavelengths are close enough together, as in this case, $d\lambda$ can be replaced by $\Delta\lambda$, the actual wavelength difference; $d\theta$ then becomes $\Delta\theta$, the quantity we seek. This gives

$$\Delta\theta = \frac{m\,\Delta\lambda}{d\cos\theta} = \frac{(1)(5.9\text{ A})}{(25{,}400\text{ A})(\cos 13.3°)} = 2.4 \times 10^{-4}\text{ radian} = 0.014°.$$

Note that although the wavelengths involve five significant figures our calculation, done this way, involves only two or three, with consequent reduction in numerical manipulation. ◀

The quantity $d\theta/d\lambda$, called the *dispersion* D of a grating, is a measure of the angular separation produced between two incident monochromatic waves whose wavelengths differ by a small wavelength interval. From this example we see that

$$D = \frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta}. \qquad (45\text{-}4)$$

45-4 Resolving Power of a Grating

To distinguish light waves whose wavelengths are close together, the principal maxima of these wavelengths formed by the grating should be as narrow as possible. Expressed otherwise, the grating should have a high *resolving power* R, defined from

$$R = \frac{\lambda}{\Delta\lambda}. \qquad (45\text{-}5)$$

Here λ is the mean wavelength of two spectrum lines that can barely be recognized as separate and $\Delta\lambda$ is the wavelength difference between them. The smaller $\Delta\lambda$ is, the closer the lines can be and still be resolved; hence the greater the resolving power R of the grating. It is to achieve a high resolving power that gratings with many rulings are constructed.

The resolving power of a grating is usually determined by the same consideration (that is, the Rayleigh criterion) that we used in Section 44-5 to determine the resolving power of a lens. If two principal maxima are to be barely resolved, they must, according to this criterion, have an angular separation $\Delta\theta$ such that the maximum of one line coincides with the first minimum of the other; see Fig. 44-11. If we apply this criterion, we can show that

$$R = Nm, \qquad (45\text{-}6)$$

where N is the total number of rulings in the grating and m is the order. As expected, the resolving power is zero for the central principal maximum ($m = 0$), all wavelengths being undeflected in this order.

Let us derive Eq. 45-6. The angular separation between two principal maxima whose wavelengths differ by $\Delta\lambda$ is found from Eq. 45-4, which we recast as

$$\Delta\theta = \frac{m\,\Delta\lambda}{d\cos\theta}. \qquad (45\text{-}4)$$

The Rayleigh criterion (Section 44-5) requires that this be equal to the angular separation between a principal maximum and its adjacent minimum. This is given from Eq. 45–3, dropping the subscript m in $\cos\theta_m$, as

$$\Delta\theta_m = \frac{\lambda}{dN\cos\theta}. \tag{45-3}$$

Equating Eqs. 45–4 and 45–3 leads to

$$R\ (= \lambda/\Delta\lambda) = Nm,$$

which is the desired relation.

▶ **Example 3.** In Example 2 how many rulings must a grating have if it is barely to resolve the sodium doublet in the third order?

From Eq. 45–5 the required resolving power is

$$R = \frac{\lambda}{\Delta\lambda} = \frac{5890\text{ A}}{(5895.9 - 5890.0)\text{A}} = 1000.$$

From Eq. 45–6 the number of rulings needed is

$$N = \frac{R}{m} = \frac{1000}{3} = 330.$$

This is a modest requirement. ◀

The resolving power of a grating must not be confused with its dispersion. Table 45–1 shows the characteristics of three gratings, each illuminated with light of $\lambda = 5890$ A, the diffracted light being viewed in the first order ($m = 1$ in Eq. 45–1).

Table 45–1

SOME CHARACTERISTICS OF THREE GRATINGS

($\lambda = 5890$ A, $m = 1$)

Grating	N	d, A	θ	R	D 10^{-3} degrees/A
A	10,000	25,400	13.3°	10,000	2.32
B	20,000	25,400	13.3°	20,000	2.32
C	10,000	13,700	25.5°	10,000	4.64

The student should verify that the values of D and R given in the table can be calculated from Eqs. 45–4 and 45–6, respectively.

For the conditions of use noted in Table 45–1, gratings A and B have the same *dispersion* and A and C have the same *resolving power*. Figure 45–9 shows the intensity patterns that would be produced by these gratings for two incident waves of wavelengths λ_1 and λ_2, in the vicinity of $\lambda = 5890$ A. Grating B, which has high resolving power, has narrow intensity maxima and is inherently capable of distinguishing lines that are much closer together in wavelength than those of Fig. 45–9. Grating C, which has high dispersion, produces twice the angular separation between rays λ_1 and λ_2 that grating B does.

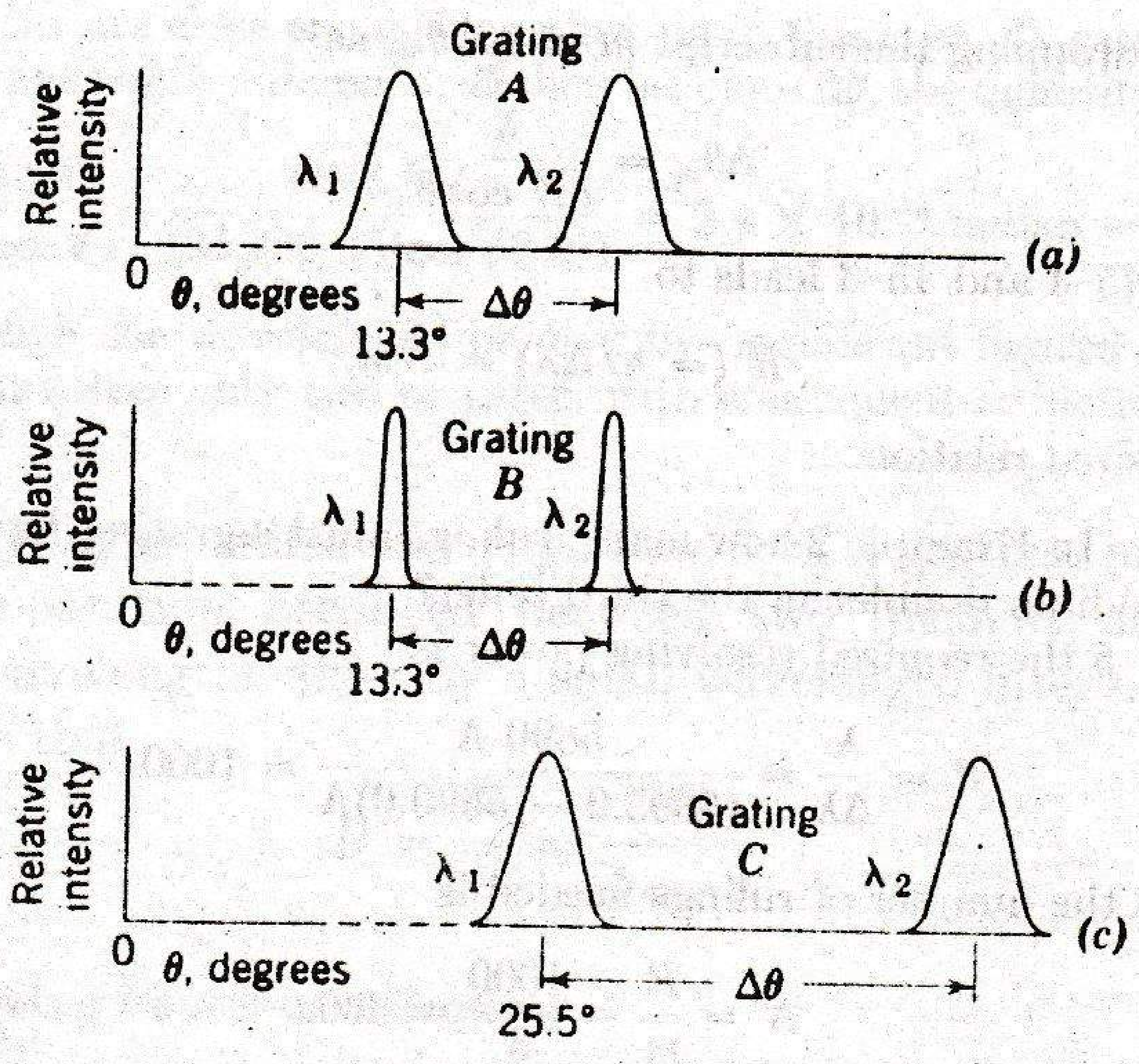

Fig. 45–9 The intensity patterns for light of wavelengths λ_1 and λ_2 near 5890 A, incident on the gratings of Table 45–1. Grating *B* has the highest resolving power and grating *C* the highest dispersion.

▶ **Example 4.** The grating of Example 1 has 8000 lines illuminated by light from a mercury vapor discharge. (*a*) What is the expected dispersion, in the third order, in the vicinity of the intense green line (λ = 5460 A)? Noting that d = 31,700 A, we have, from Eq. 45–1,

$$\theta = \sin^{-1}\frac{m\lambda}{d} = \sin^{-1}\frac{(3)(5460\text{ A})}{31{,}700\text{ A}} = \sin^{-1} 0.517 = 31.1°.$$

From Eq. 45–4 we have

$$D = \frac{m}{d\cos\theta} = \frac{3}{(31{,}700\text{ A})(\cos 31.1°)} = 1.1 \times 10^{-4}\text{ radian/A} = 6.3 \times 10^{-3}\text{ deg/A}.$$

(*b*) What is the expected resolving power of this grating in the fifth order? Equation 45–6 gives

$$R = Nm = (8000)(5) = 40{,}000.$$

Thus near λ = 5460 A a wavelength difference $\Delta\lambda$ given by Eq. 45–5, or

$$\Delta\lambda = \frac{\lambda}{R} = \frac{5460\text{ A}}{40{,}000} = 0.14\text{ A},$$

can be distinguished. ◀

45–5 X-ray Diffraction

Figure 45–10 shows how X-rays are produced when electrons from a heated filament F are accel rated by a potential difference V and strike a metal target T. X-rays are electromagnetic radiation with wavelengths of the order of 1 A. This value is to be compared to 5500 A for the center of the visible spectrum. For such small wavelengths a standard optical diffraction grating, as normally employed, cannot be used. For λ = 1 A and d = 30,000 A,

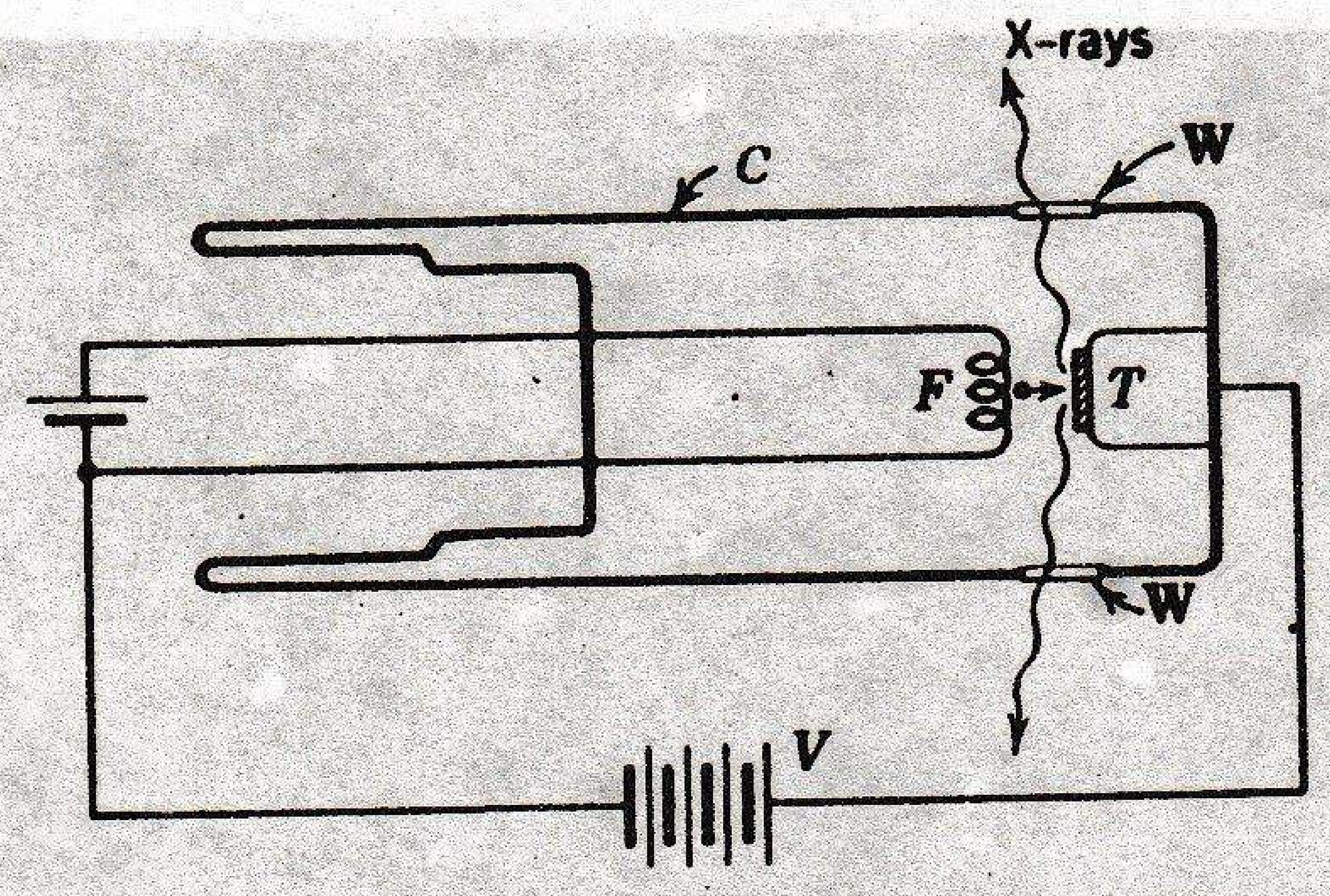

Fig. 45-10 X-rays are generated when electrons from heated filament *F*, accelerated by potential difference *V*, are brought to rest on striking metallic target *T*. *W* is a "window" —transparent to X-rays—in the evacuated metal container *C*.

for example, Eq. 45-1 shows that the first-order maximum occurs at

$$\theta = \sin^{-1}\frac{m\lambda}{d} = \sin^{-1}\frac{(1)(1\text{ A})}{3 \times 10^4\text{ A}} = \sin^{-1} 0.33 \times 10^{-4} = 0.002°.$$

This is too close to the central maximum to be practical. A grating with $d \cong \lambda$ is desirable, but, since X-ray wavelengths are about equal to atomic diameters, such gratings cannot be constructed mechanically.

In 1912 it occurred to the German physicist Max von Laue that a crystalline solid, consisting as it does of a regular array of atoms, might form a natural three-dimensional "diffraction grating" for X-rays. Figure 45-11 shows that if a collimated beam of X-rays, continuously distributed in wavelength, is allowed to fall on a crystal, such as sodium chloride, intense beams corresponding to constructive interference from the many diffracting centers

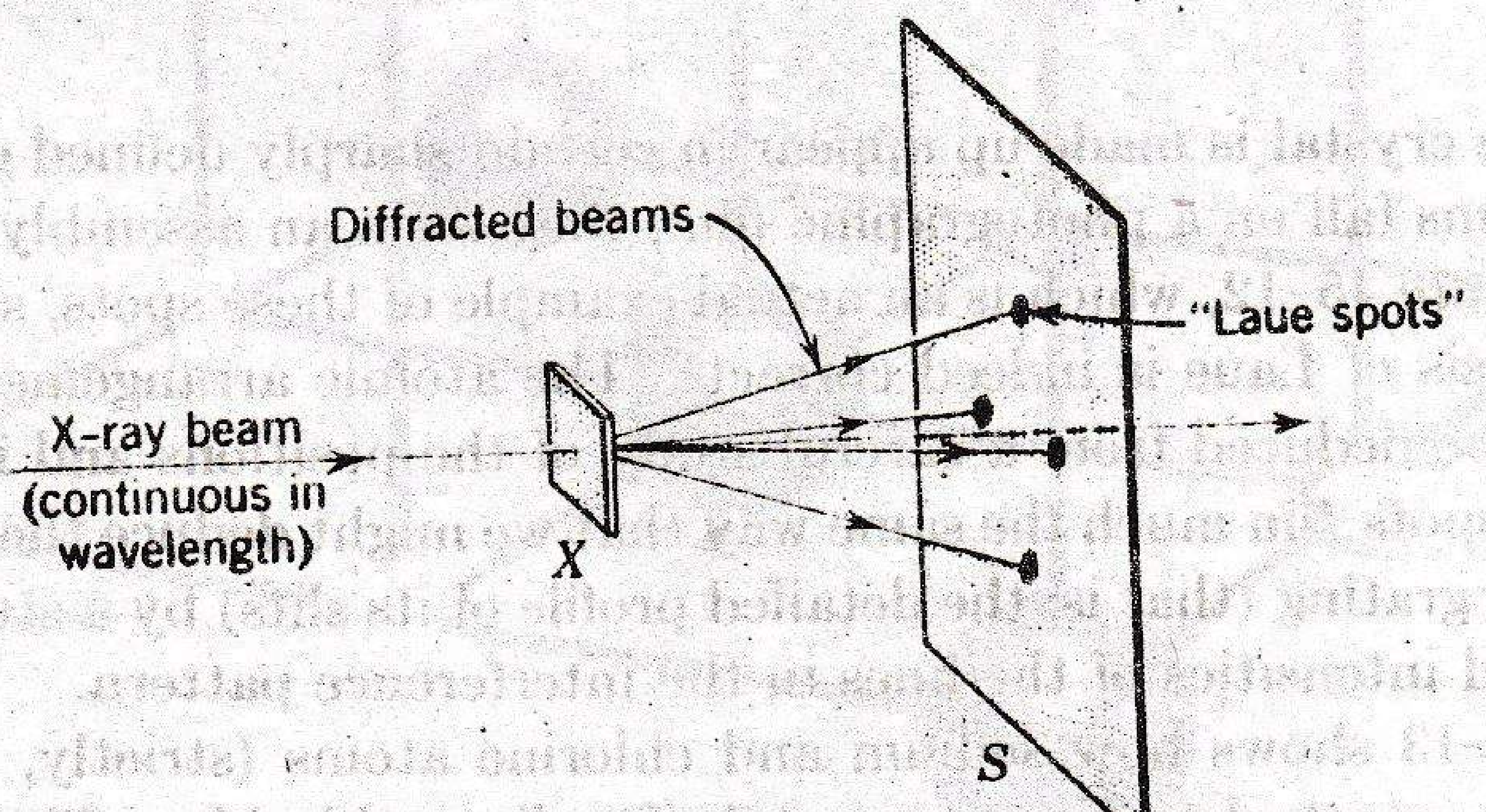

Fig. 45-11 A nonmonochromatic beam of X-rays falls on a crystal *X*, which may be NaCl. Strong diffracted beams appear in certain directions, forming a so-called Laue pattern on a photographic film *S*.

Fig. 45–12 Laue X-ray diffraction pattern from sodium chloride. A crystal of ordinary table salt was used in making this plate. (Courtesy of W. Arrington and J. L. Katz, X-ray Laboratory, Rensselaer Polytechnic Institute.)

of which the crystal is made up appear in certain sharply defined directions. If these beams fall on a photographic film, they form an assembly of "Laue spots." Figure 45–12, which is an actual example of these spots, shows that the hypothesis of Laue is indeed correct. The atomic arrangements in the crystal can be deduced from a careful study of the positions and intensities of the Laue spots * in much the same way that we might deduce the structure of an optical grating (that is, the detailed profile of its slits) by a study of the positions and intensities of the lines in the interference pattern.

Figure 45–13 shows how sodium and chlorine atoms (strictly, Na^+ and Cl^- ions) are stacked to form a crystal of sodium chloride. This pattern, which has *cubic* symmetry, is one of the many atomic arrangements exhibited

* Other experimental arrangements have supplanted the Laue technique to a considerable extent today; the principle remains unchanged, however.

by solids. The model represents the *unit cell* for sodium chloride. This is the smallest unit from which the crystal may be built up by repetition in three dimensions. The student should verify that no smaller assembly of atoms possesses this property. For sodium chloride the length of the cube edge of the unit cell is 5.62737 A.

Each unit cell in sodium chloride has four sodium ions and four chlorine ions associated with it. In Fig. 45-13 the sodium ion in the center belongs entirely to the cell shown. Each of the other twelve sodium ions shown is shared with three adjacent unit cells so that each contributes one-fourth of an ion to the cell under consideration. The total number of sodium ions is then $1 + \frac{1}{4}(12) = 4$. By similar reasoning the student can show that although there are fourteen chlorine ions in Fig. 45-13 only four are associated with the unit cell shown.

The unit cell is the fundamental repetitive diffracting unit in the crystal, corresponding to the slit (and its adjacent opaque strip) in the optical dif-

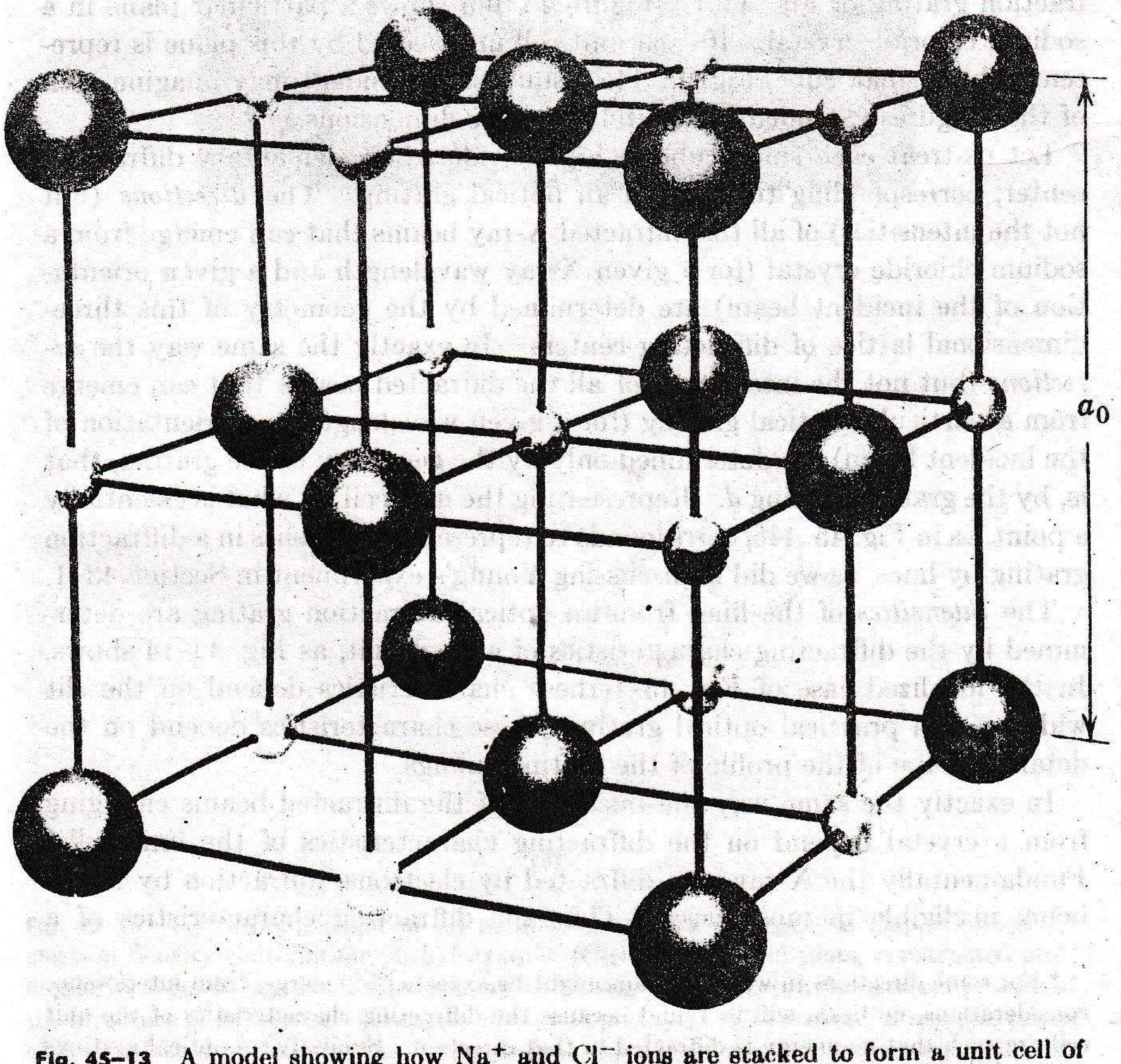

Fig. 45-13 A model showing how Na^+ and Cl^- ions are stacked to form a unit cell of NaCl. The small spheres represent sodium ions, the large ones chlorine. The edge a_0 of the (cubical) unit cell is 5.62737 A.

