Chapter A
Capacitance and Inductance

This chapter is optional.

The long road leading from the light bulb to the computer started with one very important step: the introduction of feedback into electronic circuits. Although the principle of feedback has been understood and and applied to mechanical systems for centuries, and to electrical ones since the early twentieth century, for most of us the word evokes an image of Jimi Hendrix (or some more recent guitar hero) intentionally creating earsplitting screeches, or of the school principal doing the same inadvertently in the auditorium. In the guitar example, the musician stands in front of the amp and turns it up so high that the sound waves coming from the speaker come back to the guitar string and make it shake harder. This is an example of *positive* feedback: the harder the string vibrates, the stronger the sound waves, and the stronger the sound waves, the harder the string vibrates. The only limit is the power-handling ability of the amplifier.

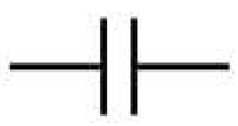
Negative feedback is equally important. Your thermostat, for example, provides negative feedback by kicking the heater off when the house gets warm enough, and by firing it up again when it gets too cold. This causes the house's temperature to oscillate back and forth within a certain range. Just as out-of-control exponential freak-outs are a characteristic behavior of positive-feedback systems, oscillation is typical in cases of negative feedback. You have already studied negative feedback extensively in *Vibrations and Waves* in the case of a mechanical system, although we didn't call it that.

A.1 Capacitance and inductance

In a mechanical oscillation, energy is exchanged repetitively between potential and kinetic forms, and may also be siphoned off in the form of heat dissipated by friction. In an electrical circuit, resistors are the circuit elements that dissipate heat. What are the electrical analogs of storing and releasing the potential and kinetic energy of a vibrating object? When you think of energy storage in an electrical circuit, you are likely to imagine a battery, but even rechargeable batteries can only go through 10 or 100 cycles before they wear out.

In addition, batteries are not able to exchange energy on a short enough time scale for most applications. The circuit in a musical synthesizer may be called upon to oscillate thousands of times a second, and your microwave oven operates at gigahertz frequencies. Instead of batteries, we generally use capacitors and inductors to store energy in oscillating circuits. Capacitors, which you've already encountered, store energy in electric fields. An inductor does the same with magnetic fields.

Capacitors



a / The symbol for a capacitor.

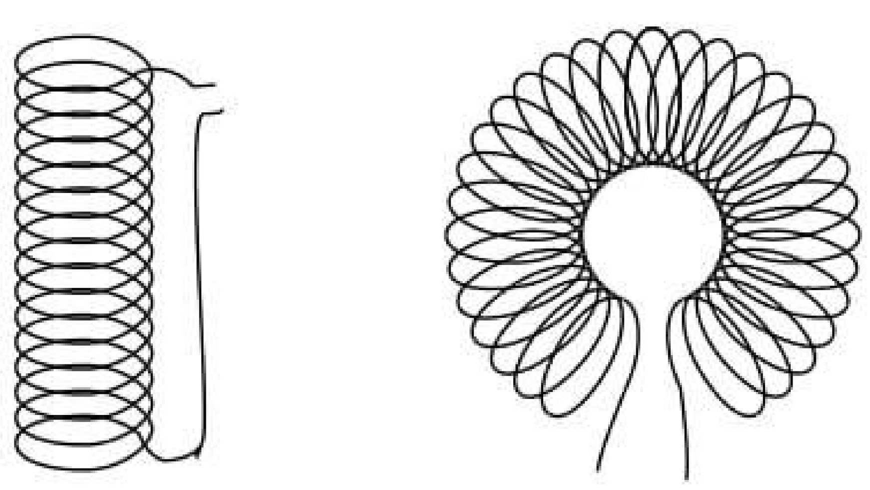
b / Some capacitors.

A capacitor's energy exists in its surrounding electric fields. It is proportional to the square of the field strength, which is proportional to the charges on the plates. If we assume the plates carry charges that are the same in magnitude, $+q$ and $-q$, then the energy stored in the capacitor must be proportional to q^2. For historical reasons, we write the constant of proportionality as $1/2C$,

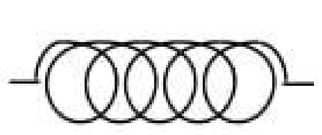
$$E_C = \frac{1}{2C}q^2 \qquad .$$

The constant C is a geometrical property of the capacitor, called its capacitance.

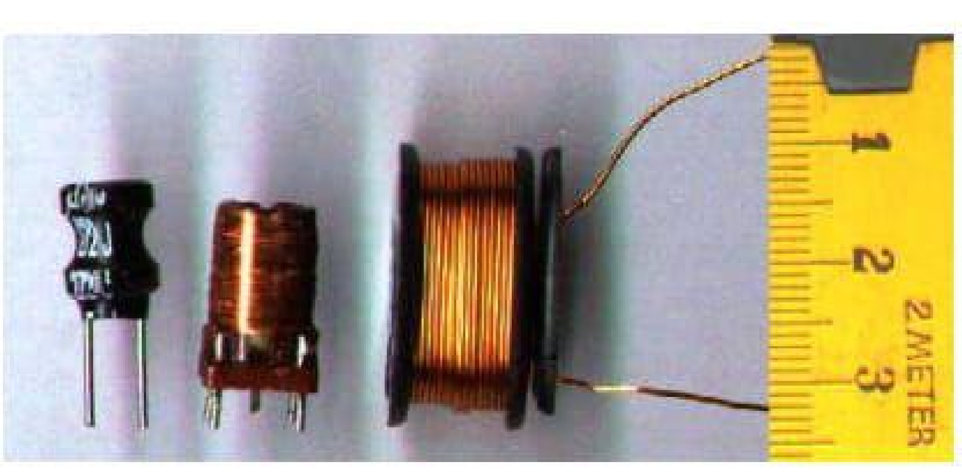
Based on this definition, the units of capacitance must be coulombs squared per joule, and this combination is more conveniently abbreviated as the farad, $1\ \text{F} = 1\ \text{C}^2/\text{J}$. "Condenser" is a less formal term for a capacitor. Note that the labels printed on capacitors often use MF to mean μF, even though MF should really be the symbol for megafarads, not microfarads. Confusion doesn't result from this nonstandard notation, since picofarad and microfarad values are the most common, and it wasn't until the 1990's that even millifarad and farad values became available in practical physical sizes. Figure a show the symbol used in schematics to represent a capacitor.



c / Two common geometries for inductors. The cylindrical shape on the left is called a solenoid.



d / The symbol for an inductor.



e / Some inductors.

Inductors

Any current will create a magnetic field, so in fact every current-carrying wire in a circuit acts as an inductor! However, this type of "stray" inductance is typically negligible, just as we can usually ignore the stray resistance of our wires and only take into account the actual resistors. To store any appreciable amount of magnetic energy, one usually uses a coil of wire designed specifically to be an inductor. All the loops' contribution to the magnetic field add together to make a stronger field. Unlike capacitors and resistors, practical inductors are easy to make by hand. One can for instance spool some wire around a short wooden dowel, put the spool inside a plastic aspirin bottle with the leads hanging out, and fill the bottle with epoxy to make the whole thing rugged. An inductor like this, in the form cylindrical coil of wire, is called a solenoid, c, and a stylized solenoid, d, is the symbol used to represent an inductor in a circuit regardless of its actual geometry.

