Chapter 5
Fields of Force

"Okay. Your duties are as follows: Get Breen. I don't care how you get him, but get him soon. That faker! He posed for twenty years as a scientist without ever being apprehended. Well, I'm going to do some apprehending that'll make all previous apprehending look like no apprehension at all. You with me?"

"Yes," said Battle, very much confused. "What's that thing you have?"

"Piggy-back heat-ray. You transpose the air in its path into an unstable isotope which tends to carry all energy as heat. Then you shoot your juice light, or whatever along the isotopic path and you burn whatever's on the receiving end. You want a few?"

"No," said Battle. "I have my gats. What else have you got for offense and defense?" Underbottam opened a cabinet and proudly waved an arm. "Everything," he said.

"Disintegraters, heat-rays, bombs of every type. And impenetrable shields of energy, massive and portable. What more do I need?"

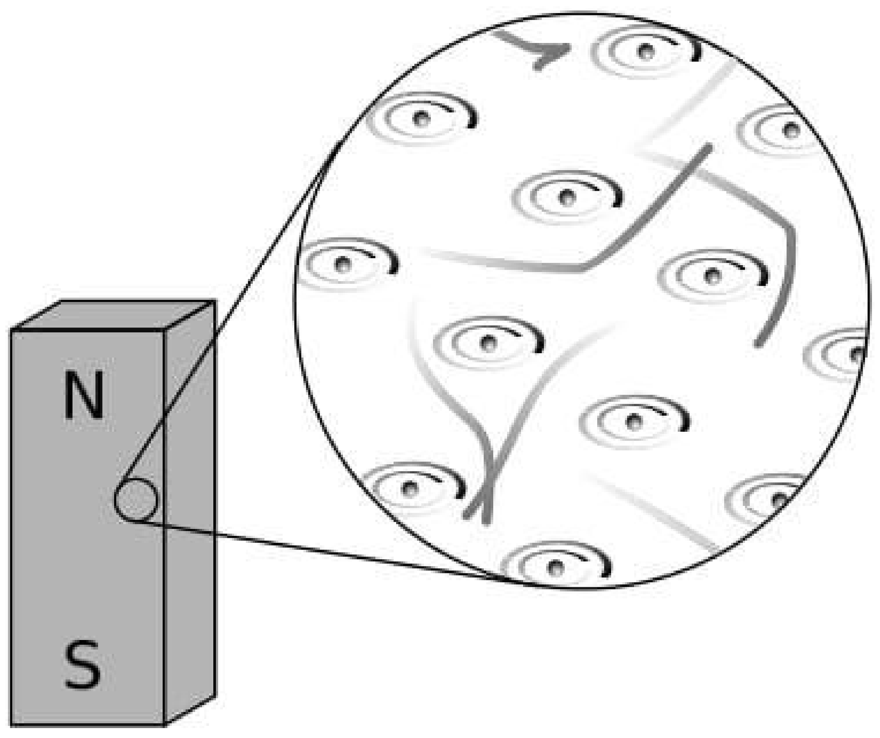
From THE REVERSIBLE REVOLUTIONS by Cecil Corwin, Cosmic Stories, March 1941. Art by Morey, Bok, Kyle, Hunt, Forte. Copyright expired.

Cutting-edge science readily infiltrates popular culture, though sometimes in garbled form. The Newtonian imagination populated the universe mostly with that nice solid stuff called matter, which was made of little hard balls called atoms. In the early twentieth century, consumers of pulp fiction and popularized science began to hear of a new image of the universe, full of x-rays, N-rays, and Hertzian waves. What they were beginning to soak up through their skins was a drastic revision of Newton's concept of a universe made of chunks of matter which happened to interact via forces. In the newly emerging picture, the universe was *made* of force, or, to be more technically accurate, of ripples in universal fields of force. Unlike the average reader of Cosmic Stories in 1941, you now possess enough technical background to understand what a "force field" really is.

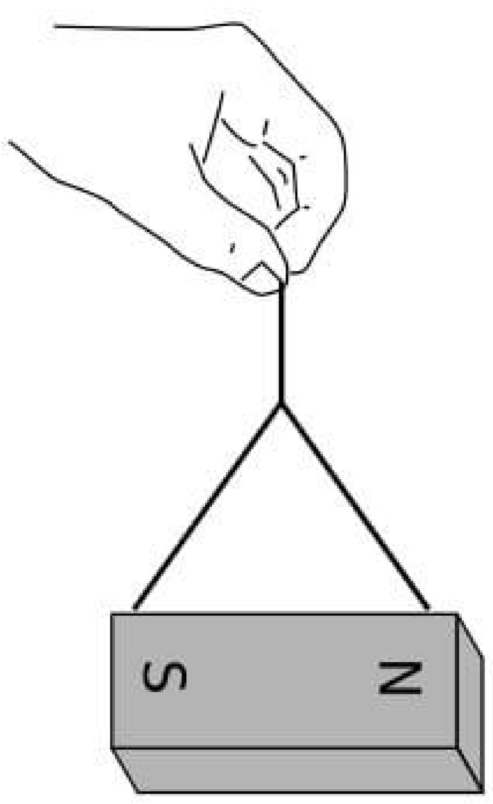
5.1 Why fields?

Time delays in forces exerted at a distance

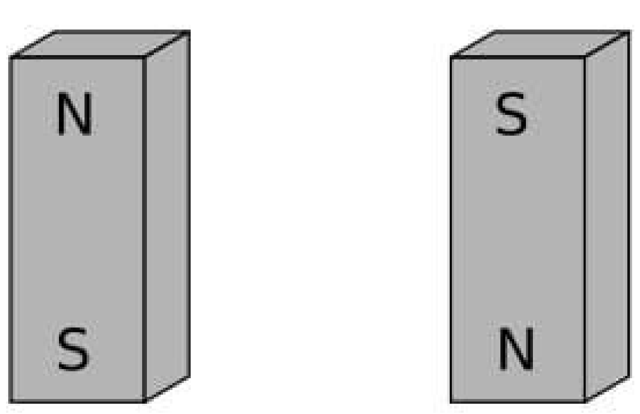
What convinced physicists that they needed this new concept of a field of force? Although we have been dealing mostly with electrical forces, let's start with a magnetic example. (In fact the main reason I've delayed a detailed discussion of magnetism for so long is that mathematical calculations of magnetic effects are handled much more easily with the concept of a field of force.) First a little background leading up to our example. A bar magnet, a, has an axis about which many of the electrons' orbits are oriented. The earth itself is also a magnet, although not a bar-shaped one. The interaction between the earth-magnet and the bar magnet, b, makes them want to line up their axes in opposing directions (in other words such that their electrons rotate in parallel planes, but with one set rotating clockwise and the other counterclockwise as seen looking along the axes). On a smaller scale, any two bar magnets placed near each other will try to align themselves head-to-tail, c.



a / A bar magnet's atoms are (partially) aligned.



b / A bar magnet interacts with our magnetic planet.



c / Magnets aligned north-south.

Now we get to the relevant example. It is clear that two people separated by a paper-thin wall could use a pair of bar magnets to signal to each other. Each person would feel her own magnet trying to twist around in response to any rotation performed by the other person's magnet. The practical range of communication would be very short for this setup, but a sensitive electrical apparatus could pick up magnetic signals from much farther away. In fact, this is not so different from what a radio does: the electrons racing up and down the transmitting antenna create forces on the electrons in the distant receiving antenna. (Both magnetic and electric forces are involved in real radio signals, but we don't need to worry about that yet.)

A question now naturally arises as to whether there is any time delay in this kind of communication via magnetic (and electric) forces. Newton would have thought not, since he conceived of physics in terms of instantaneous action at a distance. We now know, however, that there is such a time delay. If you make a long-distance phone call that is routed through a communications satellite, you should easily be able to detect a delay of about half a second over the signal's round trip of 50,000 miles. Modern measurements have shown that electric, magnetic, and gravitational forces all travel at the speed of light, 3×10^8 m/s.[1] (In fact, we will soon discuss how light itself is made of electricity and magnetism.)

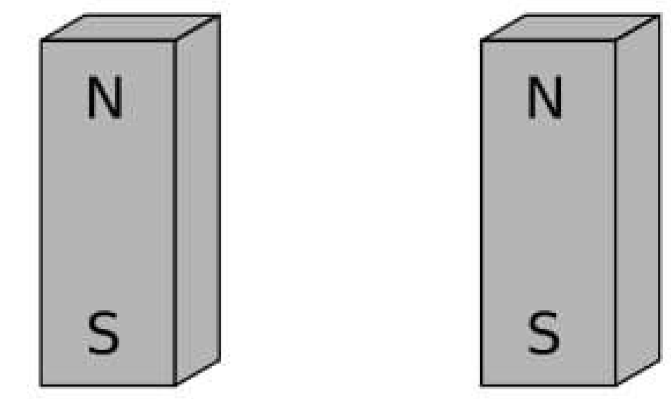
If it takes some time for forces to be transmitted through space, then apparently there is some *thing* that travels *through* space. The fact that the phenomenon travels outward at the same speed in all directions strongly evokes wave metaphors such as ripples on a pond.

More evidence that fields of force are real: they carry energy.

The smoking-gun argument for this strange notion of traveling force ripples comes from the fact that they carry energy.

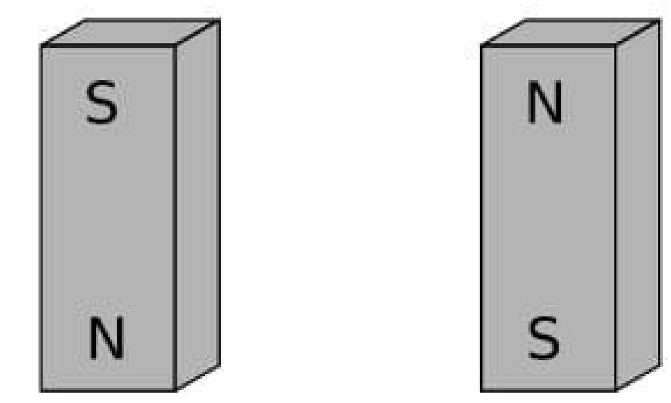
[1] As discussed in book 6 of this series, one consequence of Einstein's theory of relativity is that material objects can never move faster than the speed of light. It can also be shown that signals or information are subject to the same limit.