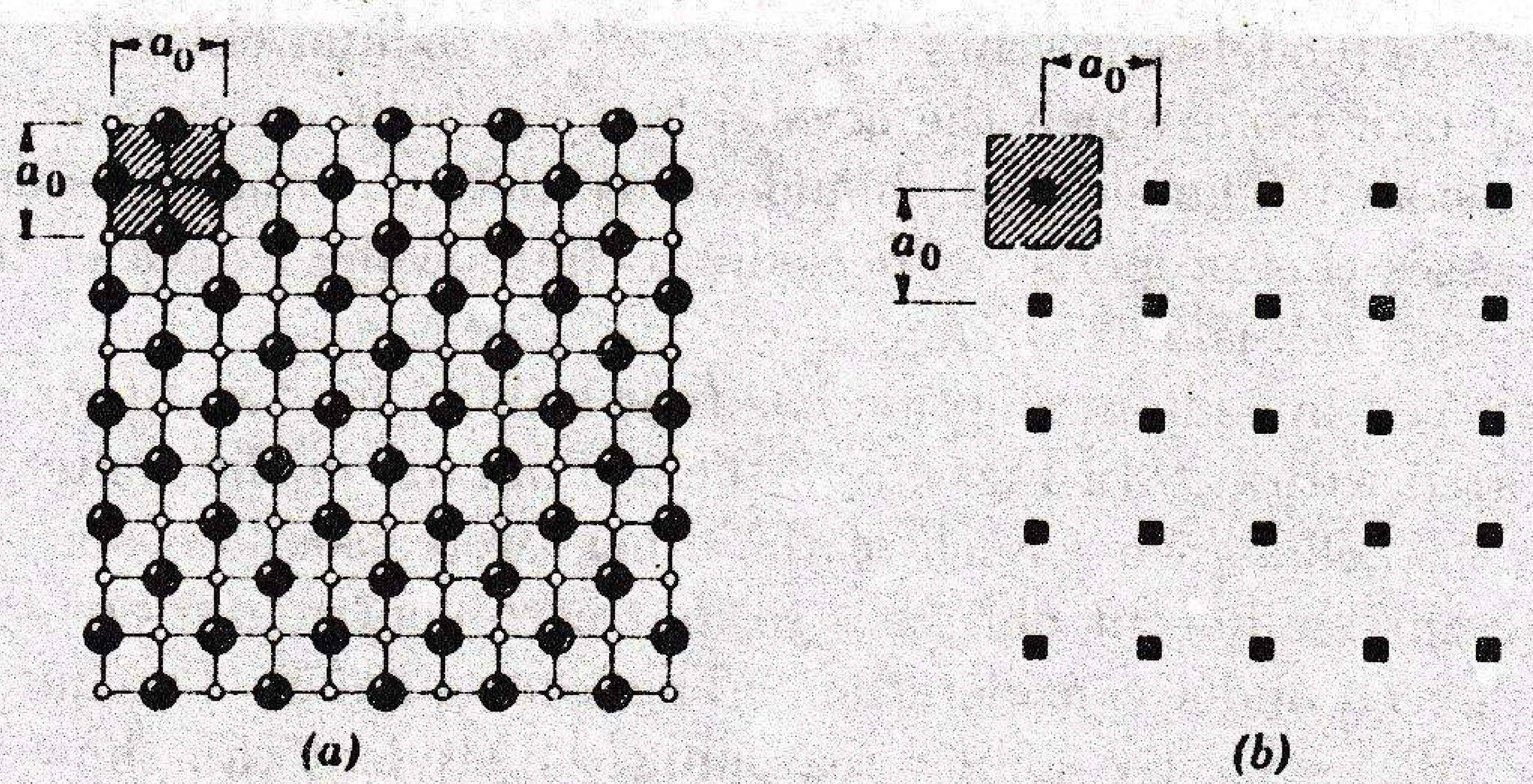

Fig. 45–14 (*a*) A section through a crystal of sodium chloride, showing the sodium and chlorine ions. (*b*) The corresponding unit cells in this section, each cell being represented by a small black square.

fraction grating of Fig. 45–1. Figure 45–14*a* shows a particular plane in a sodium chloride crystal. If each unit cell intersected by this plane is represented by a small cube, Fig. 45–14*b* results. The student may imagine each of these figures extended indefinitely in three dimensions.

Let us treat each small cube in Fig. 45–14*b* as an elementary diffracting center, corresponding to a slit in an optical grating. The *directions* (but not the intensities) of all the diffracted X-ray beams that can emerge from a sodium chloride crystal (for a given X-ray wavelength and a given orientation of the incident beam) are determined by the geometry of this three-dimensional lattice of diffracting centers. In exactly the same way the *directions* (but not the intensities) of all the diffracted beams that can emerge from a particular optical grating (for a given wavelength and orientation of the incident beam) are determined only by the geometry of the grating, that is, by the grating spacing d. Representing the unit cell by what is essentially a point, as in Fig. 45–14*b*, corresponds to representing the slits in a diffraction grating by lines, as we did in discussing Young's experiment in Section 43–1.

The *intensities* of the lines from an optical diffraction grating are determined by the diffracting characteristics of a single slit, as Fig. 44–14 shows. In the idealized case of Fig. 45–1 these characteristics depend on the slit width a. In practical optical gratings these characteristics depend on the detailed shape of the profile of the grating rulings.

In exactly the same way the *intensities* of the diffracted beams emerging from a crystal depend on the diffracting characteristics of the unit cell.* Fundamentally the X-rays are diffracted by electrons, diffraction by nuclei being negligible in most cases. Thus the diffracting characteristics of a

* For some directions in which a beam might be expected to emerge, from interference considerations, no beam will be found because the diffracting characteristics of the unit cell are such that no energy is diffracted in that direction. Similarly, in optical gratings some lines, permitted by interference considerations, may not appear if their predicted positions coincide with a null in the single-slit diffraction pattern (see Fig. 44–12).

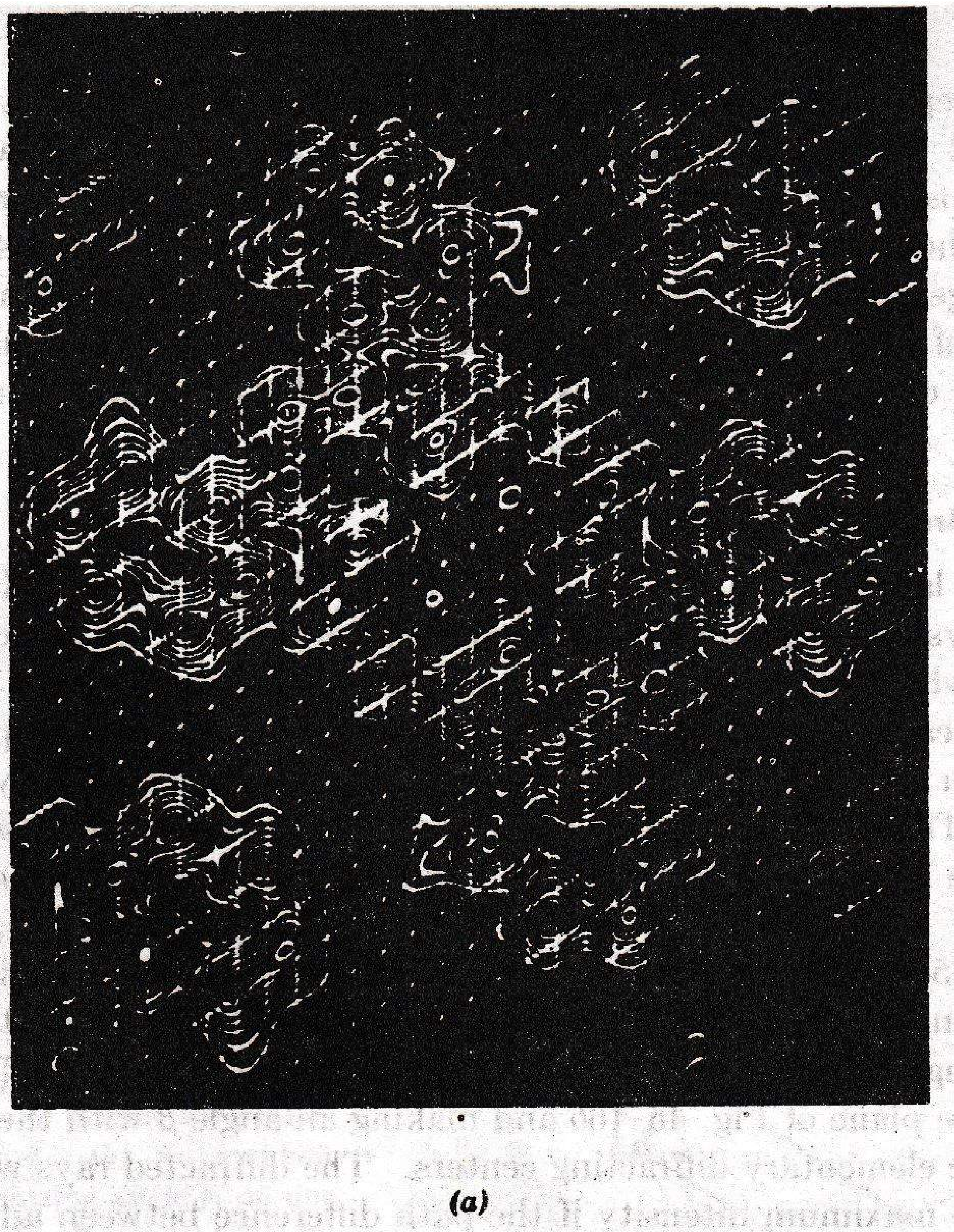

Fig. 45–15 (*a*) A photograph of an oscilloscope screen arranged to display projected electron density contours for phthalocyanine ($C_{32}H_{18}N_8$). Such plots, constructed electronically from X-ray diffraction data by an analog computer, provide a vivid picture of the structure of molecules. (Courtesy of Ray Pepinsky.) (*b*) A structural representation of the molecule phthalocyanine. The student should make a detailed comparison with (*a*), locating the various atoms identified in (*b*). Note that hydrogen atoms, which contain only a single electron, are not prominent in (*a*).

unit cell depend on how the electrons are distributed throughout the volume of the cell. By studying the *directions* of diffracted X-ray beams, we can learn the basic symmetry of the crystal. By also studying the *intensities* we can learn how electrons are distributed in the unit cell. Figure 45–15 shows the average density of electrons projected onto a particular plane through a unit cell of a crystal of phthalocyanine. This remarkable figure suggests something of the full power of X-ray methods for studying the structure of solids.

45–6 Bragg's Law

Bragg's law predicts the conditions under which diffracted X-ray beams from a crystal are possible. In deriving it, we ignore the structure of the unit cell, which is related only to the intensities of these beams. The dashed sloping lines in Fig. 45–16*a* represent the intersection with the plane of the figure of an arbitrary set of planes passing through the elementary diffracting centers. The perpendicular distance between adjacent planes is d. Many other such families of planes, with different *interplanar spacings*, can be defined.

Figure 45–16*b* shows a plane wave that lies in the plane of the figure falling on one member of the family of planes defined in Fig. 45–16*a*, the incident rays making an angle θ with the plane.* Consider a family of diffracted rays lying in the plane of Fig. 45–16*b* and making an angle β with the plane containing the elementary diffracting centers. The diffracted rays will combine to produce maximum intensity if the path difference between adjacent rays is an integral number of wavelengths or

$$ae - bd = h(\cos\beta - \cos\theta) = l\lambda \qquad l = 0, 1, 2, \ldots \qquad (45\text{–}7)$$

For $l = 0$ this leads to $\beta = \theta$,

and the plane of atoms acts like a mirror for the incident wave, no matter what the value of θ.

For other values of l, β does not equal θ, but the diffracted beam can always be regarded as being "reflected" from a *different* set of planes than that shown in Fig. 45–16*a* with a different interplanar spacing d. Since we wish to describe each diffracted beam as a "reflection" from a particular set of planes and since we are dealing in the present argument only with the particular set of planes shown in Fig. 45–16*a* we ignore all values of l other than $l = 0$ in Eq. 45–7. It can also be shown that a plane of diffracting centers acts like a mirror (that is, $\beta = \theta$) whether or not the incident wave lies in the plane of Fig. 45–16*b*.

Figure 45–16*c* shows an incident wave striking the *family* of planes, a single member of which was considered in Fig. 45–16*b*. For a single plane, mirror-like "reflection" occurs for *any* value of θ, as we have seen. To have a constructive interference in the beam diffracted from the entire family of planes

* In X-ray diffraction it is customary to specify the direction of a wave by giving the angle between the ray and the plane (the *glancing angle*) rather than the angle between the ray and the normal.

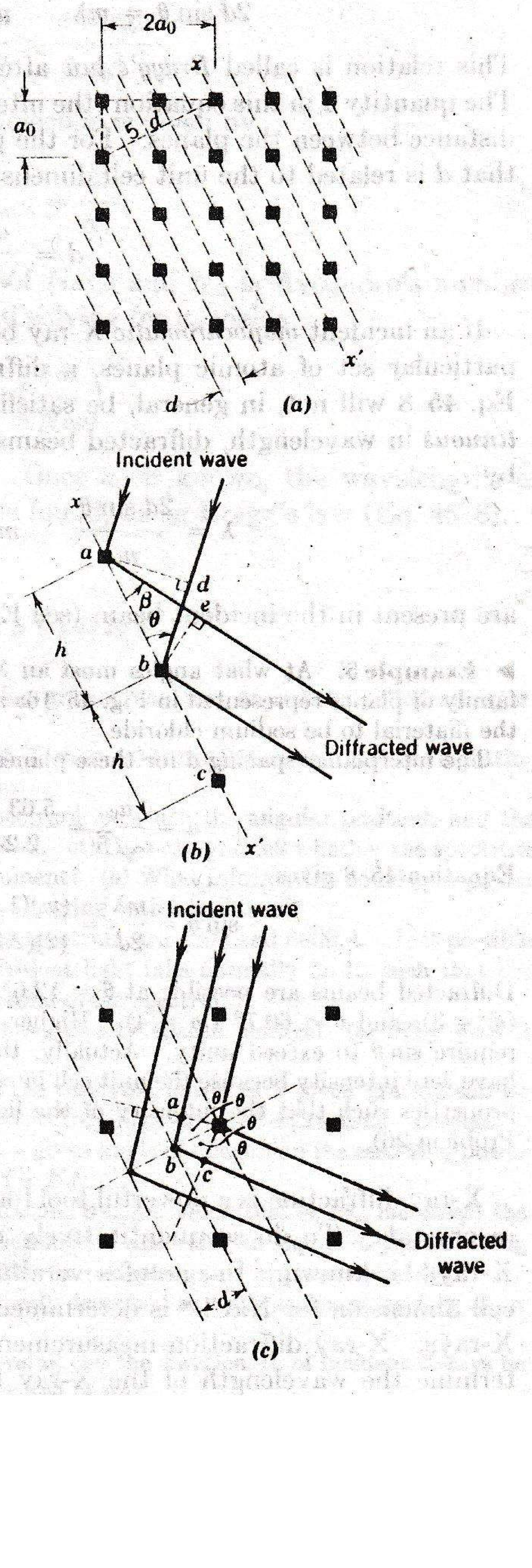

Fig. 45-16 (*a*) A section through the NaCl unit cell lattice of Fig 45-14*b*. The dashed sloping lines represent an arbitrary family of planes, with interplanar spacing *d*. (*b*) An incident wave falls, at grazing angle *θ*, on one of the planes, *xx'*, shown in (*a*). (*c*) An incident wave falls on the entire family of planes shown in (*a*). A strong diffracted wave is formed.

in the direction θ, the rays from the separate planes must reinforce each other. This means that the path difference for rays from adjacent planes (*abc* in Fig. 45-16*c*) must be an integral number of wavelengths or

$$2d \sin \theta = m\lambda \qquad m = 1, 2, 3, \ldots. \tag{45-8}$$

This relation is called *Bragg's law* after W. L. Bragg who first derived it. The quantity d in this equation (the interplanar spacing) is the perpendicular distance between the planes. For the planes of Fig. 45-16*a* analysis shows that d is related to the unit cell dimension a_0 by

$$d = \frac{a_0}{\sqrt{5}}. \tag{45-9}$$

If an incident *monochromatic* X-ray beam falls at an *arbitrary* angle θ on a particular set of atomic planes, a diffracted beam will *not* result because Eq. 45-8 will not, in general, be satisfied. If the incident X-rays are *continuous* in wavelength, diffracted beams will result when wavelengths given by

$$\lambda = \frac{2d \sin \theta}{m} \qquad m = 1, 2, 3, \ldots$$

are present in the incident beam (see Eq. 45-8).

▶ **Example 5.** At what angles must an X-ray beam with $\lambda = 1.10$ A fall on the family of planes represented in Fig. 45-16*c* if a diffracted beam is to exist? Assume the material to be sodium chloride.

The interplanar spacing d for these planes is given by Eq. 45-9 or

$$d = \frac{a_0}{\sqrt{5}} = \frac{5.63 \text{ A}}{2.24} = 2.52 \text{ A}.$$

Equation 45-8 gives

$$\sin \theta = \frac{m\lambda}{2d} = \frac{(m)(1.10 \text{ A})}{(2)(2.52 \text{ A})} = 0.218m.$$

Diffracted beams are possible at $\theta = 12.6°$ ($m = 1$), $\theta = 25.9°$ ($m = 2$), $\theta = 40.9°$ ($m = 3$), and $\theta = 60.7°$ ($m = 4$). Higher-order beams cannot exist because they require $\sin \theta$ to exceed unity. Actually, the odd-order beams ($m = 1, 3$) prove to have zero intensity because the unit cell in cubic crystals such as NaCl has diffracting properties such that the intensity of the light scattered in these orders is zero (see Problem 26). ◀

X-ray diffraction is a powerful tool for studying the arrangements of atoms in crystals. To do so quantitatively requires that the wavelength of the X-rays be known. In one of several approaches to this problem the unit cell dimension for NaCl * is determined by a method that does not involve X-rays. X-ray diffraction measurements on NaCl can then be used to determine the wavelength of the X-ray beam which in turn can be used to determine the structures of solids other than NaCl.

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* In practice, calcite ($CaCO_3$) proves to be more useful as a standard crystal for a number of technical reasons.

If ρ is the measured density of NaCl, we have, for the unit cell of Fig. 45-13,* recalling that each unit cell contains four NaCl "molecules,"

$$\rho = \frac{m}{V} = \frac{4m_{NaCl}}{a_0^3}.$$

Here m_{NaCl}, the mass of a NaCl molecule, is given by

$$m_{NaCl} = \frac{M}{N_0},$$

where M is the molecular weight of NaCl and N_0 is Avogadro's number. Combining these two equations and solving for a_0 yields

$$a_0 = \left(\frac{4M}{N_0\rho}\right)^{1/3}$$

which permits us to calculate a_0. Once a_0 is known, the wavelengths of monochromatic X-ray beams can be found, using Bragg's law (Eq. 45-8).

QUESTIONS

1. Discuss this statement: "A diffraction grating can just as well be called an interference grating."

2. For the simple spectroscope of Fig. 45-7, show (*a*) that θ increases with λ for a grating and (*b*) that θ decreases with λ for a prism.

3. You are given a photograph of a spectrum on which the angular positions and the wavelengths of the spectrum lines are marked. (*a*) How can you tell whether the spectrum was taken with a prism or a grating instrument? (*b*) What information could you gather about either the prism or the grating from studying such a spectrum?

4. Assume that the limits of the visible spectrum are 4300 and 6800 A. Is it possible to design a grating, assuming that the incident light falls normally on it, such that the first-order spectrum *barely overlaps* the second-order spectrum?

5. (*a*) Why does a diffraction grating have closely spaced rulings? (*b*) Why does it have a large number of rulings?

6. The relation $R = Nm$ suggests that the resolving power of a given grating can be made as large as desired by choosing an arbitrarily high order of diffraction. Discuss.

7. Show that at a given wavelength and a given angle of diffraction the resolving power of a grating depends only on its width W ($= Nd$).

8. According to Eq. 45-3 the principal maxima become wider (that is, $\Delta\theta_m$ increases) the higher the order m (that is, the larger θ_m becomes). According to Eq. 45-6 the resolving power becomes greater the higher the order m. Explain this apparent paradox.

9. Is the pattern of Fig. 45-12 more properly described as a diffraction pattern or as an interference pattern?

10. For a given family of planes in a crystal, can the wavelength of incident X-rays be (*a*) too large or (*b*) too small to form a diffracted beam?

11. If a parallel beam of X-rays of wavelength λ is allowed to fall on a randomly oriented crystal of any material, generally no intense diffracted beams will occur. Such beams

* This relation cannot be written down unless it is known that the structure of NaCl is cubic. This can be determined, however, by inspection of the symmetry of the spots in Fig. 45-12; the wavelength of the X-rays need not be known.

appear if (*a*) the X-ray beam consists of a continuous distribution of wavelengths rather than a single wavelength or (*b*) the specimen is not a single crystal but a finely divided powder. Explain.

12. Why cannot a simple cube of edge $a_0/2$ in Fig. 45-13 be used as a unit cell for sodium chloride?

13. How would you *measure* (*a*) the dispersion D and (*b*) the resolving power R for either a prism or a grating spectrograph.

PROBLEMS

1. Given a grating with 4000 lines/cm, how many orders of entire visible spectrum (4000–7000 A) can be produced?

2. Derive this expression for the intensity pattern for a three-slit "grating":

$$I_\theta = \tfrac{1}{9}I_m(1 + 4\cos\phi + 4\cos^2\phi),$$

where

$$\phi = \frac{2\pi d \sin\theta}{\lambda}.$$

Assume that $a \ll \lambda$ and be guided by the derivation of the corresponding double-slit formula (Eq. 43-9).

3. (*a*) Using the result of Problem 2, show that the half-width of the fringes for a three-slit diffraction pattern, assuming θ small enough so that $\sin\theta \cong \theta$, is

$$\Delta\theta \cong \frac{\lambda}{3.2d}.$$

(*b*) Compare this with the expression derived for the two-slit pattern in Problem 10, Chapter 43. (*c*) Do these results support the conclusion that for a fixed slit spacing the interference maxima become sharper as the number of slits is increased?

4. Using the result of Problem 2, show that a three-slit "grating" has only one secondary maximum. Find its location and its relative intensity.

5. A grating designed for use in the infrared region of the electromagnetic spectrum is "blazed" to concentrate all its intensity in the first order ($m = 1$) for $\lambda = 80{,}000$ A. If visible light (4000 A $< \lambda <$ 7000 A) were allowed to fall on this grating, what visual appearance would the diffracted beams present?

6. The central intensity maximum formed by a grating, along with its subsidiary secondary maxima, can be viewed as the diffraction pattern of a single "slit" whose width is that of the entire grating. Treating the grating as a single wide slit, assuming that $m = 0$, and using the methods of Section 44-4, show that Eq. 45-2 can be derived.

7. A grating has 8000 rulings/in. For what wavelengths in the visible spectrum can fifth-order diffraction be observed?

8. A diffraction grating has 5000 rulings/in., and a strong diffracted beam is noted at $\theta = 30°$. (*a*) What are the possible wavelengths of the incident light? (*b*) How could you identify them in an actual case?

9. A diffraction grating 2.0 cm wide has 6000 rulings. At what angles will maximum-intensity beams occur if the incident radiation has a wavelength of 5890 A?

10. Assume that the limits of the visible spectrum are arbitrarily chosen as 4300 and 6800 A. Design a grating that will spread the first-order spectrum through an angular range of 20°.

11. A grating has 8000 rulings/in. and is illuminated at normal incidence by white light. A spectrum is formed on a screen 30 cm from the grating. If a 1.0-cm square hole is cut in the screen, its inner edge being 5.0 cm from the central maximum, what range of wavelengths passes through the hole?

12. Assume that light is incident on a grating at an angle ψ as shown in Fig. 45–17. Show that the condition for a diffraction maximum is

$$d(\sin \psi + \sin \theta) = m\lambda \qquad m = 0, 1, 2, \ldots$$

Only the special case $\psi = 0$ has been treated in this chapter (compare Eq. 45–1).