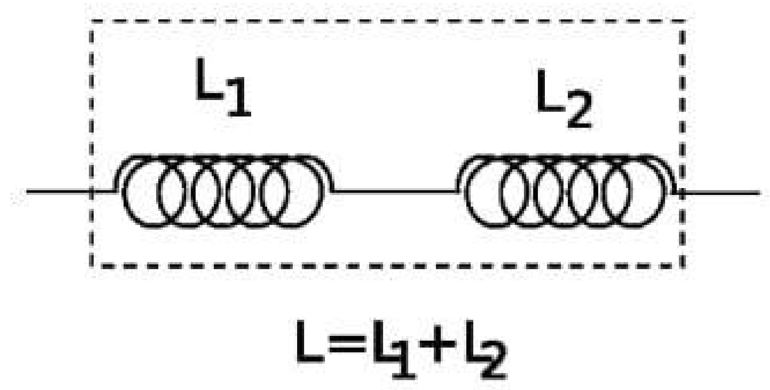
How much energy does an inductor store? The energy density is proportional to the square of the magnetic field strength, which is in turn proportional to the current flowing through the coiled wire, so the energy stored in the inductor must be proportional to I^2. We write $L/2$ for the constant of proportionality, giving

$$E_L = \frac{L}{2} I^2 \qquad .$$

As in the definition of capacitance, we have a factor of 1/2, which is purely a matter of definition. The quantity L is called the *inductance* of the inductor, and we see that its units must be joules per ampere squared. This clumsy combination of units is more commonly abbreviated as the henry, 1 henry = 1 J/A^2. Rather than memorizing this definition, it makes more sense to derive it when needed from the definition of inductance. Many people know inductors simply as "coils," or "chokes," and will not understand you if you refer to an "inductor," but they will still refer to L as the "inductance," not the "coilance" or "chokeance!"

Identical inductances in series *example 1*

If two inductors are placed in series, any current that passes through the combined double inductor must pass through both its parts. Thus by the definition of inductance, the inductance is doubled as well. In general, inductances in series add, just like resistances. The same kind of reasoning also shows that the inductance of a solenoid is approximately proportional to its length, assuming the number of turns per unit length is kept constant.

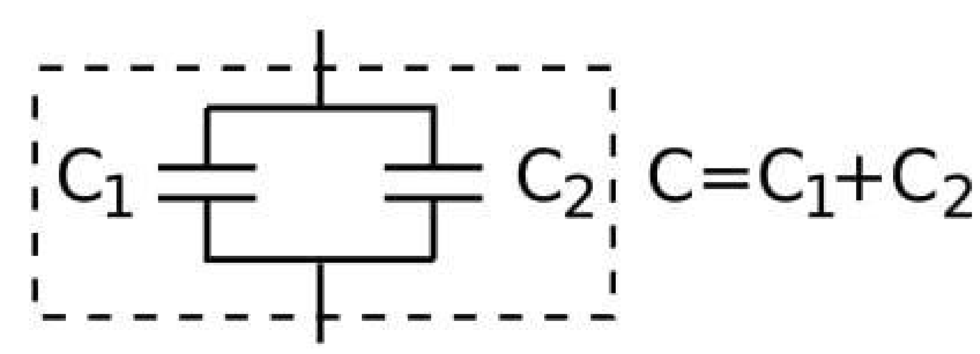


f / Inductances in series add.

Identical capacitances in parallel *example 2*

When two identical capacitances are placed in parallel, any charge deposited at the terminals of the combined double capacitor will divide itself evenly between the two parts. The electric fields surrounding each capacitor will be half the intensity, and therefore store one quarter the energy. Two capacitors, each storing one quarter the energy, give half the total energy storage. Since capacitance is inversely related to energy storage, this implies that identical capacitances in parallel give double the capacitance. In general, capacitances in parallel add. This is unlike the behavior of inductors and resistors, for which series configurations give addition.

This is consistent with the fact that the capacitance of a single parallel-plate capacitor proportional to the area of the plates. If we have two parallel-plate capacitors, and we combine them in parallel and bring them very close together side by side, we have produced a single capacitor with plates of double the area, and it has approximately double the capacitance.

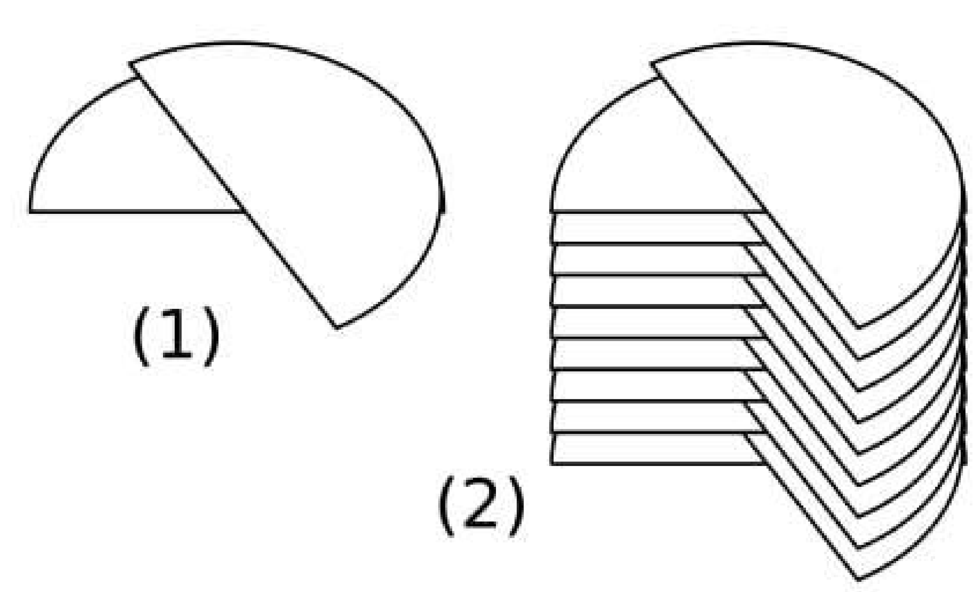


g / Capacitances in parallel add.

Inductances in parallel and capacitances in series are explored in homework problems 4 and 6.

