3

Classification of designs by symmetry group and design unit

3.1 Introduction

As described in Chapter 2, the variety of design types contained within one symmetry group is quite extensive. Consequently, to differentiate between these types requires further processes of investigation and categorisation. Bunce[1] also observed that there are restrictions in the symmetry group classification system and so developed a pattern analysis scheme of her own. In comparing the two systems with reference to her own scheme, she stated that it differs from that of symmetrical pattern classification which defines 17 classes of all-over pattern, but takes no account of the orientation of the design unit. In symmetrical classification, patterns constructed from the same basic unit and having the same notation may look very different according to the positions of reflection and rotation.[1] This point is briefly illustrated in the examples in Fig. 3.1 in which each of the six patterns may be classed in symmetry group *p*2. However, as may be observed, since a *p*2 design may be constructed on any of the five types of parallelogram lattice, the positioning of two-fold centres of rotation will vary according to each structure and thus so will the resulting design effect. The positioning and orientation of the motifs in relation to the points of symmetry also affect the appearance of the design as can be seen from the illustrations in Fig. 3.1.

The classification system devised in this chapter does concern lattice structures but the primary focus is on the symmetrical properties of the design unit inside a fundamental region. With respect to design analysis and classification, this particular aspect of a design's characteristics is generally disregarded. It is often assumed that the design unit inside each fundamental region (particularly for monotranslational symmetry groups *p*111 and *p*1*a*1 and ditranslational symmetry groups *p*1 and *pg*) is asymmetric. For example, this suggestion is made by Hann and Thomson[2] in their publication *The Geometry of Regularly Repeating Patterns* in which they comment that the most elementary border class is translation class *p*111 which is generated by translation of an asymmetrical (class *c*1) motif by a specified distance along an imaginary line known as the translation axis. Also, in a similar vein, they discuss the generation of symmetry groups *p*1*a*1, *p*1 and *pg* by applying the relevant symmetry operations to *c*1 motifs.[2] However, the motif (or in this context the 'design unit') need not necessarily be classed in this symmetry group, that is, it need not be asymmetric. The possible symmetrical properties of the design unit, which are discussed in detail later, are dependent on their positioning in relation to the unit cell and on the symmetries of the underlying design structure.

Recall that a fundamental region is any smallest area of the plane to which the generating symmetries may be applied to complete the design. In cases where the region is not bounded entirely by reflection axes and/or the exterior boundaries of the whole design, this region may be represented by a variety of different shapes. However, in the following analysis it is chosen to fit the additional criteria of containing a design unit with the highest possible order of symmetry. For example, Fig. 3.2(a) illustrates a ditranslational design, in symmetry group *p*1, based on a rectangular lattice. (Four identical points nearest to each other are chosen to establish the lattice structure.) In this pattern both A and B represent fundamental regions but A contains a design unit with the highest order of symmetry.

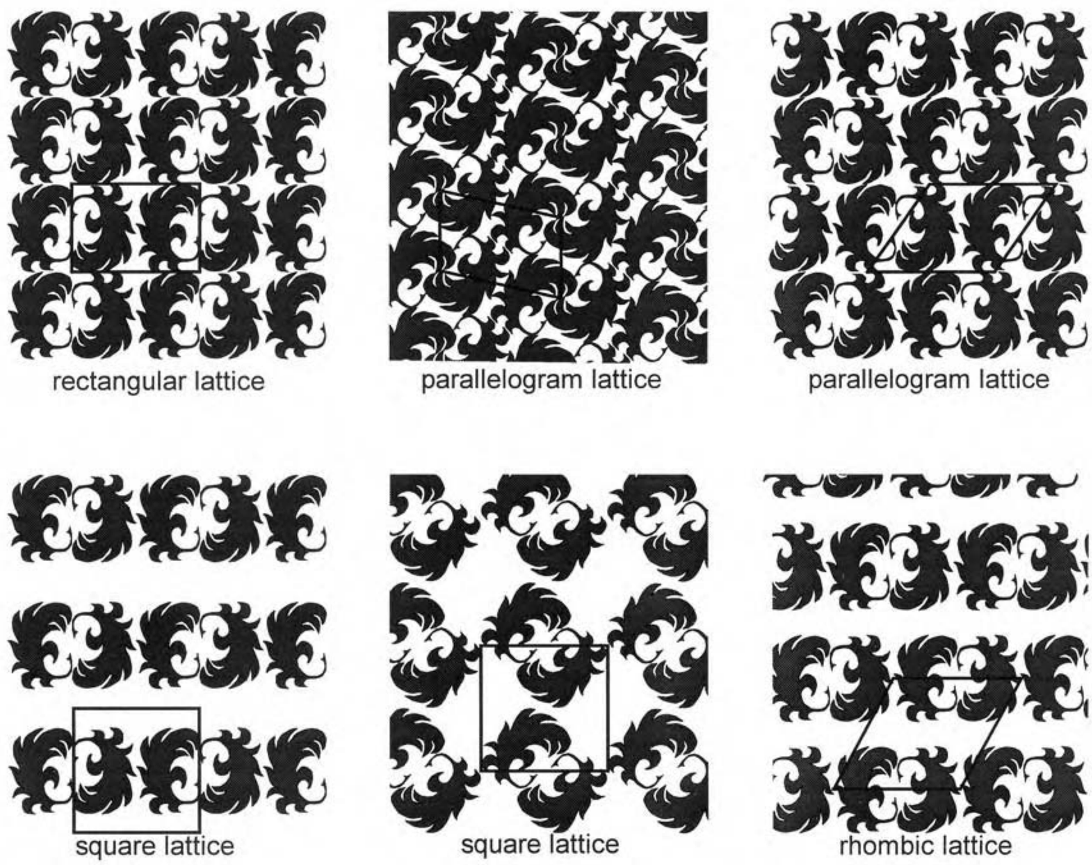


Figure 3.1 Illustrations of the effects of different structures on a symmetry group.

Although this illustrates that the construction of a $p1$ design composed of symmetric design units is possible, the initial orientation of the design unit inside a fundamental region is critical. In the case of translational designs the potential symmetrical characteristics of the design unit are dependent not only on the lattice structure of the design but also the positioning of the design units relative to the symmetries and boundaries of the unit cell. For example, reorientating the design unit and constructing the previous design on a hexagonal, rhombic, rectangular or square lattice could quite easily produce a design of a different symmetry group altogether (as shown in Fig. 3.2b(i–iv)). In these cases, had the design unit been positioned appropriately, a $p1$ design could still have been produced. Similarly, a reflectionally symmetric design unit contained within a fundamental region of a finite design must also be carefully positioned so as not to induce additional symmetries into the design structure, as shown in Fig. 3.2(c).

From one aspect, the types of design in this chapter are more intriguing than those obtained from more conventional symmetry group construction procedures. This may be due to the eye initially perceiving symmetries in the design which are normally associated with its underlying structure but which on closer observation do not influence the symmetry group classification of the design. For example, designs comprised of snowflake motifs would normally be found in patterns with a high order of symmetry and most probably of symmetry group $p6m$. This is due to the fact that this arrangement produces the most balanced and ordered appearance and the most intuitive and simple method of locking together motifs of this shape. However, with a slight tilt and adjustment of the snowflake motifs type of pattern within this, the order of a 'perfectly symmetrical' $p6m$ pattern may be reduced. Bier comments on this by saying that patterns with imperfections continually fascinate us because they confound and perplex us as they delight.[3]

The construction methods given later in this chapter account for these variations in design unit orientation by discussing the lattice types (for translational

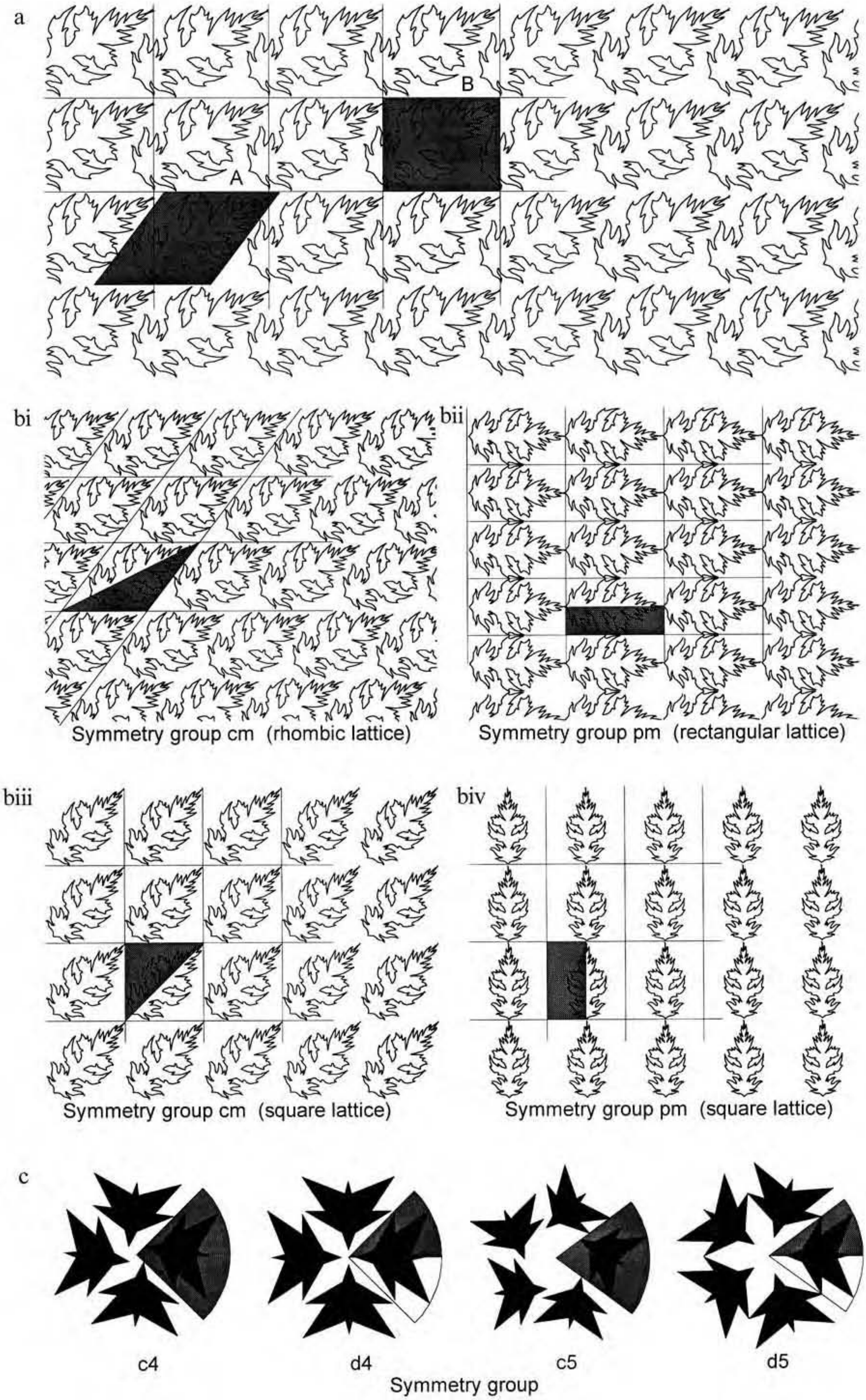


Figure 3.2 Examples illustrating (a), (b) symmetric and (c) asymmetric design units.

Table 3.1 Notation for symmetry groups of design structure and design unit

		Symmetry group of design structure and design unit	
Design class	Symmetry group of design structure	Cyclic design unit	Dihedral design unit
Finite symetry group	*cn* *dn*	*cn*(*cN*) *dn*(*cN*)	*cn*(*dN*) *dn*(*dN*)
Monotranslational symmetry group	*pyxn*	*pyxn*(*cN*)	*pyxn*(*dN*)
Ditranslational symmetry group	*pnxy*	*pnxy*(*cN*)	*pnxy*(*dN*)

N represents the number of reflection axes and/or order of rotational symmetry of the design unit.

designs), design units' symmetrical characteristics and their positioning in relation to the unit cell or fundamental region boundaries for each of the three finite, seven monotranslational and 17 ditranslational symmetry groups.

3.2 Notation

The notation for this new classification scheme has been devised by refining the symmetry group notation by including an additional bracketed finite symmetry group. The initial symbol indicates the symmetry group of the design structure and the bracketed group indicates the finite symmetry group of a design unit with the highest order of symmetry inside a fundamental region. For example, in Fig. 3.2(a) the symmetry group of the overall structure is *p*1 which gives the initial symbol. A fundamental region containing a design unit with the highest order of symmetry is marked A and the finite symmetry group of this design unit is *d*1. This gives the second symbol which is then enclosed in brackets. Amalgamating the two gives the symmetry group of the design structure and design unit *p*1(*d*1). Following this analogy, each of the symmetry groups of finite, monotranslational and ditranslational design structures may be divided into two subgroups according to the highest order of symmetry of the cyclic or dihedral group of a design unit inside a fundamental region. The form of notation for each subgroup is given in Table 3.1.

3.3 Finite designs

In Chapter 2 finite designs were divided into the two symmetry groups *cn* and *dn* depending on their dihedral and/or cyclic properties. By the previous analogy, each of these groups may be subdivided into two subgroups: *cn*(*cN*), *cn*(*dN*), *dn*(*cN*) and *dn*(*dN*). Figures 3.3 and 3.4 show schematic illustrations of these designs for $n = 1$ to 4 and $N = 1$ to 6. Further examples for a selection of these subgroups are given in Fig. 3.5.

3.4 Monotranslational designs

Monotranslational designs are divided into seven symmetry groups, each of the form *pyxn*. Again these may be subdivided into two subgroups: *pyxn*(*cN*) and *pyxn*(*dN*). Schematic illustrations (with *N* taking the three lowest possible values) are given for each symmetry group subgroup *pyxn*(*cN*) in Fig. 3.6 and for *pyxn*(*dN*) in Fig. 3.7. Further examples are given in Fig. 3.8.

3.5 Ditranslational designs

Ditranslational designs are divided into 17 symmetry groups, each of the form *pnxy*. These may also be subdivided into two subgroups: *pnxy*(*cN*) and *pnxy*(*dN*). Schematic illustrations of *pnxy*(*cN*) and *pnxy*(*dN*) subgroups are given for each symmetry group in Fig. 3.9 and 3.10, respectively. Further exam-

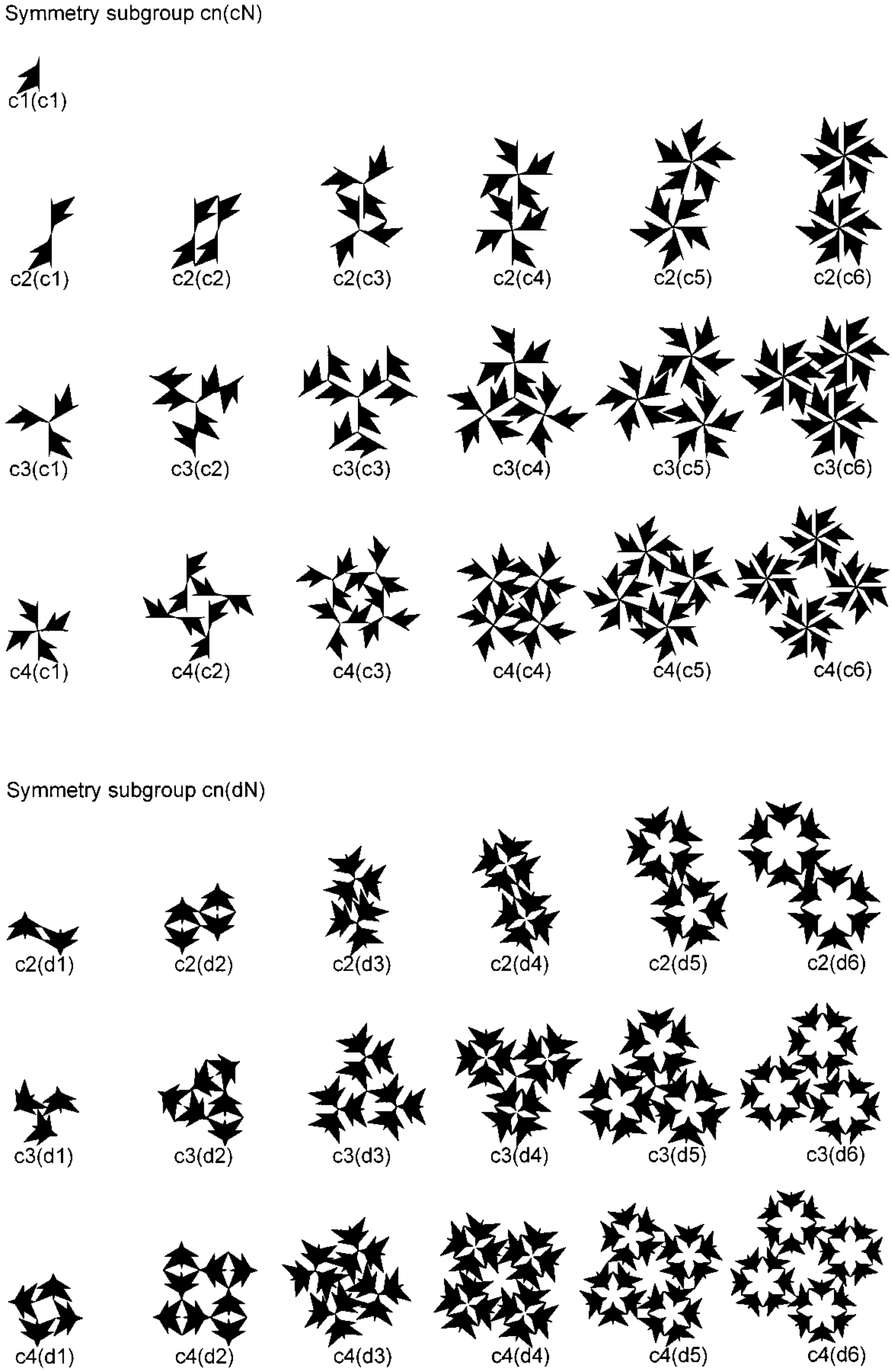


Figure 3.3 Schematic illustrations of finite design symmetry subgroups *cn*(*cN*) and *cn*(*dN*).

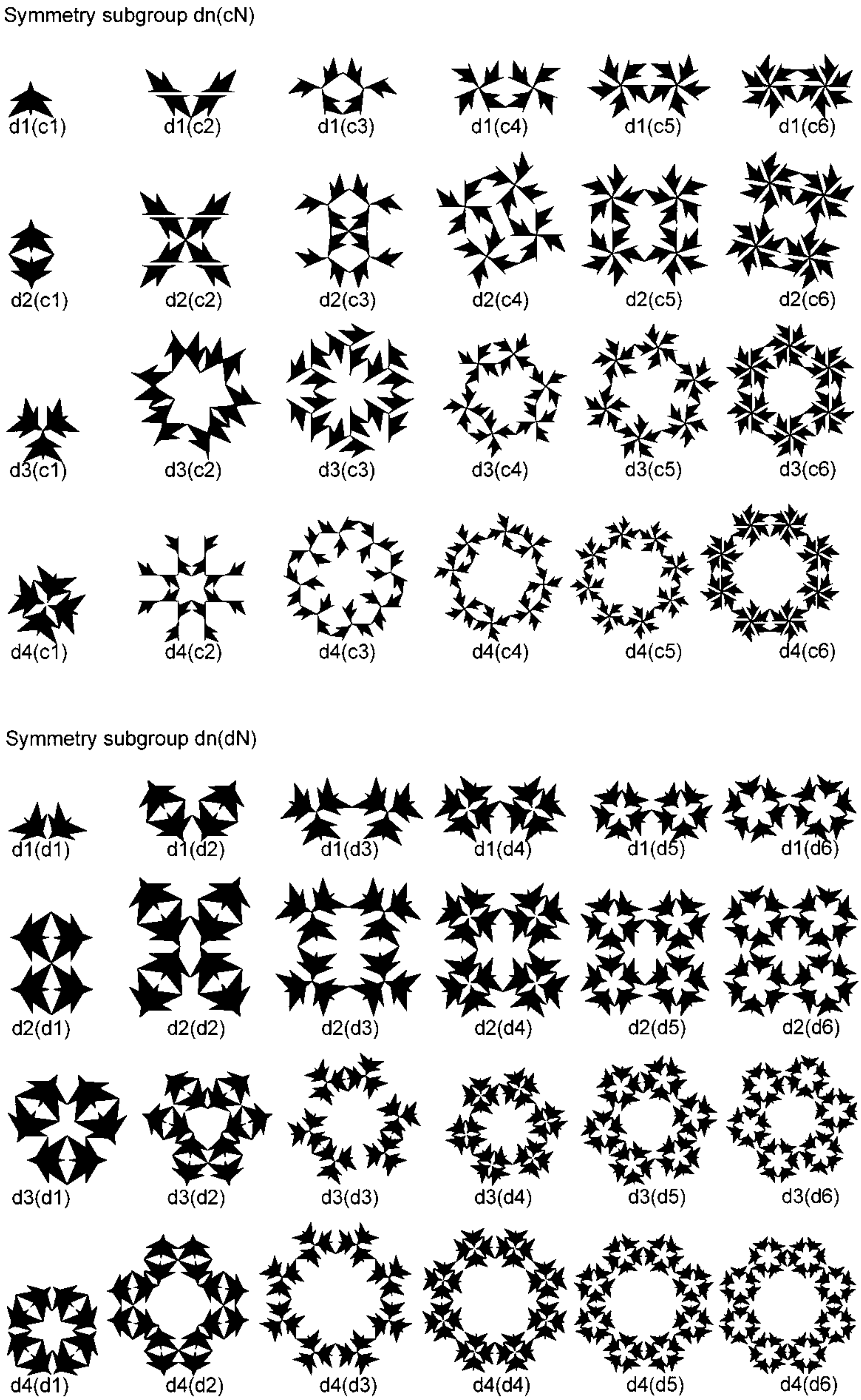


Figure 3.4 Schematic illustrations of finite design symmetry subgroups *dn*(*cN*) and *dn*(*dN*).

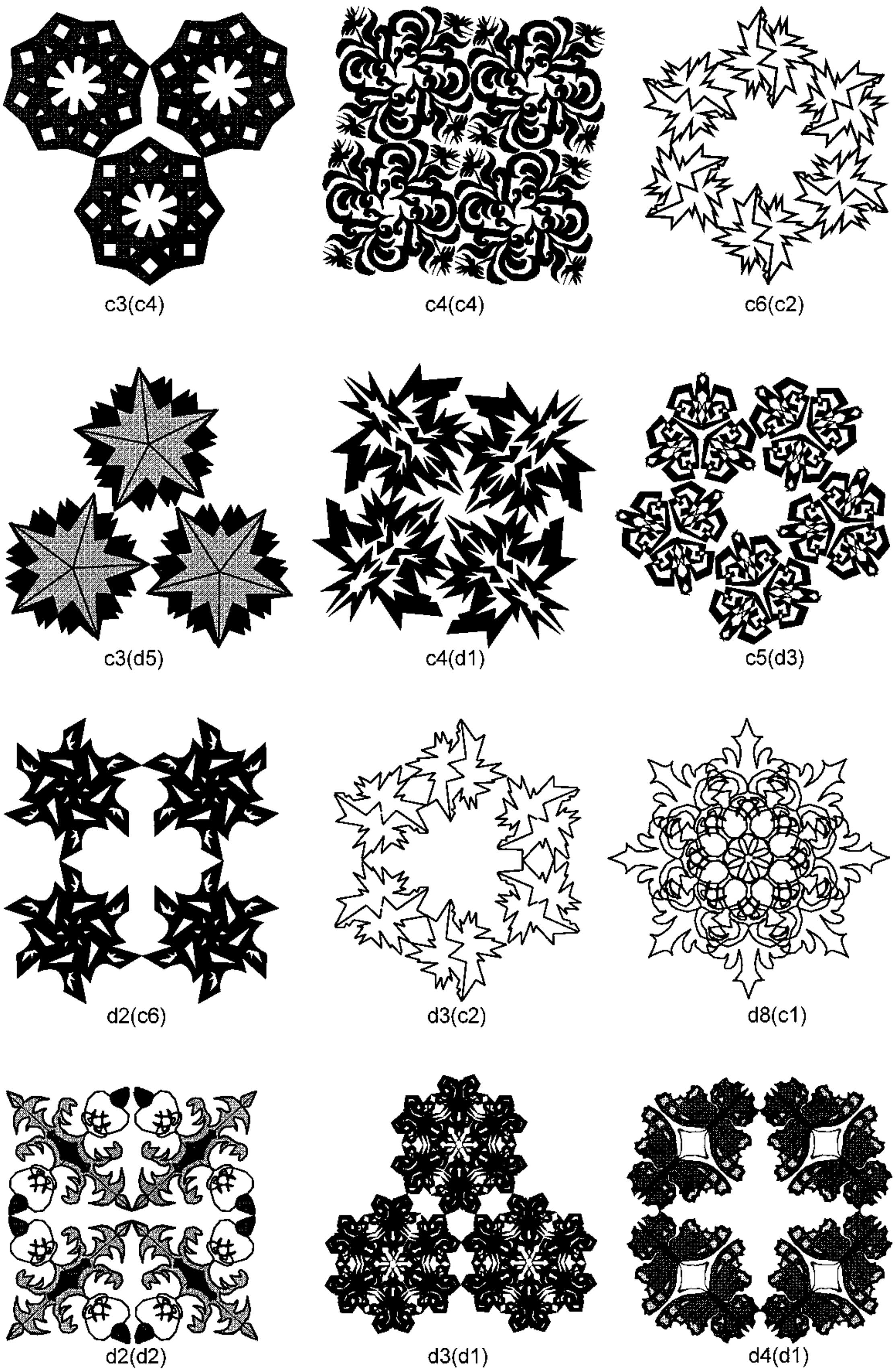


Figure 3.5 Further examples of finite design symmetry subgroups.

