10 Boundary Layer Theory and Flow Over Surfaces

10.0 INTRODUCTION

Ideal inviscid fluids do not exert any force on the surfaces over which they flow. Real fluids have viscosity. When these fluids flow over surfaces, "no slip condition" prevails. The layer near the surface has to have the same velocity as the surface. If the surface is at rest, then this layer comes to rest. The adjacent layer is retarded to a lesser extent and this proceeds to layers more removed from the surface at rest. A velocity gradient forms leading to shear force being exerted over the layers. The velocity gradient is steepest at the interface and the shear is also highest at the interface. Work is to be done to overcome the force. The equations for the analysis of the complete flow field has been formulated by Navier and Stokes. But solutions for these equations for practical boundary conditions were not available. For a long time empirical equations based on experimental results were used in designs.

The development of boundary layer theory enabled the analysis of such flows to be fairly easy. The theory was proposed by Ludwig Prandtl in 1904. He observed that **in the case of real fluids velocity gradient existed only in a thin layer near the surface. This layer was named as boundary layer**. Beyond this layer the effect of viscosity was found negligible. This was supported by measurement of velocity. The flow field now can be divided into two regions, one in which velocity gradient and shear existed and another where viscous effects are negligible. This region can be dealt with as flow of inviscid fluid or ideal fluid. In the study of flow over immersed bodies like aircraft wings the analysis can be limited to the boundary layer, instead of the field extending to long distances for the determination of forces exerted on the surface by the fluid flowing over it.

10.1 BOUNDARY LAYER THICKNESS

In the solution of the basic equations describing the flow namely continuity and momentum equations of the boundary layer, one boundary is provided by the solid surface. The need for the other boundary is met by edge of the boundary layer determined by the thickness. The determination of the velocity variation along the layer enables the determination of velocity gradient. This is made possible by these two boundary conditions. Once the velocity gradient at the surface is determined, the shear stress can be determined using the equation

$$\tau = \mu \frac{du}{dy} \tag{10.1.1}$$

This leads to the determination of resistance due to the flow.

10.1.1 Flow Over Flat Plate

The simplest situation that can be analyzed is the flow over a flat plate placed parallel to uniform flow velocity in a large flow field. The layer near the surface is retarded to rest or zero. velocity. The next layer is retarded to a lower extent. This proceeds farther till the velocity equals the free stream velocity. As the distance for this condition is difficult to determine, **the boundary layer thickness is arbitrarily defined as the distance from the surface where the velocity is 0.99 times the free stream velocity.**

There are two approaches for the analysis of the problem.

1. Exact method : Solution of the differential equations describing the flow using the boundary conditions. It is found that this method can be easily applied only to simple geometries.

2. Approximate method : Formulation of integral equations describing the flow and solving them using an assumed velocity variation satisfying the boundary conditions. This method is more versatile and results in easier solution of problems. The difference between the results obtained by the exact method and by the integral method is found to be within acceptable limits.

At present several computer softwares are available to solve almost any type of boundary, and the learner should become familiar with such softwares if he is to be current.

10.1.2 Continuity Equation

The flow of fluid over a flat plate in a large flow field is shown in Fig. 10.1.1. The flow over the top surface alone is shown in the figure.

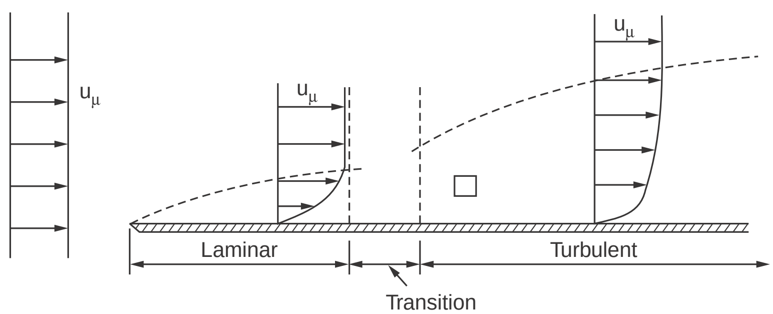

Figure 10.1.1 Formation of boundary layer over flat plate

The velocity is uniform in the flow field having a value of u_∞. Boundary layer begins to form from the leading edge and increases in thickness as the flow proceeds. This is because the viscosity effect is felt at layers more and more removed from the surface. At the earlier stages the flow is regular and layers keep their position and there is no macroscopic mixing between layers. **Momentum transfer resulting in the retarding force is by molecular diffusion**

between layers. This type of flow is called laminar flow and analysis of such flow is somewhat simpler. **Viscous effects prevail over inertial effects in such a layer**. Viscous forces maintain orderly flow. **As flow proceeds farther, inertial effects begin to prevail over viscous forces resulting in macroscopic mixing between layers. This type of flow is called turbulent flow**. Higher rates of momentum transfer takes place in such a flow. For the formulation of the differential equations an element of size $dx \times dy \times 1$ is considered.

An enlarged sectional view of the element is shown in Fig. 10.1.2.

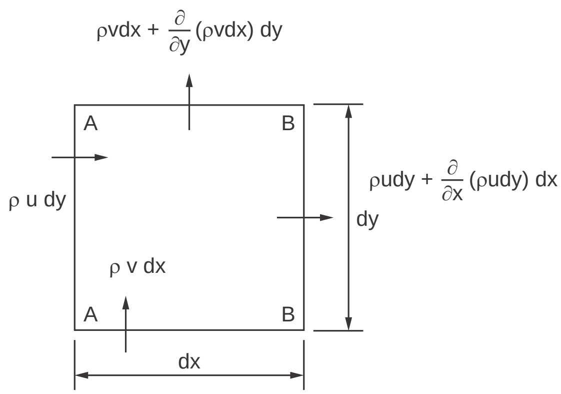

Figure 10.1.2 Enlarged view of element in the boundary layer

The assumptions are (*i*) flow is incompressible or density remains constant, (*ii*) flow is steady, (*iii*) there is no pressure gradient in the boundary layer.

Continuity equation is obtained using the principle of conservation of mass. Under steady flow conditions the net mass flow across the element should be zero. Under unsteady conditions, the net mass flow should equal the change of mass in the elemental volume considered. The values of velocities are indicated in the figure. The density of the fluid is ρ. Unit time and unit *Z* distance are assumed. **Time is not indicated in the equations.**

Flow in across face *AA*, $\rho u dy \times 1 = \rho u dy$

Flow out across face *BB*, $\rho u dy + \dfrac{\partial}{\partial x}(\rho u dy)\, dx$

Net flow in the x direction = $\dfrac{\partial(\rho u)}{\partial x}\, dxdy$

Similarly the net flow in the y direction is given by $\dfrac{\partial(\rho v)}{\partial x}\, dxdy$

Under steady conditions the sum is zero. Also for incompressible flow density is constant. Hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (10.1.2)$$

This is known as continuity equation for steady incompressible flow. If u decreases, $\dfrac{\partial u}{\partial x}$ is – ve and so $\dfrac{\partial v}{\partial y}$ should be positive. The algebraic sum of x and y directional flows is zero.

10.1.3 Momentum Equation

The equation is based on Newton's second law of motion. The net force on the surface of the element should equal the rate of change of momentum of the fluid flowing through the element. Here x directional forces are considered with reference to the element shown in Fig. 10.1.3. The flows are indicated on the figure unit time and unit Z distance are assumed. The density of the fluid is ρ

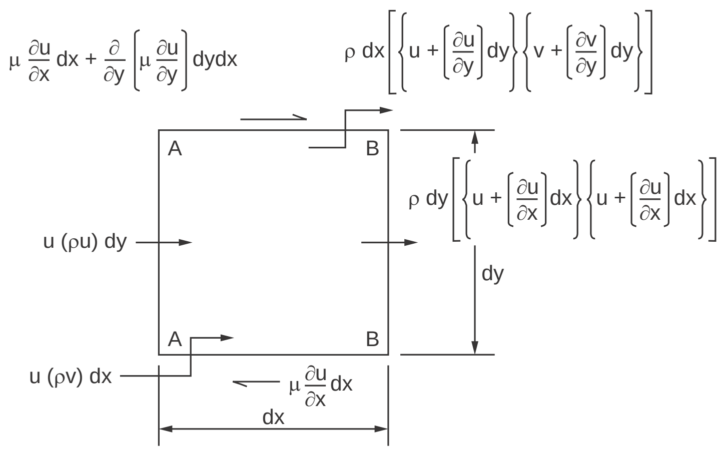

Figure 10.1.3 Momentum analysis

Consider the momentum flow in the x direction :

Across AA momentum flow = $u\,(\rho u)\,dy$

Across BB momentum flow = $u\,(\rho u)\,dy + \dfrac{\partial}{\partial x}\{u(\rho u)\,dy\}\,dx$

Taking the difference, the net flow is (as ρ is constant) (u^2 is written as $u \times u$)

$$\frac{\partial}{\partial x}[u(\rho u)dy]dx = \rho\, dxdy\left[u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\right]$$

Considering the flow in the y direction, the net x directional momentum flow is

$$\frac{\partial}{\partial y}[u(\rho v)dy]dx = \rho\, dxdy\left[u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y}\right]$$

Summing up, the net momentum flow is

$$\rho\, dxdy\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \left\{u\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]\right\}\right]$$

From continuity equation, the second set in the above equation is zero. Hence net x directional momentum flow is

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]\rho\, dxdy$$

It was assumed that no body forces or pressure forces are present. Only surface forces due to viscosity is considered.

At the bottom surface shear $= dx\,\mu \dfrac{\partial u}{\partial y}$

At the top surface shear $= dx\,\mu \dfrac{\partial u}{\partial y} + \dfrac{\partial}{\partial y}\left[\mu \dfrac{\partial u}{\partial y} dx\right] dy$

The net shear on the element is $\mu \dfrac{\partial^2 u}{\partial y^2} dxdy$, noting $\upsilon = \mu/\rho$

$$\text{Equating, and simplifying, } u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} \tag{10.1.3}$$

This is known as momentum equation for the boundary layer. υ is also called as momentum diffusivity. In case of pressure gradient along the flow $-\dfrac{1}{\rho}\dfrac{\partial P}{\partial x}$ has to added on the RHS.

10.1.4 Solution for Velocity Profile

The continuity and momentum equations should be simultaneously solved to obtain the velocity profile. The boundary conditions are

(*i*) at $y = 0$, $u = 0$, (*ii*) at $y = \delta$, $u = u_\infty$, $\dfrac{\partial u}{\partial y} = 0$

The solution for these equations was obtained by Blasius in 1908 first by converting the partial differential equation into a third order ordinary differential equation and then using numerical method.

The two new vaiables introduced were

$$\eta = y\sqrt{\frac{u_\infty}{x\upsilon}} \text{ and } f(\eta) = \psi/\sqrt{\upsilon x u_\infty} \tag{10.1.4}$$

where ψ is the stream function giving

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{10.1.5}$$

The resulting ordinary differential equation is

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0 \tag{10.1.6}$$

the boundary conditions with the new variables are

at $y = 0$, $\eta = 0$ and $\dfrac{\partial f}{\partial \eta} = 0$, at $y = \infty$, $\eta = \infty$ and $\dfrac{\partial f}{\partial \eta} = 1$

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The results where plotted with u/u_∞ as the dependent vairable and $y\sqrt{\frac{u_\infty}{vx}}$ or (η) as the independent variable resulting in a plot as shown in Fig. 10.1.4.

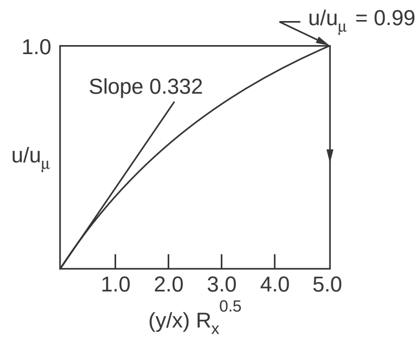

Figure 10.1.4 Velocity distribution in boundary layer

The value of $y\sqrt{\frac{u_\infty}{vx}}$ where u/u_∞ = 0.99 is found to be 5. This y value is taken as the boundary layer thickness δ as per the definition of thickness of boundary layer.

i.e. $$\delta\sqrt{u_\infty / vx} = 5, \quad \text{or} \quad \delta = \frac{5}{\sqrt{u_\infty / vx}} = \frac{5x}{\sqrt{u_\infty x / v}} \; \frac{5x}{\sqrt{Re_x}} = 5x\, Re_x^{-0.5} \quad (10.1.7)$$

This equation was more precisely solved in 1983 by Howarth. The significance of Reynolds number has already been explained under dimensional analysis as the ratio of inertia force to viscous force. Velocity gradient at the surface is of greater importance because it decides the shear on the surface at $y = 0$

$$\tau_w = \mu\frac{\partial u}{\partial y} \text{ equals the value of } \mu u_\infty \sqrt{u_\infty / vx}\,\frac{d^2 f}{d\eta^2}, \text{ at } \eta = 0$$

From the solution, at $\eta = 0$, the value of $\frac{d^2 f}{d\eta^2}$, is obtained as 0.332

Substituting this value and replacing v by μ/ρ and simplifying

$$\tau_w = 0.332\ \rho u_\infty^{\ 2}/\sqrt{Re_x} \quad (10.1.8)$$

Defining skin friction coefficient, C_{fx}, as $\tau_w/(1/2)\rho u_\infty^{\ 2}$, we obtain

$$C_{fx} = 0.664\ Re_x^{-0.5} \quad (10.1.9)$$

The average value over length L can be obtained by using

$$C_f = \frac{1}{L}\int_0^L C_{fx}\, dx = 1.328\ Re_L^{-0.5} \quad (10.1.10)$$

Not that these results are obtained for laminar flow over flat plate for $Re < 5 \times 10^5$.

