2

Classification of designs by symmetry group

2.1 Introduction

In his epic book *Pattern Design*, Day suggested that success in designing depends largely upon insight into how design works and it must be realised that the beauty of pattern is not so much due to the nature of its elements as to the right use of them as units in a rhythmic scheme.[1] There are a number of different possibilities which may be used to arrange elements in the form of a rhythmic scheme (or regularly repeating design). The geometric principles describing each different arrangement may be defined and classed by means of a distinct system. This system may then be used to compare the relationships between and properties of any one type of regularly repeating design with another.

The system used to classify designs by symmetry group is based on the geometric characteristics of the underlying structures of the designs rather than the symmetrical properties of the individual design units from which they are comprised. The arrangement of the elements, or design units (whether they are symmetric or not), determines the geometric characteristics of the design's underlying structure. These characteristics may be analysed, defined and classed in a particular group. The primary objective of this chapter is to define and explain the range of concepts, terminology and geometric principles relevant to the classification of designs by their symmetry groups. Following this an extensive range of construction techniques is described and illustrated for each group.

2.2 Symmetry and its relevance to designs

The theoretical perspectives presented in this chapter, and those following on throughout this book, are applicable to planar designs, that is, the geometric analyses and categorisations apply to designs which lie on a flat surface rather than those which occur in three dimensions. With regard to the symmetry of a design, Washburn describes it as a type of order with specific geometric parameters and that as a mathematical measure it proves useful for the classification and comparison of patterns on cultural materials.[2]

Symmetric designs give both a pleasing visual effect of balance and order, whilst also providing an element of intrigue and fascination through which the geometrical properties and structural framework are successively analysed. Davis and Hersh observed that, through intuition, the artist is often an unconscious mathematician, discovering, rediscovering, and exploring ideas of spatial arrangement, symmetry, periodicities, combinatorics and transformations and discovering, in a visual sense, theorems of geometry.[3] Thus, although rules of symmetry may be arrived at intuitively, and through artistic exploration, as stated by Washburn and Crowe, systematic classificatory schemes rather than general concepts like style can better support the process of hypothesis building.[4] Consequently, a systematic approach enables all geometric combinations and symmetric structures to be investigated and established and then used as a basis upon which to build artistic exploration.

2.3 Symmetry operations

To analyse and classify designs by symmetry group requires examination of the symmetries present in their structures. Grünbaum and Shephard give a precise mathematical definition of a symmetry as follows: 'By a symmetry of a set S we mean any isometry σ which maps S onto itself, that is $\sigma S = S$'.[5] Here the set S refers to a figure or design and this type of isometry, σ, is synonymous to a rigid motion, symmetry operation or symmetry transformation. Alternatively Washburn and Crowe describe a symmetry motion as the specific configuration of parts for each design. They go on to say that symmetry does not describe the parts, but how they are combined and arranged to make a pattern and that it concerns only one aspect of a pattern's design – its structure.[4]

Each of the isometries or symmetry motions etc., may be categorised as one of the following operations explained in Sections 2.3.1 to 2.3.6.

2.3.1 Rotational symmetry

A design has n-fold rotational symmetry about a fixed point if, when rotated in its own plane about that point through 360°/n and integral multiples of that angle, it coincides with its original position. The fixed point is called the centre of rotation, and n is an integer greater than or equal to one which corresponds to the order of rotation. After n successive rotations of 360°/n, the figure will return to its original position.

2.3.2 Translational symmetry

A design has translational symmetry if figures in it can be moved to congruent figures by a glide in any direction, whilst still keeping the same orientation. All parts of the figures move the same distance in the same direction.

2.3.3 Reflectional symmetry

A design has reflectional symmetry if it can be bisected by one or more 'mirror' axes. In this instance the portion on the left hand side of such an axis relates to the portion on the right hand side by being its mirror image. All the points on the mirror (reflection) axis remain fixed.

2.3.4 Glide–reflectional symmetry

The symmetry of glide–reflection is a motion combining a reflection and translation, along the direction of the reflection axis, consecutively. Two successive glide–reflection operations along an axis are equivalent to one unit of translation in the same direction.

In addition to these four symmetry operations, there are two other symmetries which are characteristics of every design: identity and inverse symmetry.

2.3.5 Identity symmetry

This symmetry is equivalent to no movement at all. The figure, or design, is effectively lifted up and put down in exactly the same position such that each point is mapped onto itself. Alternatively it can be thought of as a 360° rotation about a point.

2.3.6 Inverse symmetry

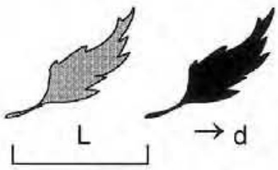
For every symmetry of a design there is another symmetry which is the reverse of it, that is, a symmetry which will take the design back to its original position. This is referred to as the 'inverse' symmetry.

Figure 2.1 shows examples of the symmetry operations described in Sections 2.3.1 to 2.3.6.

With respect to the identity symmetry, Loeb comments that any figure may be brought into self-coincidence by the operation of identification (or identity a A rotation

A rotational symmetry operation 90° clockwise about O (which is equivalent to a rotation of 270° anticlockwise about O).

b A translation

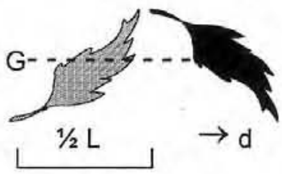


A translational symmetry operation distance L in direction d.

c A reflection

A reflectional symmetry operation about reflection axis M.

d A glide-reflection



A glide-reflectional symmetry operation distance 1/2 L about glide-reflection axis G in direction d.

e The identity symmetry

The identity symmetry operation is represented by no movement at all and is equivalent to a 360° rotation about O.

f The inverse symmetry

The inverse symmetry of **a** is a rotation 90° anticlockwise about O.

The inverse symmetry of **b** is a translation distance L in the oppposite direction, – d.

The inverse symmetry of **c** is a reflection operation about reflection axis M back to the original position.

The inverse symmetry of **d** is a glide-reflection distance 1/2 L about glide-reflection axis G in the opposite direction – d.

Key

Initial position

Position after the application of a symmetry operation

Figure 2.1 The symmetry operations.

symmetry) and that if this operation is the sole symmetric transformation of the figure, then the figure is called *asymmetric*.[6]

Conversely, in Woods' paper *The Geometrical Basis of Pattern design, Part 1*, he describes a figure as being *symmetrical* when it is possible to find two or more positions in which it can be exactly superimposed on itself and that the movement necessary to bring the figure from one such equivalent position to another is said to be a symmetry operation.[7] One of these positions refers to the identity symmetry, where the position of the figure remains unchanged, and the other one or more will correspond to one of the first four symmetry operations described above (in Sections 2.3.1 to 2.3.4).

2.4 Symmetry group

The complete set of symmetry operations, or all equivalent positions of a figure, form its *symmetry group*. A symmetry group, which is a collection of symmetry operations, has the following characteristics:

1. It always contains the identity symmetry which leaves the position of the figure unchanged.
2. For every symmetry operation which moves a figure from position A to position B, there exists an inverse operation which is able to move the figure back from position B to its original position A again.
3. Each symmetry operation in the group may be followed by another, and the resulting operation of the combination of the two is, itself, a member of the symmetry group. For example, if a design has translational symmetry and reflectional symmetry in its symmetry group, then the resultant of the two, which is a glide–reflectional symmetry, is also a member of the group. Similarly, the two operations of a horizontal translation followed by a vertical translation of a design are equivalent to the resultant which is a diagonal translation. This translation would also be a symmetry in the group of symmetries of the whole design.

Loeb describes how any symmetry group consists of symmetrical operations which themselves are *elements* of the group.[6] (Note that here the term 'element' is used to describe a symmetry motion or movement rather than the unit of the design itself that was described by Day at the beginning of this chapter.) Loeb adds that the total number of elements for all distinct equivalent positions of the figure is called the *order* of the group, for example an equilateral triangle has the order six (see Fig. 2.2). The symmetry operations, or elements, form the basis of the construction and generation of designs.

Throughout the previous definitions, the meanings of the terms 'figure' and 'design' have been taken for granted. There seems to be no distinct interpretation of these terms but further comments on each are given below.

2.5 Figures and designs

More formally, a figure is defined as either a 'superficial space enclosed by lines', an 'image', a 'diagram', an 'illustrative drawing', a 'design' or a 'pattern'. Thus the term figure has numerous meanings that could either refer to a single motif or tile, or the entire pattern or tiling generated from these single units, respectively.

With regard to a design, Washburn and Crowe define it as a specific kind of figure which admits at least one (non-trivial) isometry.[4] They therefore consider a design to be a symmetrical figure which has at least two symmetries, one of which is the identity symmetry. (In this case the identity symmetry is referred to as the non-trivial isometry.) This description implies that asymmetric patterns and irregular tilings are not designs. However, throughout this book, a design will be used to describe any form of decoration on one plane, that is, an illustration on a flat surface. (Of course, in many contexts a design may be used to represent ornament or construction in three dimensions although here, as stated above, it will be restricted to surface decoration.)

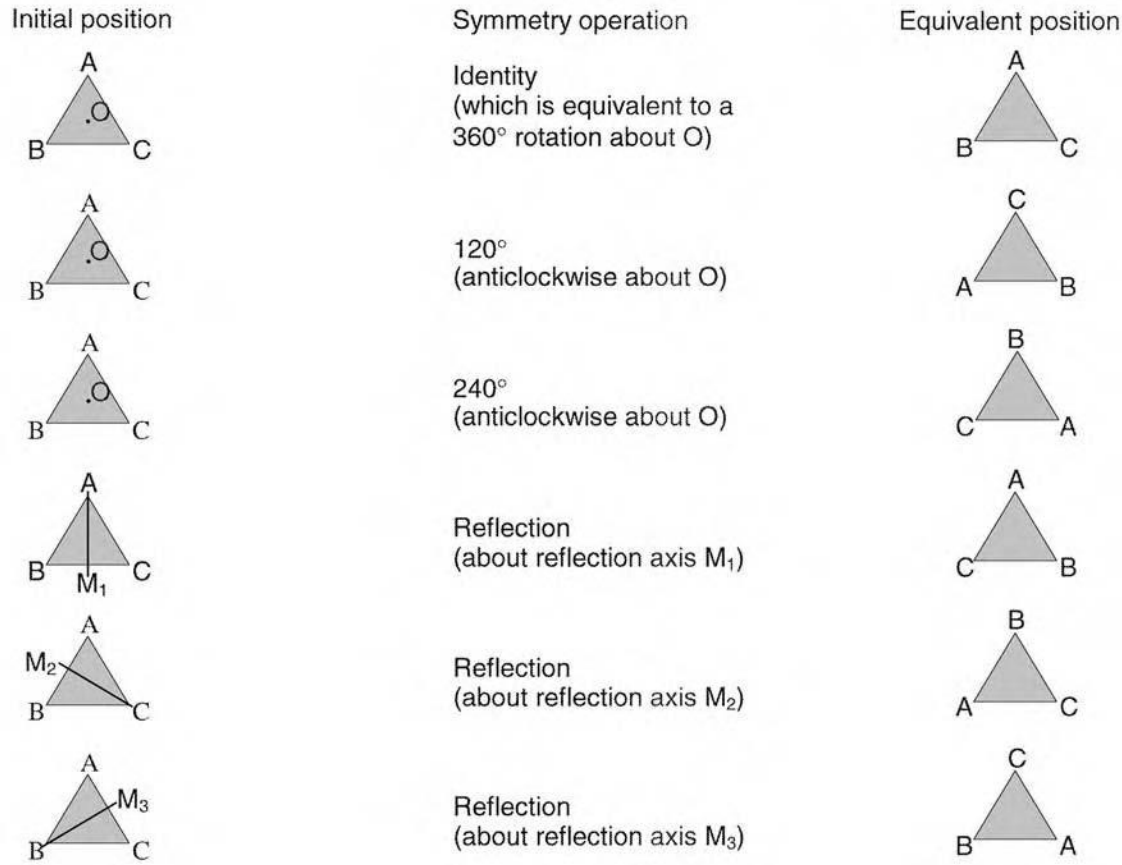


Figure 2.2 The order of symmetry.

A design may decorate a surface in a number of ways. For example a design may have no regular repetition in it at all; it may have elements in it which repeat at regular intervals around a point; it may have elements in it which regularly repeat by translational symmetry in one direction or by translational symmetry in at least two non-parallel directions. Those designs which are irregular (and therefore possess only the identity symmetry) and those which contain elements which only repeat cyclically around a point are often referred to as 'finite designs'.

2.6 Classification of finite designs

Washburn and Crowe define finite designs as those which have a central point axis around which elements can rotate or through which mirror axes can pass and that other symmetries such as translation or glide-reflection are not possible in this category.[4] Classifying finite designs by symmetry group divides them into two classes: either the cyclic symmetry group, denoted by *cn*, or the dihedral symmetry group, denoted by *dn*. Here '*n*' is used to represent a positive integer. (Note that both *cn* and *dn* designs have rotational or 'cyclic' symmetry, however, in this instance the term 'cyclic' usually refers to those designs which have only rotational symmetry.) Figures 2.3 and 2.4 show some examples of these types of design.

2.6.1 Cyclic finite designs

A cyclic design, in symmetry group *cn*, has only *n*-fold rotational symmetry about a point at its centre. After *n* consecutive rotations of 360°/*n* in one direction (either anticlockwise or clockwise) about this point, the design will return to its original position. An asymmetric unit or figure has one-fold rotational symmetry, in other words $n = 1$ and a rotation by 360°/1 (i.e. a full turn) will return the

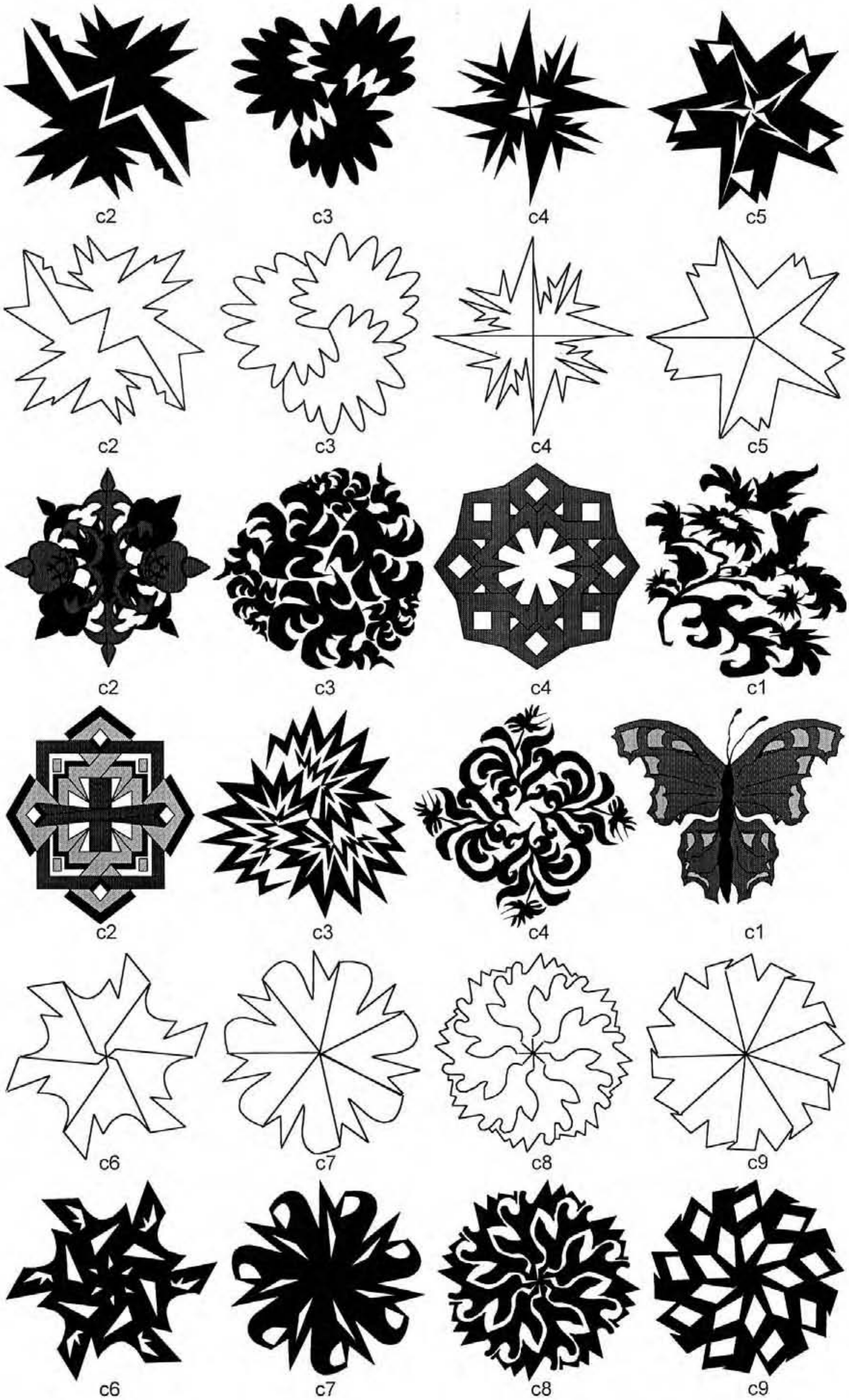


Figure 2.3 Illustrations of finite designs, symmetry group *cn*.

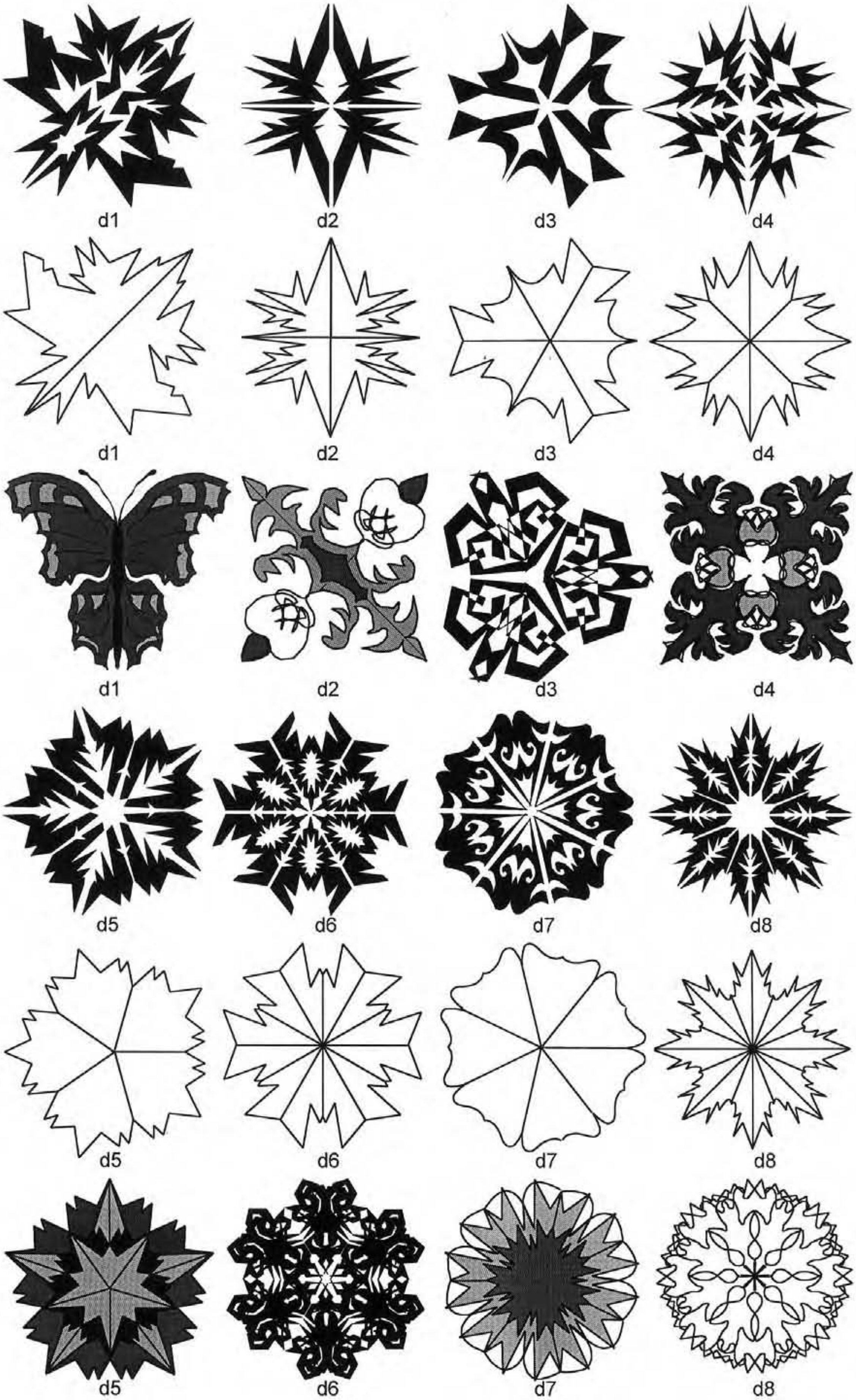


Figure 2.4 Illustrations of finite designs, symmetry group *dn*.

figure back to its original position. For asymmetric designs the centre of rotation need not necessarily be at the centre of the design (see Fig. 2.3).

2.6.2 Dihedral finite designs

A dihedral design, in symmetry group *dn*, has *n*-fold rotational symmetry about a point at its centre and also *n* reflection axes passing through that point (see Fig. 2.4).

Finite designs, *cn* and *dn*, are also referred to by Schattsneider, in her article in *Symmetry: Unifying Human Understanding*, as 'rosette designs'[8] and Loeb describes how these symmetry groups, formed only by operations which leave at least one point fixed, are called *point groups*.[8] Woods adds that this type of symmetry, centred around a point, is sometimes referred to as point symmetry or central symmetry.[9] The 'point' symmetry indicates that when symmetry groups *cn* and *dn* are rotated about their centres of rotation precisely one point remains fixed. When a design in symmetry group *dn* is reflected about a reflection axis through its centre, a whole line of points remains fixed. If n is greater or equal to two ($n \geq 2$), the reflection axes of a *dn* design intersect at a point, that being the centre of rotation.

2.7 Structure of translational designs

A design which decorates a surface by the regular repetition of a unit by translational symmetry will fall into one of two categories: (i) a monotranslational design (otherwise known as a one-dimensional design,[4] a one-sided band,[10] a strip or frieze group,[5] a border,[9] or a periodic border design[8]) or (ii) a ditranslational design (otherwise known as a two-dimensional design,[4] a wallpaper group[11] or wallpaper design,[12] a crystallographic group,[13] a periodic group,[5] a plane, a network or an all-over pattern,[7,10,14] a periodic planar design,[8] a plane group[15] or an n-dimensional space group ($n = 2$)[16]).

2.7.1 Minimum criteria of translational symmetry

A finite design has reflectional and/or rotational symmetry but no translational symmetry in its symmetry group. Washburn and Crowe define a border pattern (or in this context, a monotranslational design) as one which must satisfy the geometrical condition of having at least one unit of translation in one direction, and an all-over pattern (or in this context a ditranslational design) as one which must satisfy the geometrical condition of having at least one unit of translation in two, non-parallel, directions.[4] However, throughout this book (and in conjunction with the definitions given by Schattsneider),[8] a monotranslational design will be thought of as one which theoretically and conceptually extends to infinity in two opposite directions along a straight line and a ditranslational design will be thought of as one which extends infinitely throughout the whole plane.

2.7.2 Lattice

Every regularly repeating translational design is based on a structural framework. This is represented in the form of an array of points called a net or lattice. Woods[17] describes the construction of a ditranslational lattice as follows:

> Start with a chain of points interval a in some straight line, and . . . . . . make each of these points a point of another chain, of interval b, making an angle θ say with the first chain. We thus obtain an array of points which is such that any translation equal to a multiple of a in the direction of the first chain, or to a multiple of b in the direction of the others moves the figure into an equivalent position. Such an array is called a *net* of points, . . .

A monotranslational design is also constructed on a framework of points. In this instance the initial chain of points, interval a, in some straight line, is trans-

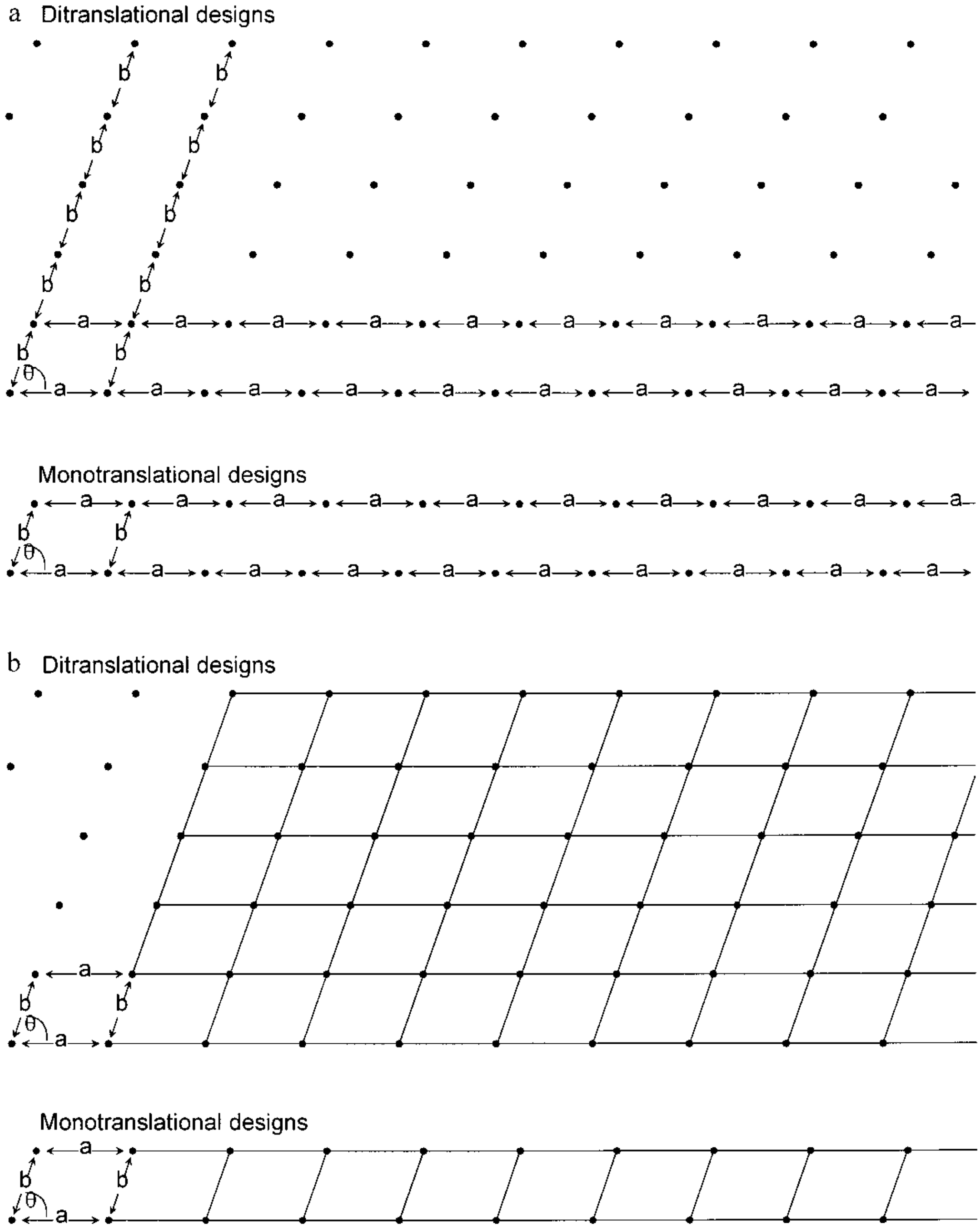


Figure 2.5 Lattice construction (a) and division of lattice points into unit cells (b).

lated at an angle θ, say by one translation. This results in two parallel lines of points upon which to base the structure of the design (see Fig. 2.5a).

2.7.3 Unit cell

Similarly, Woods describes how unit cells of a ditranslational design are constructed by drawing lines through each point of an *a*-chain parallel to *b*, and through each point of a *b*-chain parallel to *a*. The plane is divided into parallelograms, which have sides of lengths *a* and *b* and of which one angle is θ. Any such parallelogram is called a *unit cell*; it has a net point at each vertex but no others either inside or on its sides.[17]

Where monotranslational designs are concerned, parallelograms result from the division of a strip or a band rather than the division of the plane as shown in Fig. 2.5(b).

Note that a parallelogram has four straight sides: two parallel sides of length a and two parallel sides of length b. One of the angles, at which these two sets of lines intersect each other, is $\theta°$. The specific type of parallelogram is determined by the conditions held by a, b and θ. The results of different combinations of these variables are given:

1 If $a = b$ and $\theta = 90°$, the parallelogram is a square.
2 If $a = b$, the parallelogram is a rhombus.
3 If $a = b$ and $\theta = 60°$, the parallelogram is a special kind if rhombus composed of two equilateral triangles. (These types of parallelogram are associated with the 'hexagonal' lattice.)
4 If $a \neq b$, $\theta = 90°$, the parallelogram is a rectangle.
5 If $a \neq b$ and $\theta \neq 90°$, the parallelogram is a just an ordinary parallelogram (which is also referred to by Kennon[18] as a 'general parallelogram') (see Fig. 2.6).

Note that a square is a special form of a rhombus where $\theta = 90°$. A square is also a special form of a rectangle where $a = b$. However, with reference to lattice structures, each of the terms square, rhombic, rectangular, hexagonal and parallelogram is often associated with a particular type of lattice (given in Fig. 2.6) without awareness of these specific cases. For example, it is important to recognise that design types commonly associated with the rhombic lattice may also be based on square or hexagonal lattices; those associated with the rectangular lattice may be based on the square lattice; and those commonly associated with the parallelogram lattice may have any of the five types of lattice as their underlying structure.

Each cell contains one net point (on combining each piece from the four corners), hence the cell is called a *unit* cell (although Schattsneider[8] refers to it as a 'lattice unit'). The union of all the pieces of a figure enclosed within a unit cell, when rearranged in their appropriate order, fit together to form a complete motif or tile. Each unit cell of a design has the same shape and content and when successively translated in one or two directions, for a monotranslational or ditranslational design, respectively, will create the whole design. Each of the symmetry groups of the translational designs can be represented by a unit cell according to the symmetrical properties contained within it. Figure 2.7(a) and (b) shows the unit cells for the symmetry groups of monotranslational and ditranslational designs, respectively. The appropriate symmetry group is given under each unit cell, the notation for which is explained later in this chapter.

2.7.4 Group diagram

Each of the symmetry groups may also be represented by what is referred to as a 'group diagram'.[5] A group diagram shows all the symmetrical characteristics of a design's symmetry group (except translational symmetries which may be represented by vectors but which are usually omitted). In general, centres of two-, three-, four- and six-fold rotation are represented by diamonds (or ellipses), equilateral triangles, squares and regular hexagons, respectively, and glide–reflection and reflection axes are represented by bold dashed and solid straight lines. (These symbols represent the conventional notation for these symmetrical characteristics and will be used, without additional explanation, throughout the remainder of this book.) The group diagram may be incorporated into the design as shown in Fig. 2.8(a(ii)) and (b(ii)) or be separate as shown in Fig. 2.8(a(iii)) and (b(iii)). For regularly repeating translational designs, a group diagram is equivalent to filling each of the cells in a lattice with the symmetrical characteristics of its unit cell. An example of a unit cell for the pattern in Fig. 2.8(b(i)) is represented by the shaded region in Fig. 2.8(b(iii)).

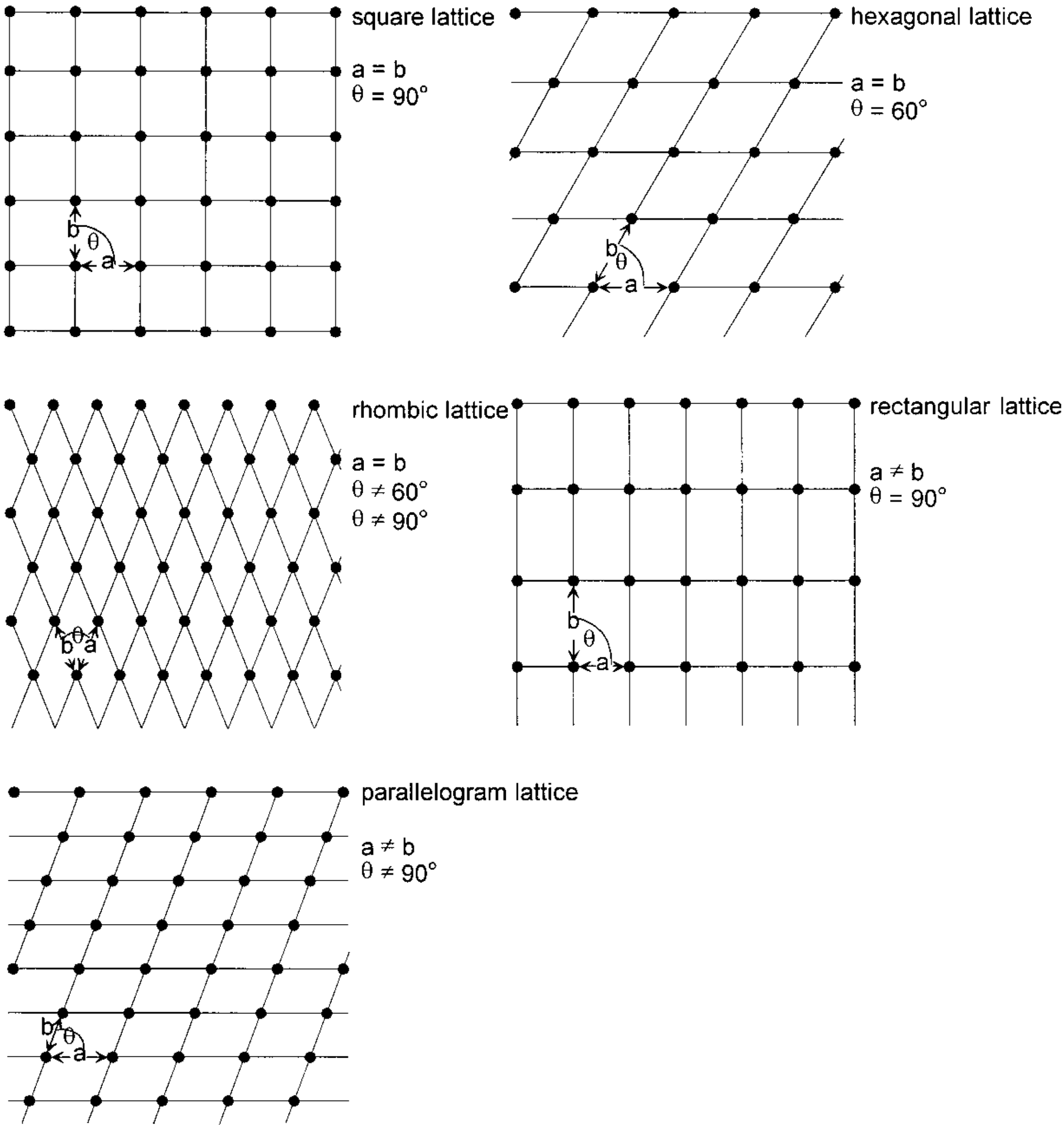


Figure 2.6 Five types of parallelogram lattice.