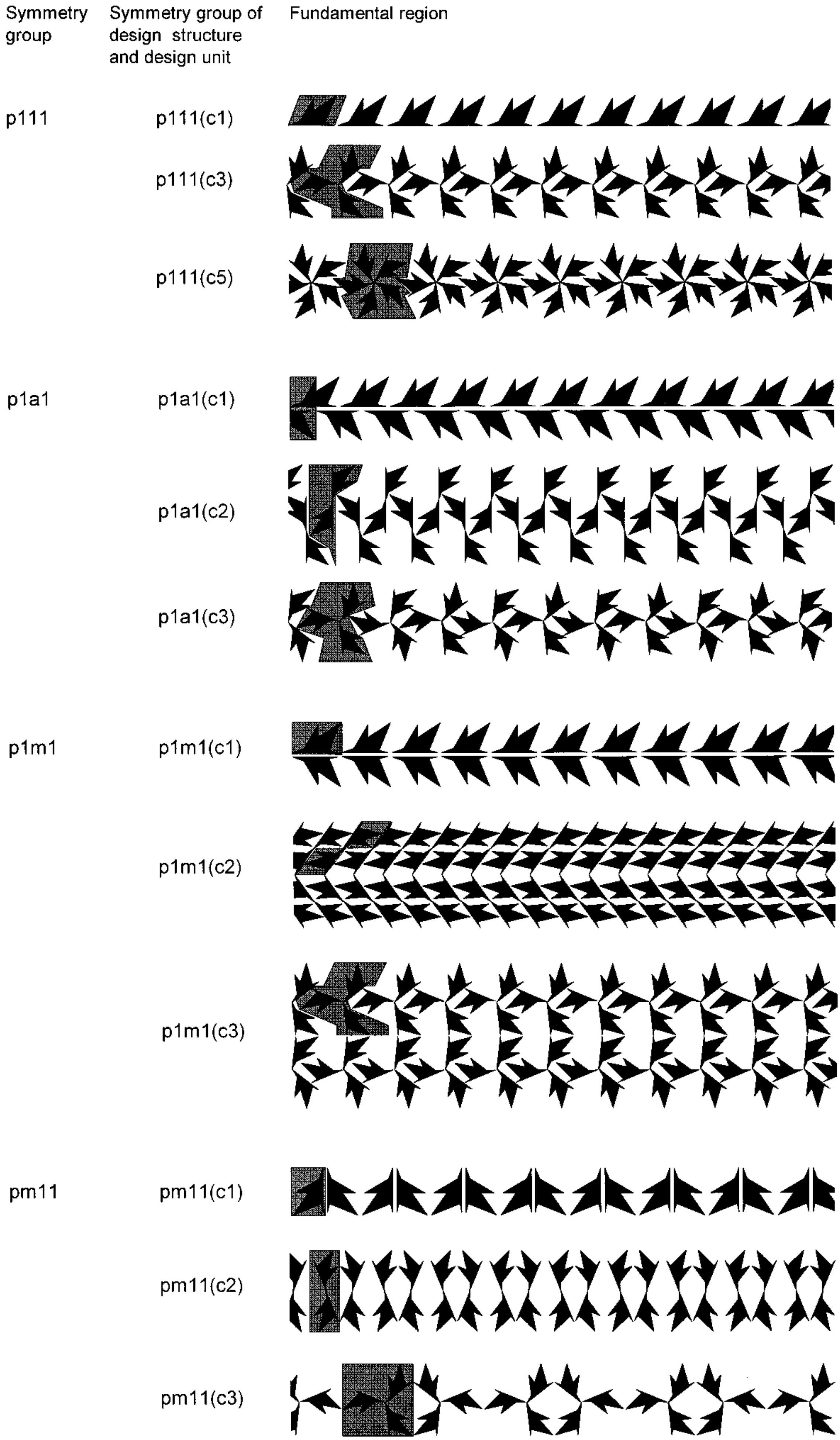


Figure 3.6 Schematic illustrations of monotranslational design symmetry subgroups *pyxn*(*cN*).

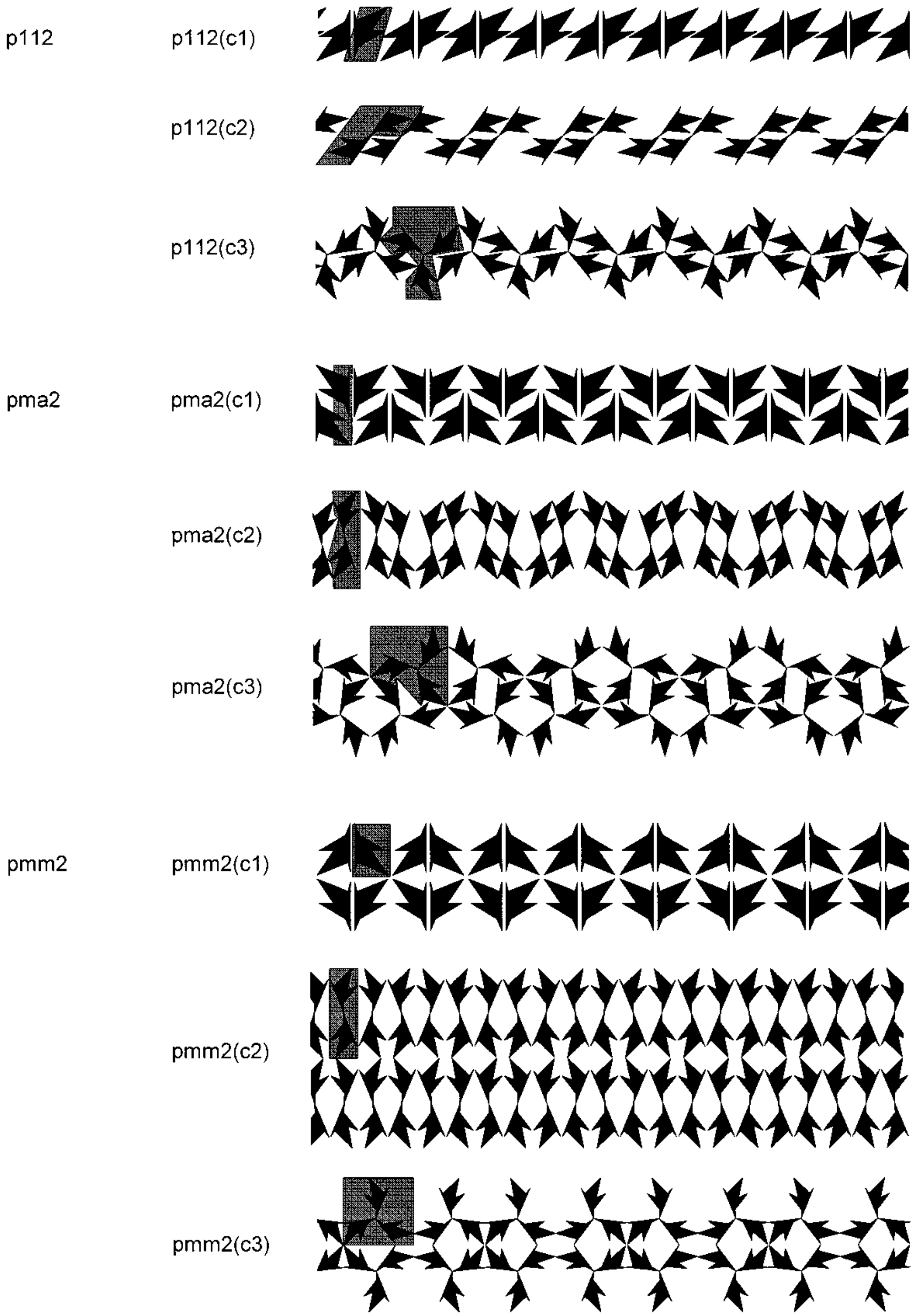


Figure 3.6 (cont.)

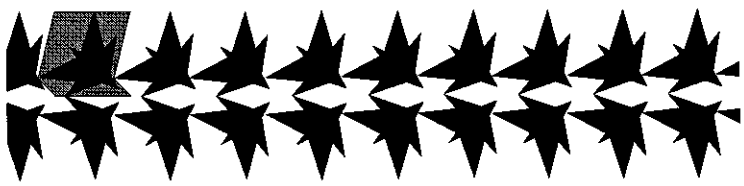
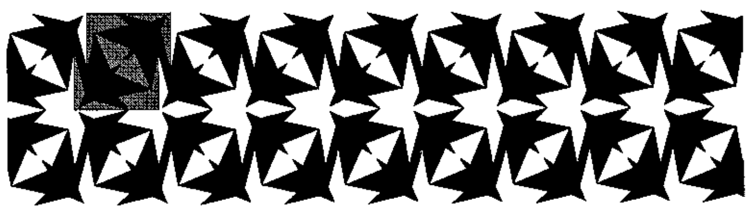
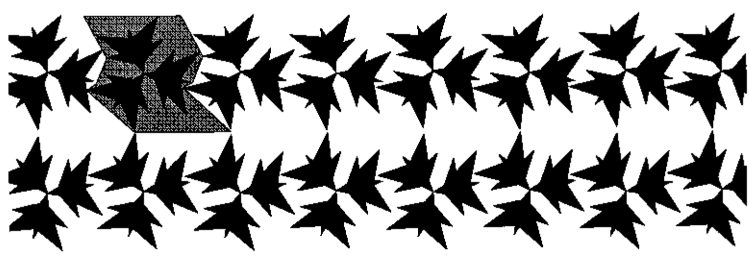
Symmetry group	Symmetry group of design structure and design unit	Fundamental region
p111	p111(d1)	
	p111(d3)	
	p111(d5)	
p1a1	p1a1(d1)	
	p1a1(d2)	
	p1a1(d3)	
p1m1	p1m1(d1)	
	p1m1(d2)	
	p1m1(d3)	
pm11	pm11(d1)	
	pm11(d2)	

Figure 3.7 Schematic illustrations of monotranslational design symmetry subgroups *pyxn*(*dN*).

pm11(d3)

p112 p112(d1)

p112(d2)

p112(d3)

pma2 pma2(d1)

pma2(d2)

pma2(d3)

pmm2 pmm2(d1)

pmm2(d2)

pmm2(d3)

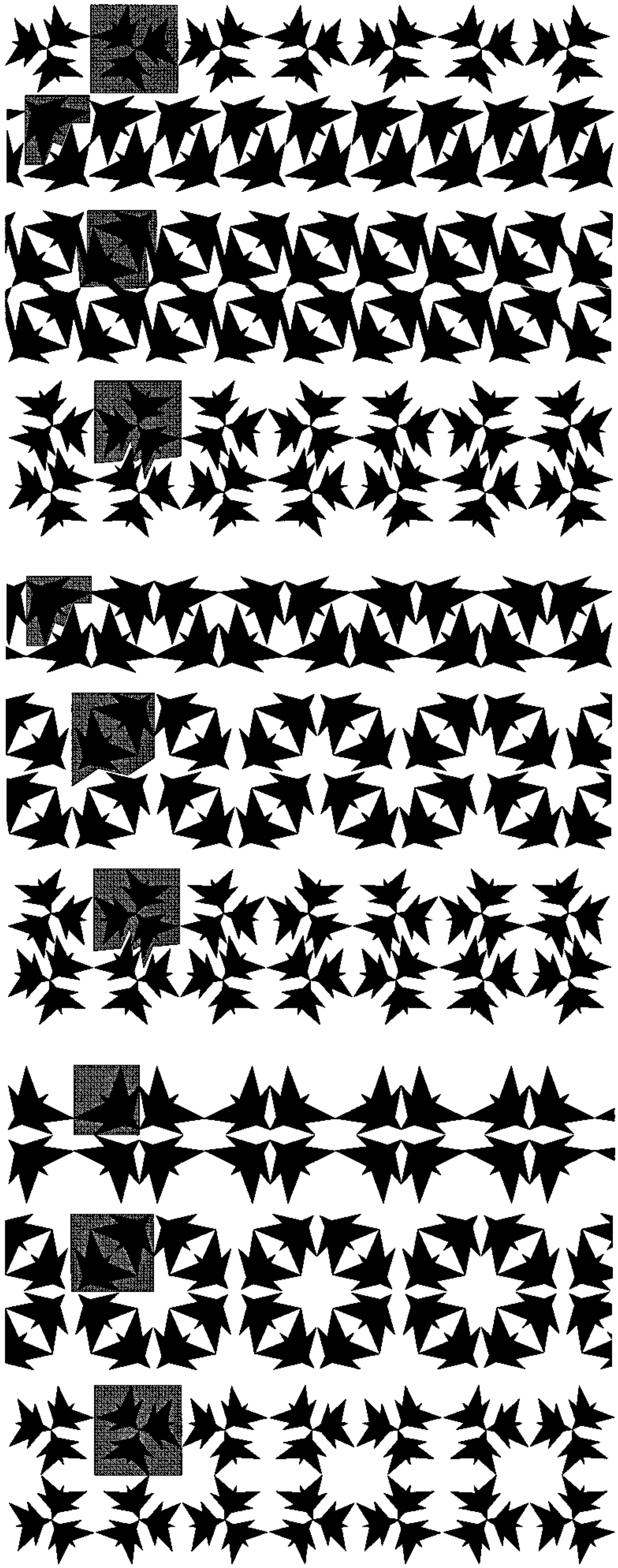


Figure 3.7 (cont.)

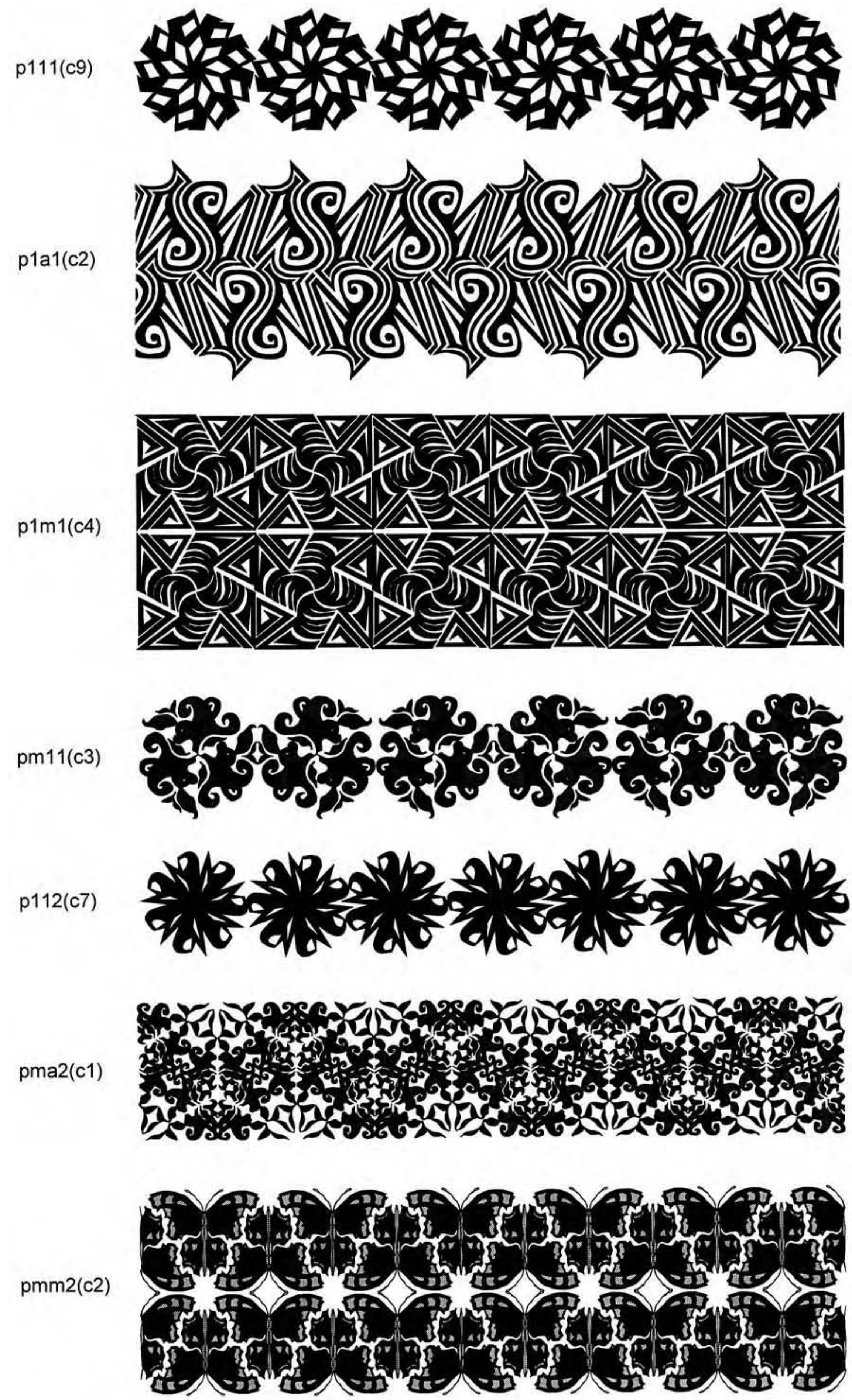


Figure 3.8 Further examples of monotranslational design symmetry subgroups.

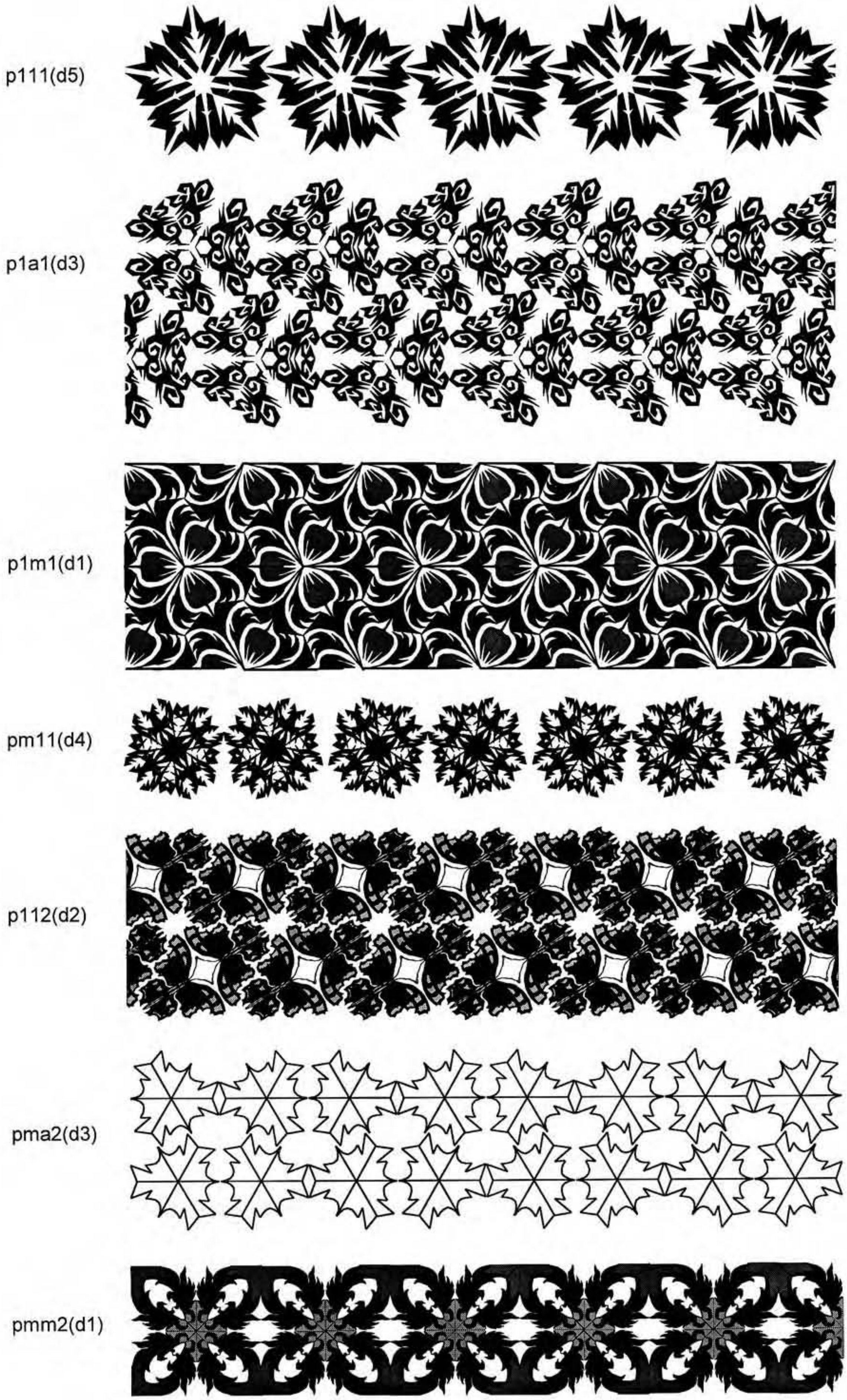


Figure 3.8 (cont.)

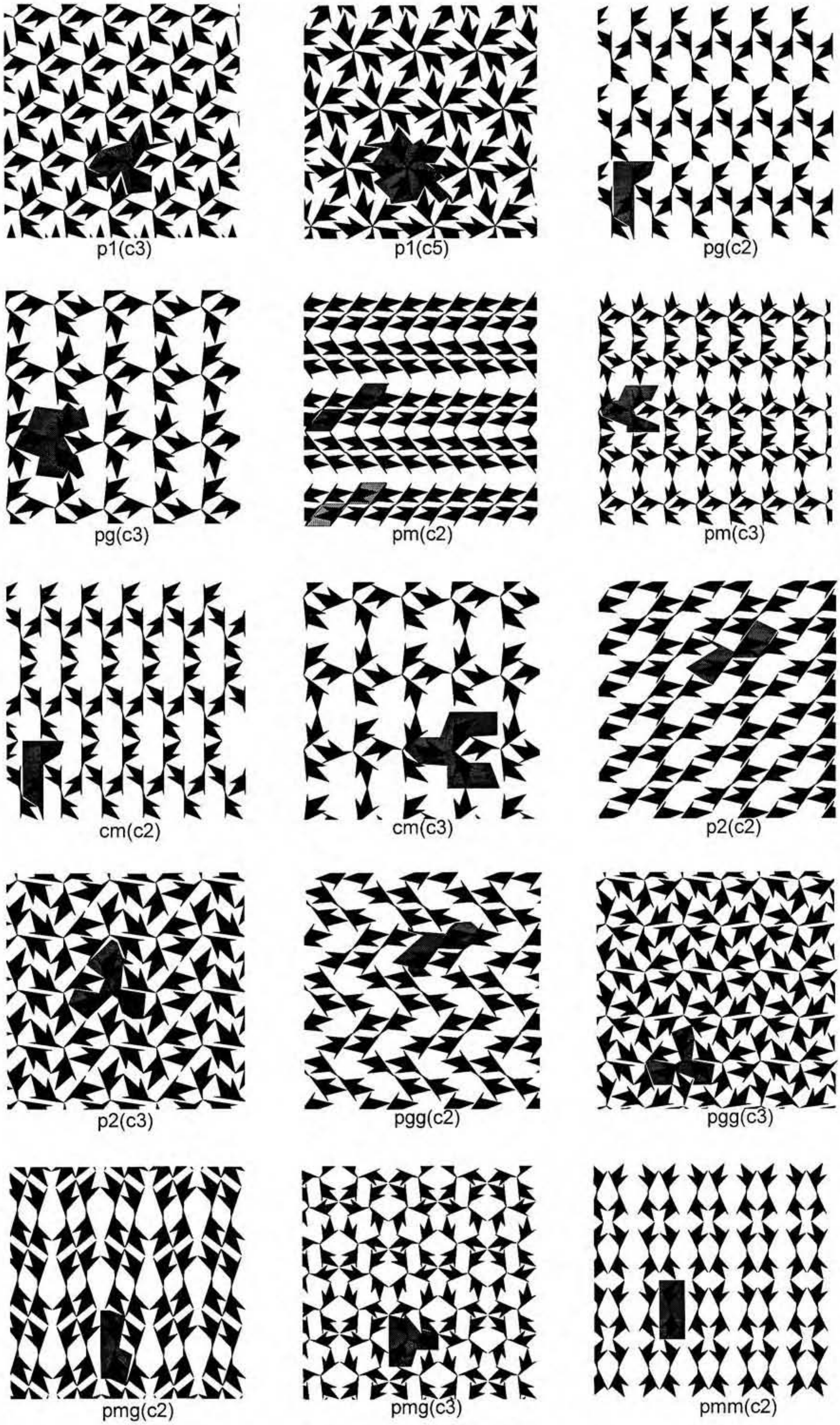


Figure 3.9 Schematic illustrations of ditranslational design symmetry subgroups *pnxy*(*cN*).