A variable capacitor *example 3*

Figure h/1 shows the construction of a variable capacitor out of two parallel semicircles of metal. One plate is fixed, while the other can be rotated about their common axis with a knob. The opposite charges on the

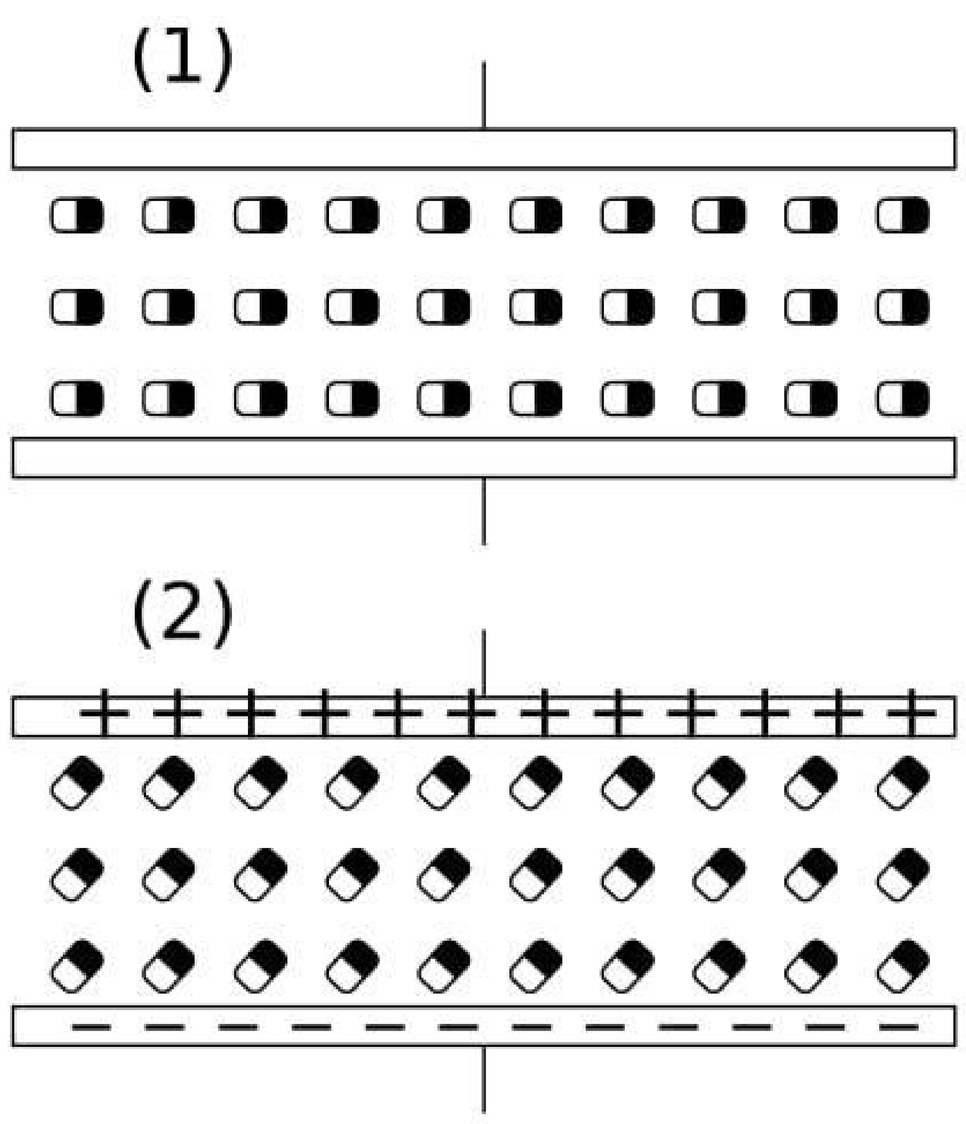


h / A variable capacitor.

two plates are attracted to one another, and therefore tend to gather in the overlapping area. This overlapping area, then, is the only area that effectively contributes to the capacitance, and turning the knob changes the capacitance. The simple design can only provide very small capacitance values, so in practice one usually uses a bank of capacitors, wired in parallel, with all the moving parts on the same shaft.

Discussion Questions

A Suppose that two parallel-plate capacitors are wired in parallel, and are placed very close together, side by side, so that their fields overlap. Will the resulting capacitance be too small, or too big? Could you twist the circuit into a different shape and make the effect be the other way around, or make the effect vanish? How about the case of two inductors in series?



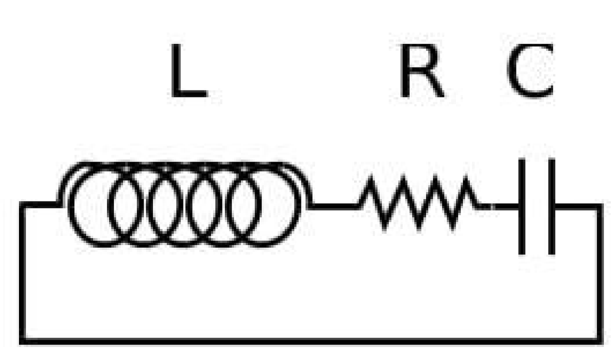
i / Discussion question B.

B Most practical capacitors do not have an air gap or vacuum gap between the plates; instead, they have an insulating substance called a dielectric. We can think of the molecules in this substance as dipoles that are free to rotate (at least a little), but that are not free to move around, since it is a solid. The figure shows a highly stylized and unrealistic way of visualizing this. We imagine that all the dipoles are intially turned sideways, (1), and that as the capacitor is charged, they all respond by turning through a certain angle, (2). (In reality, the scene might be much more random, and the alignment effect much weaker.)

For simplicity, imagine inserting just one electric dipole into the vacuum gap. For a given amount of charge on the plates, how does this affect the amount of energy stored in the electric field? How does this affect the capacitance?

Now redo the analysis in terms of the mechanical work needed in order to charge up the plates.

A.2 Oscillations



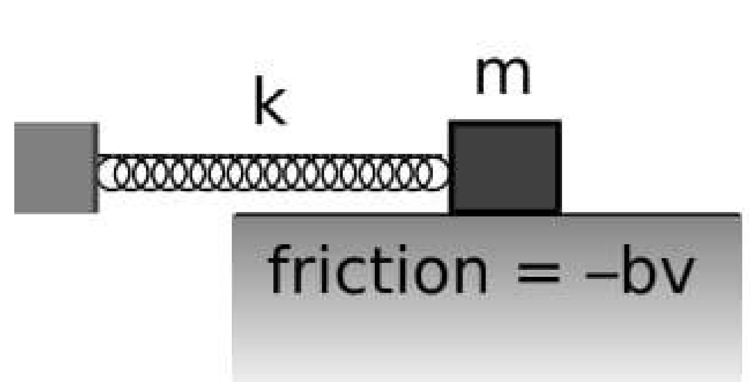
j / A series LRC circuit.

Figure j shows the simplest possible oscillating circuit. For any useful application it would actually need to include more components. For example, if it was a radio tuner, it would need to be connected to an antenna and an amplifier. Nevertheless, all the essential physics is there.

We can analyze it without any sweat or tears whatsoever, simply by constructing an analogy with a mechanical system. In a mechanical oscillator, k, we have two forms of stored energy,

$$E_{spring} = \frac{1}{2}kx^2 \qquad (1)$$

$$K = \frac{1}{2}mv^2 \quad . \qquad (2)$$



k / A mechanical analogy for the LRC circuit.

