4

Classification of discrete patterns

4.1 Introduction

As explained previously, there are numerous different ways of classifying designs. The methods in Chapters 2 and 3 identify a tiling or pattern class by the symmetry group of its design unit and/or design structure. The following classification of monomotif, discrete patterns involves the recognition not only of the symmetries of the pattern structure but also the group of symmetries in the structure which pass through a motif. This classification system (as Grünbaum and Shephard comment)[1] does have its limitations in that it is only applicable to a particular range of patterns in which there are restrictions imposed on both the characteristics of the motif and, with its repetition, the pattern it produces. These designs therefore generally exhibit a more rigid and ordered appearance compared to those of the previous two chapters, because adjacent motifs may not touch, overlap or intertwine with adjacent motifs.

As discussed in Chapter 2, a motif may possess a variety of different features. One type of pattern, resulting from the regular repetition of a motif with particular limitations on its characteristics, is referred to as a monomotif pattern.

4.2 Monomotif pattern

Grünbaum and Shephard,[1] in their classic work *Tilings and Patterns*, formally define a monomotif pattern as follows:

> A *monomotif pattern P with motif M* is a non-empty family $\{M_i \mid i \in I\}$ of sets in the plane, labelled by an index-set I, such that the following conditions hold:
> P.1 *The sets M_i are pairwise disjoint.*
> P.2 *Each M_i* is *congruent to M and called a* copy *of M.*
> P.3 *For each pair M_i, M_j of copies of the motif there is an isometry of the plane that maps P onto itself and M_i onto M_j.*

Less formally, a monomotif pattern may be thought of as one in which:

- P.1′ Each motif does not intersect or connect to (i.e. overlap or touch) any other motif.
- P.2′ Each motif is congruent to every other motif in the pattern. (Here, by congruent, as well as implying ‘direct’ congruence where the motifs are the same size and shape, a mapping from one motif to any other by reflection or glide–reflection is included in the definition. Strictly speaking, ‘congruence’ by reflection is given the term ‘indirect congruence’. It is important to note that certain authors do not include this reflective mapping in their definition of congruence, for example Shubnikov and Koptsik, when discussing whether an object is symmetric or not, define ‘geometric equality’ as either compatible equality (congruence) *or* mirror equality.[2])
- P.3′ Each motif can be mapped onto any other motif by a symmetry of the pattern.

Figure 4.1 shows some examples of monomotif and non-monomotif patterns. The explanations below discuss whether the conditions: P.1′ to P.3′ hold for each design and consequently whether each one is monomotif or not.