Example 10.1. ***Air*** *at 30° C flows over a flat plate at a free stream velocity of 5m/s.* ***Determine the boundary layer thickness*** *at distances 0.2 m, 0.5 m and 0.8 m. Also determine the skin friction coefficients, both local and average, at these locations.*

The property values for air at 30 °C are obtained from tables. $\rho = 1.165$ kg/m^3,

$$v = 16 \times 10^{-6} \text{ m}^2\text{/s}, \ \mu = 18.63 \times 10^{-6} \text{ kg/ms.}$$

$$\delta = 5x\, Re_x^{-0.5},\ C_{fx} = 0.664\, Re_x^{-0.5},\ C_{fL} = 1.328\, Re_L^{-0.5}$$

Consider 0.5 m, $$Re_x = \frac{ux}{v} = \frac{5 \times 0.5}{16 \times 10^{-6}} = 1.5625 \times 10^5 < 5 \times 10^5 \quad \therefore \quad \text{Laminar}$$

$$\therefore \qquad \delta = 6.325 \text{ mm},\ C_{fx} = 1.68 \times 10^{-3},\ C_{fL} = 3.36 \times 10^{-3}$$

The values for other distances are tabulated below.

Distance, m	Re	δ, mm	C_{fx}	C_{fL}
0.2	0.63×10^5	**4.000**	2.66×10^{-3}	5.32×10^{-3}
0.5	1.56×10^5	**6.325**	1.68×10^{-3}	3.36×10^{-3}
0.8	2.5×10^5	**8.000**	1.33×10^{-3}	2.66×10^{-3}

Note that as distance increases the local skin friction factor decreases and the average value is higher than the local value. Also note that the boundary layer thickness increases along the flow direction.

Example 10.2. ***Water*** *at 20° C flows over a flat plate at a free stream velocity of 0.2 m/s.* ***Determine the boundary layer*** *thickness and friction factors at lengths 0.2, 0.5 and 0.8 m from leading edge. The value of kinematic viscosity = 1.006×10^{-6} m^2/s, $\mu = 1.006 \times 10^{-3}$ kg/ms.*

The values calculated using equation 10.1.7, 9 and 10 are tabulated below:

Length, m	Re	δ, mm	C_{fx}	C_{fL}
0.2	0.40×10^5	**5.02**	3.33×10^{-3}	6.66×10^{-3}
0.5	0.99×10^5	**7.93**	2.11×10^{-3}	4.21×10^{-3}
0.8	1.59×10^5	**10.03**	1.67×10^{-3}	3.33×10^{-3}

Note the same trends as in Example 1. Also note that because of higher viscosity the friction values are higher.

10.1.5 Integral Method

In this case flow rate, momentum etc. in the boundary layer are determined using integration over the thickness of the boundary layer. The control volume chosen is shown in Fig. 10.1.5.

There is no flow through the face ad. (consider unit plate width)

$$\text{Flow through face} \quad ab = \int_0^H \rho u dy$$

Flow through face $cd = \int_0^H \rho u dy + \frac{d}{dx}\left[\int_0^H \rho u dy\right] dx$

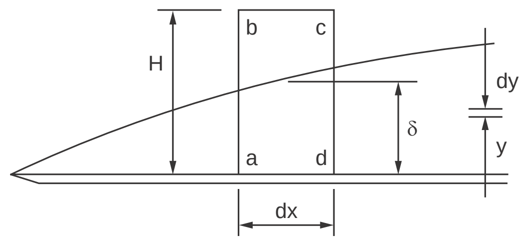

Figure 10.1.5 Boundary layer element for integral analysis

The difference should flow through *bc* as no flow is possible across *ad*.

$\therefore$ Flow through face $bc = -\frac{d}{dx}\left[\int_0^H \rho u dy\right] dx$

This is the result of continuity principle. Considering *x* directional momentum,

Momentum flow through $ab = \int_0^H u\rho u dy$

Momentum flow through $cd = \int_0^H u\rho u dy + \frac{d}{dx}\left[\int_0^H u\rho u dy\right] dx$

The mass crossing the boundary *bc* has a velocity of u_∞

Momentum flow through $bc = -\frac{d}{dx}\left[\int_0^H u_\infty \rho u dy\right] dx$

Summing up, the net momentum flow through the control volume

$$= \frac{d}{dx}\left[\int_0^H (u - u_\infty)\, \rho u dy\right] dx \qquad \text{... (1)}$$

As $(u - u_\infty)$ is zero beyond δ, the integration limit can be taken as δ instead of *H*. It is assumed that there is no pressure gradient in the boundary layer. The velocity gradient at face *bc* is zero. So the only force on the control volume surface is

$$-\tau_w\, dx = -\mu \frac{du}{dy} dx, \text{ Equating}$$

$$\frac{d}{dx}\left[\int_0^\delta (u_\infty - u)\, \rho u dy\right] = \mu \left.\frac{du}{dy}\right|_{y=0} \qquad (10.1.11)$$

or $$\frac{d}{dx}\left[u_\infty^2 \int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy\right] = v\left.\frac{du}{dy}\right|_{y=0} \tag{10.1.12}$$

This is called momentum integral equation. The boundary conditions are

at $y = 0$, $u = 0$; at $y = \delta$, $u = u_\infty$ and $\frac{du}{dy} = 0$

Also $\frac{d^2u}{dy^2} = 0$ at $y = 0$ (constant pressure gradient)

Equation 10.1.12 can be solved if a velocity profile satisfying the boundary conditions is assumed. Out of the popularly used profiles the results obtained from a cubic profile given below is in closer agreement with the exact solution.

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left[\frac{y}{\delta}\right]^3 \tag{10.1.13}$$

Substituting in equation 10.1.12

$$\frac{d}{dx}\left\{u_\infty{}^2 \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] \times \left[1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy\right\} = v\left.\frac{du}{dy}\right|_{y=0}$$

Carrying out the integration, gives

$$\frac{d}{dx}\left[\frac{39}{280}u_\infty^2\,\delta\right] = \frac{3}{2}v\frac{u_\infty}{\delta} \tag{10.1.14}$$

or $$\frac{39}{280}u_\infty^2\frac{d\delta}{dx} = \frac{3}{2}v\frac{u_\infty}{\delta}$$, Separating variables and integrating

$$\int_0^x \delta\, d\delta = \int_0^x \frac{140}{13}\frac{v}{u_\infty}dx \quad \text{at } x = 0,\ \delta = 0. \text{ This leads to}$$

$$\delta = 4.64\, x\sqrt{\frac{v}{u_\infty x}} = 4.64x/\text{Re}_x{}^{0.5} \tag{10.1.15}$$

This solution is closer to the exact solution where the constant is 5 instead of 4.64. The value of C_{fx} can be determined using the assumed velocity profile.

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left[\frac{y}{\delta}\right]^3, \quad \therefore \left.\frac{du}{dy}\right|_{y=0} = u_\infty\left[\frac{3}{2\delta}\right] \quad \therefore \tau_w = \mu u_\infty\left[\frac{3}{2\delta}\right]$$

$$C_f = \tau_w/\{(1/2)\,\rho\, u_\infty{}^2\} = \frac{3\mu u_\infty}{2\delta}\frac{2}{\rho u_\infty^2} \quad \text{As } \delta = 4.64x/\text{Re}^{1/2}$$

$$\therefore \quad C_{fx} = \frac{3\mu u_\infty}{2 \times 4.64 \times x}\text{Re}_x^{0.5}\frac{2}{\rho u_\infty^2}$$

$$C_{fx} = \frac{3}{4.64}\frac{\mu}{\rho u_\infty x}\text{Re}_x^{0.5} = 0.646/\text{Re}_x{}^{1/2} \tag{10.1.16}$$

Compared to $0.664/\mathrm{Re}_x^{1/2}$ by exact solution.

Due to flexibility this method becomes more versatile as compared to the exact method. Analysis using linear and sine function profiles illustrated under solved problems.

10.1.6 Displacement Thickness

Compared to the thickness δ in free stream, the flow in the boundary layer is reduced due to the reduction in velocity which is the result of viscous forces. In the absence of the boundary layer the flow rate that would pass through the thickness δ will be higher. The idea is illustrated in Fig. 10.1.6.

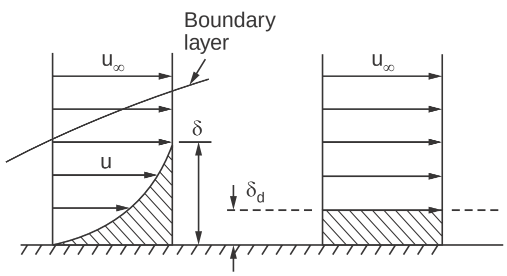

Figure 10.1.6 Displacement thickness

The reduction in volume flow is given by (for unit width)

$$= \int_0^{\delta} \rho\,(u_\infty - u)\,dy$$

If viscous forces were absent the velocity all through the thickness δ will be equal to u_∞. A thickness δ_d can be defined by equating the reduction in flow to a uniform flow with velocity u_∞ or $\rho u_\infty \delta_d$

$$\delta_d = \int_0^{\delta} \frac{(u_\infty - u)}{u_\infty}\,dy = \int_0^{\delta}\left(1 - \frac{u}{u_\infty}\right)dy \qquad (10.1.17)$$

Displacing the boundary by a distance δ_d would pass the flow in the boundary layer at free stream velocity.

Displacement thickness δ_d is the distance by which the solid boundary would have to be displaced in a frictionless flow to give the same mass flow rate as with the boundary layer.

The displacement thickness will equal δ/3. The can be shown by assuming polynomial variation for velocity u in the boundary layer. Assuming (as there are three boundary conditions) the distribution,

$$u = a + by + cy^2, \text{ with boundary conditions,}$$

(*i*) $u = 0$ at $y = 0$, (*ii*) $u = u_\infty$ at $y = \delta$ and $(du/dy) = 0$ at $y = \delta$

The first condition gives $a = 0$ and from the other two conditions

$$c = -\,u_\infty/\delta^2 \quad \text{and} \quad b = 2u_\infty/\delta$$

Hence the profile is $\dfrac{u}{u_\infty} = 2\dfrac{y}{\delta} - \left[\dfrac{y}{\delta}\right]^2$ (10.1.18)

Note that this is different from the profile previously assumed for the solution of momentum integral equation. Substituting in (10.1.17) and integrating,

$$\delta_d = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy = \int_0^\delta \left\{1 - \frac{2y}{\delta} + \left(\frac{y}{\delta}\right)\right\} dy = \left[y - \frac{y^2}{\delta} - \frac{1}{3}\frac{y^3}{\delta^2}\right]_0^\delta = (1/3)\delta$$

i.e., δ_d = δ/3 or displacement thickness equals one third of hydrodynamic boundary layer thickness. In case other profiles are adopted, this constant will be different. But this is the value nearer the Blasius solution.

Example 10.3. *Using data of problems* ***Examples 10.1 and 10.2 determine the displacement thickness*** *at the various locations. Also determine the flow out of the boundary layer in the y direction and the average values of velocity v in these sections.*

The deficit flow should go out of the top of the boundary layer. From example 10.1 **air flow at** 30 °C with free stream velocity 5 m/s, (unit width is assumed)

Distance	δ, mm	δ_d, mm	Volume flow, m³/s	$V_{(0-x)}$, m/s
0.2	4.0	**1.333**	$1.333 \times 5 \times 10^{-3}$	0.0333
0.5	6.325	**2.108**	$2.108 \times 5 \times 10^{-3}$	0.0211
0.8	8.00	**2.666**	$2.666 \times 5 \times 10^{-3}$	0.0167

The volume flow out (deficit flow) equals $\delta_d\, u_\infty$ × width, assuming 1 m width

∴ between $x = 0$ and $x = 0.2$ flow is $1.333 \times 5 \times 10^{-3}$ m³/s.

The average velocity, V = volume/area, Area = 1×0.2 m².

∴ $V = 1.333 \times 5 \times 10^{-3}/0.2 = 0.0333$ m/s. For other lengths values are tabulated above. In the case of **example 10.2, water flow** the values are given below,

Distance, m	δ, mm	δ_d, mm	flow rate, m³/s	$V_{(0-x)}$, m/s
0.2	5.02	**1.673**	3.35×10^{-4}	1.67×10^{-3}
0.5	7.93	**2.643**	5.29×10^{-4}	1.06×10^{-3}
0.8	10.03	**3.343**	6.69×10^{-4}	0.84×10^{-3}

10.1.7 Momentum Thickness

Similar to the conditions discussed in section (10.1.6) for displacement thickness, there is a reduction in momentum flow through the boundary layer as compared to the momentum flow in a thickness δ at free stream velocity.