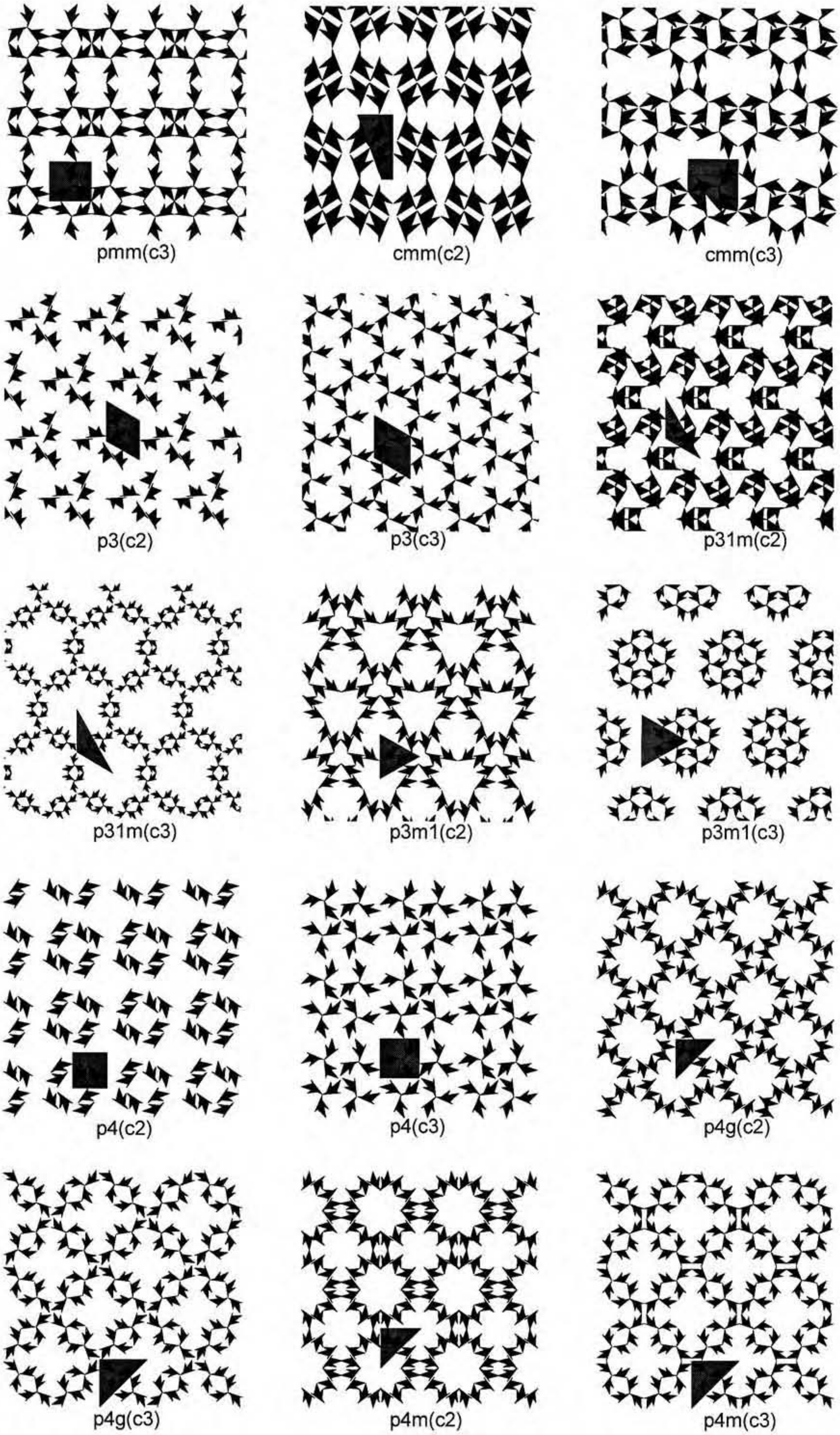


Figure 3.9 (cont.)

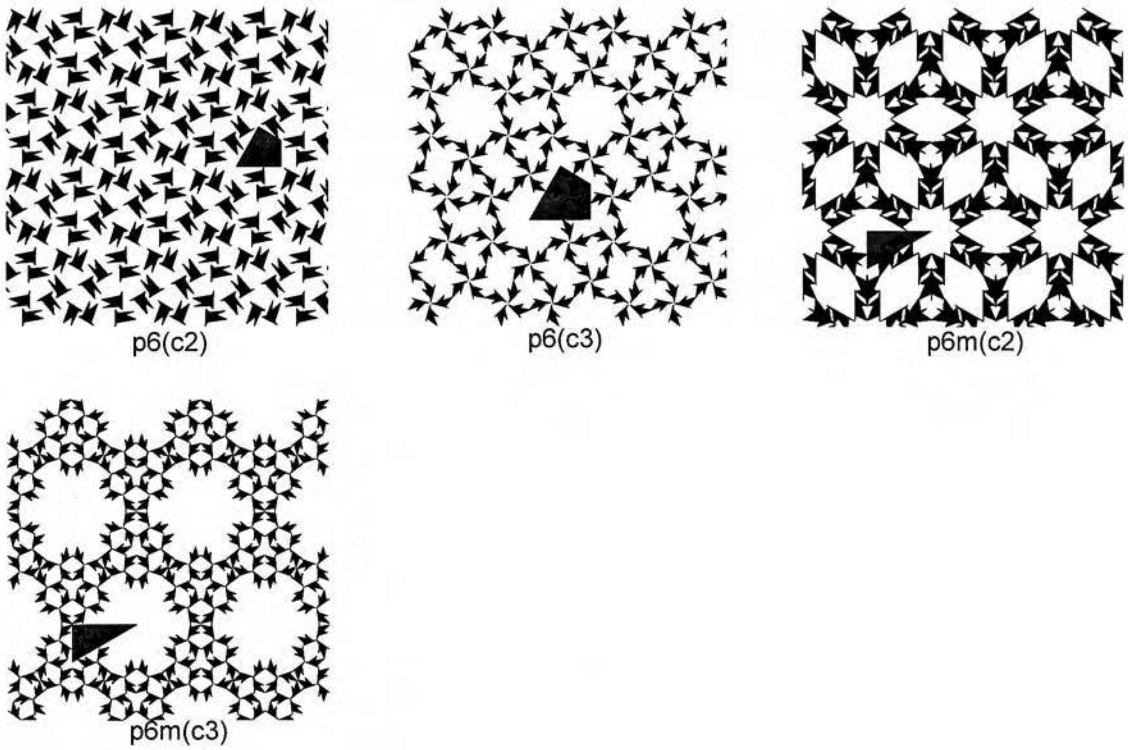


Figure 3.9 (cont.)

ples are shown in Fig. 3.11. Another interesting example which illustrates an application of this classification system is given in Fig. 3.12(b) where a projection of the structure $C_6(CH_3)_4$ displays the same symmetries as the pattern with symmetry subgroup $p2(d1)$ given in Fig. 3.12(a).

3.6 Construction of finite designs

The methods for constructing finite designs, in this classification system, are similar to the first techniques described in Chapter 2, Sections 2.11.1 and 2.11.2 for symmetry groups *cn* and *dn*, respectively. In each of the four subgroups: $cn(cN)$, $cn(dN)$, $dn(cN)$ and $dn(dN)$, the initial design unit must extend to at least one point on the circumference of the circle enclosing the design. It will also be assumed that no additional symmetries are induced into the design structure, on application of the generating symmetries to what appears to be an asymmetric design unit, such as those described in relation to a monotranslational design in Section 2.12 (and illustrated in Fig. 2.23).

In each case, a design unit having specific rotational and/or reflectional characteristics is added to a fundamental region of the finite cyclic group before applying the generating symmetries to complete the design.

3.6.1 Construction of symmetry subgroups cn(cN) and cn(dN)

To construct a finite $cn(cN)$ design, the design unit has N-fold rotational symmetry only. Its positioning inside the fundamental region is not critical. There are no limitations on the value of N except when $n = 1$, then $N = 1$ to retain the asymmetric characteristic of a $c1$-structured design. Figure 3.13(a(i)) shows an example of $cn(cN)$ design construction for $n = 3$ and $N = 3$.

To construct a $cn(dN)$ design, the design unit has N reflection axes and N-fold rotational symmetry. Its positioning inside the fundamental region is critical in that none of the N reflection axes must pass through the centre of the finite design. If this condition is not satisfied the size of the fundamental region would be reduced by half and what was originally intended to be a $cn(dN)$ design would

Table 3.2 Construction of symmetry subgroups *cn*(*cN*)

Symmetry group of design structure	Symmetry group of design unit *c*1	*c*2	*c*3	*c*4	*c*5	*c*6
*c*1	*c*1(*c*1)	—	—	—	—	—
*c*2	*c*2(*c*1)	*c*2(*c*2)	*c*2(*c*3)	*c*2(*c*4)	*c*2(*c*5)	*c*2(*c*6)
*c*3	*c*3(*c*1)	*c*3(*c*2)	*c*3(*c*3)	*c*3(*c*4)	*c*3(*c*5)	*c*3(*c*6)
*c*4	*c*4(*c*1)	*c*4(*c*2)	*c*4(*c*3)	*c*4(*c*4)	*c*4(*c*5)	*c*4(*c*6)
*c*5	*c*5(*c*1)	*c*5(*c*2)	*c*5(*c*3)	*c*5(*c*4)	*c*5(*c*5)	*c*5(*c*6)
*c*6	*c*6(*c*1)	*c*6(*c*2)	*c*6(*c*3)	*c*6(*c*4)	*c*6(*c*5)	*c*6(*c*6)

cn(*cN*) is constructable for all *N* (where *N* is a positive integer) if $n > 1$. If $n = 1$ then $N = 1$.

Table 3.3 Construction of symmetry subgroups *cn*(*dN*)

Symmetry group of design structure	Symmetry group of design unit *d*1	*d*2	*d*3	*d*4	*d*5	*d*6
*c*1	—	—	—	—	—	—
*c*2	*c*2(*d*1)*	*c*2(*d*2)*	*c*2(*d*3)*	*c*2(*d*4)*	*c*2(*d*5)*	*c*2(*d*6)*
*c*3	*c*3(*d*1)*	*c*3(*d*2)*	*c*3(*d*3)*	*c*3(*d*4)*	*c*3(*d*5)*	*c*3(*d*6)*
*c*4	*c*4(*d*1)*	*c*4(*d*2)*	*c*4(*d*3)*	*c*4(*d*4)*	*c*4(*d*5)*	*c*4(*d*6)*
*c*5	*c*5(*d*1)*	*c*5(*d*2)*	*c*5(*d*3)*	*c*5(*d*4)*	*c*5(*d*5)*	*c*5(*d*6)*
*c*6	*c*6(*d*1)*	*c*6(*d*2)*	*c*6(*d*3)*	*c*6(*d*4)*	*c*6(*d*5)*	*c*6(*d*6)*

cn(*dN*) is constructable for all *N* (where *N* is a positive integer) if $n > 1$. None of the *N* reflection axes may intersect the centre of overall design structure.
* = restrictions are imposed on the positioning and orientation of the design unit.

Table 3.4 Construction of symmetry subgroups *dn*(*cN*)

Symmetry group of design structure	Symmetry group of design unit *c*1	*c*2	*c*3	*c*4	*c*5	*c*6
*d*1	*d*1(*c*1)	*d*1(*c*2)	*d*1(*c*3)	*d*1(*c*4)	*d*1(*c*5)	*d*1(*c*6)
*d*2	*d*2(*c*1)	*d*2(*c*2)	*d*2(*c*3)	*d*2(*c*4)	*d*2(*c*5)	*d*2(*c*6)
*d*3	*d*3(*c*1)	*d*3(*c*2)	*d*3(*c*3)	*d*3(*c*4)	*d*3(*c*5)	*d*3(*c*6)
*d*4	*d*4(*c*1)	*d*4(*c*2)	*d*4(*c*3)	*d*4(*c*4)	*d*4(*c*5)	*d*4(*c*6)
*d*5	*d*5(*c*1)	*d*5(*c*2)	*d*5(*c*3)	*d*5(*c*4)	*d*5(*c*5)	*d*5(*c*6)
*d*6	*d*6(*c*1)	*d*6(*c*2)	*d*6(*c*3)	*d*6(*c*4)	*d*6(*c*5)	*d*6(*c*6)

dn(*cN*) is constructable for all $N \geq 1$, for all $n \geq 1$.

be transformed into a *dn*(*c*1) design as illustrated for $n = 4$ and $n = 5$ in Fig. 3.2(c). The four examples in Fig. 3.2(c) show *c*4(*d*1), *d*4(*c*1), *c*5(*d*1) and *d*5(*c*1) designs, respectively. An example showing the construction of a *cn*(*dN*) design is given in Fig. 3.13(a(ii)) for $n = 4$ and $N = 2$.

3.6.2 Construction of symmetry subgroups dn(cN) and dn(dN)

To construct a *dn*(*cN*) design, the design unit has *N*-fold rotational symmetry only. Its positioning inside the fundamental region, as for *cn*(*cN*) designs, is not critical. Figure 3.13(b(i)) shows an example of *dn*(*cN*) design construction for $n = 2$ and $N = 4$.

To construct a *dn*(*dN*) design, the design unit has *N* reflection axes and *N*-fold rotational symmetry. Its positioning inside the fundamental region is critical in that although one of the *N* reflection axes may pass through the centre of the overall finite design, this axis must not bisect the fundamental region since this would reduce the size of the fundamental region by half and transform what was originally intended to be a *dn*(*dN*) design into a *d*2*n*(*d*1) design. Figure 3.13(b(ii)) shows an example of *dn*(*dN*) design construction for $n = 6$ and $N = 1$.

Tables 3.2 to 3.5 summarise the information given above by indicating, for $n = 1$ to 6 and $N = 1$ to 6, whether a particular symmetry subgroup is constructable

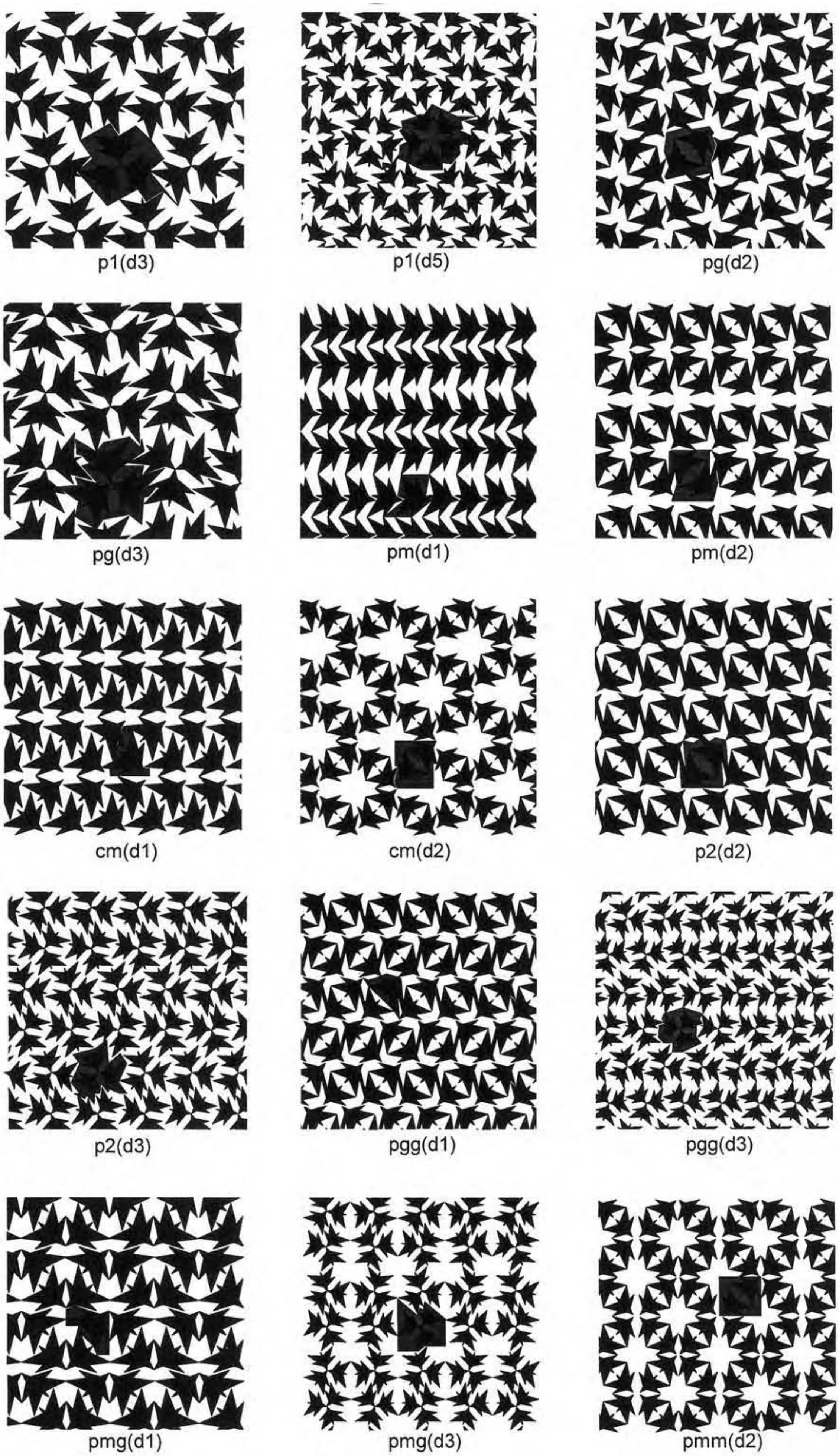


Figure 3.10 Schematic illustrations of ditranslational design symmetry subgroups *pnxy*(*dN*).

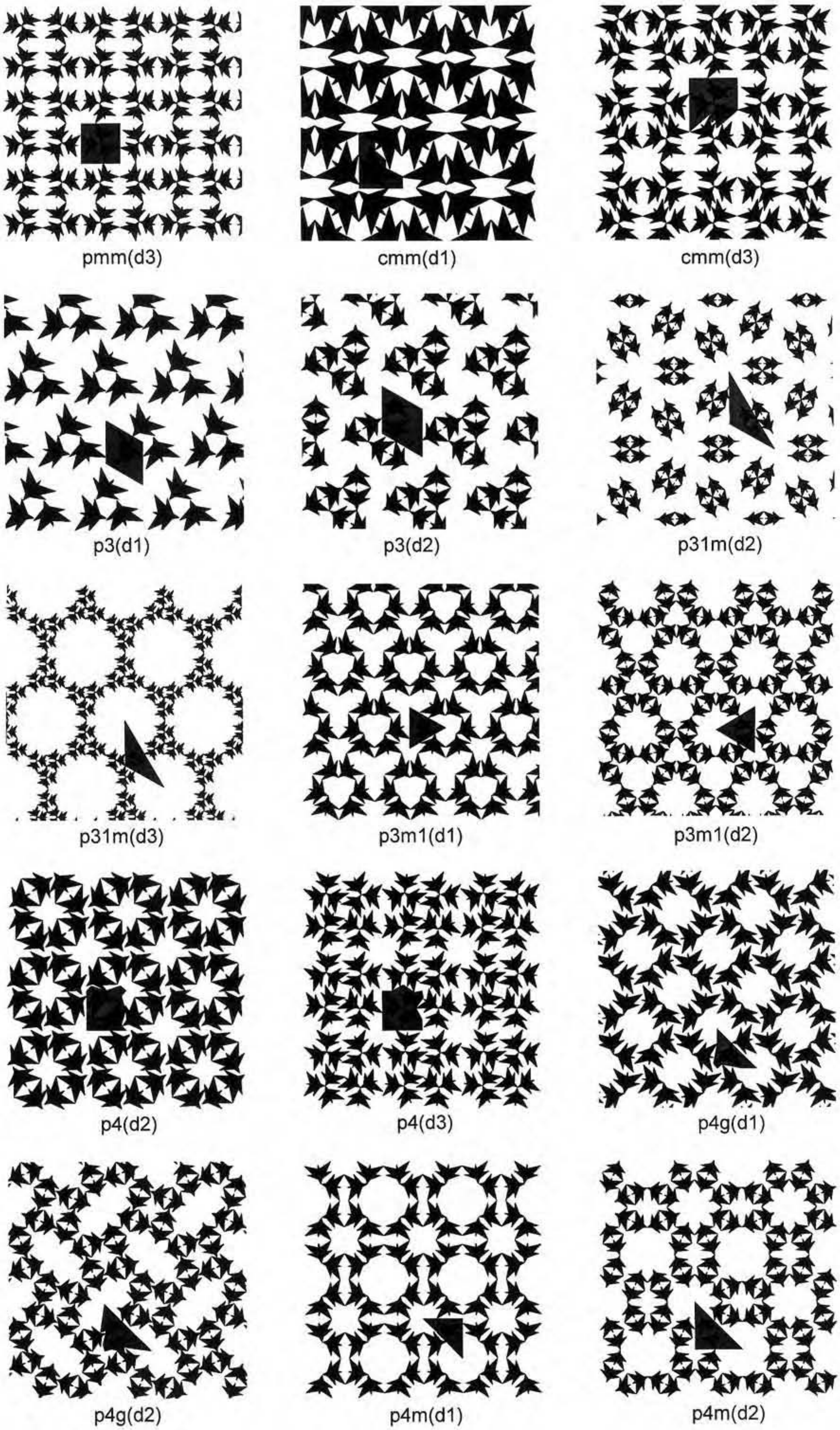


Figure 3.10 (cont.)

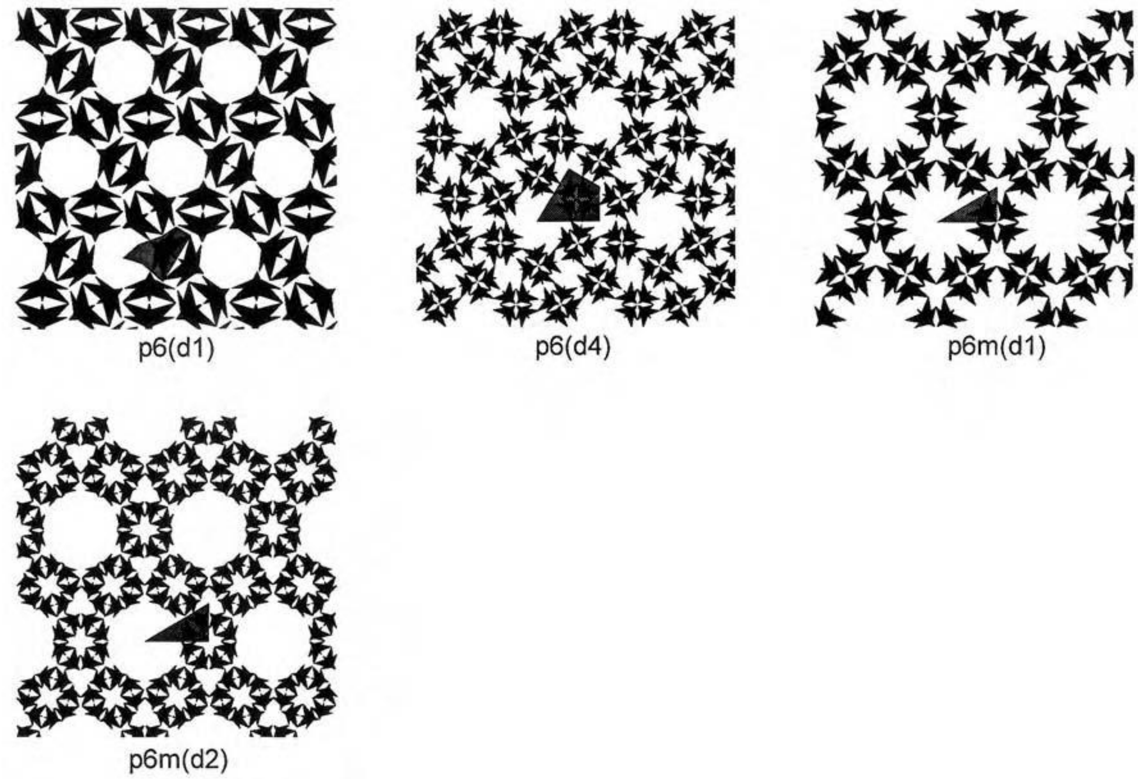


Figure 3.10 (cont.)