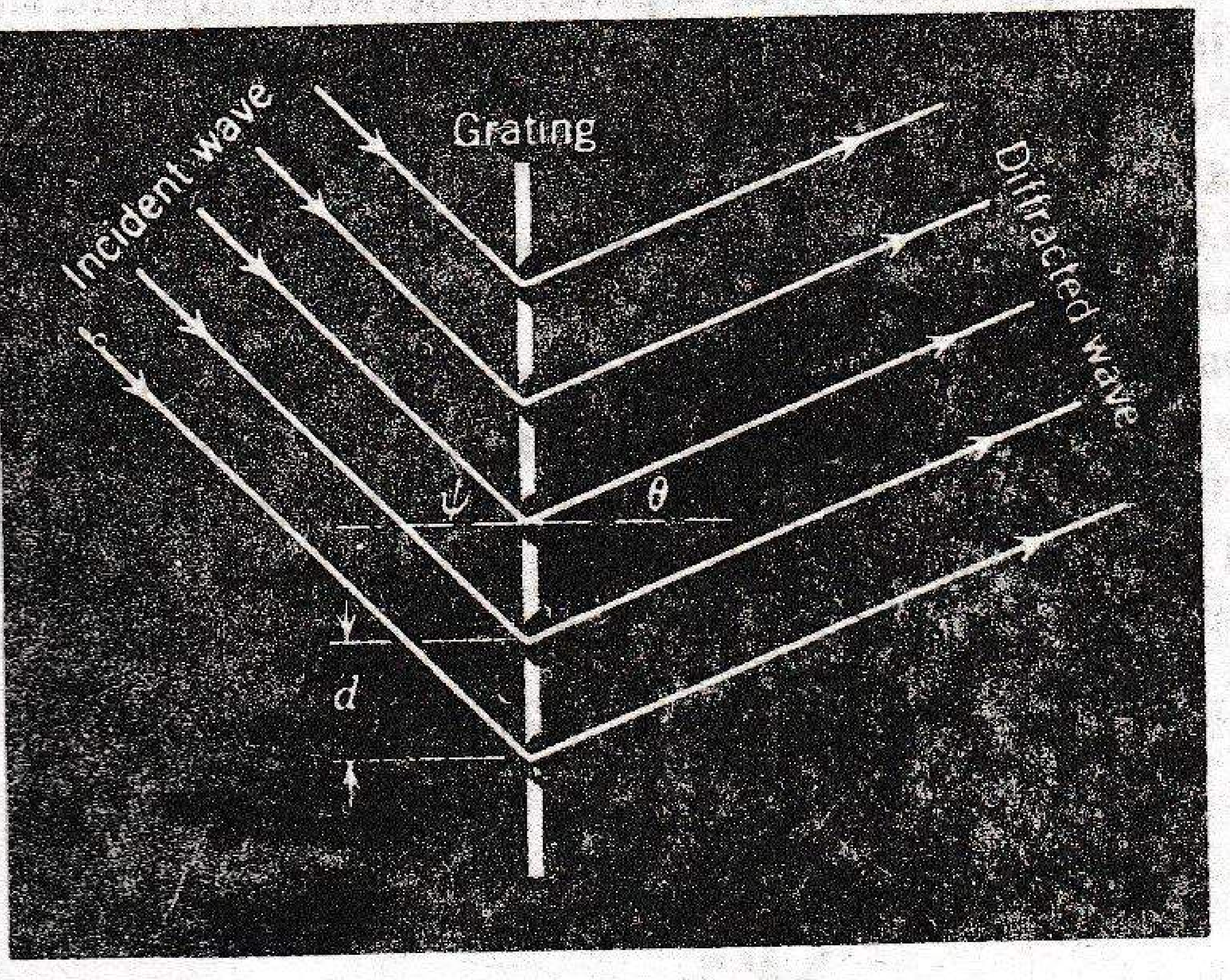

Fig. 45–17

13. Show that in a grating with alternately transparent and opaque strips of *equal* width all the even orders (except $m = 0$) are absent.

14. A transmission grating with $d = 1.50 \times 10^{-4}$ cm is illuminated at various angles of incidence by light of wavelength 6000 A. Plot as a function of angle of incidence (0 to 90°) the angular deviation of the first-order diffracted beam from the incident direction.

15. Derive Eq. 45–3, that is, the expression for $\Delta\theta_m$, the angular distance between a principal maximum of order m and either adjacent minimum.

16. A three-slit grating has separation d between adjacent slits. If the middle slit is covered up, will the half-width of the intensity maxima become broader or narrower? See Problem 3 and also Problem 10, Chapter 43.

17. A grating has 40,000 rulings spread over 3.0 in. (*a*) What is its expected dispersion D for sodium light ($\lambda = 5890$ A) in the first three orders? (*b*) What is its resolving power in these orders?

18. In a particular grating the sodium doublet (see Example 2) is viewed in third order at 80° to the normal and is barely resolved. Find (*a*) the grating spacing and (*b*) the total width of the rulings.

19. A source containing a mixture of hydrogen and deuterium atoms emits a red doublet at $\lambda = 6563$ A whose separation is 1.8 A. Find the minimum number of lines needed in a diffraction grating which can resolve these lines in the first order.

20. Show that the dispersion of a grating can be written as

$$D = \frac{\tan \theta}{\lambda}.$$

21. A grating has 6000 rulings/cm and is 6.0 cm wide. (*a*) What is the smallest wavelength interval that can be resolved in the third order at $\lambda = 5000$ A? (*b*) For this wavelength and this grating, can the resolution be improved? How?

22. Light containing a mixture of two wavelengths, 5000 A and 6000 A, is incident normally on a diffraction grating. It is desired (1) that the first and second principal maxima for each wavelength appear at $\theta \leq 30°$, (2) that the dispersion be as high as possible, *and* (3) that the third order for 6000 A be a missing order. (*a*) What is the separation

between adjacent slits? (b) What is the smallest possible individual slit width? (c) Name *all* orders for 6000 A that actually appear on the screen with the values chosen in (a) and (b).

23. Light of wavelength 6000 A is incident normally on a diffraction grating. Two *adjacent* principal maxima occur at $\sin\theta = 0.2$ and $\sin\theta = 0.3$, respectively. The fourth order is a missing order. (a) What is the separation between adjacent slits? (b) What is the smallest possible individual slit width? (c) Name *all* orders actually appearing on the screen with the values chosen in (a) and (b).

24. An optical grating with a spacing $d = 15{,}000$ A is used to analyze soft X-rays of wavelength $\lambda = 5.0$ A. The angle of incidence θ is $90° - \gamma$, where γ is a *small* angle. The first order maximum is found at an angle $\theta = 90° - 2\beta$. Find the value of β.

25. Monochromatic X-rays ($\lambda = 1.20$ A) fall on a crystal of sodium chloride, making an angle of 45° with a reference line as shown in Fig. 45-18. Through what angles must the crystal be turned to give a diffracted beam associated with the planes shown? Assume that the crystal is turned about an axis that is perpendicular to the plane of the page. Ignore the possibility (see Problem 26) that some of these beams may be of zero intensity.

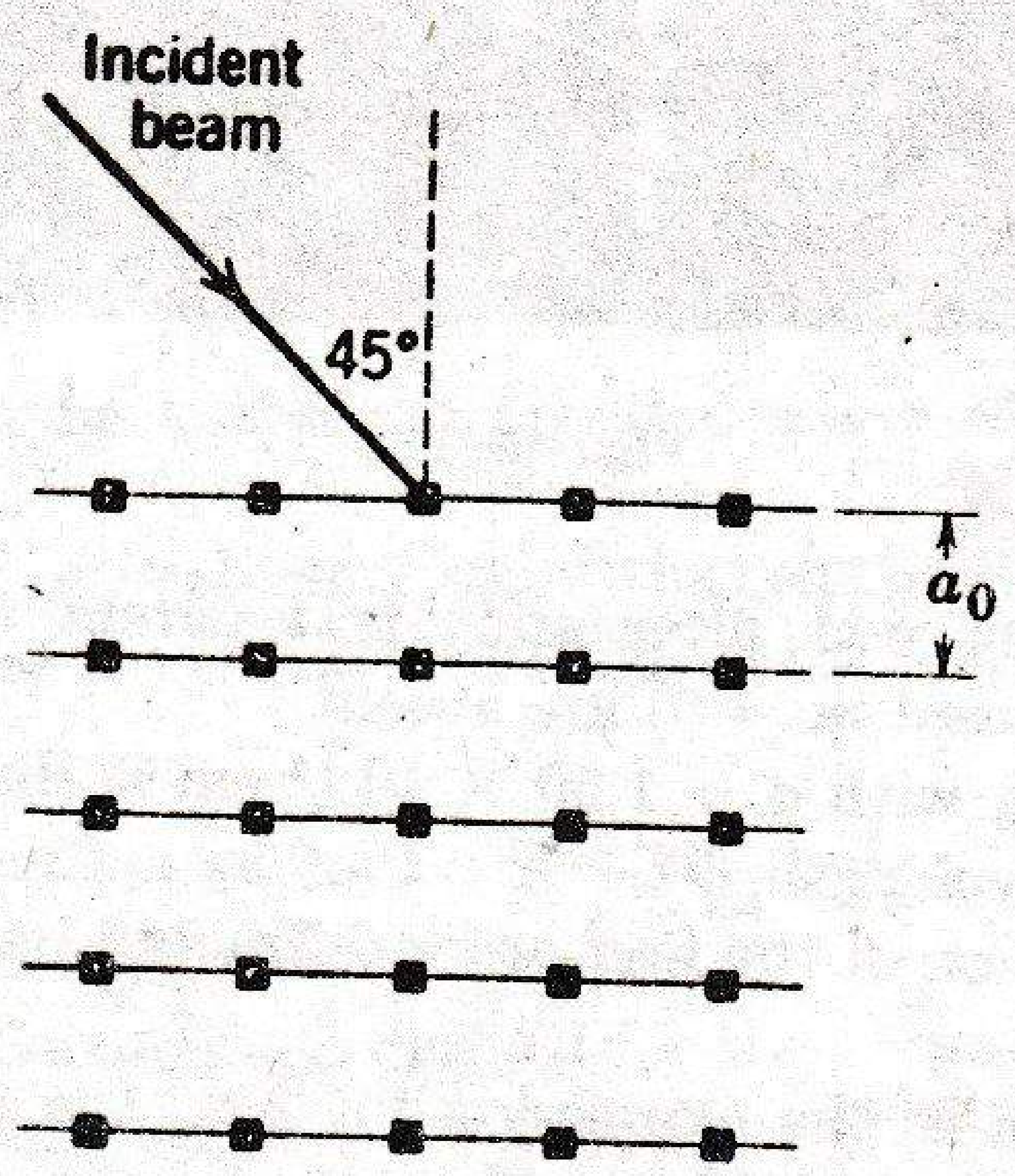

Fig. 45-18

26. *Missing orders in X-ray diffraction.* In Problem 25 the $m = 1$ beam, permitted by interference considerations, has zero intensity because of the diffracting properties of the unit cell for this geometry of beams and crystal. Prove this. (Hint: Show that the "reflection" from an atomic plane through the top of a layer of unit cells is canceled by a "reflection" from a plane through the middle of this layer of cells. All odd-order beams prove to have zero intensity.)

27. Assume that the incident X-ray beam in Fig. 45-18 is not monochromatic but contains wavelengths in a band from 0.95 to 1.30 A. Will diffracted beams, associated with the planes shown, occur? Assume $a_0 = 2.75$ A.

28. In comparing the wavelengths of two monochromatic X-ray lines, it is noted that line *A* gives a first-order reflection maximum at a glancing angle of 30° to the smooth face of a crystal. Line *B*, known to have a wavelength of 0.97 angstroms, gives a third-order reflection maximum at an angle of 60° from the same face of the same crystal. Find the wavelength of line *A*.

Polarization

CHAPTER 46

46–1 Polarization

Light, like all electromagnetic radiation, is predicted by electromagnetic theory to be a ***transverse wave***, the directions of the vibrating electric and magnetic vectors being at right angles to the direction of propagation instead of parallel to it as in a longitudinal wave. The transverse waves of Figs. 46–1 and 39–11 have the additional characteristic that they are ***plane-polarized.*** This means that the vibrations of the **E** vector are parallel to each other for all points in the wave. At any such point the vibrating **E** vector and the direction of propagation form a plane, called the ***plane of vibration;*** in a plane-polarized wave all such planes are parallel.

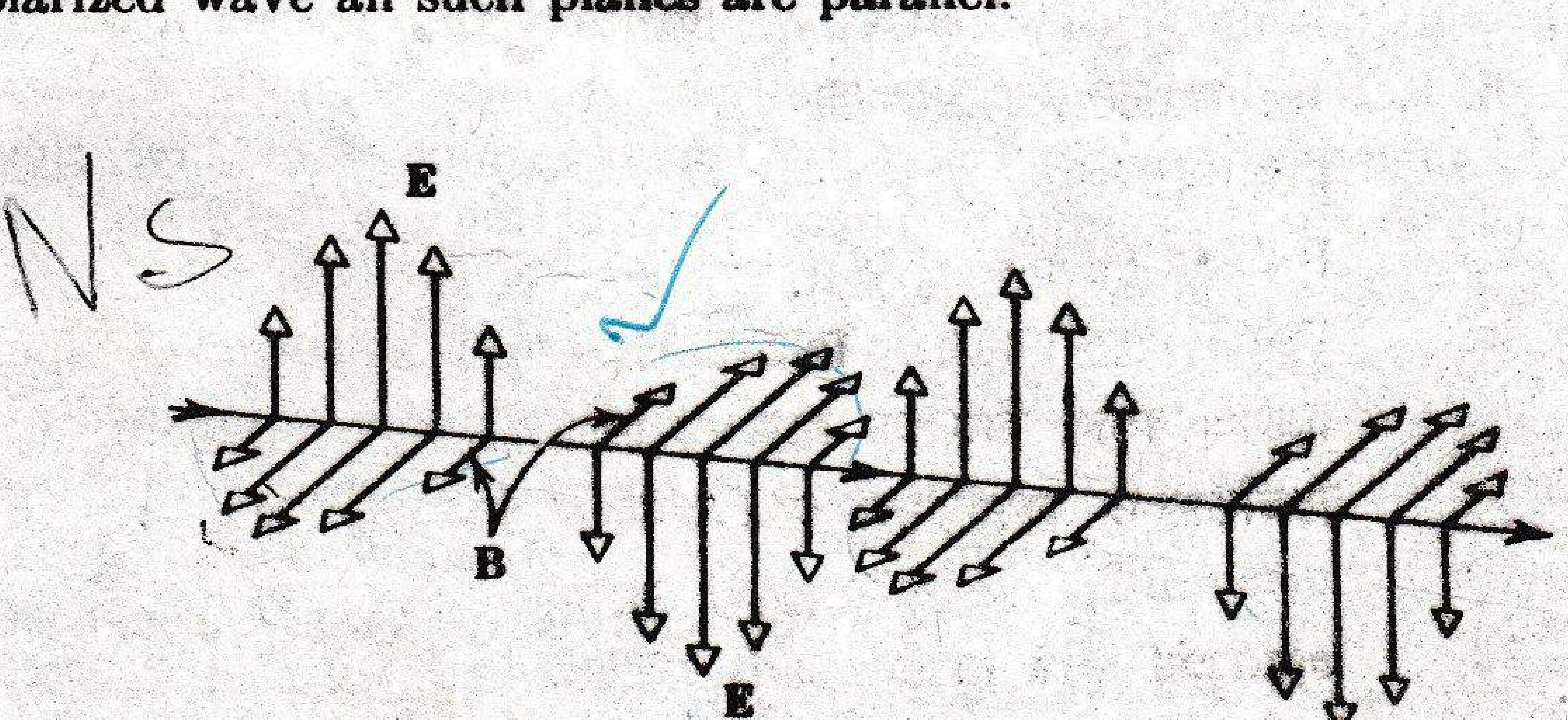

Fig. 46–1 An instantaneous "snapshot" of a plane-polarized wave showing the vectors **E** and **B** along a particular ray. The wave is moving to the right with speed *c*. The plane containing the vibrating **E** vector and the direction of propagation is a *plane of vibration*.

The transverse nature of light waves cannot be deduced from the interference or diffraction experiments so far described because longitudinal waves such as sound waves also show these effects. An experimental basis for believing that light waves are transverse was provided by Thomas Young in 1817. Two of his contemporaries, Dominique-François Arago (1786–1853) and Augustin Jean Fresnel (1788–1827), were able, by allowing a light beam to fall on a crystal of calcite, to produce two separate beams (see Section 46-4). Astonishingly, these beams, although coherent, produced no interference fringes but only a uniform illumination. Young deduced from this that light must be a transverse wave and that the planes of vibration in the two beams must be at right angles to each other. Wave disturbances that act at right angles to each other cannot show interference effects; the student is asked to prove this in Problem 9. Young's words to Arago were these:

> I have been reflecting on the possibility of giving an imperfect explanation of the affection of light which constitutes polarization without departing from the genuine doctrine of undulations. It is a principle in this theory that all undulations are simply propagated through homogeneous mediums in concentric spherical surfaces like the undulations of sound, consisting simply in the direct and retrograde motions of the particles in the direction of the radius with their concomitant condensation and rarefactions [that is, longitudinal waves]. And yet, it is possible to explain in this theory a transverse vibration, propagated also in the direction of the radius, and with equal velocity, the motions of the particles being in a certain constant direction with respect to that radius; and this is a *polarization.*

Note how Young presents the possibility of a transverse vibration as a novel idea, light having been generally—but incorrectly—assumed to be a longitudinal vibration.

In a plane-polarized transverse wave it is necessary to specify two directions, that of the wave disturbance (**E**, say) and that of propagation. In a longitudinal wave these directions are identical. In plane-polarized transverse waves, but not in longitudinal waves, we may thus expect a lack of symmetry about the direction of propagation. Electromagnetic waves in the radio and microwave range exhibit this lack of symmetry readily. Such a wave, generated by the surging of charge up and down in the dipole that forms the transmitting antenna of Fig. 46-2, has (at large distances from the dipole and at right angles to it) an electric field vector parallel to the dipole axis. When this plane-polarized wave falls on a second dipole connected to a microwave detector, the alternating electric component of the wave will cause electrons to surge back and forth in the receiving antenna, producing a reading on the detector. If we turn the receiving antenna through 90° about the direction of propagation, the detector reading drops to zero. In this orientation the electric field vector is not able to cause charge to move along the dipole axis because it points at right angles to this axis. We can reproduce the experiment of Fig. 46-2 by turning the receiving antenna of a television set (assumed an electric dipole type) through 90° about an axis that points toward the transmitting station.

Common sources of visible light differ from radio and microwave sources

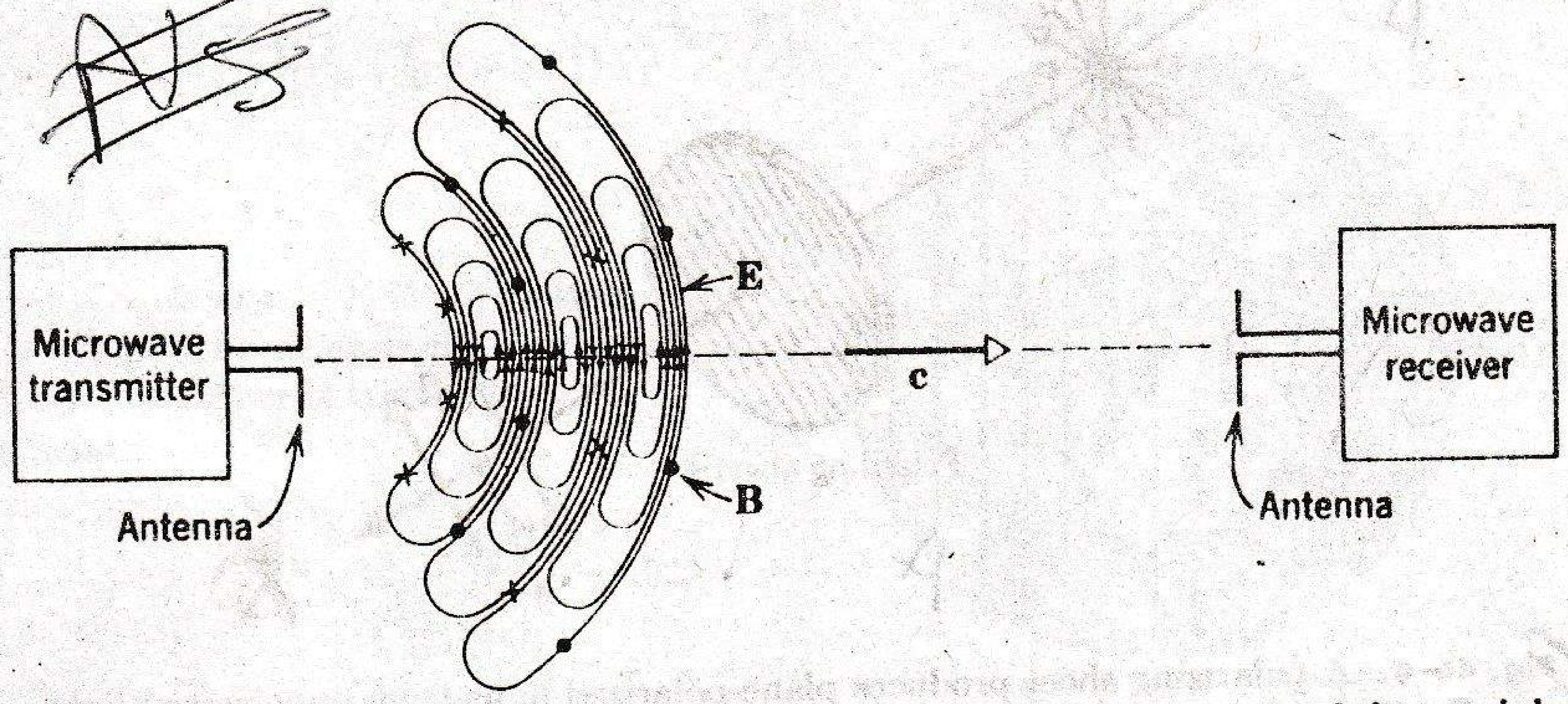

Fig. 46–2 The vectors **E** in the transmitted wave are parallel to the axis of the receiving antenna so that the wave will be detected. If the receiving antenna is rotated through 90° about the direction of propagation, no signal will be detected.

in that the elementary radiators, that is, the atoms and molecules, act independently. The light propagated in a given direction consists of independent wavetrains whose planes of vibration are randomly oriented about the direction of propagation, as in Fig. 46–3*b*. Such light, though still transverse, is *unpolarized*. The random orientation of the planes of vibration produces symmetry about the propagation direction, which, on casual study, conceals the true transverse nature of the waves. To study this transverse nature, a way must be found to unsort the different planes of vibration.