Finite designs may also be represented by a group diagram but they will only include a minimal number of symmetries. Any centres of n-fold rotation, other than those mentioned above, may be represented by regular n-sided figures or n-pointed stars.

2.7.5 Translation unit

A translation unit is a minimum area of the plane which, when successively translated in one or two non-parallel directions (for a monotranslational or ditranslational design, respectively) creates the whole design. A translation unit has the same area as a unit cell but its shape may not necessarily be a parallelogram. Thus a unit cell is a translation unit but a translation unit is not necessarily a unit cell.

In a monotranslational design the size of the translation unit is sometimes referred to as being independent in relation to the size of the unit cell. For example, Schattsneider[8] describes a translation unit (for a border design consist-

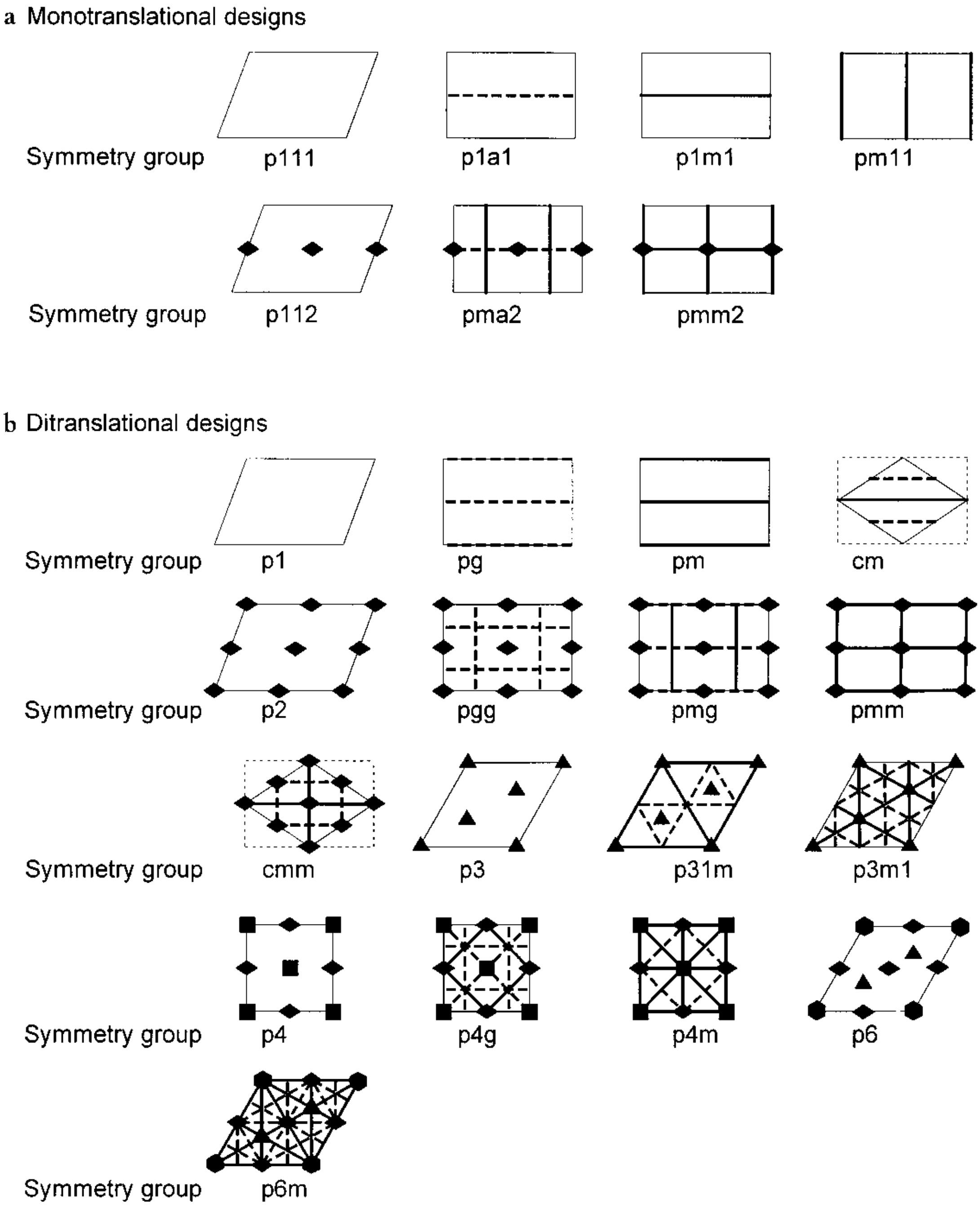


Figure 2.7 Unit cells of translational designs. ◆, 2-fold centre of rotation, ▲, 3-fold centre of rotation; ■, 4-fold centre of rotation; ⬣, 6-fold centre of rotation; ———,unit cell boundary; ...., centred double cell; ———, reflection axis; - - - - -, glide–reflection axis.

ing of non-interlocking motifs) as a smallest region which, when translated repeatedly by T and $-T$, produces the whole border design. T refers to a translation and $-T$ refers to the same translation but in the opposite direction. Similarly, she describes a translation unit for a border tiling as a minimum block of tiles which fills out the whole border by translations alone. The areas enclosed by these translation units may not necessarily fill out the whole unit cell. In some instances, it is difficult to categorise a monotranslational design as a pattern, made up of motifs, or as a tiling, made up of tiles, that is, to differentiate between a pattern and tiling. Therefore, to avoid the problem of having to categorise the type of design unit(s) enclosed within the translation unit it is simpler to regard the translation unit as having the same area as a unit cell for both ditranslational and monotranslational designs.

With this understanding, a translation unit of a monotranslational (or border) design has two sides coinciding with parts of the two parallel lines which enclose the whole design. The remaining two sides, which are also parallel to each other, may be irregular shapes instead of straight lines (which is the case for the unit cell).

The area of a translation unit of a ditranslational design is determined by the positioning not only of the adjacent motifs or tiles to the left and right, but also of those above and below it. Consequently, this area is always fixed and equal to that of the unit cell. Alternative definitions, when differentiating between the decorative components of the design, are therefore not required. The opposite edges of a translation unit are always parallel to each other but are not necessarily straight lines. Figure 2.9(a) and (b) shows examples of translation units.

2.7.6 Fundamental region

A fundamental region is also referred to as a fundamental domain, an asymmetric region[19] or a generating region.[8] It may be defined as the smallest region of the design which, when acted on repeatedly by the symmetries of its symmetry group, creates the whole design. The shape of the region is not always unique for any one design but its area is always the same. Throughout the following discussions, the figure enclosed within a fundamental region will be referred to as a 'design unit' and the separate components of the design unit will be referred to as 'design elements'.

The shape and contents of a fundamental region need not necessarily be asymmetric (which therefore implies that 'asymmetric region' is not a very suitable term for such a region). For example, see Fig. 2.10 where each shaded area represents a fundamental region. In Fig. 2.10(a), a *p*111 monotranslational design has been constructed on a rhombic parallelogram lattice of points. A fundamental region has been chosen to coincide with a unit cell in such a way that the long diagonal axis of the rhombus forms a line of reflectional symmetry coinciding with one through the motif. In Fig. 2.10(b), a *p*1 ditranslational design has been constructed on a rectangular lattice but again, the fundamental region and design unit shown both have coinciding reflectional symmetry. Thus these fundamental regions have been chosen such that their shapes and contents are symmetric rather than asymmetric. However, in cases such as these, the design unit will have no symmetries coinciding with those of the design structure.

Figure 2.10(c) illustrates a symmetrically shaped fundamental region reduced to a form with no symmetries in common with the design structure by introducing five-fold rotationally symmetric design units whose symmetries cannot possibly coincide with any regularly repeating translational design. (As stated in Hauy's theorem in 1822, it is impossible to construct a translational design with n-fold rotational symmetry in its structure if $n = 5$ or is greater than 6, because of the laws of crystallographic restriction. For example, a plane cannot be covered with interlocking regular pentagons alone without there being gaps in between them, or with regular heptagons, octagons or nonagons, etc.) Figure 2.10(d) shows another *p*1 ditranslational design constructed from the same symmetric design unit but in this instance it is contained within an asymmetric fundamental region. (Further analysis and discussion involving designs with symmetric design units are continued in more detail in Chapter 3.)

2.7.6.1 Finite designs

Any finite design may be enclosed within a circle such that its area is just big enough to enclose the extremities of the design (see Fig. 2.11(a)). Suppose the centre of the circle is labelled O. Schattsneider states that for *cn* designs a wedge (circular sector) having angle $360°/n$ at O is a minimal area in which to place the motif.[8] In this ai

ii

Group diagram

iii

bi

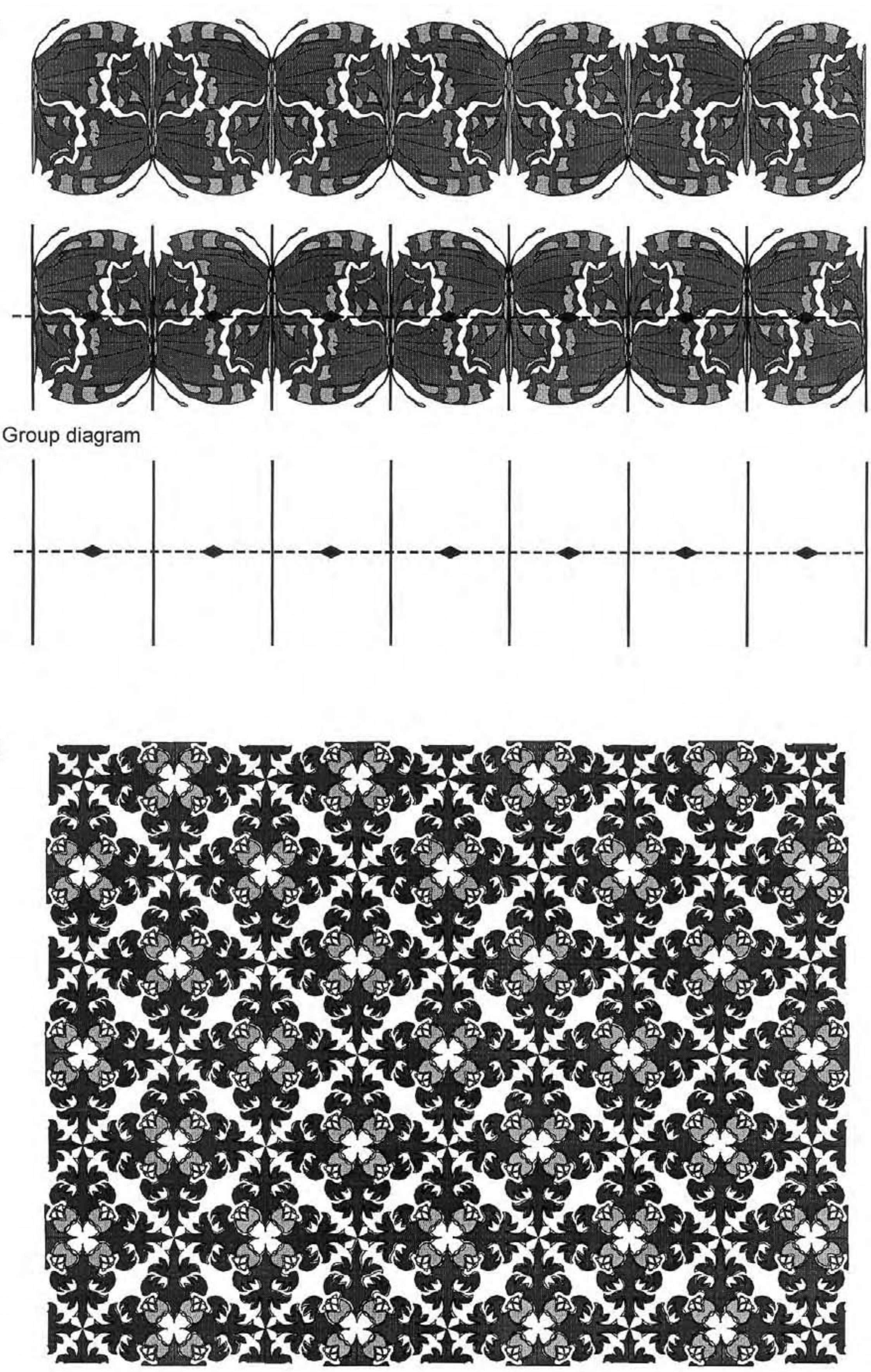


Figure 2.8 Illustrations of group diagrams.

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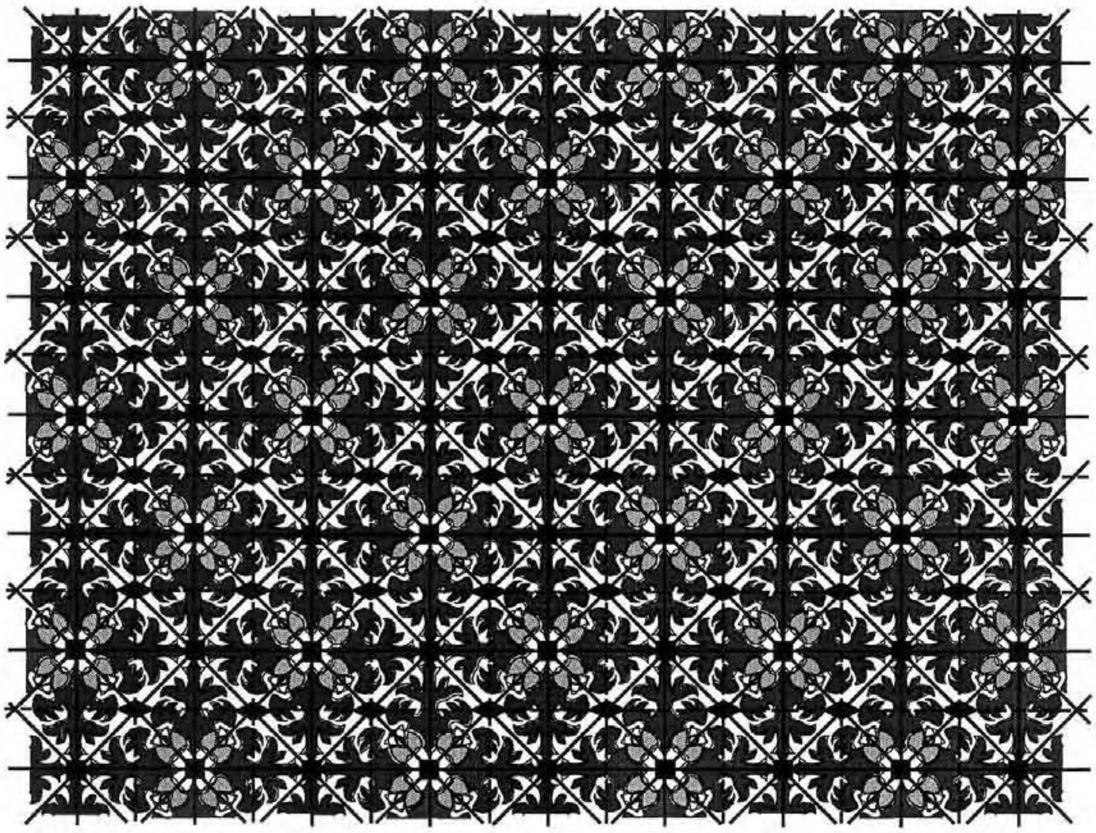


Group diagram

iii

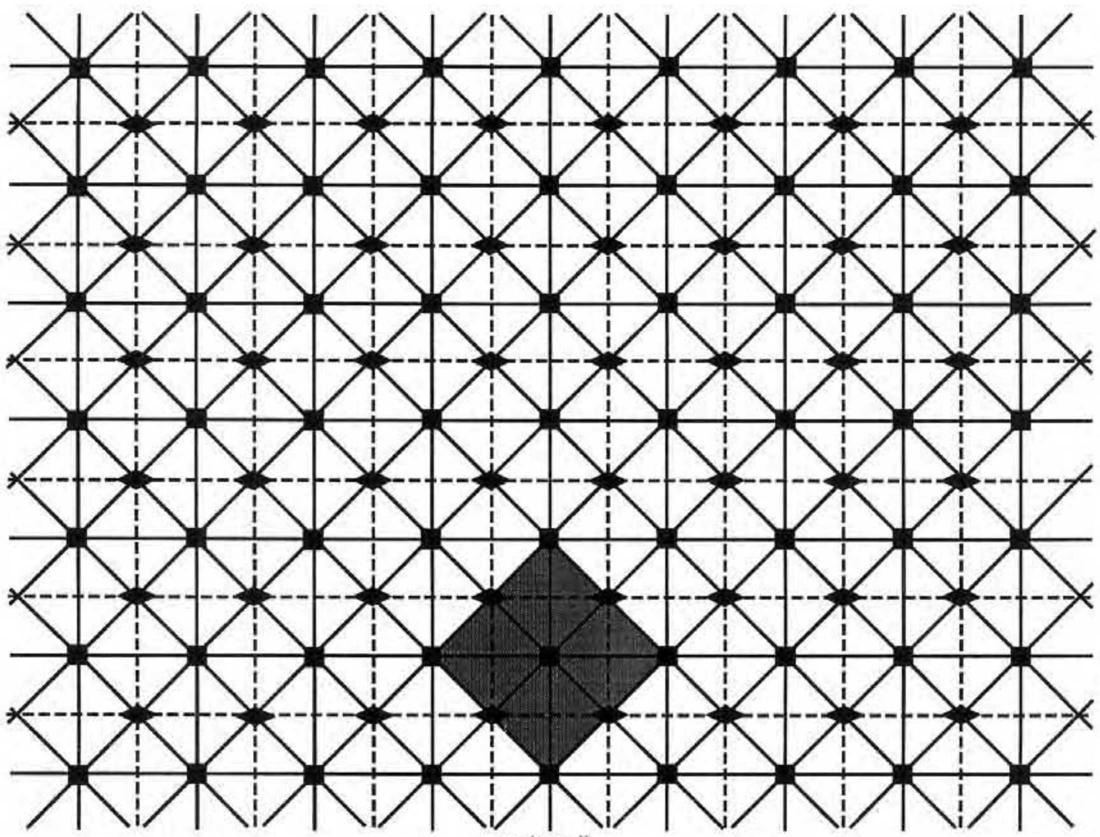


Figure 2.8 (cont.)

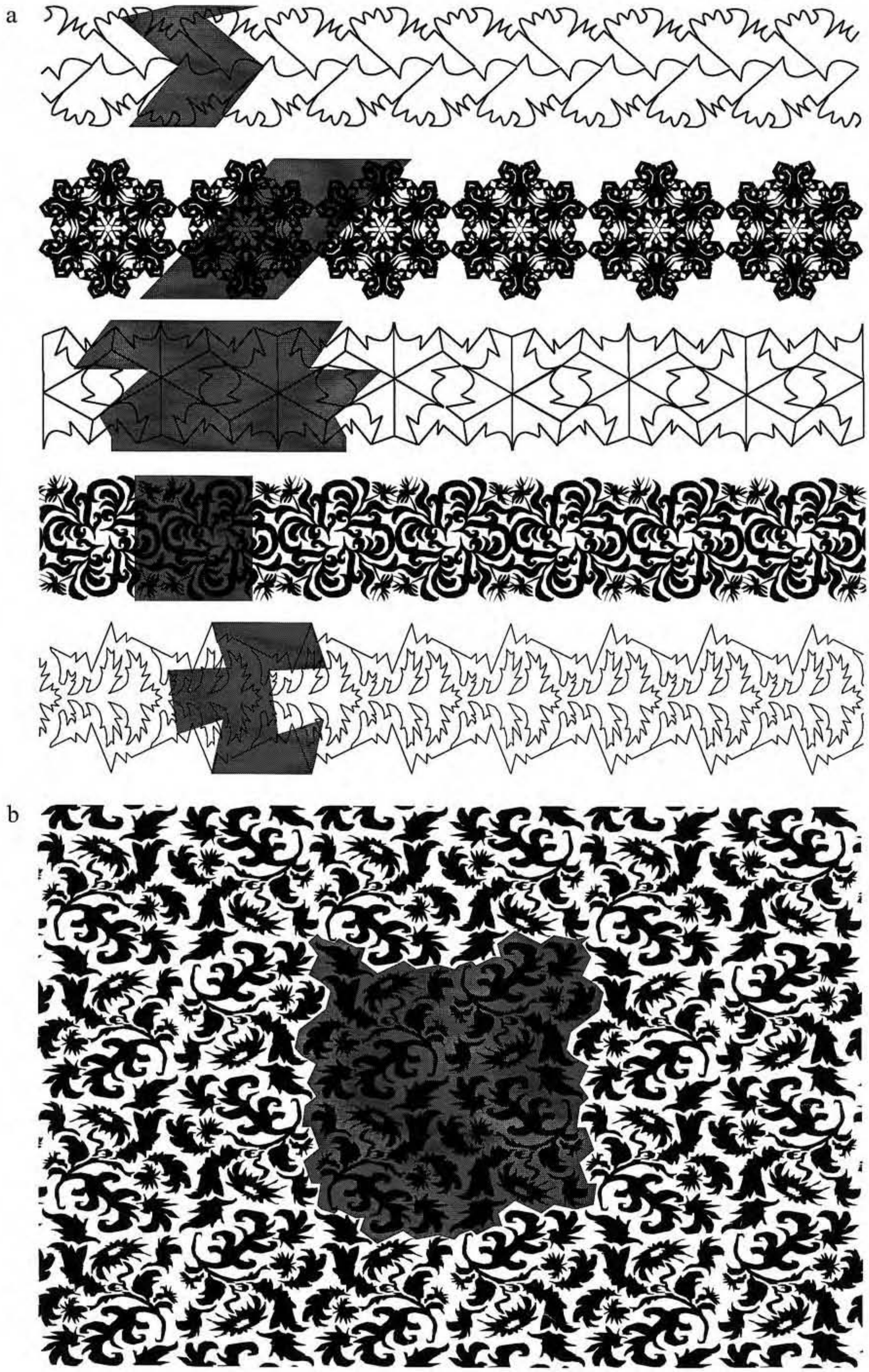


Figure 2.9 Examples of translation units of (a) monotranslational and (b) ditranslational designs.

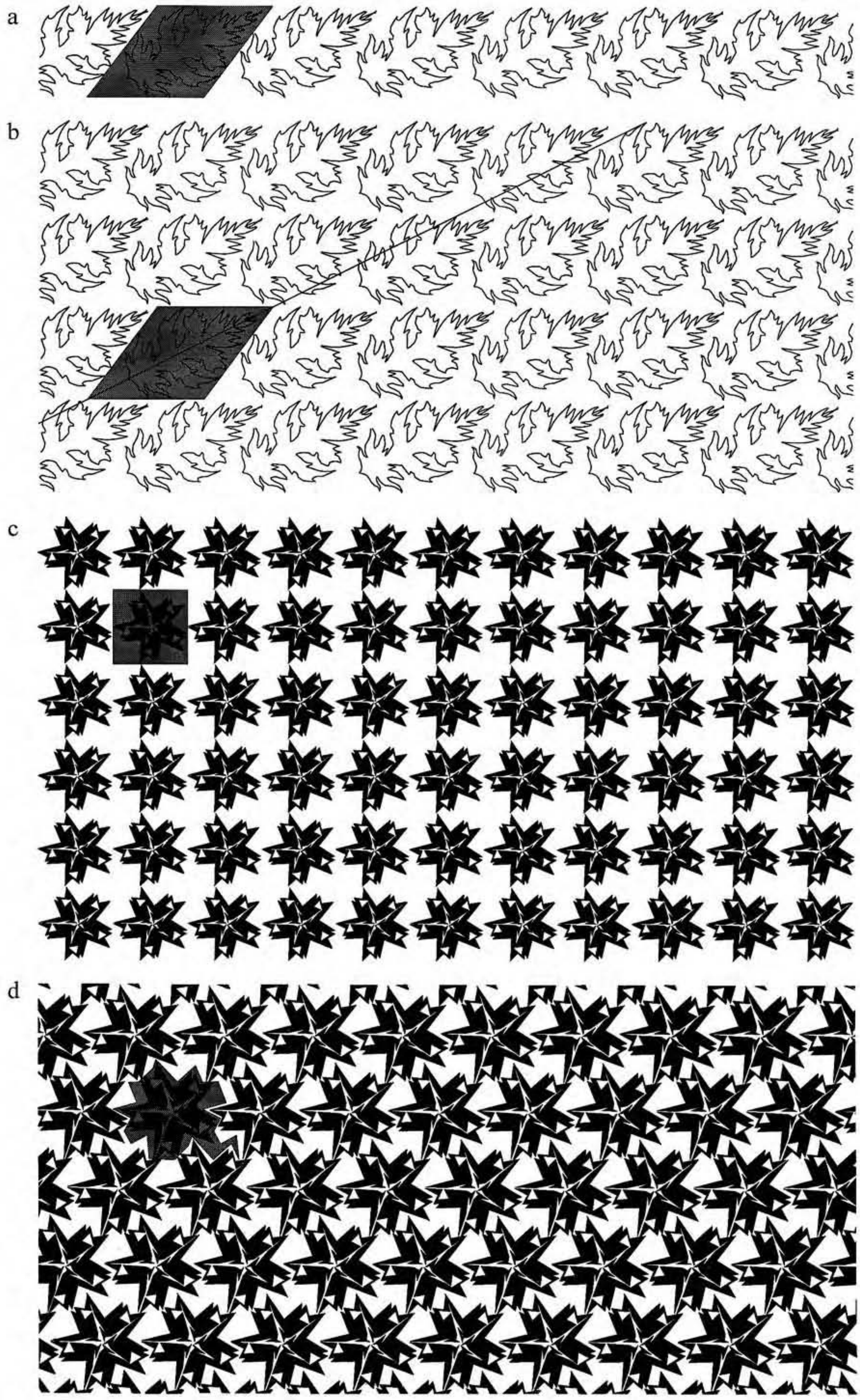


Figure 2.10 Examples of (a), (b), (c) symmetric and (d) asymmetric fundamental regions.

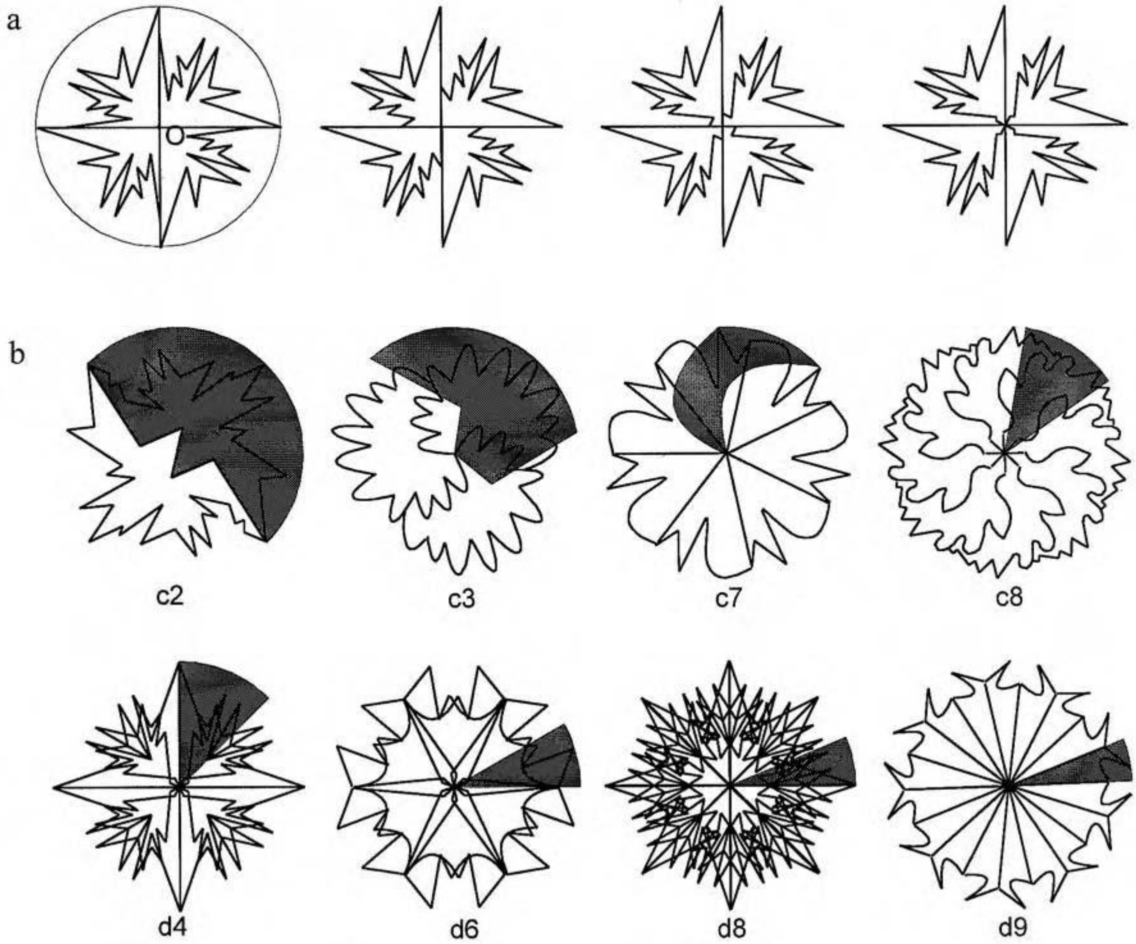


Figure 2.11 Examples of fundamental regions of finite designs.

context, 'a minimal area in which to place the motif' represents a fundamental region. For a finite tiling she describes this region as a smallest tile which, when acted on repeatedly by the generating isometries, fills out the whole tiling. She goes on to say that in designs which are obviously tilings due to the interlocking nature of the tiles, it is not necessary to consider an (artificial) circle surrounding the tiling; the edge of such a tiling provides its own well-defined encircling boundary.

However, because in some instances (as explained in the context of translation units of monotranslational designs) it is difficult to differentiate between a motif and a tile (see Fig. 2.11(a)), when referring to a finite design, whatever its form, a fundamental region will be represented in the form of a circular segment. One boundary edge will be on the circumference of the circle enclosing the design. The other two edges are straight or irregular lines, which are rotations of each other (about the centre O), and radiate outwards from the centre of the circle to its circumference. For a design in symmetry group *cn*, the area of the fundamental region will be $1/n$ of the area of the enclosing circle and for a design in symmetry group *dn* it will be $1/2n$ of the area of the enclosing circle and the two edges radiating from the centre will be straight lines. Examples of fundamental regions of finite designs are represented by the shaded areas in Fig. 2.11(b).

2.7.6.2 Monotranslational designs

Schattsneider comments that, with respect to monotranslational designs, each can be imagined as being enclosed between two parallel lines (the edges of the border). In other words, the border can be thought of as being enclosed within a strip of finite width and infinite length, and having centreline L which is equidistant from the edges.[8]

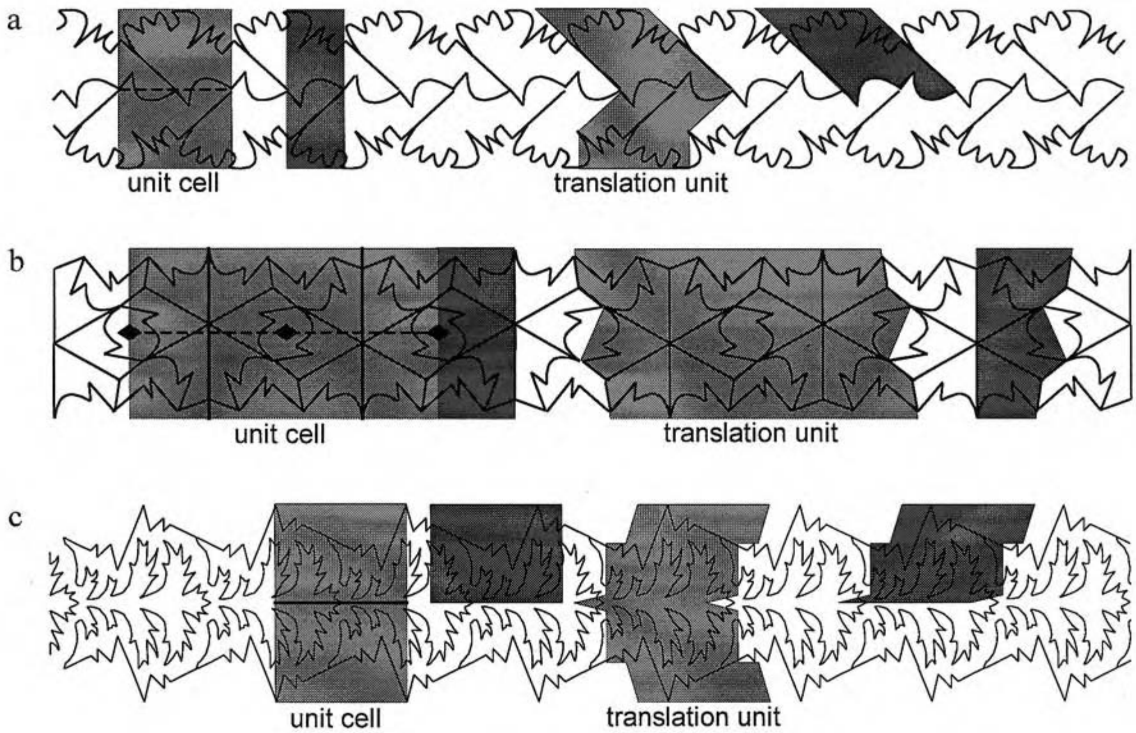


Figure 2.12 Examples of fundamental regions of monotranslational designs. Symmetry groups are (a) p1a1, (b) pma2, (c) p1m1 (see section 2.9).