Table 3.5 Construction of symmetry subgroups *dn*(*dN*)

Symmetry group of design structure	Symmetry group of design unit *d*1	*d*2	*d*3	*d*4	*d*5	*d*6
*d*1	*d*1(*d*1)*	*d*1(*d*2)*	*d*1(*d*3)*	*d*1(*d*4)*	*d*1(*d*5)*	*d*1(*d*6)*
*d*2	*d*2(*d*1)*	*d*2(*d*2)*	*d*2(*d*3)*	*d*2(*d*4)*	*d*2(*d*5)*	*d*2(*d*6)*
*d*3	*d*3(*d*1)*	*d*3(*d*2)*	*d*3(*d*3)*	*d*3(*d*4)*	*d*3(*d*5)*	*d*3(*d*6)*
*d*4	*d*4(*d*1)*	*d*4(*d*2)*	*d*4(*d*3)*	*d*4(*d*4)*	*d*4(*d*5)*	*d*4(*d*6)*
*d*5	*d*5(*d*1)*	*d*5(*d*2)*	*d*5(*d*3)*	*d*5(*d*4)*	*d*5(*d*5)*	*d*5(*d*6)*
*d*6	*d*6(*d*1)*	*d*6(*d*2)*	*d*6(*d*3)*	*d*6(*d*4)*	*d*6(*d*5)*	*d*6(*d*6)*

dn(*dN*) is constructable for all $N \geq 1$, for all $n \geq 1$ provided that none of the N reflection axes bisect a fundamental region.

and if restrictions on the symmetric characteristics of the design unit are required. A dash indicates that, using the given symmetry group of design unit, the construction of that particular symmetry subgroup is not possible. An asterisk indicates that although the symmetry subgroup is constructable, restrictions are imposed on the positioning and orientation of the design unit.

3.7 Construction of monotranslational designs

The methods for constructing monotranslational designs, in this classification system, are similar to those discussed in Section 2.12. In each of the 14 *pyxn*(*cN*) and *pyxn*(*dN*) subgroups, the design unit must extend at some point to at least one, and where possible both of the outside straight edges of the strip enclosing the overall design. It will also be assumed that no additional symmetries of the form discussed in Chapter 2, Section 2.12, are induced into the design structure on applying the generating symmetries.

The boundaries of the fundamental regions may remain as part of the design units after being used in the construction process to give a form of patterned tiling as described for design types (i) and (iv) in Chapter 2. However, this limits the possible symmetrical characteristics of the design units. Therefore, to produce a larger range of design symmetry subgroups it is more suitable to produce a design of type (ii) or (v) (see Chapter 2, Section 2.12).

Construction methods are given which may be derived by simply following the methods described previously in Chapter 2. The only extra conditions concern the symmetry of the design unit and its positioning and orientation relative to the symmetries in the underlying structure of the design. These are described in detail for each symmetry group subgroup along with a selection of examples to illustrate these restrictions.

3.7.1 Construction of symmetry subgroups pyxn(cN)

A strip is divided either into parallelogram-shaped or into alternatively shaped fundamental regions as described in Chapter 2 for monotranslational design types (i) and (iii). A design unit having specific N-fold rotational symmetry only is added to one of these regions and then mapped onto the remaining ones by applying the generating symmetries. In some instances an even-fold rotationally symmetric design unit would induce extra symmetries into the structure of the design thus increasing its order of symmetry. Consequently this would alter the symmetry group under construction. The following criteria given below, for each symmetry subgroup, relate to the conditions imposed on the initial design unit added to the strip.

3.7.1.1 Symmetry subgroup $p111(cN)$

A $p111(cN)$ design may be constructed from any N-fold rotationally symmetric design unit provided that N is an odd number. Any design unit with even-fold rotational symmetry would induce two-fold rotational symmetry into the design structure thus altering the symmetry group. An illustration showing the construction of symmetry subgroup $p111(cN)$ is given in Fig. 3.14(a) using a $c3$ design unit.

3.7.1.2 Symmetry subgroup $p1a1(cN)$

To construct a $p1a1(cN)$ design, if N is an odd number, the positioning of the design unit inside the fundamental region is not critical. If N is even, its centre of rotation must not intersect the longitudinal axis of the strip. Figure 3.14(b) shows an illustration of the construction of symmetry subgroup $p1a1(cN)$ using a $c2$ design unit. In this instance (as in the second example in Fig. 2.25(vi) showing the construction of $p1a1$) the design is based on two strips, one of which is a glide–reflection of the other. The left and right hand edges of the shaded fundamental region are translations of each other and the bottom edge is composed of two parts, the right hand side of which is a glide–reflection of the left hand side.

3.7.1.3 Symmetry subgroup $p1m1(cN)$

There are no limitations on the value of N or the positioning of the centre of rotation of the design unit within the fundamental region for the construction of a $p1m1(cN)$ design. An illustration showing the construction of symmetry subgroup $p1m1(cN)$ is given in Fig. 3.14(c) using a $c2$ design unit.

3.7.1.4 Symmetry subgroup $pm11(cN)$

To construct a $pm11(cN)$ design, if N is an odd number, the positioning of the design unit inside the fundamental region is not critical. If N is even, its centre of rotation must not lie half way between transverse axes of reflectional symmetry of the design structure. An illustration showing the construction of symmetry subgroup $pm11(cN)$ is given in Fig. 3.14(d) using a $c2$ design unit.

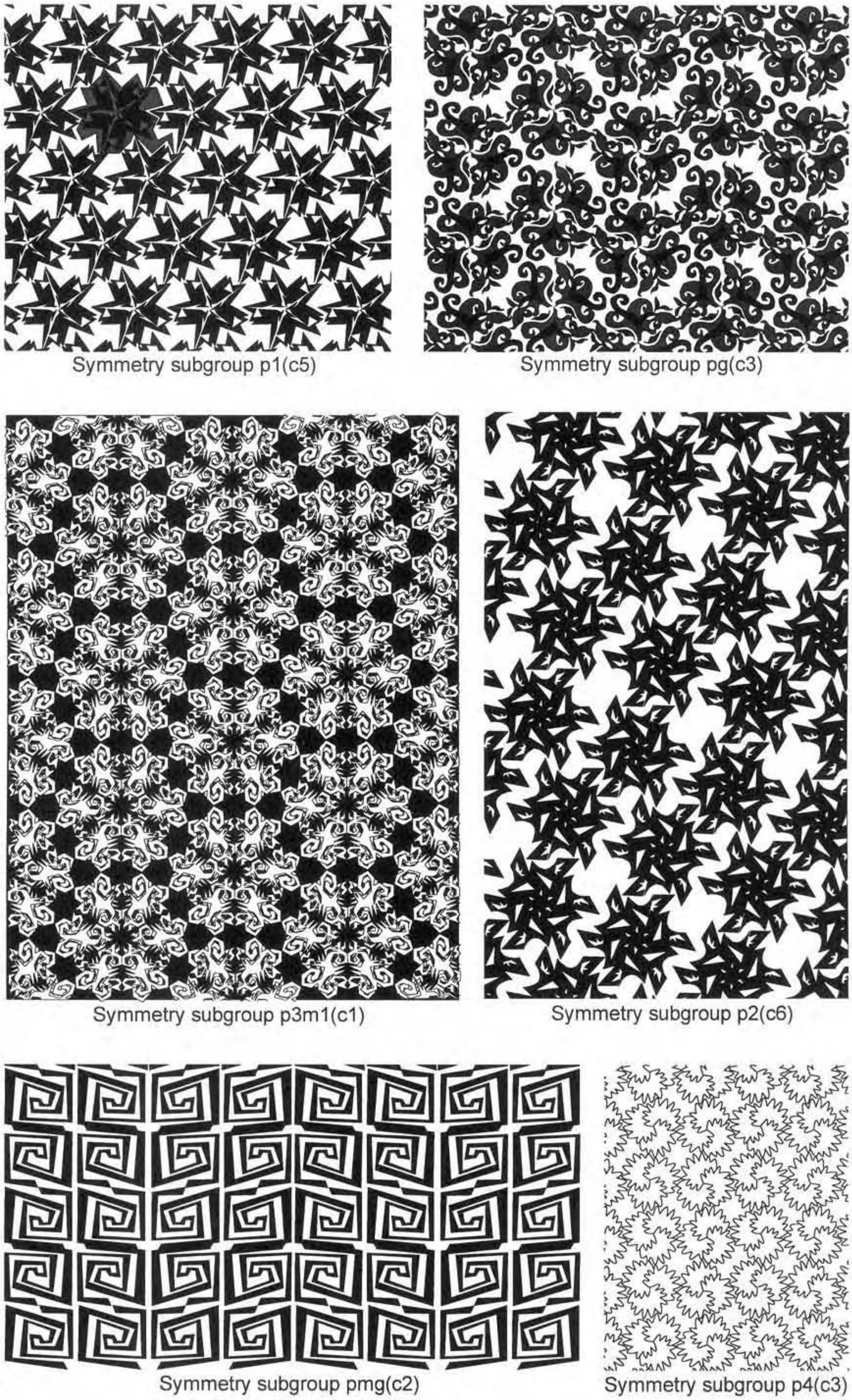
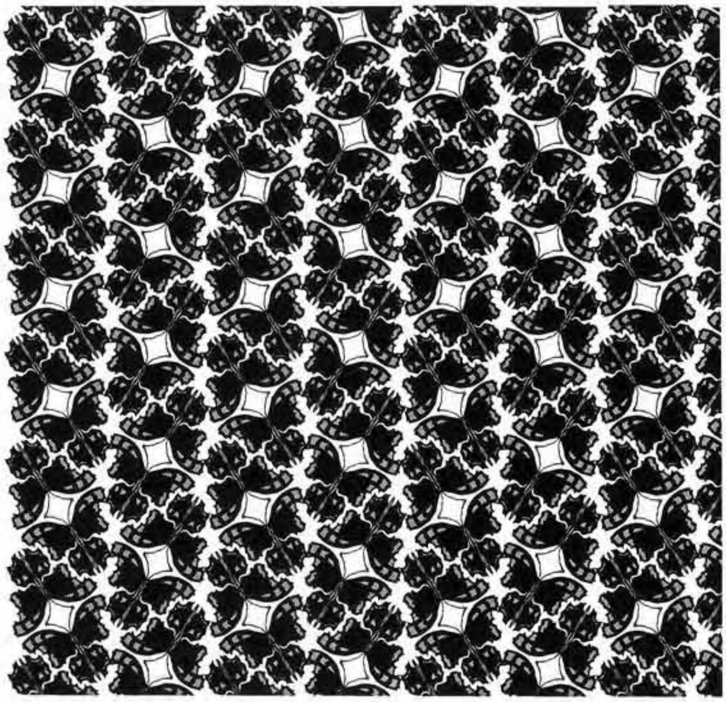
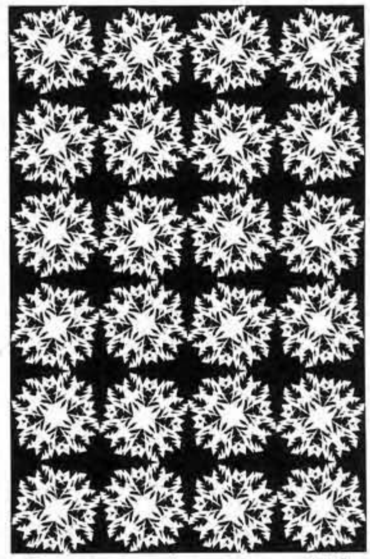


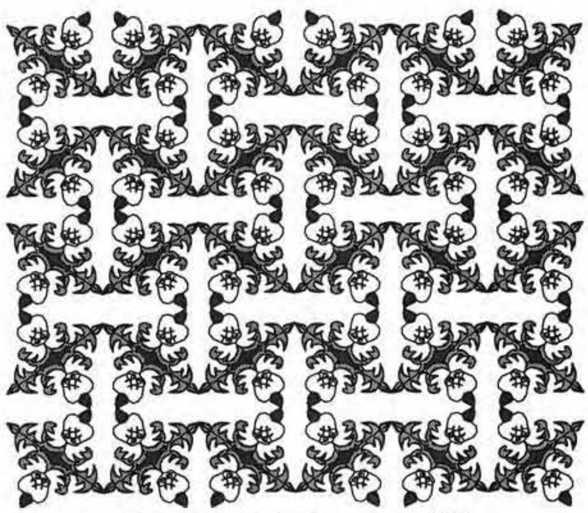
Figure 3.11 Further examples of ditranslational design symmetry subgroups.



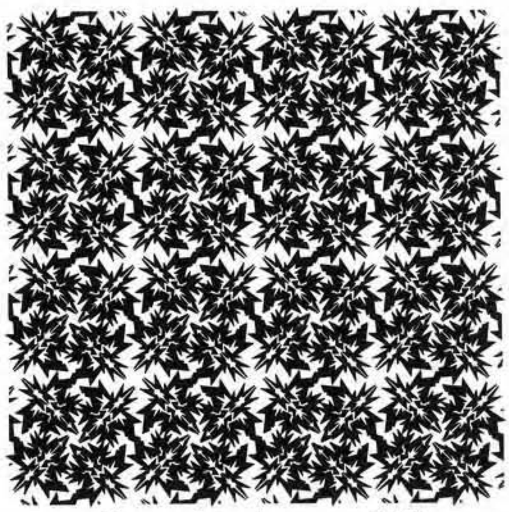


Symmetry subgroup pmg(d4)

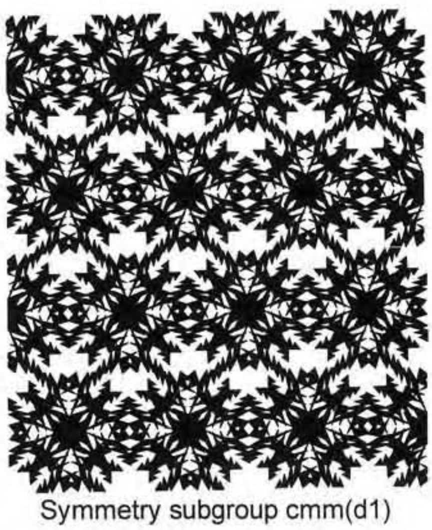
Symmetry subgroup pgg(d1)



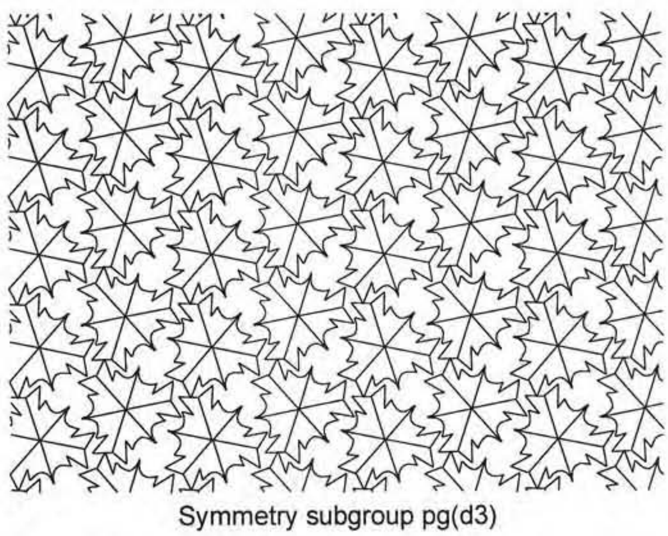
Symmetry subgroup cmm(d1)



Symmetry subgroup p4(d1)



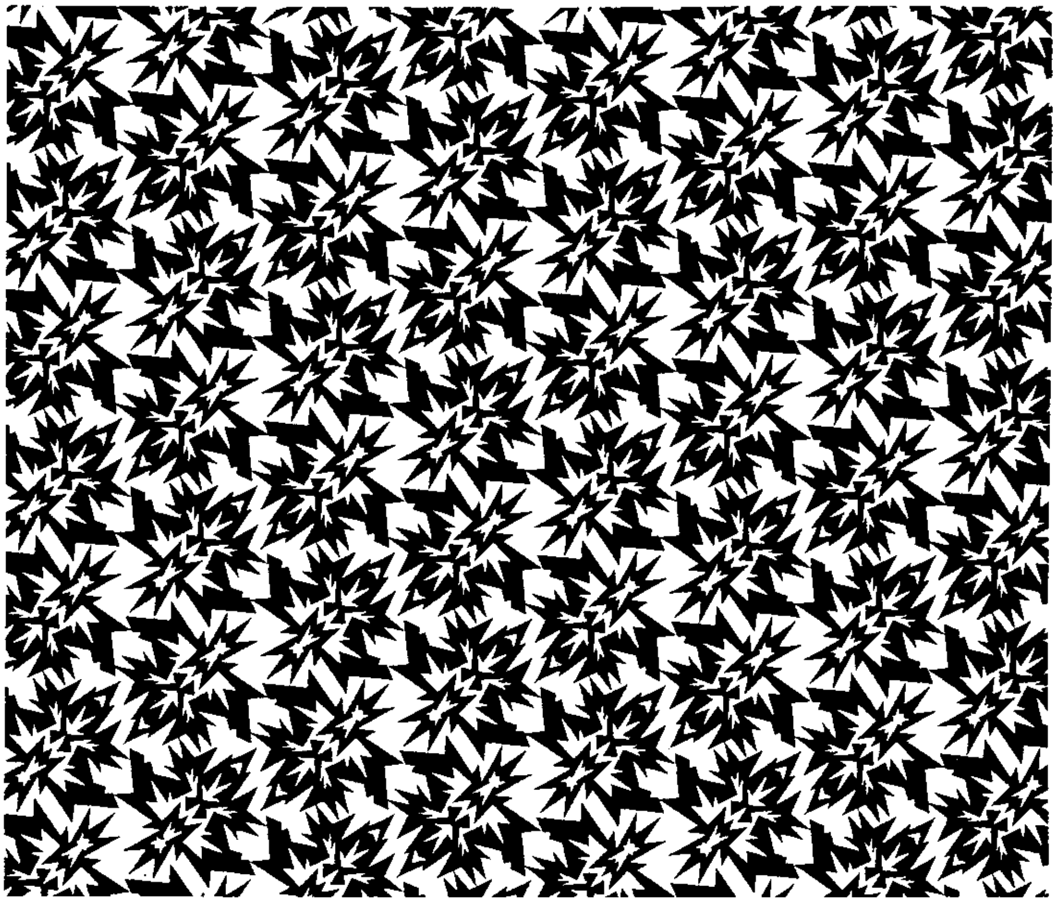
Symmetry subgroup cmm(d1)



Symmetry subgroup pg(d3)

Figure 3.11 (cont.)

a



b

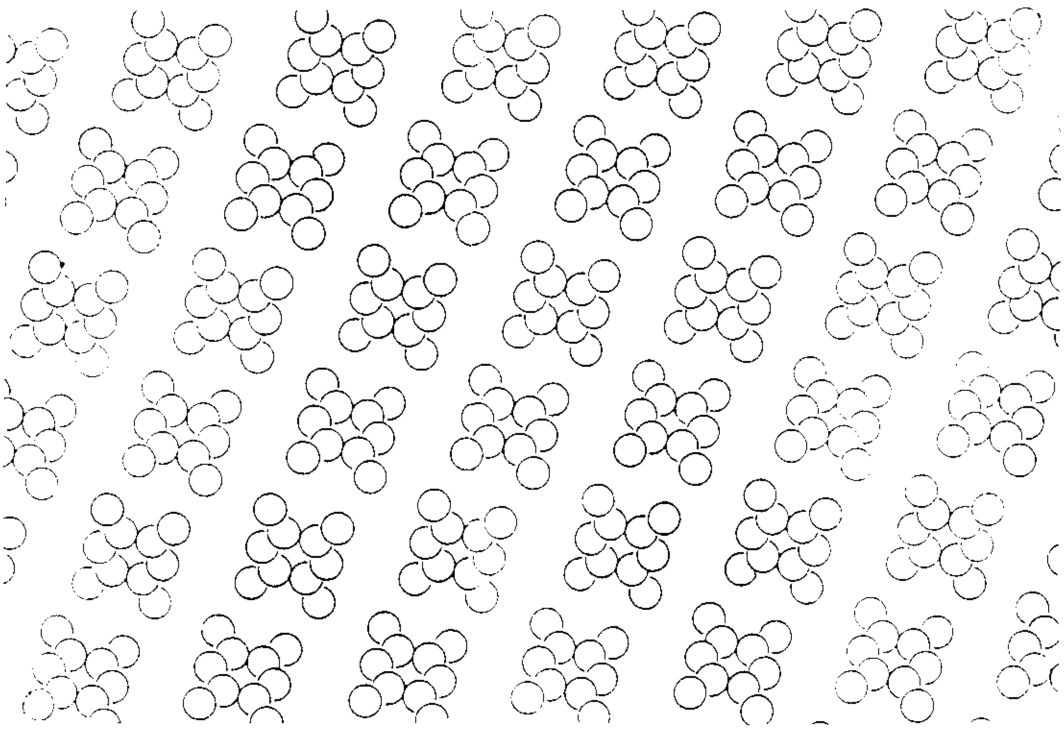


Figure 3.12 Example of symmetry subgroup *p*2(*d*1) in (a) a pattern and (b) projection of the structure $C_6(CH_3)_4$. (b) Source: Hammond C, 'Introduction to Crystallography', *Microscopy Handbooks 19*, Oxford University Press, 1990. Reproduced by permission of McGraw Hill from an original publication 1970.

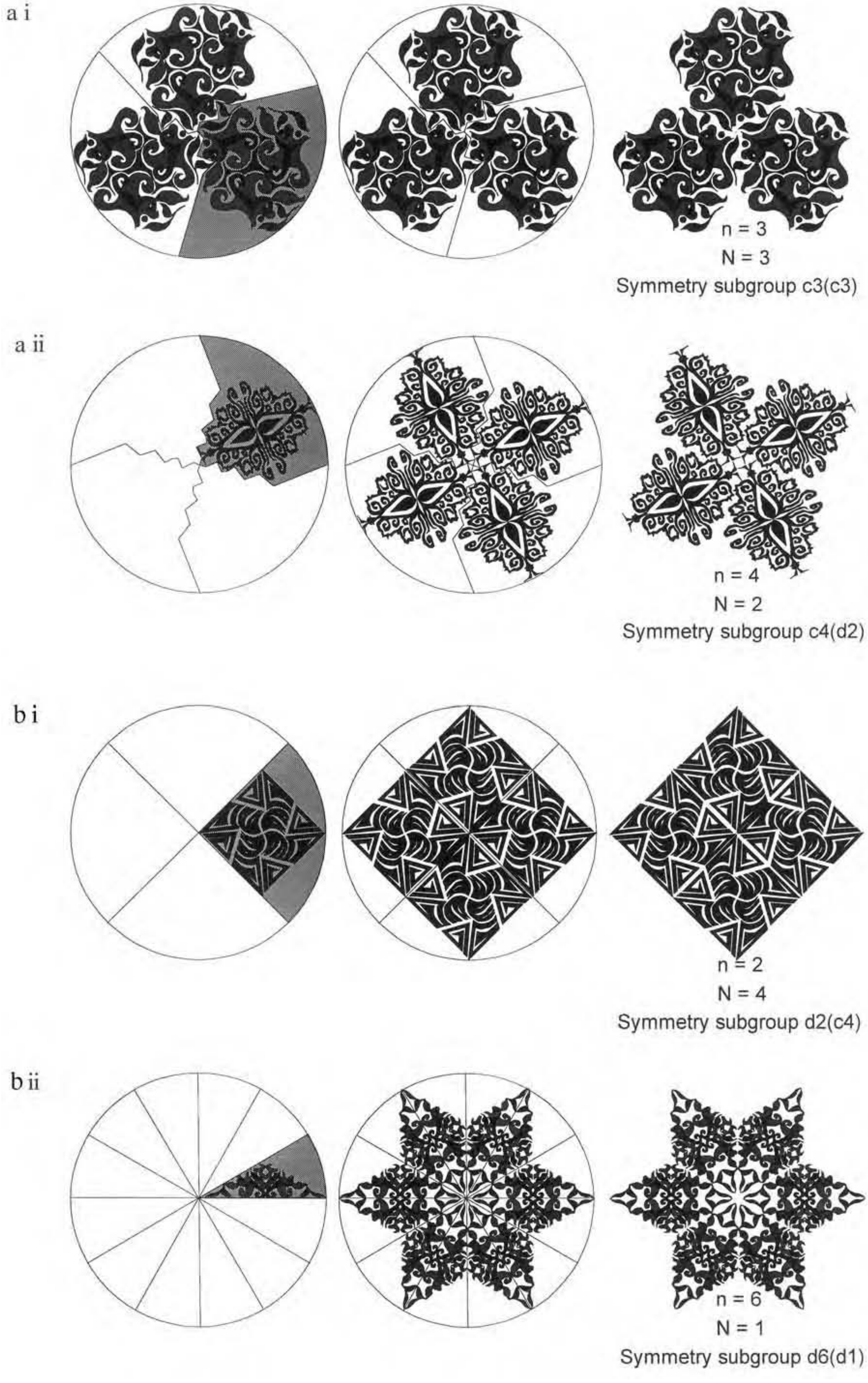


Figure 3.13 Construction of finite design symmetry groups *cn*(*cN*), *cn*(*dN*), dn(*cN*) and dn(*dN*).