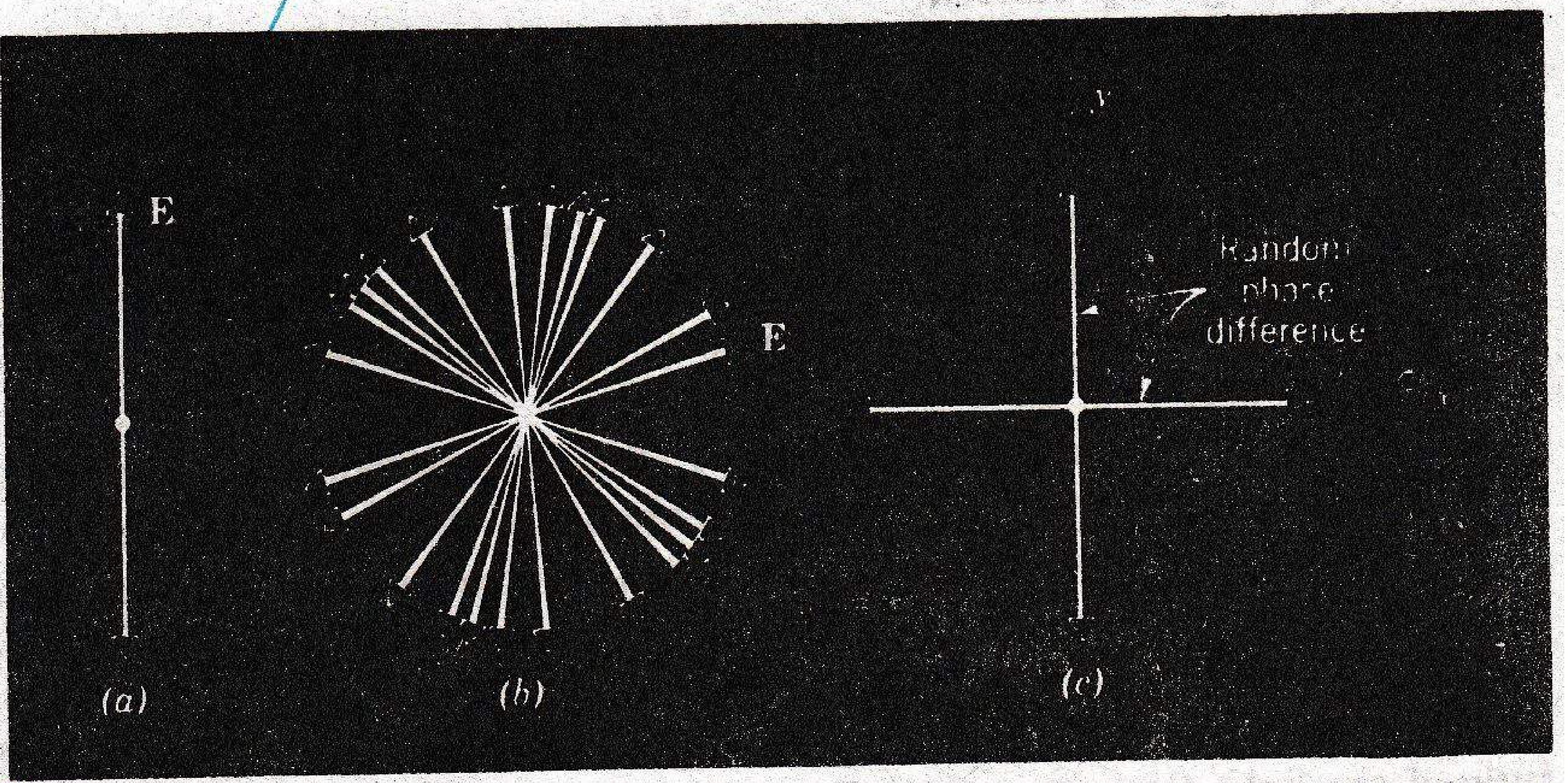

Fig. 46–3 (*a*) A plane-polarized transverse wave moving toward the reader, showing only the electric vector. (*b*) An unpolarized transverse wave viewed as a random superposition of many plane-polarized wavetrains. (*c*) A second, completely equivalent, description of an unpolarized transverse wave; here the unpolarized wave is viewed as two plane-polarized waves *with a random phase difference*. The orientation of the x and y axes about the propagation direction is completely arbitrary.

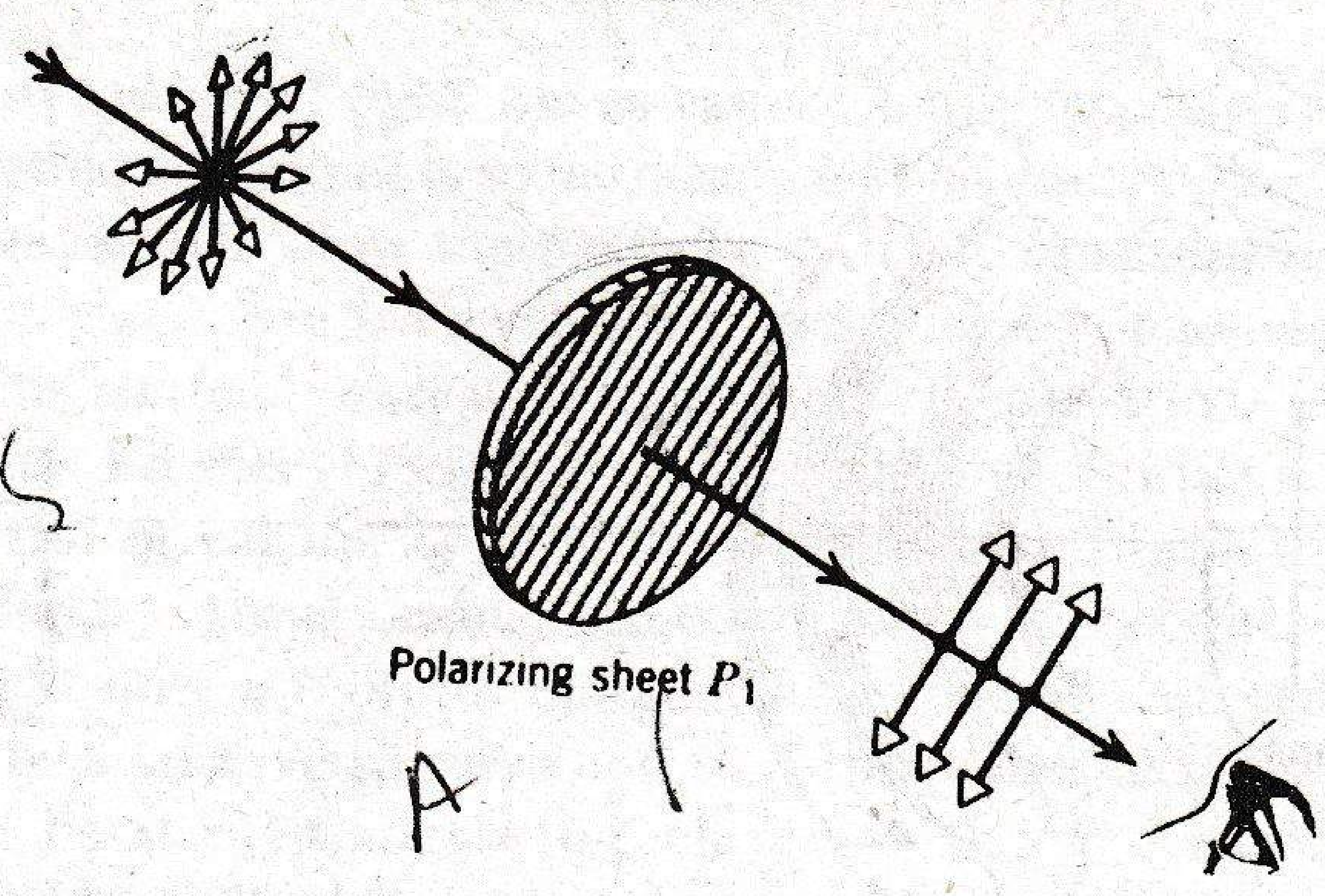

Fig. 46-4 A polarizing sheet produces plane-polarized light from unpolarized light. The parallel lines, which are not actually visible on the sheet, suggest the characteristic polarizing direction of the sheet.

46-2 Polarizing Sheets

Figure 46-4 shows unpolarized light falling on a sheet of commercial polarizing material called *Polaroid.** There exists in the sheet a certain characteristic polarizing direction, shown by the parallel lines. *The sheet will transmit only those wave-train components whose electric vectors vibrate parallel to this direction and will absorb those that vibrate at right angles to this direction.* The emerging light will be plane-polarized. This polarizing direction is established during the manufacturing process by embedding certain long-chain molecules in a flexible plastic sheet and then stretching the sheet so that the molecules are aligned parallel to each other. Polarizing sheets 2 ft wide and 100 ft long may be produced.

In Fig. 46-5 the polarizing sheet or *polarizer* lies in the plane of the page and the direction of propagation is into the page. The arrow **E** shows the plane of vibration of a randomly selected wavetrain falling on the sheet. Two vector components, $\mathbf{E}_x$ (of magnitude $E \sin \theta$) and $\mathbf{E}_y$ (of magnitude $E \cos \theta$), can replace **E**, one parallel to the polarizing direction and one at right angles to it. Only the former will be transmitted; the other is absorbed within the sheet.

Let us place a second polarizing sheet P_2 (usually called, when so used, an *analyzer*) as in Fig. 46-6. If P_2 is rotated about the direction of propagation, there are two positions, 180° apart, at which the transmitted light intensity is almost zero; these are the positions in which the polarizing directions of P_1 and P_2 are at right angles.

If the amplitude of the plane-polarized light falling on P_2 is E_m, the amplitude of the light that emerges is $E_m \cos \theta$, where θ is the angle between the polarizing directions of P_1 and P_2. Recalling that the intensity of the light

* There are other ways of producing polarized light without using this well-known commercial product. We mention some of them below.

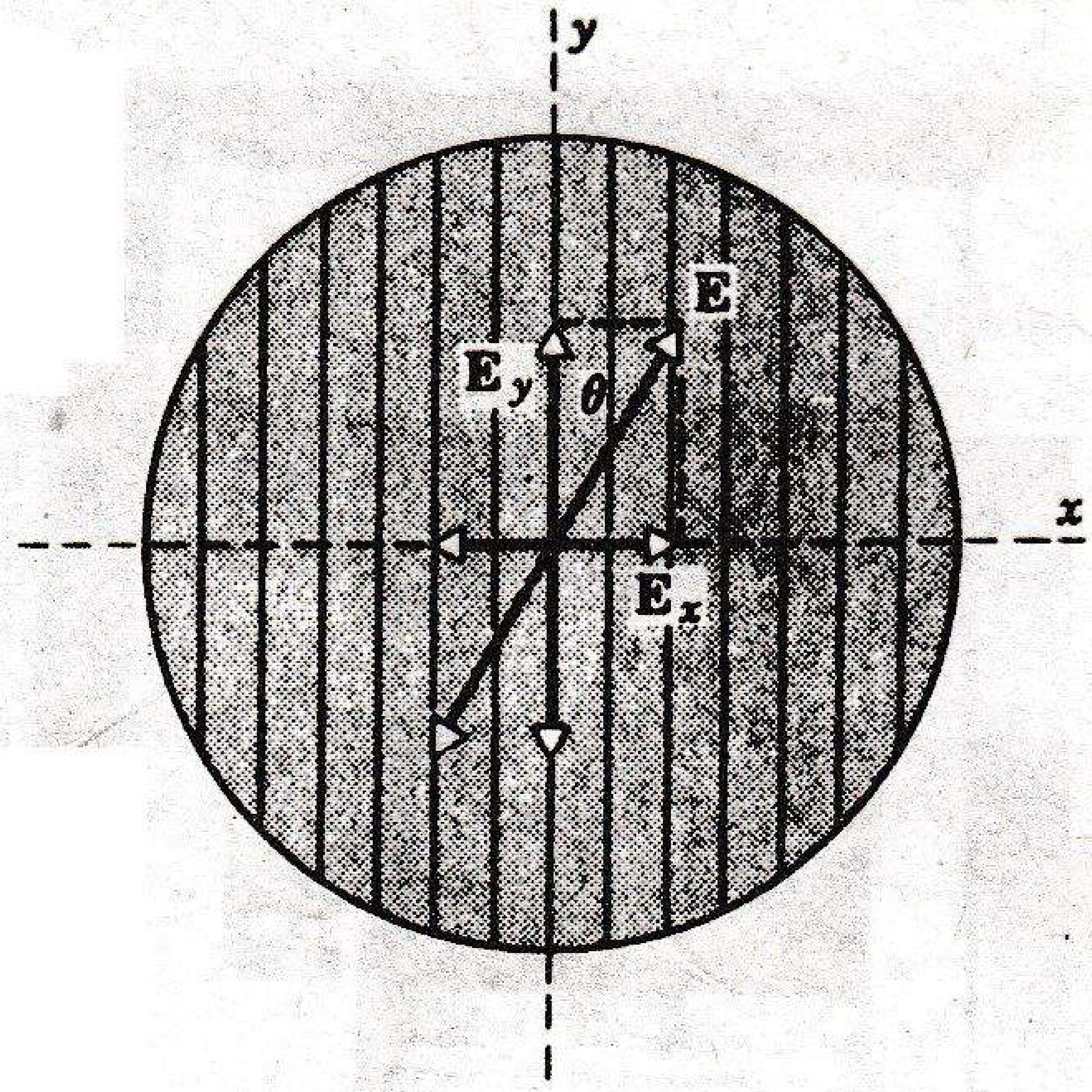

Fig. 46–5 A wavetrain **E** is equivalent to two component wavetrains $\mathbf{E}_y$ and $\mathbf{E}_x$. Only the former is transmitted by the polarizer.

beam is proportional to the square of the amplitude, we see that the transmitted intensity I varies with θ according to

$$I = I_m \cos^2 \theta, \tag{46-1}$$

in which I_m is the maximum value of the transmitted intensity. It occurs when the polarizing directions of P_1 and P_2 are parallel, that is, when $\theta = 0$ or 180°. Figure 46–7*a*, in which two overlapping polarizing sheets are in the parallel position ($\theta = 0$ or 180° in Eq. 46–1) shows that the light transmitted through the region of overlap has its maximum value. In Fig. 46–7*b* one or the other of the sheets has been rotated through 90° so that θ in Eq. 46–1 has the value 90 or 270°; the light transmitted through the region of overlap is now a minimum.

Equation 46–1, called the law of Malus, was discovered by Étienne Louis Malus (1775–1812) experimentally in 1809, using polarizing techniques other

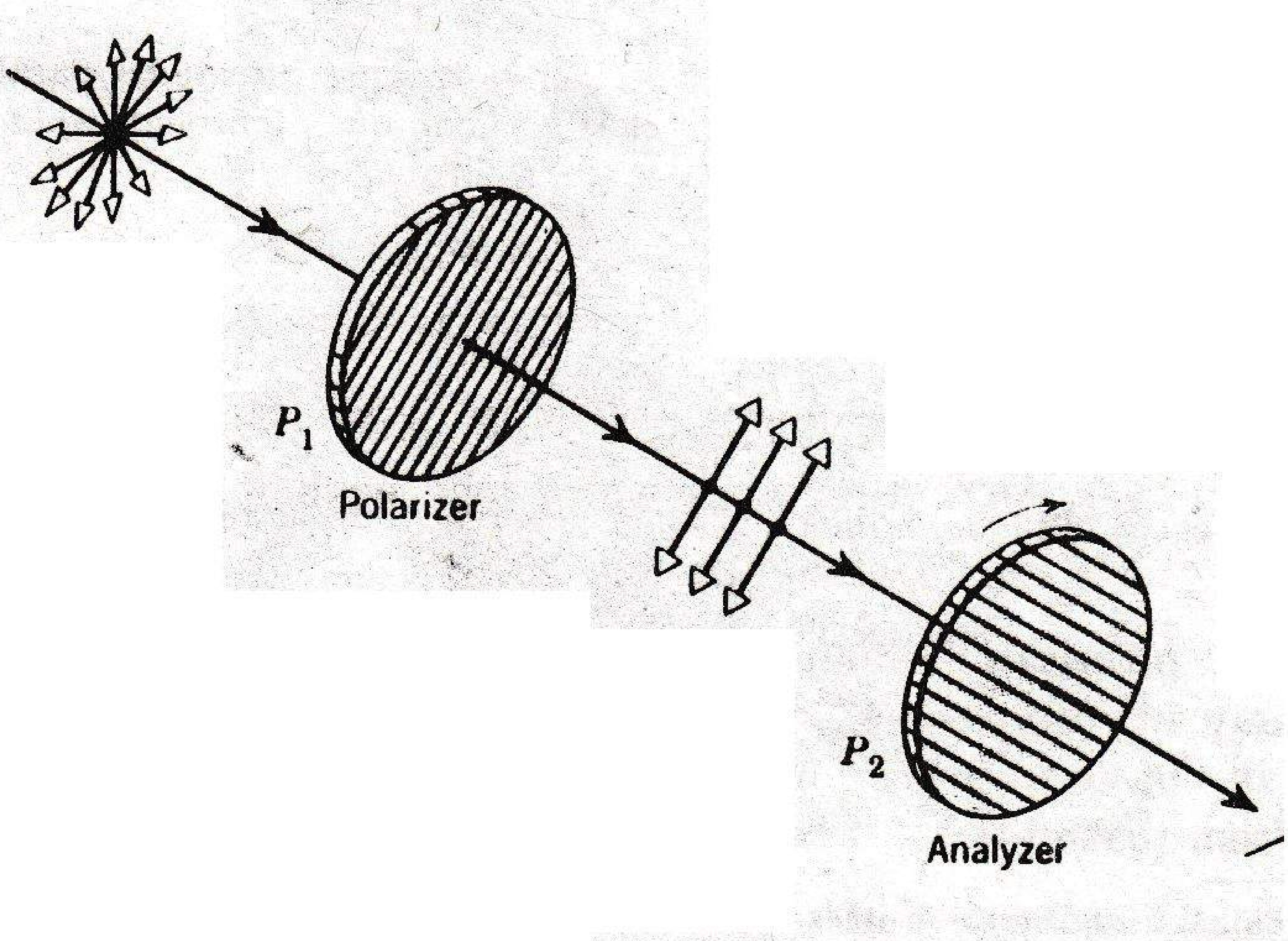

Fig. 46–6 Unpolarized light is not transmitted by crossed polarizing sheets.

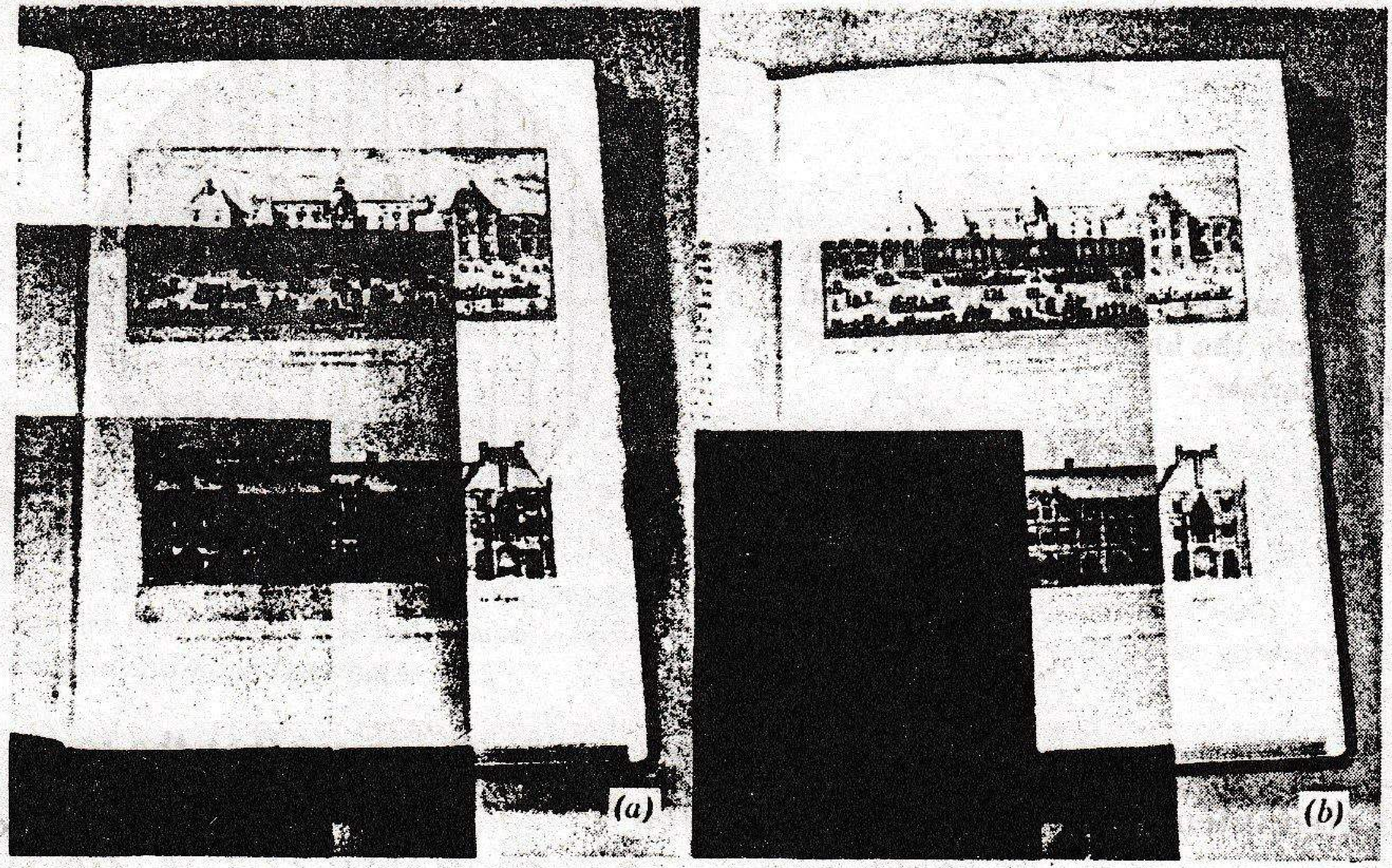

Fig. 46–7 Two square sheets of Polaroid are laid over a book. In (*a*) the axes of polarization of the two sheets are parallel and light passes through both sheets. In (*b*) one sheet has been rotated 90° and no light passes through. The book is opened to an illustration of the Luxembourg Palace in Paris. Malus discovered the phenomenon of polarization by reflection while looking at sunlight reflected off the palace windows through a calcite crystal.

than those so far described (see Section 46–3). Equation 46–1 describes precisely the lack of symmetry about the propagation direction that must be exhibited by plane-polarized transverse waves. Longitudinal waves could not possibly show such effects. Interestingly enough the human eye, under certain conditions, can detect polarized light.*

▶ **Example 1.** Two polarizing sheets have their polarizing directions parallel so that the intensity I_m of the transmitted light is a maximum. Through what angle must either sheet be turned if the intensity is to drop by one-half?

From Eq. 46–1, since $I = \frac{1}{2}I_m$, we have

$$\tfrac{1}{2}I_m = I_m \cos^2\theta$$

or

$$\theta = \cos^{-1} \pm \frac{1}{\sqrt{2}} = \pm 45°, \pm 135°.$$

The same effect is obtained no matter which sheet is rotated or in which direction. ◀

Historically polarization studies were made to investigate the nature of light. Today we reverse the procedure and deduce something about the nature of an object from the polarization state of the light emitted by or scattered from that object. It has been possible to deduce, from studies of

* The so-called *Haidinger's brushes*; the interested student is referred to *Concepts of Classical Optics*, John Strong, W. H. Freeman & Co., 1958.

the polarization of light reflected from them, that the grains of cosmic dust present in our galaxy have been oriented in the weak galactic magnetic field ($\sim 2 \times 10^{-4}$ gauss) so that their long dimension is parallel to this field. Polarization studies have shown that Saturn's rings consist of ice crystals. The size and shape of virus particles can be determined by the polarization of ultraviolet light scattered from them. Much useful information about the structure of atoms and nuclei is gained from polarization studies of their emitted radiations in all parts of the electromagnetic spectrum. Thus we have a useful research technique for structures ranging in size from a galaxy ($\sim 10^{+20}$ meters) to a nucleus ($\sim 10^{-14}$ meter). Polarized light also has many practical applications in industry and in engineering science.

46-3 Polarization by Reflection

Malus discovered in 1809 that light can be partially or completely polarized by reflection. Anyone who has watched the sun's reflection in water, while wearing a pair of sunglasses made of polarizing sheet, has probably noticed the effect. It is necessary only to tilt the head from side to side, thus rotating the polarizing sheets, to observe that the intensity of the reflected sunlight passes through a minimum.

Figure 46-8 shows an unpolarized beam falling on a glass surface. The **E** vector for each wavetrain in the beam can be resolved into two components, one perpendicular to the plane of incidence—which is the plane of Fig. 46-8—and one lying in this plane. The first component, represented by the dots, is called the *σ-component*, from the German *senkrecht*, meaning perpendicular. The second component, represented by the arrows, is called the *π-component* (for *parallel*). On the average, for completely unpolarized incident light, these two components are of equal amplitude.