In the case of a mechanical oscillator, we have usually assumed a friction force of the form that turns out to give the nicest mathematical results, $F = -bv$. In the circuit, the dissipation of energy into heat occurs via the resistor, with no mechanical force involved, so in order to make the analogy, we need to restate the role of the friction force in terms of energy. The power dissipated by friction equals the mechanical work it does in a time interval Δt, divided by Δt, $P = W/\Delta t = F\Delta x/\Delta t = Fv = -bv^2$, so

$$\text{rate of heat dissipation} = -bv^2 \quad . \qquad (3)$$

self-check A

Equation (1) has *x* squared, and equations (2) and (3) have *v* squared. Because they're squared, the results don't depend on whether these variables are positive or negative. Does this make physical sense? ▷ Answer, p. 196

In the circuit, the stored forms of energy are

$$E_C = \frac{1}{2C}q^2 \qquad (1')$$
$$E_L = \frac{1}{2}LI^2 \quad , \qquad (2')$$

and the rate of heat dissipation in the resistor is

$$\text{rate of heat dissipation} = -RI^2 \quad . \qquad (3')$$

Comparing the two sets of equations, we first form analogies between quantities that represent the state of the system at some moment in time:

$$x \leftrightarrow q$$
$$v \leftrightarrow I$$

self-check B

How is *v* related mathematically to *x*? How is *I* connected to *q*? Are the two relationships analogous? ▷ Answer, p. 196

Next we relate the ones that describe the system's permanent characteristics:

$$k \leftrightarrow 1/C$$
$$m \leftrightarrow L$$
$$b \leftrightarrow R$$

Since the mechanical system naturally oscillates with a period $T = 2\pi\sqrt{m/k}$, we can immediately solve the electrical version by analogy, giving

$$T = 2\pi\sqrt{LC} \quad .$$

Rather than period, T, and frequency, f, it turns out to be more convenient if we work with the quantity $\omega = 2\pi f$, which can be interpreted as the number of radians per second. Then

$$\omega = \frac{1}{\sqrt{LC}} \quad .$$

Since the resistance R is analogous to b in the mechanical case, we find that the Q (quality factor, not charge) of the resonance is inversely proportional to R, and the width of the resonance is directly proportional to R.

Tuning a radio receiver *example 4*

A radio receiver uses this kind of circuit to pick out the desired station. Since the receiver resonates at a particular frequency, stations whose frequencies are far off will not excite any response in the circuit. The value of R has to be small enough so that only one station at a time is picked up, but big enough so that the tuner isn't too touchy. The resonant frequency can be tuned by adjusting either L or C, but variable capacitors are easier to build than variable inductors.

A numerical calculation *example 5*

The phone company sends more than one conversation at a time over the same wire, which is accomplished by shifting each voice signal into different range of frequencies during transmission. The number of signals per wire can be maximized by making each range of frequencies (known as a bandwidth) as small as possible. It turns out that only a relatively narrow range of frequencies is necessary in order to make a human voice intelligible, so the phone company filters out all the extreme highs and lows. (This is why your phone voice sounds different from your normal voice.)

▷ If the filter consists of an LRC circuit with a broad resonance centered around 1.0 kHz, and the capacitor is 1 μF (microfarad), what inductance value must be used?

▷ Solving for L, we have

$$\begin{aligned} L &= \frac{1}{C\omega^2} \\ &= \frac{1}{(10^{-6}\ \text{F})(2\pi \times 10^3\ \text{s}^{-1})^2} \\ &= 2.5 \times 10^{-3}\ \text{F}^{-1}\text{s}^2 \end{aligned}$$

Checking that these really are the same units as henries is a little tedious, but it builds character:

$$\begin{aligned} \text{F}^{-1}\text{s}^2 &= (\text{C}^2/\text{J})^{-1}\text{s}^2 \\ &= \text{J} \cdot \text{C}^{-2}\text{s}^2 \\ &= \text{J}/\text{A}^2 \\ &= \text{H} \end{aligned}$$

The result is 25 mH (millihenries).

This is actually quite a large inductance value, and would require a big, heavy, expensive coil. In fact, there is a trick for making this kind of circuit small and cheap. There is a kind of silicon chip called an op-amp, which, among other things, can be used to simulate the behavior of an inductor. The main limitation of the op-amp is that it is restricted to low-power applications.

A.3 Voltage and Current

What is physically happening in one of these oscillating circuits? Let's first look at the mechanical case, and then draw the analogy to the circuit. For simplicity, let's ignore the existence of damping, so there is no friction in the mechanical oscillator, and no resistance in the electrical one.

Suppose we take the mechanical oscillator and pull the mass away from equilibrium, then release it. Since friction tends to resist the spring's force, we might naively expect that having zero friction would allow the mass to leap instantaneously to the equilibrium position. This can't happen, however, because the mass would have to have infinite velocity in order to make such an instantaneous leap. Infinite velocity would require infinite kinetic energy, but the only kind of energy that is available for conversion to kinetic is the energy stored in the spring, and that is finite, not infinite. At each step on its way back to equilibrium, the mass's velocity is controlled exactly by the amount of the spring's energy that has so far been converted into kinetic energy. After the mass reaches equilibrium, it overshoots due to its own momentum. It performs identical oscillations on both sides of equilibrium, and it never loses amplitude because friction is not available to convert mechanical energy into heat.

Now with the electrical oscillator, the analog of position is charge. Pulling the mass away from equilibrium is like depositing charges $+q$ and $-q$ on the plates of the capacitor. Since resistance tends to resist the flow of charge, we might imagine that with no friction present, the charge would instantly flow through the inductor (which is, after all, just a piece of wire), and the capacitor would discharge instantly. However, such an instant discharge is impossible, because it would require infinite current for one instant. Infinite current would create infinite magnetic fields surrounding the inductor, and these fields would have infinite energy. Instead, the rate of flow of current is controlled at each instant by the relationship between the amount of energy stored in the magnetic field and the amount of current that must exist in order to have that strong a field. After the capacitor reaches $q = 0$, it overshoots. The circuit has its own kind of electrical "inertia," because if charge was to stop flowing, there would have to be zero current through the inductor. But the current in the inductor must be related to the amount of energy stored in its magnetic fields. When the capacitor is at $q = 0$, all the circuit's energy is in the inductor, so it must therefore have strong magnetic fields surrounding it and quite a bit of current going through it.

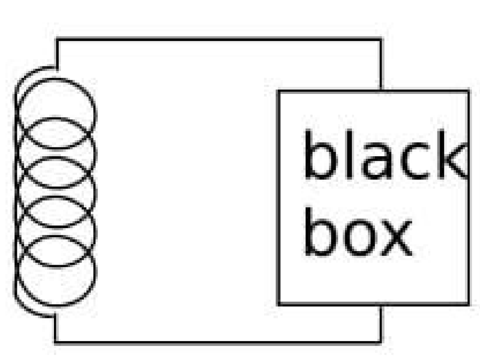
The only thing that might seem spooky here is that we used to speak as if the current in the inductor caused the magnetic field, but now it sounds as if the field causes the current. Actually this is symptomatic of the elusive nature of cause and effect in physics. It's equally valid to think of the cause and effect relationship in either way. This may seem unsatisfying, however, and for example does not really get at the question of what brings about a voltage difference across the resistor (in the case where the resistance is finite); there must be such a voltage difference, because without one, Ohm's law would predict zero current through the resistor.