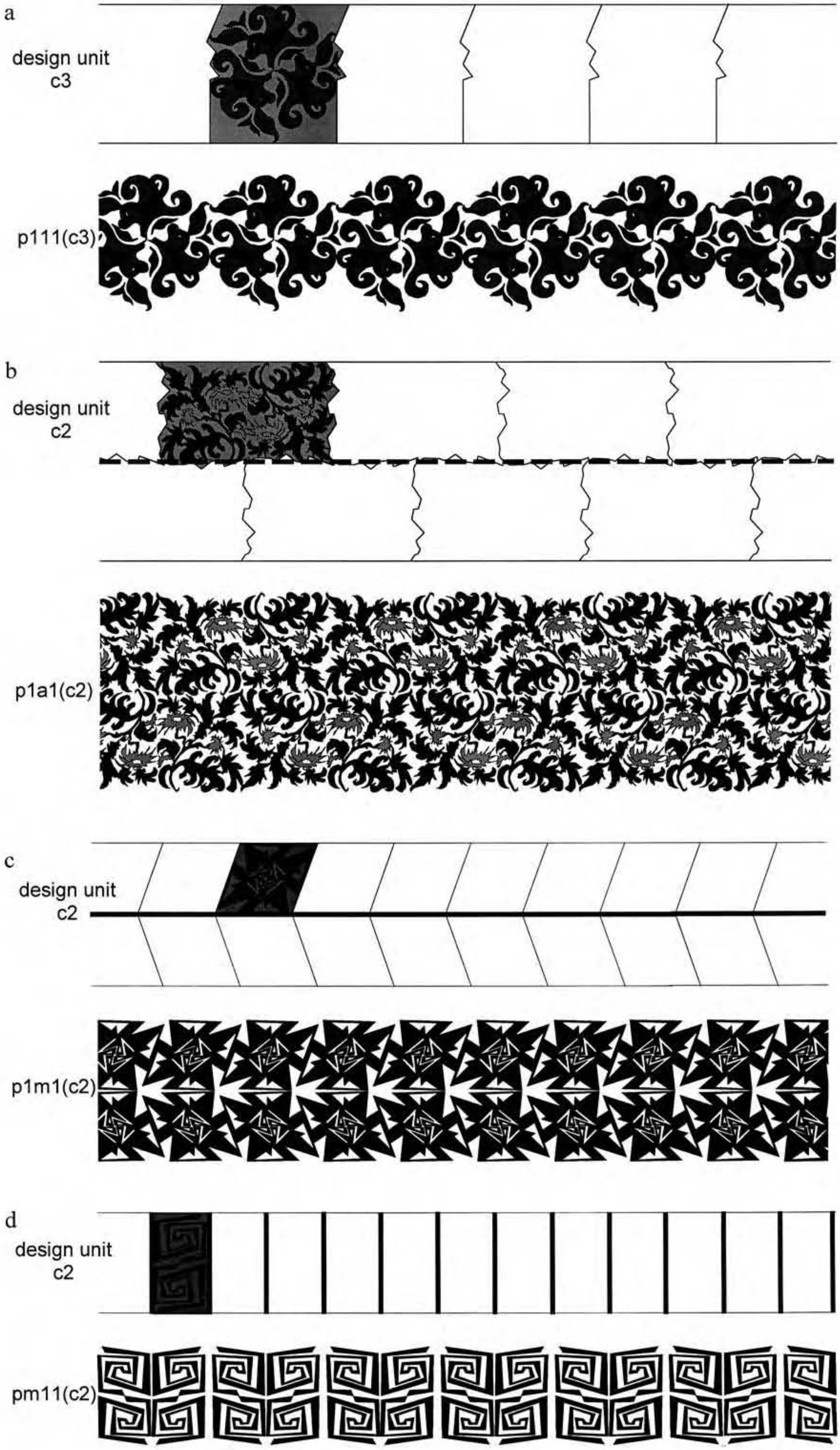


Figure 3.14 Construction of monotranslational design symmetry subgroups *pyxn*(*cN*).

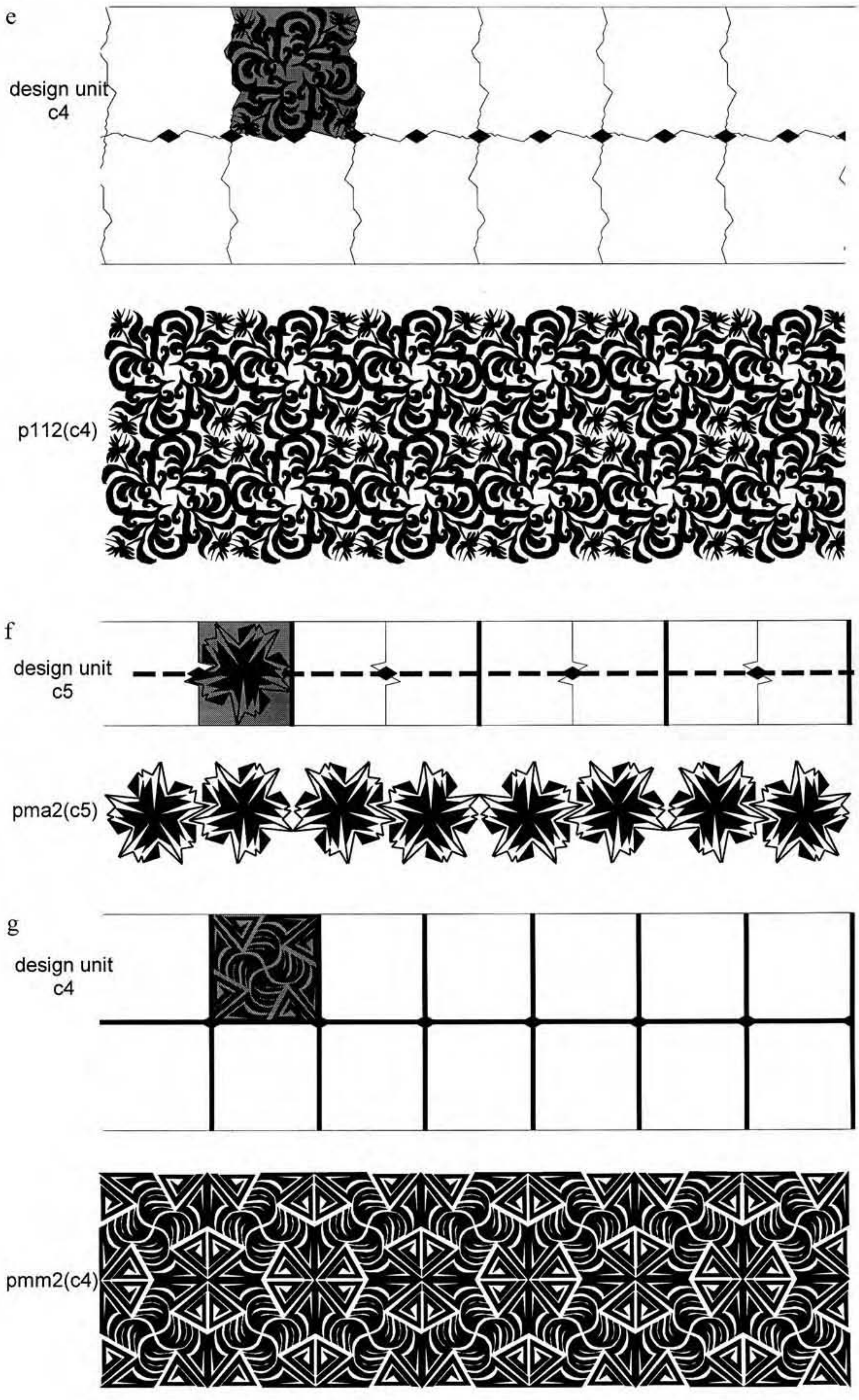


Figure 3.14 (cont.)

3.7.1.5 *Symmetry subgroup* $p112(cN)$

To construct a $p112(cN)$ design, if N is an odd number, the positioning of the design unit inside the fundamental region is not critical. If N is even, its centre of rotation must not lie on the longitudinal axis of the strip half way between centres of two-fold rotation of the design structure. An illustration showing the construction of symmetry subgroup $p112(cN)$ is given in Fig. 3.14(e) using a $c4$ design unit.

3.7.1.6 *Symmetry subgroup* $pma2(cN)$

To construct a $pma2(cN)$ design, the positioning of the design unit inside the fundamental region, for any N, is not critical. An illustration showing the construction of symmetry subgroup $pma2(cN)$ is given in Fig. 3.14(f) using a $c5$ design unit.

3.7.1.7 *Symmetry subgroup* $pmm2(cN)$

To construct a $pmm2(cN)$ design, the positioning of the design unit inside the fundamental region, for any N, is not critical. Figure 3.14(g) shows an illustration of the construction of symmetry subgroup $pmm2(cN)$ using a $c4$ design unit.

3.7.2 *The construction of symmetry subgroups pyxn(dN)*

A strip is divided either into parallelogram-shaped or into alternatively shaped fundamental regions and a design unit having N reflection axes and hence N-fold rotational symmetry is added to one of these regions and then mapped onto the remaining ones by applying the generating symmetries. The following criteria given below, for each symmetry subgroup, relate to the conditions imposed on the initial design unit added to the strip.

3.7.2.1 *Symmetry subgroup* $p111(dN)$

A $p111(dN)$ design may be constructed provided that N is an odd number. None of the reflection axes may coincide with the longitudinal axis or any transverse axis of the strip. An even number of reflection axes automatically results in a centre of even-fold rotational symmetry at their point of intersection. This would induce two-fold rotational symmetry into the design structure thus altering its symmetry group. An illustration showing the construction of symmetry subgroup $p111(dN)$ is given in Fig. 3.15(a) using a $d1$ design unit.

3.7.2.2 *Symmetry subgroup* $p1a1(dN)$

To construct a $p1a1(dN)$ design N may be any number provided that none of the reflection axes coincides with the longitudinal axis or any transverse axis of the strip. If N is even the point of intersection of the reflection axes must not coincide with the longitudinal axis of the strip. Figure 3.15(b) shows an illustration of the construction of symmetry subgroup $p1a1(dN)$ using a $d1$ design unit.

3.7.2.3 *Symmetry subgroup* $p1m1(dN)$

To construct a $p1m1(dN)$ design N may be any number provided that none of the reflection axes lies parallel to any transverse axis of the strip. An illustration showing the construction of symmetry subgroup $p1m1(dN)$ is given in Fig. 3.15(c) using a $d1$ design unit.

3.7.2.4 *Symmetry subgroup* $pm11(dN)$

To construct a $pm11(dN)$ design N may be any number provided that none of the reflection axes coincides with the longitudinal axis or lies parallel to and half way

Table 3.6 Construction of symmetry subgroups *pyxn*(*cN*)

Symmetry group of design structure	Symmetry group of design unit c1	c2	c3	c4	c5	c6
p111	p111(c1)	—	p111(c3)	—	p111(c5)	—
p1a1	p1a1(c1)	p1a1(c2)*	p1a1(c3)	p1a1(c4)*	p1a1(c5)	p1a1(c6)*
p1m1	p1m1(c1)	p1m1(c2)	p1m1(c3)	p1m1(c4)	p1m1(c5)	p1m1(c6)
pm11	pm11(c1)	pm11(c2)*	pm11(c3)	pm11(c4)*	pm11(c5)	pm11(c6)*
p112	p112(c1)	p112(c2)*	p112(c3)	p112(c4)*	p112(c5)	p112(c6)*
pma2	pma2(c1)	pma2(c2)	pma2(c3)	pma2(c4)	pma2(c5)	pma2(c6)
pmm2	pmm2(c1)	pmm2(c2)	pmm2(c3)	pmm2(c4)	pmm2(c5)	pmm2(c6)

Table 3.7 Construction of symmetry subgroups *pyxn*(*dN*)

Symmetry group of design structure	Symmetry group of design unit d1	d2	d3	d4	d5	d6
p111	p111(d1)*	—	p111(d3)*	—	p111(d5)*	—
p1a1	p1a1(d1)*	p1a1(d2)*	p1a1(d3)*	p1a1(d4)*	p1a1(d5)*	p1a1(d6)*
p1m1	p1m1(d1)*	p1m1(d2)*	p1m1(d3)*	p1m1(d4)*	p1m1(d5)*	p1m1(d6)*
pm11	pm11(d1)*	pm11(d2)*	pm11(d3)*	pm11(d4)*	pm11(d5)*	pm11(d6)*
p112	p112(d1)*	p112(d2)*	p112(d3)*	p112(d4)*	p112(d5)*	p112(d6)*
pma2	pma2(d1)*	pma2(d2)*	pma2(d3)*	pma2(d4)*	pma2(d5)*	pma2(d6)*
pmm2	pmm2(d1)*	pmm2(d2)*	pmm2(d3)*	pmm2(d4)*	pmm2(d5)*	pmm2(d6)*

between the transverse reflection axes of the design structure. Figure 3.15(d) shows an illustration of the construction of symmetry subgroup *pm*11(*dN*) using a *d*2 design unit.

3.7.2.5 Symmetry subgroup p112(dN)

To construct a *p*112(*dN*) design *N* may be any number provided that none of the reflection axes coincides with the longitudinal axis. Neither must any of them lie parallel to a transverse axis and half way between or through the two-fold centres of rotation of the design stucture. An illustration showing the construction of symmetry subgroup *p*112(*dN*) is given in Fig. 3.15(e) using a *d*3 design unit.

3.7.2.6 Symmetry subgroup pma2(dN)

To construct a *pma*2(*dN*) design *N* may be any number provided that none of the reflection axes coincides with the longitudinal axis. Neither must any transverse reflection axis of the design unit pass through a centre of two-fold rotation of the design structure. Figure 3.15(f) shows an illustration of the construction of symmetry subgroup *pma*2(*dN*) using a *d*4 design unit.

3.7.2.7 Symmetry subgroup pmm2(dN)

To construct a *pmm*2(*dN*) design, *N* may be any number provided that none of the reflection axes lies parallel to and half way between the transverse reflection axes of the design structure. An illustration showing the construction of symmetry subgroup *pmm*2(*dN*) is given in Fig. 3.15(g) using a *d*3 design unit.

Tables 3.6 and 3.7 summarise the information given above by indicating, for $n = 1$ to 6 and $N = 1$ to 6, whether a particular symmetry subgroup is constructable and if restrictions on the symmetric characteristics of the design unit are required. A dash indicates that, using the given symmetry group of design unit, the construction of that particular symmetry subgroup is not possible. An asterisk indicates that the symmetrical characteristics, positioning and orientation of the design unit inside a fundamental region are critical. In this

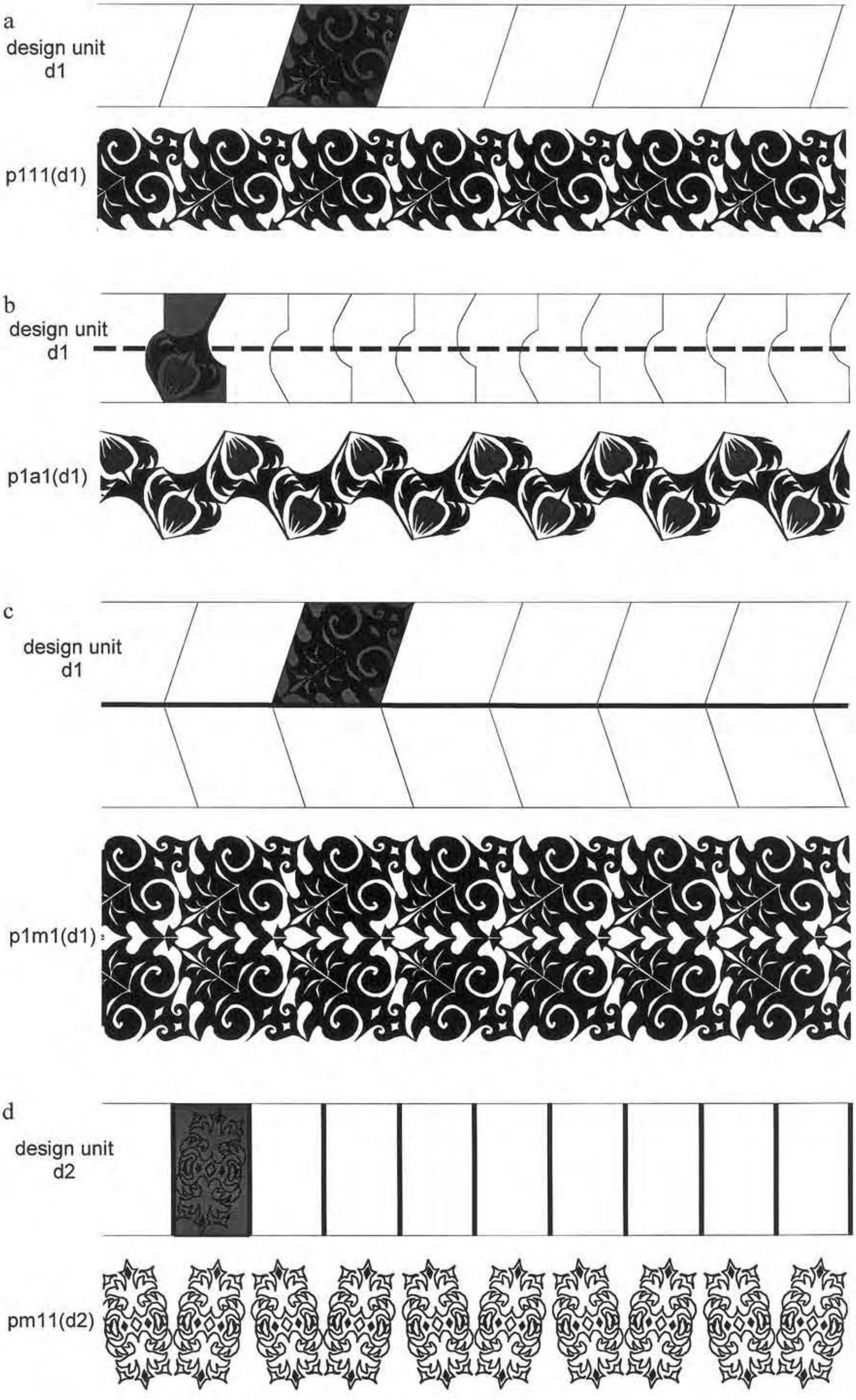


Figure 3.15 Construction of monotranslational design symmetry subgroups *pyxn*(*dN*).

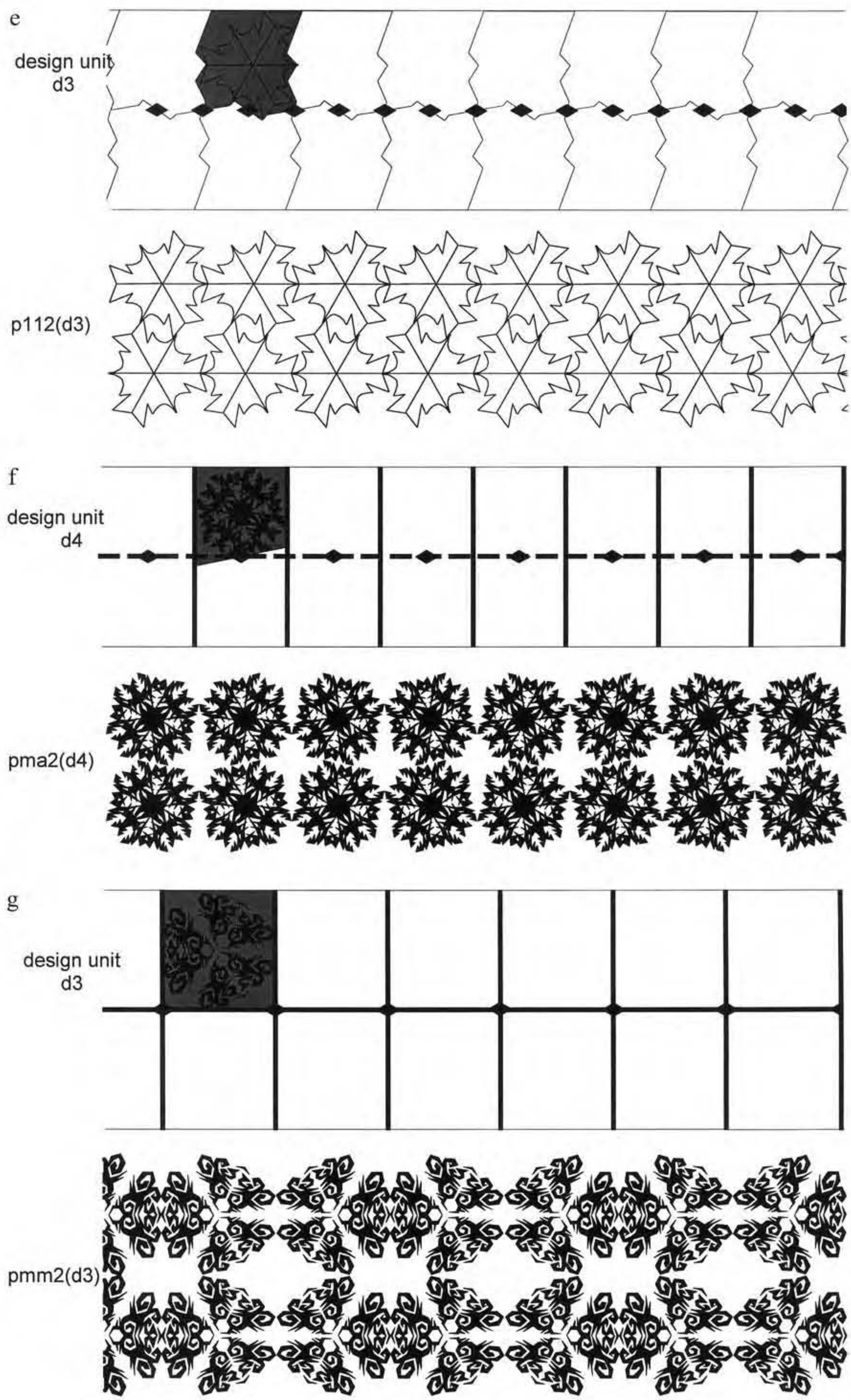


Figure 3.15 (cont.)

instance the relevant conditions, for each symmetry subgroup, have been explained above.

3.8 Construction of ditranslational designs

The methods for constructing ditranslational designs, in this classification system, are similar to those discussed in Section 2.13. Again, in each of the 34 $pnxy(cN)$ and $pnxy(dN)$ subgroups it is assumed that no additional symmetries, of the form described for monotranslational designs in Section 2.12, are induced into the design structure on applying the generating symmetries.

The extra conditions concerning the symmetry of the initial design unit and its positioning and orientation in relation to the symmetries in the underlying structure of the design are described in detail for each symmetry subgroup. When the positioning of the design unit is not critical, no further explanation is given. For symmetry groups $p1$ and $p2$, extensions have been made to include designs based on any of the five lattice structures. However, for these two cases, using a rhombic, hexagonal or ordinary parallelogram structure as a basis would require an adaptation to the construction processes if used in the context of screen printing (as described in Chapter 2). This is due to the non-perpendicular translation directions of the unit cell in these lattices. In these instances translation T_2 would be taken to be the length and direction of an oblique side of unit cell (i.e. a side which is not parallel to the longitudinal axis of the strip).

3.8.1 Construction of symmetry subgroups pnxy(cN)

The initial design unit, having specific N-fold rotational symmetry, is added to a symmetrically or asymmetrically-shaped fundamental region in a strip or double strip of unit cells. The strip and remainder of the design is completed by following the methods described in Chapter 2. In some cases (those marked with an asterisk in Table 3.8) conditions are imposed on the positioning of the centre of N-fold rotation and in others (those marked with a dash) a $pnxy(cN)$ subgroup is not constructable without altering the underlying symmetry group of the design structure. Table 3.8 is given after Section 3.8. The details of the process of construction for each $pnxy(cN)$ subgroup are discussed in the following subsections with a selection of illustrations.