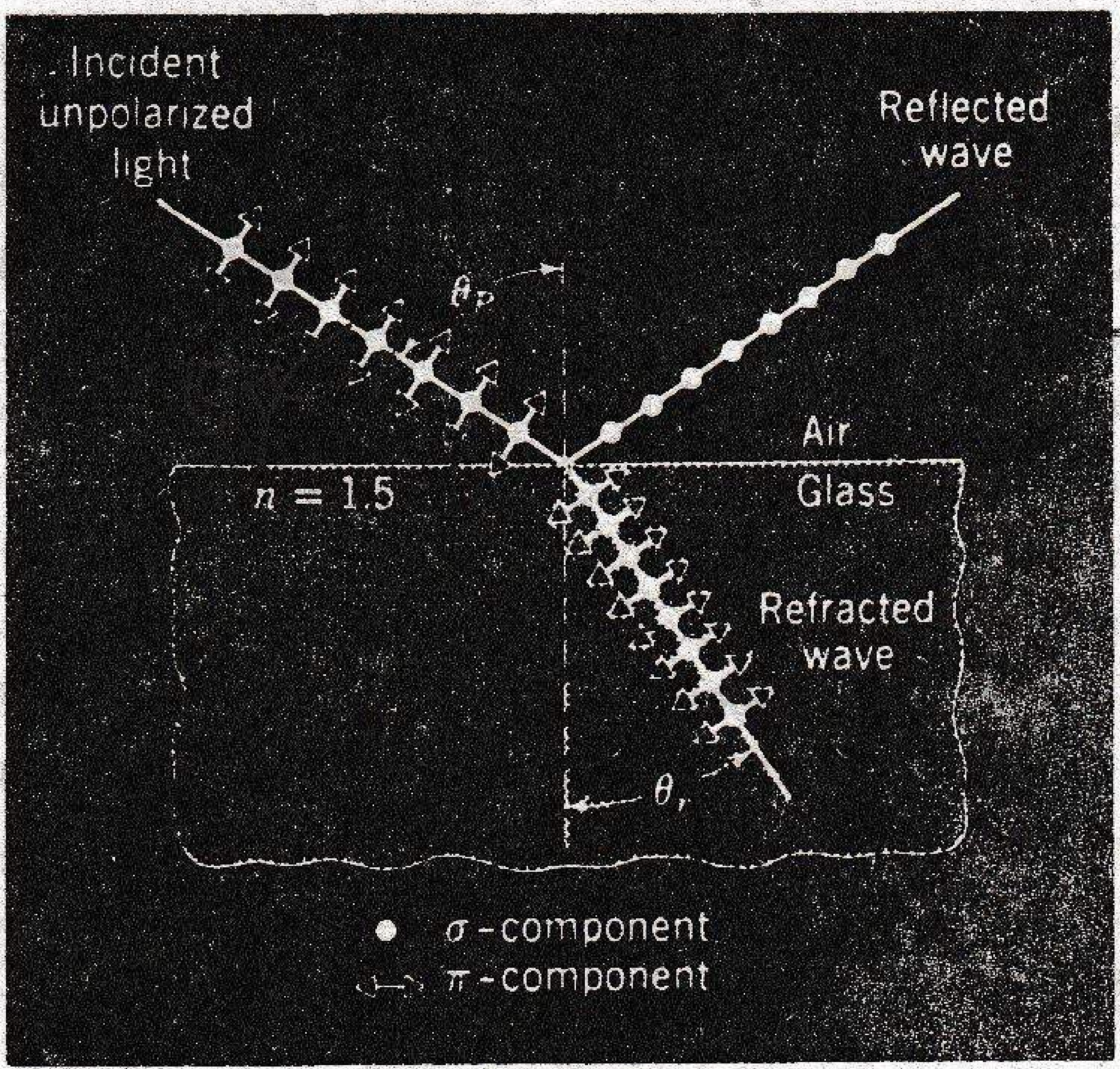

Fig. 46-8 For a particular angle of incidence θ_p, the reflected light is completely polarized, as shown. The transmitted light is partially polarized.

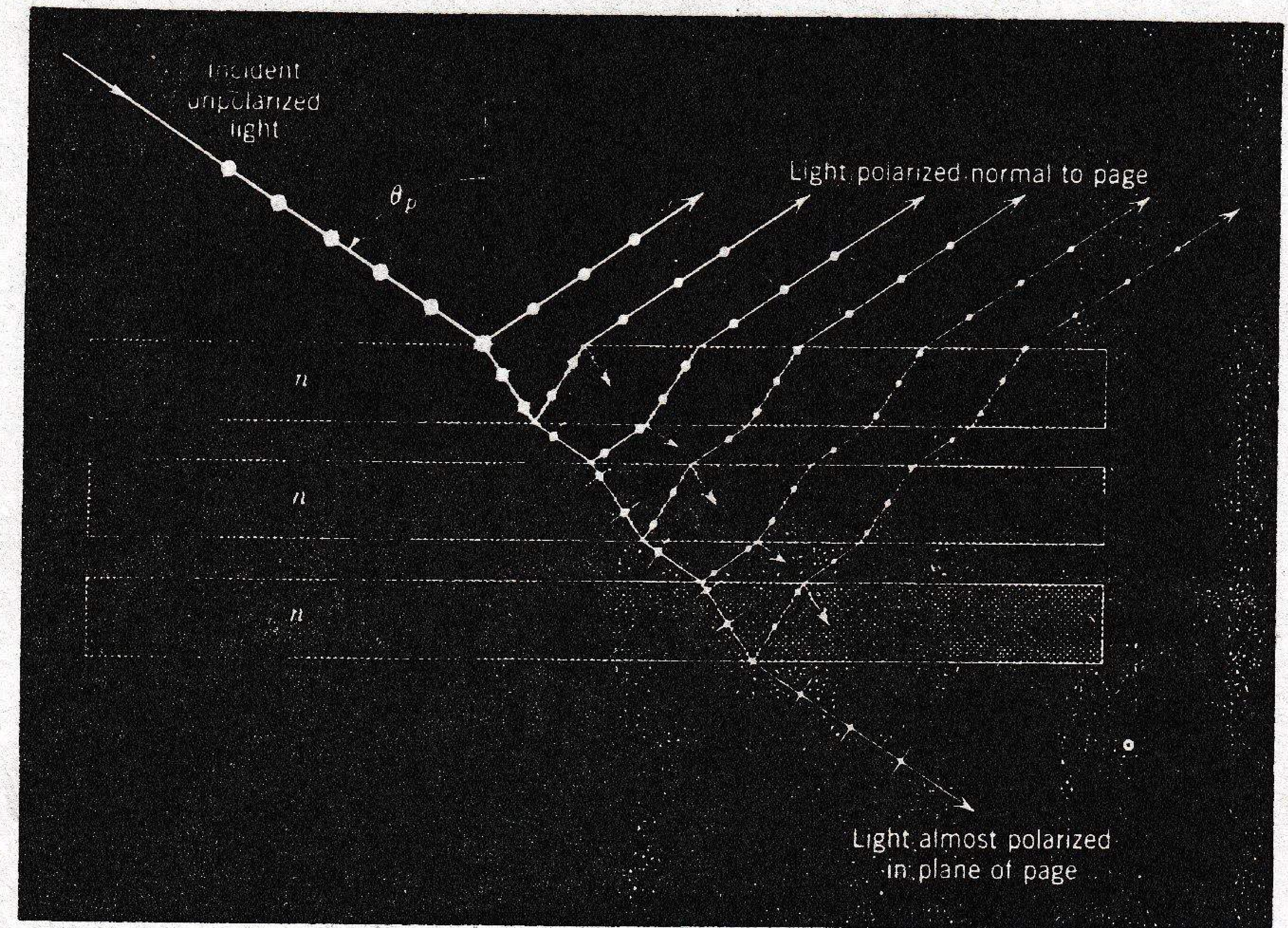

Fig. 46–9 Polarization of light by a stack of glass plates. Unpolarized light is incident on a stack of glass plates at Brewster's angle θ_p. (Polarization in the plane of the page is shown by the short lines and polarization normal to the page by the dots.) All light reflected out of the original ray is polarized normal to the page. After passing through several reflecting interfaces, the light in the original ray no longer contains any appreciable component polarized normal to the page.

Experimentally, for glass or other dielectric materials, there is a particular angle of incidence, called the *polarizing angle* θ_p, at which the reflection coefficient for the π-component is zero. This means that the beam reflected from the glass, although of low intensity, is plane-polarized, with its plane of vibration at right angles to the plane of incidence. This polarization of the reflected beam can easily be verified by analyzing it with a polarizing sheet.

The π-component at the polarizing angle is entirely refracted; the σ-component is only partially refracted. Thus the transmitted beam, which is of high intensity, is only partially polarized. By using a stack of glass plates rather than a single plate, reflections from successive surfaces occur and the intensity of the emerging reflected (σ-component) beam can be increased (see Fig. 46–9). By the same token, the σ-components are progressively removed from the transmitted beam, making it more completely π-polarized.

At the polarizing angle it is found experimentally that the reflected and the refracted beams are at right angles, or (Fig. 46–8)

$$\theta_p + \theta_r = 90°.$$

From Snell's law,

$$n_1 \sin\theta_p = n_2 \sin\theta_r.$$

Combining these equations leads to

$$n_1 \sin \theta_p = n_2 \sin (90° - \theta_p) = n_2 \cos \theta_p$$

or

$$\tan \theta_p = \frac{n_2}{n_1}, \tag{46-2}$$

where the incident ray is in medium one and the refracted ray in medium two. This can be written as

$$\tan \theta_p = n, \tag{46-3}$$

where n $(= n_2/n_1)$ is the index of refraction of medium two with respect to medium one. Equation 46-3 is known as *Brewster's law* after Sir David Brewster (1781–1868), who deduced it empirically in 1812. It is possible to prove this law rigorously from Maxwell's equations.

▶ **Example 2.** We wish to use a plate of glass (n = 1.50) as a polarizer. What is the polarizing angle? What is the angle of refraction?

From Eq. 46-3,

$$\theta_p = \tan^{-1} 1.50 = 56.3°.$$

The angle of refraction follows from Snell's law:

$$(1) \sin \theta_p = n \sin \theta_r$$

or

$$\sin \theta_r = \frac{\sin 56.3°}{1.50} = 0.555 \qquad \theta_r = 33.7°.$$ ◀

46-4 Double Refraction

In earlier chapters we assumed that the speed of light, and thus the index of refraction, is independent of the direction of propagation in the medium and of the state of polarization of the light. Liquids, amorphous solids such as glass, and crystalline solids having cubic symmetry normally show this behavior and are said to be *optically isotropic*. Many other crystalline solids are optically *anisotropic* (that is, not isotropic).*

Solids may be anisotropic in many properties. Mica cleaves readily in one plane only; a cube of crystalline graphite does not have the same electric resistance between all pairs of opposite faces; a cube of crystalline nickel magnetizes more readily in certain directions than in others, etc. If a solid is a mixture of a large number of tiny crystallites, it may appear to be isotropic because of the random orientations of the crystallites. Powdered

* Many transparent amorphous solids such as glasses and plastics become optically anisotropic when they are mechanically stressed. This fact is useful in engineering design studies in that strains in gears, bridge structures, etc., can be studied quantitatively by building plastic models, stressing them appropriately, and examining the optical anisotropy that results, using polarization techniques. The interested student should consult "Photoelasticity," a chapter by H. T. Jessop in Vol. 6 of the *Encyclopedia of Physics*, edited by H. Flugge (1958), Springer Verlag, Berlin.

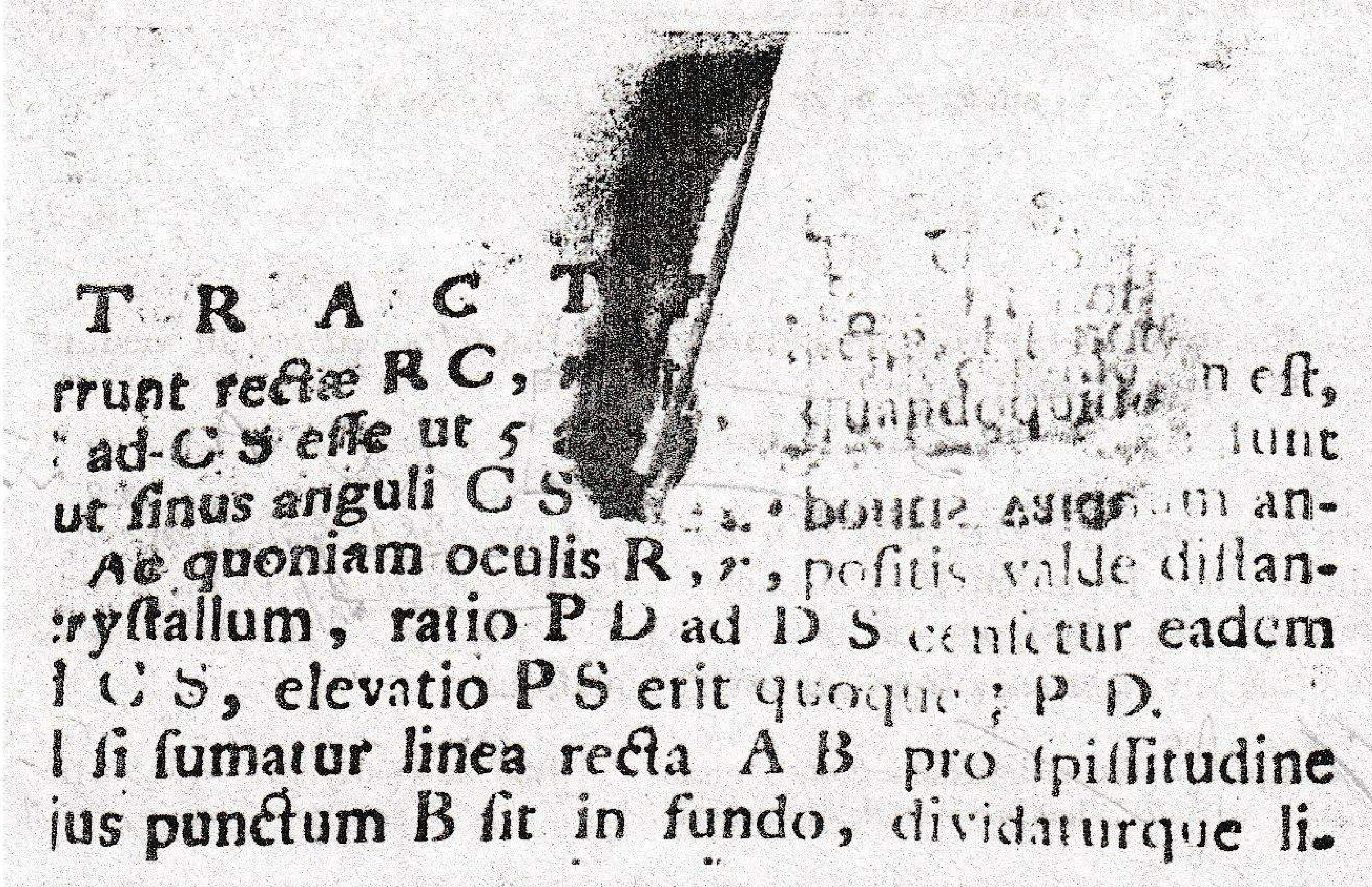

Fig. 46–10 Two images, one polarized 90° relative to the other, are formed by a calcite crystal. The book on which the crystal is lying is Huygens' *Opera Reliqua*, wherein the phenomenon of birefringence is discussed.

mica, for example, compacted to a solid mass with a binder, does not exhibit the cleavage properties that characterize the crystallites making it up.

Figure 46–10, in which a polished crystal of calcite ($CaCO_3$) is laid over some printed letters, shows the optical anisotropy of this material; *the image appears double.* Figure 46–11 shows a beam of unpolarized light falling on a calcite crystal at right angles to one of its faces. The single beam splits into two at the crystal surface. The "double-bending" of a beam transmitted through calcite, exhibited in Figs. 46–10 and 46–11, is called *double refraction.*

If the two emerging beams in Fig. 46–11 are analyzed with a polarizing sheet they are found to be plane-polarized with their planes of vibration at right angles to each other, a fact discovered by Huygens in 1678. Huygens used a second calcite crystal to investigate the polarization states of the beams labeled *o* and *e* in the figure.

If experiments are carried out at various angles of incidence, one of the beams in Fig. 46–11 (represented by the *ordinary ray*, or *o*-ray) will be found to obey Snell's law of refraction at the crystal surface, just like a ray passing from one isotropic medium into another. The second beam (represented by the *extraordinary ray*, or *e*-ray) will not. In Fig. 46–11, for example, the angle of incidence for the incident light is zero but the angle of refraction of the *e*-ray, contrary to the prediction of Snell's law, is not. In general, the *e*-ray does not even lie in the plane of incidence.

This difference between the waves represented by the *o*- and *e*-rays with respect to Snell's law can be explained in these terms:

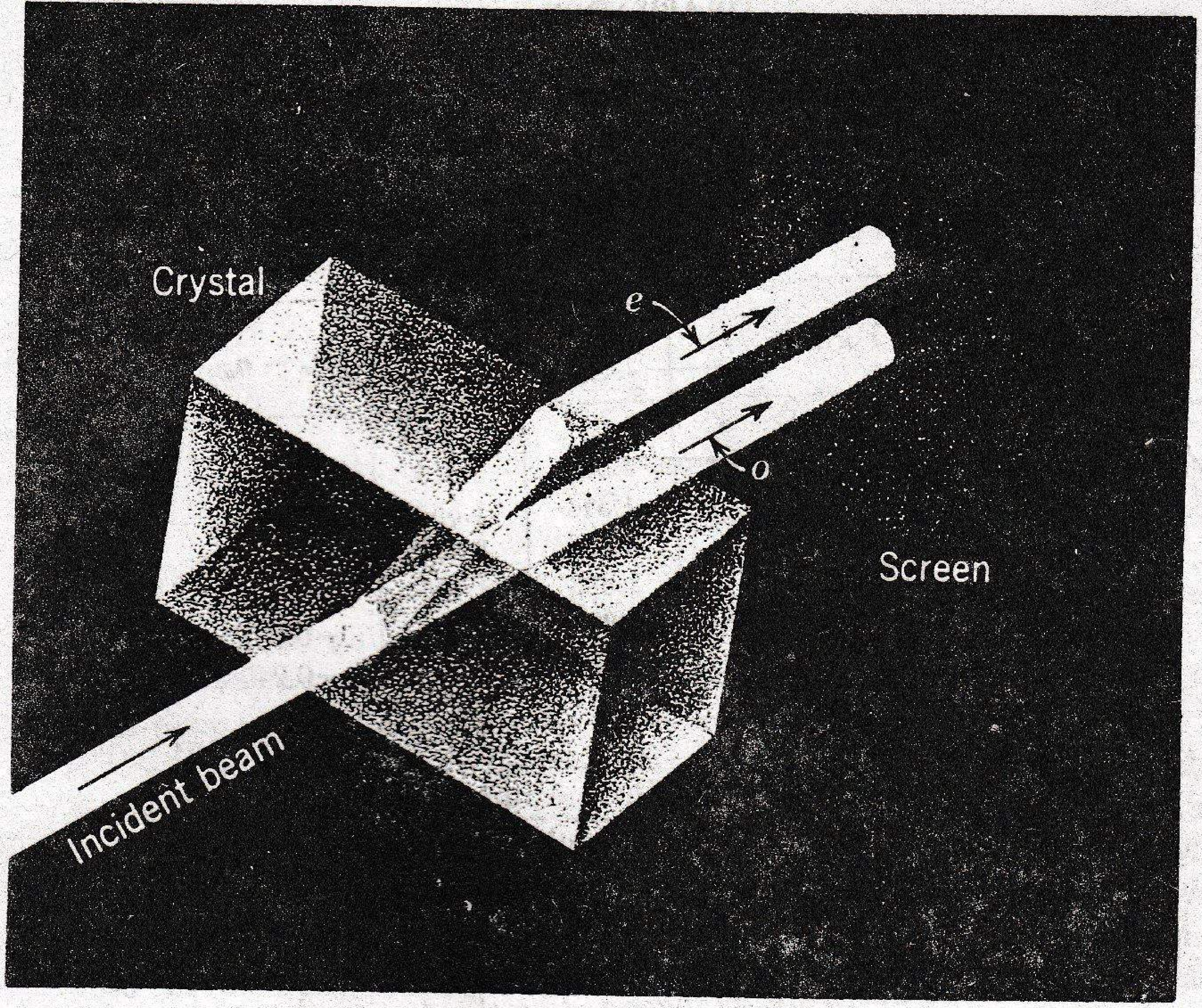

Fig. 46-11 A beam of unpolarized light falling on a calcite crystal is split into two beams which are polarized at right angles to each other.

1. The o-wave travels in the crystal with the same speed v_o in all directions. In other words, the crystal has, for this wave, a single index of refraction n_o, just like an isotropic solid.
2. The e-wave travels in the crystal with a speed that varies with direction from v_o to a larger value (for calcite) v_e. In other words, the index of refraction, defined as c/v, varies with direction from n_o to a smaller value (for calcite) n_e.

The quantities n_o and n_e are called the *principal indices of refraction* for the crystal. Problem 7 suggests how to measure them. Table 46-1 shows these indices for six doubly refracting crystals. For three of them the e-wave is slower; for the other three it is faster. Some doubly refracting crystals (mica, topaz, etc.) are more complex optically than calcite and require *three* principal indices of refraction for a complete description of their optical properties. Crystals whose basic crystal structure is cubic (see Fig. 45-13) are optically isotropic, requiring only *one* index of refraction.

The behavior for the speeds of the two waves traveling in calcite is summarized by Fig. 46-12, which shows two wave surfaces spreading out from an imaginary point light source S imbedded in the crystal. The o-wave sur-

Table 46–1

PRINCIPAL INDICES OF REFRACTION OF
SEVERAL DOUBLY REFRACTING CRYSTALS
(For sodium light, $\lambda = 5890$ A)

Crystal	Formula	n_o	n_e	$n_e - n_o$
Ice	H_2O	1.309	1.313	+0.004
Quartz	SiO_2	1.544	1.553	+0.009
Wurzite	ZnS	2.356	2.378	+0.022
Calcite	$CaCO_3$	1.658	1.486	−0.172
Dolomite	$CaO \cdot MgO \cdot 2CO_2$	1.681	1.500	−0.181
Siderite	$FeO \cdot CO_2$	1.875	1.635	−0.240

face is a sphere, as we would expect if the medium were isotropic. The *e*-wave surface is an ellipsoid of revolution about a characteristic direction in the crystal called the *optic axis*. The two wave surfaces represent light having two different polarization states. If we consider for the present only rays lying in the plane of Fig. 46–12, then (*a*) the plane of polarization for the *o*-rays is perpendicular to the figure, as suggested by the dots, and (*b*) that for the *e*-rays coincides with the plane of the figure, as suggested by the short lines. We describe the polarization states more fully at the end of this section.

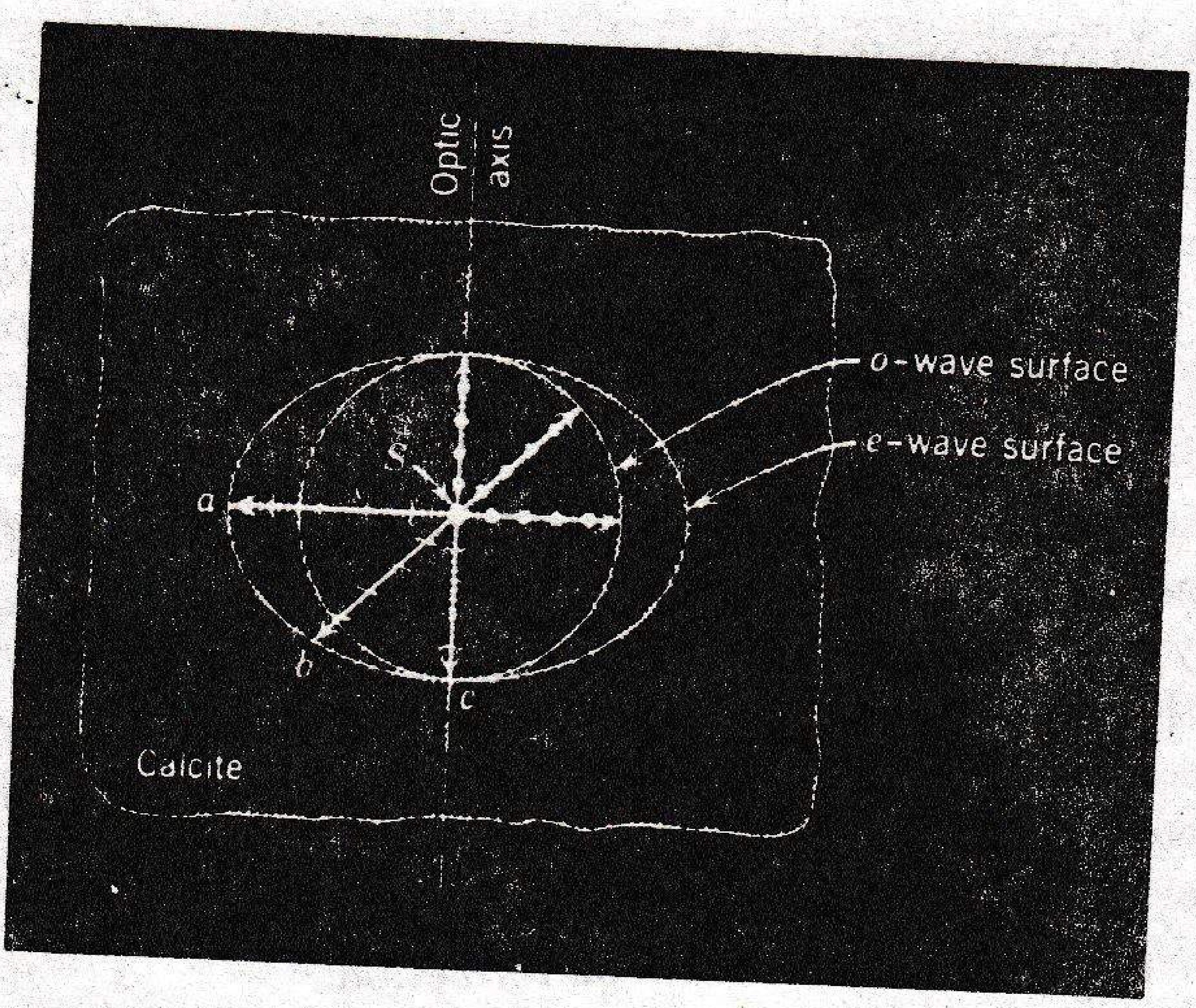

Fig. 46–12 Huygens' wave surfaces generated by a point source S imbedded in calcite. The polarization states for three *o*-rays and three *e*-rays are shown by the dots and lines, respectively. Note that in general (ray Sb) the bars representing the polarization direction are not perpendicular to the *e*-rays.