First suppose that the person holding the bar magnet on the right decides to reverse hers, resulting in configuration d. She had to do mechanical work to twist it, and if she releases the magnet, energy will be released as it flips back to c. She has apparently stored energy by going from c to d. So far everything is easily explained without the concept of a field of force.



d / The second magnet is reversed.

But now imagine that the two people start in position c and then simultaneously flip their magnets extremely quickly to position e, keeping them lined up with each other the whole time. Imagine, for the sake of argument, that they can do this so quickly that each magnet is reversed while the force signal from the other is still in transit. (For a more realistic example, we'd have to have two radio antennas, not two magnets, but the magnets are easier to visualize.) During the flipping, each magnet is still feeling the forces arising from the way the other magnet *used* to be oriented. Even though the two magnets stay aligned during the flip, the time delay causes each person to feel resistance as she twists her magnet around. How can this be? Both of them are apparently doing mechanical work, so they must be storing magnetic energy somehow. But in the traditional Newtonian conception of matter interacting via instantaneous forces at a distance, interaction energy arises from the relative positions of objects that are interacting via forces. If the magnets never changed their orientations relative to each other, how can any magnetic energy have been stored?

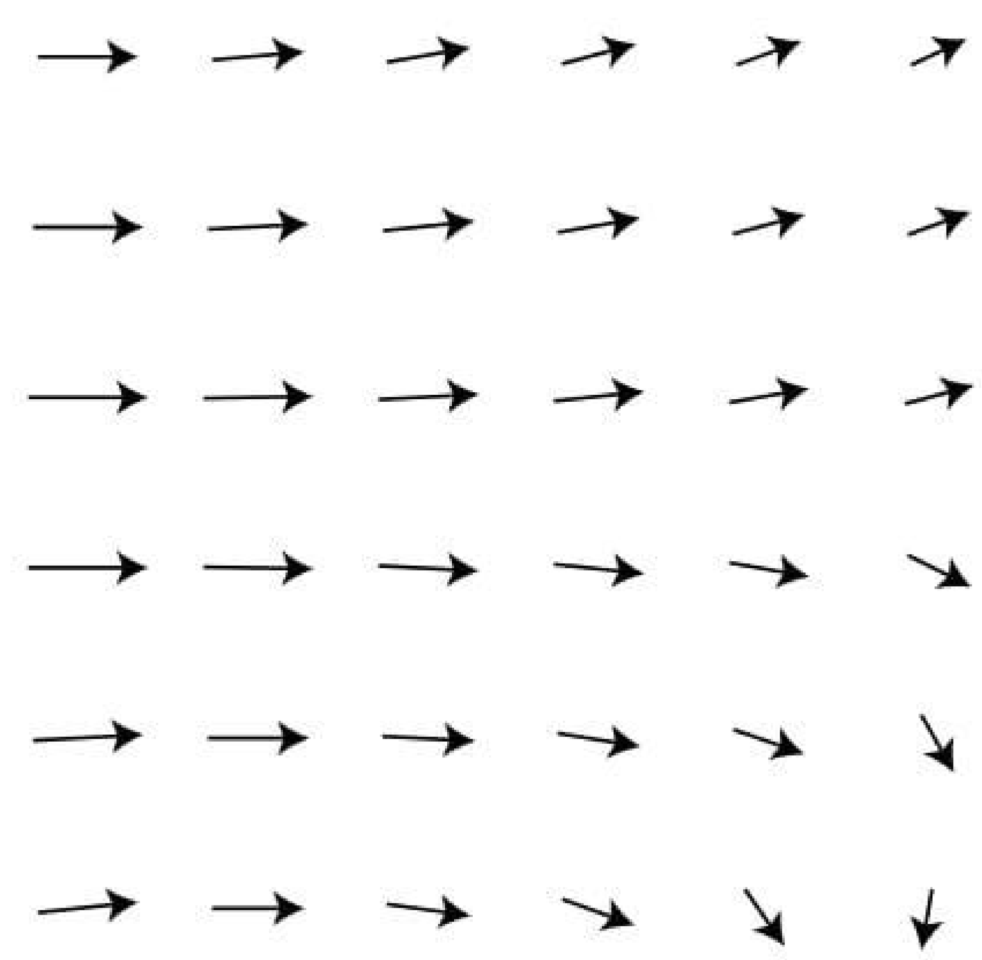


e / Both magnets are reversed.

The only possible answer is that the energy must have gone into the magnetic force ripples crisscrossing the space between the magnets. Fields of force apparently carry energy across space, which is strong evidence that they are real things.

This is perhaps not as radical an idea to us as it was to our ancestors. We are used to the idea that a radio transmitting antenna consumes a great deal of power, and somehow spews it out into the universe. A person working around such an antenna needs to be careful not to get too close to it, since all that energy can easily cook flesh (a painful phenomenon known as an "RF burn").

5.2 The gravitational field


f / The wind patterns in a certain area of the ocean could be charted in a "sea of arrows" representation like this. Each arrow represents both the wind's strength and its direction at a certain location.

Given that fields of force are real, how do we define, measure, and calculate them? A fruitful metaphor will be the wind patterns experienced by a sailing ship. Wherever the ship goes, it will feel a certain amount of force from the wind, and that force will be in a certain direction. The weather is ever-changing, of course, but for now let's just imagine steady wind patterns. Definitions in physics are operational, i.e., they describe how to measure the thing being defined. The ship's captain can measure the wind's "field of force" by going to the location of interest and determining both the direction of the wind and the strength with which it is blowing. Charting all these measurements on a map leads to a depiction of the field of wind force like the one shown in the figure. This is known as the "sea of arrows" method of visualizing a field.

Now let's see how these concepts are applied to the fundamental force fields of the universe. We'll start with the gravitational field, which is the easiest to understand. As with the wind patterns, we'll start by imagining gravity as a static field, even though the existence of the tides proves that there are continual changes in the gravity field in our region of space. Defining the direction of the gravitational field is easy enough: we simply go to the location of interest and measure the direction of the gravitational force on an object, such as a weight tied to the end of a string.

But how should we define the strength of the gravitational field? Gravitational forces are weaker on the moon than on the earth, but we cannot specify the strength of gravity simply by giving a certain number of newtons. The number of newtons of gravitational force depends not just on the strength of the local gravitational field but also on the mass of the object on which we're testing gravity, our "test mass." A boulder on the moon feels a stronger gravitational force than a pebble on the earth. We can get around this problem by defining the strength of the gravitational field as the force acting on an object, *divided by the object's mass.*

definition of the gravitational field
The gravitational field vector, $\mathbf{g}$, at any location in space is found by placing a test mass m_t at that point. The field vector is then given by $\mathbf{g} = \mathbf{F}/m_t$, where $\mathbf{F}$ is the gravitational force on the test mass.

The magnitude of the gravitational field near the surface of the earth is about 9.8 N/kg, and it's no coincidence that this number looks familiar, or that the symbol $\mathbf{g}$ is the same as the one for gravitational acceleration. The force of gravity on a test mass will equal $m_t\mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration. Dividing by m_t simply gives the gravitational acceleration. Why define a new name and new units for the same old quantity? The main reason is that it prepares us with the right approach for defining other fields.

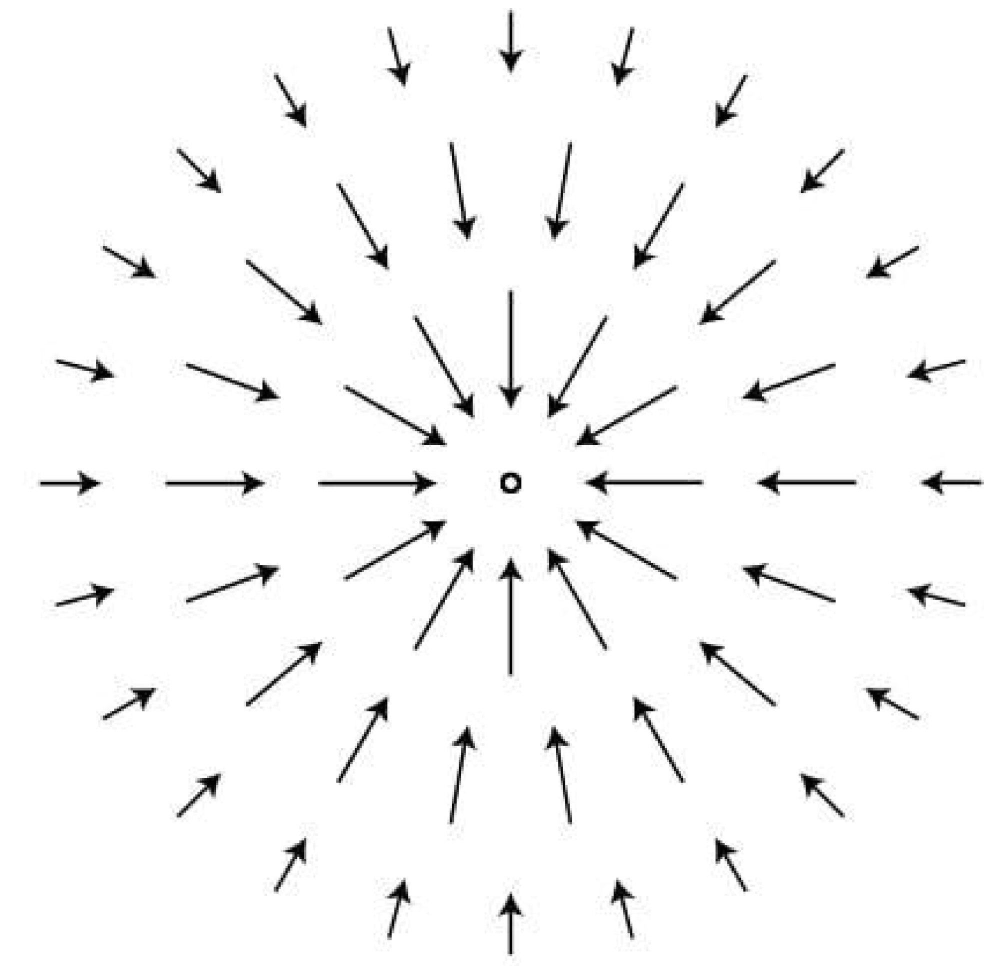
The most subtle point about all this is that the gravitational field tells us about what forces *would* be exerted on a test mass by the earth, sun, moon, and the rest of the universe, *if* we inserted a test mass at the point in question. The field still exists at all the places where we didn't measure it.

Gravitational field of the earth — *example 1*

▷ What is the magnitude of the earth's gravitational field, in terms of its mass, M, and the distance r from its center?