For monotranslational designs, as with finite designs, it is sometimes difficult to distinguish between a pattern and a tiling. To avoid this categorisation problem, it is simpler, when determining the translation unit or fundamental region, for every monotranslational design to be considered as being enclosed in a parallel-sided strip. At least one edge of the fundamental region will coincide with part of the boundary edge(s) of the strip enclosing the design, whether it is a pattern or a tiling. Each fundamental region, for both monotranslational and ditranslational designs, is a fraction of the area of the unit cell or translation unit. Examples of fundamental regions of monotranslational designs are represented by the dark shaded areas in Fig. 2.12.

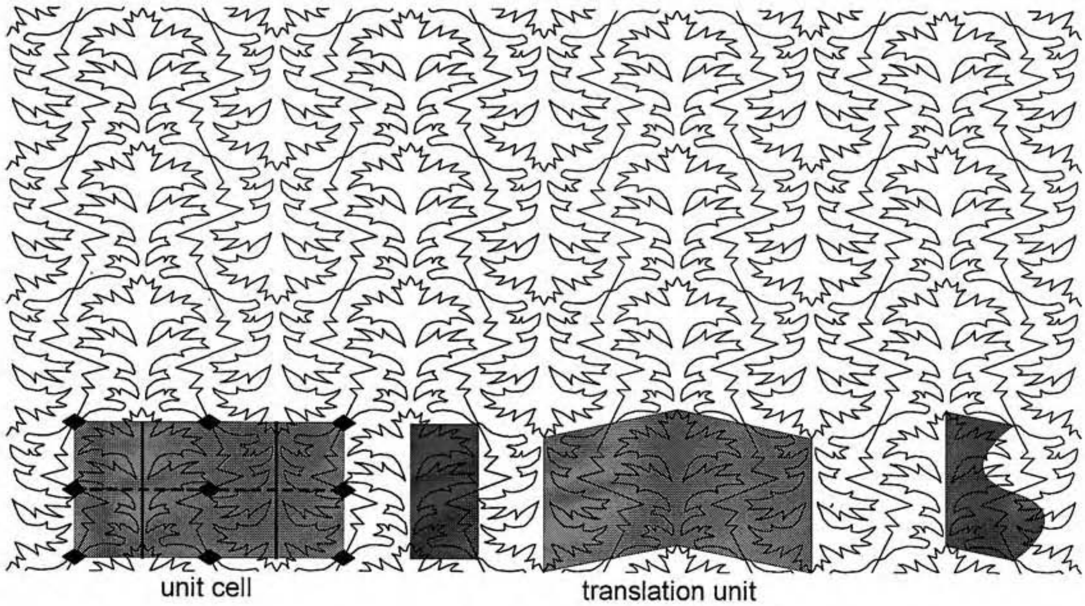
2.7.6.3 Ditranslational designs

Illustrations of fundamental regions of ditranslational designs are represented by the darker shaded areas in Fig. 2.13.

There is much ambiguity in the relevant literature with regard to the differentiation between patterns and tilings for both finite and monotranslational designs. This may be partly due to the fact that often these types of tiling design are not considered since a tiling is usually thought of as a type of pattern and/or something which covers an entire surface rather than such a limited portion of space. Similarly, ditranslational designs may be difficult to categorise strictly as a pattern or tiling. In Chapters 4 and 5, which involve finer classification systems, conditions are imposed on the characteristics of the designs in an attempt to prevent this confusion arising.

2.8 Generating functions

The symmetries which lie on the boundary of a fundamental region can be applied to that region to create the whole design. Schattsneider refers to these symmetry operations as ‘generating functions’, ‘generating symmetries’ or ‘generators’ of the design.[8] Although there could be many different symmetries a



b

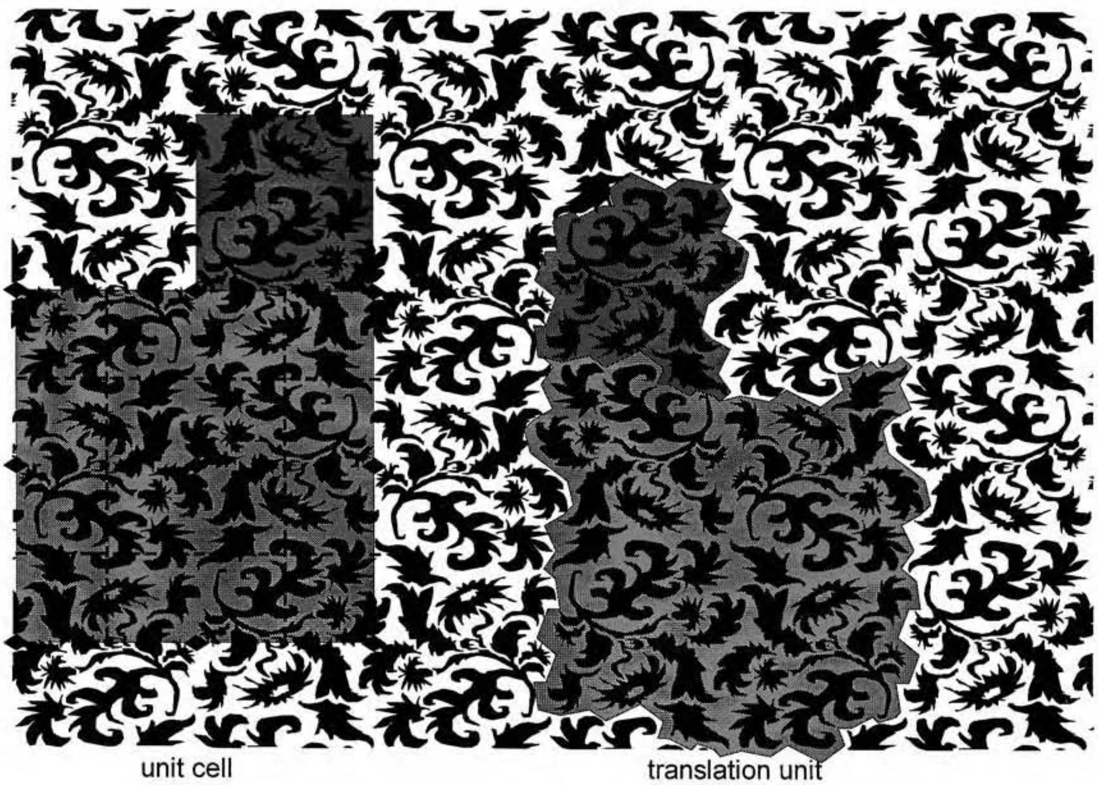


Figure 2.13 Examples of fundamental regions of ditranslational designs. Symmetry groups are (a) pmg, (b) pgg (see section 2.10).

within a design, only a selection of them may be required to generate it. The smallest set of symmetries able to do this is called the 'minimal set of generators'.[8] For example, a design in the cyclic symmetry group *cn* is generated by $n - 1$ consecutive applications, to the fundamental region, of the rotation by $360°/n$ about the centre of the design either clockwise or anticlockwise. This rotation forms the minimal set of generators (even though there is only one of them). An example illustrating the generation of a *c*3 finite design is given in Fig. 2.14(a).

On the boundary of a fundamental region of a finite design, group *dn*, there are two different reflection axes and an *n*-fold centre of rotation. However,

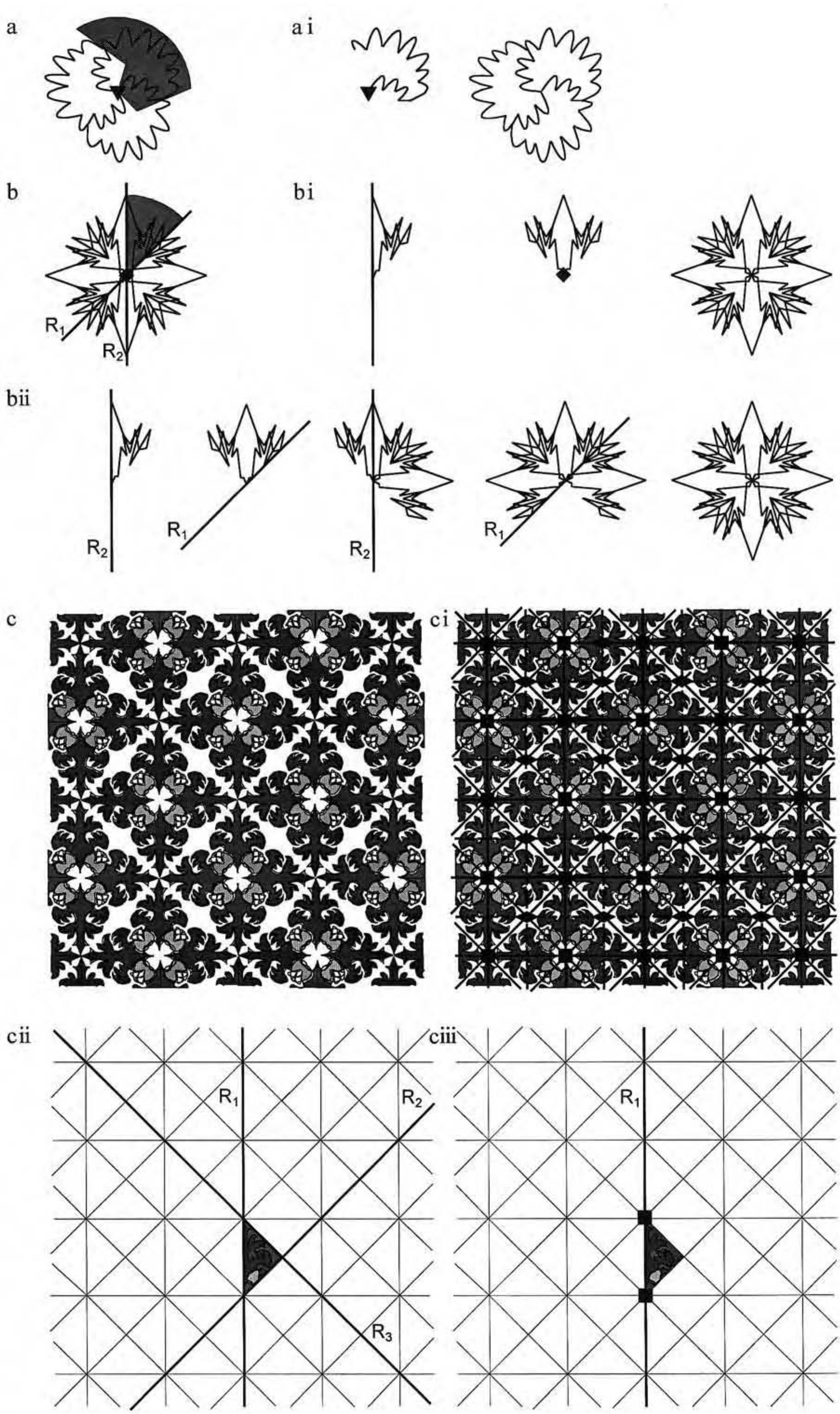


Figure 2.14 Examples of design generators. R_1 and R_2 represent two different reflection axes which may be used to generate a d4 finite design.

only two of these three symmetries are required to create the whole design: either both reflection axes or one reflection axis and an *n*-fold rotation (e.g. see the construction of a *d*4 finite design in Fig. 2.14b(i) and (ii)). On this point, Schattsneider comments how, although the *number* of isometries in a minimal set of generators for a design is unique, the *choice* of these isometries is not always unique.[8] For finite designs, a design in symmetry group *cn* requires a minimum of one generator to construct it, whereas a dihedral finite design, group *dn*, requires two.

Each fundamental region in the ditranslational design, in Fig. 2.14(c), has one centre of two-fold rotation, two centres of four-fold rotation, three different reflection axes (i.e. at three different angles) and a glide–reflection axis passing through its boundaries. (In addition, the design has translational symmetries which may be used as generators). However, only a minimal set of three of these symmetries are required to generate the whole design. For example, applying either the three reflection axes surrounding the fundamental region or the two four-fold centres of rotation and a reflection axis (as shown in Fig. 2.14c(ii) and (iii)) would complete the design, as may a number of other combinations of the symmetries in the symmetry group.

2.9 Classification of monotranslational designs

There are seven distinct symmetry groups of monotranslational designs, each of which is structured between two parallel lines of points. These points are divided into unit cells, whose shape is determined by the geometrical characteristics of the design. A *p*111 or *p*112 design may be structured on a lattice of any form of parallelogram (recall that squares, rectangles and rhombi are just special forms of parallelogram). The remaining five symmetry groups of monotranslational designs are necessarily structured on rectangular or square lattices owing to the reflectional symmetries about the transverse and longitudinal axes of the designs. (Transverse axes lie perpendicular to the longitudinal axis of the strip. The longitudinal axis coincides with the centre line L along the length of the strip enclosing the design.)

2.9.1 Notation

There is a range of different notation used by various authors to differentiate between each class of design. The more commonly used international notation takes the form of a four-term symbol, *pxyz*. However, in the context of surface-pattern design, confusion could arise because the letters *x* and *y* are symbols assigned according to symmetrical characteristics which relate to the transverse and longitudinal axes of the strip which may, conversely, be more easily associated with *y* and *x* axes, respectively. The last term, *z*, in the *pxyz* notation may be thought of (in a three-dimensional context) as being an axis perpendicular to the flat surface about which rotational symmetry occurs. However in the context of surface pattern *z* is always given a number in relation to rotational symmetry about a point. For designers, and for design classification, a more logical four term symbol, *pyxn*, seems more appropriate. The order of symbol allocation remains the same but in this case, *x* represents a symmetrical characteristic in the longitudinal *x* axis, *y* represents a symmetrical characteristic in the transverse *y* axis and *n* represents a number 1 or 2 depending upon whether or not there is two-fold rotational symmetry present. Only two-fold rotation is applicable to monotranslational designs owing to the nature of the 'stripe-like' structure of the strip, of width *W*, enclosing the design, which obviously may only be rotated by 180° for it to superimpose onto itself.

For monotranslational designs, the initial letter, '*p*', in the '*pyxn*' notation, which is common to all seven symmetry groups, stands for 'primitive' which relates to the basic unit cell. The allocation of symbols to *y*, *x* and *n* is as follows:

- $y = m$ if there is a transverse reflection axis,
  1 otherwise.
- $x = m$ if there is a longitudinal reflection axis,
  a if there is a glide–reflection axis,
  1 otherwise.
- $n = 2$ if there is two-fold rotation,
  1 otherwise.

Figure 2.15 shows schematic illustrations of the seven monotranslational symmetry groups along with their unit cells and examples of fundamental regions. Further examples are given in Fig. 2.16.

One method of determining the symmetry group of a monotranslational design is to follow a sequence of steps of analysis, which successively investigate the geometrical properties of the design. These eventually lead to the classification by symmetry group.

Washburn and Crowe, in their book *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, popularised the idea of flow diagrams to deduce the symmetry group of translational designs.[4] An alternative flow diagram, which uses a similar procedure of deduction, is given in Fig. 2.17 for the classification of monotranslational designs.

2.10 Classification of ditranslational designs

There are 17 distinct symmetry groups of ditranslational designs, each of which may be represented by a unit cell. The shape of the unit cell is determined by the

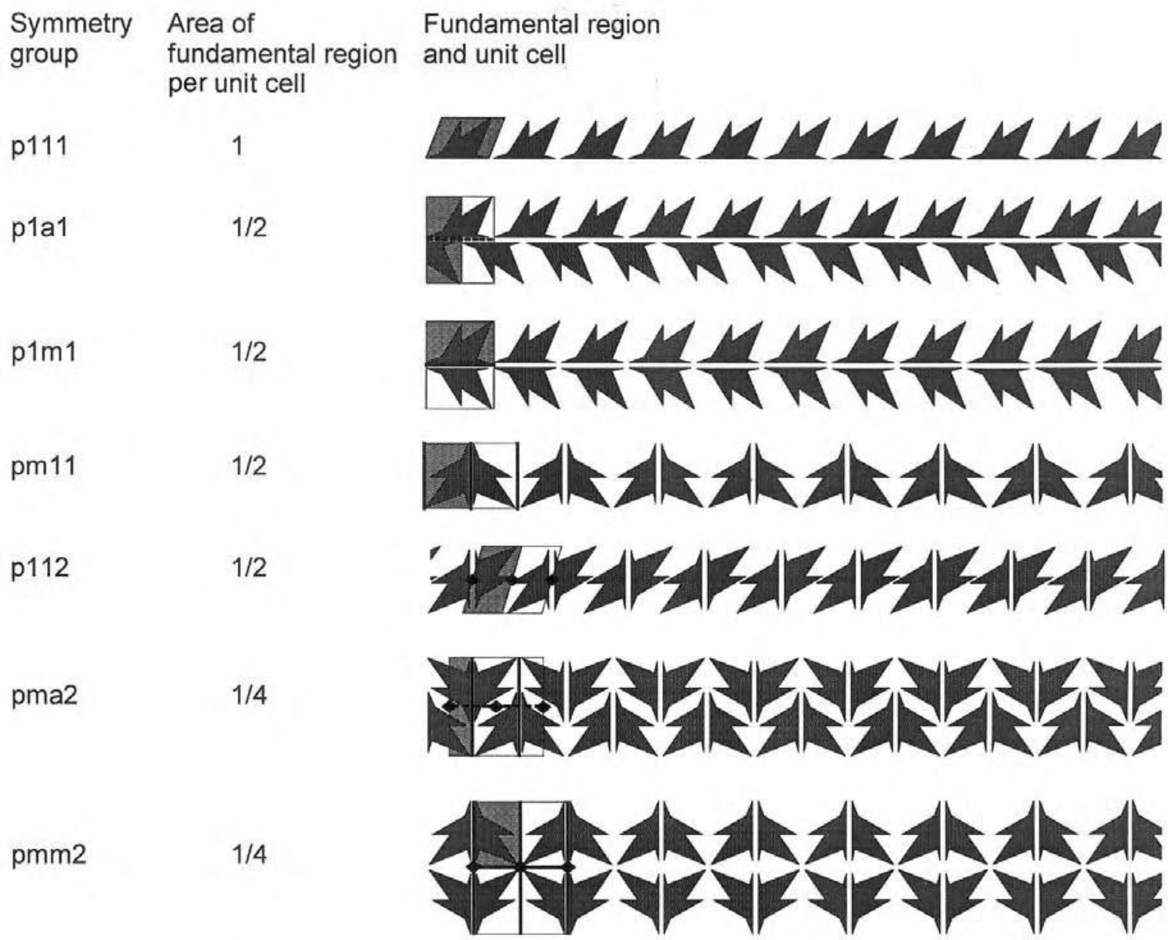


Figure 2.15 Schematic illustrations of the seven symmetry groups of monotranslational designs.

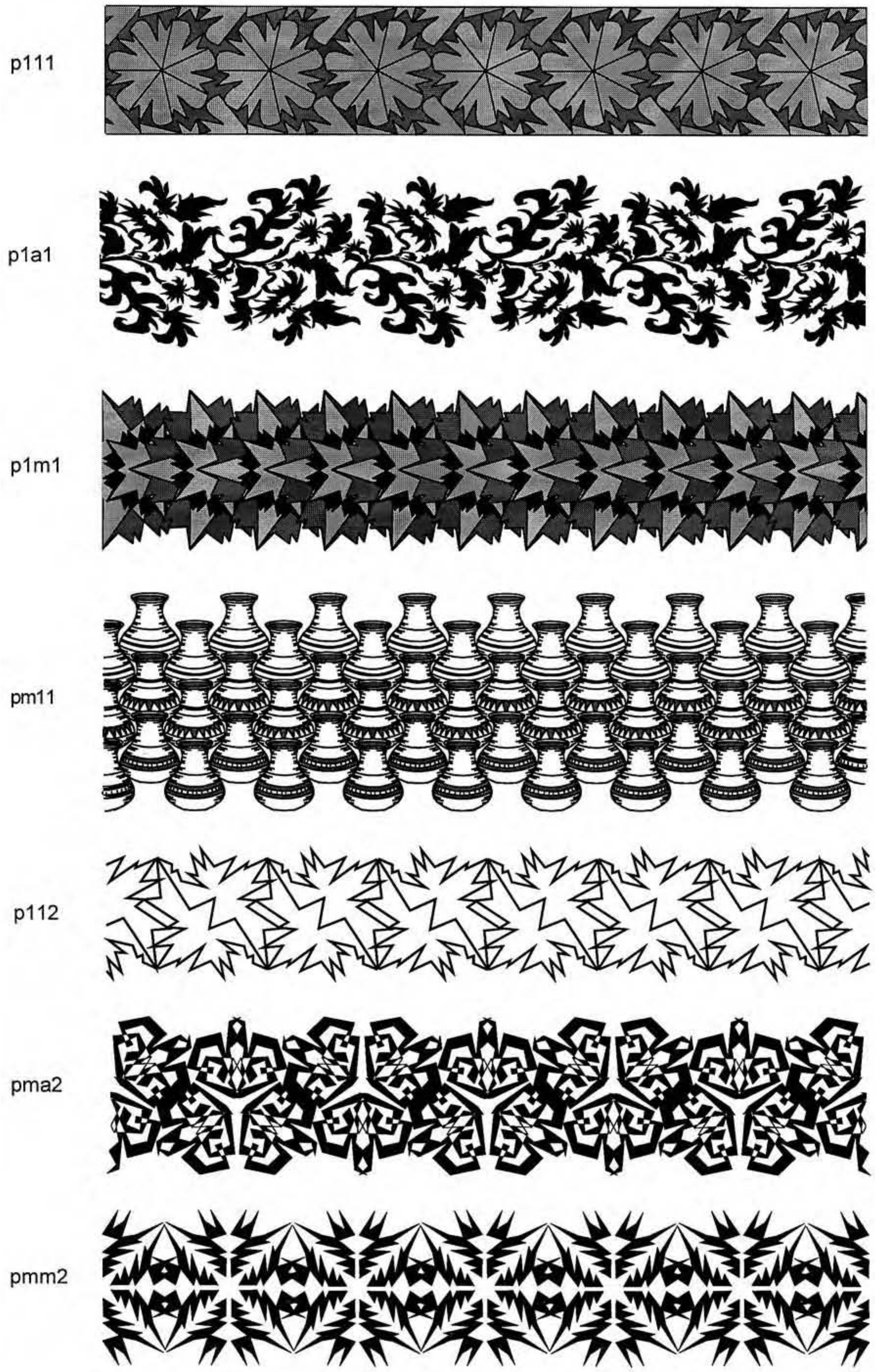


Figure 2.16 Further examples of symmetry groups of monotranslational designs.

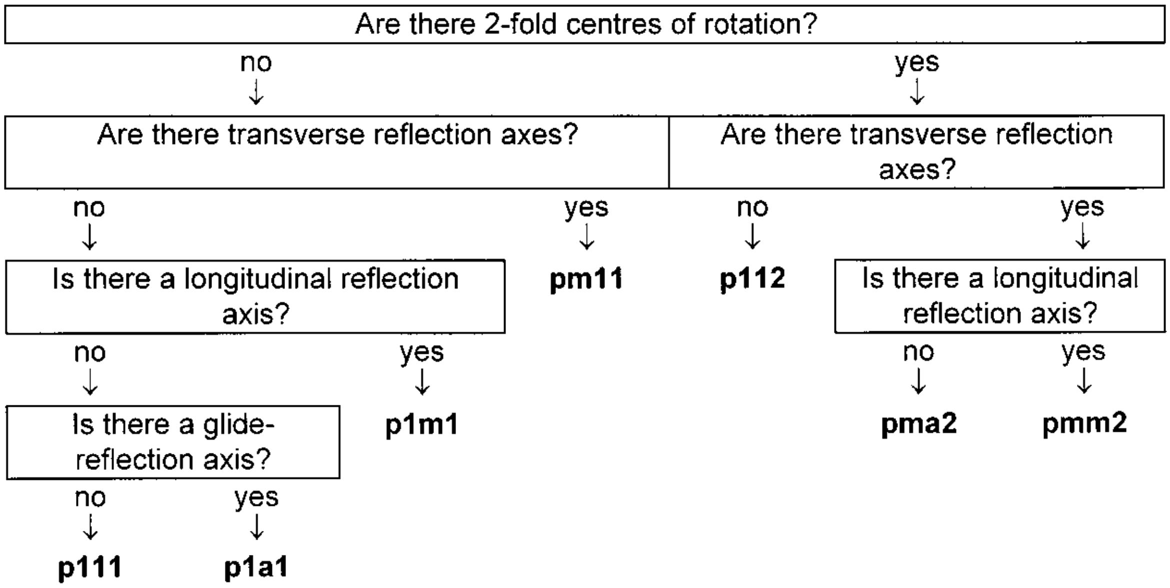


Figure 2.17 Flow diagram for symmetry group identification of monotranslational designs. Source: derived from Crowe D W and Washburn D K, *Material Anthropology: Contemporary Approaches to Material Culture*, Lanham, Maryland, University Press of America, 1987 and Rose B I and Stafford R D, 'An Elementary Course in Mathematical Symmetry', *American Mathematical Monthly*, 1981 **88** 59–64.

lattice structure of which there are five different types (as discussed in Sections 2.7.2 and 2.7.3 above).

The lattices form what are known as 'primitive' cells, containing just one net point, the vertices of which fall on rotational centres of the highest order of the design structure. However, for two particular symmetry groups, both of which are based on the rhombic lattice, a 'non-primitive' double-cell is often chosen which is twice the size and has sides parallel to the diagonals of the primitive unit cell. The double cell is referred to as the centred cell and it contains two net points, one at the centre and one divided up at the corners. These double-cells have sides parallel to reflection axes in their design structures unlike their associated primitive cells.

2.10.1 Notation

As with monotranslational designs, the universal notation will be used when classifying the seventeen symmetry groups of ditranslational design. Similarly, this takes the form of a four-term symbol which is usually denoted by *pxyz* or *cxyz* where *x*, *y* and *z* are each allocated a symbol according to the design's symmetrical properties. However, since in the use of this notation and in the context of surface-pattern design, confusion could arise because the letters *y* and *z* are symbols assigned according to the symmetrical characteristics which relate to the *x* and *y* axes and the first term '*x*' in the *pxyz* is always given a number, a new, less confusing four-term symbol is proposed, namely '*pnxy*' or '*cnxy*'. (Note that the '*nxy*' of ditranslational design notation is the reverse of the '*yxn*' of monotranslational design notation.) The order of symbol allocation remains the same but in this case '*n*' represents a number 2, 3, 4 or 6; '*x*' represents a symmetrical characteristic in relation to the *x* axis and '*y*' a symmetrical characteristic in relation to the *y* axis. The positioning of these axes for each unit cell type is given in Fig. 2.18.

The following system is used for the allocation of numbers or letters to *n*, *x* and *y* in the *pnxy*/*cnxy* notation. For 15 of 17 of the groups, the initial symbol is '*p*' which represents a primitive cell, as opposed to the remaining two cases where '*c*' represents a centred cell. The symbol '*n*' is assigned an integer *n*, where *n* is the highest order of rotation in the design (only two-, three-, four- and six-fold

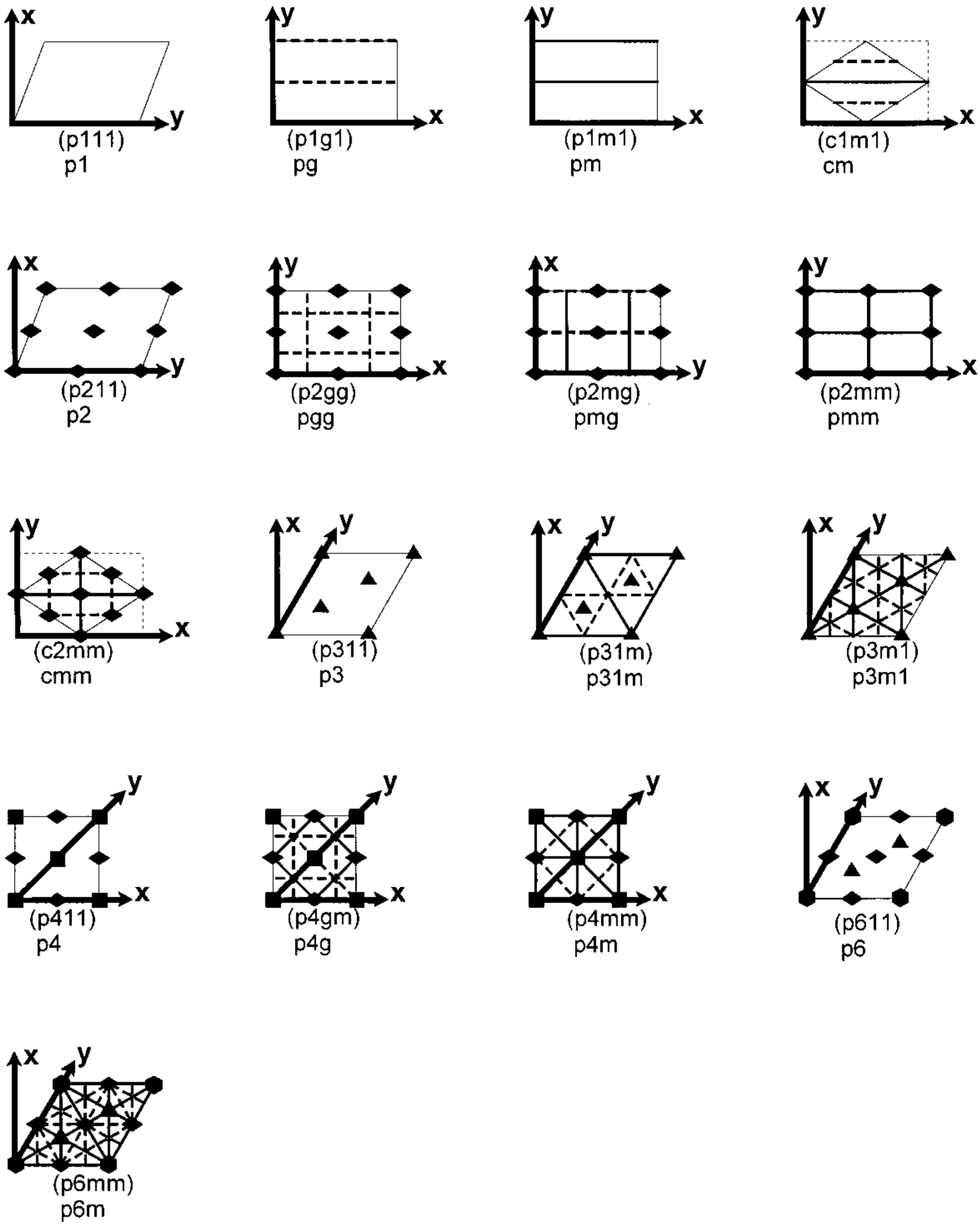


Figure 2.18 Illustrations showing the positioning of *x* and *y* axes in relation to the unit cells of ditranslational designs.

rotation are applicable to ditranslational designs.) The letter '*x*' is assigned a symbol which indicates a symmetry axis perpendicular to one side of the unit cell (or double-cell for the two particular symmetry groups); this will be called the *x* axis. Where there is reflectional symmetry, or glide–reflectional symmetry, this axis lies parallel to a line of reflection/glide–reflection. Where there are both, a reflection axis takes priority over a glide–reflection axis in assigning the correct symbols. The letter '*y*' is assigned a symbol which indicates a symmetry axis (i.e. the *y* axis) at 90°, 45° or 30° to the *x* axis depending upon whether there is two-, four-, three- or six-fold rotation present, respectively. The following system is used for the allocation of symbols to the letters *n*, *x* and *y*:

- $n = 1$ if there is no rotational symmetry,
  2 if two-fold rotational symmetry is the highest order of rotational symmetry,
  3 if three-fold rotational symmetry is the highest order of rotational symmetry,
  4 if four-fold rotational symmetry is the highest order of rotational symmetry,
  6 if six-fold rotational symmetry is the highest order of rotational symmetry.
- $x = m$ if there is a reflection axis perpendicular to one side of the unit cell, i.e. if the x axis is parallel to a reflection axis in the unit cell (see Fig. 2.18),
  g if there is a glide–reflection axis perpendicular to one side of the unit cell, i.e. if the x axis is parallel to a glide–reflection axis in the unit cell,
  1 otherwise.
- $y = m$ if there is a reflection axis at:
  – 90° to the x axis if $n = 2$,
  – 45° to the x axis if $n = 4$,
  – 30° to the x axis if $n = 3$ or 6, i.e. if the y axis is parallel to a reflection axis in the unit cell (see Fig. 2.18);
  g if there is a glide–reflection axis at:
  – 90° to the x axis if $n = 2$,
  – 45° to the x axis if $n = 4$,
  – 30° to the x axis if $n = 3$ or 6, i.e. if the y axis is parallel to a glide–reflection axis in the unit cell;
  1 otherwise.

Several of the symmetry groups are frequently represented by an abbreviated form of this notation which is indicated underneath the international notation in Fig. 2.18. This shorter form is used in subsequent references to symmetry group classification.

Figure 2.19 shows schematic illustrations of the 17 symmetry groups with their unit cells and examples of fundamental regions. Further illustrations of ditranslational designs are given in Fig. 2.20.

By a procedure similar to that described for monotranslational designs, a step by step analysis of the geometrical properties of a ditranslational design enables it to be classed as one of the 17 symmetry groups. The flow diagram in Fig. 2.21 has been derived from the one given by Washburn and Crowe.[4]

2.11 Construction of finite designs

An irregular design, classed in the finite symmetry group $c1$, possesses no symmetrical properties other than the identity symmetry and so its construction only has to conform to its overall asymmetric characteristic. A regularly repeating design may be generated by the application of a minimal set of generators to the fundamental or generating region. Alternatively they may be produced by applying the generating symmetries about a point or line through a motif such that design elements overlap each other. In this instance it must be ensured that the overlapped design elements are not obscured but form part of the design.

Symmetric finite designs may be constructed in a variety of different ways. The most suitable is dependent on the exact nature of the design type required. The first method, discussed in this section (for both symmetry groups cn and dn), initially involves constructing a circle with radius R, where R is chosen such that the resulting circle just encloses the extremities of the design. Any design unit added to a fundamental region must extend to at least one point on the circumference of this circle, otherwise the circle segment does not satisfy the definition given for a fundamental region.