Voltage, then, is what is really missing from our story so far.

Let's start by studying the voltage across a capacitor. Voltage is electrical potential energy per unit charge, so the voltage difference between the two plates of the capacitor is related to the amount by which its energy would increase if we increased the absolute values of the charges on the plates from q to $q + \Delta q$:

$$\begin{aligned} V_C &= (E_{q+\Delta q} - E_q)/\Delta q \\ &= \frac{\Delta E_C}{\Delta q} \\ &= \frac{\Delta}{\Delta q}\left(\frac{1}{2C}q^2\right) \\ &= \frac{q}{C} \end{aligned}$$

Many books use this as the definition of capacitance. This equation, by the way, probably explains the historical reason why C was defined so that the energy was *inversely* proportional to C for a given value of C: the people who invented the definition were thinking of a capacitor as a device for storing charge rather than energy, and the amount of charge stored for a fixed voltage (the charge "capacity") is proportional to C.



l / The inductor releases energy and gives it to the black box.

In the case of an inductor, we know that if there is a steady, constant current flowing through it, then the magnetic field is constant, and so is the amount of energy stored; no energy is being exchanged between the inductor and any other circuit element. But what if the current is changing? The magnetic field is proportional to the current, so a change in one implies a change in the other. For concreteness, let's imagine that the magnetic field and the current are both decreasing. The energy stored in the magnetic field is therefore decreasing, and by conservation of energy, this energy can't just go away — some other circuit element must be taking energy from the inductor. The simplest example, shown in figure l, is a series circuit consisting of the inductor plus one other circuit element. It doesn't matter what this other circuit element is, so we just call it a black box, but if you like, we can think of it as a resistor, in which case the energy lost by the inductor is being turned into heat by the resistor. The junction rule tells us that both circuit elements have the same current through them, so I could refer to either one, and likewise the loop rule tells us $V_{inductor} + V_{black\ box} = 0$, so the two voltage drops have the same absolute value, which we can refer to as V. Whatever the black box is, the rate at which it is taking energy from the inductor is given by $|P| = |IV|$, so

$$\begin{aligned} |IV| &= \left|\frac{\Delta E_L}{\Delta t}\right| \\ &= \left|\frac{\Delta}{\Delta t}\left(\frac{1}{2}LI^2\right)\right| \\ &= \left|LI\frac{\Delta I}{\Delta t}\right| \qquad , \end{aligned}$$

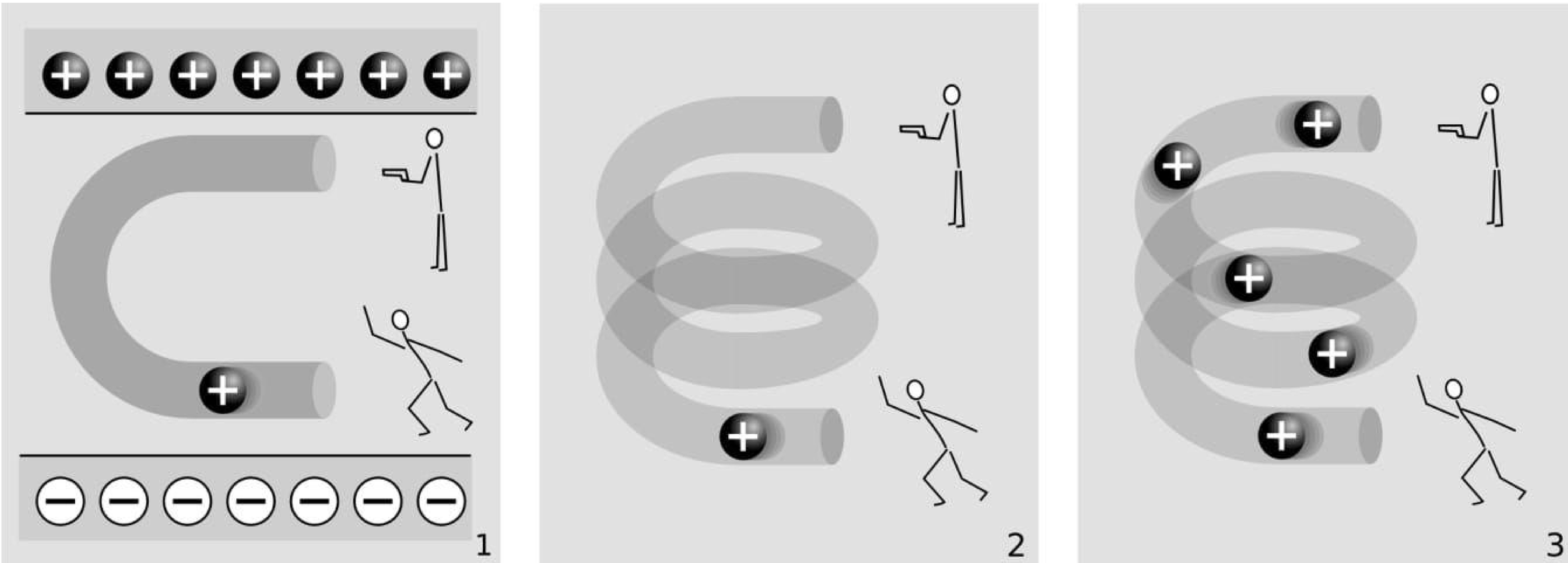
or

$$|V| = \left|L\frac{\Delta I}{\Delta t}\right| \qquad ,$$

which in many books is taken to be the definition of inductance. The direction of the voltage drop (plus or minus sign) is such that the inductor resists the change in current.

There's one very intriguing thing about this result. Suppose, for concreteness, that the black box in figure l is a resistor, and that the inductor's energy is decreasing, and being converted into heat in the resistor. The voltage drop across the resistor indicates that it has an electric field across it, which is driving the current. But where is this electric field coming from? There are no charges anywhere that could be creating it! What we've discovered is one special case of a more general principle, the principle of induction: a changing magnetic field creates an electric field, which is in addition to any electric field created by charges. (The reverse is also true: any electric field that changes over time creates a magnetic field.) Induction forms the basis for such technologies as the generator and the transformer, and ultimately it leads to the existence of light, which is a wave pattern in the electric and magnetic fields. These are all topics for chapter 6, but it's truly remarkable that we could come to this conclusion without yet having learned any details about magnetism.

The cartoons in figure m compares electric fields made by charges, 1, to electric fields made by changing magnetic fields, 2-3. In m/1, two physicists are in a room whose ceiling is positively charged and whose floor is negatively charged. The physicist on the bottom throws a positively charged bowling ball into the curved pipe. The physicist at the top uses a radar gun to measure the speed of the ball as it comes out of the pipe. They find that the ball has slowed down by the time it gets to the top. By measuring the change in the ball's kinetic energy, the two physicists are acting just like a voltmeter. They conclude that the top of the tube is at a higher voltage than the bottom of the pipe. A difference in voltage indicates an



m / Electric fields made by charges, 1, and by changing magnetic fields, 2 and 3.

electric field, and this field is clearly being caused by the charges in the floor and ceiling.