▷ Substituting $|\mathbf{F}| = GMm_t/r^2$ into the definition of the gravitational field, we find $|\mathbf{g}| = GM/r^2$. This expression could be used for the field of any spherically symmetric mass distribution, since the equation we assumed for the gravitational force would apply in any such case.

Sources and sinks

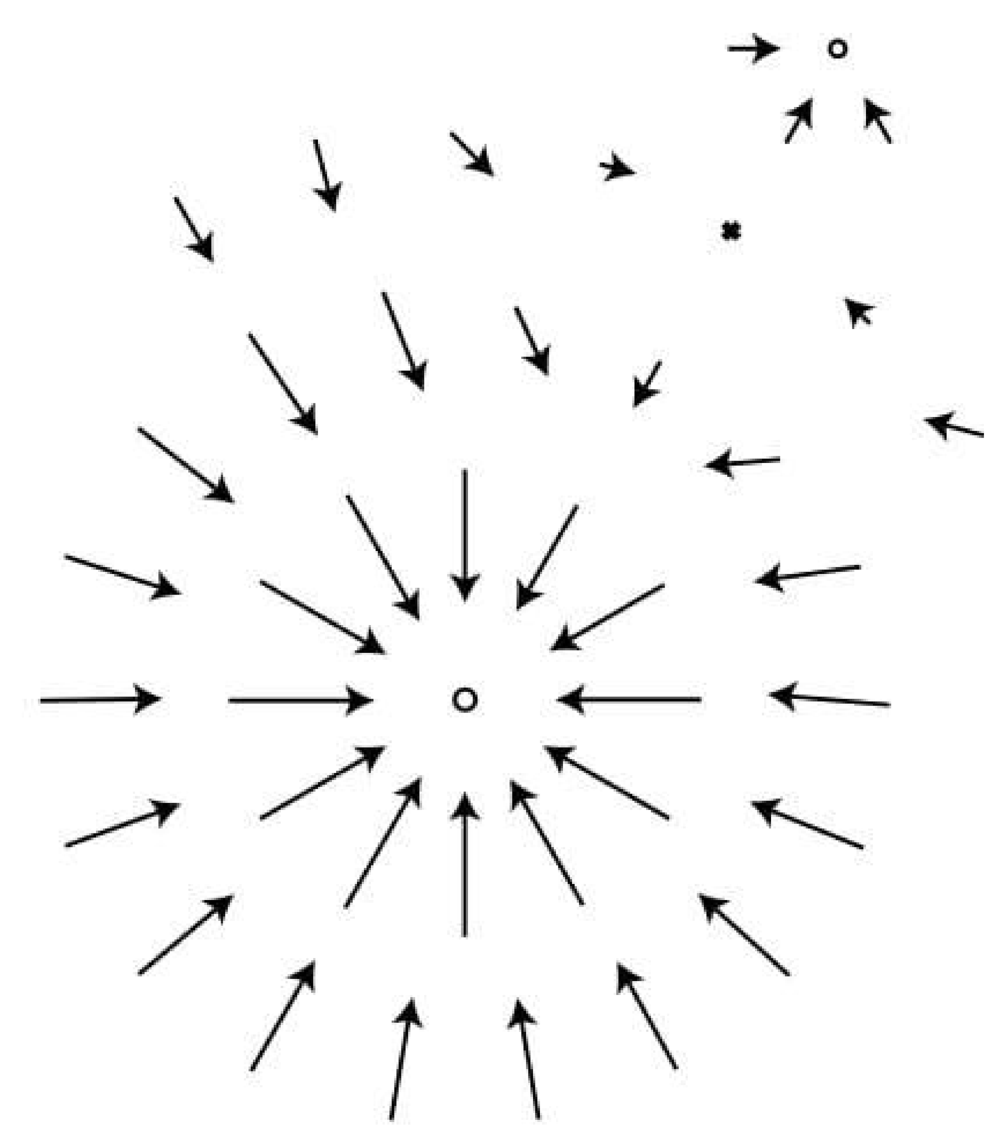


g / The gravitational field surrounding a clump of mass such as the earth.

If we make a sea-of-arrows picture of the gravitational fields surrounding the earth, g, the result is evocative of water going down a drain. For this reason, anything that creates an inward-pointing field around itself is called a sink. The earth is a gravitational sink. The term "source" can refer specifically to things that make outward fields, or it can be used as a more general term for both "outies" and "innies." However confusing the terminology, we know that gravitational fields are only attractive, so we will never find a region of space with an outward-pointing field pattern.

Knowledge of the field is interchangeable with knowledge of its sources (at least in the case of a static, unchanging field). If aliens saw the earth's gravitational field pattern they could immediately infer the existence of the planet, and conversely if they knew the mass of the earth they could predict its influence on the surrounding gravitational field.

Superposition of fields



h / The gravitational fields of the earth and moon superpose. Note how the fields cancel at one point, and how there is no boundary between the interpenetrating fields surrounding the two bodies.

A very important fact about all fields of force is that when there is more than one source (or sink), the fields add according to the rules of vector addition. The gravitational field certainly will have this property, since it is defined in terms of the force on a test mass, and forces add like vectors. Superposition is an important characteristics of waves, so the superposition property of fields is consistent with the idea that disturbances can propagate outward as waves in a field.

Reduction in gravity on Io due to Jupiter's gravity — *example 2*

▷ The average gravitational field on Jupiter's moon Io is 1.81 N/kg. By how much is this reduced when Jupiter is directly overhead? Io's orbit has a radius of 4.22×10^8 m, and Jupiter's mass is 1.899×10^{27} kg.

▷ By the shell theorem, we can treat the Jupiter as if its mass was all concentrated at its center, and likewise for Io. If we visit Io and land at the point where Jupiter is overhead, we are on the same line as these two centers, so the whole problem can be treated one-dimensionally, and vector addition is just like scalar addition. Let's use positive numbers for downward fields (toward the center of Io) and negative for upward ones. Plugging the appropriate data into the expression derived in example 1, we find that the Jupiter's contribution to the field is -0.71 N/kg. Superposition says that we can find the actual gravitational field by adding up the fields created by Io and Jupiter: $1.81 - 0.71$ N/kg = 1.1 N/kg. You might think that this reduction would create some spectacular effects, and make Io an exciting tourist destination. Actually you would not detect any difference if you flew from one side of Io to the other. This is because your body and Io both experience Jupiter's gravity, so you follow the same orbital curve through the space around Jupiter.

Gravitational waves

A source that sits still will create a static field pattern, like a steel ball sitting peacefully on a sheet of rubber. A moving source will create a spreading wave pattern in the field, like a bug thrashing on the surface of a pond. Although we have started with the gravitational field as the simplest example of a static field, stars and planets do more stately gliding than thrashing, so gravitational waves are not easy to detect. Newton's theory of gravity does not describe gravitational waves, but they are predicted by Einstein's general theory of relativity. J.H. Taylor and R.A. Hulse were awarded the Nobel Prize in 1993 for giving indirect evidence that Einstein's waves actually exist. They discovered a pair of exotic, ultra-dense stars called neutron stars orbiting one another very closely, and showed that they were losing orbital energy at the rate predicted by Einstein's theory.

i / The part of the LIGO gravity wave detector at Hanford Nuclear Reservation, near Richland, Washington. The other half of the detector is in Louisiana.

A Caltech-MIT collaboration has built a pair of gravitational wave detectors called LIGO to search for more direct evidence of gravitational waves. Since they are essentially the most sensitive vibration detectors ever made, they are located in quiet rural areas, and signals will be compared between them to make sure that they were not due to passing trucks. The project began operating at full sensitivity in 2005, and is now able to detect a vibration that causes a change of 10^{-18} m in the distance between the mirrors at the ends of the 4-km vacuum tunnels. This is a thousand times less than the size of an atomic nucleus! There is only enough funding to keep the detectors operating for a few more years, so the physicists can only hope that during that time, somewhere in the universe, a sufficiently violent cataclysm will occur to make a detectable gravitational wave. (More accurately, they want the wave to arrive in our solar system during that time, although it will have been produced millions of years before.)

5.3 The electric field

Definition

The definition of the electric field is directly analogous to, and has the same motivation as, the definition of the gravitational field:

> **definition of the electric field**
> The electric field vector, $\mathbf{E}$, at any location in space is found by placing a test charge q_t at that point. The electric field vector is then given by $\mathbf{E} = \mathbf{F}/q_t$, where $\mathbf{F}$ is the electric force on the test charge.

Charges are what create electric fields. Unlike gravity, which is always attractive, electricity displays both attraction and repulsion. A positive charge is a source of electric fields, and a negative one is a sink.

The most difficult point about the definition of the electric field is that the force on a negative charge is in the opposite direction compared to the field. This follows from the definition, since dividing a vector by a negative number reverses its direction. It's as though we had some objects that fell upward instead of down.

j / Example 3.

self-check A

Find an equation for the magnitude of the field of a single point charge Q. ▷ Answer, p. 195

Superposition of electric fields *example 3*

▷ Charges q and $-q$ are at a distance b from each other, as shown in the figure. What is the electric field at the point P, which lies at a third corner of the square?

▷ The field at P is the vector sum of the fields that would have been created by the two charges independently. Let positive x be to the right and let positive y be up.

Negative charges have fields that point at them, so the charge $-q$ makes a field that points to the right, i.e., has a positive x component. Using the answer to the self-check, we have

$$E_{-q,x} = \frac{kq}{b^2}$$
$$E_{-q,y} = 0 \quad .$$

Note that if we had blindly ignored the absolute value signs and plugged in $-q$ to the equation, we would have incorrectly concluded that the field went to the left.

By the Pythagorean theorem, the positive charge is at a distance $\sqrt{2}b$ from P, so the magnitude of its contribution to the field is $E = kq/2b^2$.

Positive charges have fields that point away from them, so the field vector is at an angle of 135° counterclockwise from the x axis.