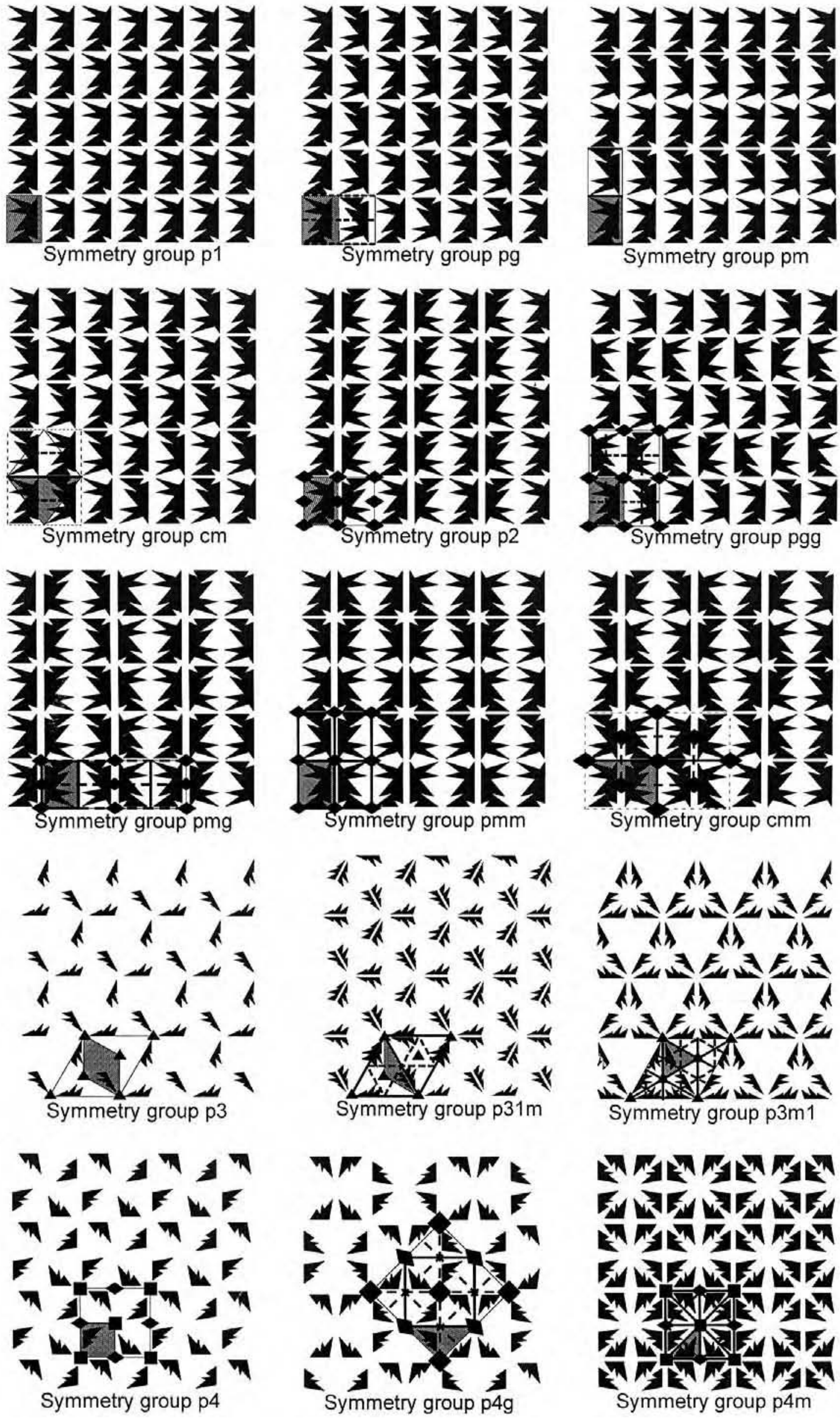
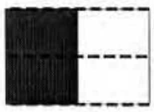
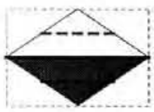
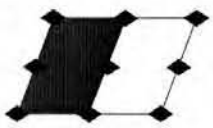
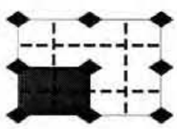
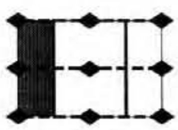
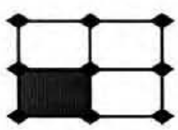
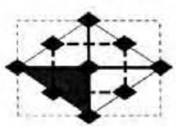


Figure 2.19 Schematic illustrations of the 17 symmetry groups of ditranslational designs.

Symmetry group p6

Symmetry group p6m

Symmetry group	Area of fundamental region per unit cell	fundamental region and unit cell
p1	1	
pg	1/2	
pm	1/2	
cm	1/2	
p2	1/2	
pgg	1/4	
pmg	1/4	
pmm	1/4	
cmm	1/4	

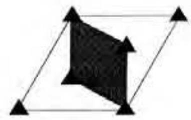
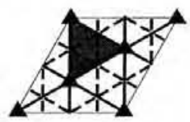
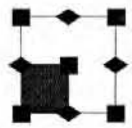
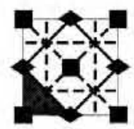
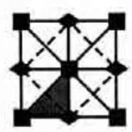
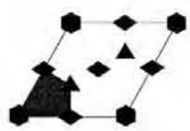
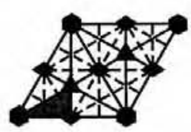
Symmetry group	Area of fundamental region per unit cell	Fundamental region and unit cell
p3	1/3	
p31m	1/6	
p3m1	1/6	
p4	1/4	
p4g	1/8	
p4m	1/8	
p6	1/6	
p6m	1/12	

Figure 2.19 (cont.)

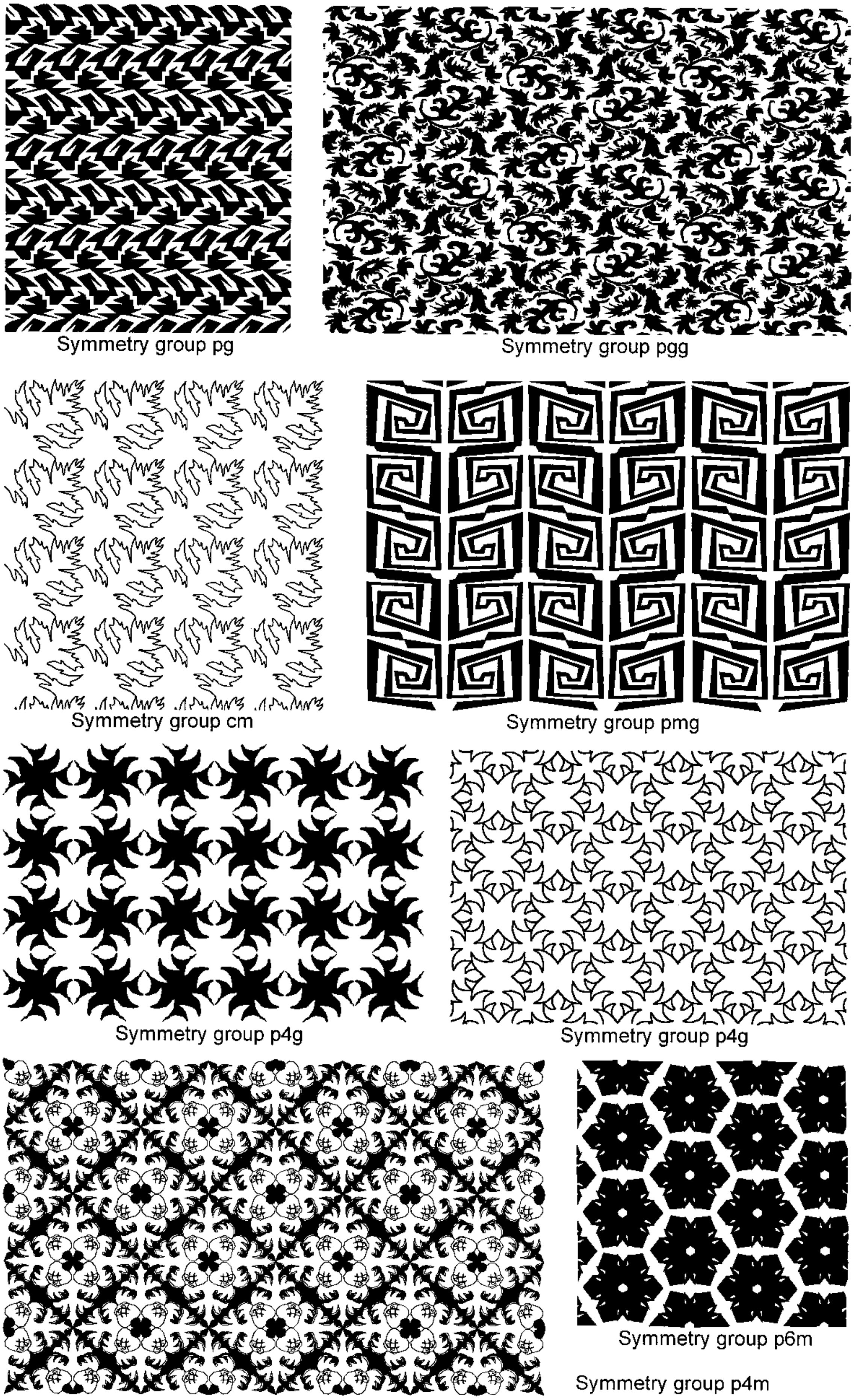


Figure 2.20 Further examples of symmetry groups of ditranslational designs.

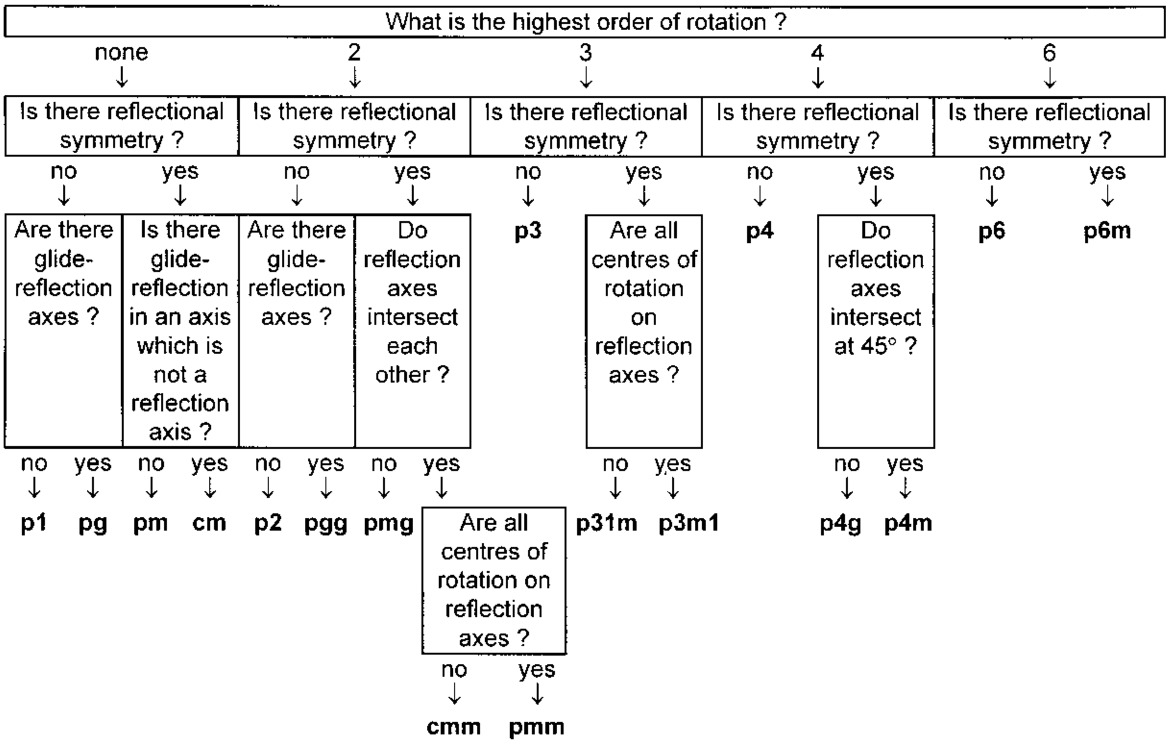


Figure 2.21 Flow diagram for symmetry group identification of ditranslational designs. Source: derived from Schattsneider D, 'The Plane Symmetry Groups: Their Recognition and Notation', *American Mathematical Monthly*, 1978 **85** 439–450.

2.11.1 Symmetry group cn

To construct a finite design, of symmetry group *cn*, the circle is divided into *n* fundamental regions as described in Section 2.7.6.1 above. A design unit (which has no reflection axis passing through the centre of the circle) is added to one fundamental region and then mapped onto the remaining fundamental regions, to complete the design, by applying rotational symmetry (as described in Section 2.8).

Alternatively a *cn* design may be constructed by $n - 1$ applications of the *n*-fold rotational symmetry about a point passing through or close to a motif. This may result in overlapping design elements and a more intricate design. (Note that, as stated above, each consecutive design unit must not conceal any parts of the previous one(s) otherwise the final result will be asymmetric.) Illustrations of these two methods of *cn* design construction are given in Fig. 2.22(a(i)) and (a(ii)).

2.11.2 Symmetry group dn

To construct a finite design, symmetry group *dn*, the circle is divided into 2*n* fundamental regions as described in Section 2.7.6.1 above. A design unit (which has no reflection axis passing through the centre of the circle) is added to one fundamental region and then mapped onto the remaining fundamental regions, to complete the design, by applying the generating symmetries (as described in Section 2.8). Examples are given in Fig. 2.22(b(i)) for $n = 3$ and $n = 2$.

Alternatively a *dn* design may be derived from a *cn* or *dn*/2 (where *n* is even) design by superimposition. Applying a reflectional symmetry about an axis passing through the centre of rotation of a *cn* design will produce a *dn* design as shown in Fig. 2.22(b(ii)) for $n = 4$. Applying a rotation of 360°/*n* to a copy of *dn*/2 and then superimposing the two *dn*/2 designs such that their centres of rotation coincide will produce a *dn* design. For example, in Fig. 2.22(b(iii)) a *d*4 design has been constructed from a *d*2 design and its rotation by 360°/4 = 90°. Similarly, in

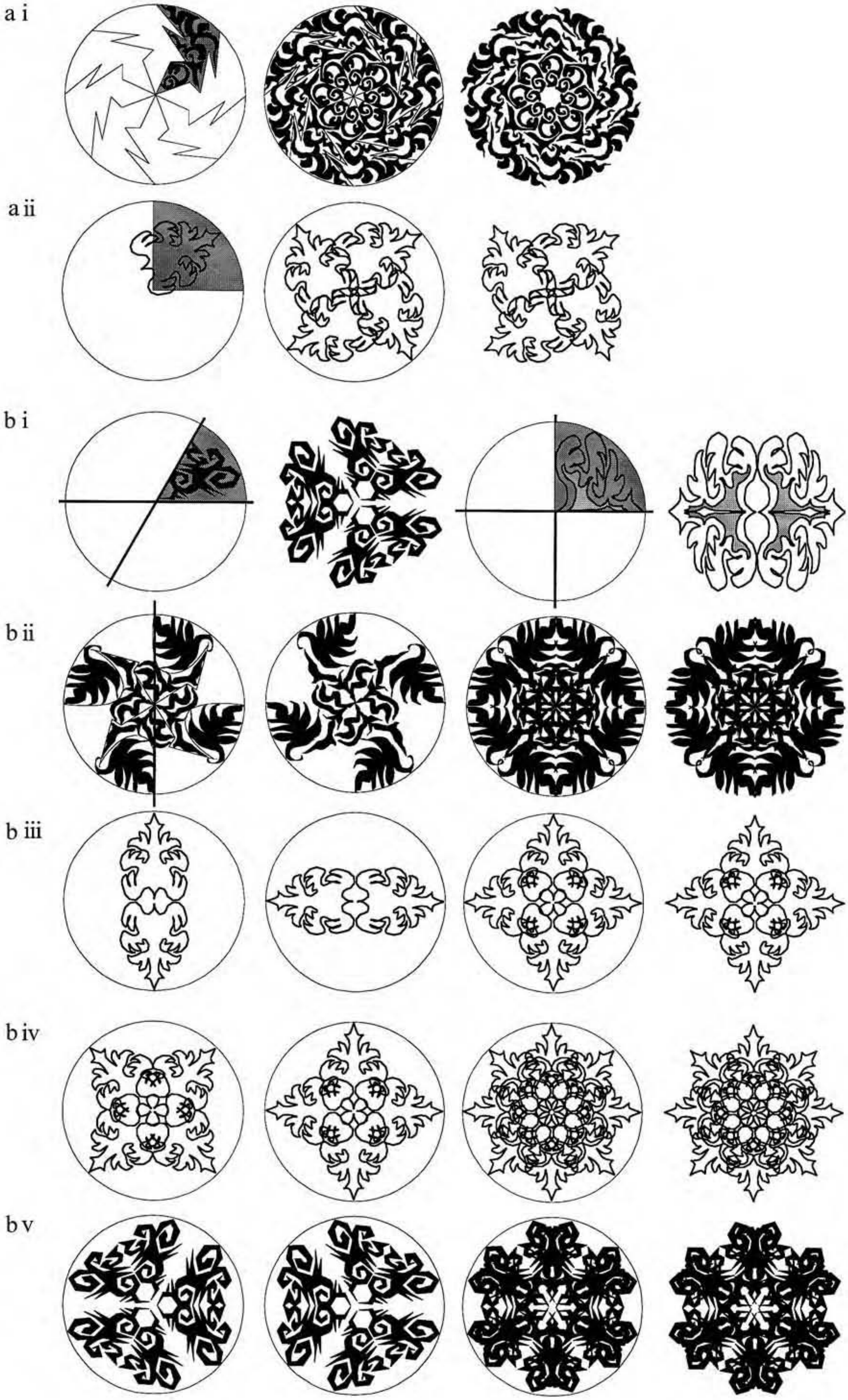


Figure 2.22 Construction of finite design symmetry groups (a) *cn* and (b) *dn*.

Fig. 2.22(b(iv)) and (b(v)), *d*8 and *d*6 designs have been constructed from *d*4 and *d*3 designs, respectively. (Again, by this method, superimposing one design onto another must not conceal any parts of the one underneath.)

2.12 Construction of monotranslational designs

The construction of a monotranslational design begins with a strip, width *W*, which is based on the lattice of two parallel lines of points as described in Section 2.9 above. Each fundamental region will have at least one boundary edge coinciding with a portion of one or both of the two parallel lines outlining this strip. The initial design unit added to a fundamental region must touch at least one point on one or both of these boundaries where possible, otherwise this area does not satisfy the definition given for a fundamental region.

In this section, construction techniques are illustrated for six different design types. These are denoted by type (i) to type (vi) and each is built upon the structure of the previous type. Type (i) forms the basis of the most simple form of construction for each symmetry group. The design types fall into the categories whose characteristics have been summarised below.

1 Design type (i): a strip is divided into parallelogram-shaped fundamental regions. Design elements are added to one and then mapped onto all equivalent positions in the strip by applying the generating symmetries of the symmetry group. In each case the boundaries of the fundamental region are included as part of the design unit.
2 Design type (ii): this is derived from type (i) by removing the boundaries of the parallelogram-shaped fundamental regions chosen for type (i).
3 Design type (iii): the initial division of a strip into parallelogram-shaped fundamental regions, as described for design type (i), is altered by exchanging a straight edge of a fundamental region for an asymmetric one. This edge is then mapped to all equivalent positions in the strip by applying the generating symmetries. The sides of the fundamental regions coinciding with the parallel edges of the strip and those coinciding with reflection axes cannot be altered and will be referred to as 'fixed' edges. This gives a more interlocking type of tiling design. For symmetry groups *pm*11 and *pmm*2, where the boundaries of the fundamental regions lie either on reflection axes and/or on the outside edges of the strip, this alteration is not possible and therefore design type (iii), and consequently types (iv) to (vi), are not constructable. Conversely, there may be one, two or three ways of producing interlocking tiles from design type (i) depending on the number of different 'sets' of fundamental region edges. These are discussed in detail for each symmetry group.
4 Design type (iv): this is derived from type (iii) by adding design elements to one fundamental region and then mapping them onto the remaining ones by applying the generating symmetries of the symmetry group. This produces a patterned interlocking tiling design.
5 Design type (v): this is derived by removing the boundaries of the fundamental regions chosen for type (iv). If the design elements are initially chosen to extend towards the boundaries of the fundamental regions (for type (iv)), each motif appears to interlock with its neighbouring ones, to a lesser or greater degree, depending on the nature of the initial tiling design. This construction often forms the most visually pleasing type of the six varieties discussed in this section owing to the resulting appearance of continuity in the design structure.
6 Design type (vi): this is formed, where possible, by first dividing the strip into symmetrical shaped fundamental regions (not coinciding with those of type (i)). Design elements are added to one tile and then mapped onto all the equivalent positions in the strip by applying the generating symmetries. The design elements inside the initial fundamental region, if symmetric, must be suitably positioned so as not to add any extra reflective or rotational symmetry to the structure of the design.

It should be noted that the initial fundamental region (including its design unit) must not have any symmetries coinciding with those of the structure of the strip, otherwise the symmetry group will be altered or the size of the fundamental region reduced. The symmetries of the strip are two-fold centres of rotation and transverse reflection axes at any point along its longitudinal axis, and longitudinal reflectional symmetry with the reflection axis coincides with the centre line L. However, this still allows the boundaries of each fundamental region to be parallelogram shaped and be included as part of a design unit provided that, together with the design elements inside them, they do not have any symmetries coinciding with the strip (e.g. design types (i) and (vi)). Conversely, if the boundaries of the fundamental regions are asymmetric and chosen to be part of the design unit, the design elements inside them may have symmetries in common with the strip because overall each fundamental region is asymmetric (e.g. specific forms of type (iv)). This circumstance, although not discussed in further detail in this chapter, may be observed in the $p111$ design shown in Fig. 2.24(iv(b)), where the design elements inside the fundamental region have two-fold rotational symmetry.

The design descriptions for types (i) and (ii), for each symmetry group, are clearly shown in the following illustrations without further explanation. Similarly, types (iv) and (v) are simply derived from type (iii). For design type (v) the design unit will be taken to be asymmetric to avoid further complication. Design types (iii) and (vi) require additional definition, for each symmetry group, which is given below.

Symmetrically shaped design units are discussed in detail in the classification and construction methods in Chapter 3. For simplicity, in the majority of construction methods discussed in this chapter, the design unit will be taken to be asymmetric. In the following examples T_1, when referred to, represents a translation parallel to the longitudinal axis of the strip and distance equal to the length of a side of a unit cell coinciding with the strip edges. G represents a glide–reflection in the same direction, about the longitudinal centre line L, but of length $1/2T_1$. In the illustrations throughout this section, the dark shaded area represents a fundamental region and the figure section number represents the design type, for example Fig. 2.25(vi) represents a design type (vi).

It is also assumed that no symmetries are induced into the structure by, for example, the translation of what initially appears to be an asymmetric translation unit (as shown in Fig. 2.23). Here, the fundamental region is chosen to contain an

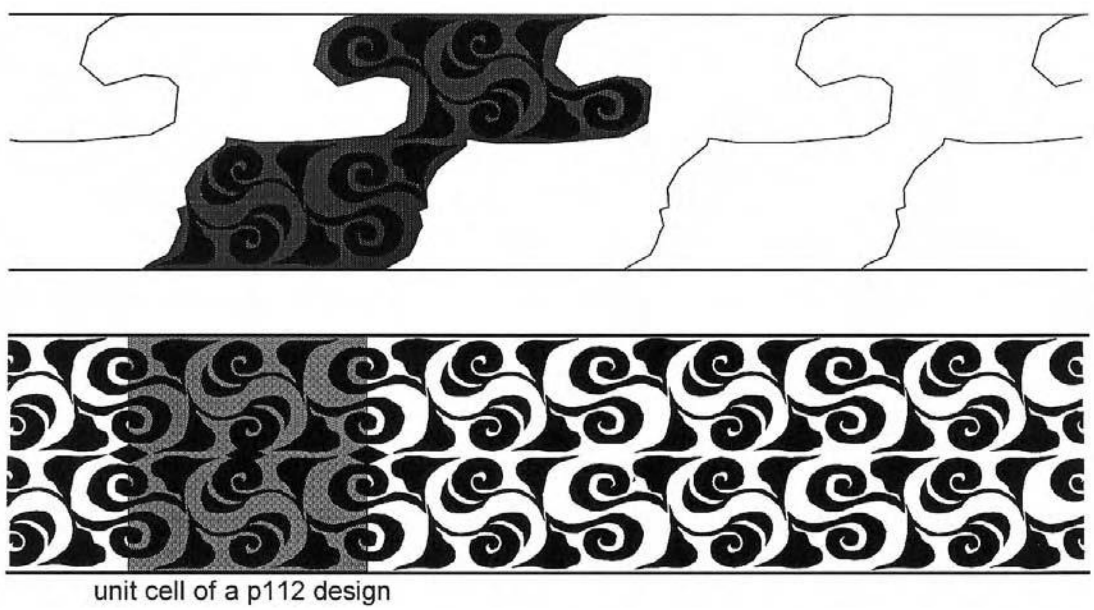


Figure 2.23 Example of an asymmetric fundamental region unsuitable for the construction of a $p111$ monotranslational design.

asymmetric design unit but on its translation, for the construction of a $p111$ design, a $p112$ design is formed. However, to construct a different symmetry group to the one planned by this method is a fairly unlikely occurrence.

*2.12.1 Symmetry group p*111

To construct design type (iii), for symmetry group $p111$, one of a fundamental region's edges, not coinciding with the boundaries of the strip, is replaced with an asymmetric one which is then used to replace all the 'unfixed' edges by applying consecutive translations of T_1. To construct design type (vi), the parallelogram-shaped fundamental regions of type (i) may be replaced by fundamental regions having either two-fold rotational or longitudinal reflectional symmetry. A design unit is then added to a fundamental region and mapped onto the remaining ones by applying T_1. Figure 2.24 shows some examples of design types (i) to (vi) for symmetry group $p111$.

*2.12.2 Symmetry group p*1*a*1

To construct design type (iii), one of the two 'unfixed' edges of a fundamental region is replaced by an asymmetric one which is then used to replace all the equivalent edges by applying glide–reflection *G*. To construct design type (vi) the parallelogram-shaped fundamental regions of type (i) may be replaced by a strip of fundamental regions that has longitudinal reflectional symmetry only. Alternatively the fundamental regions may form two strips inside the monotranslational design, one of which is a glide–reflection of the other. In this case the shape of each fundamental region may be two-fold rotationally, transversely and/or longitudinally reflectively symmetric. A design unit is then added to a fundamental region and mapped onto the remaining ones by applying *G*. Figure 2.25 shows some examples of design types (i) to (vi) for symmetry group $p1a1$.

*2.12.3 Symmetry group p*1*m*1

To construct design type (iii) one of the two 'unfixed' edges of a fundamental region is replaced by an asymmetric one which is then used to replace all the equivalent edges by applying a reflection about the longitudinal axis and translations of T_1. To construct design type (vi) the parallelogram-shaped fundamental regions of type (i) may be replaced by fundamental regions that have either two-fold rotational or longitudinal reflectional symmetry. A design unit is then added to a fundamental region and mapped onto the remaining ones by applying the generating symmetries. Figure 2.26 shows some examples of design types (i) to (vi) for symmetry group $p1m1$.

*2.12.4 Symmetry group pm*11

For symmetry group $pm11$, all four sides of the fundamental region are fixed since they fall on reflection axes or the edges of the strip enclosing the design. Therefore none of the design types (iii) to (vi) are constructable. Figure 2.27 shows some examples of design types (i) and (ii) for symmetry group $pm11$.

*2.12.5 Symmetry group p*112

There are two ways of constructing a type (iii) design, from type (i), for symmetry group $p112$. Because there are two different centres of two-fold rotation in a unit cell, R_1 and R_2, the asymmetric replacement lines which meet at these points may be different too. One case of design type (iii) occurs when one straight edge of a fundamental region, passing through R_1 say, remains fixed and the one passing through R_2 is altered (see the first two examples in Fig. 2.28(iii)). The replacement edge need not necessarily have the same end points but it must retain the two-fold rotational symmetry passing through its centre.

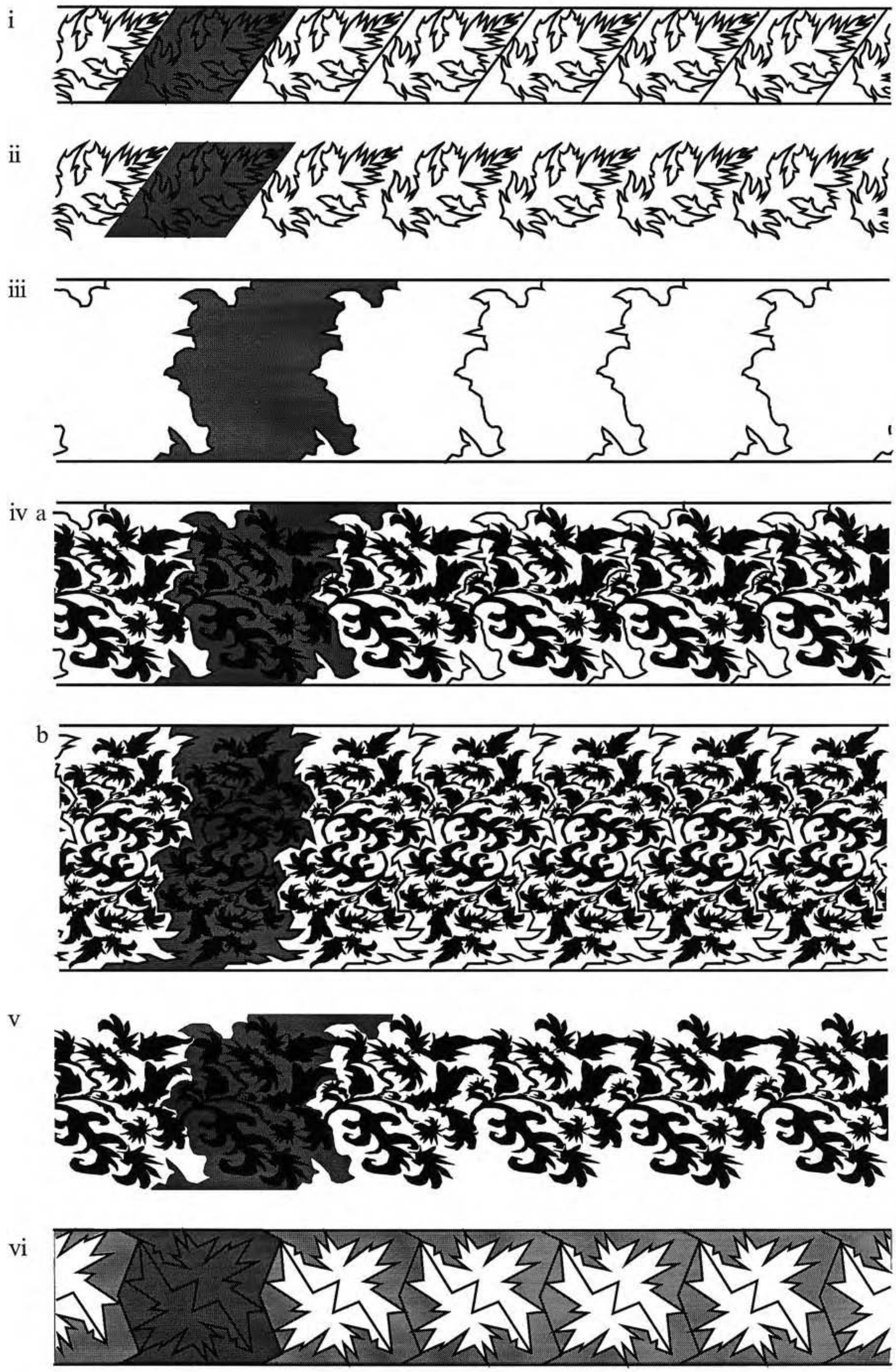


Figure 2.24 Construction of symmetry group $p111$.

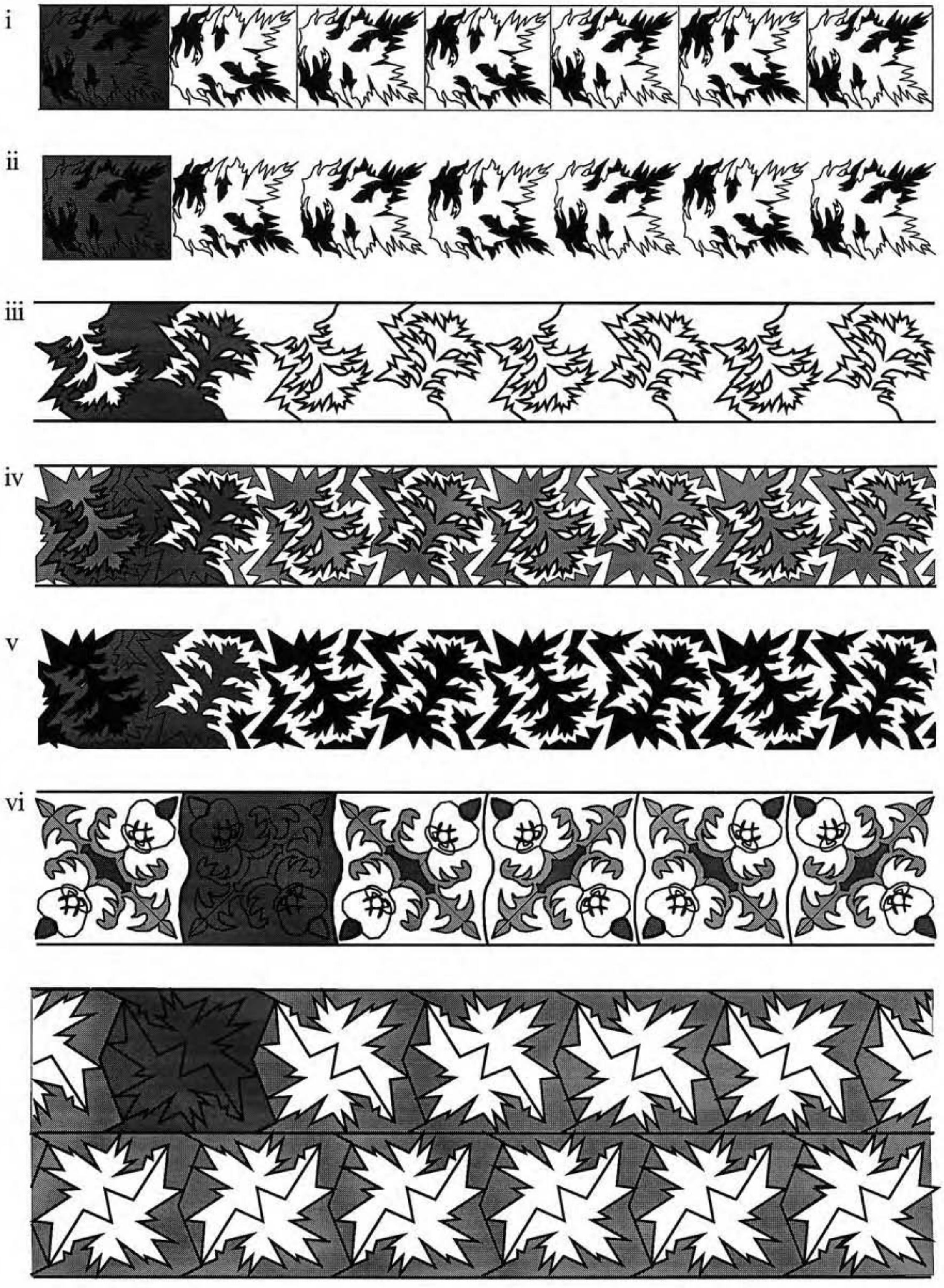


Figure 2.25 Construction of symmetry group *p1a1*.