The thickness which at free stream velocity will have the same momentum flow as the dificit flow is called momentum thickness. The deficit flow at any thin layer at y of thickness dy is (for unit width) $\rho\,(u_\infty - u)\,dy$

Momentum for this flow is $\rho u\,(u_\infty - u)\,dy$

Hence the deficit momentum = $\int_0^\delta \rho u\,(u_\infty - u)\,dy$

Considering δ_m as momentum thickness,

$$\delta_m\ \rho u_\infty\ u_\infty = \int_0^\delta \rho u\,(u_\infty - u)\,dy$$

$$\delta_m = \int_0^\delta \left[\frac{u}{u_\infty} - \left(\frac{u}{u_\infty}\right)^2\right] dy = \int_0^\delta \frac{u}{u_\infty}\left[1 - \frac{u}{u_\infty}\right] dy \qquad (10.1.19)$$

The concept of reduction in momentum is shown in Fig. 10.1.7.

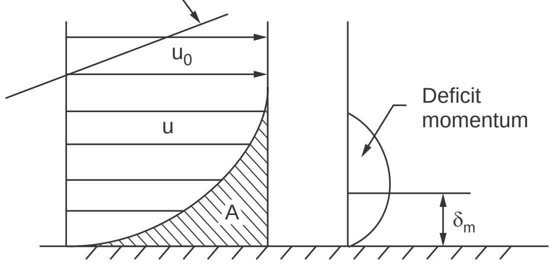

Figure 10.1.7 Momentum thickness

The value of momentum thickness is generally taken as 1/7th of boundary layer thickness in laminar flow. The value will vary with the assumption about velocity distribution. For example if the velocity profiles as in the previous article is used, then

$$\frac{u}{u_\infty} = 2\,\frac{y}{\delta} - \left[\frac{y}{\delta}\right]^2 \quad \text{substituting in 10.1.19 and simplyifying}$$

$$\delta_m = \int_0^\delta \left[2\,\frac{y}{\delta} - 5\left[\frac{y}{\delta}\right]^2 + 4\left[\frac{y}{\delta}\right]^3 - \left[\frac{y}{\delta}\right]^4\right] dy$$

$$= \delta - \frac{5}{3}\delta + \delta - \frac{1}{5}\delta = \frac{2}{15}\delta = \left(\frac{1}{7.5}\right)\delta$$

10.2 TURBULENT FLOW

As flow preceeds farther along the flat plate, inertia forces begin to prevail and viscous forces are unable to keep the flow in an orderly way. **Reynolds number is the ratio of inertia force to viscous force. As inertia force increases Reynolds number increases and the flow becomes turbulent**. Generally the limiting Reynolds number for laminar flow over flat plate is taken as 5×10^5 (for internal flow the critical Reynolds number is 2000).

Turbulent flow is characterized by the variation of velocity with time at any location. The velocity at any location at any time, can be represented by

$$u = \bar{u} + u'$$

where u is the instantaneous velocity, $\bar{u}$ is the average over time and u' is the fluctuating component. The flow is steady as u' is constant at any location. An accurate velocity profile known as universal velocity profile, having different distributions at different heights is available. However it is too cmplex for use with integral method at our level of discussion.

One seventh power law has been adopted as a suitable velocity distribution for turbulent flow.

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \tag{10.2.1}$$

Substituting in the integral momentum equation 10.1.2, boundary layer thickness is obtained as

$$\delta = 0.382\, x/\mathrm{Re}_x^{\,0.2} \tag{10.2.2}$$

For combined laminar and turbulent flow,

$$\delta_L = (0.381x/\mathrm{Re}_L^{\,0.2}) - (10256/\mathrm{Re}_L) \tag{10.2.2 a}$$

The friction coefficient is obtianed as

$$C_{fx} = 0.0594/\mathrm{Re}_x^{\,0.2} \tag{10.2.3}$$

for combined laminar turbulent flow

$$C_{fL} = 0.074\mathrm{Re}^{-0.2} - 1742\mathrm{Re}_L^{\,-1} \tag{10.2.4}$$

Displacement thickness is obtained as $\delta_d = \delta/8$

Example 10.4. ***Water flows*** *at a velocity of 1.2 m/s over a flat plate 1.2 m long. Assume 1/7th power law and* ***determine the boundary layer thickness and displacement thickness.*** *Compare the values with values calculated using laminar flow correlations.*

$$v = 1.006 \times 10^{-6}\ m^2/s.$$

$$\mathrm{Re} = \frac{ux}{v} = \frac{1.2 \times 1.2}{1.006 \times 10^{-6}} = 1.43 \times 10^6 > 5 \times 10^5 \text{ So the flow is turbulent}$$

$$\boldsymbol{\delta_L} = 0.382x/Re_L^{\,0.2} = 0.0269\ m \text{ or } \mathbf{26.9\ mm}$$

$$\delta_d = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy = \int_0^\delta \left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy = \left[y - \frac{7}{8}\frac{y^{1+\frac{1}{7}}}{\delta^{1/7}}\right]_0^\partial = \frac{1}{8}\delta$$

$$\delta_{dL} = 26.9/8 = 3.37 \text{ mm}, \; C_f = 0.0594/\mathrm{Re}^{0.2} = 0.003488$$

$$\tau_w = C_f (1/2)\, \rho\, u_\infty^{\,2} = (0.003488/2) \times 1000 \times 1.2^2 = 2.51 \text{ N/m}^2$$

In case laminar flow correlations were used:

$$\delta = 5x/\mathrm{Re}^{0.5} = 0.005 \text{ m} \quad \text{or} \quad 5.0 \text{ mm (about 1/5th)}$$

$$\delta_d = \delta/7 = 0.72 \text{ mm}, \; C_f = 0.664/\mathrm{Re}^{0.5} = 5.55 \times 10^{-4}$$

$$\tau = 5.55 \times 10^{-4} \times 0.5 \times 1000 \times 1.2^2 = 0.40 \text{ N/m}^2$$

The boundary layer is thicker and shear stress is higher in turbulent flow.

10.3 FLOW SEPARATION IN BOUNDARY LAYERS

Boundary layer is formed in the case of flow of real fluids. Viscous forces exist in such flows. The shear stress at the wall is given by

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0}, \text{ The wall shear cannot be zero. Hence at } y = 0,\ \frac{du}{dy}$$

cannot be zero. This means that the velocity gradient at the wall cannot be zero.

Separation of flow is said to occur when the direction of the flow velocity near the surface is opposed to the direction of the free stream velocity, which means $(du/dy) \leq 0$. Such a situation does not arise when there is no pressure gradient opposed to the flow direction, *ie.,* the pressure downstream of flow is higher compared to the pressure upstream. An example is subsonic diffuser. In the direction of flow the pressure increases. The increase in area along the flow causes a pressure rise.

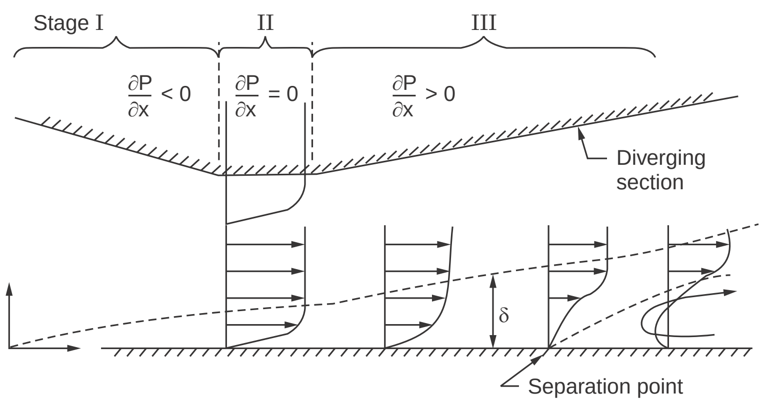

Figure 10.3.1 Flow separation

If (dp/dx) increases to the extent that it can overcome the shear near the surface, then separation will occur. Such a pressure gradient is called adverse pressure gradient. In the case of incompressible flow in a nozzle a favorable pressure gradient exists. Separation will not occur in such flows. In the case of diverging section of a diffuser, separation can occur if the rate of area increase is large. This is shown in Fig. 10.3.1. In turbulent flow, the momentum near the surface is high compared to laminar flow. Hence turbulent layer is able to resist separation better than laminar layer.

In the case of flow over spheres, cylinders, blunt bodies, airfoils etc., there is a change in flow area due to the obstruction and hence an adverse pressure gradient may be produced. Simple analytical solutions are not available to determine exactly at what conditions separation will occur. Experimental results are used to predict such conditions.

10.3.1 Flow Around Immersed Bodies – Drag and Lift

When fluid flows around a body or the body moves in a fluid there is a relative motion between the fluid and the body. The body will experience a force in such a situation. In the case of a flat plate positioned parallel to the direction of the flow, the force is parallel to the surface.

But generally in the case of blunt bodies, the force will neither be paraller nor perpendicular to the surface. The force can be resolved into two components one parallel to the flow and the other perpendicular to the flow. The former may be called shear force and the other, the pressure force.

The component parallel to the direction of motion is called drag force F_D and the component perpendicular to the direction of motion is called lift force, F_L. Determination of these forces is very important in many applications, an obvious example being aircraft wings. Simple analytical methods are found to be insufficient for the determination of such forces. So experimentally measured coefficients are used to compute drag and lift.

10.3.2 Drag Force and Coefficient of Drag

Drag is the component of force acting parallel to the direction of motion. Using the method of dimensional analysis the drage force can be related to flow Reynolds number by

$$\frac{F_D}{\rho AV^2} = f\,(\text{Re}) \tag{10.3.1}$$

For generality velocity is indicated as V

Defining coefficient of drag as the ratio of drag to dynamic pressure, it is seen that

$$C_D = f\,(\text{Re}),$$

$$C_D = \frac{F_D}{(1/2)\,\rho\,AV^2} \tag{10.3.2}$$

This applies to viscous drag only. In case wave drag is encountered, then

$$C_D = f\,(Re,\ Fr) \tag{10.3.3}$$

If compressibility effect is to be considered

$$C_D = f\,(Re,\ M) \tag{10.3.4}$$

Friction coefficient over flat plate in laminar flow, at a location was defined by $C_{fx} = \tau_w /(1/2)\,\rho\,A\,V^2 = 0.664/{Re_x}^{0.5}$. Over a given length the average value is obtained as twice this value. For a flat plate of length L, in laminar flow

$$C_D = 1.328/{Re_L}^{0.5} \tag{10.3.5}$$

In turbulent flow in the range $5 \times 10^5 > Re < 10^7$

$$C_D = 0.074/\,{Re_L}^{0.2} \tag{10.3.6}$$

For Re_L up to 10^9, an empirical correlation due to Schlichting is

$$C_D = 0.455/(\log Re_L)^{2.58} \tag{10.3.7}$$

For combined laminar and turbulent flow in the range $5 \times 10^5 > Re < 10^7$

$$C_D = \frac{0.074}{\text{Re}_L^{0.2}} - \frac{1740}{\text{Re}_L} \tag{10.3.8}$$

For the range $5 \times 10^5 > Re < 10^9$

$$C_D = \frac{0.455}{(\log \text{Re}_L)^{2.58}} - \frac{1610}{\text{Re}_L} \tag{10.3.9}$$

The values of C_D for laminar flow is in the range 0.002 to 0.004.

Example 10.5. *A ship having a wetted perimeter of 50 m and length of 140 m is to travel at 5 m/s.* ***Determine the power required to overcome the skin friction****. Assume kinematic viscosity ν = 1.4 × 10^{-6} m^2/s. Density 1025 kg/m^3*

$$Re = 5 \times 140/1.4 \times 10^{-6} = 0.5 \times 10^9,$$

So the equation applicable is 10.3.9

$$C_D = \frac{0.455}{(\log 0.5 \times 10^9)^{2.58}} - \frac{1610}{0.5 \times 10^9} = 1.719 \times 10^{-3}$$

$$F_D = C_D A(1/2)\,\rho u^2 = (1.7179 \times 10^{-3})\,(1/2) \times 140 \times 50 \times 1025 \times 5^2 \text{ N}$$

$$= \mathbf{0.154 \times 10^6\ N}$$

$$\therefore \quad \textbf{Power} = F_D\, u = 0.154 \times 10^6 \times 5 = 0.77 \times 10^6 \text{ W} = \mathbf{0.77\ MW}$$

10.3.3 Pressure Drag

When flow is perpendicular to blunt objects, like a plate or a disk, shear does not contribute to drag force. The drag is then mainly due to pressure difference between the faces. So it is called pressure drag. The drag coefficient is based on the frontal area (or projected area) of the object. In the case of airfoils the plan area is the basis for drag coefficient. The drag coefficient for same geometries are shown in Table 10.3.1 below. These are applicable for Re > 10^3.

Table 10.3.1 Drag coefficients for various shapes

Shape	C_D
Square plate	1.18
Rectangle 1:5	1.20
Cube	1.05
Disk	1.17
Hemisphere facing flow	1.42
Parachute	1.20
Hemisphere facing downstream	0.38

It may be seen that the coefficient of pressure drag is independent of Reynolds number.