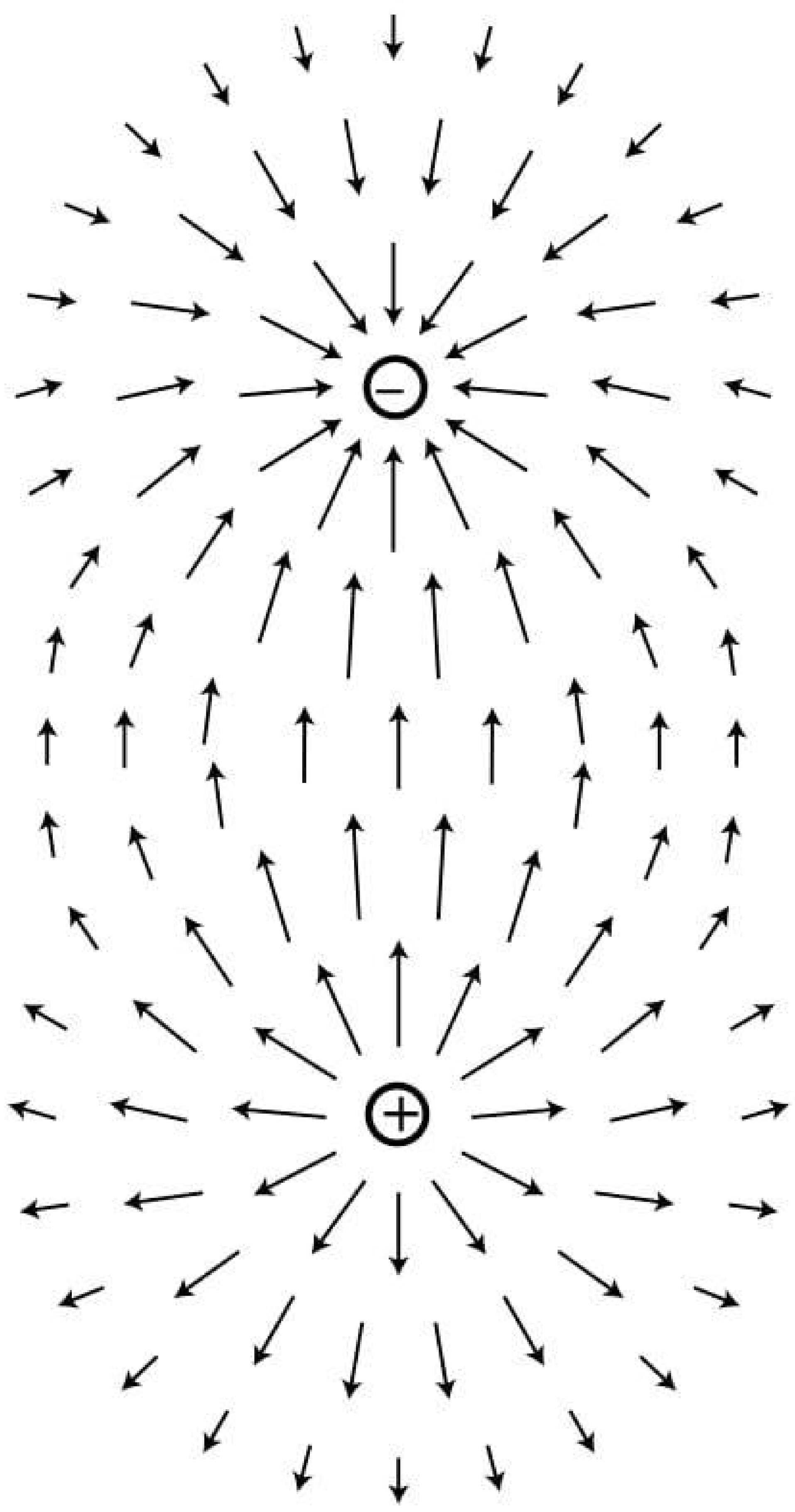
$$E_{q,x} = \frac{kq}{2b^2}\cos 135^\circ = -\frac{kq}{2^{3/2}b^2}$$

$$E_{q,y} = \frac{kq}{2b^2}\sin 135^\circ = \frac{kq}{2^{3/2}b^2}$$

The total field is

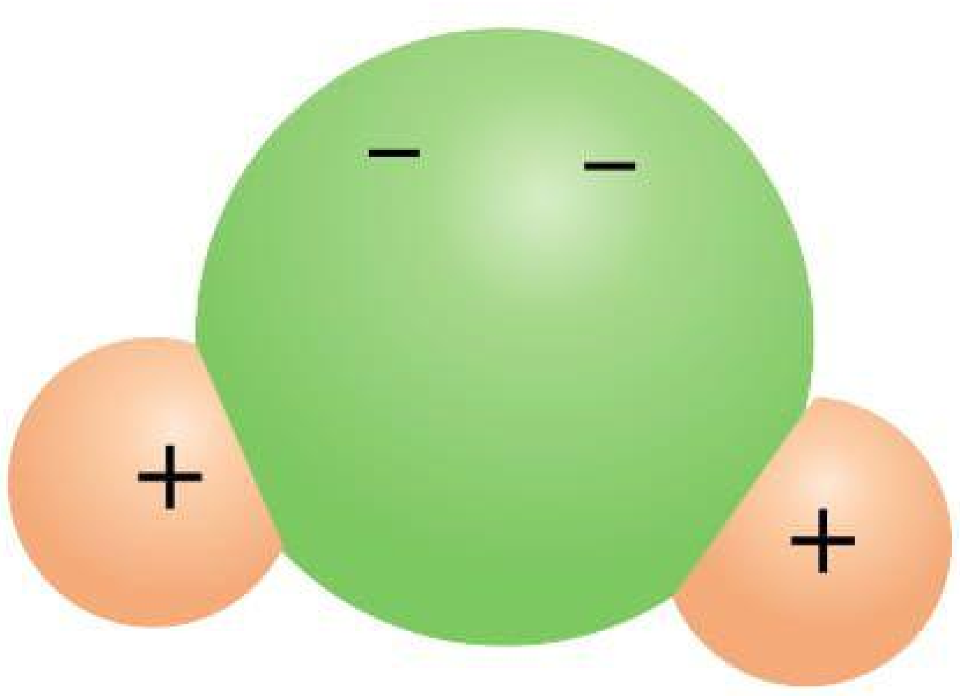
$$E_x = \left(1 - 2^{-3/2}\right)\frac{kq}{b^2}$$

$$E_y = \frac{kq}{2^{3/2}b^2}$$

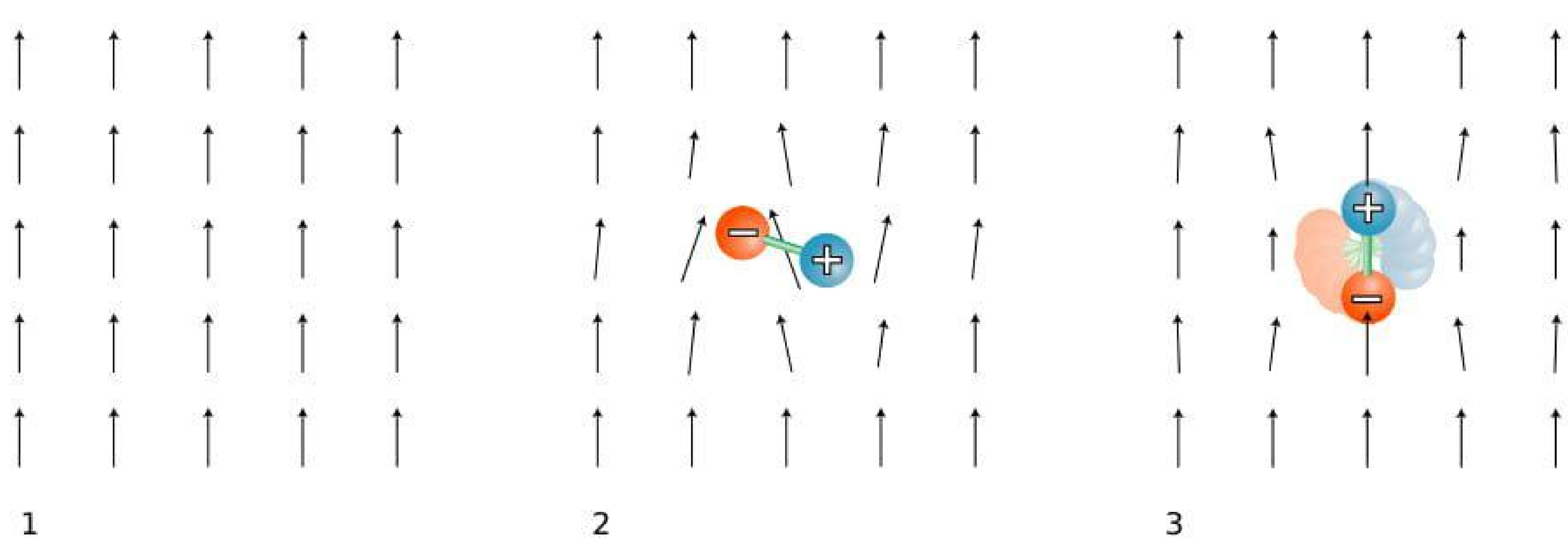

k / A dipole field. Electric fields diverge from a positive charge and converge on a negative charge.

Dipoles

The simplest set of sources that can occur with electricity but not with gravity is the *dipole*, consisting of a positive charge and a negative charge with equal magnitudes. More generally, an electric dipole can be any object with an imbalance of positive charge on one side and negative on the other. A water molecule, l, is a dipole because the electrons tend to shift away from the hydrogen atoms and onto the oxygen atom.


l / A water molecule is a dipole.

Your microwave oven acts on water molecules with electric fields. Let us imagine what happens if we start with a uniform electric field, m/1, made by some external charges, and then insert a dipole, m/2, consisting of two charges connected by a rigid rod. The dipole disturbs the field pattern, but more important for our present purposes is that it experiences a torque. In this example, the positive charge feels an upward force, but the negative charge is pulled down. The



m / 1. A uniform electric field created by some charges "off-stage." 2. A dipole is placed in the field. 3. The dipole aligns with the field.

result is that the dipole wants to align itself with the field, m/3. The microwave oven heats food with electrical (and magnetic) waves. The alternation of the torque causes the molecules to wiggle and increase the amount of random motion. The slightly vague definition of a dipole given above can be improved by saying that a dipole is any object that experiences a torque in an electric field.

What determines the torque on a dipole placed in an externally created field? Torque depends on the force, the distance from the axis at which the force is applied, and the angle between the force and the line from the axis to the point of application. Let a dipole consisting of charges $+q$ and $-q$ separated by a distance ℓ be placed in an external field of magnitude $|\mathbf{E}|$, at an angle θ with respect to the field. The total torque on the dipole is

$$\begin{aligned}\tau &= \frac{\ell}{2}q|\mathbf{E}|\sin\theta + \frac{\ell}{2}q|\mathbf{E}|\sin\theta \\ &= \ell q|\mathbf{E}|\sin\theta \quad .\end{aligned}$$

(Note that even though the two forces are in opposite directions, the torques do not cancel, because they are both trying to twist the dipole in the same direction.) The quantity is called the dipole moment, notated D. (More complex dipoles can also be assigned a dipole moment — they are defined as having the same dipole moment as the two-charge dipole that would experience the same torque.)