The other case occurs when both edges joining or passing through R_1 and R_2 are exchanged leaving the fundamental region having just one straight edge along the outside edge of the strip (see the third example in Fig. 2.28(iii)). One of these edges, through R_1 say, must meet the parallel boundaries of the strip whereas the other through R_2 could meet the boundaries of the strip or join at a point on the new edge through R_1. (If both of the new edges meet the boundaries of the strip,

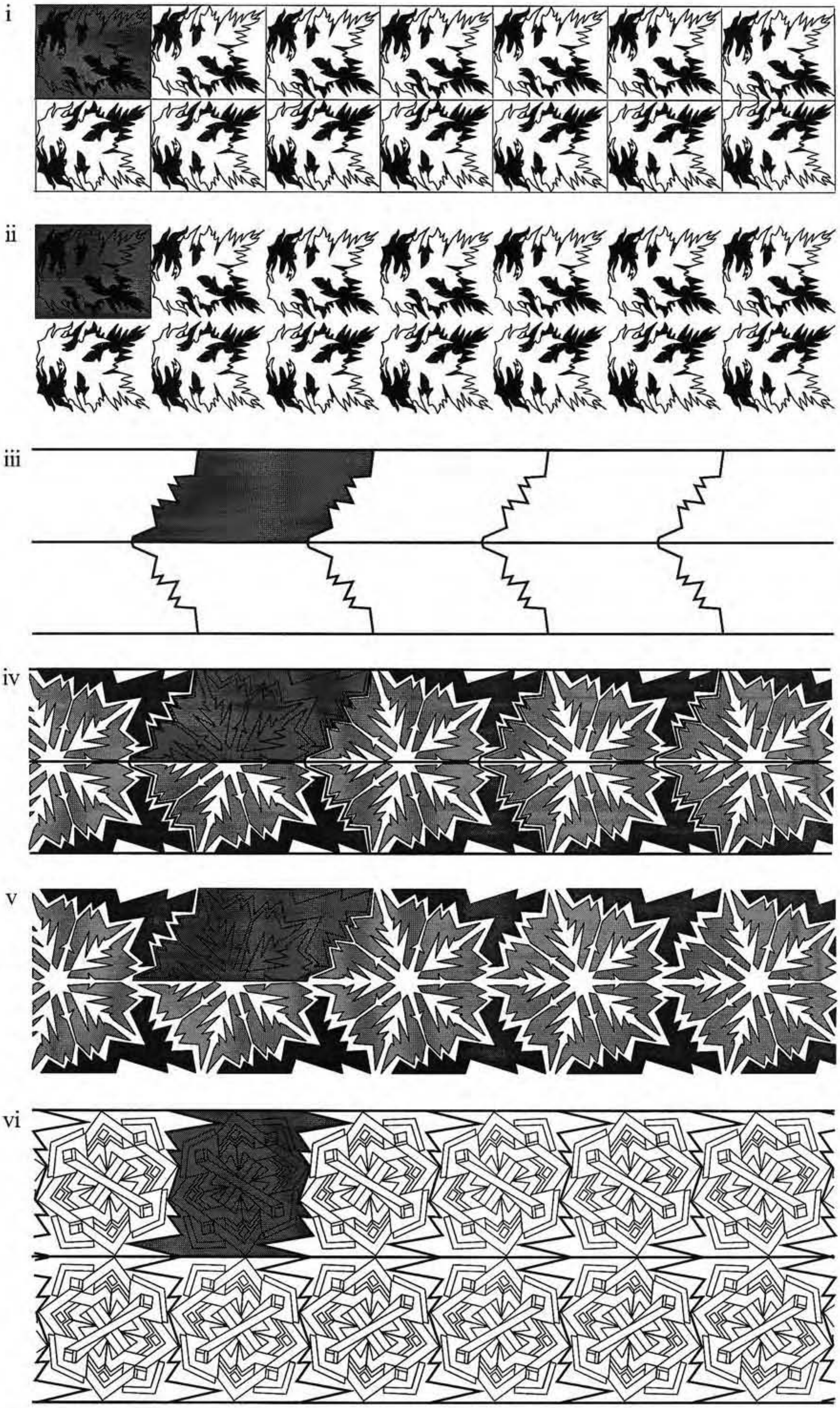


Figure 2.26 Construction of symmetry group $p1m1$.

i

ii

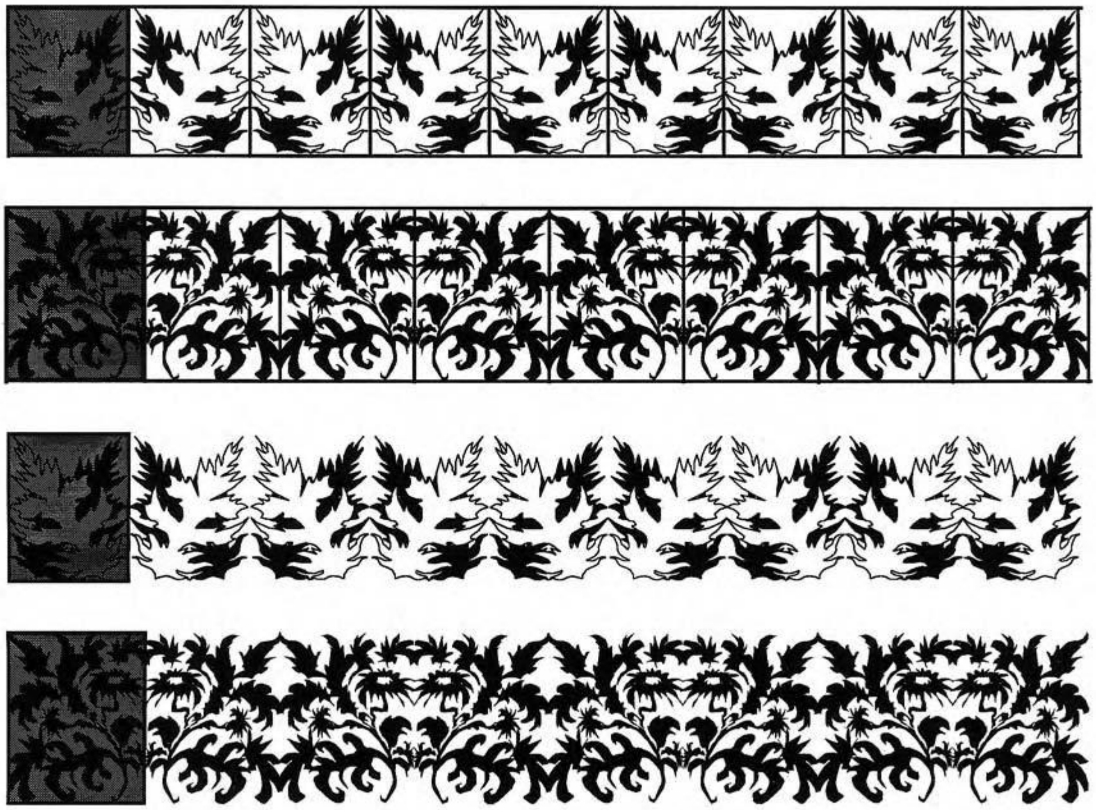


Figure 2.27 Construction of symmetry group *pm*11.

the fundamental region will have two straight sections occurring on opposite sides of the strip.) In each of these cases the new fundamental region edges replace all equivalent ones by applying the generating symmetries. To construct design type (vi) the fundamental regions may only have two-fold rotational symmetry or longitudinal reflectional symmetry as shown in Fig. 2.28(vi). A design unit is then added to a fundamental region and mapped onto the remaining ones by applying the rotational symmetries in the design structure. Some examples of design types (i) to (v) for symmetry group *p*112 are given in Fig. 2.28(i) to (v), respectively.

2.12.6 Symmetry group pma2

To construct design type (iii), three out of four of the edges of each fundamental region remain fixed. The only alterable fundamental region boundary has a centre of two-fold rotation at its centre. Thus, although the replacement for this edge may have its end points positioned differently from the straight line it is replacing, it must still have two-fold rotational symmetry about this point. To construct design type (vi) the only symmetrical alternative to rectangular (or square)-shaped fundamental regions for a *pma*2 design is isosceles triangle-shaped ones. These may be constructed provided that the initial monotranslational design is structured on a rectangular lattice where each rectangle is composed of two squares (i.e. the unit cell has width W (coinciding with the width of the strip) and length $2W$). Since the symmetries of these triangles do not induce any additional symmetrical characteristics in the structure of the strip, any symmetric or asymmetric design unit can be added to a triangle and mapped

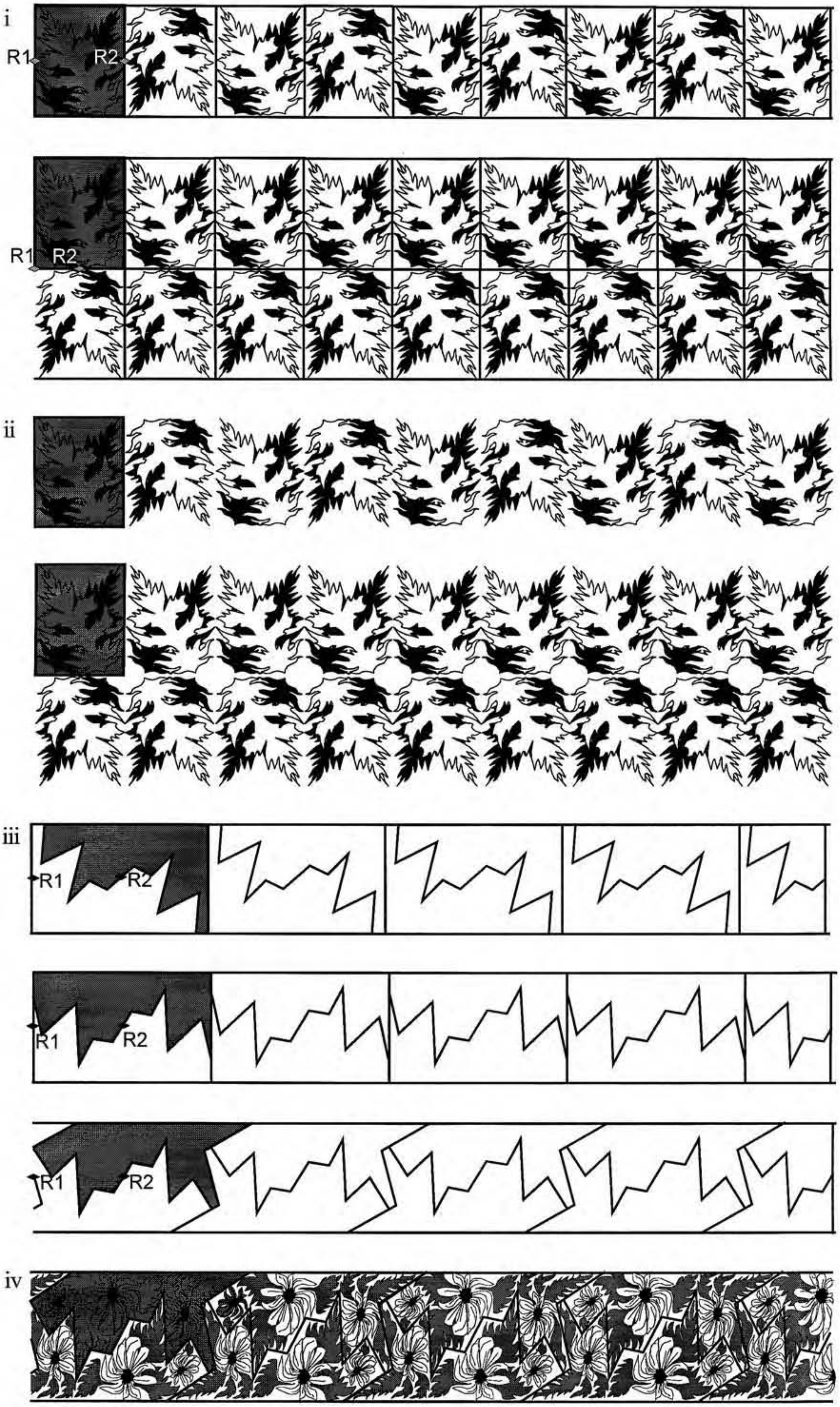


Figure 2.28 Construction of symmetry group *p*112.

v

vi

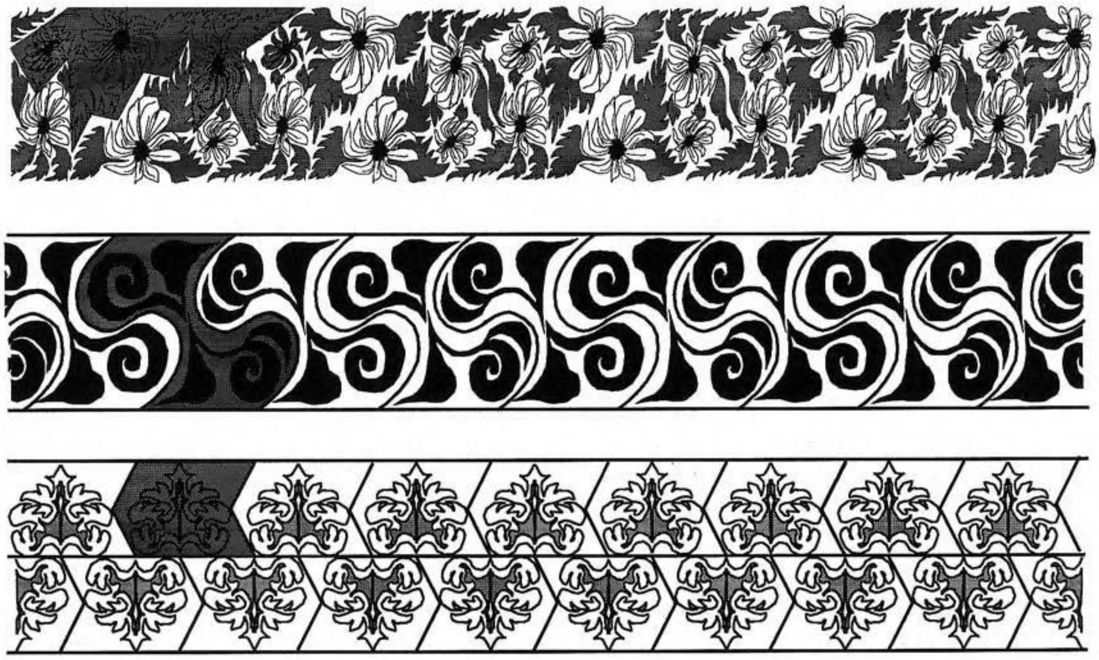


Figure 2.28 (cont.)

onto the remaining ones by applying a set of generators. Figure 2.29 shows some examples of design types (i) to (vi) for symmetry group *pma*2.

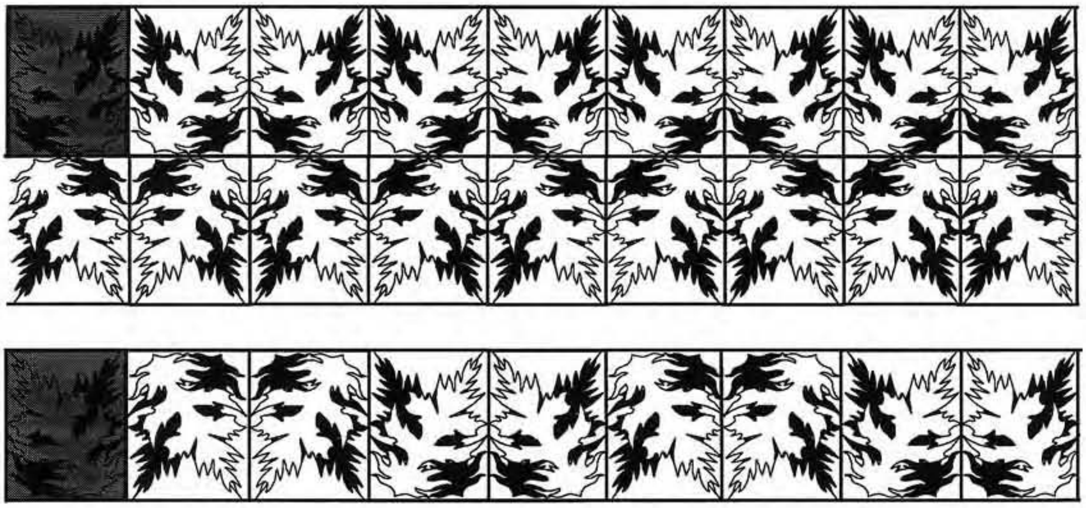
2.12.7 Symmetry group pmm2

For symmetry group *pmm*2, all four sides of the fundamental region are fixed since they fall on reflection axes or the boundaries of the strip enclosing the design. Therefore none of the remaining design types (iii) to (vi) are constructable. Figure 2.30 shows some examples of design types (i) and (ii) for symmetry group *pmm*2.

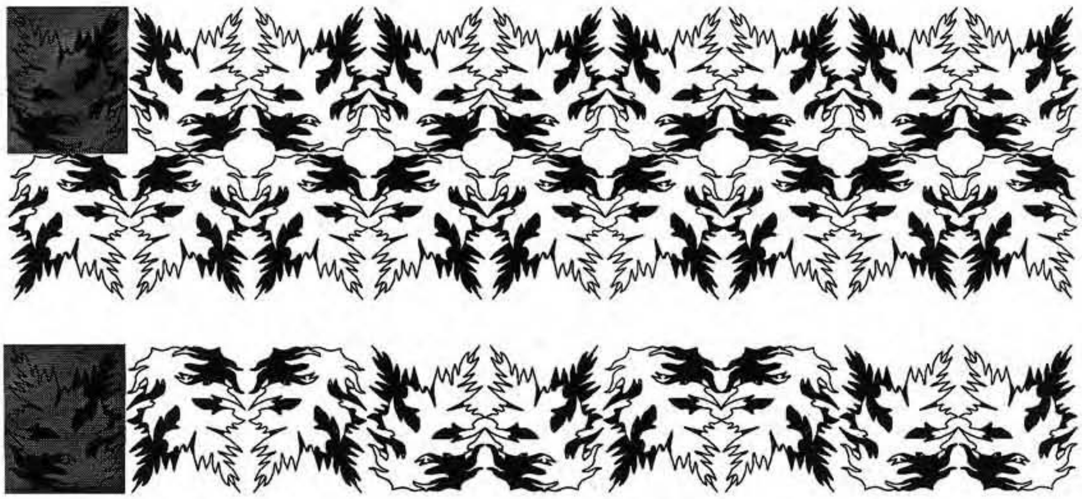
2.13 Construction of ditranslational designs

There are numerous different methods which may be used to decorate a plane with a given design symmetry group, for example a tiling, a patterned tiling or a pattern. A tiling/pattern may consist of equally or differently shaped tiles/motifs and in addition the motifs of a pattern may either interlock, join or be separate from each other. The following sections describe a selection of construction techniques for different design types analogous to those described for monotranslational designs. By initially dividing the plane into a tiling of fundamental regions it is possible to produce numerous topologically differing design effects (which relate to the interlocking nature of the design, the details of which are discussed in Chapter 5). Only the simpler ones will be outlined in the following sections. For example, in Fig. 2.31 there are two tilings of fundamental regions both of which may be used in the construction of a *p*1 design. However, the resulting appearance of the design, when the 'tile'/fundamental region boundaries are removed, will differ owing to the interlocking relationship between each of the fundamental regions and its neighbours. For a *p*1 design there are only two topological ways of forming a tiling of fundamental regions but for some of the other symmetry groups the possibilities are numerous.

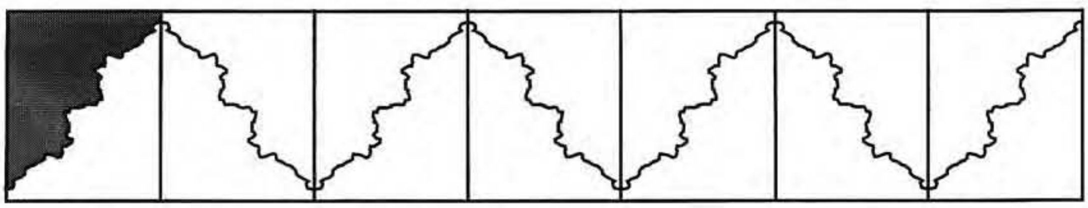
One method of producing a ditranslational design would be to apply, successively, a minimal set of generators to a suitably decorated fundamental region. This would then gradually fill out the whole design. Alternatively, Stevens, in his book *A Handbook of Regular Patterns*, describes a process whereby any asymi



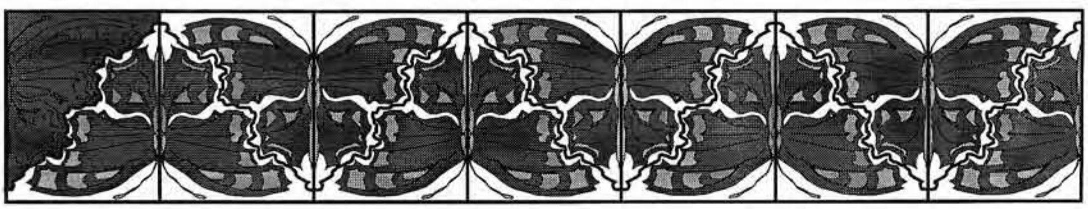
ii



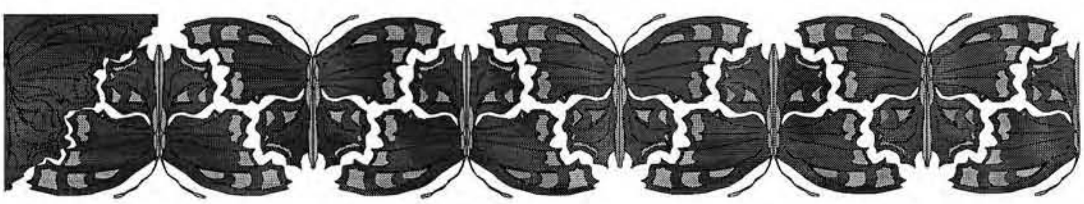
iii



iv



v



vi

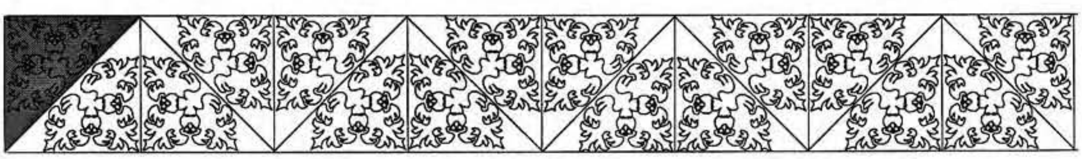


Figure 2.29 Construction of symmetry group *pma*2.

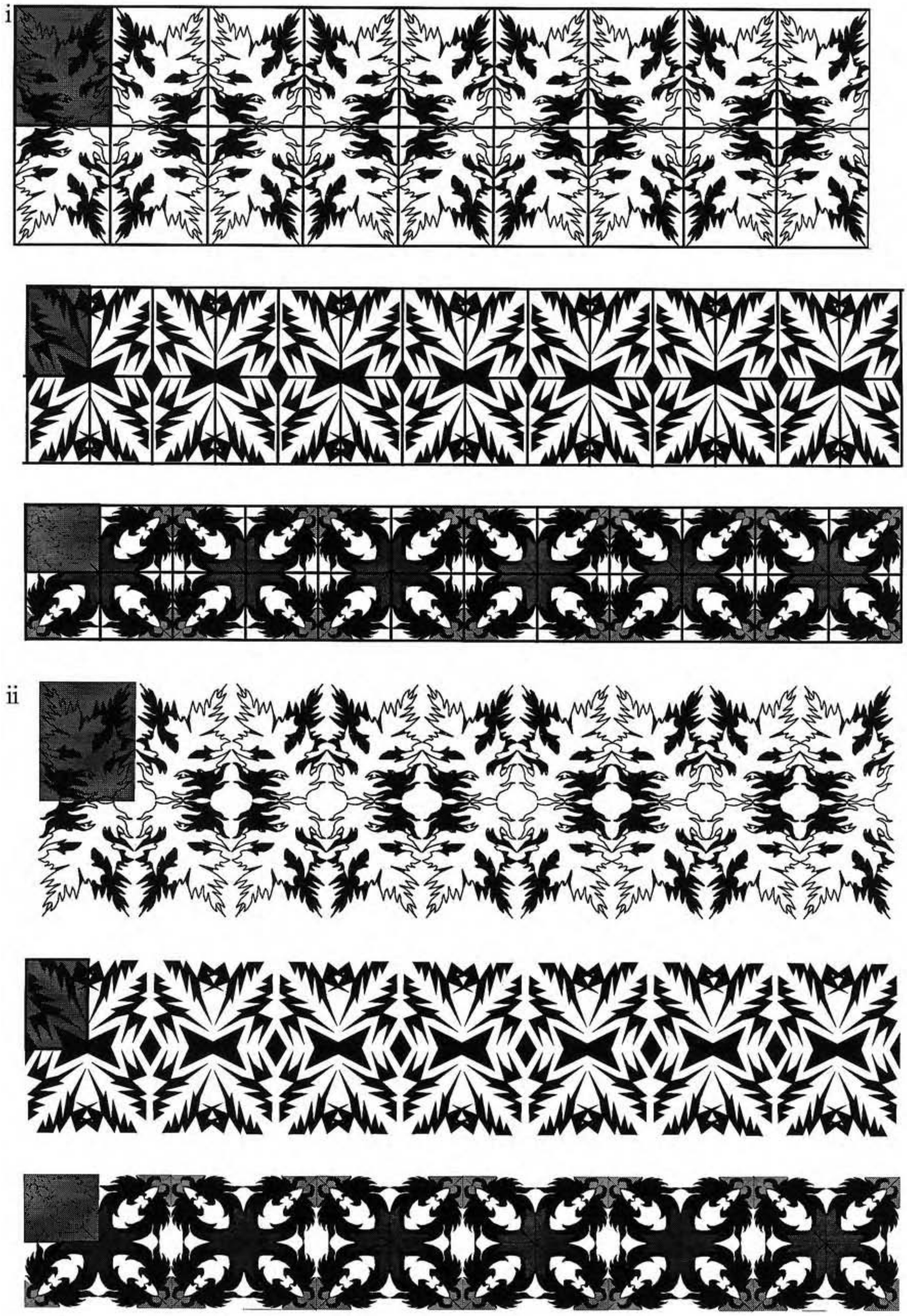


Figure 2.30 Construction of symmetry group *pmm*2.

metrical motif can be stacked with itself to create seven linear bands (monotranslational designs) and 17 planar patterns (ditranslational designs).[20]

In a similar vein, Bunce describes how panel or band patterns can be used to build up a design.[21] Following this construction method a monotransla-

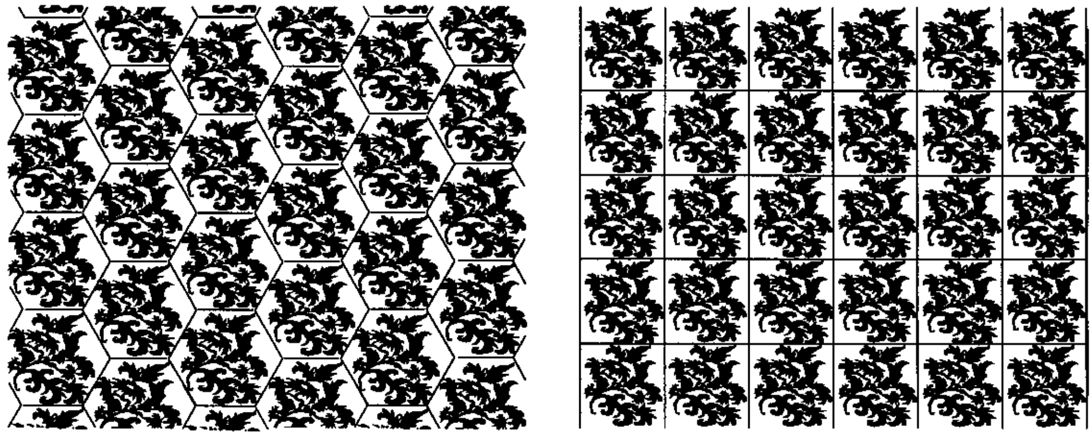


Figure 2.31 Examples of possible tiling structures for symmetry group *p*1.

tional design is translated at unit intervals in a direction of θ° to its longitudinal axis. This technique is, effectively, equivalent to consecutively placing strips, of width W, adjacent to each other to cover the plane. Bunce goes on to say that since panel designs are usually based on the symmetrical division of a defined area, when repeated, they exhibit a regular grid appearance. However, although a formal, rigid-structured, grid-like appearance may result in the overall design, this property may be reduced by altering the characteristics of the initial 'band pattern' or monotranslational design from which it is constructed.

This construction procedure enables all 17 symmetry groups of ditranslational design to be constructed by a process which may be suitably adapted for screen printing (e.g. for textile or paper printing). (For this application, the most suitable value of θ is 90°.) In each construction method the top boundary edge of the initial 'tiled' strip (or double strip, where stated) is removed before applying the consecutive translational symmetries perpendicular to its longitudinal axis. By employing this technique, ditranslational symmetry groups *p*1 to *pmm* may be constructed from the seven monotranslational designs discussed in Section 2.12. Symmetry groups with three-, four- and six-fold rotational symmetry may also be formed by this method but the initial monotranslational design (which, though, may be classified as one of the seven symmetry groups) requires specific additional geometrical characteristics in its structure before applying translational symmetries perpendicular to its longitudinal axis.

In the construction techniques discussed below, reference is made to three translational symmetry operations: T_1 parallel to the longitudinal axis of the initial monotranslational design and distance equal to the length of a unit cell; T_2 parallel to the side of a unit cell (not to the longitudinal axis) and distance equal to the side's length (for rectangular and square lattices, this length is W); T_3 perpendicular to the longitudinal axis and distance $2W$, twice the width of a strip of unit cells. For some symmetry groups, a reflectional symmetry M is applied to the initial monotranslational design about an axis coinciding with the top edge of the strip (which produces a double strip), before consecutive applications of translation T_3. For $p3xy$ and $p6xy$ designs, reflection M is applied to a tiled strip of fundamental regions, before adding design elements, to establish the correct structure upon which to build the design. Reference is also made to a glide–reflection G which is parallel to T_1 and of a distance equal to half its length.

Although, as described previously, symmetry groups *p*1 and *p*2 may be based on any form of parallelogram lattice, in this section their structures are restricted to rectangular ones. Alternative structures will be described in more detail in Chapter 5. Also, to avoid complication, when exchanging fundamental region edges for asymmetric ones, as described for the type (iii) monotranslational designs, it is assumed that the end points of the edges remain fixed. Construction methods of six different ditranslational design types (analogous to those for monotranslational designs) are discussed for each symmetry group. A general description of each is given below.

1 Design type (i): the first design type involves initially constructing a monotranslational design type (i) that has triangular, parallelogram or, for symmetry group $p6$, kite-shaped fundamental region boundaries, in other words the fundamental region is chosen, where possible, to be a symmetrical portion of the unit cell. Reflectional symmetry M may be applied to this design (which is stated for each symmetry group where applicable) and then the top edge of the strip is removed before applying consecutively the translational symmetries T_2 or T_3.
2 Design type (ii): this is derived from monotranslational design type (i) by removing the boundaries of the fundamental regions/'tiles' before applying reflection M and/or translational symmetries, T_2 or T_3. This reduces a patterned tiling to a pattern which may appear to have a more 'grid-like' appearance owing to the straight edges chosen for the fundamental region boundaries.
3 Design type (iii): this is derived from monotranslational design type (iii). Some or all of the edges of the fundamental regions are altered before removing the top edge of the strip and then applying reflection M and/or the translational symmetries T_2 or T_3. This gives a more interlocking type of tiling design. As with monotranslational designs, the new edges must be positioned so as not to overlap with each other on application of the generating symmetries. For symmetry groups where each of the edges of the fundamental regions lies on a reflection axis, this alteration is not possible and therefore design type (iii), and consequently types (iv) to (vi), are not constructable. Conversely, there may be one, two or three ways of producing interlocking tiles from the initial monotranslational design depending on the number of different 'sets' of fundamental region edges. These are discussed in detail below for each symmetry group.
4 Design type (iv): this is derived from the monotranslational design used to construct ditranslational design type (iii). Design elements are added to one fundamental region and then mapped onto the remaining ones in the strip before applying reflection M and/or consecutive translations of T_2 or T_3 of the symmetry group. This produces a patterned interlocking tiling design.
5 Design type (v): this is derived by initially removing the boundaries of the fundamental regions chosen for the monotranslational design type (iv) before applying reflection M and/or consecutive translations of T_2 or T_3. If the design elements are initially chosen to extend towards the boundaries of the fundamental regions (for type (iv)), each motif appears to interlock with its neighbouring motifs resulting in a design with a more continuous and therefore less disjointed appearance.
6 Design type (vi): this is formed, where possible, by dividing the initial strip into symmetrical-shaped fundamental regions (not coinciding with those of type (i)). This design construction method is only discussed for symmetry groups $p1xy$, $p2xy$ and $p4xy$. Figure 2.32 shows examples of a selection (but not all) of the possible tiling structures suitable for this design type. These structures illustrate some of the simplest forms of tilings composed of tiles with two- and four-fold rotational symmetry and longitudinal or transverse reflectional symmetry in relation to the sides of the initial strip. Design elements are added to one tile and then mapped onto all the equivalent positions in the strip before applying reflectional and/or the translational symmetries. A further version of design type (v) (interlocking motifs without tile boundaries) may be derived from type (vi). However, the design elements inside the fundamental regions must not induce any additional symmetries into the design structure on removal of these 'tile' boundaries.