Example 10.6. *A drag chute is used to slowdown a car with a mass 1800 kg travelling at 60 m/s. The value of coefficient of drag for the car is 0.32 and frontal area is 1.1 m^2. The chute is of 1.8 m* ***diameter and drag coefficient is*** *1.2 Density of air = 1.2 kg/m^3. Determine the speed after 50 secs. Also determine the time for the speed to reach 20 m/s.*

The total drag force at any instant for the car and the chute is given by (subscript C refers to car and P refers to parachute)

$$F_D = \frac{1}{2}\rho u^2\,[C_{DC} A_C + C_{DP} A_P] \text{ and this force acts to decelerate the car.}$$

Force = mass × Acceleration = $m(du/dt)$ ∴ (du/dt) = force/mass

$$\therefore \quad \frac{du}{dt} = \frac{F_D}{m} = -\frac{k}{m} u^2 \text{ where } k = \frac{\rho}{2}\,[C_{DC} A_C + C_{DP} A_P]$$

Separating variables and integrating,

$$\int_{u_0}^{u} \frac{du}{u^2} = -\frac{k}{m}\int_0^t dt$$

$$\therefore \quad \frac{1}{u_0} - \frac{1}{u} = -\frac{k}{m}\,t$$

$$\therefore \quad u = \frac{u_0}{1+\left(\dfrac{k}{m}\right)u_0 t} \qquad \text{... }(A)$$

$$k = \frac{1.2}{2}\,[(0.32 \times 1.1) + (1.2 \times \pi \times 1.8^2/4)] = 2.0433$$

∴ $(k/m)u_0 = (2.0433 \times 60)/1800 = 0.06811$

∴ (*i*) After 50 seconds, $\mathbf{u} = 60/(1 + 0.06811 \times 50)$ = **13.62 m/s**

(*ii*) For $u = 20$, $20 = 60/(1 + 0.06811 \times t)$ ∴ **t = 29.36 seconds**

The distance travelled can be obtained by integrating *u dt*.

$$\therefore \quad s = \int_0^t u\,dt = \int_0^t \frac{u_0\,dt}{1+(k/m)u_0 t} = \frac{u_0}{(k/m)u_0}\ln\,[1 + (k/m)u_0 t] = (m/k)\ln\,[1 + (k/m)u_0 t]$$

At $t = 50$ sec $s = \dfrac{1800}{2.0433}\ln(1 + 0.06811 \times 50) = 1306$ m

At $t = 29.36$ sec $s = \dfrac{1800}{2.0433}\ln(1 + 0.06811 \times 29.36) = 968$ m

10.3.4 Flow Over Spheres and Cylinders

In these cases both pressure and friction drag contribute to the total drag. The flow separation at the rear and formation of wake contributes to the pressure drag. The flow pattern and the variation of drag coefficient is shown in Fig. 10.3.2. It may be noted that the coefficient of drag is nearly constant from $Re = 10^3$ to 5×10^5. From experiments the boundary layer in the forward portion is found to be laminar in this range. Separation is found to occur at about mid section and a wide wake is found to exist with pressure in the wake below that at the front.

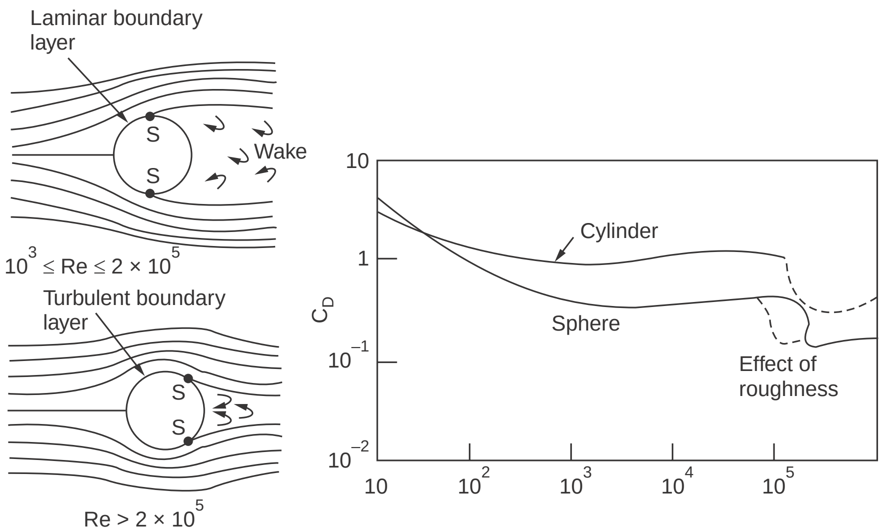

Figure 10.3.2 Flow separation in flow over cylinder/sphere

There is a sharp drop in the value of C_D after the critical Reynolds number. The flow in the forward side is found to turn turbulent and separation moves downstream and wake is now narrow, reducing the net pressure drag leading to the abrupt decrease in the drag coefficient. Turbulent layer has a higher momentum near the surface resisting separation.

Separation can be reduced by streamlining the body shape, reducing the pressure drag. This generally increases the area thus increasing friction drag. An optimum streamlined shape is the one which gives minimum total drag. Stream lining is now adopted not only for aircrafts but almost for all transport vehicles.

Example 10.7. *A model of a bathysphere 50 mm diameter is towed under water at a speed of 1 m/s.* ***Determine the tension in the towline****. Density of water = 1020 kg/m³. Kinematic viscosity = 1.006 × 10⁻⁶ m²/s*

$$Re = uD/\nu = 1 \times 0.05/1.006 \times 10^{-6} = 4.97 \times 10^4$$

From graph (Fig. 10.3.2) C_D is read as 0.45

$$\therefore \quad \mathbf{F_D} = C_D\,(1/2)\,\rho\,Au^2 = \frac{0.45}{2} \times 1020 \times \frac{\pi \times 0.05^2}{4} \times 1^2 = \mathbf{0.45\ N}$$

10.3.5 Lift and Coefficient of Lift

The force on an immersed body moving in a fluid can be resolved into two components. The component along the flow direction is called drag. The component perpendicular to the flow direction is called lift. The lift on airfoil is an example. The coefficient of lift is defined by

$$C_L = \frac{F_L}{(1/2)\,\rho\,Au^2} \tag{10.3.10}$$

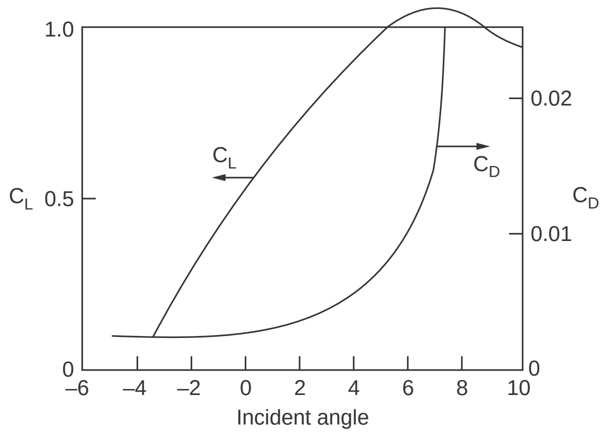

Figure 10.3.3 Variation of Lift and Drag on an airfoil

Lift is of interest mainly in the design of airfoil sections. Airfoil blade shapes are also used in turbomachines. The lift and drag coefficients depend on the Reynolds number and angle of attack. The angle between the airfoil chord and the flow direction is called angle of attack. The chord of an airfoil is the line joining the leading edge and the trailing edge. The planform area (the maximum projected area) is used in the definition of lift and drag coefficients. A typical plot of the variation of lift and drag coefficients with angle of attack for a specified Reynolds number is shown in Fig. 10.3.3.

For each airfoil section such plots are available. Flow separation will result in sudden drop in the lift, known as stall. Presently computer softwares are available for the design of airfoil sections with a very high ratio of lift to drag. These data are for long spans and corrections should be made as per the aspect ratio defined by b^2/A_p. where b is the span length and A_p is the planform area. This will equal the ratio (span/chord) as, $A_p = bc$. The lift to drag ratio varies from 20 to 40 with the lower value applicable for small planes.

10.3.6 Rotating Sphere and Cylinder

In order to reduce skin friction in flow over surfaces, particularly curved surfaces boundary layer control is used. One method of boundary layer control is by the use of moving surfaces at locations where separation may start. This is difficult to apply due to mechanical restrictions. However this principle is used in sports like baseball, golf, cricket and tennis where spin is applied to control the trajectory of the ball. Spin also provides significant aerodynamic lift to increase the distance travelled by the ball. Spin can also be used to obtain a curved path of travel for the ball.

Spin alters the pressure distribution and also the location of boundary layer separation. For spin along the flow direction, separation is delayed on the upper surface and it occurs earlier in the lower surface. Pressure is reduced on the upper surface and is increased on the lower surface and the wake is deflected downwards.

The coefficients of lift and drag are found to be a function of ω D/2u called spin ratio.

Example 10.8. ***Show*** *using dimensional analysis that the* ***lift and drag coefficients are functions of spin ratio and Reynolds number****.*

The variables affecting the phenomenon are listed below. As C_L and C_D are dimensionless, these are not listed.

No	Variable	Unit	Dimension
1	Linear Velocity, u	m/s	L/T
2	Radius, R	m	L
3	Angular velocity, ω	Radians/s	$1/T$
4	Kinematic viscosity, v	m^2/s	L^2/T

There are four variables and two dimensions, namely L and T. Hence two π terms can be identified. Choosing linear velocity and radius as repeating variables

Let $$\pi_1 = \omega\, u^a R^b \text{ or } L^0 T^0 = \frac{1}{T}\frac{L^a}{T^a} L^b$$

$\therefore$ $$a + b = 0, -1 - a = 0 \quad \therefore \quad a = -1 \quad \therefore \quad b = 1$$
$$\pi_1 = \omega R/u = \omega R/2u, \text{ called } \textbf{spin ratio.}$$

Let $$\pi_2 = vu^a R^b \quad \text{or} \quad L^0T^0 = \frac{L^2}{T}\frac{L^a}{T^a} L^b, \quad \therefore \quad 2 + a + b = 0, -1 - a = 0$$

$\therefore$ $$a = -1 \quad b = -1 \quad \pi_2 = \frac{v}{uR} \quad \text{or} \quad \frac{uD}{v}, \text{ Reynolds number.}$$

Hence $$\mathbf{C_L = f\left[\frac{\omega D}{2u}, \frac{uD}{v}\right] \text{ and } C_D = f\left[\frac{\omega D}{2u}, \frac{uD}{v}\right]}$$

The variation of C_L and C_D are found to be influenced more by spin ratio than Reynolds number. The trend is shown in Fig. Ex. 10.8. In the case of cylinders the area for definition of C_L and C_D is $L \times D$

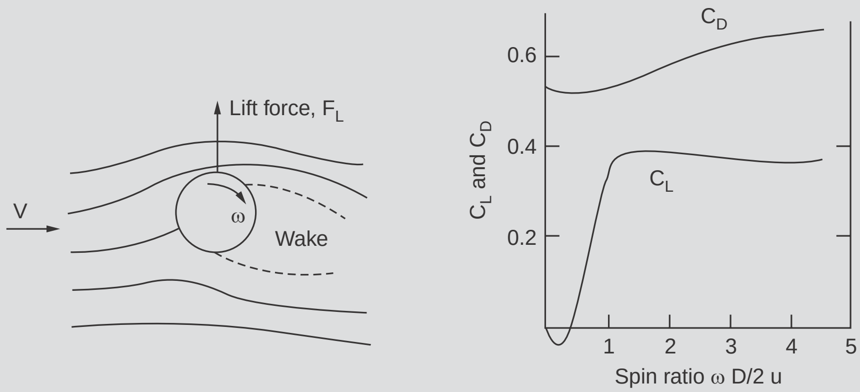

Figure Ex. 10.8 Variation of Lift and Drag with spin ratio

A force perpendicular to both direction of motion and the spin axis is created during the flight. This is known as MAGNUS effect. This can cause drift in the flight path.