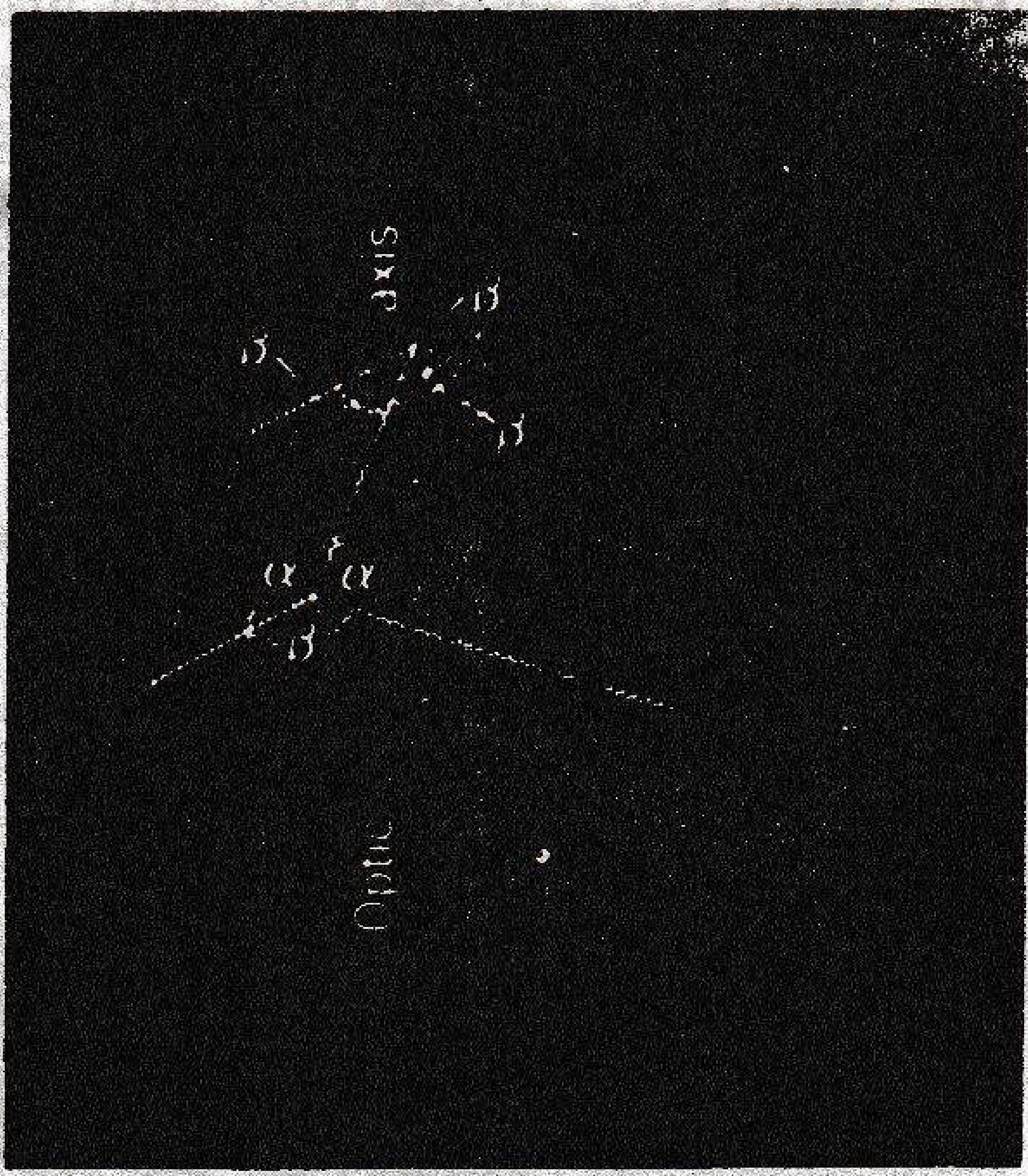

Fig. 46-13 A calcite crystal; α is 78° 13′; β is 102° 21′.

Figure 46-13, which shows a typical calcite crystal that may be obtained by cleavage from a naturally occurring crystal, shows how to locate the optic axis. The edges of calcite crystals may have any lengths but the angles at which the edges intersect always have one or another of two values, 78° 13′ or 102° 21′. The optic axis is found by erecting a line at either of the two corners where three obtuse angles meet (the "blunt" corners), making equal angles with the crystal edges. *Any line in the crystal parallel to this line is also an optic axis.*

We can use Huygens' principle to study the propagation of light waves in doubly refracting crystals. Figure 46-14*a* shows the special case in which unpolarized light falls at normal incidence on a calcite slab cut from a crystal in such a way that the optic axis is normal to the surface. Consider a wavefront that, at time $t = 0$, coincides with the crystal surface. Following Huygens, we may let any point on this surface serve as a radiating center for a double set of Huygens' wavelets, such as those in Fig. 46-12. The plane of tangency to these wavelets represents the new position of this wavefront at a later time t. The incident beam in Fig. 46-14*a* is propagated through the crystal without deviation at speed v_o. The beam emerging from the slab will have the same polarization character as the incident beam. The calcite slab, in these special circumstances only, behaves like an isotropic material, and no distinction can be made between the *o*- and the *e*-waves.

Figure 46-14*b* shows two views of another special case, namely, unpolarized incident light falling at right angles on a slab cut so that the optic axis is parallel to its surface. In this case also the incident beam is propagated without deviation. However, we can now identify *o*- and *e*-waves that travel through the crystal with different speeds, v_o and v_e, respectively. These waves are polarized at right angles to each other.

Some doubly refracting crystals have the interesting property called *dichroism*, in which one of the polarization components is strongly absorbed within the crystal.

the other being transmitted with little loss. Dichroism, illustrated in Fig. 46–15, is the basic operating principle of the commercial Polaroid sheet. The many small crystallites, imbedded in a plastic sheet with their optic axes parallel, have a polarizing action equivalent to that of a single large crystal slab.

Figure 46–14*c* shows unpolarized light falling at normal incidence on a calcite slab cut so that its optic axis makes an arbitrary angle with the crystal surface. Two spatially separated beams are produced, as in Fig. 46–11. They travel through the crystal at different speeds, that for the *o*-wave being v_o and that for the *e*-wave being intermediate between v_o and v_e.

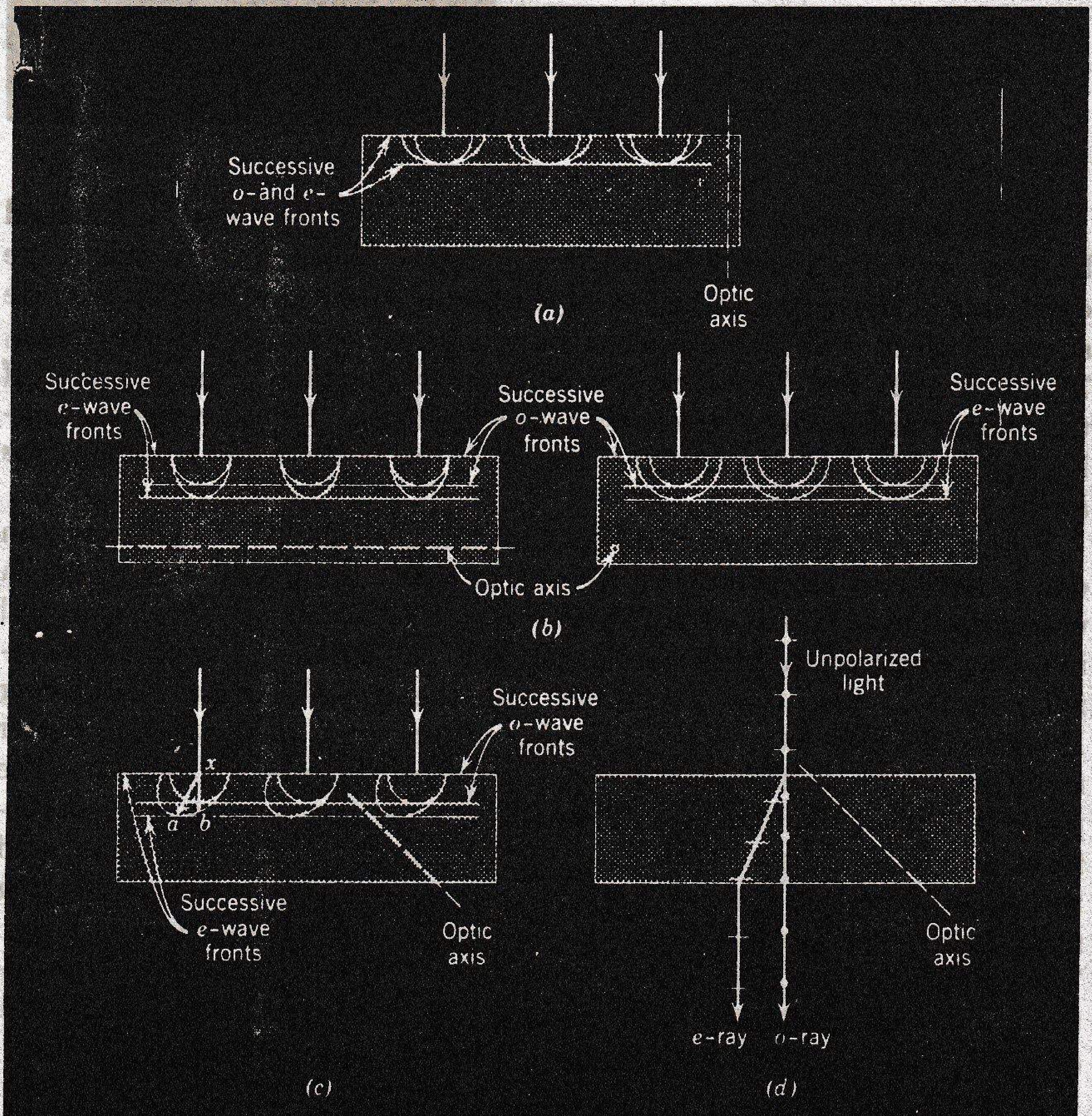

Fig. 46–14 Unpolarized light falls at normal incidence on slabs cut from a calcite crystal. The Huygens' wavelets are appropriate sections of the figure of revolution about the optic axis represented by Fig. 46–15. (*a*) No double refraction or speed difference occurs. (*b*) No double refraction occurs but there is a speed difference. (*c*) Both double refraction and a speed difference occur. (*d*) Same as (*c*) but showing the polarization states and the emerging rays.

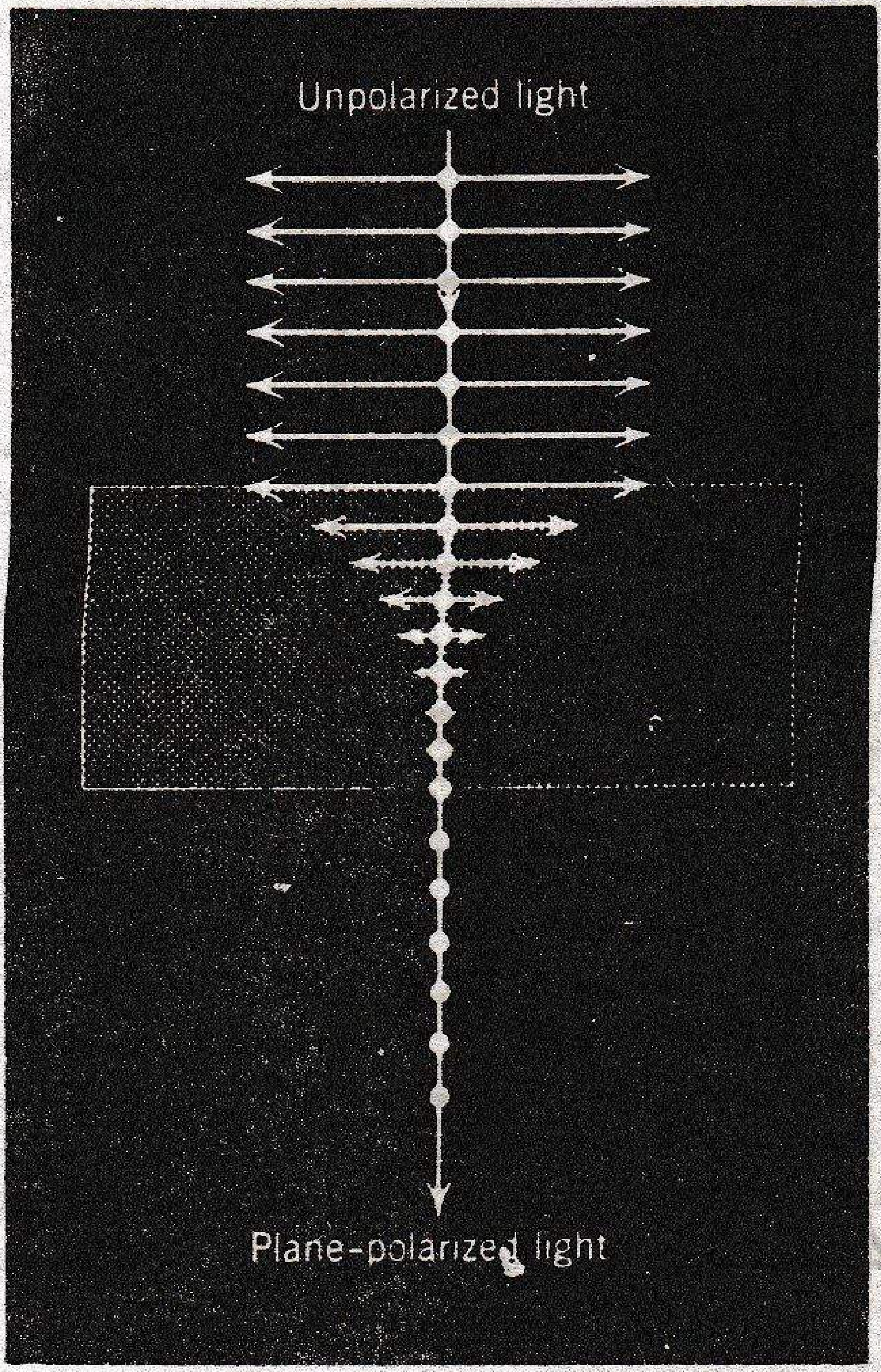

Fig. 46–15 Showing the absorption of one polarization component inside a dichroic crystal of the type used in Polaroid sheets.

Note that ray *xa* represents the shortest *optical* path for the transfer of light energy from point *x* to the *e*-wavefront. Energy transferred along any other ray, in particular along ray *xb*, would have a longer transit time, a consequence of the fact that the speed of *e*-waves varies with direction.* Figure 46–14*d* represents the same case as Fig. 46–14*c*. It shows the rays emerging from the slab, as in Fig. 46–11, and makes clear that the emerging beams are polarized at right angles to each other, that is, they are *cross-polarized*.

We now seek to understand, in terms of the atomic structure of optically anisotropic crystals, how cross-polarized light waves with different speeds can exist. Light is propagated through a crystal by the action of the vibrating **E** vectors of the wave on the electrons in the crystal. These electrons, which experience electrostatic restoring forces if they are moved from their equilibrium positions, are set into forced periodic oscillation about these positions and pass along the transverse wave disturbance that constitutes the light wave. The strength of the restoring forces may be measured by a force constant *k*, as for the simple harmonic oscillator discussed in Chapter 15 (see Eq. 15–4).

In optically isotropic materials the force constant *k* is the same for all directions of displacement of the electrons from their equilibrium positions. In doubly refracting crystals, however, *k* varies with direction. For electron displacements that lie in a plane at right angles to the optic axis *k* has the constant value k_o, no matter how the displacement is oriented in this plane. For displacements parallel to the

* The student who has not previously read Section 41–6 on Fermat's principle may care to do so now.

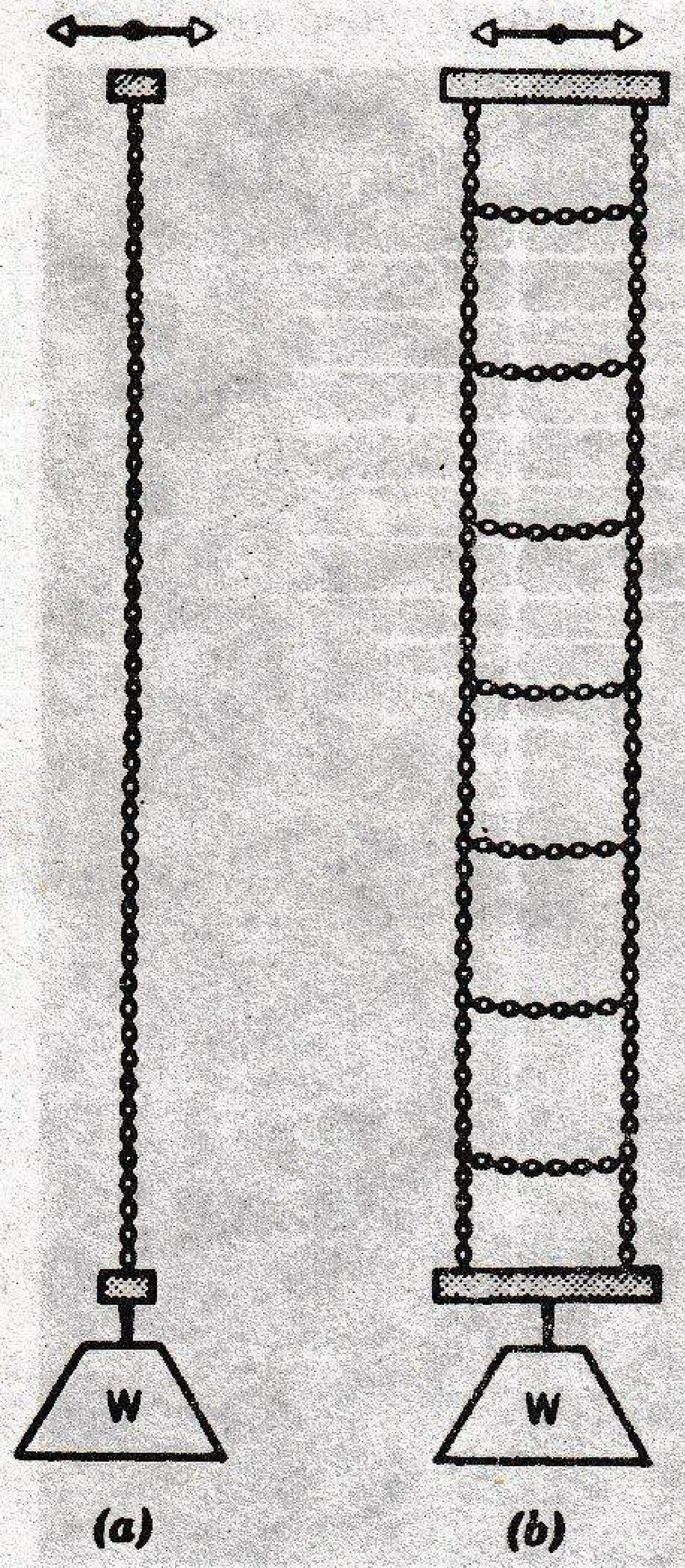

Fig. 46–16 Two views of a one-dimensional mechanical model for double refraction.

optic axis, k has the larger value (for calcite) k_e.* Note carefully that the speed of a wave in a crystal is determined by the direction in which the **E** vectors vibrate and *not* by the direction of propagation. It is the transverse **E**-vector vibrations that call the restoring forces into play and thus determine the wave speed. Note too that the stronger the restoring force, that is, the larger k, the faster the wave. For waves traveling along a stretched cord, for example, the restoring force for the transverse displacements is determined by the tension F in the cord. Equation 19–12 shows that an increase in F means an increase in the wave speed v.

Figure 46–16, a long weighted "tire chain" supported at its upper end, provides a one-dimensional mechanical analogy for double refraction. It applies specifically to *o*- and *e*-waves traveling at right angles to the optic axis, as in Fig. 46–14*b*. If the supporting block is oscillated, as in Fig. 46–16*a*, a transverse wave travels along the chain with a certain speed. If the block is oscillated lengthwise, as in Fig. 46–16*b*, another transverse wave is also propagated. The restoring force for the second wave is greater than for the first, the chain being more rigid in the plane of Fig. 46–16*b* than in the plane of Fig. 46–16*a*. Thus the second wave travels along the chain with a greater speed.

In the language of optics we would say that the speed of a transverse wave in the chain depends on the orientation of the plane of vibration of the wave. If we oscillate the top of the chain in a random way, the wave disturbance at a point along the chain can be described as the sum of two waves, polarized at right angles and traveling with different speeds. This corresponds exactly to the optical situation of Fig. 46–14*b*.

* For doubly refracting crystals with $n_e > n_o$ (see Table 46–1) k for displacements parallel to the optic axis is *smaller* than for those at right angles to it. Also, for crystals with three principal indices of refraction, there will be three principal force constants. Such crystals have two optic axes and are called *biaxial*. The crystals listed in Table 46–1 have only a single optic axis and are called *uniaxial*.

For waves traveling parallel to the optic axis, as in Fig. 46–14*a*, or for waves in optically isotropic materials, the appropriate mechanical analogy is a single weighted hanging chain. Here there is only one speed of propagation, no matter how the upper end is oscillated. The restoring forces are the same for all orientations of the plane of polarization of waves traveling along such a chain.

These considerations allow us to understand more clearly the polarization states of the light represented by the double-wave surface of Fig. 46–12. For the (spherical) *o*-wave surface, the **E**-vector vibrations must be everywhere at right angles to the optic axis. If this is so, the same force constant k_o will always be operative, and the *o*-waves will travel with the same speed in all directions. More specifically, if we draw a ray in Fig. 46–12 from *S* to the *o*-wave surface, considered three-dimensionally (that is, as a sphere), the **E**-vector vibrations will always be at right angles to the plane defined by this ray and the optic axis. Thus these vibrations will always be at right angles to the optic axis.

For the (ellipsoidal) *e*-wave surface, the **E**-vector vibrations in general have a component parallel to the optic axis. For rays such as *Sa* in Fig. 46–12 or for the *e*-rays of Fig. 46–14*b*, the vibrations are completely parallel to this axis. Thus a relatively strong force constant (in calcite) k_e is operative, and the wave speed v_e will be relatively high. For *e*-rays, such as *Sb* in Fig. 46–12, the parallel component of the **E**-vector vibrations is less than 100%, so that the corresponding wave speed will be less than v_e. For ray *Sc*, in Fig. 46–12, the parallel component is zero, and the distinction between *o*- and *e*-rays disappears.