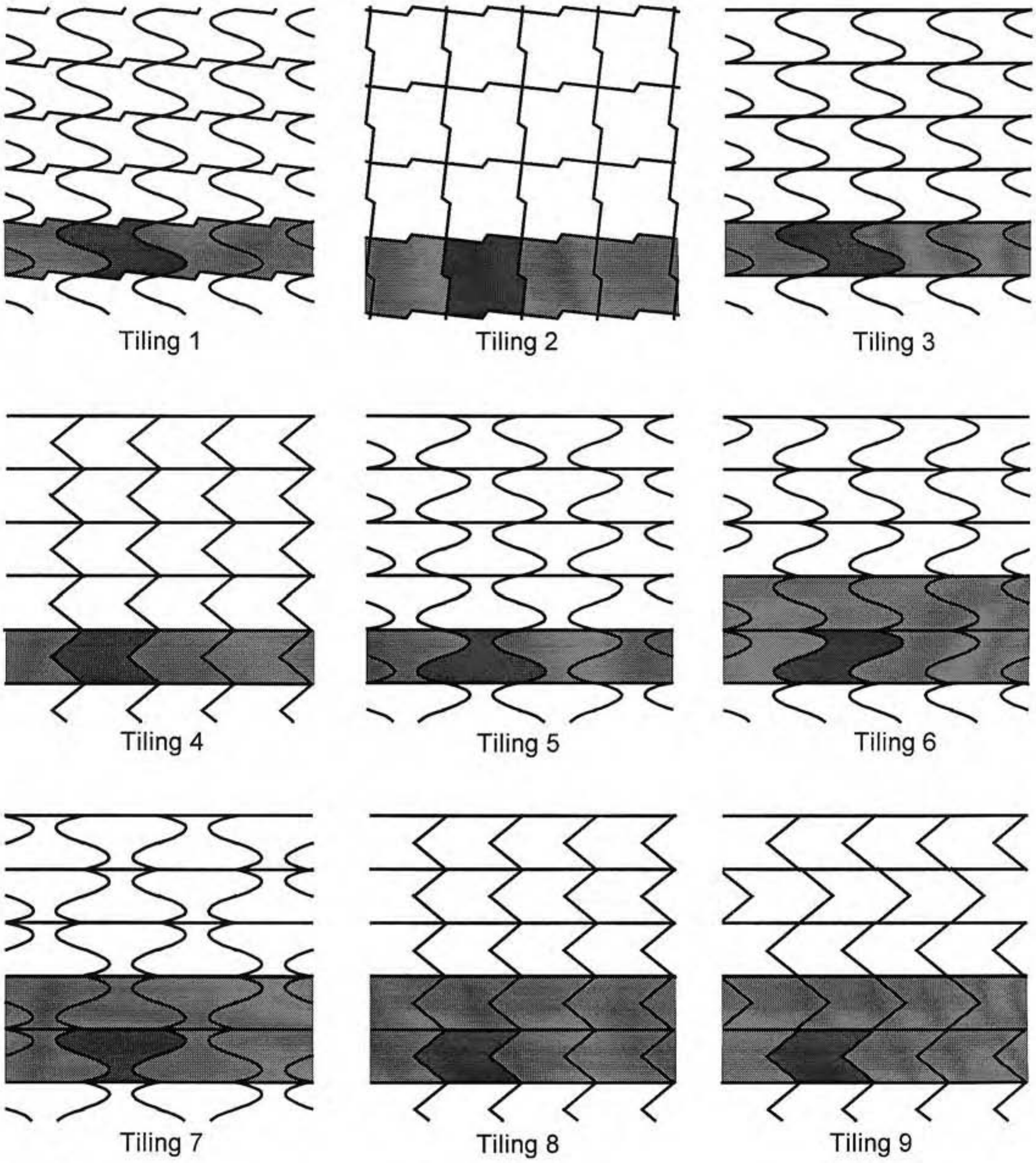


Figure 2.32 Examples of possible tiling structures for type (vi) ditranslational designs.

This gives a general outline of the construction technique for design types (i) to (vi) for each group of designs $p1xy$, $p2xy$, $p3xy$, $p4xy$ and $p6xy$. The positioning of symmetrical design units is critical therefore in the ditranslational design construction methods in this chapter, for simplicity, the design unit is generally taken to be asymmetric. It is also assumed that no additional symmetries are induced into the design structure on translating the unit cell or translation unit, such as those described in relation to monotranslational designs in Section 2.12 and illustrated in Fig. 2.23. For symmetry groups where design types (i), (ii) and (iii) are simply derived from consecutive applications of translation T_2 to an associated monotranslational design, no further explanation is given. Illustrations of all six design types are given for symmetry group $p1$ but only a selection of examples are shown for subsequent symmetry groups. Any additional versions of design type (iii) are described for each symmetry group although the design types (iv) and (v) which may be derived from type (iii) (by an analogous method for monotranslational designs) are not. Design type (vi) (where reference is made to the tilings in Fig. 2.32) is self-explanatory for each symmetry group, from the description given above.

In the examples throughout the remainder of this chapter, the light shaded area represents the initial monotranslational design (or two adjacent monotranslational designs) which is either translated at unit intervals of $W(T_2)$ or, where stated, at unit intervals of $2W(T_3)$ at 90° to the longitudinal axis of the strip. The darker area in the strip represents a fundamental region. Note that although tiling and patterned tiling designs may be constructed for screen printing it may prove more difficult to register tile boundaries. As a result of this, for printing purposes, design types (ii) and (v) are most appropriate. In each of the illustrations in the following figures the section number represents the design type, for example Fig. 2.33(iii) represents design type (iii).

2.13.1 Symmetry groups p1xy and c1xy

There are four ditranslational symmetry groups of the form $p1xy$ and $c1xy$ which are abbreviated to $p1$, pg, pm and cm.

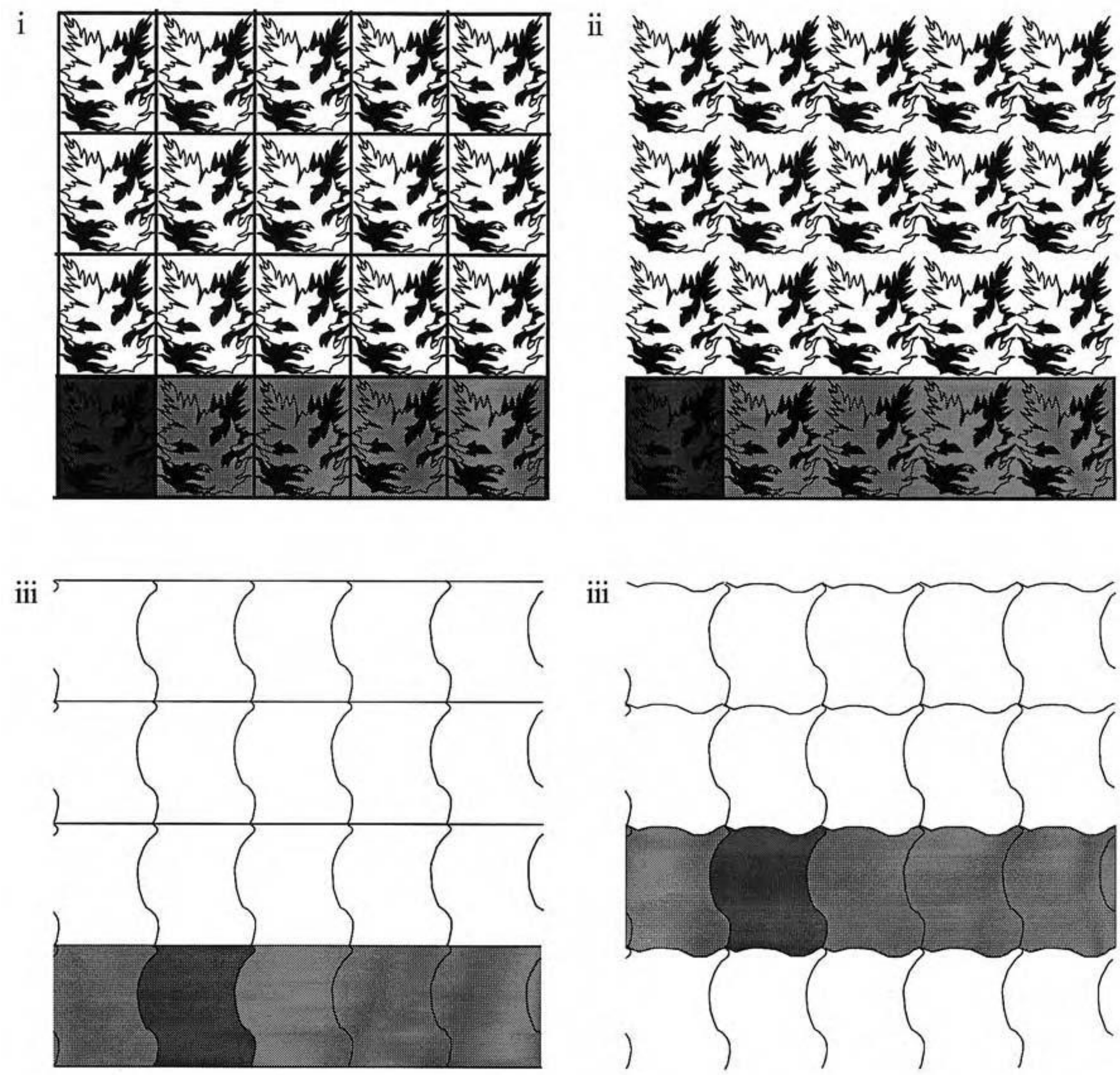


Figure 2.33 Construction of symmetry group *p*1.

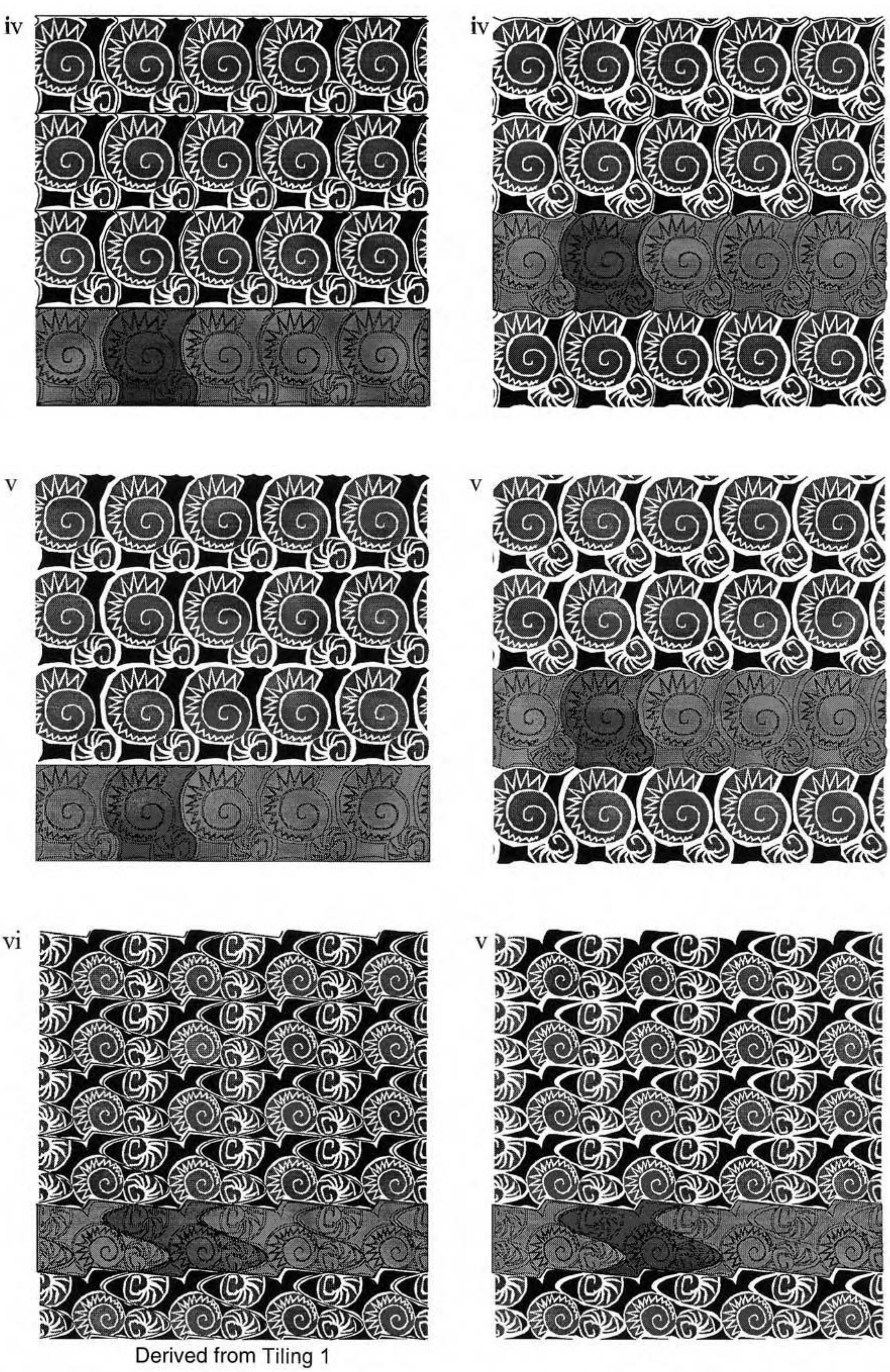


Figure 2.33 (cont.)

2.13.1.1 *Symmetry group p1*

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of $p111$, as shown in Fig. 2.33(i), (ii) and (iii). A second version of type (iii) may be constructed by replacing a straight edge of a fundamental region on the bottom edge of the strip by an asymmetric one and then using it to replace each adjacent edge, in the longitudinal direction, by repeatedly applying T_1. The top straight edge is removed and then T_2 is applied at unit intervals. An illustration is given in the second example of Fig. 2.33(iii). Design type (vi) may be constructed from any of the tilings 1, 2, 3 or 4.

2.13.1.2 *Symmetry group pg*

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of $p1a1$. A second version of type (iii) may be constructed by replacing a straight edge of a fundamental region on the bottom edge of the strip by an asymmetric one and then using it to replace each adjacent edge, in the longitudinal direction, by the repeated application of glide–reflection G. The top straight edge is removed and then T_2 is applied at unit intervals (see Fig. 2.34(iii)). Design type (vi) may be constructed from either of the tilings 4 and 5.

2.13.1.3 *Symmetry group pm*

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of $p1m1$. (A *pm* ditranslational design may also be constructed by applying the same translations to a $pm11$ monotranslational design which, in the context of printing, results in reflection axes occurring parallel to the warp/length of the fabric/paper as opposed to them being parallel to the weft/width if constructed from the initial monotranslational design $p1m1$). Symmetry group *pm* has only one form of design type (iii) because two edges of each fundamental region fall on reflection axes, occurring on the boundaries of the strip, which cannot be altered (see Fig. 2.35). Design type (vi) may be constructed from tilings 6 and 8.

2.13.1.4 *Symmetry group cm*

Design type (i), for symmetry group *cm*, is constructed by first applying reflection M to a $p1a1$ monotranslational design to give a strip with width $2W$. Consecutive translations of T_3 are then applied to this double strip to complete the patterned tiling design. Design types (ii) and (iii) are constructed by applying the same operations to types (ii) and (iii) of monotranslational design $p1a1$. Symmetry group *cm* has only one form of design type (iii) because two edges of each fundamental region fall on reflection axes, occurring on the boundary and longitudinal axis of the strip, which cannot be altered (see Fig. 2.36). Design type (vi) may be constructed from either of the tilings 7 and 8.

2.13.2 *Symmetry groups p2xy and c2xy*

There are five ditranslational symmetry groups of the form $p2xy$ or $c2xy$ which are abbreviated to $p2$, *pgg*, *pmg*, *pmm* and *cmm*.

2.13.2.1 *Symmetry group p2*

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and

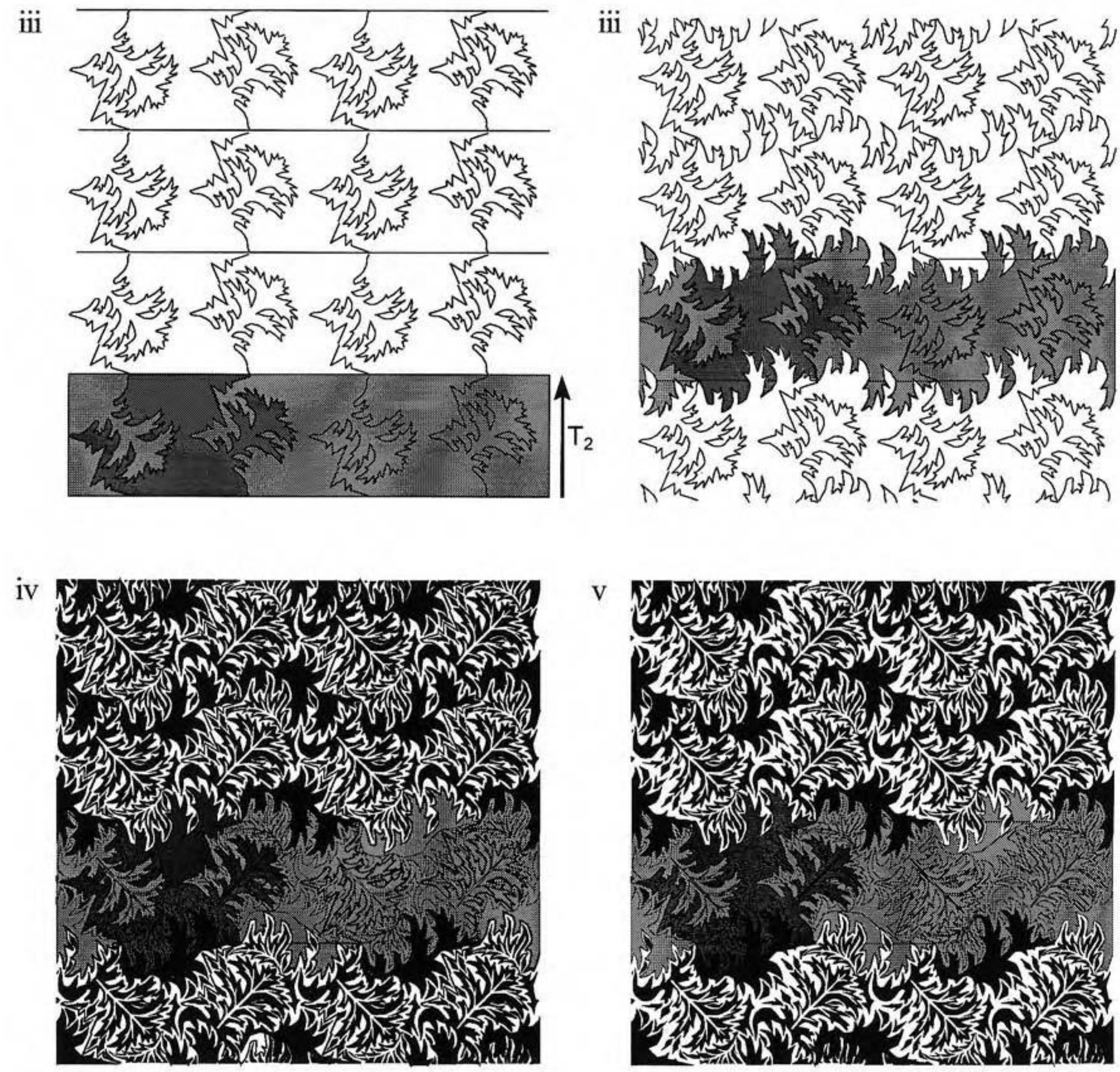


Figure 2.34 Construction of symmetry group *pg*.

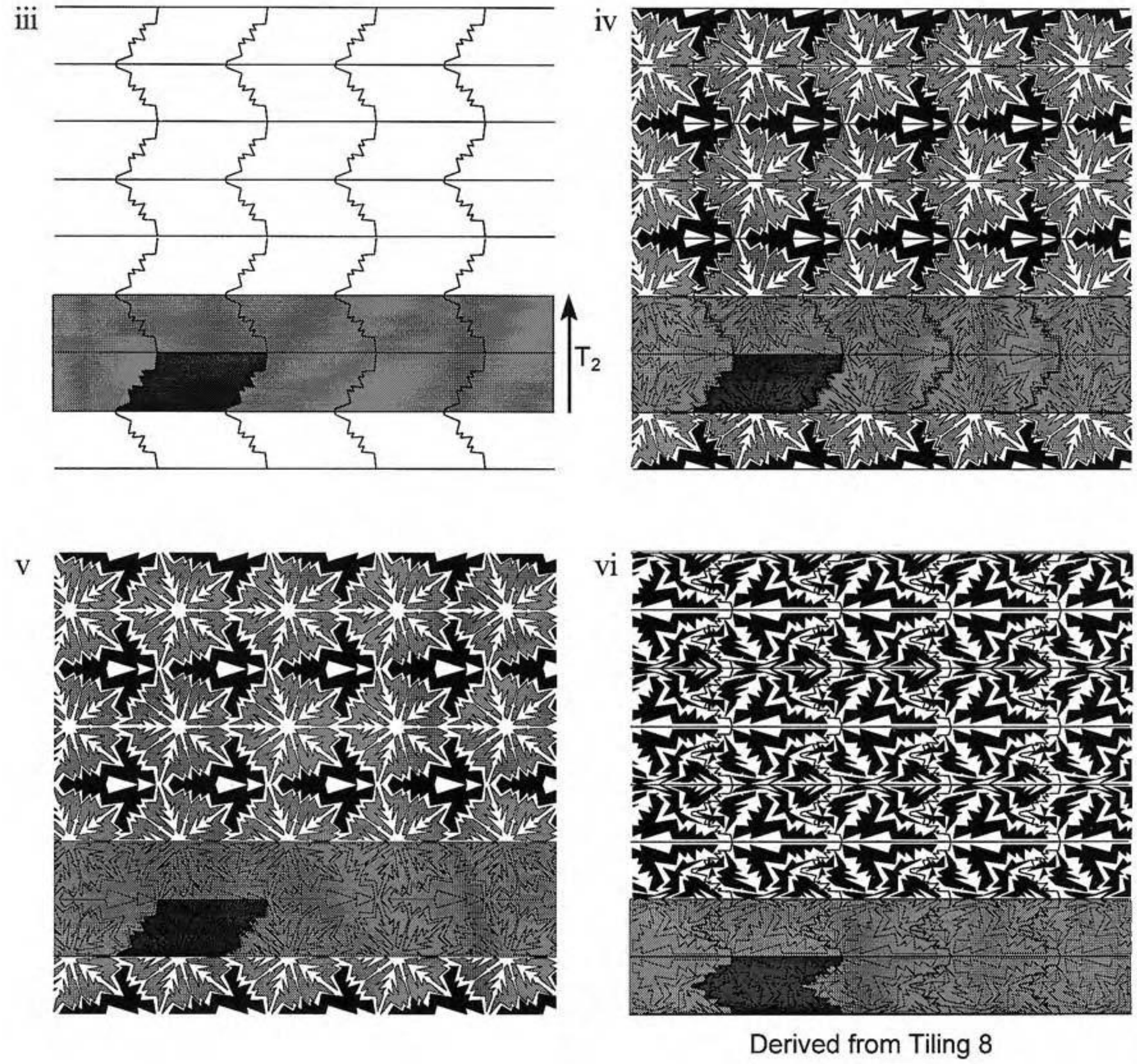


Figure 2.35 Construction of symmetry group *pm*.

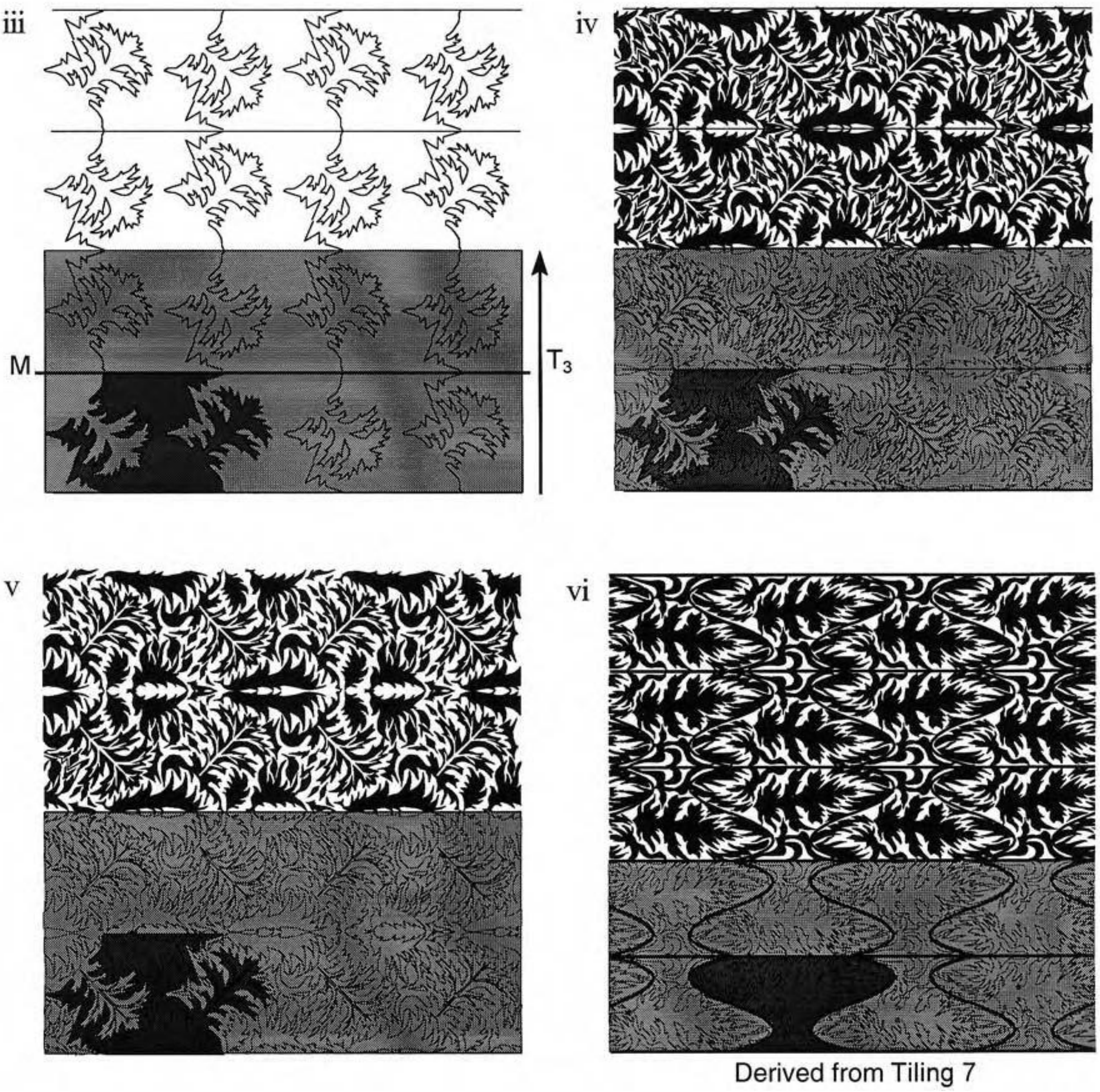


Figure 2.36 Construction of symmetry group *cm*.

(iii) of $p112$. An additional version of type (iii) may be constructed by replacing a straight edge of a unit cell on the bottom edge of the strip by one having two-fold rotational symmetry. It is then used to replace each adjacent edge, in the longitudinal direction, by repeatedly applying T_1. The top straight edge is removed and then T_2 is applied at unit intervals as shown in Fig. 2.37(iii). Design type (vi) may be constructed from any of the tilings 1, 2, 3 or 5. A $p2$ design may also be constructed from tiling 7 or tiling 9 although in these cases, the single $p112$ or $p111$ strip is two-fold rotated about the midpoint of a top edge or top corner of a fundamental region, respectively, to form a double strip, width $2W$, before consecutive applications of T_3 (see Fig. 2.37).

2.13.2.2 Symmetry group pgg

A *pgg* ditranslational design may be constructed by repeatedly applying the translation, T_3, to either two $p112$ monotranslational designs, one of which is a glide–reflection of the other, or to two $p1a1$ monotranslational designs, one of

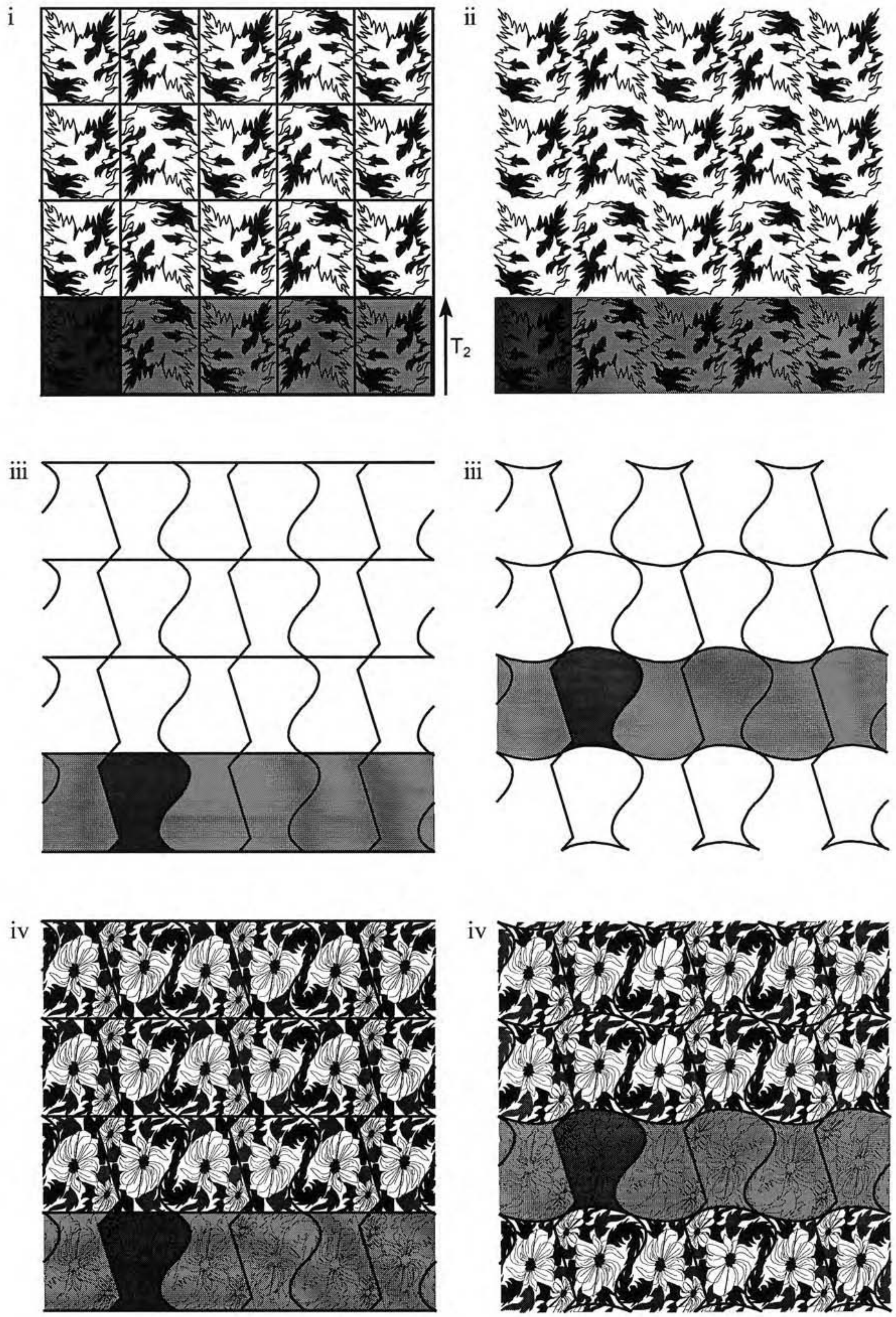


Figure 2.37 Construction of symmetry group *p*2.