3.8.1.1 Symmetry subgroups $p1xy(cN)$ and $c1xy(cN)$

Symmetry subgroup $p1(cN)$

A $p1(cN)$ design may be constructed on any of the five types of lattice however, its choice is significant in relation to the limitations on the possible values that N can take. If N is even, whatever the underlying lattice structure, there is no $p1(cN)$ subgroup. If N is an odd number which is a multiple of three, there is no $p1(cN)$ subgroup on a hexagonal lattice. For each of the four remaining lattices, a $p1(cN)$ subgroup is constructable for any odd number from a $p111(cN)$ monotranslational design. An illustration showing the construction of symmetry subgroup $p1(c7)$ is given in Fig. 3.16(a).

Symmetry subgroup $pg(cN)$

A $pg(cN)$ design may be constructed on either a rectangular or square lattice. If N is even, the N-fold centre of rotation of the design unit must not lie on the glide–reflection axis or at a point, perpendicular distance $1/4W$ from the glide–reflection axis of the initial $p1a1$ monotranslational design. An illustration showing the construction of symmetry subgroup $pg(c2)$ is given in Fig. 3.16(b).

Symmetry subgroup $pm(cN)$

A $pm(cN)$ design may be constructed on either a rectangular or square lattice. If N is even, the N-fold centre of rotation of the design unit must not lie at a point,

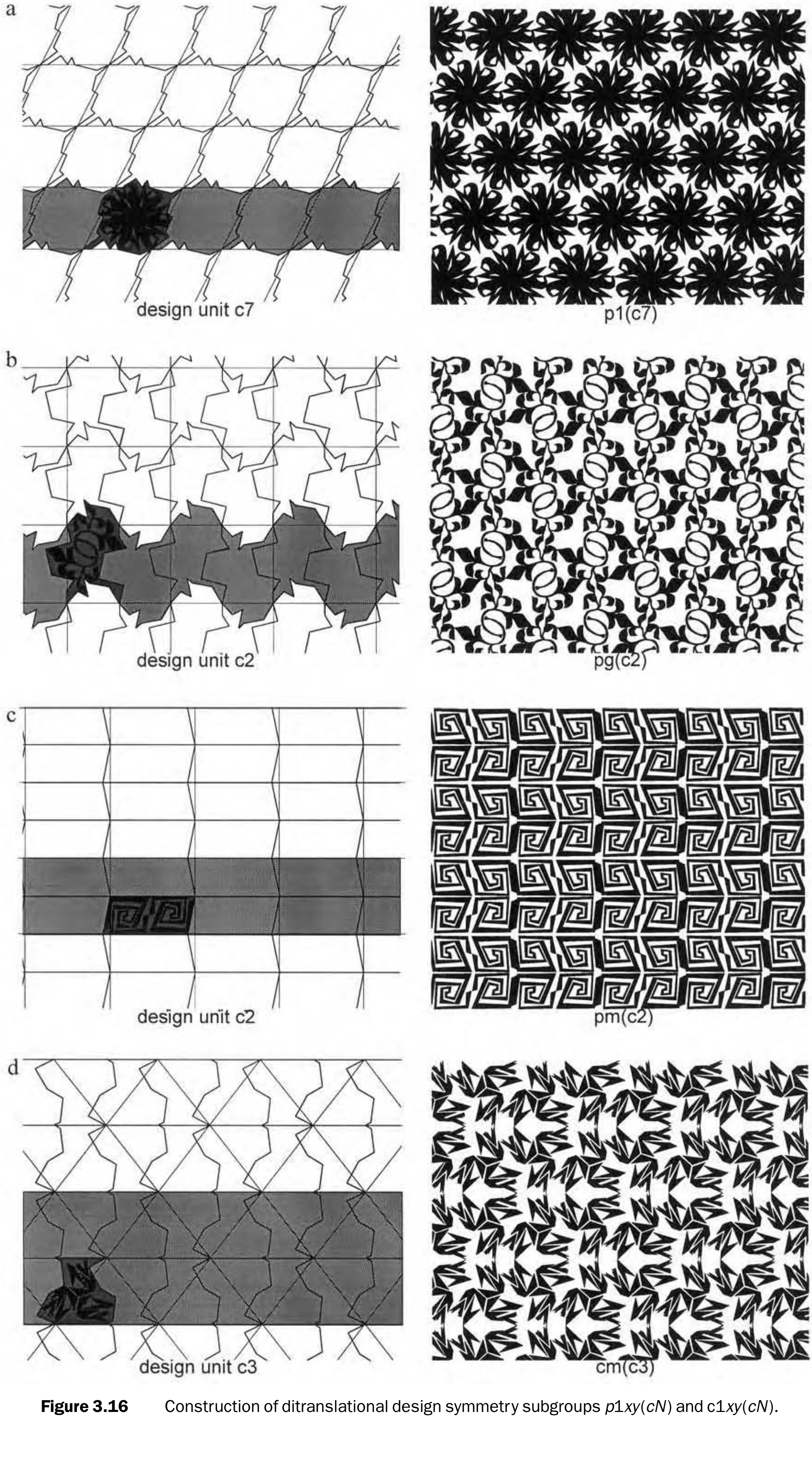


Figure 3.16 Construction of ditranslational design symmetry subgroups $p1xy(cN)$ and $c1xy(cN)$.

perpendicular distance $1/4W$ from the reflection axis of the initial $p1m1$ monotranslational design. An illustration showing the construction of symmetry subgroup $pm(c2)$ is given in Fig. 3.16(c).

Symmetry subgroup $cm(cN)$

A $cm(cN)$ design may be constructed on either a square, rhombic or hexagonal lattice. Following a similar method of construction to that of symmetry group *cm* described in Section 2.13.1 (and illustrated in Fig. 2.36), if N is even, the N-fold centre of rotation of the design unit must not lie at a point, on the glide–reflection axis of the initial $p1a1$ monotranslational design. Furthermore, if N is a multiple of three and the design is constructed on a hexagonal lattice, the N-fold centre of rotation of the design unit must not lie at a point, $1/6W$ from the glide–reflection axis of the initial $p1a1$ monotranslational design. An illustration showing the construction of symmetry subgroup $cm(c3)$ is given in Fig. 3.16(d).

3.8.1.2 Symmetry subgroups $p2xy(cN)$ and $c2xy(cN)$

Symmetry subgroups $p2(cN)$

A $p2(cN)$ design may be constructed on any of the five types of lattice. If N is an even number, the N-fold centre of rotation of the design unit must not lie at a point half way along a straight line joining adjacent centres of two-fold rotation of the underlying structure. If N is an odd number which is a multiple of three, there is no $p2(cN)$ subgroup on a hexagonal lattice if the N-fold centre of rotation is positioned at the centre of one of the two equilateral triangles of the unit cell. For each of the four remaining lattices, a $p2(cN)$ subgroup is constructable for any odd number. An illustration showing the construction of symmetry subgroup $p2(c3)$ is given in Fig. 3.17(a).

Symmetry subgroup $pgg(cN)$

A $pgg(cN)$ design may be constructed on either a rectangular or square lattice. If N is even, the N-fold centre of rotation of the design unit must not lie at a point half way along a straight line joining adjacent centres of two-fold rotation of the underlying structure of the initial $p112$ monotranslational design. An illustration showing the construction of symmetry subgroup $pgg(c2)$ is given in Fig. 3.17(b) on a rectangular lattice.

Symmetry subgroup $pmg(cN)$

A $pmg(cN)$ design may be constructed on either a rectangular or square lattice. There are no limitations imposed on the value or positioning of the N-fold centre of rotation of the design unit in the initial $pma2$ monotranslational design. An illustration showing the construction of symmetry subgroup $pmg(c5)$ is given in Fig. 3.17(c).

Symmetry subgroup $pmm(cN)$

A $pmm(cN)$ design may be constructed on either a rectangular or square lattice. If N is even, the N-fold centre of rotation of the design unit must not lie at a point at the centre of a fundamental region in the initial $pmm2$ monotranslational design. An illustration showing the construction of symmetry subgroup $pmm(c4)$ is given in Fig. 3.17(d).

Symmetry subgroup $cmm(cN)$

A $cmm(cN)$ design may be constructed on either a square, rhombic or hexagonal lattice. There are no limitations imposed on the value of N or the positioning of the N-fold centre of rotation of the design unit in the initial $pma2$ monotranslational design. An illustration showing the construction of symmetry subgroup *cmm*($c2$) is given in Fig. 3.17(e).

3.8.1.3 Symmetry subgroups p3*xy*(*cN*)

Symmetry subgroup *p*3*xy*(*cN*)
A *p*3(*cN*) design may be constructed on a hexagonal lattice only. If N is even, the centre of rotation of the design unit must not lie on the point of intersection of the two diagonals of the unit cell (which is equivalent to the centre of a rhombic fundamental). An illustration showing the construction of symmetry subgroup *p*3(*c*2) is given in Fig. 3.18(a).

Symmetry subgroup *p*31*m*(*cN*)
A *p*31*m*(*cN*) design may be constructed on a hexagonal lattice only. There are no limitations imposed on the value of N or the positioning of the N-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*31*m*(*c*2) is given in Fig. 3.18(b).

Symmetry subgroup *p*3*m*1(*cN*)
A *p*3*m*1(*cN*) design may be constructed on a hexagonal lattice only. If N is a multiple of three, the centre of rotation of the design unit must not lie at the centre of the equilateral triangular shaped fundamental region. An illustration showing the construction of symmetry subgroup *p*3*m*1(*c*4) is given in Fig. 3.18(c).

3.8.1.4 Symmetry subgroups p4*xy*(*cN*)

Symmetry subgroups *p*4(*cN*)
A *p*4(*cN*) design may be constructed on a square lattice only. If N is even, the centre of rotation of the design unit must not lie at the centre of a square fundamental region (which is equivalent to the mid-point of a straight line joining adjacent centres of four-fold rotation). An illustration showing the construction of symmetry subgroup *p*4(*c*4) is given in Fig. 3.19(a).

Symmetry subgroup *p*4*g*(*cN*)
A *p*4*g*(*cN*) design may be constructed on a square lattice only. There are no limitations imposed on the value of N or the positioning of the N-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*4*g*(*c*2) is given in Fig. 3.19(b).

Symmetry subgroup *p*4*m*(*cN*)
A *p*4*m*(*cN*) design may be constructed on a square lattice only. There are no limitations imposed on the value of N or the positioning of the N-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*4*m*(*c*3) is given in Fig. 3.19(c).

3.8.1.5 Symmetry subgroups p6*xy*(*cN*)

Symmetry subgroup *p*6(*cN*)
A *p*6(*cN*) design may be constructed on a hexagonal lattice only. There are no limitations imposed on the value of N or the positioning of the N-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*6(*c*5) is given in Fig. 3.20(a).

Symmetry subgroup *p*6*m*(*cN*)
A *p*6*m*(*cN*) design may be constructed on a hexagonal lattice only. There are no limitations imposed on the value of N or the positioning of the N-fold centre of

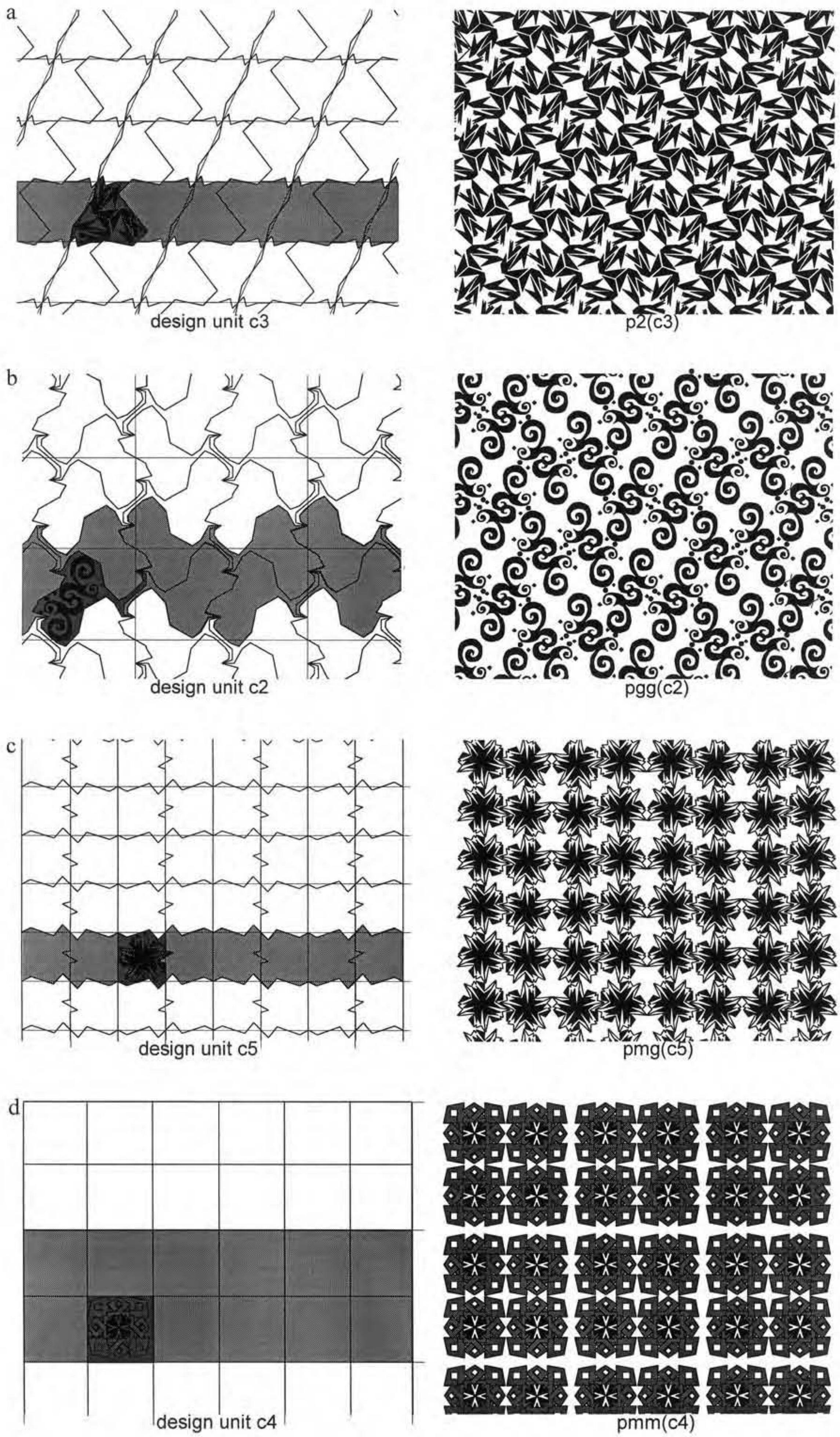


Figure 3.17 Construction of ditranslational design symmetry subgroups *p*2*xy*(*cN*) and *c*2*xy*(*cN*).

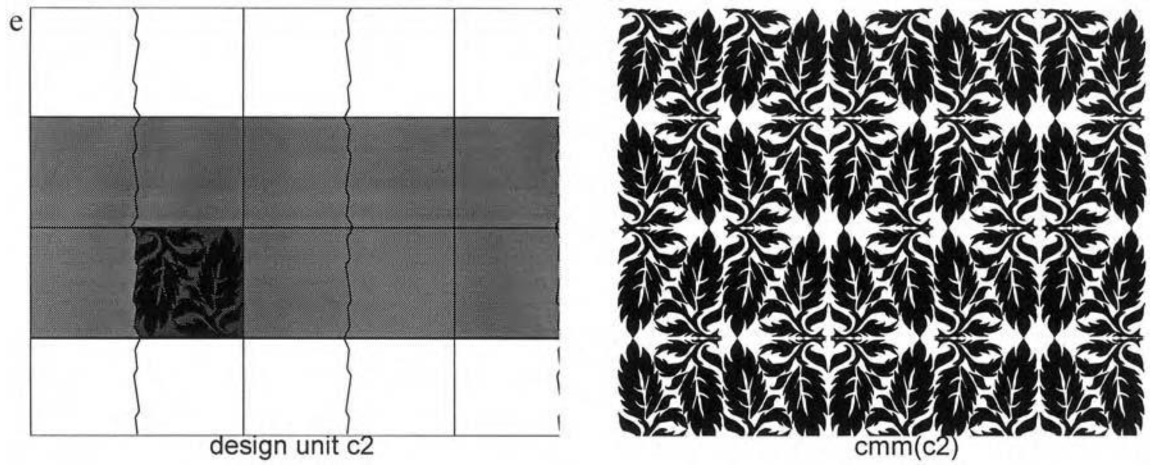


Figure 3.17 (cont.)

rotation of the design unit. An illustration showing the construction of symmetry subgroup $p6m(c2)$ is given in Fig. 3.20(b).

3.8.2 Construction of symmetry subgroups pnxy(dN)

The initial design unit, having N reflection axes and N-fold rotational symmetry, is added to a symmetrically or asymmetrically-shaped fundamental region in a strip or double strip of unit cells. The strip and remainder of the design is completed by following the methods described in Chapter 2. In some cases (those marked with an asterisk in Table 3.9) conditions are imposed on the positioning of the N reflection axes and in others (those marked with a dash) a $pnxy(dN)$ subgroup is not constructable without altering the underlying symmetry group of the design structure. Table 3.9 is given after Section 3.8. The details of the process of construction for each $pnxy(dN)$ subgroup are discussed in the following subsections with a selection of illustrations.

3.8.2.1 Symmetry subgroups $p1xy(dN)$ and $c1xy(dN)$

Symmetry subgroup $p1(dN)$

A $p1(dN)$ design may be constructed on any of the five types of lattice. However, the choice of lattice is significant in relation to the limitations on the possible values that N can take. If N is even, whatever the underlying lattice structure, there is no $p1(dN)$ subgroup. If N is an odd number which is a multiple of three, there is no $p1(dN)$ subgroup on a hexagonal lattice. For any other odd number, the design unit must have no reflection axes either parallel or perpendicular to the sides or diagonals of the unit cell of a hexagonal lattice. For each of the four remaining lattices, a $p1(dN)$ subgroup is constructable for any odd number provided that the following conditions are satisfied: on a rectangular lattice, the design unit must have no reflection axes parallel to the sides of the unit cell; on a square lattice it must have no reflection axes parallel to the sides or diagonals of the unit cell; on a rhombic lattice it must have no reflection axes parallel to the diagonals of the unit cell; on an ordinary parallelogram lattice there are no further limitations beyond N being odd. An illustration showing the construction of symmetry subgroup $p1(dN)$ is given in Fig. 3.21(a) for $N = 3$.

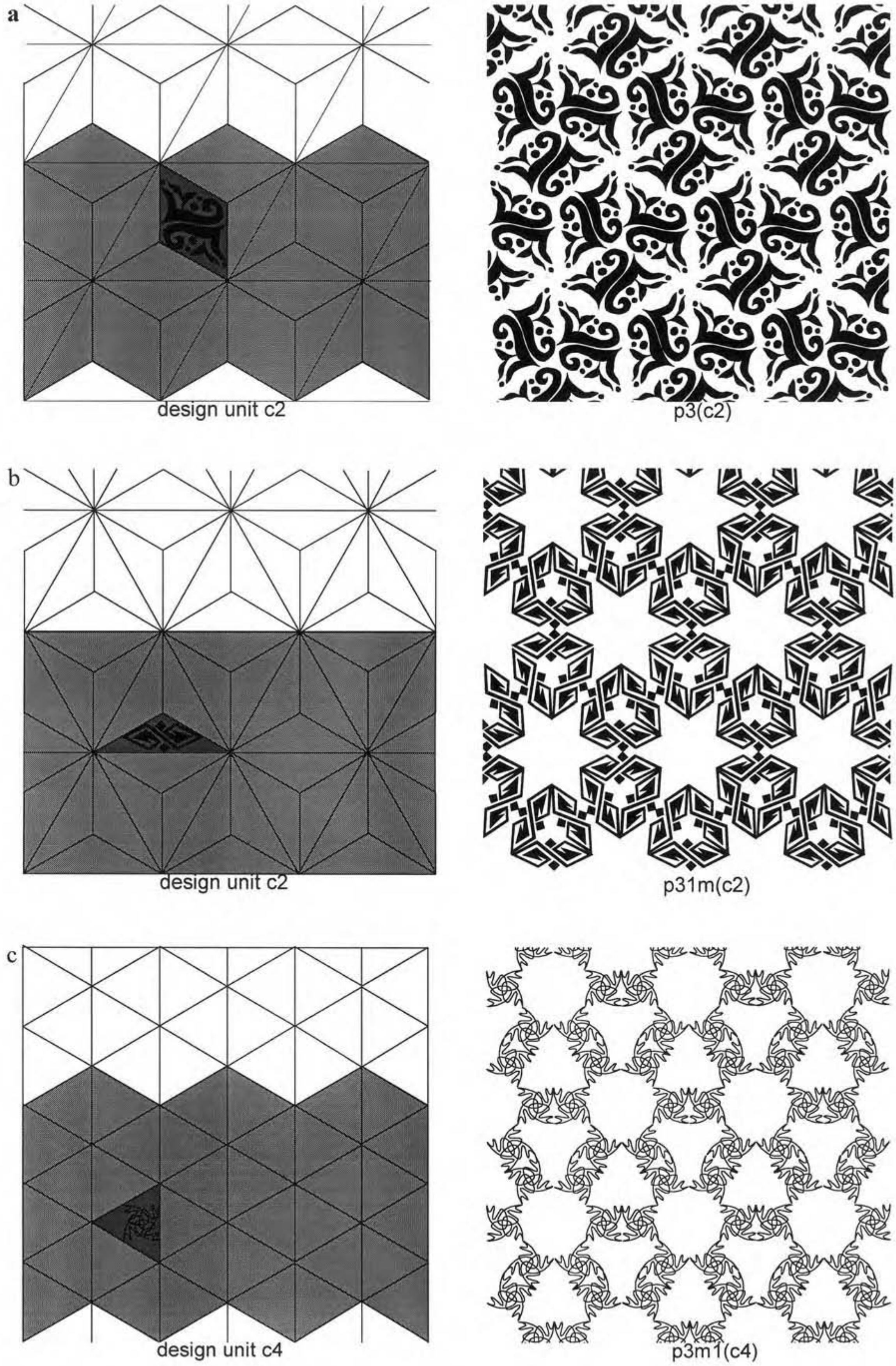


Figure 3.18 Construction of ditranslational design symmetry subgroups *p*3*xy*(*cN*).

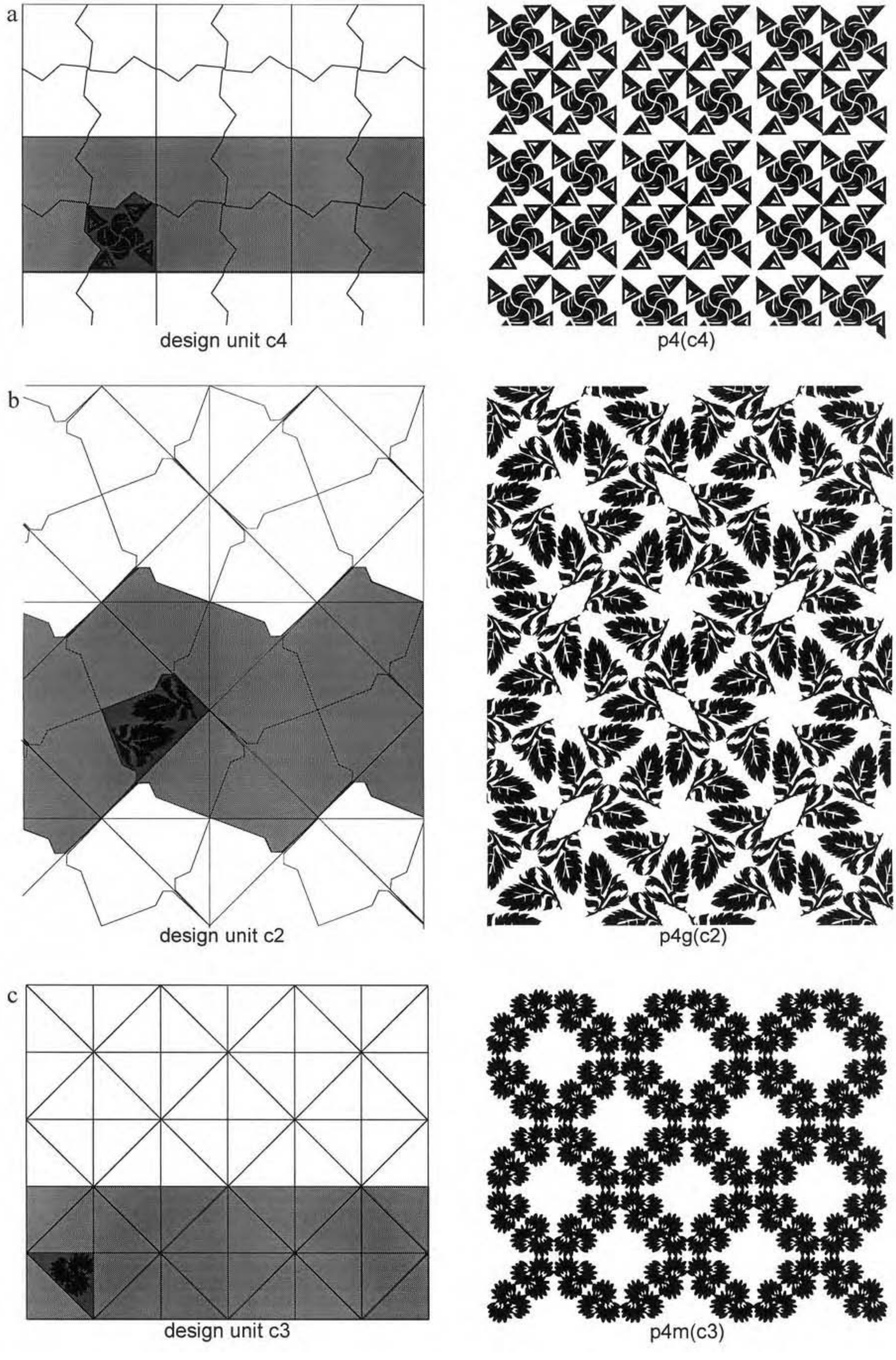


Figure 3.19 Construction of ditranslational design symmetry subgroups $p4xy(cN)$.