46–5 Circular Polarization

Let plane-polarized light of angular frequency ω ($= 2\pi\nu$) fall at normal incidence on a slab of calcite cut so that the optic axis is parallel to the face of the slab, as in Fig. 46–17. The two waves that emerge will be plane-polarized at right angles to each other, and, if the incident plane of vibration is at 45° to the optic axis, they will have equal amplitudes. Since the waves travel through the crystal at different speeds, there will be a phase difference ϕ between them when they emerge from the crystal. If the crystal thickness is chosen so that (for a given frequency of light) $\phi = 90°$, the slab is called a *quarter-wave plate*. The emerging light is said to be *circularly polarized.*

In Section 15–7 we saw that the two emerging plane-polarized waves just described (vibrating at right angles with a 90° phase difference) can be represented as the projections on two perpendicular axes of a vector rotating with

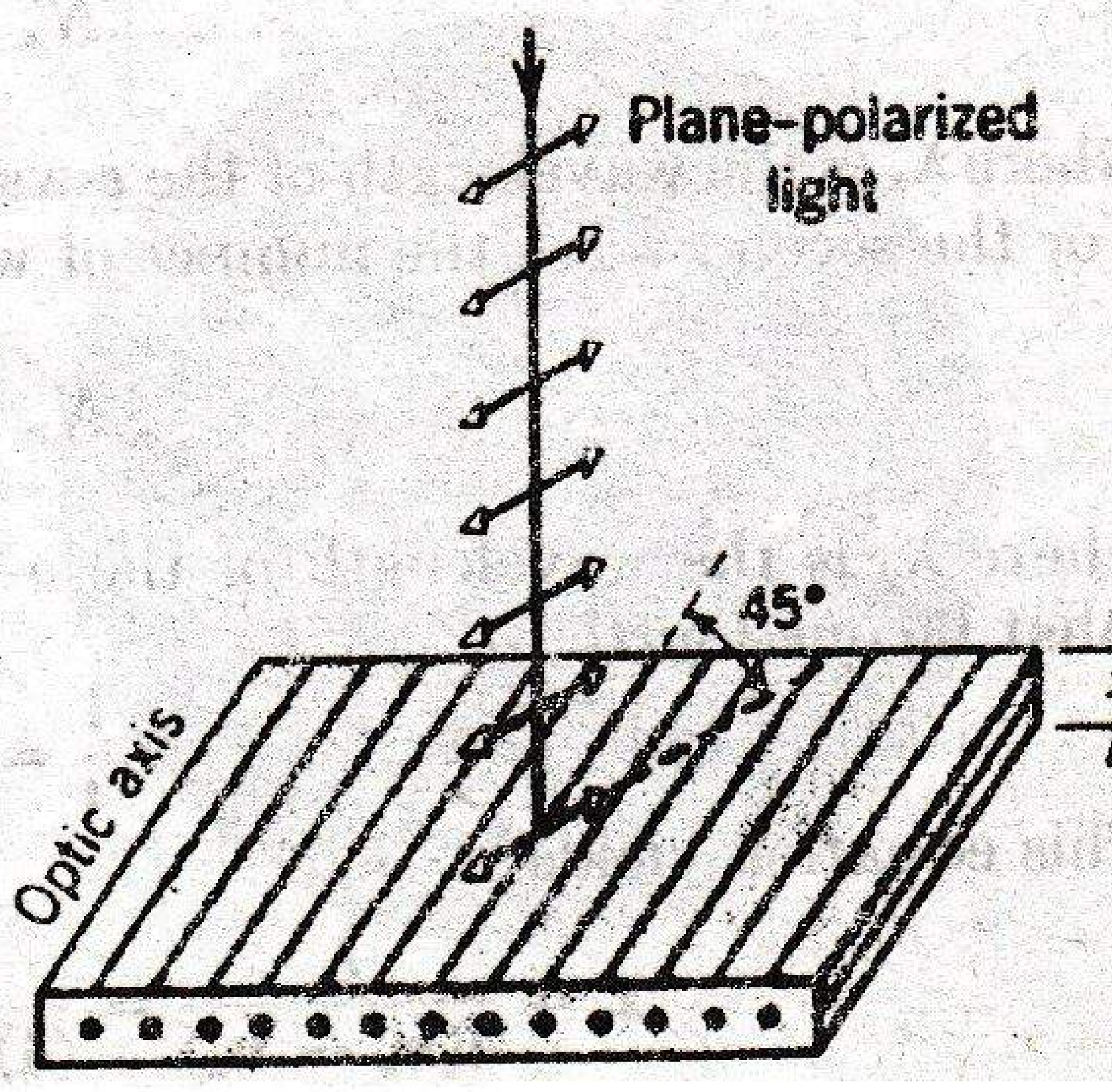

Fig. 46–17 Plane-polarized light falls on a doubly refracting slab of thickness *x* cut with its optic axis parallel to the surface. The plane of vibration of the incident light is oriented to make an angle of 45° with the optic axis.

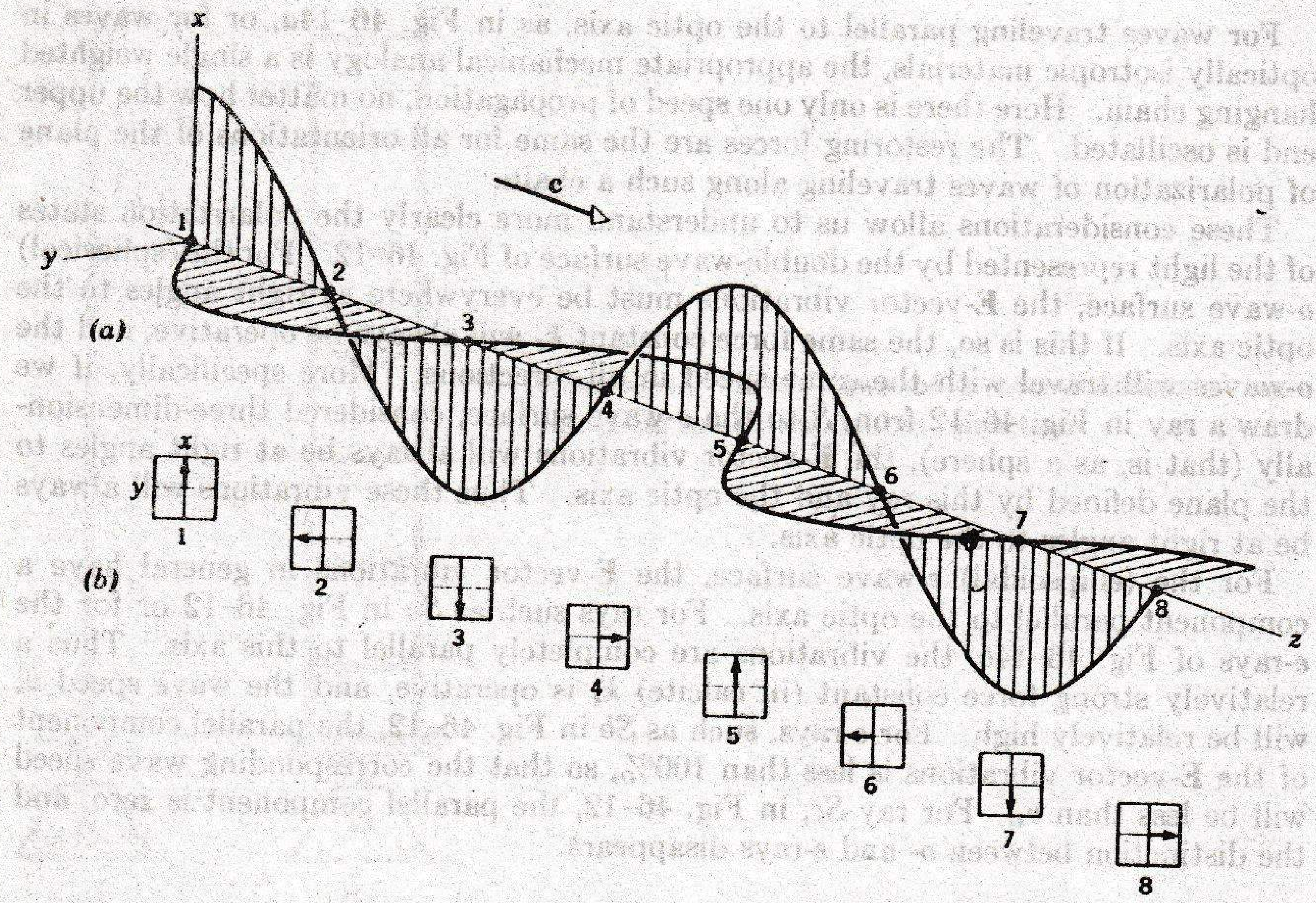

Fig. 46–18 (*a*) Two plane-polarized waves of equal amplitude and at right angles to each other are moving in the *z* direction. They differ in phase by 90°; where one wave has maximum values, the other is zero. (*b*) Views of the resultant amplitude of the approaching wave as seen by observers located at the positions shown on the *z* axis. Note that each observer will see the resultant vector rotate clockwise with time.

angular frequency ω about the propagation direction. These two descriptions of circularly polarized light are completely equivalent. Figure 46–18 clarifies the relationship between these two descriptions.

▶ **Example 3.** A quartz quarter-wave plate is to be used with sodium light (λ = 5890 A). What must its thickness be?

Two waves travel through the slab at speeds corresponding to the two principal indices of refraction given in Table 46–1 (n_e = 1.553 and n_o = 1.544). If the crystal thickness is x, the number of wavelengths of the first wave contained in the crystal is

$$N_e = \frac{x}{\lambda_e} = \frac{xn_e}{\lambda},$$

where λ_e is the wavelength of the *e*-wave in the crystal and λ is the wavelength in air For the second wave the number of wavelengths is

$$N_o = \frac{x}{\lambda_o} = \frac{xn_o}{\lambda},$$

where λ_o is the wavelength of the *o*-wave in the crystal. The difference $N_e - N_o$ must be one-fourth, or

$$\frac{1}{4} = \frac{x}{\lambda}(n_e - n_o).$$

This equation yields

$$x = \frac{\lambda}{4(n_e - n_o)} = \frac{5890 \text{ A}}{(4)(1.553 - 1.544)} = 0.016 \text{ mm}.$$

This plate is rather thin; most quarter-wave plates are made from mica, splitting the sheet to the correct thickness by trial and error.

Example 4. A beam of circularly polarized light falls on a polarizing sheet. Describe the emerging beam.

The circularly polarized light, as it enters the sheet, can be represented by

$$E_x = E_m \sin \omega t$$

and

$$E_y = E_m \cos \omega t,$$

where x and y represent arbitrary perpendicular axes. These equations correctly represent the fact that a circularly polarized wave is equivalent to two plane-polarized waves with equal amplitude and a 90° phase difference.

The resultant amplitude in the incident circularly polarized wave is

$$E_{cp} = \sqrt{E_x^2 + E_y^2} = \sqrt{E_m^2 (\sin^2 \omega t + \cos^2 \omega t)} = E_m,$$

an expected result if the circularly polarized wave is represented as a rotating vector. The resultant intensity in the incident circularly polarized wave is proportional to E_m^2, or

$$I_{cp} \propto E_m^2. \tag{46-4}$$

Let the polarizing direction of the sheet make an arbitrary angle θ with the x axis as shown in Fig. 46-19. The instantaneous value of the plane-polarized wave transmitted by the sheet is

$$\begin{aligned} E &= E_y \sin \theta + E_x \cos \theta \\ &= E_m \cos \omega t \sin \theta + E_m \sin \omega t \cos \theta \\ &= E_m \sin (\omega t + \theta). \end{aligned}$$

The intensity of the wave transmitted by the sheet is proportional to E^2, or

$$I \propto E_m^2 \sin^2 (\omega t + \theta).$$

The eye and other measuring instruments respond only to the average intensity $\bar{I}$, which is found by replacing $\sin^2 (\omega t + \theta)$ by its average value over one or more cycles ($= \frac{1}{2}$), or

$$\bar{I} \propto \tfrac{1}{2} E_m^2.$$

Comparison with Eq. 46-4 shows that inserting the polarizing sheet reduces the intensity by one-half. The orientation of the sheet makes no difference, since θ does not

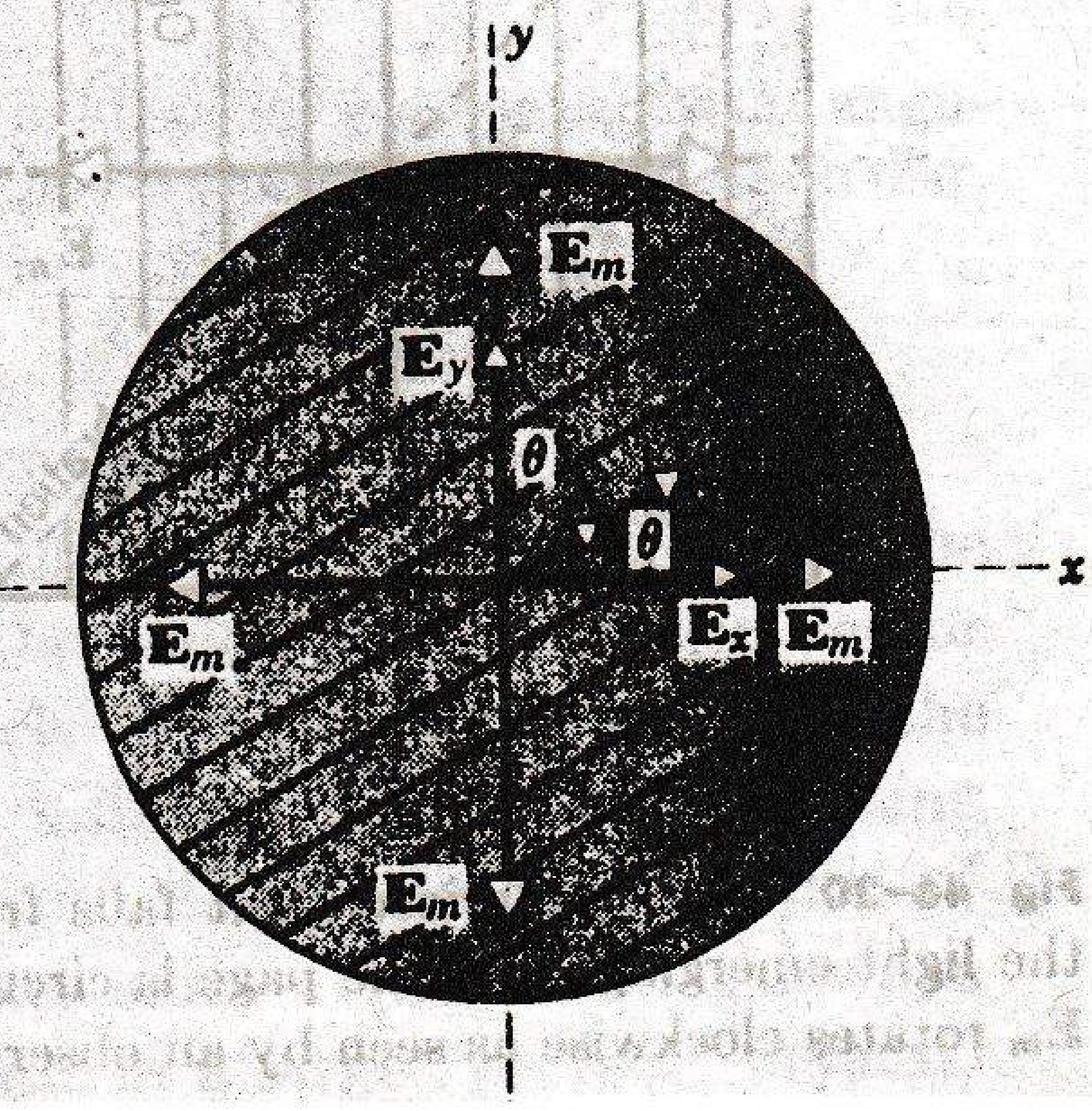

Fig. 46-19 Circularly polarized light falls on a polarizing sheet. $\mathbf{E}_x$ and $\mathbf{E}_y$ are instantaneous values of the two components, their maximum values being $\mathbf{E}_m$.

appear in this equation; this is to be expected if circularly polarized light is represented by a rotating vector, all azimuths about the propagation direction being equivalent. Inserting a polarizing sheet in an *unpolarized* beam has just the same effect, so that a simple polarizing sheet cannot be used to distinguish between unpolarized and circularly polarized light.

Example 5. A beam of light is thought to be circularly polarized. How may this be verified?

Insert a quarter-wave plate. If the beam is circularly polarized, the two components will have a phase difference of 90° between them. The quarter-wave plate will introduce a further phase difference of ±90° so that the emerging light will have a phase difference of either zero or 180°. In either case the light will now be *plane-polarized* and can be made to suffer complete extinction by rotating a polarizer in its path.

Does the quarter-wave plate have to be oriented in any particular way to carry out this test?

Example 6. A plane-polarized light wave of amplitude E_0 falls on a calcite quarter-wave plate with its plane of vibration at 45° to the optic axis of the plate, which is taken as the y axis; see Fig. 46–20. The emerging light will be circularly polarized. In what direction will the rotating electric vector appear to rotate? The direction of propagation is out of the page.

The wave component whose vibrations are parallel to the optic axis (the e-wave) can be represented as it emerges from the plate as

$$E_y = (E_0 \cos 45°) \sin \omega t = \frac{1}{\sqrt{2}} E_0 \sin \omega t = E_m \sin \omega t.$$

The wave component whose vibrations are at right angles to the optic axis (the o-wave) can be represented as

$$E_x = (E_0 \sin 45°) \sin (\omega t - 90°) = -\frac{1}{\sqrt{2}} E_0 \cos \omega t = -E_m \cos \omega t,$$

the 90° phase shift representing the action of the quarter-wave plate. Note that E_x

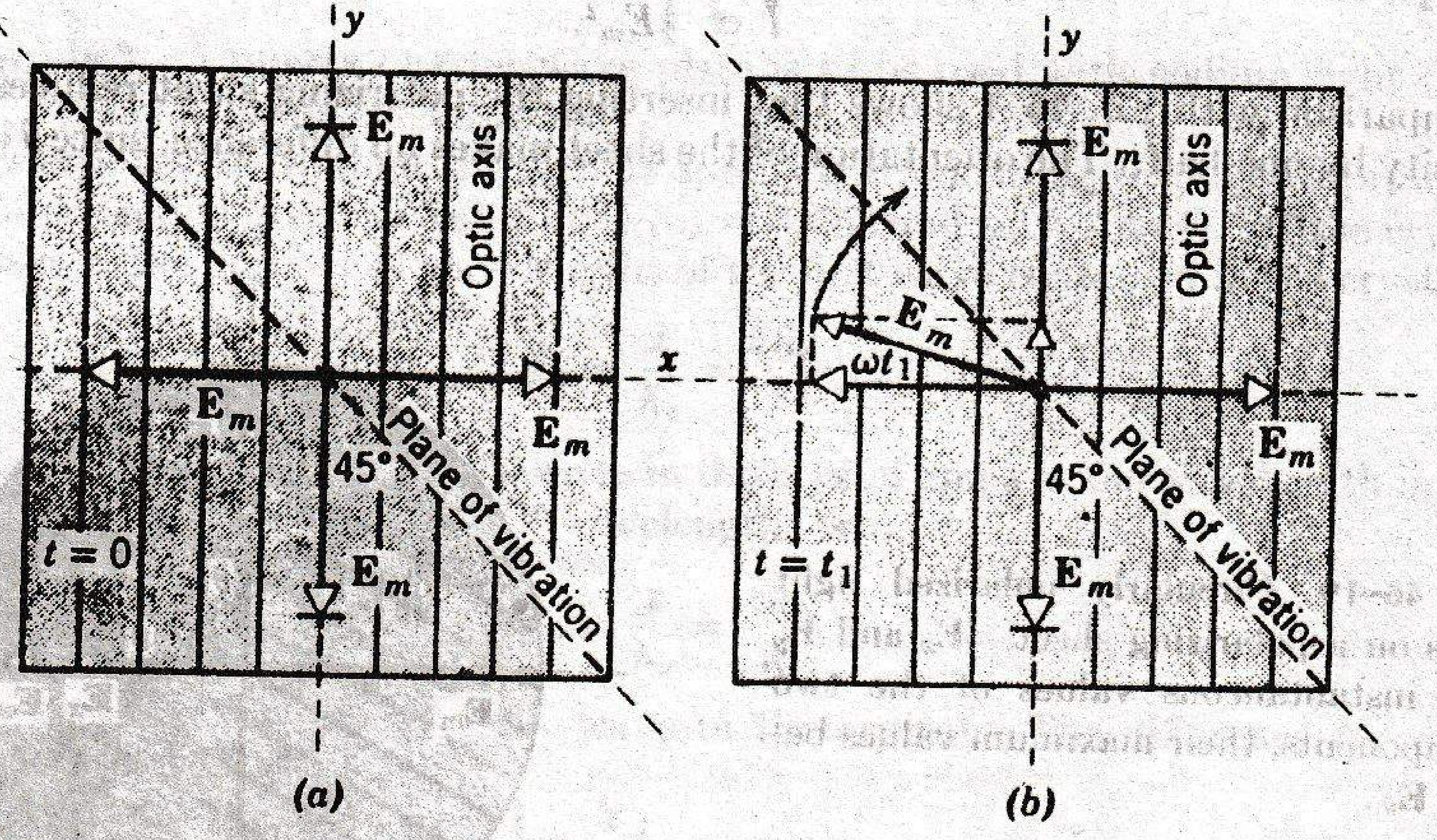

Fig. 46–20 Plane-polarized light falls from behind on a quarter-plate oriented so that the light emerging from the page is circularly polarized. In this case the electric vector $\mathbf{E}_m$ rotates clockwise as seen by an observer facing the light source.

reaches its maximum value one-fourth of a cycle *later* than E_y does, for, in calcite, wave E_x (the *o*-wave) travels *more slowly* than wave E_y (the *e*-wave).

To decide the direction of rotation, let us locate the tip of the rotating electric vector at two instants of time, (*a*) $t = 0$ and (*b*) a short time t_1 later chosen so that ωt_1 is a small angle. At $t = 0$ the coordinates of the tip of the rotating vector (see Fig. 46-20*a*) are

$$E_y = 0 \quad \text{and} \quad E_x = -E_m.$$

At $t = t_1$ these coordinates become, approximately,

$$E_y = E_m \sin \omega t_1 \cong E_m(\omega t_1)$$

$$E_x = -E_m \cos \omega t_1 \cong -E_m.$$

Figure 46-20*b* shows that the vector that represents the emerging circular polarized light is rotating clockwise; by convention such light is called *right-circularly polarized*, the observer always being considered to face the light source.

If the plane of vibration of the incident light in Fig. 46-20 is rotated through $\pm 90°$, the emerging light will be *left-circularly polarized*. ◄

46-6 Angular Momentum of Light

That light waves can deliver *linear momentum* to an absorbing screen or to a mirror is in accord with classical electromagnetism, with quantum physics, and with experiment. The facts of circular polarization suggest that light so polarized might also have *angular* momentum associated with it. This is indeed the case; once again the prediction is in accord with classical electromagnetism and with quantum physics. Experimental proof was provided in 1936 by Beth, who showed that when circularly polarized light is produced in a doubly refracting slab the slab experiences a reaction torque.

The angular momentum carried by light plays a vital role in understanding the emission of light from atoms and of γ-rays from nuclei. If light carries away angular momentum as it leaves the atom, the angular momentum of the residual atom must change by exactly the amount carried away; otherwise the angular momentum of the isolated system *atom plus light* will not be conserved.

Classical and quantum theory both predict that if a beam of circularly polarized light is completely absorbed by an object on which it falls, an angular momentum given by

$$L = \frac{U}{\omega} \tag{46-5}$$

is transferred to the object, where U is the amount of absorbed energy and ω the angular frequency of the light. The student should verify that the dimensions in Eq. 46-5 are consistent.

46-7 Scattering of Light

A light wave, falling on a transparent solid, causes the electrons in the solid to oscillate periodically in response to the time-varying electric vector of the incident wave. The wave that travels through the medium is the resultant of the incident wave and of the radiations from the oscillating electrons. The resultant wave has a maximum intensity in the direction of the incident beam, falling off rapidly on either side. The lack of sideways scattering, which would be essentially complete in a large "perfect" crystal, comes about because the oscillating charges in the medium act cooperatively or coherently.

When light passes through a gas, we find much more sideways scattering. The oscillating electrons in this case, being separated by relatively large distances and not being bound together in a rigid structure, act independently rather than cooperatively. Thus the rigid cancellation of wave disturbances that are not in the forward direction is less likely to occur; there is more sideways scattering.

Light scattered sideways from a gas can be wholly or partially polarized, even though the incident light is unpolarized. Figure 46–21 shows an unpolarized beam moving upward on the page and striking a gas atom at *a*. The electrons at *a* will oscillate in response to the electric components of the incident wave, their motion being equivalent to two oscillating dipoles whose axes are represented by the arrow and the dot at *a*. An oscillating dipole does not radiate along its own line of action. Thus an observer at *b* would receive no radiation from the dipole represented by the arrow at *a*. The radiation reaching him would come entirely from the dipole represented by the dot at *a*; thus this radiation would be plane-polarized, the plane of vibration passing through the line *ab* and being normal to the page.