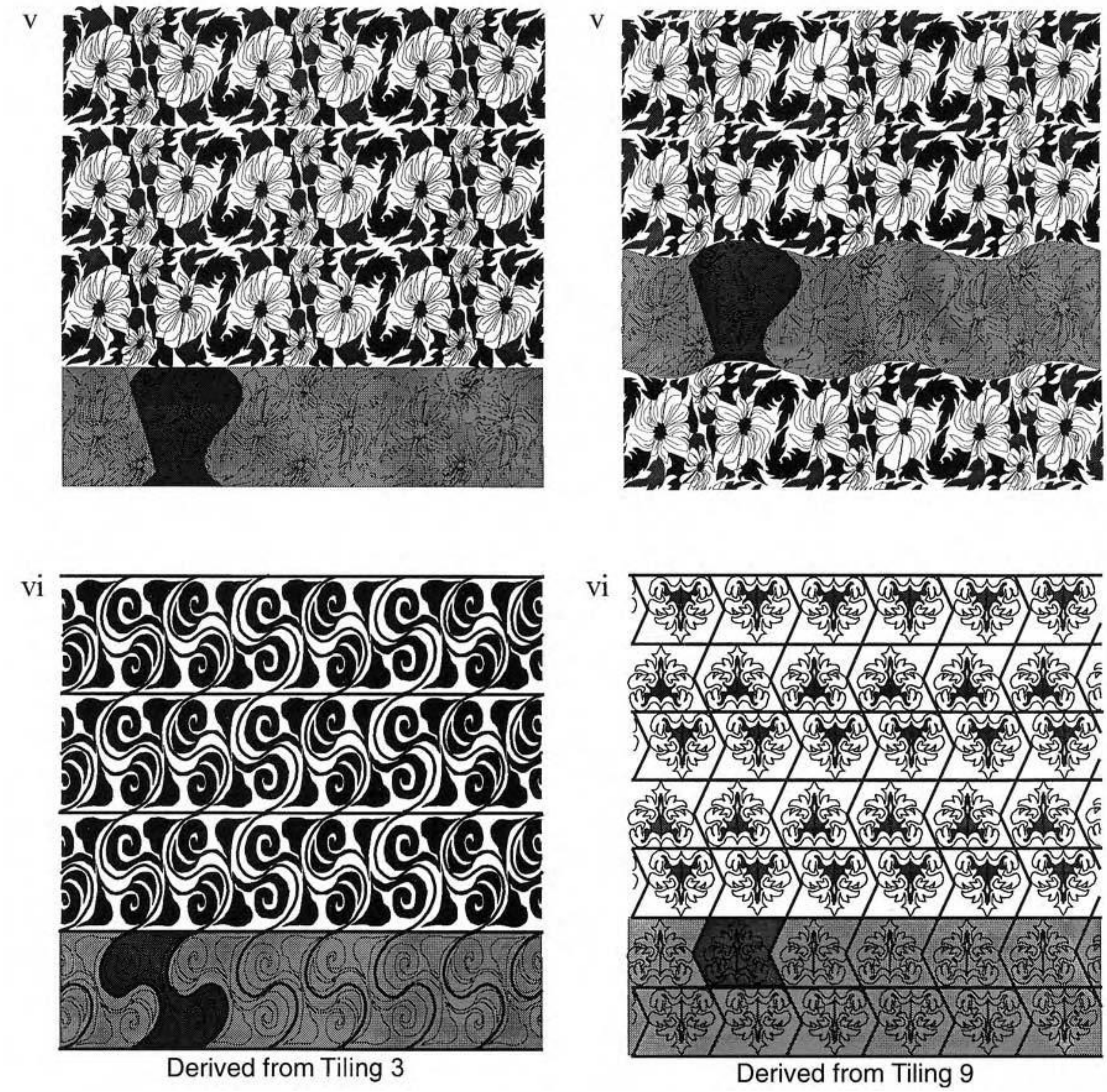


Figure 2.37 (cont.)

which is a two-fold rotation of the other. The first of these two possibilities is discussed for each design type below. Design types (i), (ii) and (iii) may each be constructed by consecutive applications of translation T_3 to a double strip, width $2W$, which has been derived from the corresponding monotranslational design types (i), (ii) and (iii) of $p112$, respectively. In each case, the double strip consists of two $p112$ monotranslational designs, one of which is a glide–reflection of the other. The glide–reflection axis coincides with a straight edge of the strip and its distance is equal to half the length of translation T_1 (see Fig. 2.38). An additional version of type (iii) may be constructed by replacing a straight edge of a fundamental region, on the bottom edge of the double strip, by an asymmetric one. It is then used to replace each adjacent edge, in the longitudinal direction, by repeatedly applying glide–reflection G. The central straight longitudinal axis of the double strip is exchanged for one which is a two-fold rotation, of the new bottom edge of the strip, about a centre of rotation occurring on the boundary of a fundamental region (as shown in Fig. 2.38(iii)). The top straight edge is removed and then T_3 is applied at unit intervals. Design type (vi) may be constructed from either a double strip of tiling 5 or tiling 6.

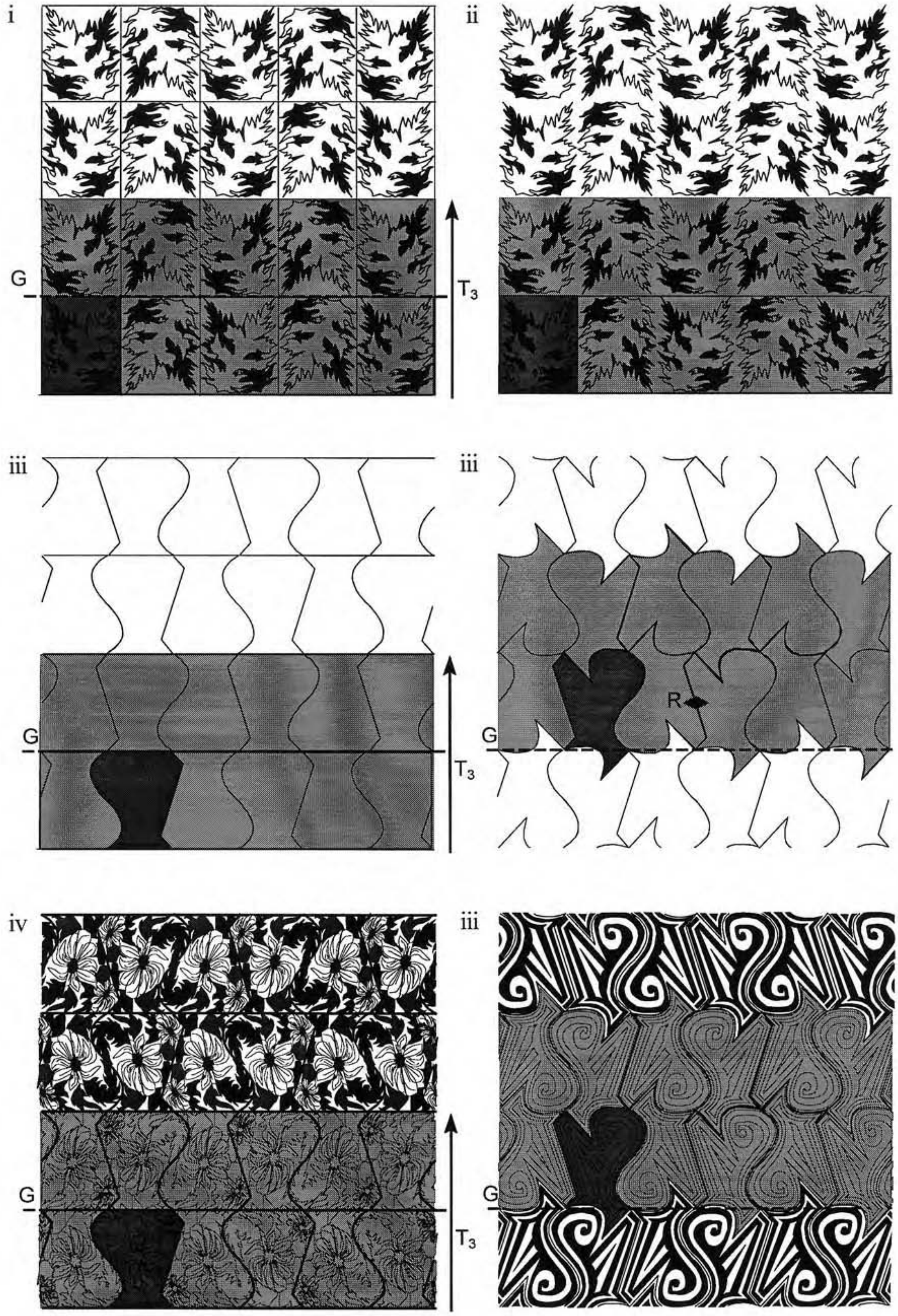


Figure 2.38 Construction of symmetry group *pgg*.

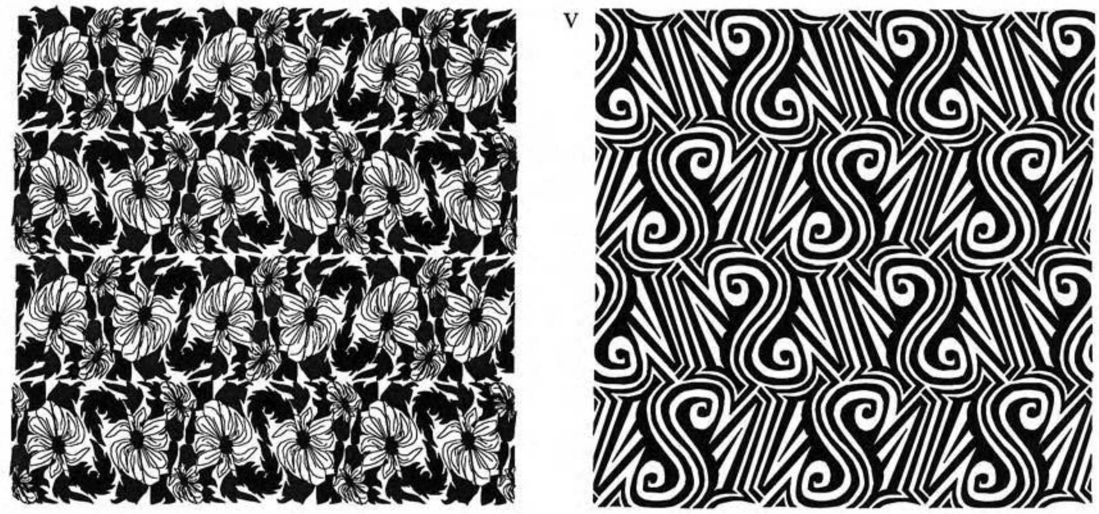


Figure 2.38 (cont.)

2.13.2.3 Symmetry group pmg

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of *pma*2. (This results in the reflection axes occurring parallel to the warp/length of the fabric/paper.) An additional version of type (iii) may be constructed by replacing a straight edge of a fundamental region, on the bottom edge of the strip, by an asymmetric one. It is then used to replace each adjacent edge, in the longitudinal direction, by the repeated application of alternating two-fold rotation and transverse reflection passing through the corners of each fundamental region (see Fig. 2.39(iii)). (The axes about which it is reflected coincide with those in the monotranslational *pma*2 structure.) The top straight edge is removed and then T_2 is applied at unit intervals. Design type (vi) cannot be constructed from any of the tilings 1 to 9 owing to the limitations caused by the reflection axes occurring in the structure of the design.

2.13.2.4 Symmetry group pmm

Design types (i) and (ii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i) and (ii) of *pmm*2. Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design. Figure 2.40 shows some examples of design types (i) and (ii) for *pmm*.

2.13.2.5 Symmetry group cmm

A *cmm* ditranslational design may be constructed by repeatedly applying the translation T_3, to either two *pma*2 monotranslational designs, one of which is a reflection of the other, or to two *pmm*2 monotranslational designs, one of which is a glide–reflection of the other. The first of these two possibilities is discussed for each design type below. Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_3 to a double strip, width $2W$, of the corresponding monotranslational design types (i), (ii) and (iii) of *pma*2, respectively. The double strip is constructed by applying reflection M to a *pma*2 monotranslational design (see Fig. 2.41(iii)). Ditranslational symmetry group *cmm* has only one form of design type (iii) which is derived by altering the fundamental region edges which pass through a centre of rotation. This is because two edges of each fundamental region fall on reflection axes, occurring on the boundary and longi-

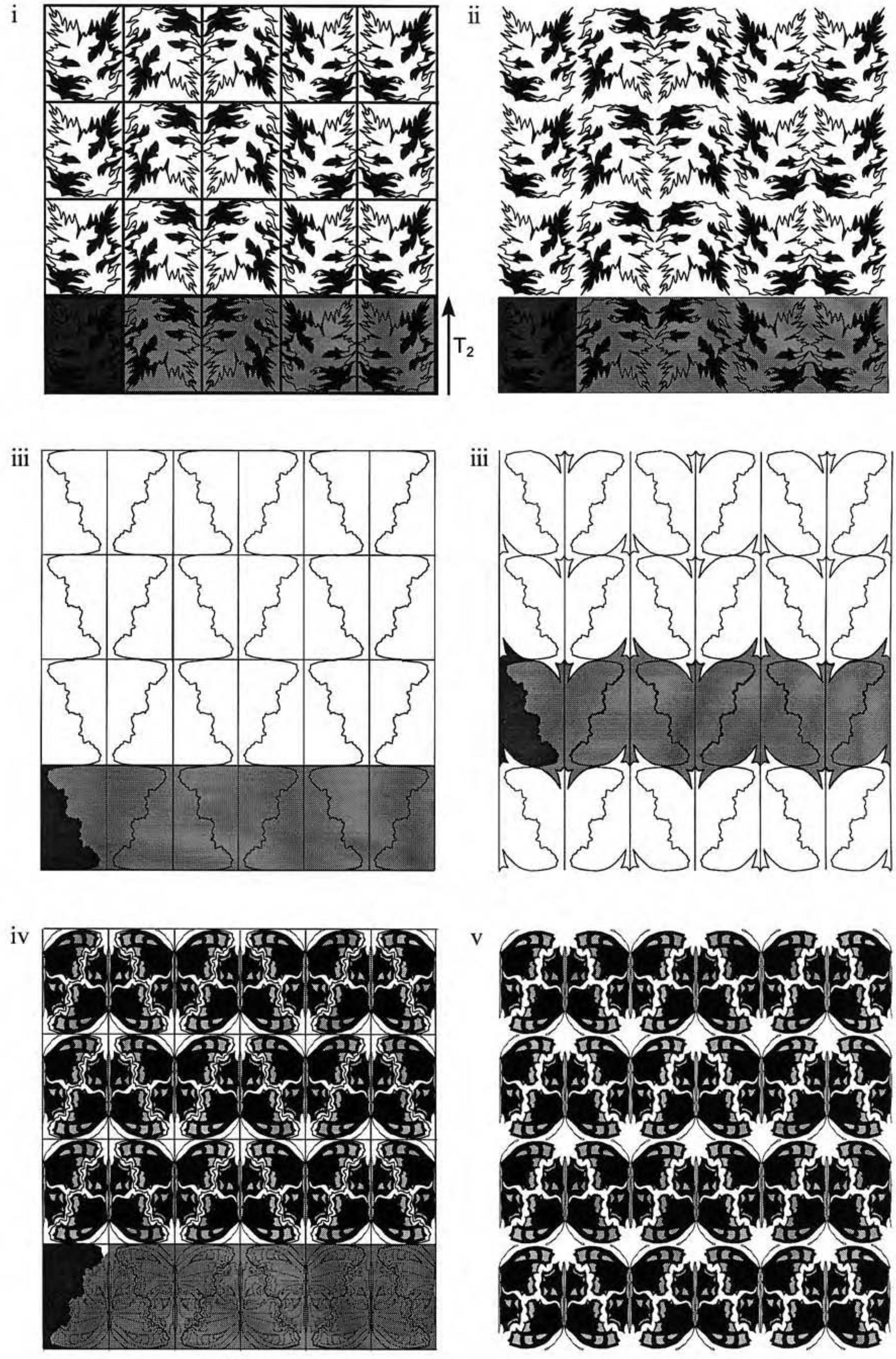


Figure 2.39 Construction of symmetry group *pmg*.

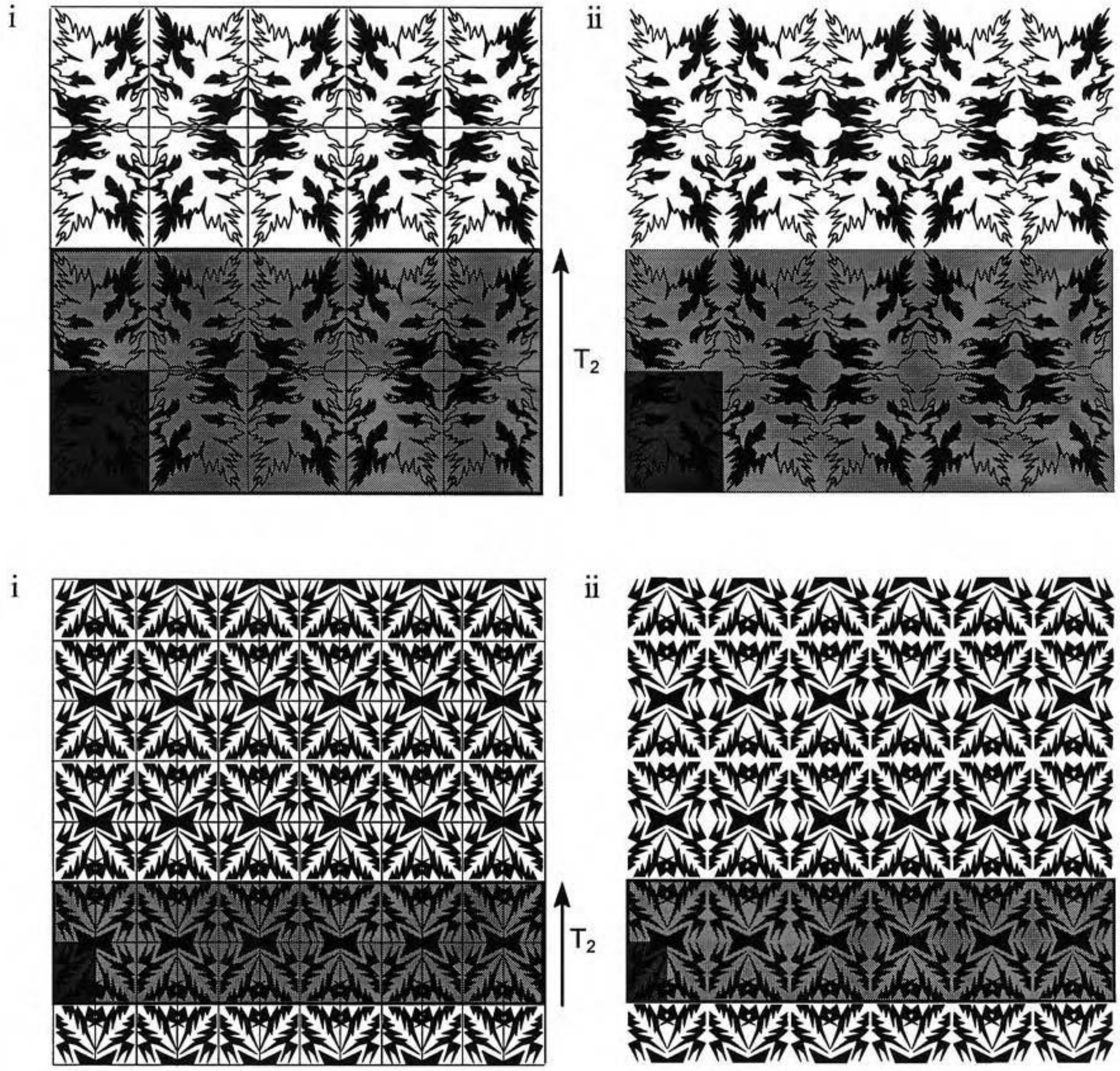


Figure 2.40 Construction of symmetry group *pmm*.

tudinal axis of the strip, which cannot be altered. Two examples of this form are illustrated in Fig. 2.41(iii). Design type (vi) cannot be constructed from any of the tilings 1 to 9 owing to the limitations caused by the reflection axes occurring in the structure of the design.

2.13.3 Symmetry groups p4xy

There are three ditranslational symmetry groups of the form $p4xy$ which are abbreviated to $p4$, $p4g$ and $p4m$. Each of these symmetry groups is based on a square lattice, therefore the initial strip used to construct these designs is divided into square parallelograms each of which represents a unit cell. These are then divided into fundamental regions which, as a strip of a ditranslational design, have reflectional and/or four-fold rotational symmetries occurring on their boundaries. These symmetries are not a property of a monotranslational design, however they are referred to when filling out the initial strip pattern. On applying these symmetries, design elements which are mapped onto positions outside the structure of the initial monotranslational design are not included.

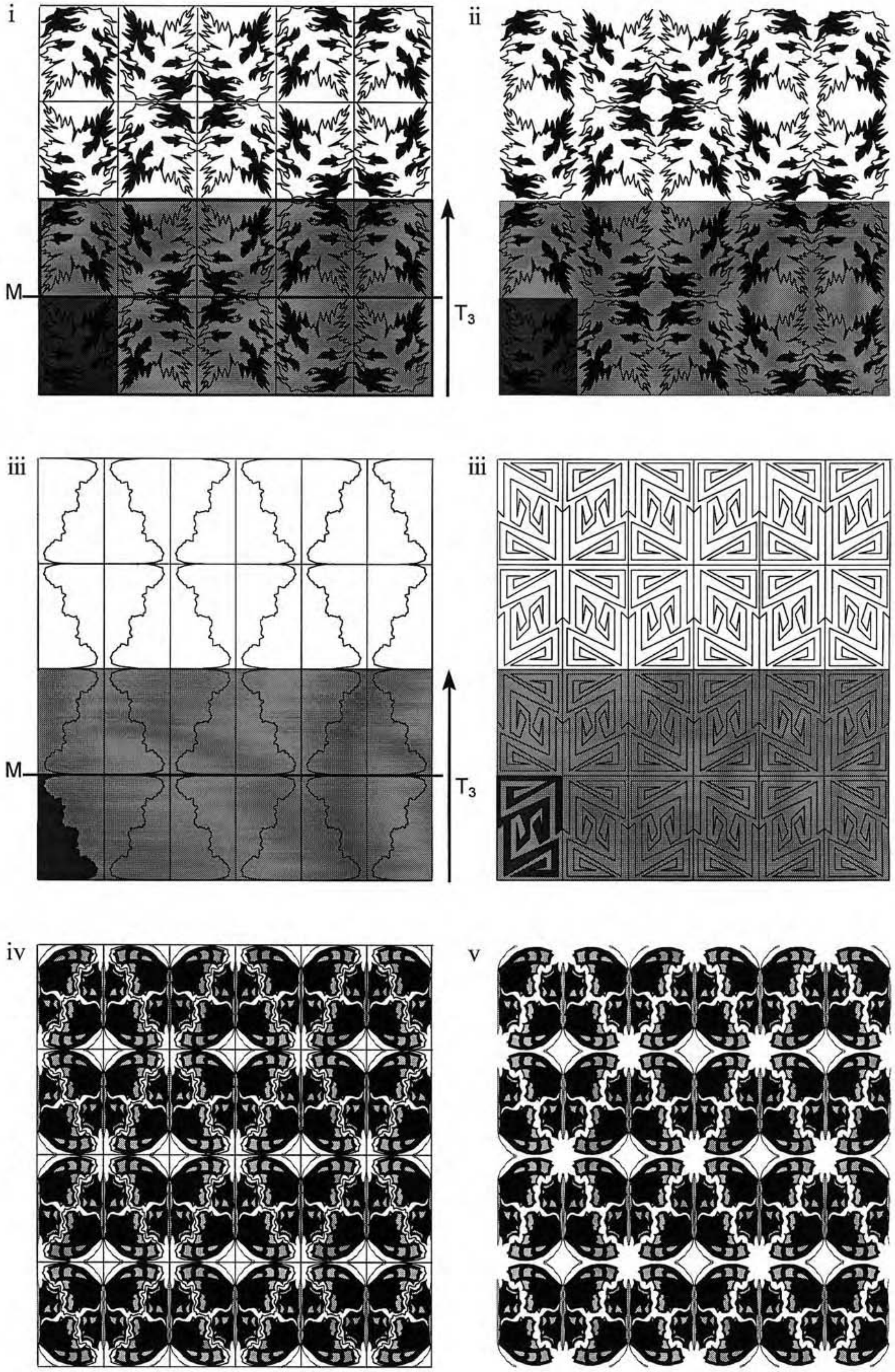


Figure 2.41 Construction of symmetry group *cmm*.

2.13.3.1 *Symmetry group p*4

The construction of a design of type (i) requires the division of each unit cell in the strip into four square fundamental regions. This strip, if associated with a *p*4 design, will have alternating centres of two- and four-fold rotational symmetry occurring through the longitudinal axis of the strip at corners of fundamental regions (see Fig. 2.42). Applying one of these four-fold rotational symmetries to design elements inside a square fundamental region will complete a unit cell which, on repeated application of T_1 will form a monotranslational design type (i). If the top straight edge of the strip is removed and then T_2 is applied at unit intervals, a ditranslational design type (i) is formed and if the boundaries of the tiles are removed, this gives a type (ii) *p*4 design. Design type (iii) may be produced by replacing one of the boundaries of a square fundamental region, joining a centre of four-fold rotation with the straight edge of the strip, by an asymmetric one and then mapping it onto all equivalent positions in a unit cell and the remainder of the strip as described above (see the first example in Fig. 2.42(iii)). Then T_2 is applied at unit intervals. An alternative version of type (iii) may be constructed by replacing an edge joining a two-fold centre of rotation to the boundary of the strip in addition to the previous alteration. This edge is mapped onto all equivalent positions down the centre of the strip and is used to replace the bottom edge (as shown in the second example of Fig. 2.42(iii)). The top straight edge is removed and then T_2 is applied at unit intervals. Design type (vi) may be constructed from a double strip of tiling 2 which is based in a square lattice.

2.13.3.2 *Symmetry group p*4*g*

Design type (i) is constructed by dividing each unit cell in the strip into eight isosceles triangle fundamental regions as shown in Fig. 2.43(i). For a *p*4*g* design, the diagonals represent axes of reflectional symmetry and so are fixed. At their points of intersection are centres of two-fold rotation and, in each case, half way between adjacent two-fold centres of rotation, in the longitudinal direction, is a centre of four-fold rotational symmetry. Applying a reflection and one of these four-fold rotational symmetries to design elements inside an isosceles triangle-shaped fundamental region will decorate a unit cell which, on repeated application of T_1 will complete a monotranslational design type (i). If the top straight edge of the strip is removed and then T_2 is applied at unit intervals, this forms ditranslational design type (i) and if the boundaries of the tiles are removed, this gives a type (ii) *p*4*g* design. Design type (iii) may be produced by replacing one of the boundaries of an isosceles triangle fundamental region, joining a centre of four-fold rotation with the straight edge of the strip, by an asymmetric one and then mapping it onto equivalent positions as shown in Fig. 2.43(iii). The top straight edge is removed and then T_2 is applied at unit intervals. An alternative version of type (iii) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design. Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32.

2.13.3.3 *Symmetry group p*4*m*

Design type (i) is constructed by dividing each unit cell in the strip into eight isosceles triangle fundamental regions by the method described for *p*4*g*. For a *p*4*m* design, each of these diagonal, transverse and longitudinal lines represents an axis of reflectional symmetry and so is fixed. Applying a diagonal reflectional symmetry and a four-fold rotation to design elements inside an isosceles triangle-shaped fundamental region completes a unit cell. Consecutive applications of T_1 will then generate a monotranslational design type (i). If the top straight edge of the strip is removed and then T_2 is applied at unit intervals, this forms ditranslational design type (i) and if the boundaries of the tiles are removed, this gives a type (ii) *p*4*m* design (see Fig. 2.44). Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the

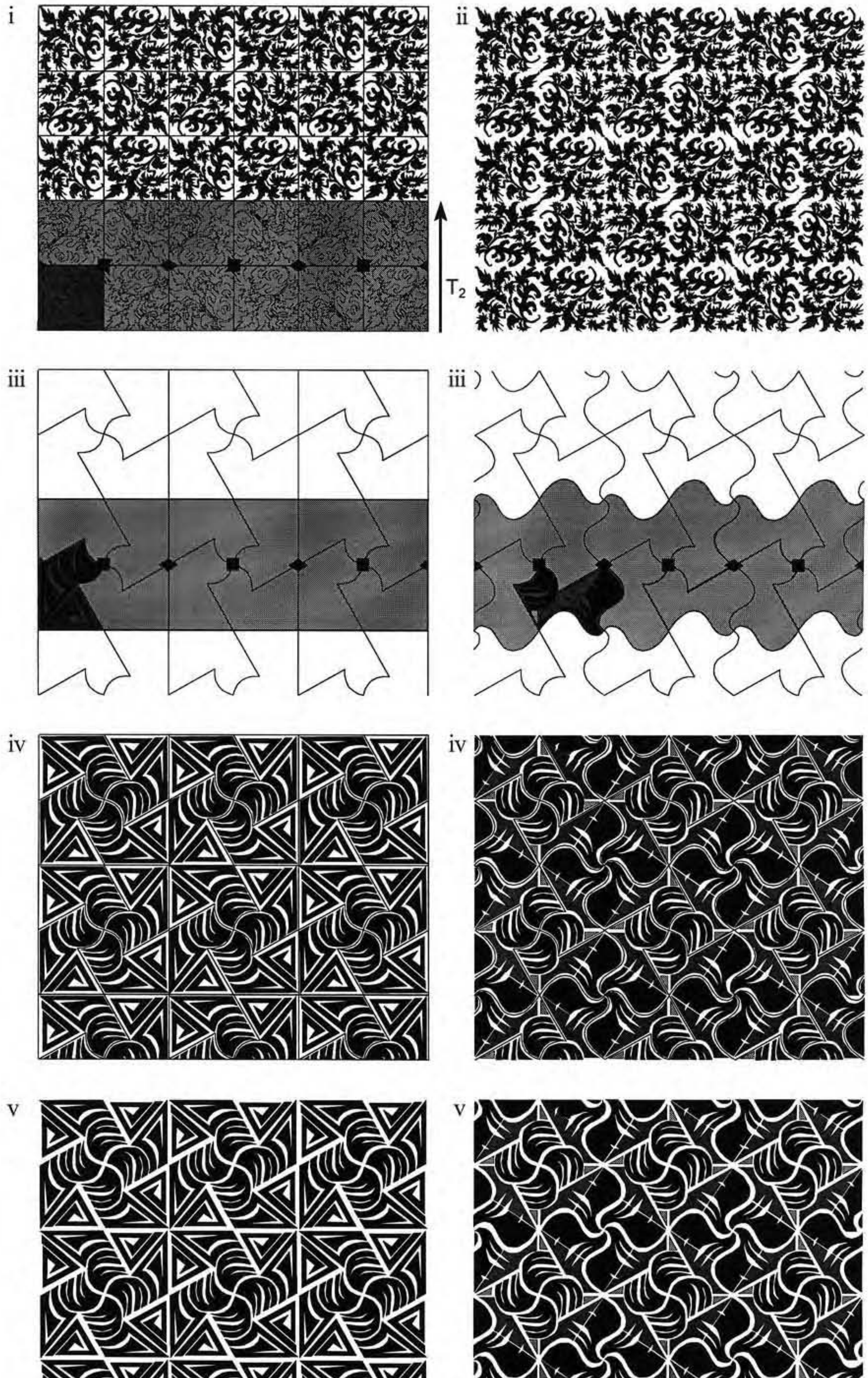


Figure 2.42 Construction of symmetry group *p*4.

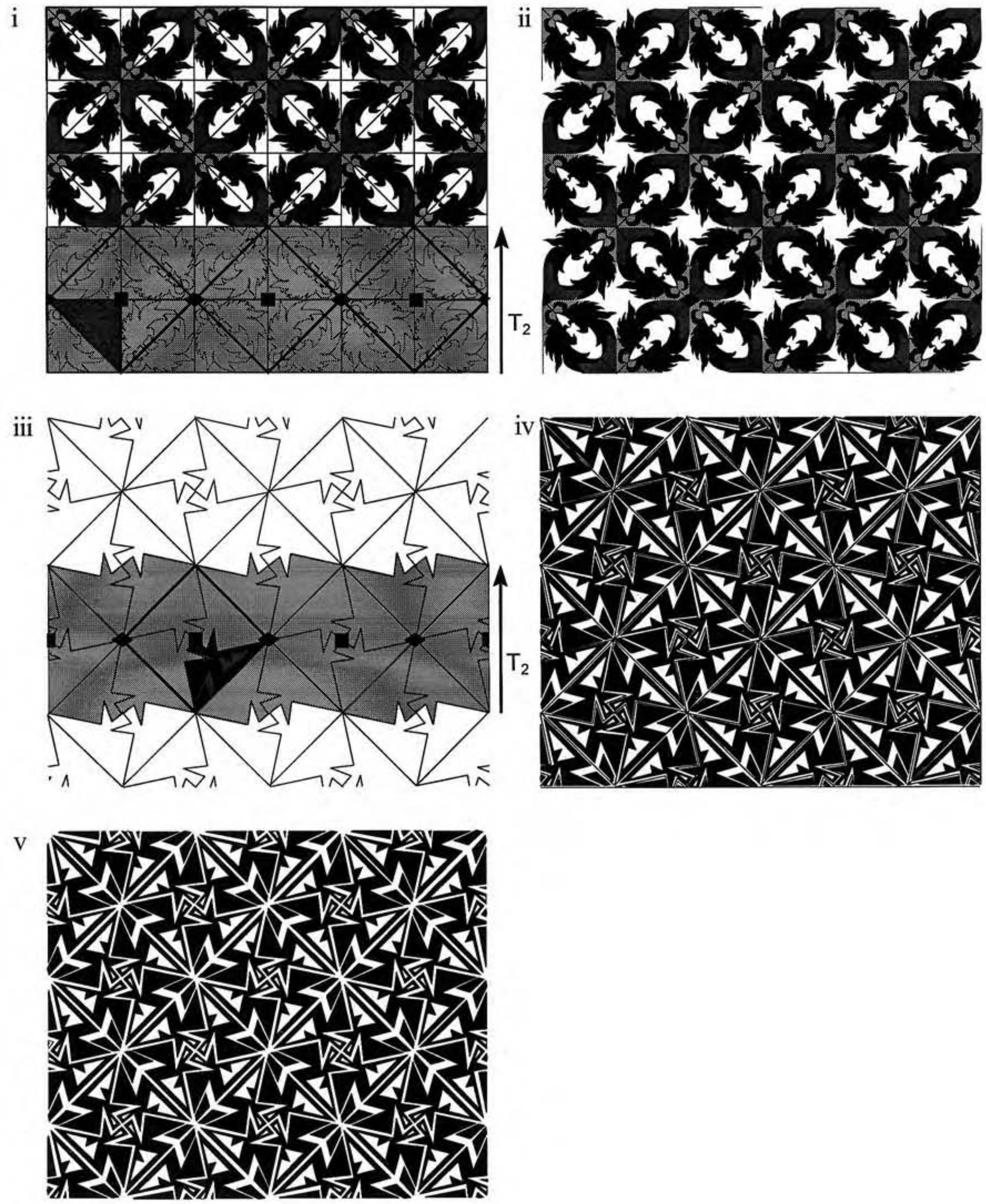


Figure 2.43 Construction of symmetry group *p*4*g*.

design. Although, as shown in the first two examples of Fig. 2.44, a straight-sided strip may be used to construct this type of pattern, in the context of screen printing it is inappropriate to dissect a motif. In the third and fourth examples of Fig. 2.44 a more suitable translation area is represented which may be consecutively translated by T_2.