SOLVED PROBLEMS

Problem 10.1 *Assuming **linear velocity variation in the boundary layer** and using linear momentum integral equation, determine the thickness of the boundary layer. Also determine the friction coefficient and the displacement and momentum thicknesses.*

Momentum integral equation is

$$\frac{d}{dx}\left[\int_0^\delta u(u_\infty - u)dy\right] = \nu \left.\frac{du}{dy}\right|_{y=0}. \text{ As } \frac{u}{u_\infty} = \frac{y}{\delta} \quad \therefore \quad u = \frac{u_\infty y}{\delta},$$

$$\frac{du}{dy} = \frac{u_\infty}{\delta} \quad \therefore \tau = \frac{\mu u_\infty}{\delta}, \text{ Considering the integral part}$$

$$\int_0^\delta \left[\frac{u_\infty^2}{\delta}y - \frac{u_\infty^2}{\delta^2}y^2\right]dy = \left[\frac{u_\infty^2\delta}{2} - \frac{u_\infty^2\delta}{3}\right] = \frac{1}{6}u_\infty^2\delta$$

$$\therefore \qquad \frac{d}{dx}\left[\frac{u_\infty^2}{6}\delta\right] = \nu\frac{u_\infty}{\delta} \text{ or } \frac{u_\infty^2}{6}\frac{d\delta}{dx} = \nu\frac{u_\infty}{\delta}$$

Separating variables and integrating, $\delta\, d\delta = (6\nu/u_\infty)\, dx$

$$\delta^2 = (12\nu x)/u_\infty = 12x^2/(\nu/u_\infty x) = 12x^2/\text{Re}_x$$

$$\therefore \qquad \boldsymbol{\delta = 3.464x/\text{Re}_x^{0.5}},$$

The constant is 3.464 instead of 5 in the exact solution

$$\mathbf{C_{fx}} = \frac{\tau}{(1/2)\rho u_\infty^2} = \frac{2\mu u_\infty}{\rho u_\infty^2\delta} = \frac{2\nu\,\text{Re}_x^{0.5}}{u_\infty 3.464x} = \mathbf{0.577/Re_x^{0.5}}$$

The displacement thickness

$$\delta_d = \int_0^\delta\left[1 - \frac{u}{u_\infty}\right]dy = \int_0^\delta\left[1 - \frac{y}{\delta}\right]dy = (1/2)\delta \text{ or } \delta/2. \text{ As against } \delta/3$$

Momentum thickness is given by

$$\delta_m = \int_0^\delta \frac{u}{u_\infty}\left[1 - \frac{u}{u_\infty}\right]dy = \int_0^\delta\left[\frac{y}{\delta} - \frac{y^2}{\delta^2}\right]dy = \frac{1}{2}\delta - \frac{1}{3}\delta = \frac{1}{6}\delta$$

By the exact solution, $\delta_m = (1/7)\delta$

Problem 10.2 *Assuming **second degree velocity distribution in** the boundary layer determine using the integral momentum equation, the thickness of boundary layer friction coefficient, displacement and momentum thicknesses.*

Let $u = a + by + cy^2$. The boundary conditions are $u = 0$ at $y = 0$,

At $y = \delta$, $\frac{du}{dy} = 0$, and $u = u_\infty$. This gives $\frac{u}{u_\infty} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$ (Refer 10.1.18)

Substituting in the integral momentum equation,

$$\frac{d}{dx}\left[\int_0^{\delta} u(u_\infty - u)dy\right] = v\left.\frac{du}{dy}\right|_{y=0},$$

$$u = u_\infty\left[2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right], \left.\frac{du}{dy}\right|_{y=0} = 2u_\infty/\delta,\ \tau = 2\mu u_\infty/\delta$$

Considering the integral part,

$$u_\infty^2\int_0^{\delta}\left[\left(\frac{u}{u_\infty}\right) - \left(\frac{u}{u_\infty}\right)^2\right]dy = u_\infty^2\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 - 4\left(\frac{y}{\delta}\right)^2 + 4\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right]dy$$

$$= u_\infty^2\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - 5\left(\frac{y}{\delta}\right)^2 + 4\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right]dy$$

$$= u_\infty^2\left[\delta - \frac{5}{3}\delta + \delta - \frac{1}{5}\delta\right] = \frac{2}{15}u_\infty^2\delta \qquad \text{... }(A)$$

Substituting

$$\frac{d}{dx}\left[\frac{2}{15}u_\infty^2\delta\right] = 2vu_\infty/\delta \quad \text{or} \quad \frac{2}{15}u_\infty^2 = \frac{d\delta}{dx} = 2\,vu_\infty/\delta$$

$\therefore$ $\delta d\delta = 15(v/u_\infty)dx$ Integrating

$$\delta^2 = 30vx/u_\infty,\ 30\left(\frac{v}{u_\infty x}\right)x^2 = 30x^2/\text{Re}_x,\ \boldsymbol{\delta = 5.477x/\text{Re}_x^{0.5}}$$

Note that the constant is 5.477 as against 5 by exact solution.

As $\tau = 2\mu u_\infty/\delta$ and $\delta = 5.477\,x/\text{Re}_x^{0.5}$

$$\mathbf{C_{fx}} = \frac{\tau}{(1/2)\rho u_\infty^2} = \frac{4\mu u_\infty}{\rho u_\infty^2\delta} = \frac{4v}{u_\infty\delta} = \frac{4v\,\text{Re}_x^{0.5}}{5.477\,xu_\infty} = \mathbf{0.73/Re_x^{0.5}}$$

instead of $0.644/\text{Re}_x^{0.5}$

$$\delta_d = \int_0^{\delta}\left[1 - \frac{u}{u_\infty}\right]dy = \int_0^{\delta}\left[1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right]dy = \frac{\delta}{3}$$

$$\delta_m = \int_0^{\delta}\left[\frac{u}{u_\infty} - \left(\frac{u}{u_\infty}\right)^2\right]dy = \frac{2}{15}\delta \quad \text{(see equation } A\text{)}$$

Problem 10.3 *Assuming the velocity distribution in the boundary layer as* $\frac{u}{u_\infty} = sin\left(\frac{\pi y}{2\delta}\right)$ *(in the range* $0 \le y \le \delta$*, and* $u/u_\infty = 1$ *beyond* δ*) determine the thickness of the boundary layer, using integral momentum method (Refer equation 10.1.12).*

$$\frac{u}{u_\infty} = \sin\left(\frac{\pi y}{2\delta}\right), \frac{du}{dy} = \frac{\pi}{2\delta}\cos\frac{\pi y}{2\delta} \text{ at } y = 0,\ \frac{du}{dy} = u_\infty\ (\pi/2\delta)$$

$$\frac{d}{dx}\left[\int_0^\delta u(u_\infty - u)dy\right] = v\left.\frac{du}{dy}\right|_{y=0}$$

Considering the integral part and substituting the velocity distribution,

$$u_\delta^2 \int_0^\delta \left[\left(\frac{u}{u_\infty}\right) - \left(\frac{u}{u_\infty}\right)^2\right] dy = u_\delta^2 \int_0^\delta \left[\sin\frac{\pi y}{2\delta} - \sin^2\frac{\pi y}{2\delta}\right] dy$$

Noting $$\int \sin^2 ax = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$= u_\infty^2 \left[-\frac{2\delta}{\pi}\cos\frac{\pi y}{2\delta} - \frac{y}{2} + \frac{\delta}{2\pi}\sin\frac{\pi y}{2\pi}\right]_0^\delta$$

$$= u_\infty^2 \left[0 - \frac{\delta}{2} + 0\right] - \left[-\frac{2\delta}{\pi} - 0 + 0\right] = 0.1366 \times u_\infty{}^2 \times \delta \qquad \text{... }(A)$$

$$\frac{d}{dx}[u_\infty^2 \times 0.1366\,\delta] = \frac{\pi}{2\delta}\,u_\infty v,\ [u_\infty^2 \times 0.1366]\frac{d\delta}{dx} = \frac{\pi}{2\delta}\,u_\infty v$$

$\therefore$ $$\delta\, d\delta = \frac{\pi}{2 \times 0.1366}\frac{v}{u_\infty}dx$$

Integrating $$\frac{\delta^2}{2} = \frac{\pi}{2 \times 0.1366}\frac{vx}{u_\infty} \text{ or } \boldsymbol{\delta = 4.8x/Re_x{}^{0.5}}$$

$\mathbf{C_{fx}} = \tau_w/(1/2)\rho\, u_\infty{}^2$, $\tau_w = \mu\,(du/dy)$, at $y = 0$, $\tau_w = \mu\, u_\infty\, \pi/2\delta$

$\therefore$ $$\mathbf{C_{fx}} = \frac{2\mu\, u_\infty\, \pi}{2\delta\rho u_\infty^2} = \frac{\pi}{\delta}\frac{v}{u_\infty} = \frac{\pi}{4.8}\frac{v}{u_\infty x}\mathrm{Re}_x^{0.5} = \mathbf{0.655/Re_x{}^{0.5}}$$

$$\delta_d = \int_0^\delta \left[1 - \frac{u}{u_\infty}\right] dy = \int_0^\delta \left[1 - \sin\left(\frac{\pi y}{2\delta}\right)\right] dy = \left[\delta + \frac{2\delta}{\pi}\cos\frac{\pi y}{\delta}\right]_0^\delta$$

$= [\delta + 0] - [0 + (2\delta/\pi)] = 0.3625\ \delta = \delta/2.76$, instead of $\delta/3$

$\delta_m = 0.1366\ \delta$, or $\delta/7.32$ (refer result A)

Problem 10.4 *Using the* ***cubic velocity profile*** *determine upto a length* ***L the flow out of the boundary*** *layer in terms of the boundary layer thicknes.*

The free stream flow for thickness of δ is $\rho\, u_\infty\, \delta$.

Assuming cubic velocity profile,

$$u = u_\infty\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]$$

Mass flow through the boundary layer

$$= \int_0^\delta \rho u dy = \int_0^\delta \rho u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy = \rho u_\infty \left[\frac{3}{2}\frac{y^2}{\delta} - \frac{1}{8}\frac{y^4}{\delta^3}\right]_0^\delta = \frac{5}{8}\rho u_\infty\,\delta$$

$\therefore$ Mass flow out of the boundary layer = (1–(5/8)) $\rho\, u_\infty\, \delta$ = 3/8 $\rho\, u_\infty\, \delta$ or displacement thickness times the free stream flow. (Nota : δ_d = (3/8) δ for cubic profile)

The average velocity in the y direction can be obtained by dividing the volume flow by area *i.e.,* 1 × x for unit width. **Volume flow out of the boundary**

$$\mathbf{v} = (3/8)\, u_\infty\, \delta, \text{ velocity} = \frac{3}{8} u_\infty \frac{4.64x}{\text{Re}_L^{0.5}} \frac{1}{x} = \frac{1.74 u_\infty}{\text{Re}_L^{0.5}}$$

This will be low as Reynolds number will be high.

Consider the data from example (10.1) Air flow, u_∞ = 5 m/s, at a distance 0.5 m,

$$\text{Re} = 1.56 \times 10^5 \quad \therefore \mathbf{v} = (1.74 \times 5)/(1.56 \times 10^5)^{0.5} = \mathbf{0.022\ m/s}$$

This can also be calculated in a round about way using the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. The value of $\frac{\partial u}{\partial x}$ can be obtained from the assumed profile and then equated to $-\frac{\partial v}{\partial y}$. Integrating the same between 0 and δ the same result will be obtained. [Refer Problem 10.6].

Problem 10.5 *Using the continuity and momentum equations show that at y = 0,* $-\frac{\partial^3 u}{\partial y^3} = 0$. *Deduce from the above that the cubic profile is approximate.*

Consider the x directional momentum equation, $u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \upsilon \frac{\partial^2 u}{\partial y^2}$. Differentiating with resect to y,

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} = \upsilon \frac{\partial^3 u}{\partial y^3}, \text{ Simplifying.}$$

$$\frac{\partial u}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} = \upsilon \frac{\partial^3 u}{\partial y^3}$$

The first term is zero due to continuity equation. At y = 0, u = 0 and v = 0. Hence the second and third terms are also zero. So $\frac{\partial^3 u}{\partial y^3}$ should be zero.

Consider the cubic profile:

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\therefore \quad \frac{\partial u}{\partial y} = u_\infty \left[\frac{3}{2}\frac{1}{\delta} - \frac{3}{2}\frac{y^2}{\delta^3}\right] \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = u_\infty \left[-\frac{6}{2}\frac{y}{\delta^3}\right] \frac{\partial^3 u}{\partial y^2} = -3u_\infty/\delta^3$$

This is not zero. Hence profile assume is approximate.

Problem 10.6 *Derive a general expression* ***for the y directional velocity at a location*** *x in the boundary layer in flow over a flat plate. Indicate at what y location this will be maximum. Assume cubic velocity variation.*

Consider continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

$$u = u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \right], \ \delta = 5x/\text{Re}_x^{1/2} = \frac{5xv^{1/2}}{u_\infty^{1/2}\, x^{1/2}} = cx^{1/2}$$

where $c = [5v^{1/2}/u_\infty^{1/2}]$ substituting and putting $c_1 = 3u_\infty/2\,\delta$ and $c_2 = u_\infty/2\,\delta^3$, velocity expression reduces to $u = c_1\, y\, x^{-1/2} - c_2\, y^3\, x^{-3/2}$, $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -[c_1 \times (-1/2) \times y\, x^{-3/2} + (3/2)\, c_2\, y^3\, x^{-5/2}]$

Integrating w.r.t. y, and substituting for c_1 and c_2

$$v = \frac{3}{8c}\frac{u_\infty}{x^{3/2}} y^2 - \frac{3u_\infty}{16c^3}\frac{y^4}{x^{5/2}}, \text{ Substituting for } c$$

$$v = \frac{3}{8}\frac{u_\infty u_\infty^{1/2} y^2}{5v^{1/2} x^{3/2}} - \frac{3 \times u_\infty^{3/2}}{16 \times 125 v^{3/2}}\frac{u_\infty y^4}{x^{5/2}}$$

Substituting $\frac{5xv^{1/2}}{u_\infty^{1/2}\, x^{1/2}} = \delta$

$$v = \frac{3}{8}u_\infty \frac{y^2}{\delta x} - \frac{3}{16}u_\infty \frac{y^4}{\delta^3 x} = \frac{3}{8}\frac{u_\infty}{\delta x}\left[y^2 - \frac{1}{2}\frac{y^4}{\delta^2} \right]$$

(Check for dimensional consistency : dimensions of $y^2/\delta x$ and $y^4/\delta^3 x$ cancel and v has the same unit as u_∞)

Maximum value occurs when $\frac{\partial v}{\partial y} = 0$.

$$\frac{\partial}{\partial y}\left[y^2 - \frac{1}{2\delta^2} y^4 \right] = 2y - \frac{1}{2\delta^2} 4y^3. \text{ Equating to zero and solving } y = \delta$$

This is physically explainable as the total flow in y direction should occur at

$$y = \delta. \text{ Velocity at } y = \delta \text{ is } \mathbf{v_{\delta x}} = \frac{3}{16}\frac{u_\infty \delta}{x} = \mathbf{0.87}\, \frac{\mathbf{u_\infty}}{\mathbf{Re_x^{0.5}}}$$

Total mass flow when integrated over the length will equal (3/8) $\rho\, u_\infty\, \delta_L$ (Refer Problem 10.4).