In m/2, there are no charges anywhere in the room except for the charged bowling ball. Moving charges make magnetic fields, so there is a magnetic field surrounding the helical pipe while the ball is moving through it. A magnetic field has been created where there was none before, and that field has energy. Where could the energy have come from? It can only have come from the ball itself, so the ball must be losing kinetic energy. The two physicists working together are again acting as a voltmeter, and again they conclude that there is a voltage difference between the top and bottom of the pipe. This indicates an electric field, but this electric field can't have been created by any charges, because there aren't any in the room. This electric field was created by the change in the magnetic field.

The bottom physicist keeps on throwing balls into the pipe, until the pipe is full of balls, m/3, and finally a steady current is established. While the pipe was filling up with balls, the energy in the magnetic field was steadily increasing, and that energy was being stolen from the balls' kinetic energy. But once a steady current is established, the energy in the magnetic field is no longer changing. The balls no longer have to give up energy in order to build up the field, and the physicist at the top finds that the balls are exiting the pipe at full speed again. There is no voltage difference any more. Although there is a current, $\Delta I/\Delta t$ is zero.

Discussion Questions

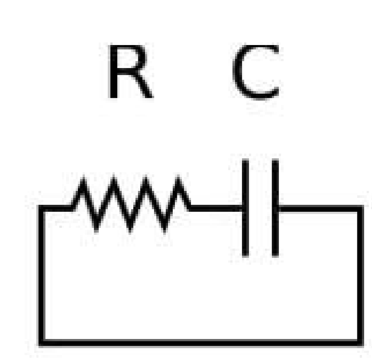
A What happens when the physicist at the bottom in figure m/3 starts getting tired, and decreases the current?

A.4 Decay

Up until now I've soft-pedaled the fact that by changing the characteristics of an oscillator, it is possible to produce non-oscillatory behavior. For example, imagine taking the mass-on-a-spring system and making the spring weaker and weaker. In the limit of small k, it's as though there was no spring whatsoever, and the behavior of the system is that if you kick the mass, it simply starts slowing down. For friction proportional to v, as we've been assuming, the result is that the velocity approaches zero, but never actually reaches zero. This is unrealistic for the mechanical oscillator, which will not have vanishing friction at low velocities, but it is quite realistic in the case of an electrical circuit, for which the voltage drop across the resistor really does approach zero as the current approaches zero.

Electrical circuits can exhibit all the same behavior. For simplicity we will analyze only the cases of LRC circuits with $L = 0$ or $C = 0$.

The RC circuit



n / An RC circuit.

We first analyze the RC circuit, n. In reality one would have to "kick" the circuit, for example by briefly inserting a battery, in order to get any interesting behavior. We start with Ohm's law and the equation for the voltage across a capacitor:

$$V_R = IR$$
$$V_C = q/C$$

The loop rule tells us

$$V_R + V_C = 0 \qquad ,$$

and combining the three equations results in a relationship between q and I:

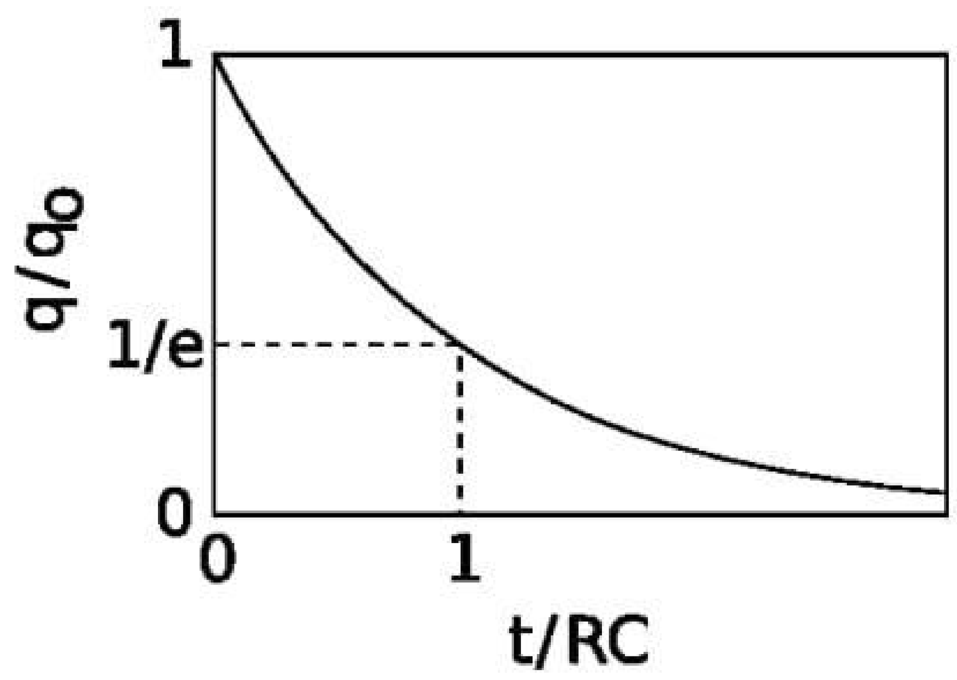
$$I = -\frac{1}{RC}q$$

The negative sign tells us that the current tends to reduce the charge on the capacitor, i.e. to discharge it. It makes sense that the current is proportional to q: if q is large, then the attractive forces between the $+q$ and $-q$ charges on the plates of the capacitor are large, and charges will flow more quickly through the resistor in order to reunite. If there was zero charge on the capacitor plates, there would be no reason for current to flow. Since amperes, the unit of current, are the same as coulombs per second, it appears that the quantity RC must have units of seconds, and you can check for yourself that this is correct. RC is therefore referred to as the time constant of the circuit.

How exactly do I and q vary with time? Rewriting I as $\Delta q/\Delta t$, we have

$$\frac{\Delta q}{\Delta t} = -\frac{1}{RC}q \qquad .$$

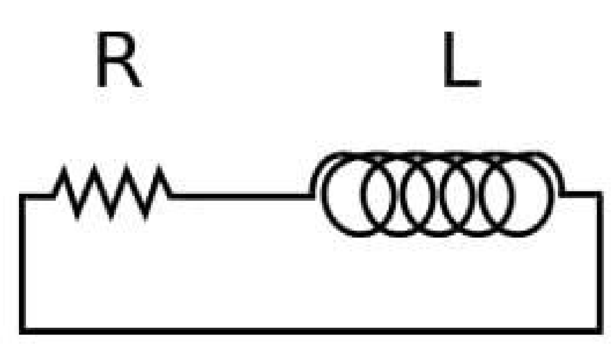
This equation describes a function $q(t)$ that always gets smaller over time, and whose rate of decrease is big at first, when q is big, but gets smaller and smaller as q approaches zero. As an example of this type of mathematical behavior, we could imagine a man who has 1024 weeds in his backyard, and resolves to pull out half of them every day. On the first day, he pulls out half, and has 512 left. The next day, he pulls out half of the remaining ones, leaving 256. The sequence continues exponentially: 128, 64, 32, 16, 8, 4, 2, 1. Returning to our electrical example, the function $q(t)$ apparently needs to be an exponential, which we can write in the form ae^{bt}, where $e = 2.718...$ is the base of natural logarithms. We could have written it with base 2, as in the story of the weeds, rather than base e, but the math later on turns out simpler if we use e. It doesn't make sense to plug a number that has units into a function like an exponential, so bt must be unitless, and b must therefore have units of inverse seconds. The number b quantifies how fast the exponential decay is. The only physical parameters of the circuit on which b could possibly depend are R and C, and the only way to put units of ohms and farads together to make units of inverse seconds is by computing $1/RC$. Well, actually we could use $7/RC$ or $3\pi/RC$, or any other unitless number divided by RC, but this is where the use of base e comes in handy: for base e, it turns out that the correct unitless constant *is* 1. Thus our solution is



o / Over a time interval RC, the charge on the capacitor is reduced by a factor of e.

$$q = q_o \exp\left(-\frac{t}{RC}\right) \qquad .$$

The number RC, with units of seconds, is called the RC time constant of the circuit, and it tells us how long we have to wait if we want the charge to fall off by a factor of $1/e$.



p / An RL circuit.