Dipole moment of a molecule of NaCl gas *example 4*

▷ In a molecule of NaCl gas, the center-to-center distance between the two atoms is about 0.6 nm. Assuming that the chlorine completely steals one of the sodium's electrons, compute the magnitude of this molecule's dipole moment.

▷ The total charge is zero, so it doesn't matter where we choose the origin of our coordinate system. For convenience, let's choose it to be at one of the atoms, so that the charge on that atom doesn't contribute to the dipole moment. The magnitude of the dipole moment is then

$$\begin{aligned}D &= (6\times10^{-10}\text{ m})(e) \\ &= (6\times10^{-10}\text{ m})(1.6\times10^{-19}\text{ C}) \\ &= 1\times10^{-28}\text{ C}\cdot\text{m}\end{aligned}$$

Alternative definition of the electric field

The behavior of a dipole in an externally created field leads us to an alternative definition of the electric field:

> **alternative definition of the electric field**
> The electric field vector, E, at any location in space is defined by observing the torque exerted on a test dipole D_t placed there. The direction of the field is the direction in which the field tends to align a dipole (from $-$ to $+$), and the field's magnitude is $|\mathbf{E}| = \tau/D_t\sin\theta$.

The main reason for introducing a second definition for the same concept is that the magnetic field is most easily defined using a similar approach.

Voltage related to electric field

Voltage is potential energy per unit charge, and electric field is force per unit charge. We can therefore relate voltage and field if we start from the relationship between potential energy and force,

$$\Delta PE = -Fd \quad , \qquad \text{[assuming constant force and motion parallel to the force]}$$

and divide by charge,

$$\Delta PE = -Fd \quad , \qquad \text{[assuming constant force and motion parallel to the force]}$$

giving

$$\Delta V = -Ed \quad , \qquad \text{[assuming constant force and motion parallel to the force]}$$

In other words, the difference in voltage between two points equals the electric field strength multiplied by the distance between them. The interpretation is that a strong electric field is a region of space where the voltage is rapidly changing. By analogy, a steep hillside is a place on the map where the altitude is rapidly changing.

Field generated by an electric eel — *example 5*

▷ Suppose an electric eel is 1 m long, and generates a voltage difference of 1000 volts between its head and tail. What is the electric field in the water around it?

▷ We are only calculating the amount of field, not its direction, so we ignore positive and negative signs. Subject to the possibly inaccurate assumption of a constant field parallel to the eel's body, we have

$$|\mathbf{E}| = \frac{\Delta V}{\Delta x} = 1000\ \text{V/m} \quad .$$

Relating the units of electric field and voltage — *example 6*

From our original definition of the electric field, we expect it to have units of newtons per coulomb, N/C. The example above, however, came out in volts per meter, V/m. Are these inconsistent? Let's reassure ourselves that this all works. In this kind of situation, the best strategy is usually to simplify the more complex units so that they involve only mks units and coulombs. Since voltage is defined as electrical energy per unit charge, it has units of J/C:

$$\frac{\text{V}}{\text{m}} = \frac{\text{J/C}}{\text{m}} = \frac{\text{J}}{\text{C}\cdot\text{m}} \quad .$$

To connect joules to newtons, we recall that work equals force times distance, so J = N · m, so

$$\frac{\mathrm{V}}{\mathrm{m}} = \frac{\mathrm{N}\cdot\mathrm{m}}{\mathrm{C}\cdot\mathrm{m}}$$
$$= \frac{\mathrm{N}}{\mathrm{C}}$$

As with other such difficulties with electrical units, one quickly begins to recognize frequently occurring combinations.

Discussion Questions

A In the definition of the electric field, does the test charge need to be 1 coulomb? Does it need to be positive?

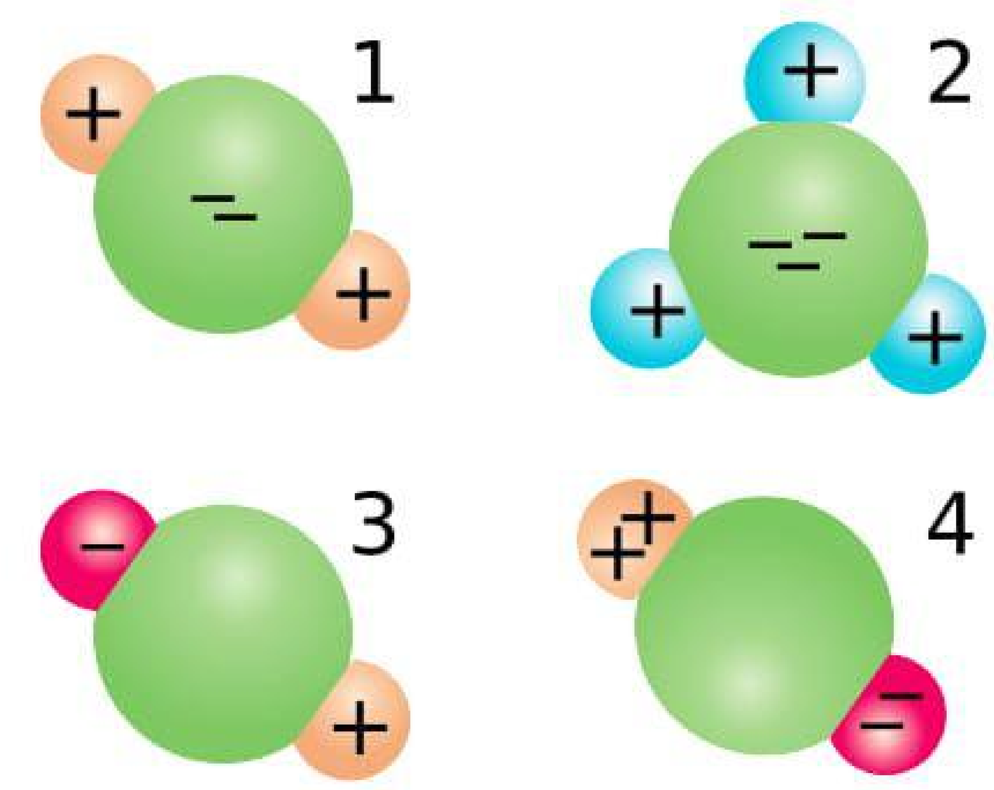
B Does a charged particle such as an electron or proton feel a force from its own electric field?

C Is there an electric field surrounding a wall socket that has nothing plugged into it, or a battery that is just sitting on a table?

D In a flashlight powered by a battery, which way do the electric fields point? What would the fields be like inside the wires? Inside the filament of the bulb?

E Criticize the following statement: "An electric field can be represented by a sea of arrows showing how current is flowing."

F The field of a point charge, $|\mathbf{E}| = kQ/r^2$, was derived in the self-check above. How would the field pattern of a uniformly charged sphere compare with the field of a point charge?



n / Discussion question H.

G The interior of a perfect electrical conductor in equilibrium must have zero electric field, since otherwise the free charges within it would be drifting in response to the field, and it would not be in equilibrium. What about the field right at the surface of a perfect conductor? Consider the possibility of a field perpendicular to the surface or parallel to it.

H Compare the dipole moments of the molecules and molecular ions shown in the figure.

I Small pieces of paper that have not been electrically prepared in any way can be picked up with a charged object such as a charged piece of tape. In our new terminology, we could describe the tape's charge as inducing a dipole moment in the paper. Can a similar technique be used to induce not just a dipole moment but a charge?

J The earth and moon are fairly uneven in size and far apart, like a baseball and a ping-pong ball held in your outstretched arms. Imagine instead a planetary system with the character of a double planet: two planets of equal size, close together. Sketch a sea of arrows diagram of their gravitational field.

5.4 ∫ Voltage for Nonuniform Fields

The calculus-savvy reader will have no difficulty generalizing the field-voltage relationship to the case of a varying field. The potential energy associated with a varying force is

$$\Delta PE = -\int F \, \mathrm{d}x \quad , \quad \text{[one dimension]}$$

so for electric fields we divide by q to find

$$\Delta V = -\int E \, \mathrm{d}x \quad , \quad \text{[one dimension]}$$

Applying the fundamental theorem of calculus yields

$$E = -\frac{\mathrm{d}V}{\mathrm{d}x} \quad . \quad \text{[one dimension]}$$

Voltage associated with a point charge *example 7*

▷ What is the voltage associated with a point charge?

▷ As derived previously in self-check A on page 129, the field is

$$|\mathbf{E}| = \frac{kQ}{r^2}$$

The difference in voltage between two points on the same radius line is

$$\begin{aligned} \Delta V &= -\int \mathrm{d}V \\ &= -\int E_x \mathrm{d}x \end{aligned}$$

In the general discussion above, x was just a generic name for distance traveled along the line from one point to the other, so in this case x really means r.

$$\begin{aligned} \Delta V &= -\int_{r_1}^{r_2} E_r \mathrm{d}r \\ &= -\int_{r_1}^{r_2} \frac{kQ}{r^2} \mathrm{d}r \\ &= \left.\frac{kQ}{r}\right]_{r_1}^{r_2} \\ &= \frac{kQ}{r_2} - \frac{kQ}{r_1} \quad . \end{aligned}$$