Problem 10.7 *The shear at a location 2 m from the leading edge of a flat plate was measured as 2.1 N/m². Assuming the flow to be turbulent from the start determine if air at 20°C was flowing over the plate (i) the velocity of air (ii) the boundary layer thickness and (iii) the velocity at 15 mm above the plate.* $\rho = 1.205$ *kg/m³*, $v = 15.06 \times 10^{-6}$ *m²/s*

Using equation (10.2.3) $C_{fx} = 0.0594/\text{Re}_x^{0.2}$, $\tau_w = C_{fx}\,(1/2)\,\rho\, u^2$,

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$$\text{Equating } 2.1 = \frac{0.0594 \times (15.06 \times 10^{-6})^{0.2}}{u_\infty^{0.2}\, 2^{0.2}} \times \frac{1}{2} \times 1.205\, u_\infty^{\,2}, \text{ Solving,}$$

$$u_\infty^{\,1.8} = 621.04 \text{ or } u_\infty = 35.623 \text{ m/s}$$

$$\text{Re} = 35.623 \times 2/15.06 \times 10^{-6} = 4.808 \times 10^6,$$

Turbulent hence the use of equation (10.2.3) is justified. Using equation (10.2.1),

$$\delta = 0.382x/\text{Re}_x^{\,0.2} = 0.382 \times 2/(4.808 \times 10^6)^{0.2}$$

$$= 0.0352 \text{ m or } \delta = 35.2 \text{ mm}$$

If the velocity profile is assumed as

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

$\therefore$ $\mathbf{u} = 35.623\ (15/35.2)^{1/7} = \mathbf{32.05\ m/s}$

Problem 10.8 ***Determine the length at which the flow over a flat plate will turn turbulent for air****, water and engine oil if the flow velocity is 3 m / s. Also determine the boundary layer thickness at the location. Temperature of the fluid = 20°C. The kinematic viscosity and density of the fluids are :*

S. No	Density, kg/m³	Kinematic viscosity	L_{cv}, m	δ, mm
Air	1.205	15.06×10^{-6}	**2.51**	**17.7**
Water	1000	1.006×10^{-6}	**0.17**	**1.2**
Engine oil	888	901×10^{-6}	**150**	**1061**

The flow turns turbulent at Re = 5×10^5

(1) Air : $5 \times 10^5 = 3 \times L_a/15.06 \times 10^{-6}$ $\therefore$ $L_a = 2.51$ m

$$\delta = 5x/\text{Re}_x^{\,0.5} = 5 \times 2.51/(5 \times 10^5)^{0.5} = 0.0177 \text{ m}$$

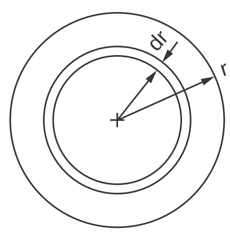
(2) Water : $5 \times 10^5 = 3 \times L_w/1.006 \times 10^{-6}$ $\therefore$ $L_w = 0.1677$ m

$$\delta = 5x/\text{Re}_x^{\,0.5} = 5 \times 0.1677/(5 \times 10^5)^{0.5} = 0.0012 \text{ m}$$

(3) Engine oil : $5 \times 10^5 = 3 \times L_o/901 \times 10^{-6}$ $\therefore$ $L_o = 150.17$ m

$$\delta = 5x/\text{Re}_x^{\,0.5} = 5 \times 150.17/(5 \times 10^5)^{0.5} = 1.061 \text{ m}$$

Problem 10.9 *The pressure distribution on the front and back surfaces of a thin disk of radius, R oriented perpendicular to a fluid stream was measured and the pressure coefficient has been correlated as below.*



Front side : $C_P = 1 - (r/R)^6$. Rear surface : $C_P = -0.42$

Determine the drag coefficient for the disk.

$$C_P = \Delta P/(1/2)\,\rho\, AV^2$$

$\therefore$ $\Delta P = C_P(1/2)\,\rho\, AV^2 = C_P\, 2\pi\, rdr\, (\rho\, V^2/2)$

Consider a small strip of width dr at a radius r. The force on the area = $\Delta P \times 2\pi\, rdr$

$$F_D = \int_0^R \Delta P \times 2\pi r dr = \frac{1}{2}\rho V^2 \int_0^R C_p\ 2\pi\ r dr = \frac{1}{2}\rho V^2 \int_0^R \left[1 - \left(\frac{r}{R}\right)^6\right] 2\pi r dr$$

$$= \frac{1}{2} 2\pi \times \rho V^2 \left[\frac{r^2}{2} - \frac{r^8}{8R^6} \right]_0^R = \frac{1}{2} 2\pi \times \rho V^2 \left[\frac{R^2}{2} - \frac{R^2}{8} \right]$$

$$= \frac{1}{2} \rho V^2 \pi R^2 \, [3/4] = (1/8)\, \rho\, AV^2$$

$$\mathbf{C_D} = F_D/(1/2)\, \rho\, AV^2 \quad \therefore \mathbf{C_D} = 3/4 = \mathbf{0.75}$$

On the otherside, the pressure is independent of radius $\therefore C_D = C_p$ and it is in the opposite direction

$$\therefore \qquad C_p = 0.75 + 0.42 = \mathbf{1.17}$$

Problem 10.10 *Air flows along a triagular plate as shown in Fig. P. 10.10. Determine the shear force on both sides of the plate. Assume air temperature. as 20°C. ρ = 1.205 kg/m³, kinematic viscosity is 15.06 × 10⁻⁶ m²/s.*

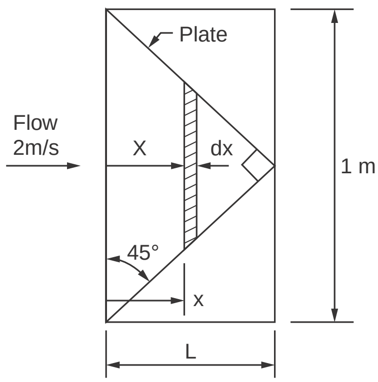


Figure P.10.10 Problem model

Considering the maximum length of 0.5 m

$$Re = 2 \times 0.5/15.06 \times 10^{-6} = 0.66 \times 10^5 \quad \therefore \quad \text{flow is laminar}$$

$$\tau_x = 0.332\, \rho\, u^2/\mathrm{Re}^{0.5} = 0.332\, \rho\, u^2 v^{1/2}/u^{1/2} x^{1/2}$$

$$= 0.332\, \rho\, u^{1.5} v^{1/2} x^{-1/2}$$

$$= 0.332 \times 1.205 \times 2^{1.5} \times (15.06 \times 10^{-6})^{0.5} x^{-1/2}$$

$$= 4.97 \times 10^{-3} x^{1/2}$$

Considering a strip of width dx at a distance x from base, and assuming the length of base as $2L$, height will be L.

$$dA = \frac{L - x}{L} \times 2L \times dx = 2(L - x)dx, \text{ Force on the strip}$$

$$dF = \tau_x dA = 4.97 \times 10^{-3} \times 2(L - x)x^{-1/2} dx$$

Integrating between $x = 0$ to $x = L$

$$F = 9.94 \times 10^{-3} \left[\frac{Lx^{1/2}}{0.5} - \frac{x^{1.5}}{1.5} \right]_0^L = 9.94 \times 10^{-3} \times \frac{4}{3} \times L^{1.5}$$

Here **L = 0.5 m.** $\therefore$ $\mathbf{F = 4.68 \times 10^{-3}\ N}$

Check for dimensional homogeneity.

$$F = \text{const } \rho\, u^{1.5} v^{1/2} L^{1.5},$$

$$N = \text{const } \frac{kg}{m^3} \frac{m^{1.5}}{s^{1.5}} \frac{m}{s^{0.5}} m^{1.5} = \text{kgm/s}^2 = \text{N},$$

Hence checks.

Problem 10.11 *A water ski is 1.2 m long and 0.2 m wide and moves in water at 10 m/s. the water temperature is 20°C.* ***Determine the viscous drag*** *approximating it as a flat plate.*

$$v = 1.006 \times 10^{-6} \text{ m}^2/\text{s}.\ \rho = 1000 \text{ kg/m}^3,$$

$$Re = 1.2 \times 10/1.006 \times 10^{-6} = 11.93 \times 10^6$$

∴ The flow is turbulent considering combined laminar and turbulent flows.

$$C_{fL} = 0.074\text{Re}_L^{-0.2} - 1742\text{Re}_L^{-1} = 2.99 \times 10^{-3}, \text{ Drag} = C_{fL}\ (1/2)\rho\, u^2 \Delta$$

$$\textbf{Drag} = (1/2)\ 1000 \times 10^2 \times 1.2 \times 0.2 \times 2.99 \times 10^{-3} = \mathbf{35.88\ N}$$

Power required considering 2 skis, $P = 2 \times 35.88 \times 10 = 717.6$ W

Problem 10.12 *In a power plant located near the sea* ***a chimney*** *of 1.2 m diameter and 35 m height has been installed. During a cyclone the wind reaches velocity in the range of 60 kmph.* ***Determine the moment at the base of the chimney.***

$$\rho = 1.2 \text{ kg/m}^3,\ v = 17.6 \times 10^{-6} \text{ m}^2/\text{s},\ u = 600000/3600 = 16.67 \text{ m/s}$$

$$Re = 16.67 \times 1.2/17.6 \times 10^{-6} = 1.14 \times 10^6$$

From graph for circular cylinder C_D is read as 0.35

∴ $$F_D = C_D\, \rho\, Au^2/2 = 0.35 \times 1.23 \times 35 \times 1.2 \times 16.66^2/2 = 5022.5 \text{ N}$$

As this is a uniform force, it can be taken to act at the mid point.

∴ **Moment** = 5022.5 × 35/2 = 87893 Nm or **87.893 kNm.**

Problem 10.13 *A overhead water tank is in the shape of a sphere of 12 m diameter and is supported by a 30 m tall tower of circular section of diameter 2 m.* ***Determine the moment at the base*** *caused by the aerodynamic force due to cyclonic wind of speed 100 kmph. Assume density of air as 1.205 kg/m³ and kinematic viscosity as 15.06 × 10⁻⁶ m²/s.*

For the spherical portion : $$Re = 12 \times \frac{100 \times 1000}{3600} \times \frac{1}{1506 \times 10^{-6}} = 2.21 \times 10^7$$

The value of C_D is read as 0.19 from graph by extrapolation.

For the cylindrical portion : $$Re = 2 \times \frac{100 \times 1000}{3600} \times \frac{1}{1506 \times 10^{-6}} = 3.689 \times 10^6$$

The value of C_D is read as 0.40 from graph by extrapolation.

$$F_D = C_D\,(1/2)\,\rho\, AV^2,\ M = F_D \times \text{distance},$$

$$V = 100 \times 1000/3600 = 27.78 \text{ m/s}.$$

For the spherical portion

$$M = (30 + 6) \times 0.19 \times (1/2) \times 1.205 \times (\pi \times 12^2/4) \times 27.78^2$$

$$= 359.6 \times 10^3 \text{ Nm}$$

For the cylindrical portion

$$M = 15 \times 0.4 \times (1/2) \times 1.205 \times 2 \times 30 \times 27.78^2 = 167.4 \times 10^3 \text{ Nm}$$

Total moment $= (359.6 + 167.4) \times 10^3 = \mathbf{527.0 \times 10^3\ Nm}$

Problem 10.14 *A parachute moves down at a speed of 6 m/s. The mass of the chute and the jumper is 120 kg.* ***Determine the minimum diameter of the chute****. Density of air = 1.23 kg/m³.*

For parachute $C_D = 1.2$, (Refer table 10.3.1)

Net force = 120 × 9.81 N

$$120 \times 9.81 = 1.2 \times (1/2) \times 1.23 \times (\pi D^2/4) \times 6^2$$ Solving **D = 7.51 m**

Problem 10.15 *Hail stones that are formed in thunder clouds are sopported by the drag due to the air draft upwards and will begin to fall when the size reaches a critical value.* ***Estimate the velocity upwards so that hailstones begin to fall*** *when the diameter reaches a value of 40 mm. The density and dynamic viscosity of air at the altitude of 5000 m where the stones are formed are 0.7364 kg/m³ and 1.628 × 10⁻⁵ kg/ms. Hailstone is assumed to be in the shape of a sphere with a density of 940 kg/m³.*

The drag force should be just less than the gravity force when the hailstone begins to fall. At the limiting condition thses can be taken as equal. Other body forces like buoyancy forces are negligible.

Drag force = C_D (1/2) ρ Au^2, Gravity force = ρ Vg, V being the volume.