The RL circuit

The RL circuit, p, can be attacked by similar methods, and it can easily be shown that it gives

$$I = I_o \exp\left(-\frac{R}{L}t\right) \qquad .$$

The RL time constant equals L/R.

Death by solenoid; spark plugs *example 6*

When we suddenly break an RL circuit, what will happen? It might seem that we're faced with a paradox, since we only have two forms of energy, magnetic energy and heat, and if the current stops suddenly, the magnetic field must collapse suddenly. But where does the lost magnetic energy go? It can't go into resistive heating of the resistor, because the circuit has now been broken, and current can't flow!

The way out of this conundrum is to recognize that the open gap in the circuit has a resistance which is large, but not infinite. This large resistance causes the RL time constant L/R to be very small. The current thus continues to flow for a very brief time, and flows straight across the air gap where the circuit has been opened. In other words, there is a spark!

We can determine based on several different lines of reasoning that the voltage drop from one end of the spark to the other must be very large. First, the air's resistance is large, so $V = IR$ requires a large voltage. We can also reason that all the energy in the magnetic field is being dissipated in a short time, so the power dissipated in the spark, $P = IV$, is large, and this requires a large value of V. (I isn't large — it is decreasing from its initial value.) Yet a third way to reach the same result is to consider the equation $V_L = \Delta I/\Delta t$: since the time constant is short, the time derivative $\Delta I/\Delta t$ is large.

This is exactly how a car's spark plugs work. Another application is to electrical safety: it can be dangerous to break an inductive circuit suddenly, because so much energy is released in a short time. There is also no guarantee that the spark will discharge across the air gap; it might go through your body instead, since your body might have a lower resistance.

Discussion Questions

A A gopher gnaws through one of the wires in the DC lighting system in your front yard, and the lights turn off. At the instant when the circuit becomes open, we can consider the bare ends of the wire to be like the plates of a capacitor, with an air gap (or gopher gap) between them. What kind of capacitance value are we talking about here? What would this tell you about the RC time constant?

A.5 Impedance

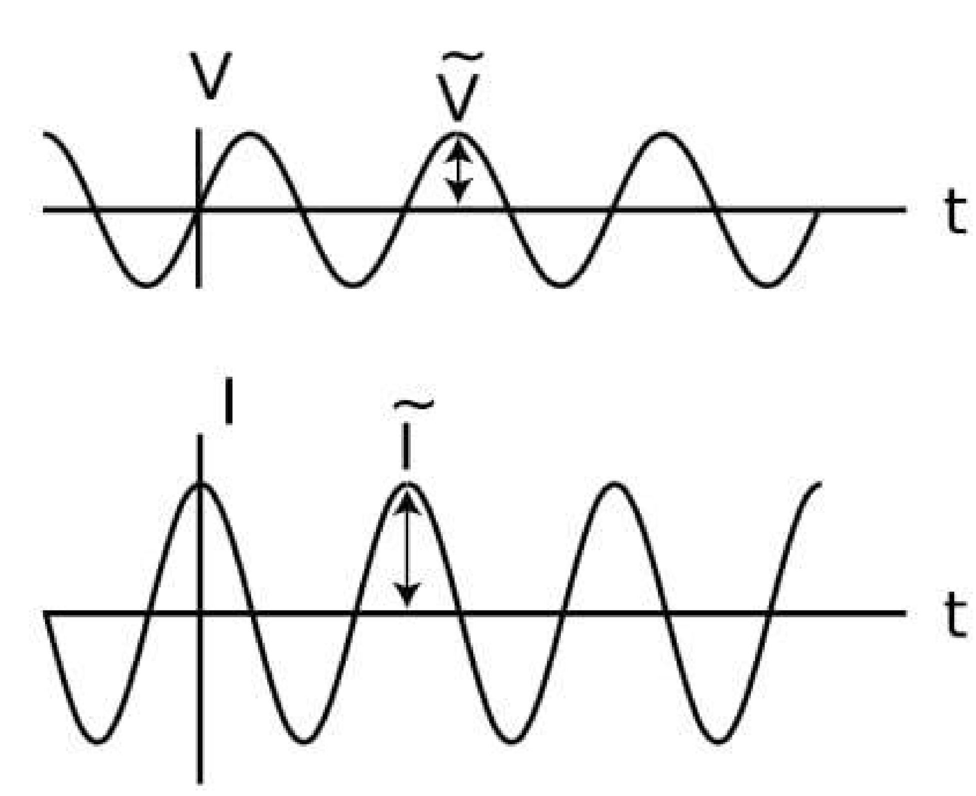
So far we have been thinking in terms of the free oscillations of a circuit. This is like a mechanical oscillator that has been kicked but then left to oscillate on its own without any external force to keep the vibrations from dying out. Suppose an LRC circuit is driven with a sinusoidally varying voltage, such as will occur when a radio tuner is hooked up to a receiving antenna. We know that a current will flow in the circuit, and we know that there will be resonant behavior, but it is not necessarily simple to relate current to voltage in the most general case. Let's start instead with the special cases of LRC circuits consisting of only a resistance, only a capacitance, or only an inductance. We are interested only in the steady-state response.

The purely resistive case is easy. Ohm's law gives

$$I = \frac{V}{R} \quad .$$

In the purely capacitive case, the relation $V = q/C$ lets us calculate

$$\begin{aligned} I &= \frac{\Delta q}{\Delta t} \\ &= C\frac{\Delta V}{\Delta t} \quad . \end{aligned}$$



q / In a capacitor, the current is 90° ahead of the voltage in phase.