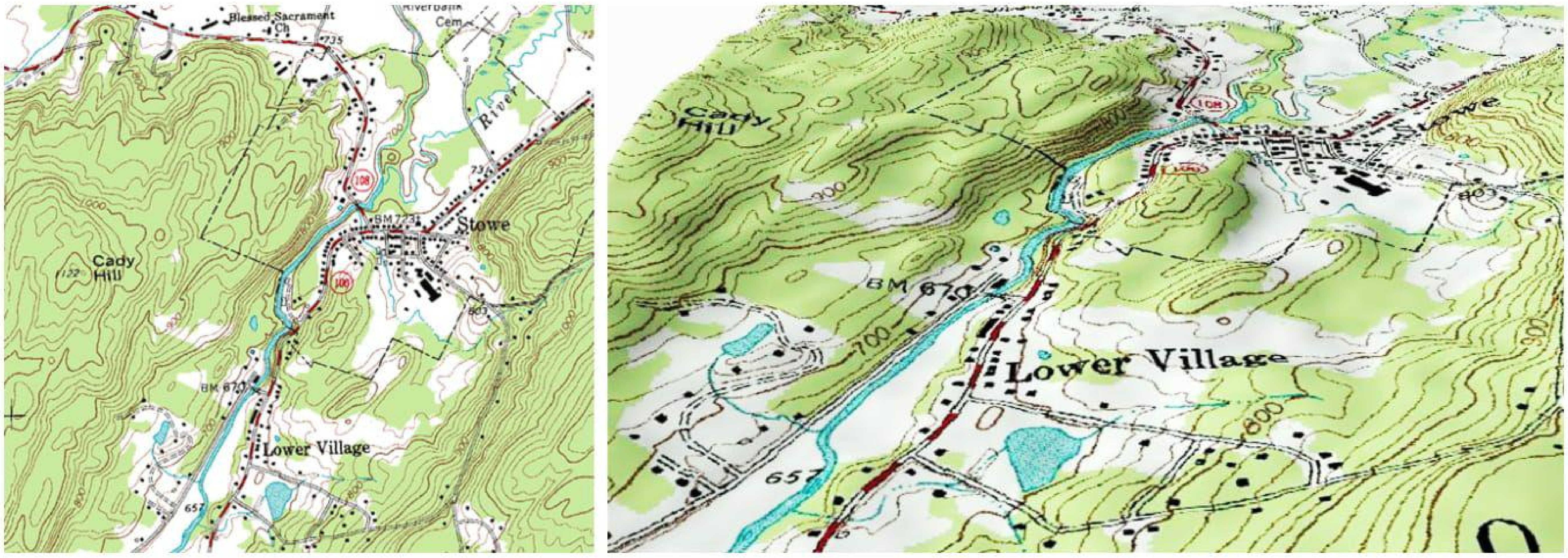
The standard convention is to use $r_1 = \infty$ as a reference point, so that the voltage at any distance r from the charge is

$$V = \frac{kQ}{r} \quad .$$

The interpretation is that if you bring a positive test charge closer to a positive charge, its electrical energy is increased; if it was released, it would spring away, releasing this as kinetic energy.

self-check B

Show that you can recover the expression for the field of a point charge by evaluating the derivative $E_x = -\mathrm{d}V/\mathrm{d}x$. ▷ Answer, p. 195



o / Left: A topographical map of Stowe, Vermont. From one constant-height line to the next is a height difference of 200 feet. Lines far apart, as in the lower village, indicate relatively flat terrain, while lines close together, like the ones to the west of the main town, represent a steep slope. Streams flow downhill, perpendicular to the constant-height lines. Right: The same map has been redrawn in perspective, with shading to suggest relief.

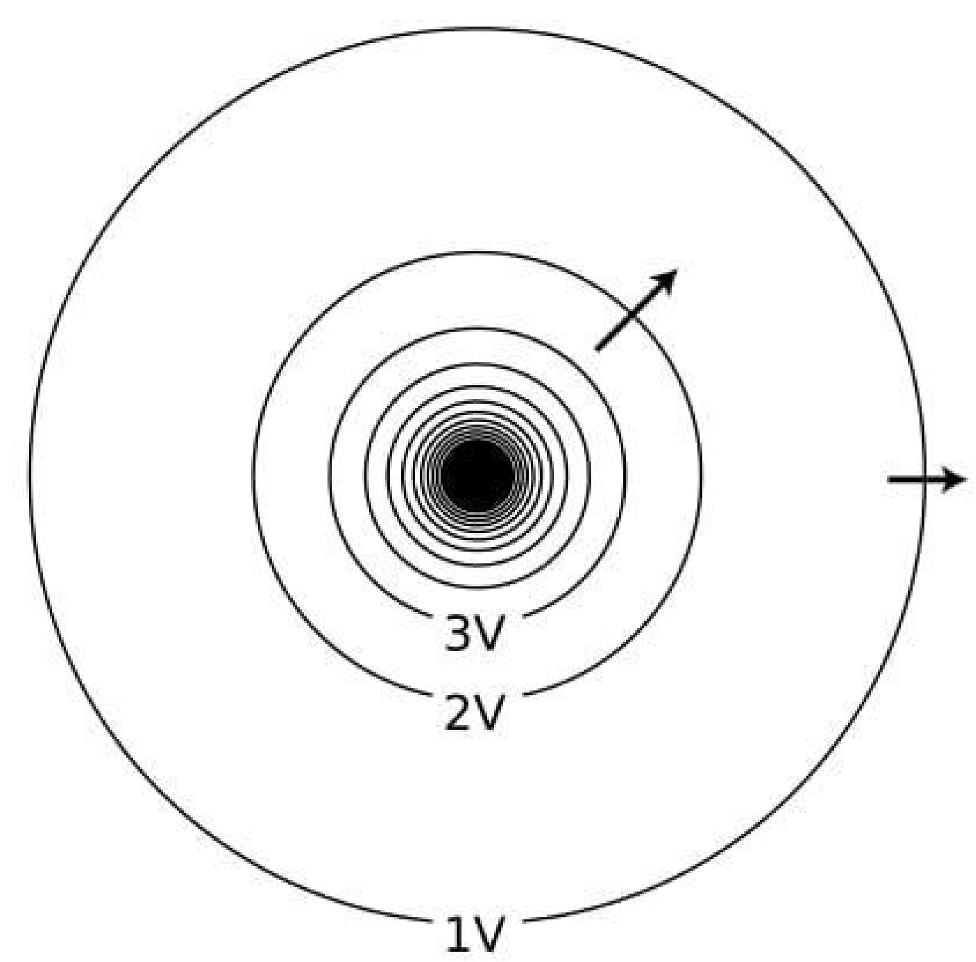
5.5 Two or Three Dimensions

The topographical map shown in figure o suggests a good way to visualize the relationship between field and voltage in two dimensions. Each contour on the map is a line of constant height; some of these are labeled with their elevations in units of feet. Height is related to gravitational potential energy, so in a gravitational analogy, we can think of height as representing voltage. Where the contour lines are far apart, as in the town, the slope is gentle. Lines close together indicate a steep slope.

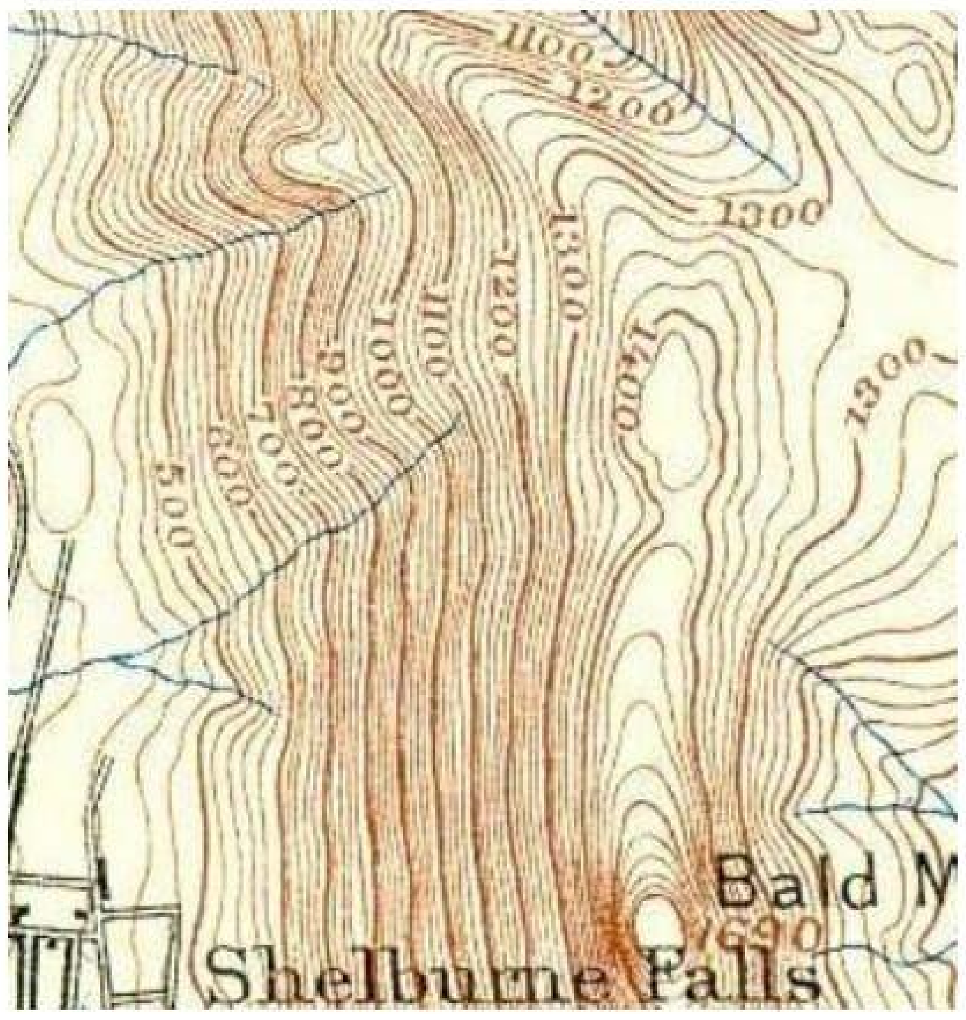
If we walk along a straight line, say straight east from the town, then height (voltage) is a function of the east-west coordinate x. Using the usual mathematical definition of the slope, and writing V for the height in order to remind us of the electrical analogy, the slope along such a line is $\Delta V/\Delta \mathrm{x}$. If the slope isn't constant, we either need to use the slope of the $V - x$ graph, or use calculus and talk about the derivative $\mathrm{d}V/\mathrm{d}x$.

What if everything isn't confined to a straight line? Water flows downhill. Notice how the streams on the map cut perpendicularly through the lines of constant height.

It is possible to map voltages in the same way, as shown in figure p. The electric field is strongest where the constant-voltage curves are closest together, and the electric field vectors always point perpendicular to the constant-voltage curves.



p / The constant-voltage curves surrounding a point charge. Near the charge, the curves are so closely spaced that they blend together on this drawing due to the finite width with which they were drawn. Some electric fields are shown as arrows.



q / Self-check C.