Observers at *c* and *d* would detect partially polarized light, since the dipole represented by the arrow at *a* would radiate somewhat in these directions. Observers viewing the transmitted or the back-scattered light would not detect any polarization effects because both dipoles at *a* would radiate equally in these two directions.

A familiar example is the scattering of sunlight by the molecules of the earth's atmosphere. If the atmosphere were not present, the sky would appear black except when we looked directly at the sun. This has been verified

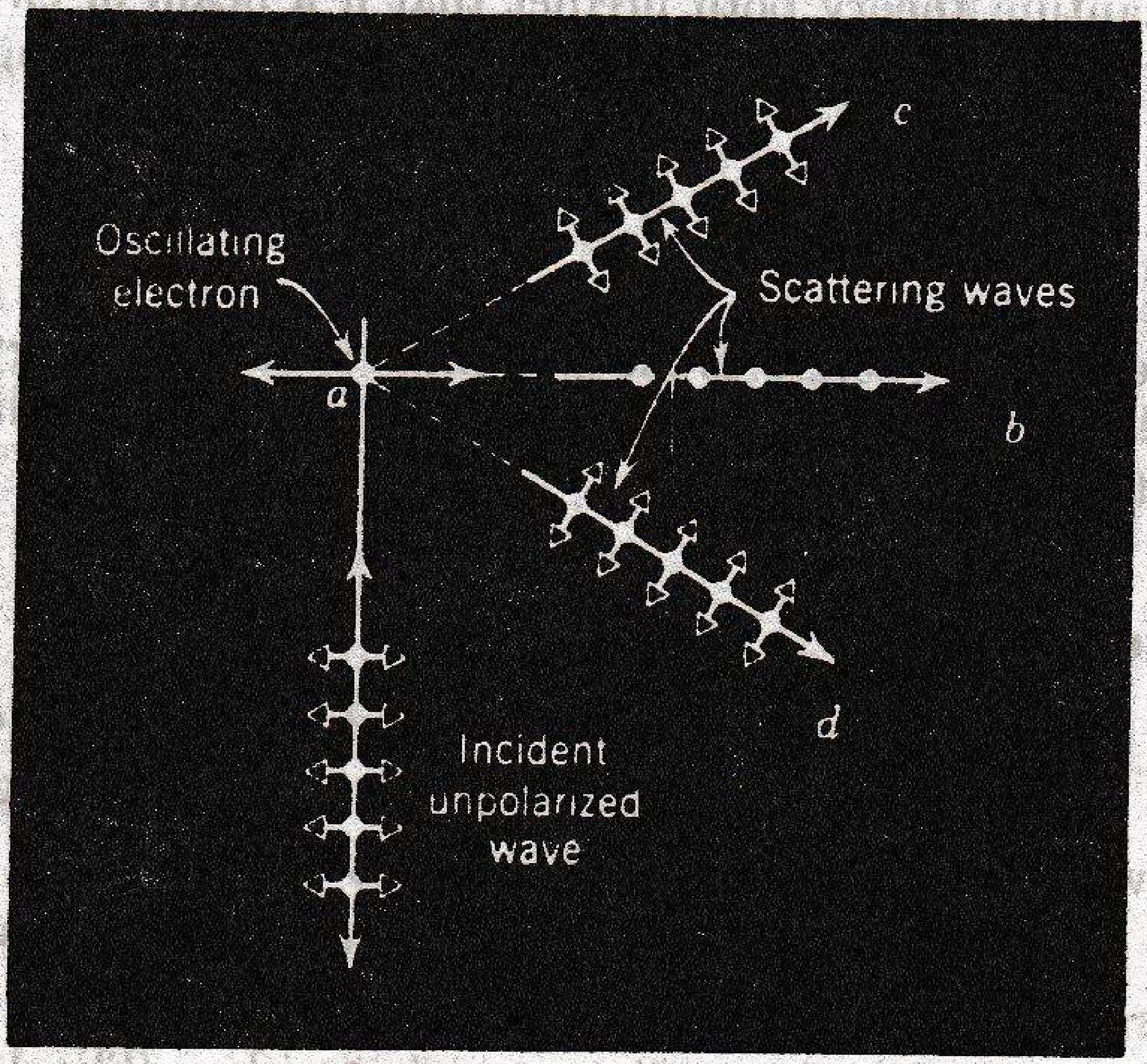

Fig. 46–21 Light is polarized either partially (*c* and *d*) or completely (*b*) by scattering from a gas molecule at *a*.

by measurements made in rockets and satellites above the atmosphere. We can easily check with a polarizer that the light from the cloudless sky is at least partially polarized. This fact is used in polar exploration in the so-called *solar compass*. In this device we establish direction by noting the nature of the polarization of the scattered sunlight. As is well known, magnetic compasses are not useful in these regions. It has been learned * that bees orient themselves in their flights between their hive and the pollen sources by means of polarization of the light from the sky; bees' eyes contain built-in polarization-sensing devices.

It still remains to be explained why the light scattered from the sky is predominantly blue and why the light received directly from the sun—particularly at sunset when the length of the atmosphere that it must traverse is greatest—is red. The cross section of an atom or molecule for light scattering depends on the wavelength, blue light being scattered more effectively than red light. Since the blue light is largely scattered, the transmitted light will have the color of normal sunlight with the blues largely removed; it is therefore more reddish in appearance.

The fact that the scattering cross section for blue light is higher than that for red light can be made reasonable. An electron in an atom or molecule is bound there by strong restoring forces. It has a definite natural frequency, like a small mass suspended in space by an assembly of springs. The natural frequency for electrons in atoms and molecules is usually in a region corresponding to violet or ultraviolet light. When light is allowed to fall on such bound electrons, it sets up forced oscillations at the frequency of the incident light beam. In mechanical resonant systems it is possible to "drive" the system most effectively if we impress on it an external force whose frequency is as close as possible to that of the natural resonant frequency. In the case of light the blue is closer to the natural resonant frequency of the bound electron than is the red light. Therefore, we would expect the blue light to be more effective in causing the electron to oscillate, and thus it will be more effectively scattered.

46-8 Double Scattering

When X-rays were discovered in 1898, there was much speculation whether they were waves or particles. In 1906 they were established as transverse waves by Charles Glover Barkla (1877–1944) by means of a polarization experiment.

When the unpolarized X-rays strike scattering block S_1 in Fig. 46–22, they set the electrons into oscillatory motion. The considerations of the preceding section require that the X-rays scattered toward the second block be plane-polarized as shown in the figure. Let this wave be scattered from the second scattering block, and let us examine the radiation scattered from it by rotating a detector D in a plane at right angles to the line joining the blocks. The electrons will oscillate parallel to each other, and the positions of maximum and zero intensity will be as shown. A plot of detector reading as a function of the angle ϕ supports the hypothesis that X-rays are transverse waves. If the X-rays were a stream of particles or a longitudinal wave, these effects could by no means be so readily understood. Thus Barkla's important experiment established that X-rays are a part of the electromagnetic spectrum.

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* See *Scientific American*, July 1955, and *Bees: Their Vision, Chemical Sense, and Language*, K. von Frisch, Cornell University Press, 1950.

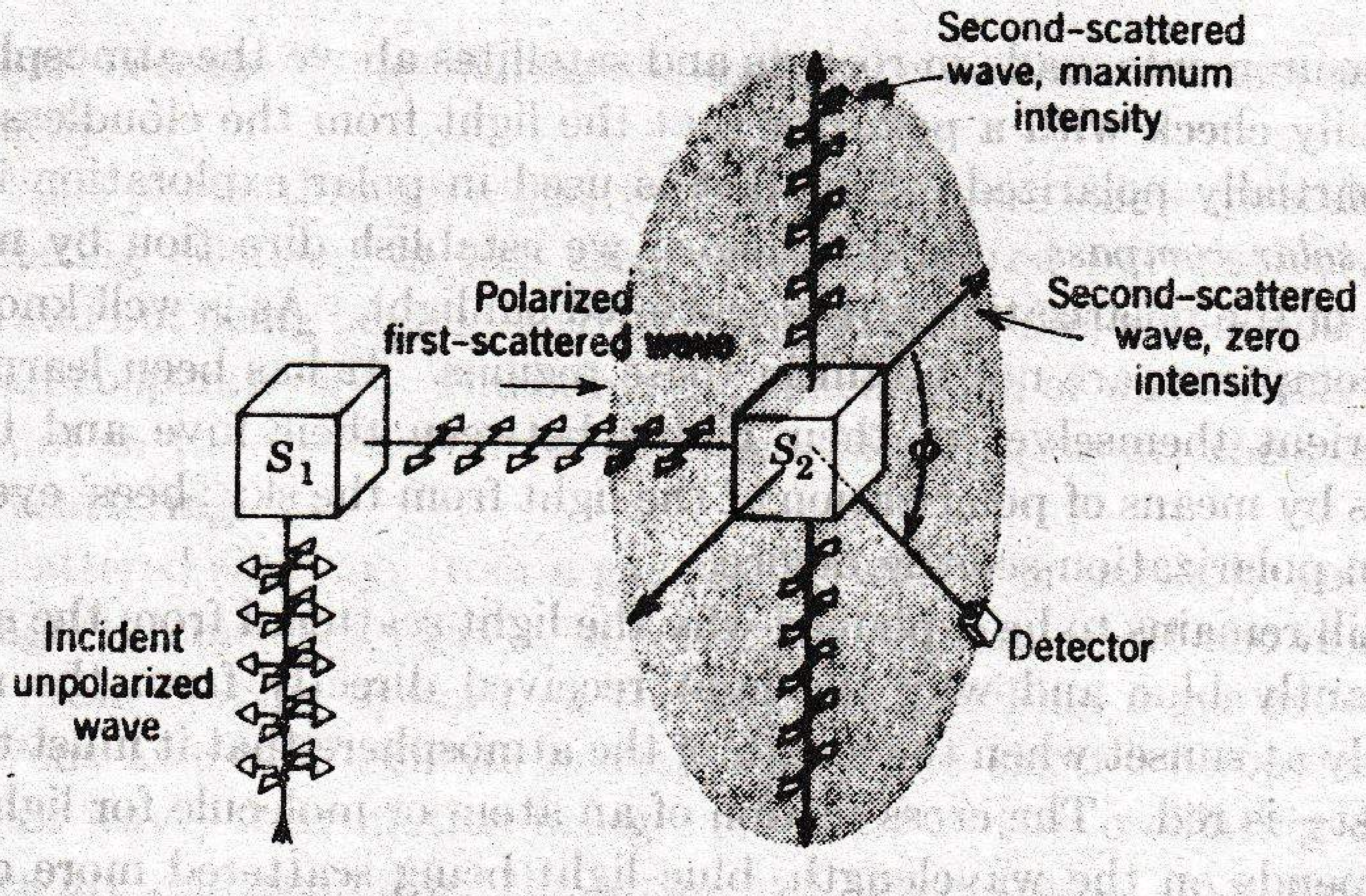

Fig. 46–22 A double-scattering experiment used by Barkla to show that X-rays are transverse waves.

In his later studies the student will learn that beams of particles such as electrons, protons, and pions can be viewed as waves. Scattering (including double scattering) techniques are often used to investigate the polarization characteristics of such beams.

QUESTIONS

1. Why do sunglasses made of polarizing materials have a marked advantage over those that simply depend on absorption effects?

2. Unpolarized light falls on two polarizing sheets so oriented that no light is transmitted. If a third polarizing sheet is placed between them, can light be transmitted?

3. Can polarization by reflection occur if the light is incident on the interface from the side with the higher index of refraction (glass to air, for example)?

4. Is the optic axis of a doubly refracting crystal simply a line or a direction in space? Has it a direction sense, like an arrow? What about the characteristic direction of a polarizing sheet?

5. Devise a way to identify the polarizing direction of a sheet of Polaroid.

6. If ice is doubly refracting (see Table 46–1), why don't we see two images of objects viewed through an ice cube?

7. From Table 46–1, would you expect a quarter-wave plate made from calcite to be thicker than one made from quartz?

8. Does the *e*-wave in doubly refracting crystals always travel at a speed given by c/n_e?

9. In Fig. 46–14*a* and *b* describe qualitatively what happens if the incident beam falls on the crystal with an angle of incidence that is not zero. Assume in each case that the incident beam remains in the plane of the figure.

10. Devise a way to identify the direction of the optic axis in a quarter-wave plate.

11. If plane-polarized light falls on a quarter-wave plate with its plane of vibration making an angle of (*a*) 0° or (*b*) 90° with the axis of the plate, describe the transmitted light. (*c*) If this angle is arbitrarily chosen, the transmitted light is called *elliptically polarized*; describe such light.

12. What would be the action of a *half-wave plate* (that is, a plate twice as thick as a quarter-wave plate) on (*a*) plane-polarized light (assume the plane of vibration to be at 45° to the optic axis of the plate), (*b*) circularly polarized light, and (*c*) unpolarized light?

13. You are given an object which may be (*a*) a disk of grey glass, (*b*) a polarizing sheet, (*c*) a quarter-wave plate, or (*d*) a half-wave plate (see Question 12). How could you identify it?

14. Can a plane-polarized light beam be represented as a sum of two circularly polarized light beams of opposite rotation? What effect has changing the phase of one of the circular components on the resultant beam?

15. How can a right-circularly polarized light beam be transformed into a left-circularly polarized beam?

16. Could (*a*) a radar beam and (*b*) a sound wave in air be circularly polarized?

17. A beam of light is said to be unpolarized, plane-polarized, or circularly polarized. How could you choose among them experimentally?

18. A parallel beam of light is absorbed by an object placed in its path. Under what circumstances will (*a*) linear momentum and (*b*) angular momentum be transferred to the object?

19. When observing a clear sky through a polarizing sheet, one finds that the intensity varies by a factor of two on rotating the sheet. This does not happen when one views a cloud through the sheet. Can you devise an explanation?

PROBLEMS

1. Unpolarized light falls on two polarizing sheets placed one on top of the other. What must be the angle between the characteristic directions of the sheets if the intensity of the transmitted light is (*a*) one-third the maximum intensity of the transmitted beam or (*b*) one-third the intensity of the incident beam? Assume that the polarizing sheet is ideal, that is, that it reduces the intensity of unpolarized light by exactly 50%.

2. An unpolarized beam of light is incident on a group of four polarizing sheets which are lined up so that the characteristic direction of each is rotated by 30° clockwise with respect to the preceding sheet. What fraction of the incident intensity is transmitted?

3. Describe the state of polarization represented by these sets of equations:

$$(a)\quad E_x = E\sin(kz - \omega t)$$
$$E_y = E\cos(kz - \omega t),$$

$$(b)\quad E_x = E\cos(kz - \omega t)$$
$$E_y = E\cos\left(kz - \omega t + \frac{\pi}{4}\right),$$

$$(c)\quad E_x = E\sin(kz - \omega t)$$
$$E_y = -E\sin(kz - \omega t).$$

4. (*a*) At what angle of incidence will the light reflected from water be completely polarized? (*b*) Does this angle depend on the wavelength of the light?

5. Calculate the range of polarizing angles for white light incident on fused quartz. Assume that the wavelength limits are 4000 and 7000 A and use the dispersion curve of Fig. 41-2.

6. A narrow beam of unpolarized light falls on a calcite crystal cut with its optic axis as shown in Fig. 46-23. (*a*) For t = 1.0 cm and for θ_i = 45°, calculate the perpendicular distance between the two emerging rays x and y. (*b*) Which is the *o*-ray and which the *e*-ray? (*c*) What are the states of polarization of the emerging rays? (*d*) Describe what

happens if a polarizer is placed in the incident beam and rotated. (Hint: Inside the crystal the **E**-vector vibrations for one ray are always perpendicular to the optic axis and for the other ray they are always parallel. The two rays are described by the indices n_o and n_e; *in this plane* each ray obeys Snell's law.)

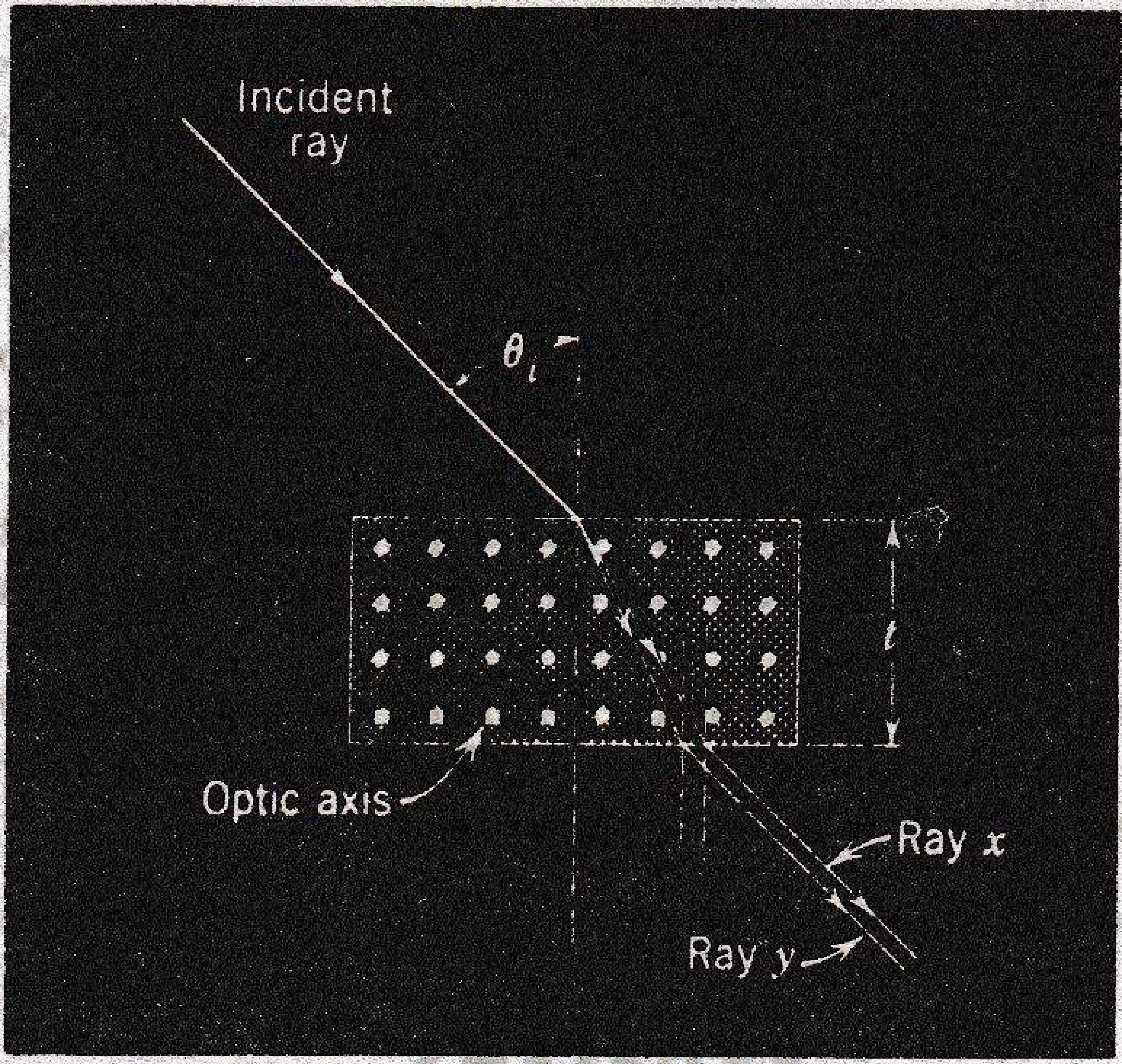

Fig. 46–23

7. A prism is cut from calcite so that the optic axis is parallel to the prism edge as shown in Fig. 46–24. Describe how such a prism might be used to measure the two principal indices of refraction for calcite. (Hint: See hint in Problem 6; see also Example 3, Chapter 41.)

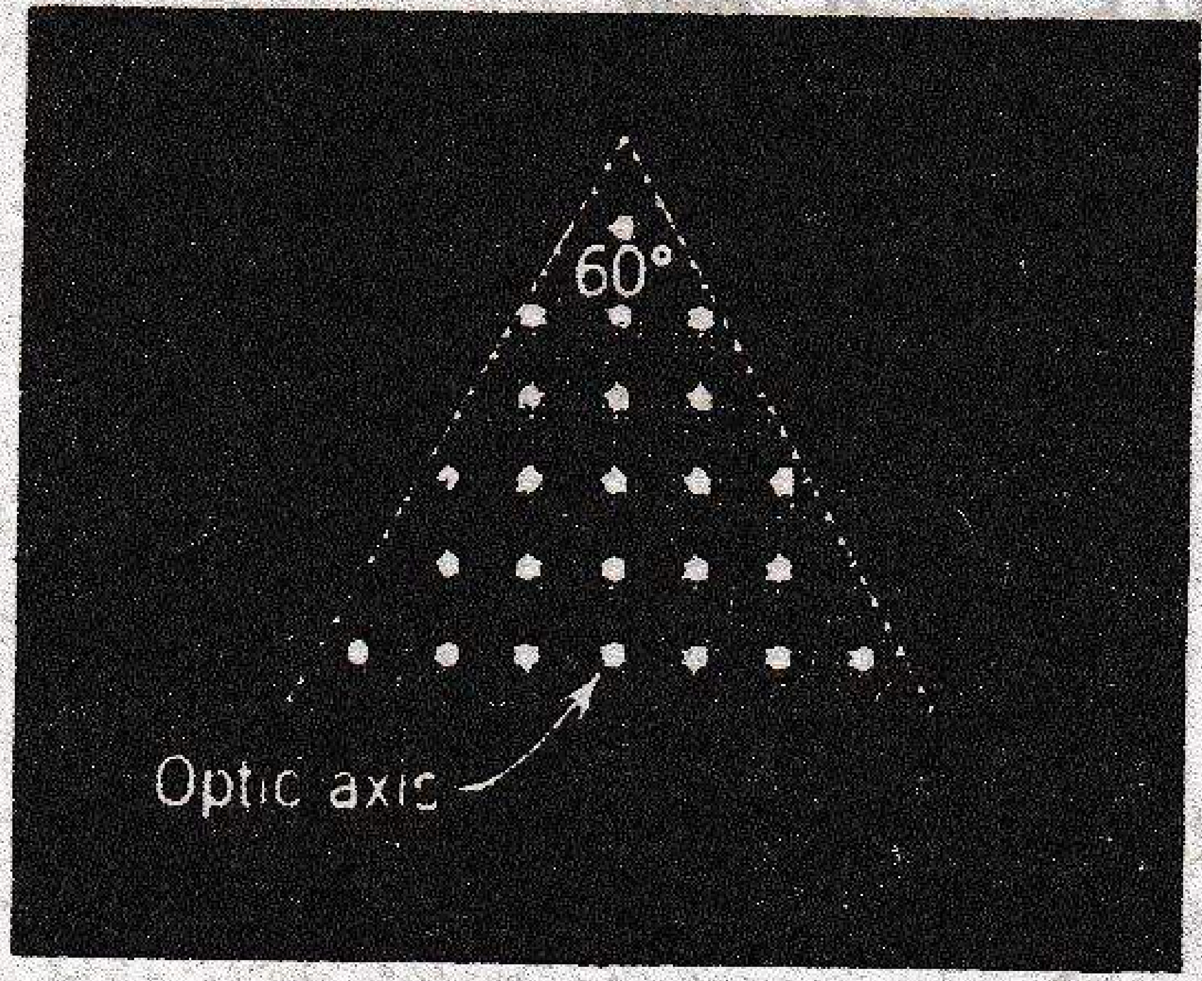

Fig. 46–24

8. How thick must a sheet of mica be if it is to form a quarter-wave plate for yellow light (λ = 5890 A)? Mica cleaves in such a way that the appropriate indices of refraction, for transmission at right angles to the cleavage plane, are 1.6049 and 1.6117.

9. Prove that two plane-polarized light waves of equal amplitude, their planes of vibration being at right angles to each other, cannot produce interference effects. (Hint: Prove that the intensity of the resultant light wave, averaged over one or more cycles of oscillation, is the same no matter what phase difference exists between the two waves.)

10. Show that in a parallel beam of circularly polarized light the angular momentum per unit volume L_v is given by

$$L_v = \frac{P}{\omega c},$$

where P is the power per unit area (watts/cm^2, say) of the beam. Start from Eq. 46–5.

11. Assume that a parallel beam of circularly polarized light whose intensity is 100 watts is absorbed by an object. At what rate is angular momentum transferred to the object? If the object is a flat disk of diameter 5.0 mm and mass 1.0×10^{-2} gm, after how long a time (assuming it is free to rotate about its axis) would it attain an angular speed of 1.0 rev/sec? Assume a wavelength of 5000 A.