Equating and substituting the values,

$$C_D = (1/2) \times 0.7364 \times \pi \times 0.02^2\, u^2 = (4/3) \times \pi \times 0.02^3 \times 9.81 \times 940$$

$$\therefore \quad C_D\, u^2 = 667.85$$

C_D depends on Reynolds number which cannot be calculated without the value of velocity. Looking at the graph for C_D for spheres, the value is about 0.45 for $Re = 10^3$ to 5×10^5. Substituting this value, **u = 38.52 m/s or 138.7 kmph.**

$$Re = 38.52 \times 0.04/1.628 \times 10^{-5} = 0.94 \times 10^5.$$

Hence the assumed value of C_D is acceptable.

Problem 10.16 *A stirrer is constructed as shown in Fig. P. 10.16. The dimensions are indicated in the figure. The stirrer speed is 90 rpm.* ***Determine the torque*** *on the shaft and also* ***the power required****. Assume the vessel is large. Neglect the drag on the rod and the shaft. Density of the fluid is 1025 kg/m³.*

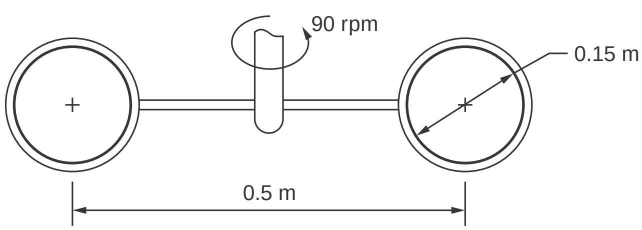


Figure P. 10.16 Stirrer details

For circular plate $C_D = 1.17$

Linear speed of the disk $= \dfrac{\pi DN}{60} = \dfrac{\pi \times 0.5 \times 90}{60} = 4.7124$ m/s

$$C_D = F_D/(1/2)\,\rho\, AV^2,\ A = \pi \times 0.15^2/4$$

$$\therefore \quad F_D = 1.17 \times (1/2) \times 1025 \times (\pi \times 0.15^2/4) \times 4.7124^2 = 235.31 \text{ N}$$

Torque = Force × torque arm = 235.30 × 0.5 = **117.65 Nm.**

Power = 2π NT/60 = (2π × 90 × 117.65)/60 = **1109 W**

Problem 10.17 *An anemometer has hemispherical cups of 80 mm dia with an arm distance from the post to center of 130 mm. If due ot fiction, the cups starts rotating at a wind speed of 3 m/s.* ***Determine the starting torque****. Consider density of air as 1.23 kg/m*^{3}.

The coefficient of drag when the cup faces the wind is 1.42. The coefficient of drag on the back = 0.38.

$\therefore$ Net coefficient = 1.42 – 0.38 = 1.04

$\therefore$ Force = $C_D A\rho\, V^2/2$, Torque = Force × torque arm, Substituting

Starting torque $= 1.04 \times \pi \times \dfrac{0.08^2}{4} \times \dfrac{1.23}{2} \times 3^2 \times 0.13 = \mathbf{3.76 \times 10^{-3}}$ **Nm**

Problem 10.18 ***Determine the wind force*** *on the antenna shown in Figure P. 10.18. All the components face the wind blowing at 100 kmph. ρ = 1.205 kg/m*^{3}*, kinematic viscosity is 15.06 ×* 10^{-6} *m*^{2}*/s.*

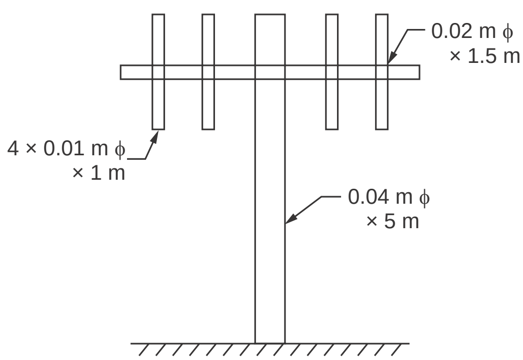


Figure P. 10.18 Antenna details

Velocity of wind = 100000/3600 = 27.78 m/s

The value of Reynolds number is given by

Re = 27.78 × 0.04/15.06 × 10^{-6} = 7.38 × 10^4 for 40 mm rod and 1.84 × 10^4 for 10 mm rod. At this value C_D is about 1.4 for both cases.

$$\mathbf{F_D} = 1.4 \times \frac{1.205}{2} \times 27.78^2\ [(5 \times 0.04) + (1.5 \times 0.02) + (4 \times 0.01)]$$

$$= \mathbf{175.7\ N}$$

Problem 10.19 *The total mass of an aircraft is 70000 kg. The wing area is 160 m². If the craft travels at 600 kmph,* ***determine the lift coefficient****. Neglect the compressibility effect. Air density at the flight conditions is 0.85 kg/m³.*

The lift force should be equal the weight at steady flight. Lift force F_L is given by

$$F_L = C_L A(1/2)\,\rho\, V^2, \text{ flight speed, } V = 600000/3600 = 166.7 \text{ m/s}$$

$$70000 \times 9.81 = C_L \times 160 \times (1/2) \times 0.85 \times 166.7^2 \quad \therefore \mathbf{C_L = 0.3635}.$$

Problem 10.20 *In championship tennis, balls are hit at speeds exceeding 100 kmph and good amount of spin.* ***Calculate the aerodynamic lift on ball and radius of curvature of path*** *in the vertical plane, when the ball is hit at a speed of 108 kmph and a top spin of 8000 rpm. The ball diameter is 0.064 m and mass is 0.057 kg. For air density = 1.165 kg/m³ and kinematic viscosity is 16 × 10⁻⁶ m²/s.*

The lift force depends on the spin ratio and Reynolds number. Top spin causes downward force. Spin ratio = $\omega D/2u$.

$$u = 108000/3600 = 30 \text{ m/s}, \ \omega = 8000 \times 2\pi/60 = 837.76 \text{ radians/s}$$

$$\therefore \quad \text{Spin ratio} = (837.76 \times 0.064/2 \times 30) = 0.8936$$

$$Re = 30 \times 0.064/16 \times 10^{-6} = 1.2 \times 10^5$$

By interpolation in Fig. 10.3.4, page 341 C_L is read as 0.25

$$\therefore \quad \textbf{Lift force} = 0.25 \times \pi \times \frac{0.064^2}{4} \times \frac{1}{2} \times 1.165 \times 30^2 = \mathbf{0.4216\ N},$$

This force acts downwards due to top spin.

$$\text{gravity force} = 0.057 \times 9.81 = 0.5592 \text{ N} \quad \therefore \text{Total force} = 0.9808 \text{ N}$$

Equating it to the z directional acceleration

$$F = mu^2/R \text{ where } R \text{ is the radius of the path in the vertical plane.}$$

$$\mathbf{R} = 0.057 \times 30^2/0.9808 = \mathbf{52.3\ m}$$

In case only gravity force acts, then $R = 0.057 \times 30^2/0.5592 = 91.7$ m

The ball comes down sharply due to the top spin.

Problem 10.21 *A table tennis ball of mass 2.5 grams and a diameter of 38 mm is hit with a velocity of 12 m/s, with a back spin ω.* ***Determine the value of back spin for the ball to travel in a horizontal path****, not dropping due to gravity. Density and kinematic viscosity of air are 1.165 kg/m³ and 16 × 10⁻⁶ m²/s.*

For this situation, the force due to gravity should equal the lift force. *i.e.,*

$$mg = C_L(1/2)\,\rho\, u^2 A$$

$$\frac{2.5}{1000} \times 9.81 = C_L \times \frac{1}{2} \times 1.165 \times 12^2 \times \pi \times \frac{0.038^2}{4} \quad \therefore C_L = 0.2578$$

From the graph for C_L vs spin ratio, (Fig. 10.3.4) the value of spin ratio is read as 0.91

$$\frac{\omega D}{2u} = 0.91, \quad \therefore \quad \boldsymbol{\omega} = \mathbf{574.7\ rad/s\ or\ 5488\ rpm}$$

If the velocity is more, for this spin the ball will rise. If the velocity is less then the ball will travel in an arc.

Problem 10.22 *A cork ball 0.3 m diameter with specific gravity 0.21 is tied on the bed of a river. At a certain time it rests at 30° to the horizontal due to the flow.* ***Determine the velocity of flow.***

The forces on the cork ball are shown in Fig. P. 10.22.

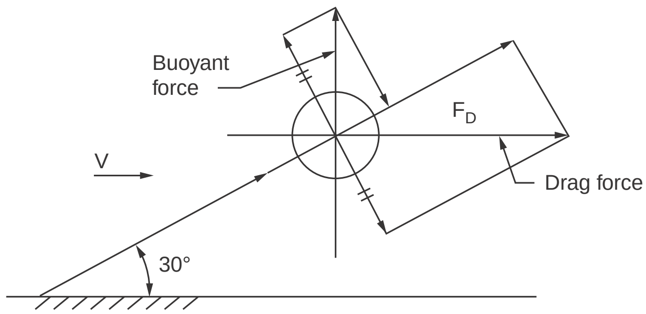


Figure P. 10.22 Force diagram

At equilibrium the components along the rope (at 30° to the horizontal) is taken up by the rope. The components perpendicular to this line should balance.

$\therefore \qquad F_D \cos 60 = F_b \cos 30$

F_b = Buoyant force = difference in density × volume × g

$$= 790 \times \frac{4}{3}\pi \times 0.15^3 \times 9.81 \text{ N} = 109.56 \text{ N}$$

$\therefore \qquad F_b \cos 30 = 94.88$ N. $\quad \therefore F_D = 94.88/\cos 60 = 189.66$ N

$F_D = C_D\,(1/2)\,\rho\, AV^2$, $\boldsymbol{C_D}$ **= 0.45 for sphere**

$$189.66 = \frac{0.45}{2} \times 1000 \times \pi \times \frac{0.3^2}{4} V^2, \text{ Solving } \mathbf{V = 3.45\ m/s}$$

Reynolds number $= \dfrac{3.45 \times 0.3}{1.06 \times 10^{-6}} = 0.98 \times 10^6$

For this value $\mathbf{C_D = 0.2}$ Corresponding **V = 5.18 m/s**

Further iteration is necessary as the new value of Re = 1.47×10^6 and $C_D = 0.35$.

Problem 10.23 *Air flows in a square duct of side 0.6 m with a velocity of 3 m/s. The displacement thickness in meter is given by* $\delta_d = 0.0039\, x^{0.5}$ *where x is the distance along the flow.* ***Determine the velocity outside the boundary layer at a distance of 30 m. Density of air*** *= 1.2 kg/m³.*

The flow with boundary layer can be taken as flow at the free stream velocity with the boundary moved by a distance equal to the displacement thickness.

At 30 m, displacement thickness is $\delta_d = 0.0039 \times 30^{0.5} = 0.02136$ m

The side of the square is reduced by twice this thickness.

∴ Length of side considering displacement thickness is

$$L_d = (0.6 - 2 \times 0.02136) = 0.5573 \text{ m}$$

Equating the volume flow rate $0.6 \times 0.6 \times 3 = 0.5573^2 \times V_2$

∴ $\mathbf{V_2 = 3.48 \text{ m/s}}$

The pressure drop can be calculated for the flow outside the boundary layer as

$$\Delta P = (1/2)\,\rho\,(V_2^2 - V_1^2) = (1/2) \times 1.2\,[3.48^2 - 3^2] = 1.87 \text{ N/m}^2$$

OBJECTIVE QUESTIONS

O Q. 10.1 Fill in the blanks:

1. In flow over surfaces, fluid at the surface takes on the velocity of the body as a result of _______________ condition.
2. The study of non viscous fluid flow is called _______________.
3. Equations describing the complete flow field are know as _______________ equations.
4. The effect of viscosity is important only in a thin layer adjacant to the surface called _______________.
5. The flow outside the boundary layer can be treated as _______________ flow.
6. Velocity gradient exists only in the _______________.
7. The forces which are important in the boundary layer are _______________.
8. In ideal flwo the forces that are important are _______________.
9. The pressure gradient at the surface causes _______________ on the surface.
10. Initially _______________ flow prevails in the boundary layer.

Answers

(1) no slip (2) Theoretical hydrodynamics. (3) Navier-Stokes (4) boundary layer (5) Ideal fluid (6) boundary layer (7) Inertia and viscous forces (8) pressure and inertia (9) shear stress (10) Laminar

O Q. 10.2 Fill in the blanks:

1. Mass and momentum flow in laminar boundary layer is only at the_______________ level.
2. The two methods of analysis of boundary layer flow are _______________.
3. Macroscopic mixing between layers occurs in _______________.
4. The ratio of inertia force to viscous force is called _______________ number.
5. Turbulent flow over a flat plate is generally taken to start at a Reynolds number of _______________.
6. In laminar flow viscous forces are _______________ compared to inertia forces.
7. In turbulent flow viscous forces are _______________ compared to inertia force.
8. Boundary layer separation occurs when there is an _______________ pressure gradient.
9. Lift is the component of the total force on a body immersed in a flow in the _______________ direction.
10. Drag is the component of the total force on a body immersed in a flow in the _______________ direction.