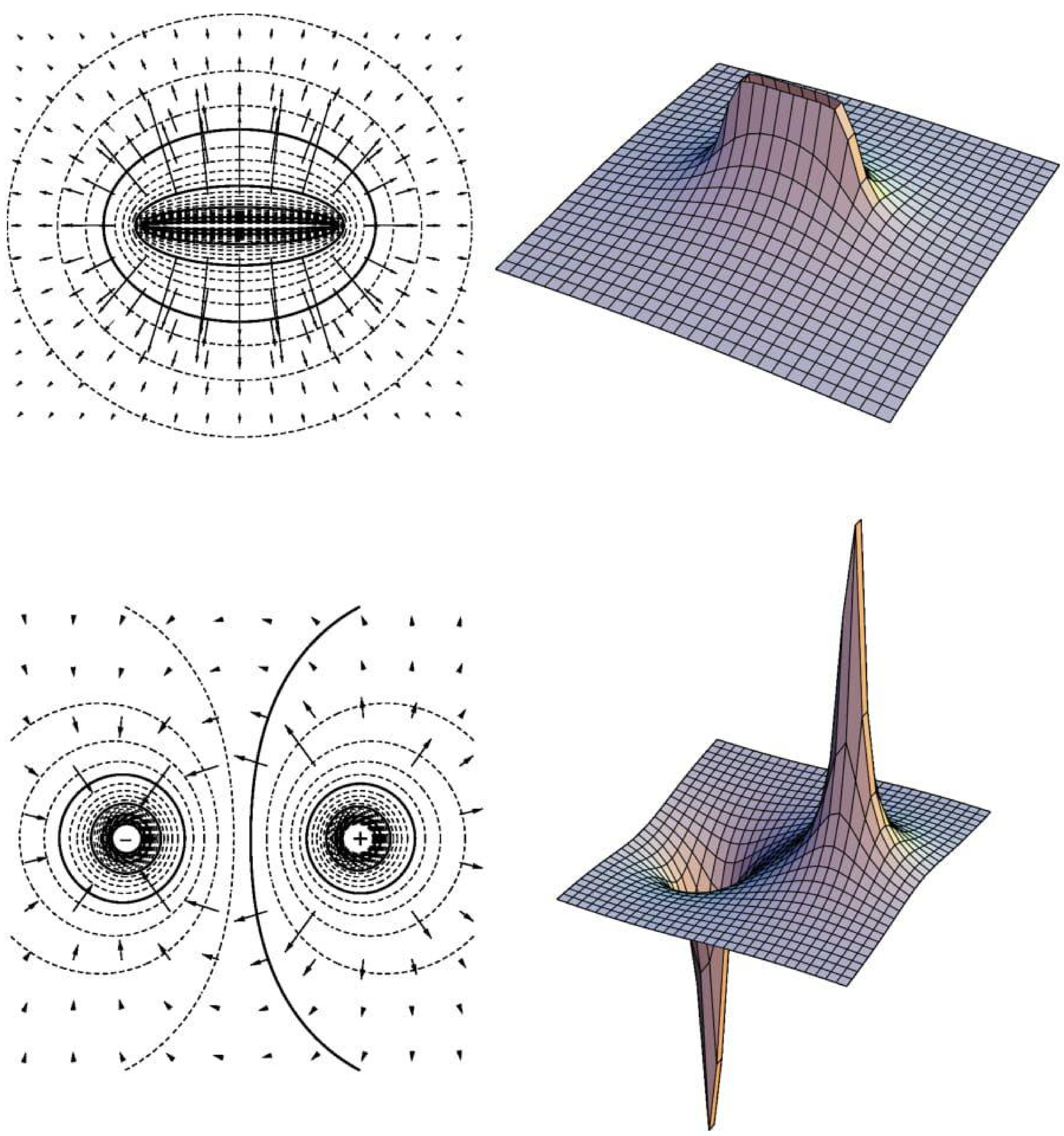
Figure r shows some examples of ways to visualize field and voltage patterns.

Mathematically, the calculus of section 5.4 generalizes to three dimensions as follows:

$$E_x = -\mathrm{d}V/\mathrm{d}x$$
$$E_y = -\mathrm{d}V/\mathrm{d}y$$
$$E_z = -\mathrm{d}V/\mathrm{d}z$$

self-check C

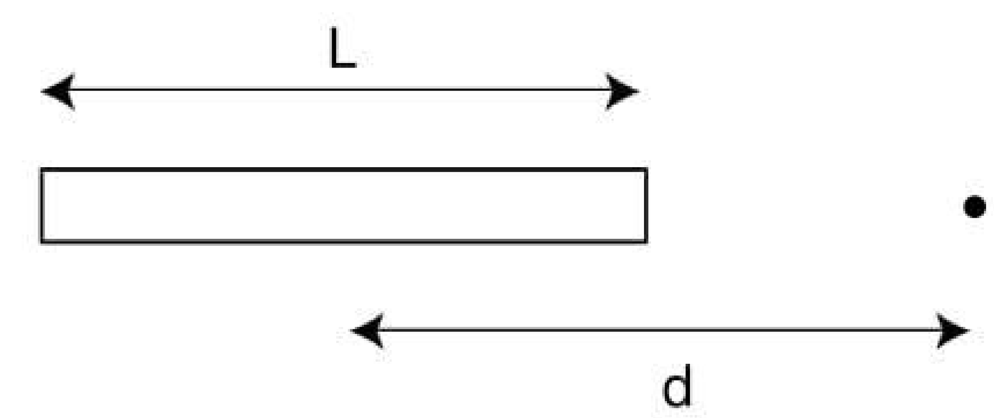
Imagine that the topographical map in figure q represents voltage rather than height. (a) Consider the stream the starts near the center of the map. Determine the positive and negative signs of dV/dx and dV/dy, and relate these to the direction of the force that is pushing the current forward against the resistance of friction. (b) If you wanted to find a lot of electric charge on this map, where would you look? ▷ Answer, p. 196



r / Two-dimensional field and voltage patterns. Top: A uniformly charged rod. Bottom: A dipole. In each case, the diagram on the left shows the field vectors and constant-voltage curves, while the one on the right shows the voltage (up-down coordinate) as a function of x and y. Interpreting the field diagrams: Each arrow represents the field at the point where its tail has been positioned. For clarity, some of the arrows in regions of very strong field strength are not shown Î¡ they would be too long to show. Interpreting the constant-voltage curves: In regions of very strong fields, the curves are not shown because they would merge together to make solid black regions. Interpreting the perspective plots: Keep in mind that even though we're visualizing things in three dimensions, these are really two-dimensional voltage patterns being represented. The third (up-down) dimension represents voltage, not position.

5.6 $\int\star$ Electric Field of a Continuous Charge Distribution

Charge really comes in discrete chunks, but often it is mathematically convenient to treat a set of charges as if they were like a continuous fluid spread throughout a region of space. For example, a charged metal ball will have charge spread nearly uniformly all over its surface, and in for most purposes it will make sense to ignore the fact that this uniformity is broken at the atomic level. The electric field made by such a continuous charge distribution is the sum of the fields created by every part of it. If we let the "parts" become infinitesimally small, we have a sum of an infinite number of infinitesimal numbers, which is an integral. If it was a discrete sum, we would have a total electric field in the x direction that was the sum of all the x components of the individual fields, and similarly we'd have sums for the y and z components. In the continuous case, we have three integrals.



s / Example 8.

Field of a uniformly charged rod — *example 8*

▷ A rod of length L has charge Q spread uniformly along it. Find the electric field at a point a distance d from the center of the rod, along the rod's axis.

▷ This is a one-dimensional situation, so we really only need to do a single integral representing the total field along the axis. We imagine breaking the rod down into short pieces of length dz, each with charge dq. Since charge is uniformly spread along the rod, we have $dq = \lambda dz$, where $\lambda = Q/L$ (Greek lambda) is the charge per unit length, in units of coulombs per meter. Since the pieces are infinitesimally short, we can treat them as point charges and use the expression kdq/r^2 for their contributions to the field, where $r = d - z$ is the distance from the charge at z to the point in which we are interested.

$$\begin{aligned} E_z &= \int \frac{kdq}{r^2} \\ &= \int_{-L/2}^{+L/2} \frac{k\lambda dz}{r^2} \\ &= k\lambda \int_{-L/2}^{+L/2} \frac{dz}{(d-z)^2} \end{aligned}$$

The integral can be looked up in a table, or reduced to an elementary form by substituting a new variable for $d - z$. The result is

$$\begin{aligned} E_z &= k\lambda \left(\frac{1}{d-z}\right)_{-L/2}^{+L/2} \\ &= \frac{kQ}{L}\left(\frac{1}{d-L/2} - \frac{1}{d+L/2}\right) \end{aligned} \quad .$$

For large values of d, this expression gets smaller for two reasons: (1) the denominators of the fractions become large, and (2) the two fractions become nearly the same, and tend to cancel out. This makes sense, since the field should get weaker as we get farther away from the charge. In fact, the field at large distances must approach kQ/d^2, since from a great distance, the rod looks like a point.

It's also interesting to note that the field becomes infinite at the ends of the rod, but is not infinite on the interior of the rod. Can you explain physically why this happens?