If the voltage varies as, for example, $V(t) = \tilde{V}\sin(\omega t)$, then the current will be $I(t) = \omega C\tilde{V}\cos(\omega t)$, so the maximum current is $\tilde{I} = \omega C\tilde{V}$. By analogy with Ohm's law, we can then write

$$\tilde{I} = \frac{\tilde{V}}{Z_C} \quad ,$$

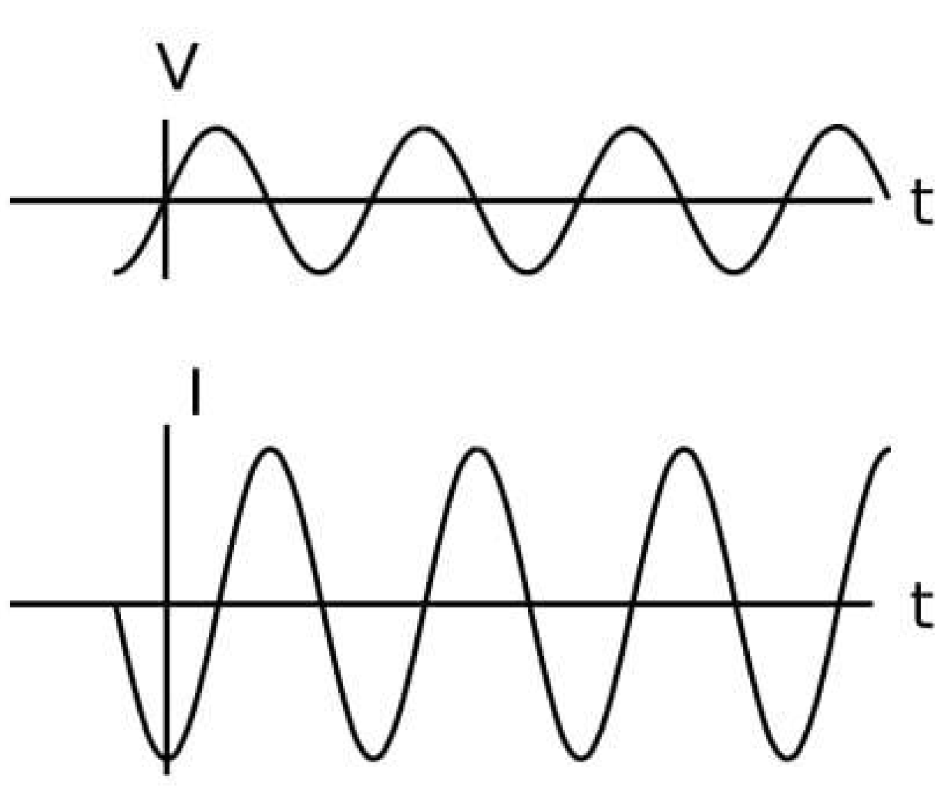
where the quantity

$$Z_C = \frac{1}{\omega C} \quad , \qquad \text{[impedance of a capacitor]}$$

having units of ohms, is called the *impedance* of the capacitor at this frequency. Note that it is only the *maximum* current, $\tilde{I}$, that is proportional to the *maximum* voltage, $\tilde{V}$, so the capacitor is not behaving like a resistor. The maxima of V and I occur at different times, as shown in figure q. It makes sense that the impedance becomes infinite at zero frequency. Zero frequency means that it would take an infinite time before the voltage would change by any amount. In other words, this is like a situation where the capacitor has been connected across the terminals of a battery and been allowed to settle down to a state where there is constant charge on both terminals. Since the electric fields between the plates are constant, there is no energy being added to or taken out of the field. A capacitor that can't exchange energy with any other circuit component is nothing more than a broken (open) circuit.

self-check C

Why can't a capacitor have its impedance printed on it along with its capacitance? ▷ Answer, p. 196



r / The current through an inductor lags behind the voltage by a phase angle of 90°.

Similar math gives

$$Z_L = \omega L \qquad \text{[impedance of an inductor]}$$

for an inductor. It makes sense that the inductor has lower impedance at lower frequencies, since at zero frequency there is no change in the magnetic field over time. No energy is added to or released from the magnetic field, so there are no induction effects, and the inductor acts just like a piece of wire with negligible resistance. The term "choke" for an inductor refers to its ability to "choke out" high frequencies.

The phase relationships shown in figures q and r can be remembered using my own mnemonic, "eVIL," which shows that the voltage (V) leads the current (I) in an inductive circuit, while the opposite is true in a capacitive one. A more traditional mnemonic is "ELI the ICE man," which uses the notation E for emf, a concept closely related to voltage.

Low-pass and high-pass filters *example 7*

An LRC circuit only responds to a certain range (band) of frequencies centered around its resonant frequency. As a filter, this is known as a bandpass filter. If you turn down both the bass and the treble on your stereo, you have created a bandpass filter.

To create a high-pass or low-pass filter, we only need to insert a capacitor or inductor, respectively, in series. For instance, a very basic surge protector for a computer could be constructed by inserting an inductor in series with the computer. The desired 60 Hz power from the wall is relatively low in frequency, while the surges that can damage your computer show much more rapid time variation. Even if the surges are not sinusoidal signals, we can think of a rapid "spike" qualitatively as if it was very high in frequency — like a high-frequency sine wave, it changes very rapidly.

Inductors tend to be big, heavy, expensive circuit elements, so a simple surge protector would be more likely to consist of a capacitor in *parallel* with the computer. (In fact one would normally just connect one side of the power circuit to ground via a capacitor.) The capacitor has a very high impedance at the low frequency of the desired 60 Hz signal, so it siphons off very little of the current. But for a high-frequency signal, the capacitor's impedance is very small, and it acts like a zero-impedance, easy path into which the current is diverted.

The main things to be careful about with impedance are that (1) the concept only applies to a circuit that is being driven sinusoidally, (2) the impedance of an inductor or capacitor is frequency-dependent, and (3) impedances in parallel and series don't combine according to the same rules as resistances. It is possible, however, to get get around the third limitation, as discussed in subsection .

Discussion Question

A Figure q on page 179 shows the voltage and current for a capacitor. Sketch the *q-t* graph, and use it to give a physical explanation of the phase relationship between the voltage and current. For example, why is the current zero when the voltage is at a maximum or minimum?

B Relate the features of the graph in figure r on page 180 to the story told in cartoons in figure m/2-3 on page 176.

Problems

Key

√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 If an FM radio tuner consisting of an LRC circuit contains a 1.0 μH inductor, what range of capacitances should the variable capacitor be able to provide? √

2 (a) Show that the equation $V_L = L\,\Delta I/\Delta t$ has the right units.
(b) Verify that RC has units of time.
(c) Verify that L/R has units of time.

3 Find the energy stored in a capacitor in terms of its capacitance and the voltage difference across it. √

4 Find the inductance of two identical inductors in parallel.

5 The wires themselves in a circuit can have resistance, inductance, and capacitance. Would "stray" inductance and capacitance be most important for low-frequency or for high-frequency circuits? For simplicity, assume that the wires act like they're in *series* with an inductor or capacitor.

6 (a) Find the capacitance of two identical capacitors in series.
(b) Based on this, how would you expect the capacitance of a parallel-plate capacitor to depend on the distance between the plates?

7 Find the capacitance of the surface of the earth, assuming there is an outer spherical "plate" at infinity. (In reality, this outer plate would just represent some distant part of the universe to which we carried away some of the earth's charge in order to charge up the earth.) √

8 Starting from the relation $V = L\Delta I/\Delta t$ for the voltage difference across an inductor, show that an inductor has an impedance equal to $L\omega$.