2.13.4 Symmetry groups p3xy

There are three ditranslational symmetry groups of the form $p3xy$ which are abbreviated to $p3$, $p31m$ and $p3m1$. The translations used in the construction methods

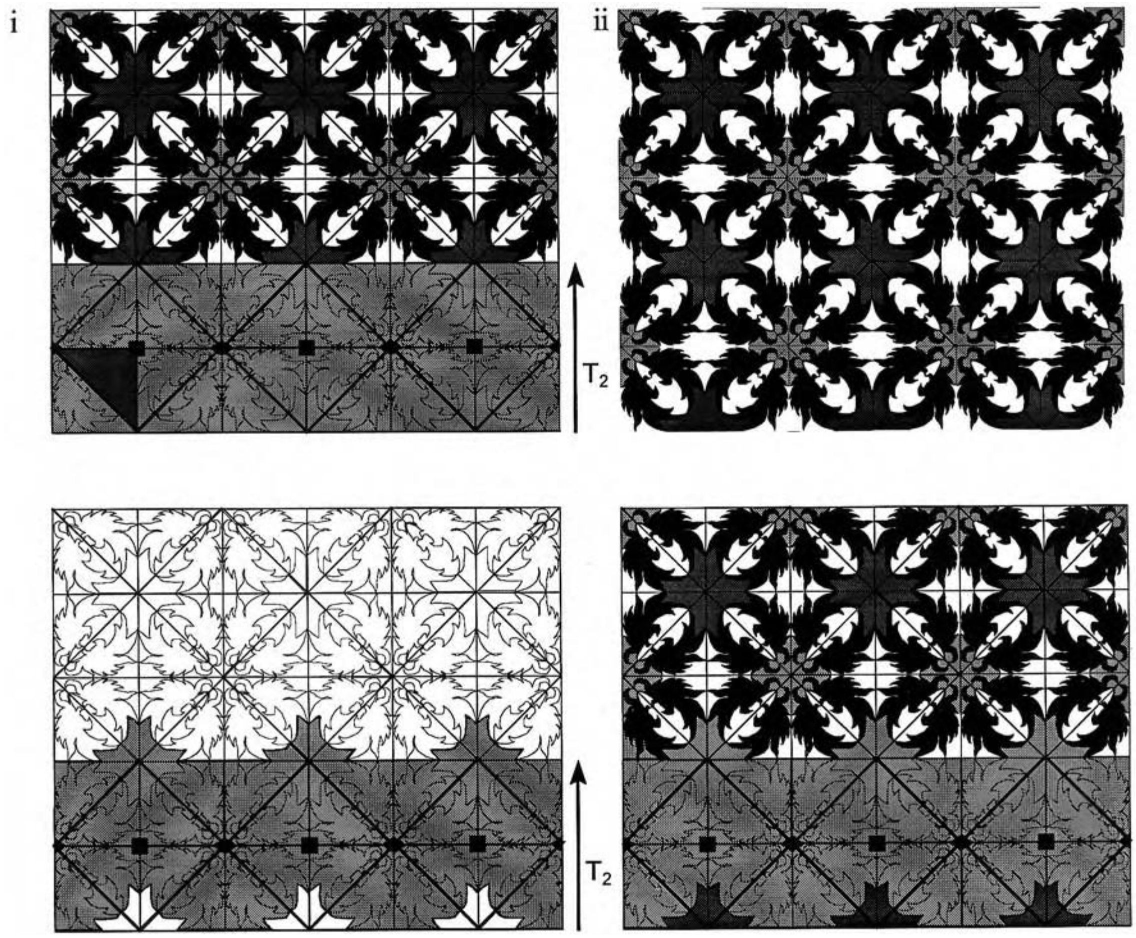


Figure 2.44 Construction of symmetry group $p4m$.

for $p3xy$ (and $p6xy$) are T_1 and T_3. A $p3xy$ ditranslational design may be constructed by repeated application of the translation, T_3, to two strips of unit cells or translation units. In cases where unit cell boundaries do not coincide with fundamental region boundaries, two strips of translation units are consecutively translated by T_3. In each of the design types discussed below, the initial monotranslational design is based on a strip of unit cells of a hexagonal lattice, width W. This is initially divided into rhombi and isosceles triangles before applying reflection M to produce a double strip, width $2W$, with the correct structure upon which to build the design. Again, as for $p4xy$ designs, symmetries occurring in the ditranslational design are used to fill out the double strip although they may not occur in the monotranslational design structure. Design elements which are mapped onto positions outside the structure of the initial 'double-strip' monotranslational design are not included since these are accounted for by translation T_3.

2.13.4.1 Symmetry group p3

Design type (i) is constructed by first dividing a strip into rhombic fundamental regions whose vertices fall on centres of three-fold rotation (as shown in Fig. 2.45). After removing the straight edges of this strip and applying reflection M to this design a new monotranslational tiling design is formed, width $2W$. Design elements are added to one rhombus which may then be mapped onto the remaining complete ones in the shaded area by applying the three-fold rotational symmetries which occur within the edges of the double strip. By applying one set of three-fold rotational symmetries which occurs at a perpendicular distance

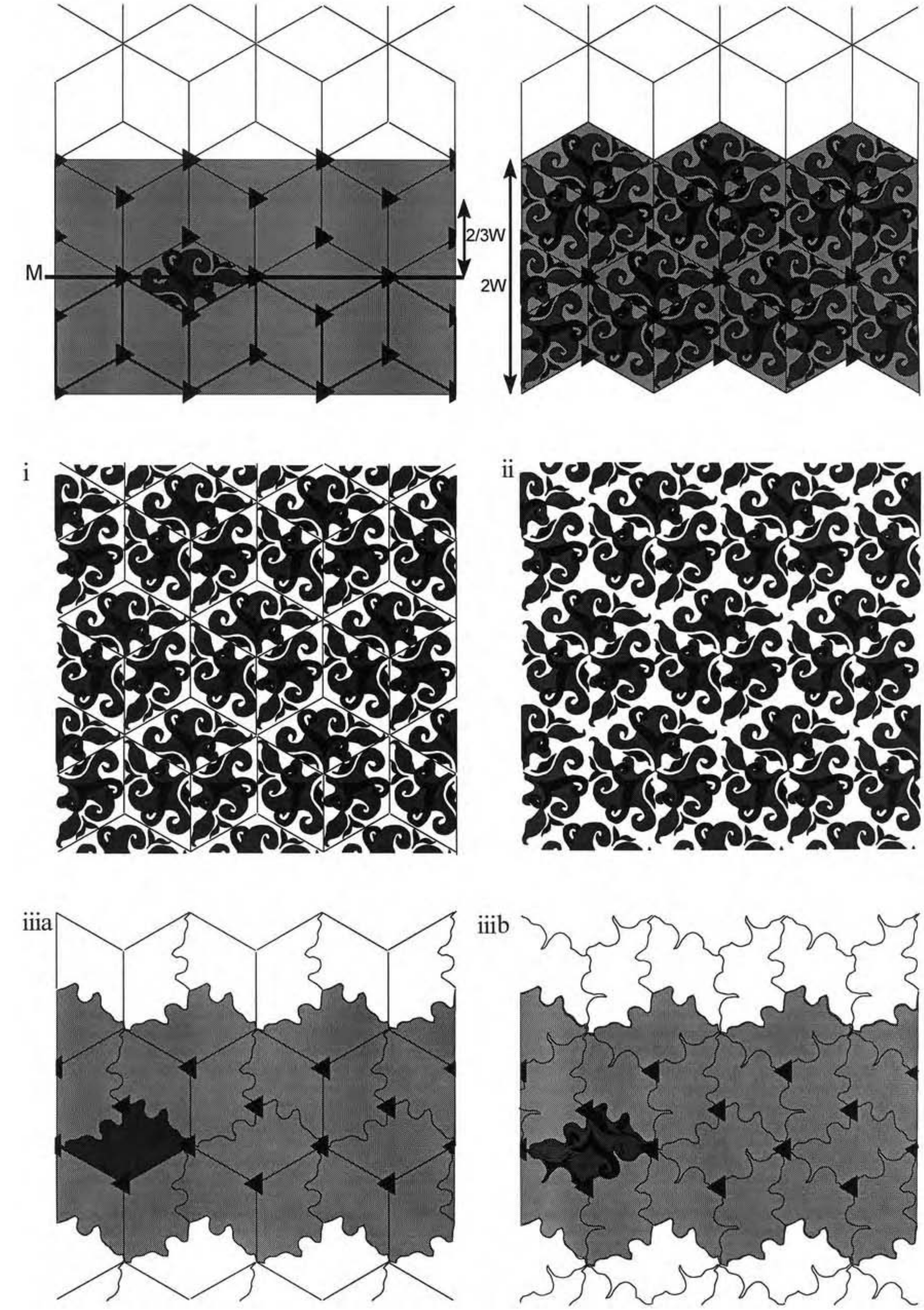


Figure 2.45 Construction of symmetry group $p3$.

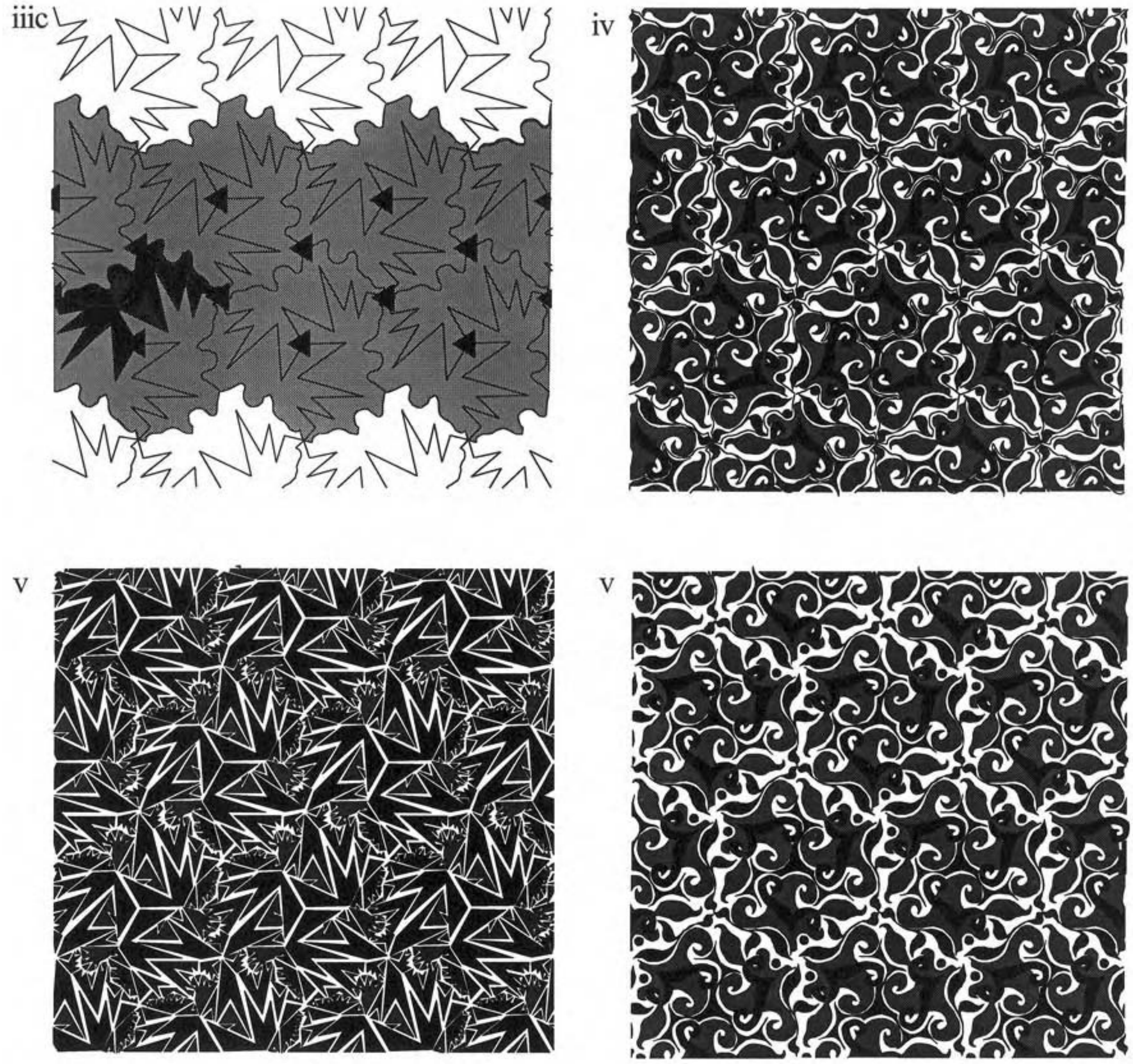


Figure 2.45 (cont.)

$2/3W$ from the longitudinal axis of the double strip, adds a line of design units not contained within the straight-edged double strip. The double strip of hexagonal translational units is then consecutively translated by T_3 to form design type (i) (as shown in the first example in Fig. 2.45). Design type (ii) is constructed by removing the rhombic fundamental region boundaries. There are two possibilities for tiling design type (iii). If one edge of a fundamental region is replaced by an asymmetric one and then mapped onto all equivalent positions in the double strip, there still remains another set of edges forming a hexagonal structure (see Fig. 2.45iii(a)). One of these edges may also be exchanged for an asymmetric one and mapped onto all equivalent positions as shown in Fig. 2.45(iii(b)) and (iii(c)). The strip is then translated by consecutive applications of T_3. Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32.

2.13.4.2 Symmetry group $p31m$

Design type (i) is constructed by first dividing a strip into rhombi as described above and then bisecting them into fundamental regions by adding a long diagonal to each one (as shown in the first example in Fig. 2.46). These diagonals form a tiling of equilateral triangles all of whose edges fall on axes of reflectional

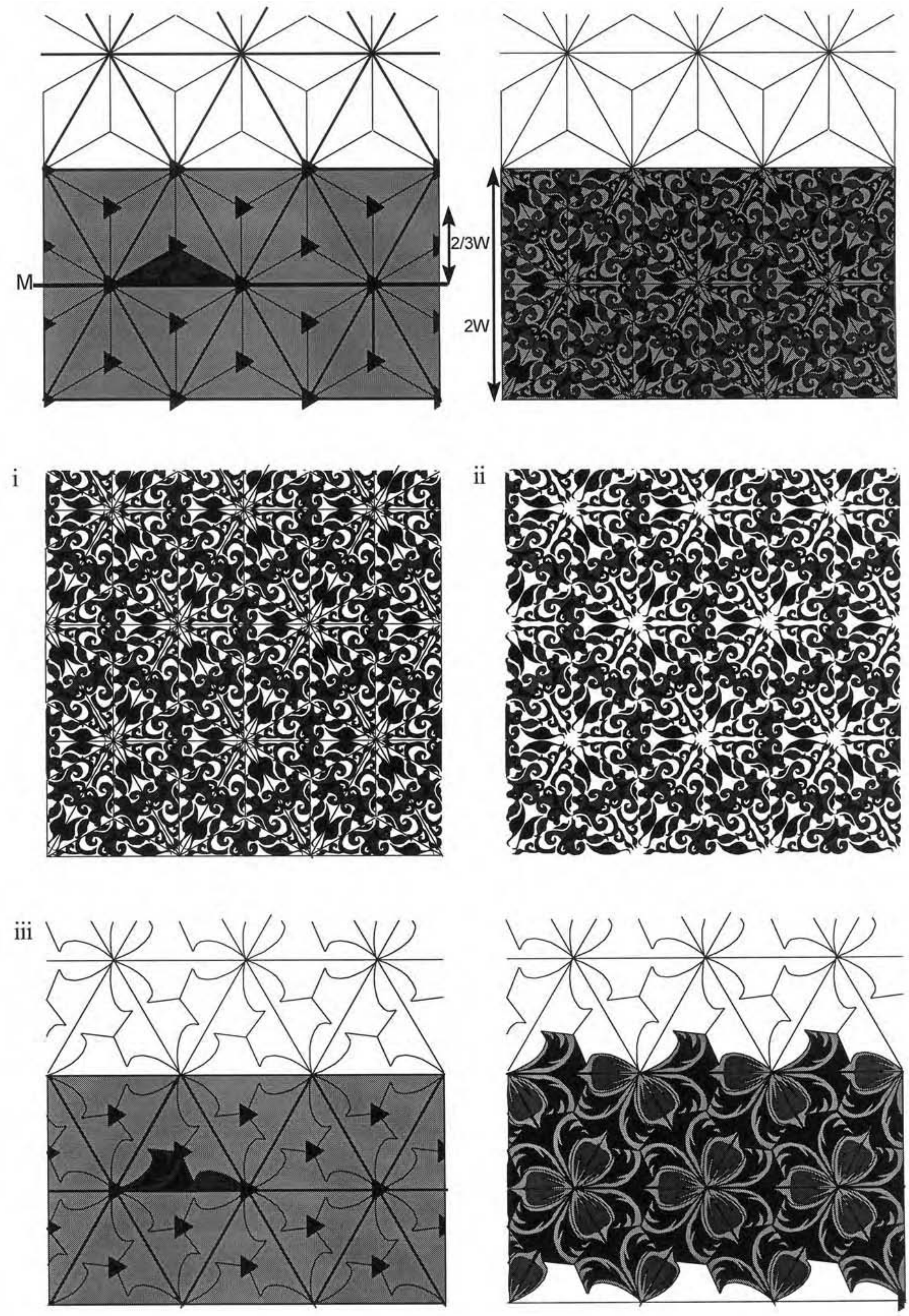
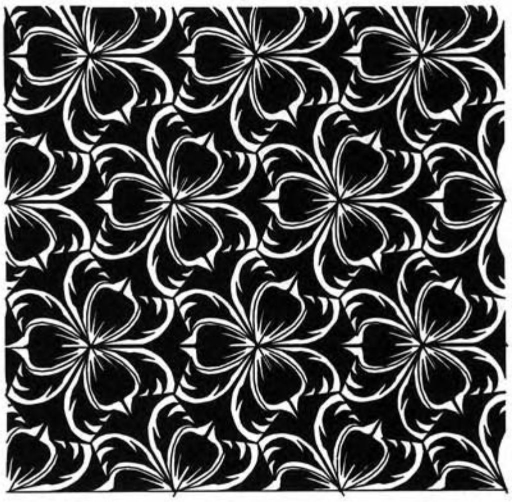


Figure 2.46 Construction of symmetry group *p*31*m*.

iv



v

Figure 2.46 (cont.)

symmetry and so are fixed. By applying reflection M to this strip, a new mono-translational tiling design is formed, width $2W$. Design elements inside one triangle may be mapped onto the remaining ones in the double strip by first applying a three-fold rotation and a reflectional symmetry to complete a unit cell; then by applying T_1 at unit intervals to complete a single strip; finally by applying a reflection M. One outside edge of the double strip is removed before consecutively translating it by T_3 to form design type (i). Design type (ii) is constructed by removing the triangular fundamental region boundaries as shown in Fig. 2.46(ii).

Construction of design type (iii), where only a selection of the edges of the fundamental regions interlock, is possible for a $p31m$ design since although some edges fall on reflection axes and so are fixed, others do not. Type (iii) may be constructed by replacing an edge, joining two centres of three-fold rotation positioned at the centre and vertex of an equilateral triangle, by an asymmetric one; mapping it to all equivalent positions in the double strip; removing one exterior edge of the double strip and then translating the strip by consecutive applications of T_3 (see Fig. 2.46(iii)). Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32. The second example given in Fig. 2.46(iii) illustrates a more suitable translation strip, for that particular design shown, which avoids dissecting motifs.

2.13.4.3 Symmetry group p3m1

Design type (i) is constructed by first dividing a strip into rhombi as described above and then bisecting them into fundamental regions by adding a short diagonal to each one. This divides the strip into equilateral triangles whose sides all fall on axes of reflectional symmetry and so are fixed (see Fig. 2.47). Removing the straight edges of the strip and applying reflection M to this design forms a mono-translational tiling design, width $2W$. Design elements inside one triangle may be mapped onto the remaining ones inside a double strip of hexagonal translation units as shown in the second example in Fig. 2.47. Consecutive applications of translation T_3 are then applied to it to form design type (i). Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design.

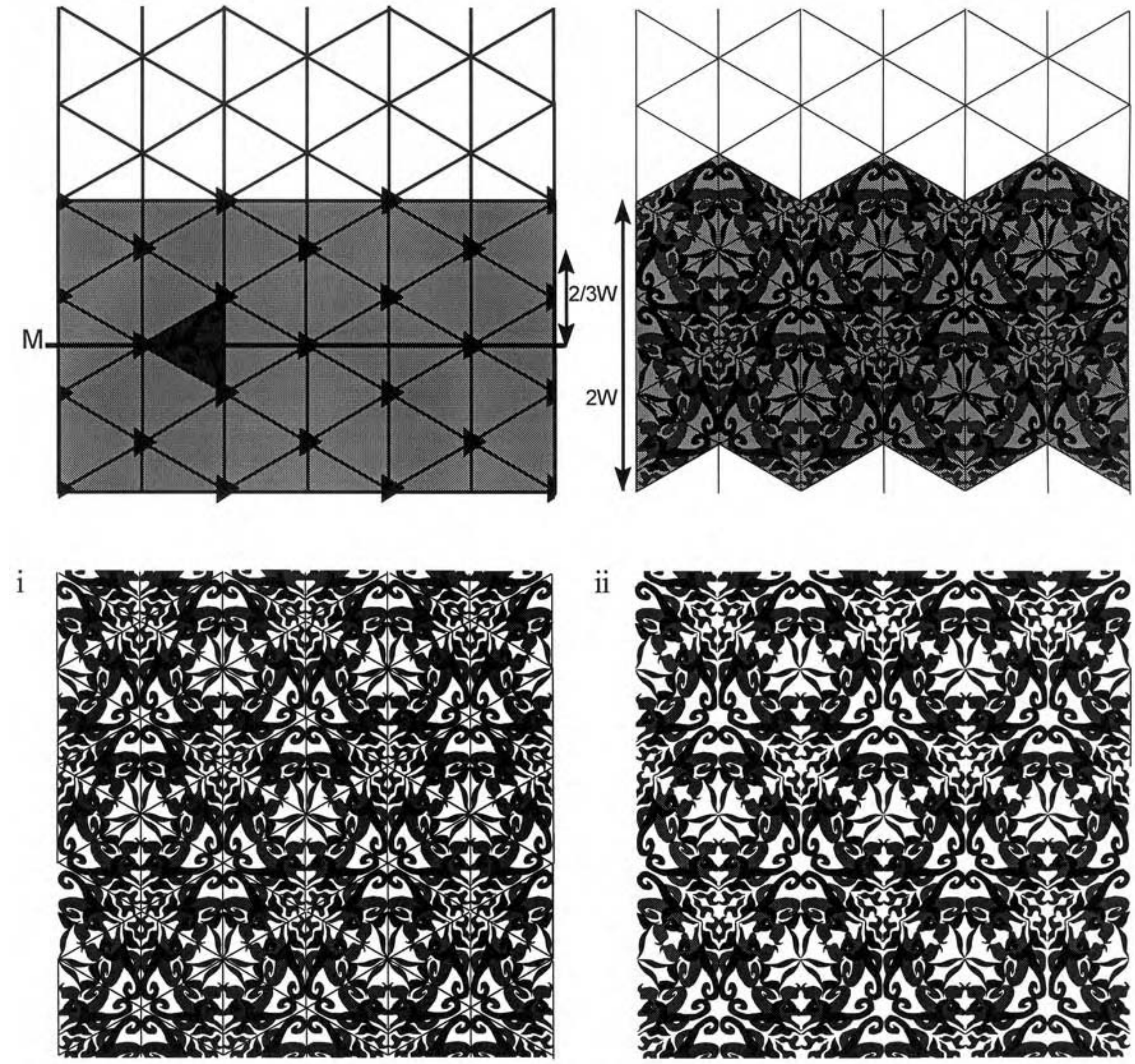


Figure 2.47 Construction of symmetry group *p3m1*.

2.13.5 Symmetry groups p6xy

There are two ditranslational symmetry groups of the form *p6xy* which are abbreviated to *p6* and *p6m*. A *p6xy* ditranslational design may be constructed by repeated application of the translation, T_3, to two strips of unit cells or translation units. In each of the design types discussed below, like *p3xy* designs, the initial monotranslational design is based on a strip of unit cells of a hexagonal lattice, width W.

2.13.5.1 Symmetry group p6

Design type (i) is constructed by first dividing a strip, width W, into kite-shaped fundamental regions whose vertices fall on centres of two-, three- and six-fold rotation (as shown in the first example in Fig. 2.48). A reflection M applied to this design forms a new monotranslational tiling design of width $2W$. Design elements inside one kite shape may be mapped onto the remaining ones by applying the two-, three- and six-fold rotational symmetries which occur within the double strips outside edges. After removing one outside edge of the double strip it is then consecutively translated by T_3 to form design type (i). There are two methods of constructing design type (iii), where only a selection of the edges of the funda-

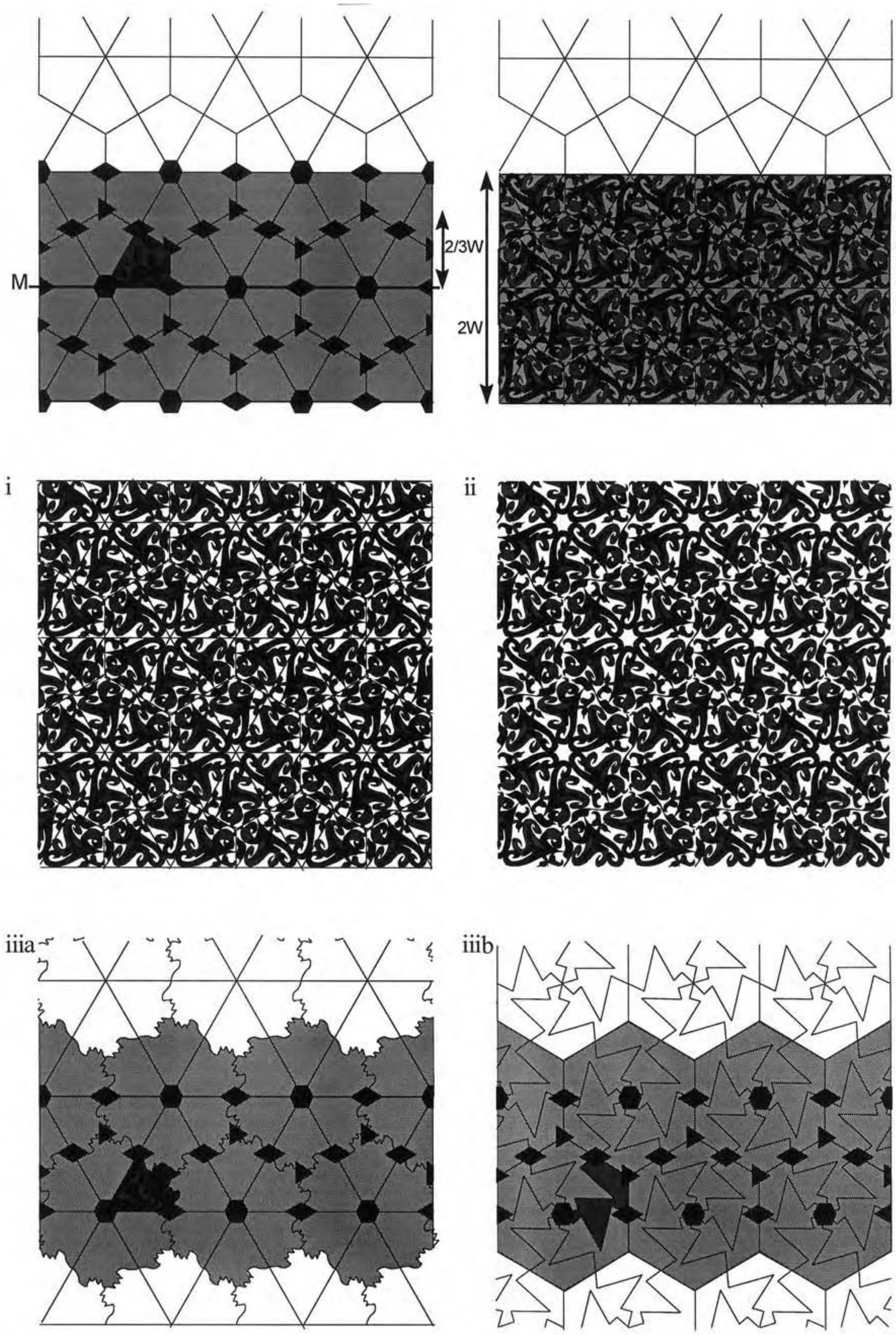


Figure 2.48 Construction of symmetry group $p6$.

mental regions interlock. Either the straight lines joining centres of two- and six-fold rotation remain fixed (which forms an equilateral triangular tiling) and lines joining centres of two- and three-fold rotation are exchanged or vice versa (which forms a hexagonal tiling). Examples of tiling designs resulting from these alterations are given in Fig. 2.48(iii(a)) and (iii(b)). Alternatively both of these two sets of edges may be replaced (see Fig. 2.48iii(c)). Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32.

2.13.5.2 Symmetry group p6m

Design type (i) is constructed by first dividing a strip into kite-shaped *p*6 fundamental regions, as described above, and then bisecting each by adding a long diagonal. This divides the strip into right-angled triangles whose sides all fall on axes of reflectional symmetry and so are fixed. A reflection M is applied to this design to form a new monotranslational tiling design, width $2W$ (see Fig. 2.49, construction of type (ii) in two stages). Design elements inside one triangle may be mapped onto the remaining ones by applying reflectional symmetries which occur within the edges of the double strip. One outside edge of the double strip is removed before consecutively translating it by T_3 to form design type (i). Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design.

2.14 Summary

The classification system discussed in this chapter is applicable to all forms of regularly repeating finite, monotranslational and ditranslational designs. It begins with explanations of the fundamental concepts which form the basis of subsequent classification systems throughout the remainder of this book. Finite, monotranslational and ditranslational designs are classified and constructed by symmetry group and extensively illustrated by schematic and more decorative forms of illustrations.

Because there are such a vast number of possible design characteristics in one symmetry group, only a selection of construction methods have been explained in detail. For example, throughout each of the ditranslational construction techniques discussed in the previous sections, the emphasis has been placed on the initial structure being based upon a tiling of specific fundamental region boundaries. This criteria restricts, to a certain extent, the interlocking relationship of the design units. No particular attention has been paid to the symmetrical properties of the individual design units or motifs within the design structure either. These characteristics are discussed in more detail in Chapters 3, 4 and 5. The formation of a tiling of fundamental regions, as shown in design type (iii), will be used as a basis for some of the construction methods discussed in these following chapters.

Throughout the descriptions of ditranslational design construction methods, reference has been made to screen printing. The initial monotranslational design, width W, or width $2W$ where specified (or an integral number of these widths) may be treated as the translation strip which is incorporated onto the length of the screen. (To print the design the screen is then translated at unit intervals perpendicular to the strip.) Where motifs are split along fundamental region edges, as shown for symmetry group *p*4*m*, a more suitable translation strip may be devised. For some symmetry groups, such as *pm*, the construction techniques have been discussed with the reflection axes having a particular orientation in relation to the warp or weft (or length and width) of the fabric (or paper) in connection with screen printing. However, should these axes be required to be perpendicular to the ones discussed it is only necessary to take a translation strip with longitudinal axis perpendicular to the ones illustrated in the construction examples.

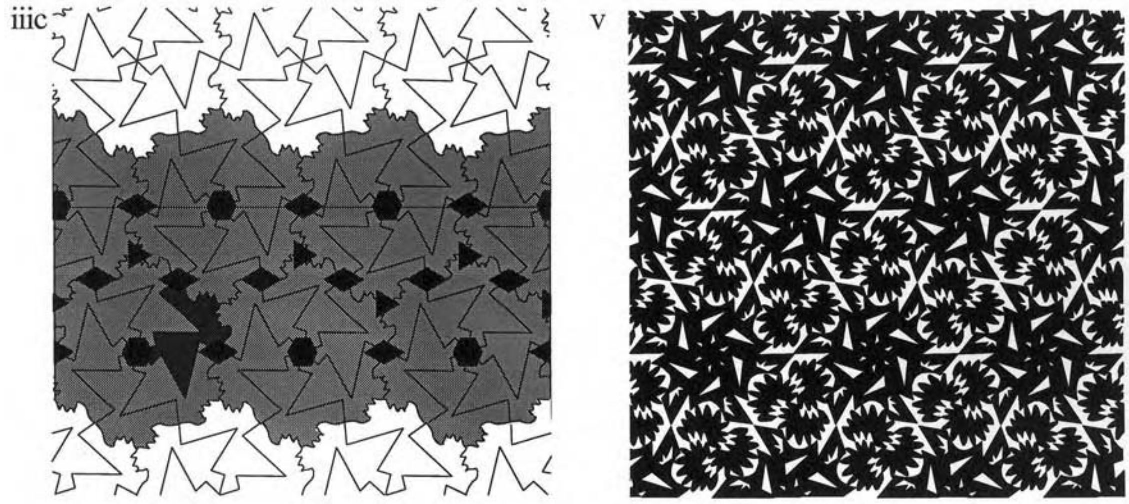


Figure 2.48 (cont.)

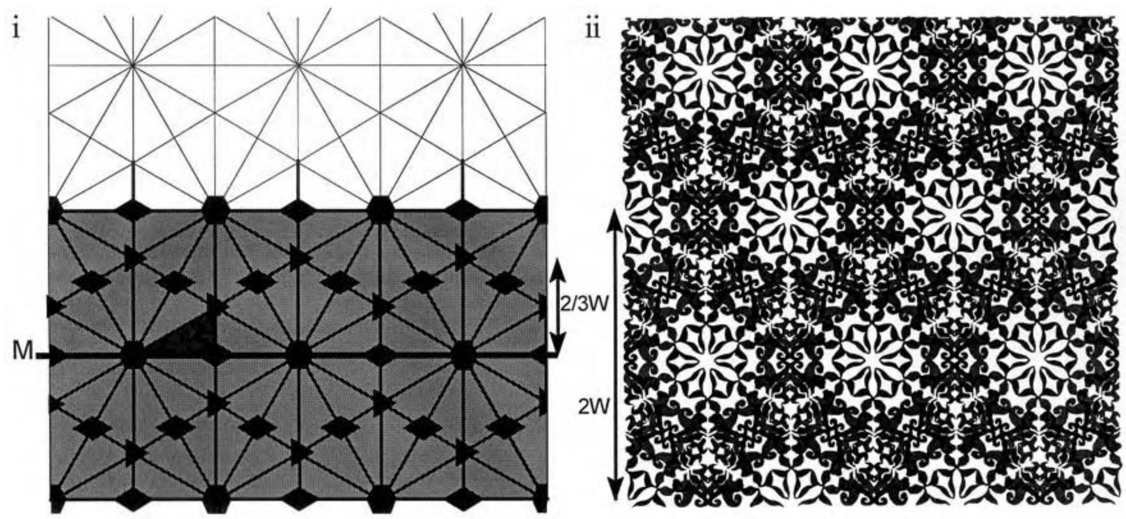


Figure 2.49 Construction of symmetry group *p*6*m*.

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