Summary

Selected Vocabulary

field	a property of a point in space describing the forces that would be exerted on a particle if it was there
sink-SHARED .	a point at which field vectors converge
source	a point from which field vectors diverge; often used more inclusively to refer to points of either convergence or divergence
electric field . . .	the force per unit charge exerted on a test charge at a given point in space
gravitational field	the force per unit mass exerted on a test mass at a given point in space
electric dipole . .	an object that has an imbalance between positive charge on one side and negative charge on the other; an object that will experience a torque in an electric field

Notation

g	the gravitational field
E	the electric field
D	an electric dipole moment

Other Terminology and Notation

d, p, m	other notations for the electric dipole moment

Summary

Newton conceived of a universe where forces reached across space instantaneously, but we now know that there is a delay in time before a change in the configuration of mass and charge in one corner of the universe will make itself felt as a change in the forces experienced far away. We imagine the outward spread of such a change as a ripple in an invisible universe-filling *field of force.*

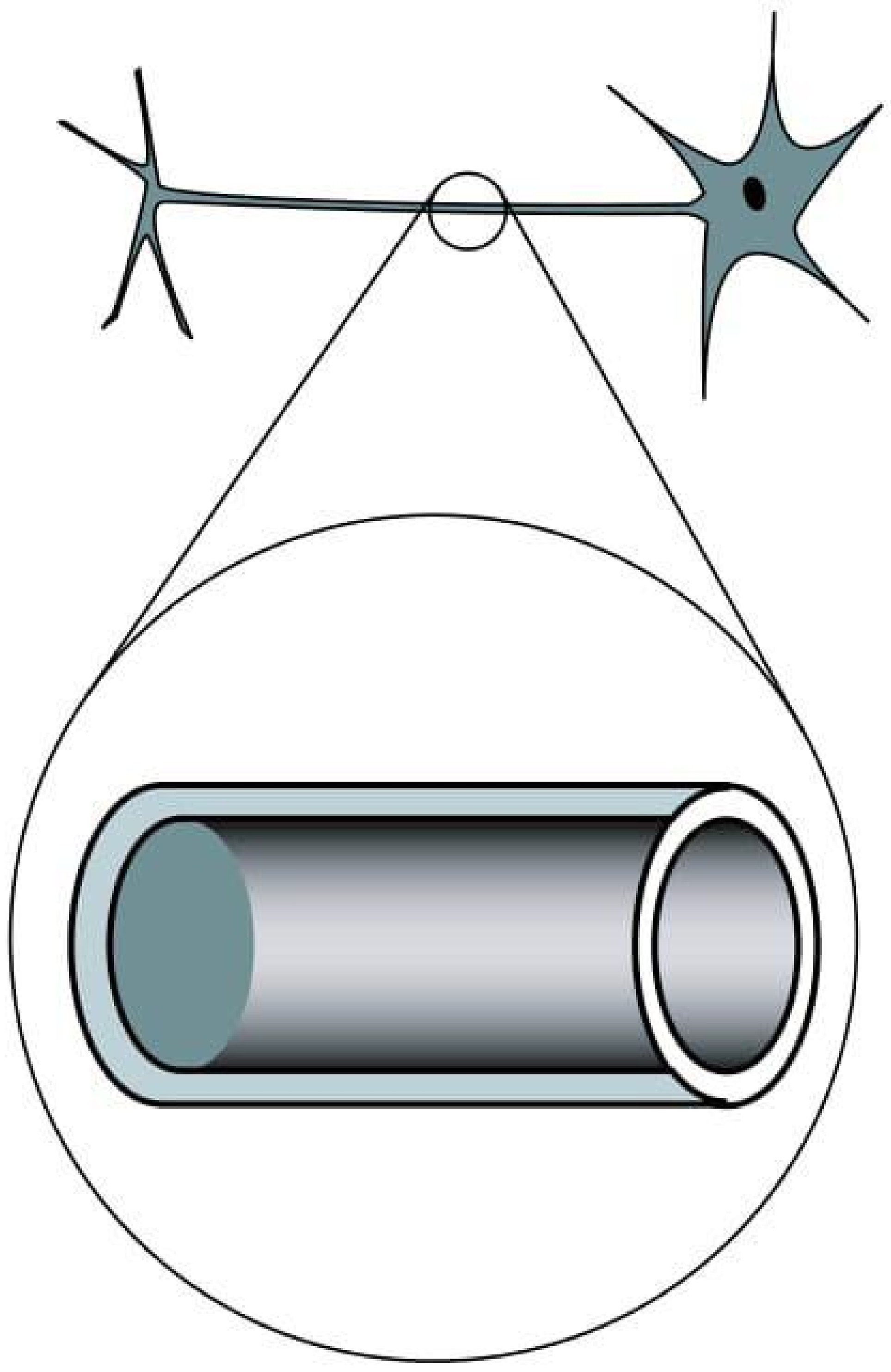
We define the *gravitational field* at a given point as the force per unit mass exerted on objects inserted at that point, and likewise the *electric field* is defined as the force per unit charge. These fields are vectors, and the fields generated by multiple sources add according to the rules of vector addition.

When the electric field is constant, the voltage difference between two points lying on a line parallel to the field is related to the field by the equation $\Delta V = —Ed$, where d is the distance between the two points.

Problems

Key

- ✓ A computerized answer check is available online.
- $\int$ A problem that requires calculus.
- ⋆ A difficult problem.



Problem 1.

1 In our by-now-familiar neuron, the voltage difference between the inner and outer surfaces of the cell membrane is about $V_{out} - V_{in} = -70$ mV in the resting state, and the thickness of the membrane is about 6.0 nm (i.e. only about a hundred atoms thick). What is the electric field inside the membrane? ✓

2 The gap between the electrodes in an automobile engine's spark plug is 0.060 cm. To produce an electric spark in a gasoline-air mixture, an electric field of 3.0×10^6 V/m must be achieved. On starting a car, what minimum voltage must be supplied by the ignition circuit? Assume the field is uniform. ✓
(b) The small size of the gap between the electrodes is inconvenient because it can get blocked easily, and special tools are needed to measure it. Why don't they design spark plugs with a wider gap?

3 (a) At time $t = 0$, a positively charged particle is placed, at rest, in a vacuum, in which there is a uniform electric field of magnitude E. Write an equation giving the particle's speed, v, in terms of t, E, and its mass and charge m and q. ✓
(b) If this is done with two different objects and they are observed to have the same motion, what can you conclude about their masses and charges? (For instance, when radioactivity was discovered, it was found that one form of it had the same motion as an electron in this type of experiment.)

4 Show that the magnitude of the electric field produced by a simple two-charge dipole, at a distant point along the dipole's axis, is to a good approximation proportional to D/r^3, where r is the distance from the dipole. [Hint: Use the approximation $(1+\epsilon)^p \approx 1 + p\epsilon$, which is valid for small ϵ.] ⋆

5 Given that the field of a dipole is proportional to D/r^3 (see previous problem), show that its voltage varies as D/r^2. (Ignore positive and negative signs and numerical constants of proportionality.) $\int$

6 A carbon dioxide molecule is structured like O-C-O, with all three atoms along a line. The oxygen atoms grab a little bit of extra negative charge, leaving the carbon positive. The molecule's symmetry, however, means that it has no overall dipole moment, unlike a V-shaped water molecule, for instance. Whereas the voltage of a dipole of magnitude D is proportional to D/r^2 (problem 5), it turns out that the voltage of a carbon dioxide molecule along its axis equals k/r^3, where r is the distance from the molecule and k

is a constant. What would be the electric field of a carbon dioxide molecule at a distance r? ∫

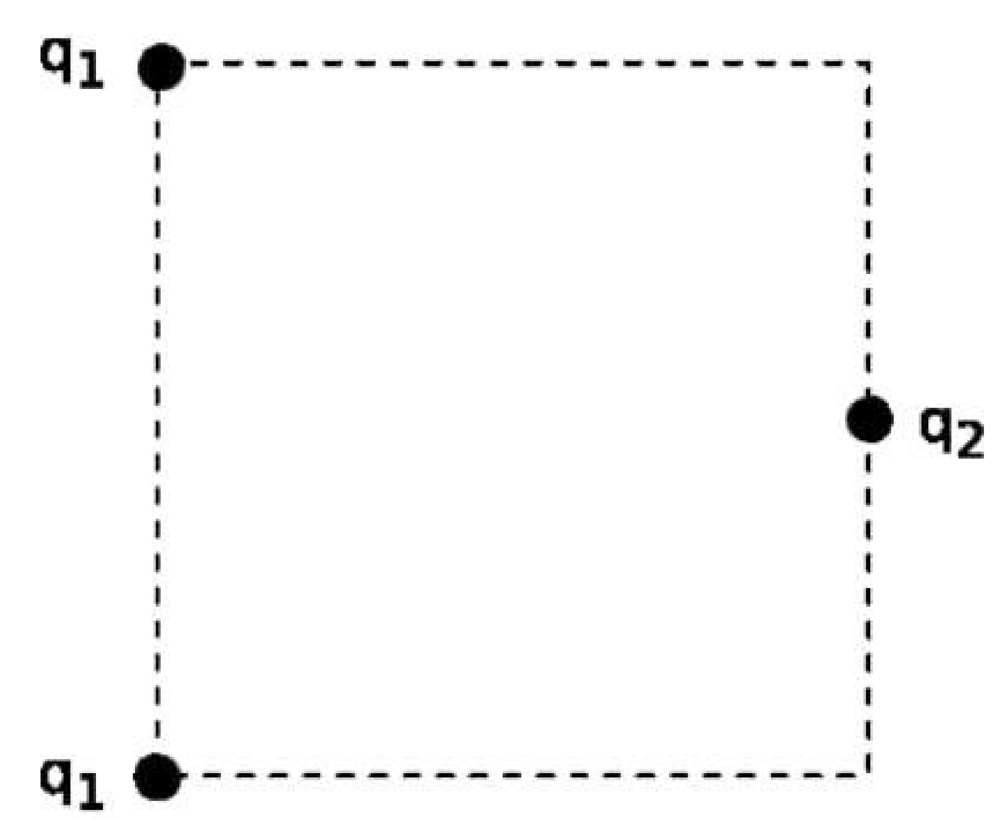
7 A proton is in a region in which the electric field is given by $E = a + bx^3$. If the proton starts at rest at $x_1 = 0$, find its speed, v, when it reaches position x_2. Give your answer in terms of a, b, x_2, and e and m, the charge and mass of the proton. ✓ ∫

8 Consider the electric field created by a uniform ring of total charge q and radius b. (a) Show that the field at a point on the ring's axis at a distance a from the plane of the ring is $kqa(a^2 + b^2)^{-3/2}$. (b) Show that this expression has the right behavior for $a = 0$ and for a much greater than b. ⋆

9 Consider the electric field created by an infinite uniformly charged plane. Starting from the result of problem 8, show that the field at any point is $2\pi k\sigma$, where σ is the density of charge on the plane, in units of coulombs per square meter. Note that the result is independent of the distance from the plane. [Hint: Slice the plane into infinitesimal concentric rings, centered at the point in the plane closest to the point at which the field is being evaluated. Integrate the rings' contributions to the field at this point to find the total field.] ▷ Solution, p. 197 ∫

10 Consider the electric field created by a uniformly charged cylinder that extends to infinity in one direction. (a) Starting from the result of problem 8, show that the field at the center of the cylinder's mouth is $2\pi k\sigma$, where σ is the density of charge on the cylinder, in units of coulombs per square meter. [Hint: You can use a method similar to the one in problem 9.] (b) This expression is independent of the radius of the cylinder. Explain why this should be so. For example, what would happen if you doubled the cylinder's radius? ∫

11 Three charges are arranged on a square as shown. All three charges are positive. What value of q_2/q_1 will produce zero electric field at the center of the square? ▷ Solution, p. 197



Problem 11.