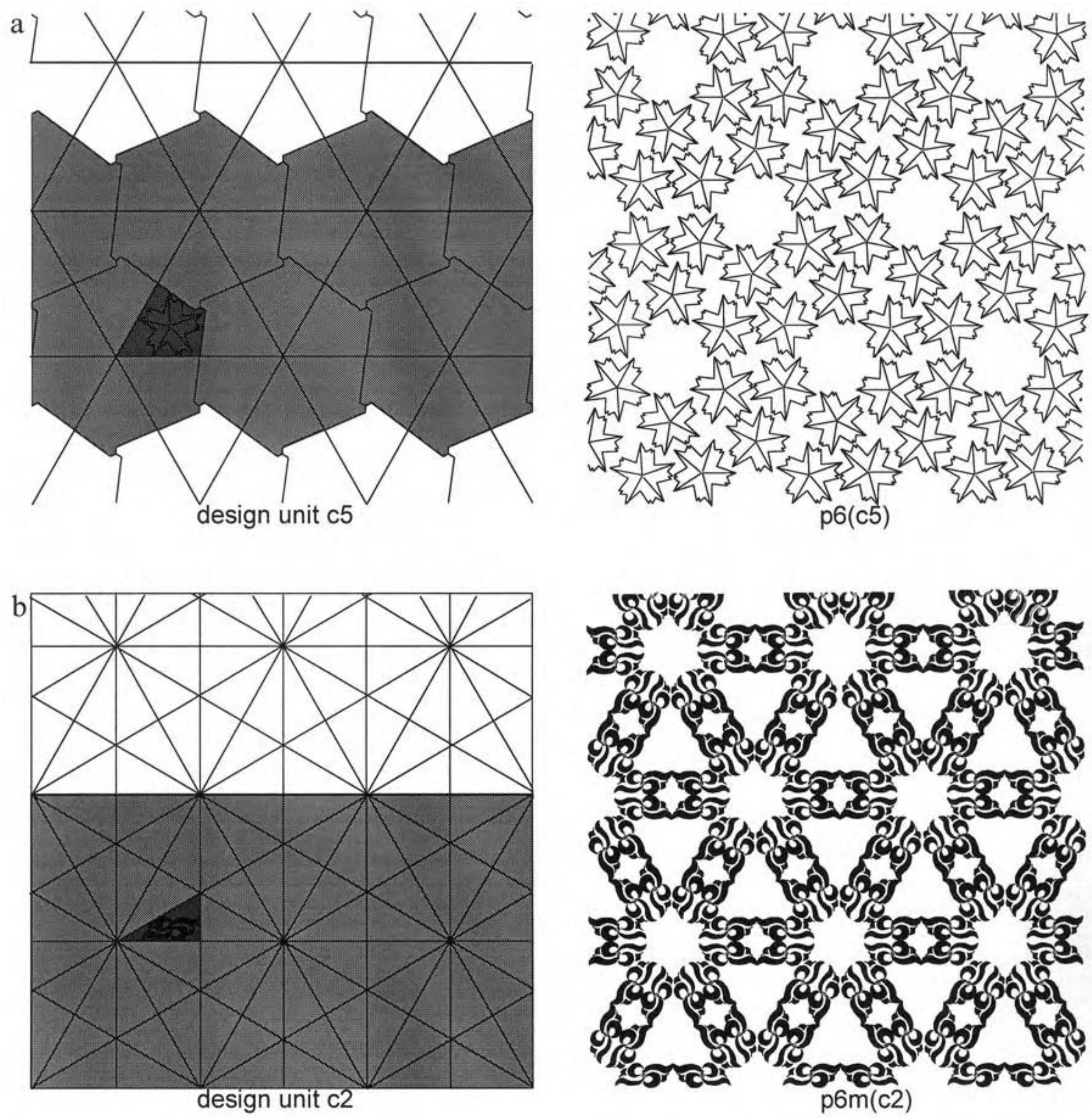


Figure 3.20 Construction of ditranslational design symmetry subgroups *p*6*xy*(*cN*).

Symmetry subgroup *pg*(*dN*)
A *pg*(*dN*) design may be constructed on either a rectangular or square lattice. Any reflection axis of the design unit must not be perpendicular to, nor be parallel to and coincide with nor be at perpendicular distance 1/4*W* from the glide–reflection axis of the initial *p*1*a*1 monotranslational design. Nor must the centre of rotation of a *dN* design unit lie at a point of perpendicular distance 1/4*W* from the glide–reflection axis. An illustration showing the construction of symmetry subgroup *pg*(*dN*) is given in Fig. 3.21(b) for $N = 1$.

Symmetry subgroup *pm*(*dN*)
A *pm*(*dN*) design may be constructed on either a rectangular or square lattice. Any reflection axis of the design unit must not be perpendicular to, nor be parallel to and at perpendicular distance 1/4*W* from the reflection axis of the initial *p*1*m*1 monotranslational design. Nor must the centre of rotation of a *dN* design unit lie at a point of perpendicular distance 1/4*W* from the reflection axis if *N* is even. An illustration showing the construction of symmetry subgroup *pm*(*dN*) is given in Fig. 3.21(c) for $N = 3$.

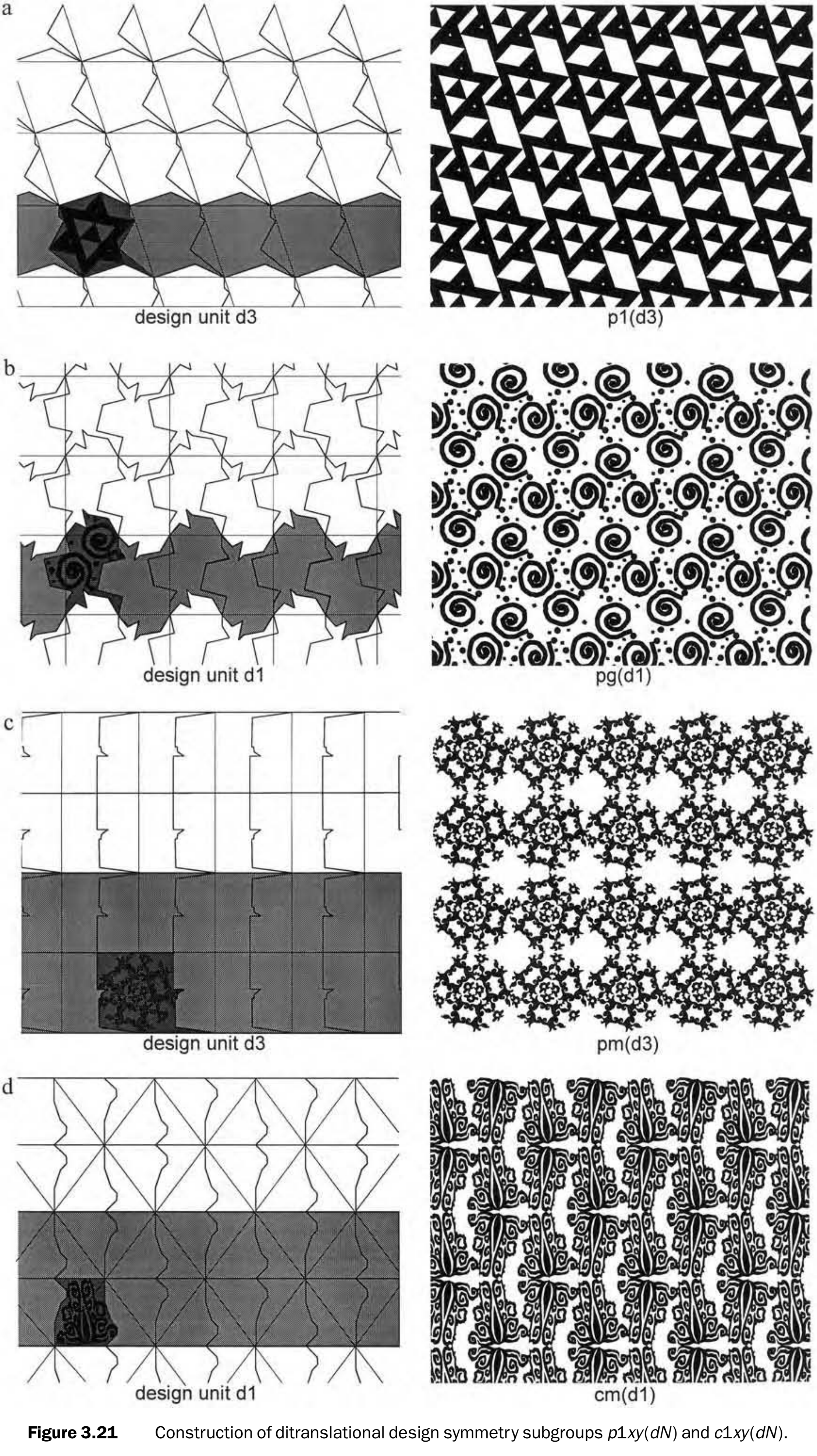


Figure 3.21 Construction of ditranslational design symmetry subgroups $p1xy(dN)$ and $c1xy(dN)$.

Symmetry subgroup *cm*(*dN*)

A *cm*(*dN*) design is based on either a square, rhombic or hexagonal lattice. However the following method of construction is similar to that described for symmetry group *cm* in Section 2.13.1 (and illustrated in Fig. 2.36) where the initial strip is a *p*1*a*1 design based on a square or rectangular lattice. If N is even, none of the N reflection axes of the design unit must lie parallel to, nor must their point of intersection lie at a point on, the glide–reflection axis of the initial monotranslational design of unit cells. Nor must a reflection axis, for any N, lie perpendicular to the glide–reflection axis of the initial *p*1*a*1 monotranslational design. An illustration showing the construction of symmetry subgroup *cm*(*dN*) is given in Fig. 3.21(d) for $N = 1$.

3.8.2.2 Symmetry subgroups p2xy(dN) and c2xy(dN)

Symmetry subgroups *p*2(*dN*)

A *p*2(*dN*) design may be constructed on any of the five types of lattice. For a design structured on an ordinary parallelogram lattice there are no restrictive conditions imposed on a design unit having one reflection axis, provided that the unit cell is not comprised of two rhombic parallelograms (in which case a reflection axis must not coincide with a diagonal of a rhombic-shaped fundamental region). If the design unit has more than one reflection axis the intersection of the reflection axes forms a centre of rotation which must not be positioned half way along a straight line joining adjacent centres of two-fold rotation.

For a rectangular lattice, any reflection axes may not be positioned parallel to the longitudinal axis of the original monotranslational design or perpendicular to this axis and pass through a point half way along a straight line joining adjacent centres of two-fold rotation. These conditions must also hold for a design structured on a square lattice with the additional requirements that the design unit may have no reflection axes coinciding with the diagonals of the unit cell.

For the rhombic lattice the design unit must have no reflection axes coinciding with the diagonals of the unit cell. This condition must also hold for the hexagonal lattice together with the design unit having no reflection axes bisecting one of the two equilateral triangles which make up a unit cell of the initial *p*112 monotranslational design. An illustration showing the construction of symmetry subgroup *p*2(*dN*) is given in Fig. 3.22(a) for $N = 3$.

Symmetry subgroup *pgg*(*dN*)

A *pgg*(*dN*) design may be constructed on either a rectangular or square lattice. For a rectangular fundamental region or one based on a rectangle, if $N = 1$ the reflection axis of the design unit must not pass through a point positioned half way along a straight line joining adjacent centres of two-fold rotation and be parallel to the sides of a unit cell. If $N \geq 2$ no reflection axes may be parallel to the sides of a unit cell. For a square fundamental region or one based on a square, the design unit must satisfy these same conditions as well as having no reflection axes passing through a point positioned half way along a straight line joining adjacent centres of two-fold rotation and parallel to the diagonals of a fundamental region in the initial *p*112 monotranslational design. An illustration showing the construction of symmetry subgroup *pgg*(*dN*) is given in Fig. 3.22(b) for $N = 2$.

Symmetry subgroup *pmg*(*dN*)

A *pmg*(*dN*) design may be constructed on either a rectangular or square lattice. In either case, with a rectangular fundamental region or one based on a rectangle (or a square fundamental region or one based on a square), any of the reflection axes of the design unit must not lie on the glide–reflection axis nor parallel to it and at a distance $1/4W$ from it in the initial *pma*2 monotranslational design. Neither must any lie perpendicular to the glide–reflection axis and be positioned half way between adjacent reflection axes in the initial strip (in which case a reflection axis would pass through a centre of two-fold rotation). An illustration showing the construction of symmetry subgroup *pmg*(dN) is given in Fig. 3.22(c) for $N = 1$.

Symmetry subgroup *pmm*(dN)

A *pmm*(dN) design may be constructed on either a rectangular or square lattice. For a rectangular lattice, the design unit must have no reflection axes half way between and parallel to lines of reflectional symmetry of the underlying structure. For a square lattice the design unit must satisfy these same conditions and have no reflection axes coinciding with the diagonals of the square fundamental region. For both lattice structures, the point of intersection of the reflection axes (for $N \geq 2$) forms a centre of rotation which, for even N, must not be positioned at the centre of the fundamental region. An illustration showing the construction of symmetry subgroup *pmm*(dN) is given in Fig. 3.22(d) for $N = 1$.

Symmetry subgroup *cmm*(dN)

A *cmm*(dN) design may be constructed on either a square, rhombic or hexagonal lattice. However, the pma2 monotranslational design from which a *cmm*(dN) design may be constructed is based on a square or rectangular lattice. If the fundamental region is rectangular, the initial design unit may have no reflection axes either coinciding with the glide–reflection axis or perpendicular to it and half way between adjacent reflection axes in the monotranslational design structure. If the fundamental region is square (or the fundamental region is derived from a square) the design unit must satisfy these same conditions, and if N is even, there must be no reflection axes coinciding with the diagonals of the square fundamental region in the initial *pma*2 monotranslational design. An illustration showing the construction of symmetry subgroup *cmm*(dN) is given in Fig. 3.22(e) for $N = 2$.

3.8.2.3 Symmetry subgroups p3*xy*(*dN*)

Symmetry subgroup *p*3(dN)

A *p*3(dN) design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes coinciding with the diagonals of the unit cell. If N is even, the point of intersection of the reflection axes must not coincide with the point of intersection of the diagonals of the unit cell. An illustration showing the construction of symmetry subgroup *p*3(dN) is given in Fig. 3.23(a) for $N = 1$.

Symmetry subgroup *p*31*m*(dN)

A *p*31*m*(dN) design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes which bisect the isosceles triangle-shaped fundamental region. An illustration showing the construction of symmetry subgroup *p*31*m*(dN) is given in Fig. 3.23(b) for $N = 1$.

Symmetry subgroup *p*3*m*1(dN)

A *p*3*m*1(dN) design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes which bisect an equilateral triangle-shaped fundamental region or a centre of n-fold rotation, where n is a multiple of three, passing through the centre of this triangle. An illustration showing the construction of symmetry subgroup *p*3*m*1(dN) is given in Fig. 3.23(c) for $N = 4$.

3.8.2.4 Symmetry subgroups p4*xy*(*dN*)

Symmetry subgroup *p*4(dN)

A *p*4(dN) design may be constructed on a square lattice only. The design unit inside the fundamental region must have no reflection axes coinciding with the

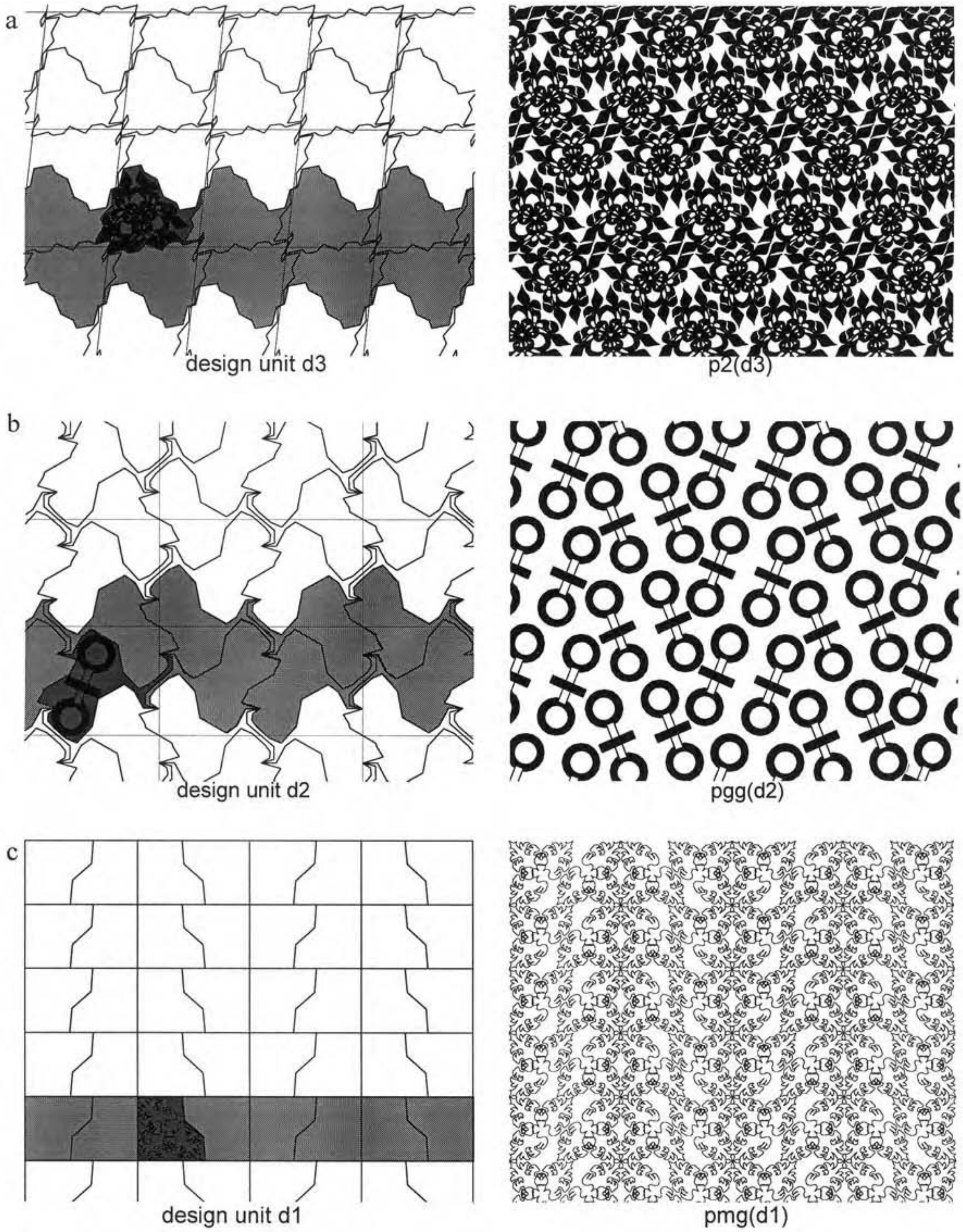


Figure 3.22 Construction of ditranslational design symmetry subgroups $p2xy(dN)$ and $c2xy(dN)$.

diagonals of the unit cell. Also, if N is even, the intersection of the reflection axes must not coincide with the centre of the square fundamental region (or the equivalent position for a fundamental region derived from a square). An illustration showing the construction of symmetry subgroup $p4(dN)$ is given in Fig. 3.24(a) for $N = 3$.

Symmetry subgroup $p4g(dN)$

A $p4g(dN)$ design may be constructed on a square lattice only. The design unit must have no reflection axes coinciding with the diagonals of the unit cell. An

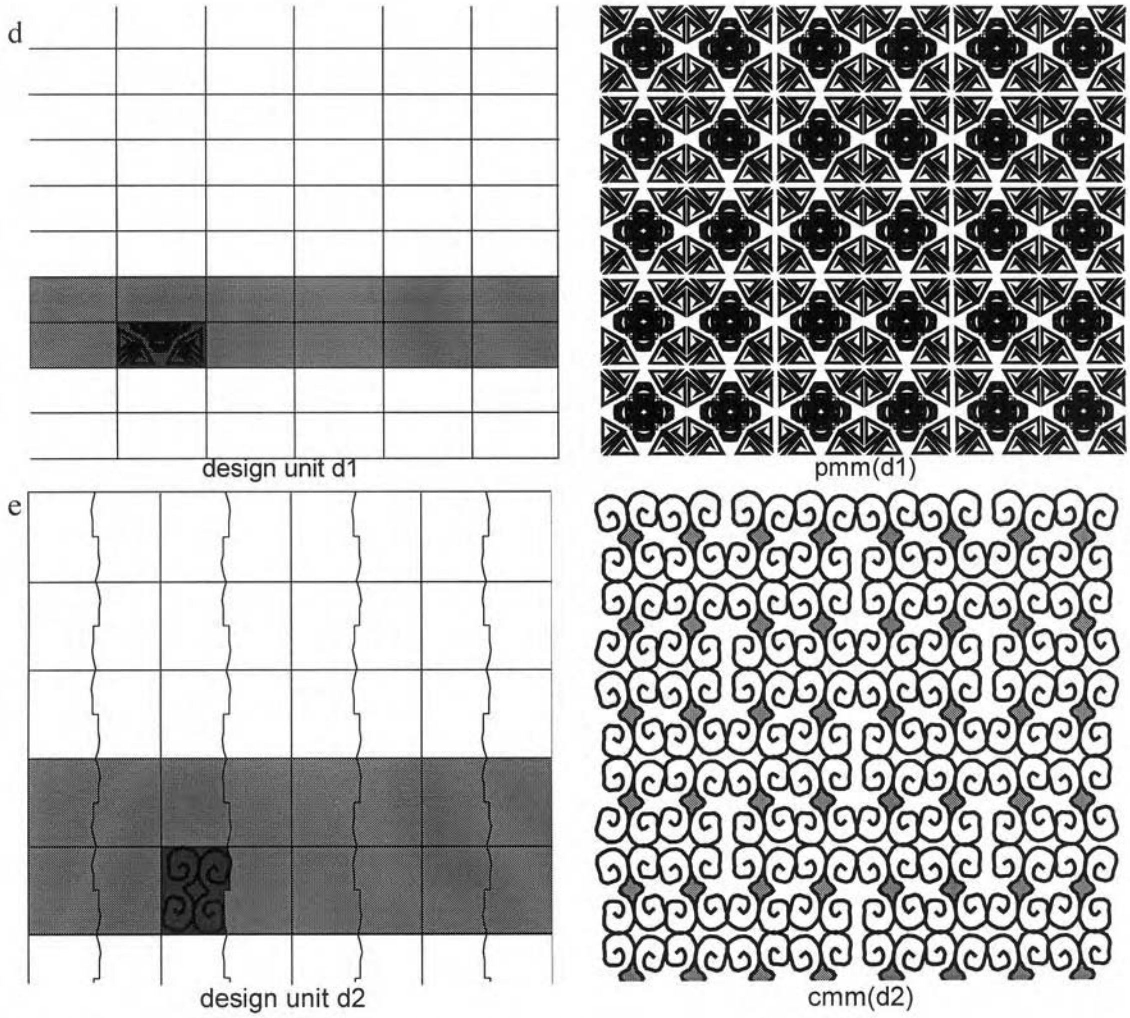


Figure 3.22 (cont.)

illustration showing the construction of symmetry subgroup $p4g(dN)$ is given in Fig. 3.24(b) for $N = 2$.