Answers

1. microscopic 2. exact differential, approximate integral. 3. Turbulent flow. 4. Reynolds 5. 5×10^5 6. Larger 7. Smaller 8. Adverse 9. Perpendicular 10. Flow

O Q. 10.3 Fill in the blanks:

1. The force perpendicular to both the flow direction and the axis of rotation of an object in flow is known as ______________ effect.
2. The drift of a shell fired is due to ______________.
3. Rotating cylinders were proposed to propel ships by the use of ______________.
4. The coefficient of lift is the ratio between ______________ and ______________.
5. The coefficcient of drag is the ratio between ______________ and ______________.
6. The coefficient of lift on an airfoil ______________ with angle of incidence upto a limit and then ______________.
7. Coefficient of lift on an airfoil decreases when ______________ occurs.
8. The coefficient of lift has values about ______________.
9. Coefficient of drag has values in the range ______________.
10. Top spin ______________ the length of travel of a ball.

Answers

1, 2, 3 Magnus effect 4. Lift force, dynamic force 5. Drag force, dynamic force 6. Increases, decreases 7. Separation 8. 1.0, 9. 0.1 10. Shortens

O Q. 10.4 Fill in the blanks:

1. The layer thickness which will have the same flow rate as the boundary layer with free stream velocity is called ______________.
2. The layer thickness which will have the same flow momentum as the boundary layer with free stream velocity is called ______________.
3. The drag due to boundary layer sparation is called ______________ drag.
4. The pressure gradient which will delay boundary layer seperation is called ______________ gradient.
5. The pressure gradient which will induce boundary layer seperation is called ______________ gradient.
6. The angle between the flow direction and the chord of an airfoil is called ______________.
7. The lift and drag coefficient on a spinning sphere is dependent on ______________ defined as $\omega D/2V$, V being the forward velocity.
8. In the range of Reynolds numbers 10^3 to 10^5, the drag coefficient on a cylinder or sphere is ______________.
9. The velocity profile in laminar flow follows nearly a ______________ polynomial.
10. The velocity profile in turbulent flow can be represented by ______________ power law.

Answers

1. Displacement thickness 2. Momentum thickness 3. Pressure 4. Favourable 5. Adverse 6. Angle of incidence 7. Spin ratio 8. Nearly constant 9. Cubic 10. One seventh

O Q. 10.5 Fill in the blanks:

1. The distance from the wall where the velocity is 99% of its asymptotic limit is known as ______________ of a boundary layer.

2. In laminar boundary layer, the momentum transfer is by _______________.
3. A boundary layer in which there is macroscopic mixing is called_______________.
4. In flow of real fluids the viscous effects can be considered to be confined to the _______________.
5. In flow over a flat plate the boundary layer undergoes transition when the value of Reynolds number is about _______________.
6. The velocity gradient in turbulent boundary layer will be _______________ than in the laminar boundary layer.
7. In a turbulent boundary layer over a smooth plate, there exits a thin layer in which velocity variation is linear is called as _______________.
8. The phenomenon of boundary layer separation takes place at adverse _______________ gradient.
9. The disturbed region downstream of boundary layer separation is known as _______________.
10. At the separation point of boundary layer, the velocity gradient will be _______________.

Answers

1. Thickness 2. Molecular diffusion 3. Turbulent boundary layer 4. Boundary layer 5. 5×10^5 6. greater 7. Laminar sub layer 8. Pressure 9. Wake 10. Zero

O Q. 10.6 Fill in the blanks with increases or decreases:

1. Boundary layer thickness _______________ along the flow direction.
2. Local velocity within a boundary layer _______________ towards the boundary surface.
3. Shear stress within the boundary layer _______________ towards the boundary surface.
4. Reynolds number _______________ in the direction of flow over flat plate.
5. Turbulence _______________ in the direction of flow.
6. Drag coefficient on a plate _______________ with increase in Reynolds number.
7. Local shear stress coefficient _______________ with increase in Reynolds number.
8. Reynolds number _______________ with increase in kinematic viscosity of a fluid.
9. Drag force _______________ with increase in free stream velocity.
10. Boundary layer thickness _______________ with increase in Reynolds number.

Answers

Increases : 1, 3, 4, 5, 9 Decreases : 2, 6, 7, 8, 10

O Q. 10.7 State True to False:

1. The velocity gradient in the boundary layer is maximum at the top edge of the layer.
2. The flow above the boundary layer can be treated as invicid flow.
3. The velocity gradient will be zero at the top plane of the boundary layer.
4. Displacement thickness is the thickness by which the plane is to be moved up, so that the flow equals the flow at free stream velocity.
5. Rate of momentum transfers will be higher in turbulent flow.
6. Momentum thickness will be larger compared to displacement thickness.
7. Boundary layer thickness will be smaller compared to displacement thickness.
8. Displacement thickness is defined by $\dfrac{1}{u_\infty^2}\displaystyle\int_0^\delta (uu_\infty - u^2)dy$.
9. Momentum thickness is given by $\displaystyle\int_0^\delta \left(1-\frac{u}{u_\infty}\right)dy$.

10. Friction coefficient $C_f = \tau_w/\rho\, u_\infty^2$.
11. In turbulent flow $C_f \propto \text{Re}^{0.2}$.
12. In laminar flow $C_f \propto \text{Re}^{-0.5}$.
13. In turbulent flow $C_f \propto \text{Re}^{-0.2}$.
14. Boundary layer thickness in laminar flow is proportional to x.
15. Boundary layer thickness in laminar flow is proportional to $x^{1/2}$.
16. In turbulent flow boundary layer thickness is proportional to $x^{0.8}$.

Answers

True : 2, 3, 4, 5, 12, 13, 15, 16

False : 1, 6, 7, 8, 9, 10, 11, 14

O Q.10.8 Define

1. Boundary layer thickness 2. Momentum thickness 3. Displacement thickness 4. Coefficient of friction 5. Laminar flow 6. Turbulent flow 7. Drag coefficient 8. Pressure drag 9. Flow separation 10. Chord, wing span of an airfoil. 11. Lift coefficient 12. Drag coefficient 13. Angle of incidence 14. Spin ratio 15. Magnus effect.

EXERCISE PROBLEMS

E 10.1 Assuming that air at 20 °C flows over a flat with a free stream velocity of 6 m/s, determine the velocity at 0.5 m and 0.8 m at a distance of 6 mm from plate surface (*i*) Assuming cubic profile and (*ii*) linear profile.

$\upsilon = 15.06 \times 10^{-6}$ m²/s $\rho = 1.205$ kg/m³, In case flow is turbulent use 1/7th power law.

E 10.2 Derive an expression for the displacement thickness and Momentum thickness in flow over a flat plate assuming $\dfrac{u}{u_\infty} = \sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$.

E 10.3 Compare the velocities for flow over a flat plate to turn turbulent at a distance of 0.6 m for (*i*) air (*ii*) water and (*iii*) engine oil, all at 20 °C.

E 10.4 Assuming momentum thickness to be constant at the transition point whether laminar or turbulent flow correlation is used, find the ratio of laminar boundary layer thickness to turbulent boundary layer thickness. Assume parabolic profile for laminar correlation $\dfrac{u}{u_\infty} = 2\dfrac{y}{\delta} - \left(\dfrac{y}{\delta}\right)^2$ and 1/7th power law variation for turbulent correlation.

(δ_t/δ_L = 144/105)

E 10.5 Determine the ratio of δ_m/δ_d in the turbulent region of flow over a flat plate, assuming 1/7th power law. **(δ_m/δ_d = 0.0972)**

E 10.6 In a flow of air over a flat plate at a distance 20 cm from leading edge, the boundary layer thickness was measured as 5.7 mm the free stream velocity being 25 m/s. If the velocity profile is given as $\dfrac{u}{u_\infty} = 2\dfrac{y}{\delta} - \left(\dfrac{y}{\delta}\right)^2 + C$, calculate the value of C. Also determine the displacement and momentum thicknesses at this section. **(δ_d = 1.9 mm, δ_m = 0.76 mm)**

E 10.7 In a wind tunnel of square section of side 80 mm, the inlet velocity is 30 m/s. At 0.3 m from inlet the displacement thickness is 1 mm. Determine the change in pressure between the sections. **(59 N/m^2)**

E 10.8 In a wind tunnel square section of side 0.305 m, the boundary layer thickness increases from 9.5 mm to 12.7 mm. At the first location, the velocity of air is 18.3 m/s. The static pressure is – 22.9 mm of water column (gauge). Assuming turbulent flow conditions and 1/7th power law velocity profile, determine the velocity at the second section. Also determine the change in static pressure. **(18.4 m/s, ΔP = 2.19 N/m^2)**

E 10.9 In a water tunnel the freestream velocity is 1.6 m/s. A plate 0.3 m long and 1 m wide is placed parallel to the flow. (0.3 m along flow). Determine the total viscous force on the plate. Use the results of cubic velocity profile assumption. **(1.62 N)**

E 10.10 Show that for flow over a flat plate, drag force upto a length L on one side is given by $F_D = \rho u_\infty^2 \delta_{mL} b$. where b is the width, u_∞ free stream velocity and δ_{mL} momentum thickness, For u_∞ = 2 m/s, and L = 0.3 m, b = 1 m, determine the total drag. **(0.81 N)**

E 10.11 Determine the velocity of water if the drag on a sphere of 12 cm dia is 5 N. The water is at 10 °C. **(V = 1.29 m/s)**

E 10.12 Asuming $(u/u_\infty) = \left(\frac{y}{\delta}\right)^{1/9}$, derive an expression for boundary layer thickness at a distance x from leading edge. **(δ = 0.287 x/R$^{1/6}$)**

E 10.13 Determine the friction drag on an airship 100 m long and 20 m diameter when it travels at 130 kmph. The condition of air is 25 °C and 0.9 bar. **(8370 N)**

E 10.14 A large truck weighing 45000 N is be air dropped using a large parachute, the value of C_D being 1.2. If the terminal speed is 10 m/s in air at 1 bar and 20 °C, determine the diameter of the chute. **(30 m)**

E 10.15 Describe the types of drag when a disc is held parallel to flow direction and perpendicular to flow direction.

E 10.16 An open C section of 30 cm dia and 8 m length is held with concave side facing the flow at 1 m/s of water. Determine the drag on the section. **(2.76 kN)**

E 10.17 A spherical balloon of helium of diameter 3 m is held tied to a rope. The wind flows at 20 kmph. Determine the angle of inclination of the cable. The pressures and temperatures on the inside and outside are 1 bar and 20 °C.

E 10.18 An advertisement board 3 m dia is exposed to a 100 kmph normal wind. Determine the total force on the board. **(3.6 kN)**

E 10.19 A high speed car with a frontal area of 1 m^2 and drag coefficient C_D = 0.3 travelling at 100 m/s is to be decelerated using a drag chute of 2 m dia with C_D = 1.2. Determine the speed 10 seconds after the chute is deployed. Density of air = 1.2 kg/m^3. **(45 m/s)**

E 10.20 A wind tunnel of 1 m square section is 6 m long. Air at 20 °C flows at 30 m/s. In order to maintain the velocity constant, the walls are to be slightly slanted outward. This is to compensate for the growth of the displacement thickness. Determine the angle of slant between 2 m and 4 m distances from entry.

E 10.21 A thin circular plate is held parallel to a flow with velocity u_∞. Derive an expression for the drag assuming the flow to be similar to that on a flat plate.

E 10.22 A flat plate of lenght L and width equal to the boundary layer thickness is held parallel to the flow direction. Derive an expression for the drag on the plate. (*i*) Assume linear velocity profile. (*ii*) Assume cubic profile and (*iii*) Assume 1/7th power law.

E 10.23 A ship 150 m long has a rough wetted area of 5000 m^2. To overcome friction the power required was 5152 kW at a speed of 15 kmph. Determine the reduction in power required if the surface was smooth. Take properties of water at 20 °C.

E 10.24 Determine the frequency of vortex shedding by a wire 5 mm dia, at a wind speed of 30 m/s. The density of air 1.2 kg/m^3 and $\mu = 18 \times 10^{-6}$ kg/ms. **(1.2 kHz)**

E 10.25 The tension of the rope holding a kite is 20 N. When the rope is at 45° to the horizontal. The kite is a square of side 6 m , and weights 5 N. The angle made by the kite is 10° with horizontal. The wind speed is 4 kmph. Air density is 1.2 kg/m^3. Calculate the lift and drag coeficients. **(C_L = 0.72, C_D = 0.53)**

E 10.26 A spherical ballon of 1.5 m dia filled with helium when let go ascended at 6.27 m/s in air. The weight of the empty ballon was 50 N. Determine the density of helium in the balloon. Density of air = 1.2 kg/m^3. **(0.2 kg/m^3)**

E 10.27 A chute carrying a bomb totally weighs 1 kN. If C_D = 1.2 and rate of descent is 6 m/s determine the diameter of chute. Density of air = 1.22 kg/m^3 **(7 m)**

E 10.28 Determine the terminal velocity and the maximum diameter of a spherical particle with a relative density of 1.8 that will settle in water with a density of 1000 kg/m^3. Dynamic viscosity = 1.2×10^{-3} Ns/m^2. Use stokes law. **(V = 8.1×10^{-3} m/s, d = 0.149 mm)**

E 10.29 Derive the expression for terminal velocity of a particle $V = D^2(f_g - f)/18\mu$, when f_g = gravity force and f is bouyant force N/m^3.

E 10.30 A small stone of dia 5 mm and relative density 2.8 falls in oil of relative density 0.9. Estimate the terminal velocity.μ = 0.3 Pas. **(8.63×10^{-2}m/s)**