Symmetry subgroup $p4m(dN)$

A $p4m(dN)$ design may be constructed on a square lattice only. The design unit must have no reflection axes which bisect the fundamental region. An illustration showing the construction of symmetry subgroup $p4m(dN)$ is given in Fig. 3.24(c) for $N = 1$.

3.8.2.5 Symmetry subgroups p6xy(dN)

Symmetry subgroup $p6(dN)$

A $p6(dN)$ design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes coinciding with the long diagonal of the unit cell. An illustration showing the construction of symmetry subgroup $p6(dN)$ is given in Fig. 3.25(a) for $N = 3$.

Symmetry subgroup $p6m(dN)$

A $p6m(dN)$ design may be constructed on a hexagonal lattice only. The design unit may have reflection axes at any position within the fundamental region. An

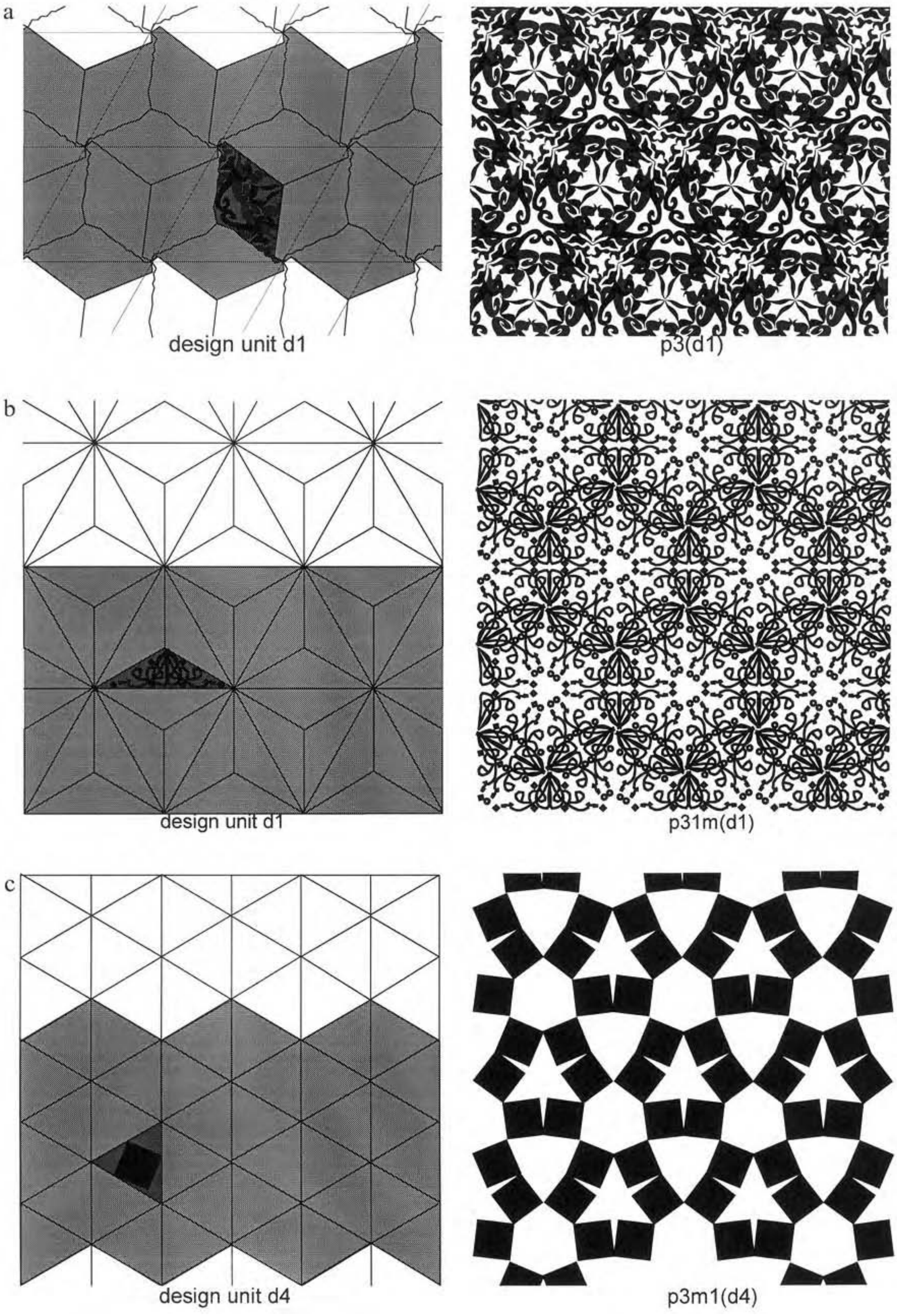


Figure 3.23 Construction of ditranslational design symmetry subgroups $p3xy(dN)$.

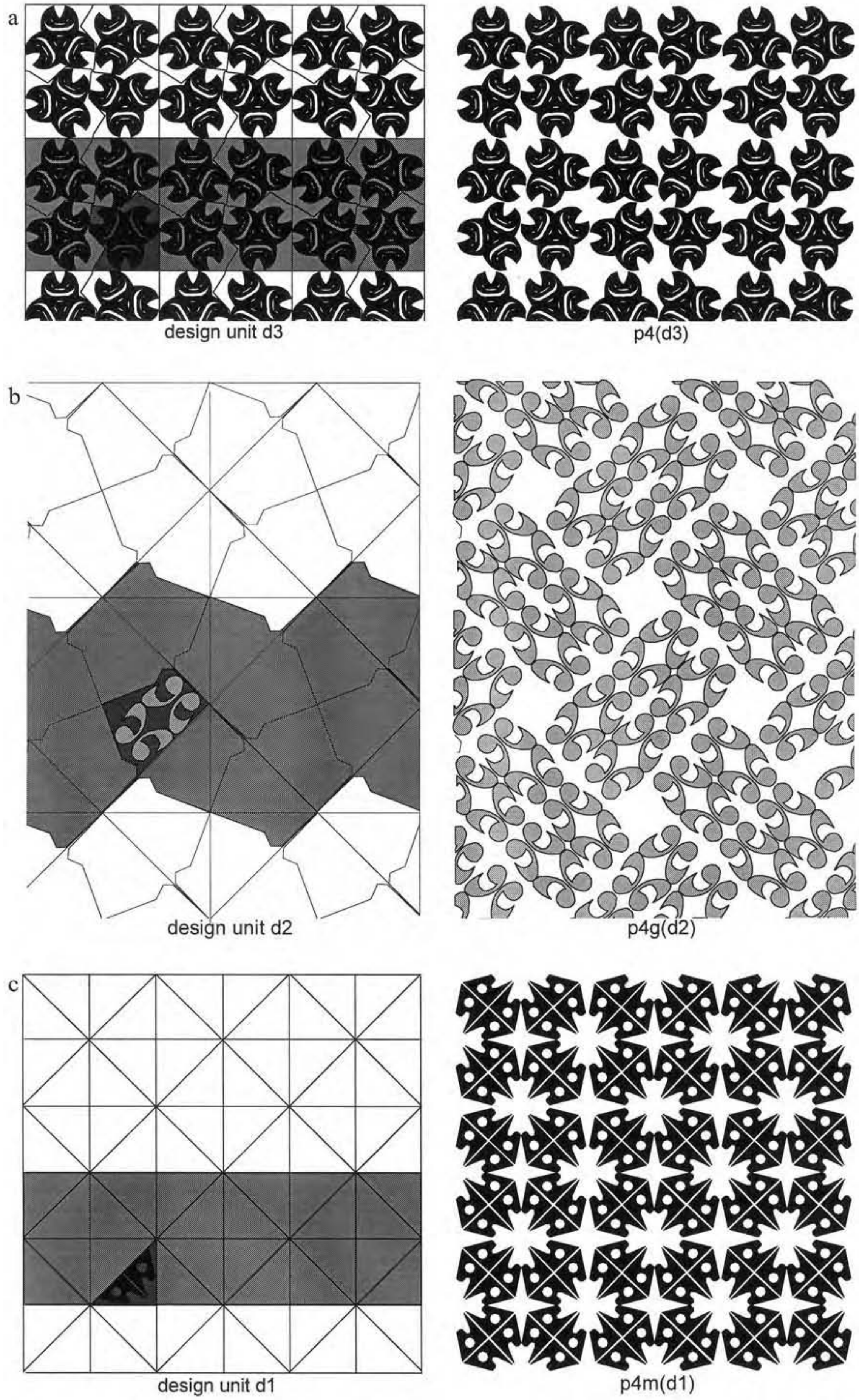


Figure 3.24 Construction of ditranslational design symmetry subgroups $p4xy(dN)$.

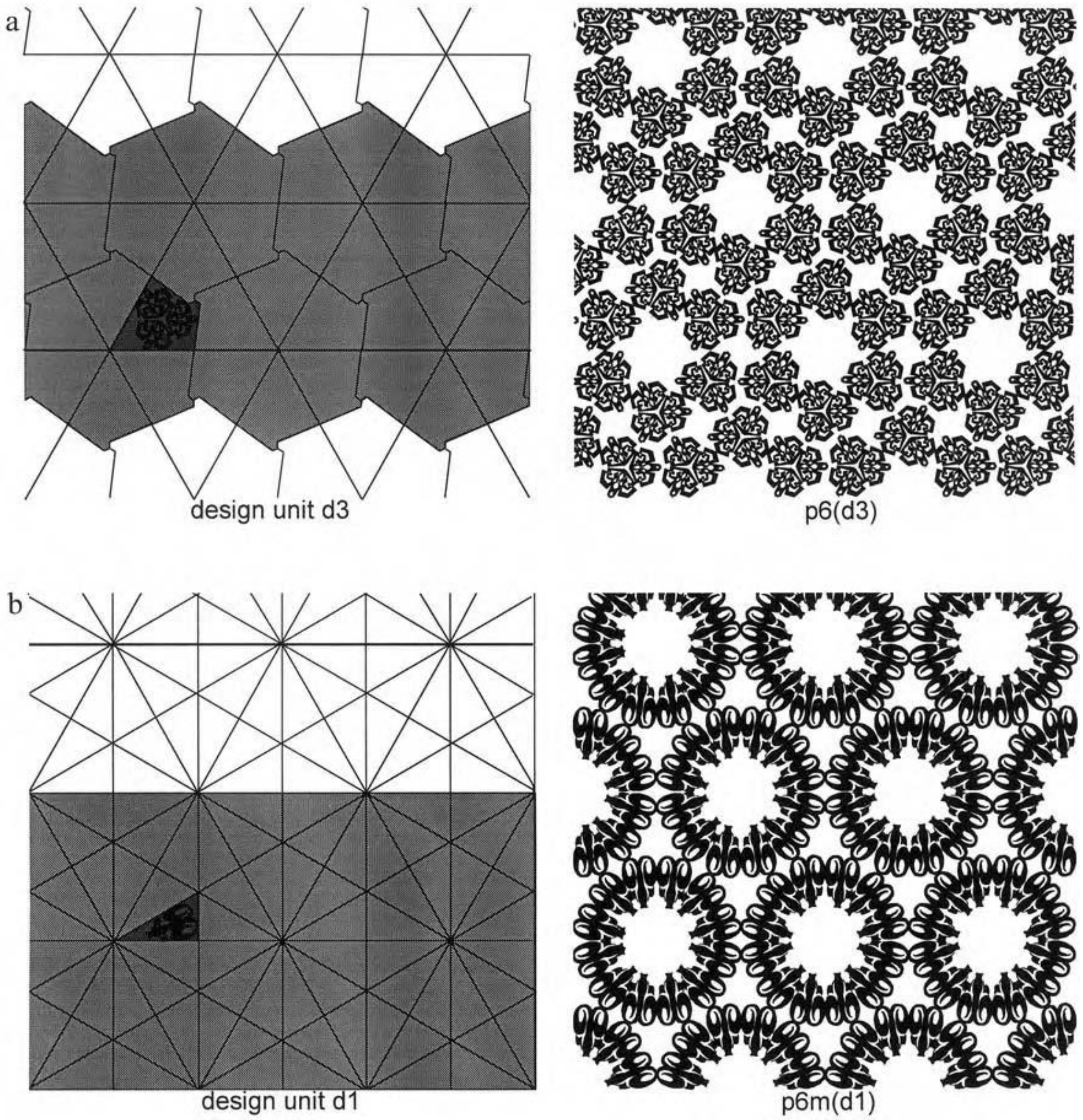


Figure 3.25 Construction of ditranslational design symmetry subgroups *p6xy(dN)*.

illustration showing the construction of symmetry subgroup $p6m(dN)$ is given in Fig. 3.25(b) for $N = 1$.

The construction of a design symmetry subgroup, as explained previously, is dependent on a variety of factors. Tables 3.8 and 3.9 indicate, for $N = 1$ to 6, whether a particular symmetry subgroup is constructable and if there are restrictions on the positioning and symmetric characteristics of the design unit.

3.9 Summary

The classification by symmetry group of design structure and design unit provides a new approach to design analysis. Some of the designs constructed from this classification system seem to exhibit a more 'chaotic' or 'random' appearance depending on the symmetry group and subgroup. (For example $p2(c3)$ in Fig. 3.17, $pg(d1)$ in Fig. 3.21 and $pgg(d2)$, $p2(d3)$, $cmm(d2)$ in Fig. 3.22 all seem to display what could be described as 'organised chaos'.) However, this hypothesis needs to be clarified with further investigation and illustration.

Table 3.8 Construction of symmetry subgroups *pnxy*(*cN*)

Symmetry group of design structure	Lattice type	Symmetry group of design unit c1	c2	c3	c4	c5	c6
p1	parallelogram	p1(c1)	—	p1(c3)	—	p1(c5)	—
	rectangular	p1(c1)	—	p1(c3)	—	p1(c5)	—
	square	p1(c1)	—	p1(c3)	—	p1(c5)	—
	rhombic	p1(c1)	—	p1(c3)	—	p1(c5)	—
	hexagonal	p1(c1)	—	—	—	p1(c5)	—
pg	rectangular	pg(c1)	pg(c2)*	pg(c3)	pg(c4)*	pg(c5)	pg(c6)*
	square	pg(c1)	pg(c2)*	pg(c3)	pg(c4)*	pg(c5)	pg(c6)*
pm	rectangular	pm(c1)	pm(c2)*	pm(c3)	pm(c4)*	pm(c5)	pm(c6)*
	square	pm(c1)	pm(c2)*	pm(c3)	pm(c4)*	pm(c5)	pm(c6)*
cm	square	cm(c1)	cm(c2)*	cm(c3)	cm(c4)*	cm(c5)	cm(c6)*
	rhombic	cm(c1)	cm(c2)*	cm(c3)	cm(c4)*	cm(c5)	cm(c6)*
	hexagonal	cm(c1)	cm(c2)*	cm(c3)	cm(c4)*	cm(c5)	cm(c6)*
p2	parallelogram	p2(c1)	p2(c2)*	p2(c3)	p2(c4)*	p2(c5)	p2(c6)*
	rectangular	p2(c1)	p2(c2)*	p2(c3)	p2(c4)*	p2(c5)	p2(c6)*
	square	p2(c1)	p2(c2)*	p2(c3)	p2(c4)*	p2(c5)	p2(c6)*
	rhombic	p2(c1)	p2(c2)*	p2(c3)	p2(c4)*	p2(c5)	p2(c6)*
	hexagonal	p2(c1)	p2(c2)*	p2(c3)*	p2(c4)*	p2(c5)	p2(c6)*
pgg	rectangular	pgg(c1)	pgg(c2)*	pgg(c3)	pgg(c4)*	pgg(c5)	pgg(c6)*
	square	pgg(c1)	pgg(c2)*	pgg(c3)	pgg(c4)*	pgg(c5)	pgg(c6)*
pmg	rectangular	pmg(c1)	pmg(c2)	pmg(c3)	pmg(c4)	pmg(c5)	pmg(c6)
	square	pmg(c1)	pmg(c2)	pmg(c3)	pmg(c4)	pmg(c5)	pmg(c6)
pmm	rectangular	pmm(c1)	pmm(c2)*	pmm(c3)	pmm(c4)*	pmm(c5)	pmm(c6)*
	square	pmm(c1)	pmm(c2)*	pmm(c3)	pmm(c4)*	pmm(c5)	pmm(c6)*
cmm	square	cmm(c1)	cmm(c2)	cmm(c3)	cmm(c4)	cmm(c5)	cmm(c6)
	rhombic	cmm(c1)	cmm(c2)	cmm(c3)	cmm(c4)	cmm(c5)	cmm(c6)
	hexagonal	cmm(c1)	cmm(c2)	cmm(c3)	cmm(c4)	cmm(c5)	cmm(c6)
p3	hexagonal	p3(c1)	p3(c2)*	p3(c3)	p3(c4)*	p3(c5)	p3(c6)*
p31m	hexagonal	p31m(c1)	p31m(c2)	p31m(c3)	p31m(c4)	p31m(c5)	p31m(c6)
p3m1	hexagonal	p3m1(c1)	p3m1(c2)	p3m1(c3)*	p3m1(c4)	p3m1(c5)	p3m1(c6)*
p4	square	p4(c1)	p4(c2)*	p4(c3)	p4(c4)*	p4(c5)	p4(c6)*
p4g	square	p4g(c1)	p4g(c2)	p4g(c3)	p4g(c4)	p4g(c5)	p4g(c6)
p4m	square	p4m(c1)	p4m(c2)	p4m(c3)	p4m(c4)	p4m(c5)	p4m(c6)
p6	hexagonal	p6(c1)	p6(c2)	p6(c3)	p6(c4)	p6(c5)	p6(c6)
p6m	hexagonal	p6m(c1)	p6m(c2)	p6m(c3)	p6m(c4)	p6m(c5)	p6m(c6)

Throughout this chapter a classification system has been developed with the primary aim of encouraging awareness amongst designers of the possible symmetrical characteristics a motif may have within the fundamental region of a symmetry group. Notation has been devised to account for the categories of design under discussion. Explanations and illustrations have been given to promote understanding of the concepts involved. Schematic and additional illustrations have been shown for a wide range of the designs which may be categorised in this classification system. Construction techniques have been discussed for finite, monotranslational and ditranslational designs for all values

Table 3.9 Construction of symmetry subgroups *pnxy*(*dN*)

Symmetry group of design structure	Lattice type	Symmetry group of design unit *d*1	*d*2	*d*3	*d*4	*d*5	*d*6
*p*1	parallelogram	*p*1(*d*1)	—	*p*1(*d*3)	—	*p*1(*d*5)	—
	rectangular	*p*1(*d*1)*	—	*p*1(*d*3)*	—	*p*1(*d*5)*	—
	square	*p*1(*d*1)*	—	*p*1(*d*3)*	—	*p*1(*d*5)*	—
	rhombic	*p*1(*d*1)*	—	*p*1(*d*3)*	—	*p*1(*d*5)*	—
	hexagonal	*p*1(*d*1)*	—	—	—	*p*1(*d*5)*	—
pg	rectangular	*pg*(*d*1)*	*pg*(*d*2)*	*pg*(*d*3)*	*pg*(*d*4)*	*pg*(*d*5)*	*pg*(*d*6)*
	square	*pg*(*d*1)*	*pg*(*d*2)*	*pg*(*d*3)*	*pg*(*d*4)*	*pg*(*d*5)*	*pg*(*d*6)*
pm	rectangular	*pm*(*d*1)*	*pm*(*d*2)*	*pm*(*d*3)*	*pm*(*d*4)*	*pm*(*d*5)*	*pm*(*d*6)*
	square	*pm*(*d*1)*	*pm*(*d*2)*	*pm*(*d*3)*	*pm*(*d*4)*	*pm*(*d*5)*	*pm*(*d*6)*
cm	square	*cm*(*d*1)*	*cm*(*d*2)*	*cm*(*d*3)*	*cm*(*d*4)*	*cm*(*d*5)*	*cm*(*d*6)*
	rhombic	*cm*(*d*1)*	*cm*(*d*2)*	*cm*(*d*3)*	*cm*(*d*4)*	*cm*(*d*5)*	*cm*(*d*6)*
	hexagonal	*cm*(*d*1)*	*cm*(*d*2)*	*cm*(*d*3)*	*cm*(*d*4)*	*cm*(*d*5)*	*cm*(*d*6)*
*p*2	parallelogram	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	rectangular	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	square	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	rhombic	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	hexagonal	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
pgg	rectangular	*pgg*(*d*1)*	*pgg*(*d*2)*	*pgg*(*d*3)*	*pgg*(*d*4)*	*pgg*(*d*5)*	*pgg*(*d*6)*
	square	*pgg*(*d*1)*	*pgg*(*d*2)*	*pgg*(*d*3)*	*pgg*(*d*4)*	*pgg*(*d*5)*	*pgg*(*d*6)*
pmg	rectangular	*pmg*(*d*1)*	*pmg*(*d*2)*	*pmg*(*d*3)*	*pmg*(*d*4)*	*pmg*(*d*5)*	*pmg*(*d*6)*
	square	*pmg*(*d*1)*	*pmg*(*d*2)*	*pmg*(*d*3)*	*pmg*(*d*4)*	*pmg*(*d*5)*	*pmg*(*d*6)*
pmm	rectangular	*pmm*(*d*1)*	*pmm*(*d*2)*	*pmm*(*d*3)*	*pmm*(*d*4)*	*pmm*(*d*5)*	*pmm*(*d*6)*
	square	*pmm*(*d*1)*	*pmm*(*d*2)*	*pmm*(*d*3)*	*pmm*(*d*4)*	*pmm*(*d*5)*	*pmm*(*d*6)*
cmm	square	*cmm*(*d*1)*	*cmm*(*d*2)*	*cmm*(*d*3)*	*cmm*(*d*4)*	*cmm*(*d*5)*	*cmm*(*d*6)*
	rhombic	*cmm*(*d*1)*	*cmm*(*d*2)*	*cmm*(*d*3)*	*cmm*(*d*4)*	*cmm*(*d*5)*	*cmm*(*d*6)*
	hexagonal	*cmm*(*d*1)*	*cmm*(*d*2)*	*cmm*(*d*3)*	*cmm*(*d*4)*	*cmm*(*d*5)*	*cmm*(*d*6)*
*p*3	hexagonal	*p*3(*d*1)*	*p*3(*d*2)*	*p*3(*d*3)*	*p*3(*d*4)*	*p*3(*d*5)*	*p*3(*d*6)*
*p*31*m*	hexagonal	*p*31*m*(*d*1)*	*p*31*m*(*d*2)*	*p*31*m*(*d*3)*	*p*31*m*(*d*4)*	*p*31*m*(*d*5)*	*p*31*m*(*d*6)*
*p*3*m*1	hexagonal	*p*3*m*1(*d*1)*	*p*3*m*1(*d*2)*	*p*3*m*1(*d*3)*	*p*3*m*1(*d*4)*	*p*3*m*1(*d*5)*	*p*3*m*1(*d*6)*
*p*4	square	*p*4(*d*1)*	*p*4(*d*2)*	*p*4(*d*3)*	*p*4(*d*4)*	*p*4(*d*5)*	*p*4(*d*6)*
*p*4*g*	square	*p*4*g*(*d*1)*	*p*4*g*(*d*2)*	*p*4*g*(*d*3)*	*p*4*g*(*d*4)*	*p*4*g*(*d*5)*	*p*4*g*(*d*6)*
*p*4*m*	square	*p*4*m*(*d*1)*	*p*4*m*(*d*2)*	*p*4*m*(*d*3)*	*p*4*m*(*d*4)*	*p*4*m*(*d*5)*	*p*4*m*(*d*6)*
*p*6	hexagonal	*p*6(*d*1)*	*p*6(*d*2)*	*p*6(*d*3)*	*p*6(*d*4)*	*p*6(*d*5)*	*p*6(*d*6)*
*p*6*m*	hexagonal	*p*6*m*(*d*1)	*p*6*m*(*d*2)	*p*6*m*(*d*3)	*p*6*m*(*d*4)	*p*6*m*(*d*5)	*p*6*m*(*d*6)

of N, the number of reflection axes and/or order of rotation of the design unit.

The disordered or chaotic appearance of particular designs within this classification system may occur because the symmetries of the design unit do not coincide with ones in the design structure. (However, the proof of this suggestion is beyond the realms of discussion of this book.) Conversely, if their symmetries do pass through ones in the structure, this presents yet another view of possible design characteristics from which a further classification system may be derived. This classification is discussed in detail in Chapter 4.

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