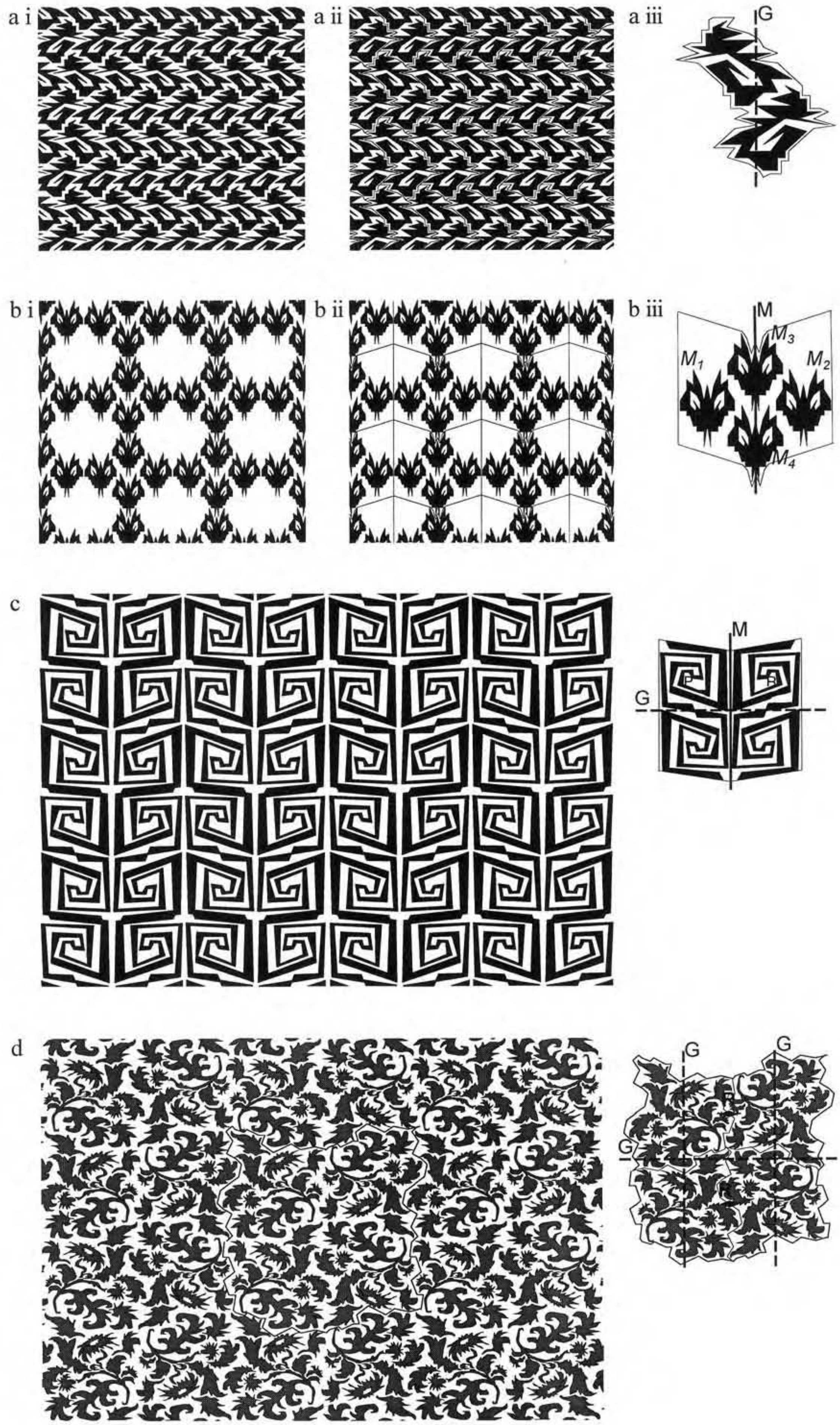


Figure 4.1 Examples of (a), (c), (d) monomotif and (b) non-monomotif patterns.

In Fig. 4.1(a(i)):

- P.1′ none of the motifs overlap or touch any other motif
- P.2′ each motif is congruent to every other motif and
- P.3′ the only symmetry of the pattern other than translational symmetry is glide–reflectional symmetry which, by itself, will generate the whole design.

The easiest way to test if condition P.3′ holds is to examine a translation unit. Figure 4.1(a(ii)) illustrates one way of dividing the pattern into translation units and Fig. 4.1(a(iii)) shows the symmetries of the design passing through one of these translation units. Consider the translation unit in Fig. 4.1(a(iii)). If each motif can be mapped onto any other motif inside it by an isometry of the pattern (in this case a glide–reflection about axis G) then by subsequent unit translations of this translation unit, any motif can be mapped onto any other. In this instance, condition P.3′ is satisfied, so together with P.1′ and P.2′, this implies that the pattern is monomotif.

In Fig. 4.1(b),

- P.1′ none of the motifs overlap or touch any other motif
- P.2′ each motif is congruent to every other motif and
- P.3′ the only symmetry of the pattern other than translational symmetry is reflectional symmmetry. However, applying this symmetry to any one motif will not generate the whole design as explained below.

Consider the translation unit in Fig. 4.1(b(iii)). If each motif can be mapped onto any other motif inside it by an isometry of the pattern then condition P.3′ is satisfied. Let the motifs inside this translation unit be labelled M_1, M_2, M_3 and M_4 as shown. M_1 can be mapped onto M_2 by reflectional symmetry about reflection axis M but not to either M_3 or M_4. This implies that each motif cannot be mapped onto any other one by a symmetry of the pattern therefore, condition P.3′ is not satisfied and so the pattern in Fig. 4.1(b(i)) is not monomotif.

Figures 4.1(c) and (d) show some further illustrations of monomotif patterns with examples of translation units. In Fig. 4.1(c) a motif is taken to be a continuous vertical strip comprising a two-fold rotationally symmetric, wavy line. In Fig. 4.1(d) the motif is one quarter of the translation unit and consists of flowers, stalks and leaves. In each case the pattern satisfies all three conditions, P.1′ to P.3′; therefore they are both monomotif.

In addition to the monomotif conditions, further restrictions may be imposed on the motif characteristics which result in the pattern being discrete.

4.3 Discrete pattern

A formal definition given by Grünbaum and Shephard[1] stated that:

> . . . a pattern is *discrete* if the following conditions hold:
> DP.1 *The motif M is a bounded and connected set.*
> DP.2 *For some i there is an open set E_i which contains the copy M_i of the motif but does not meet any other copy of the motif; that is, $M_j \cap E_i = \emptyset$ for all $j \in I$ such that $j \neq i$.*

In a more accessible context for designers, these conditions may be thought of as follows:

- DP.1′ (i) the motif is bounded, i.e. it is finite and does not continue endlessly in any direction.
  (ii) the motif is a connected set, i.e. all parts of the motif are joined together to form one piece only.
- DP.2′ each motif may be contained within a tile such that no other adjacent motif intersects that tile or its boundaries.

Figure 4.2 illustrates some discrete and non-discrete patterns and explanations follow which discuss whether the conditions DP.1′ and DP.2′ hold for each design. First though, it is important to note that the definition of a discrete

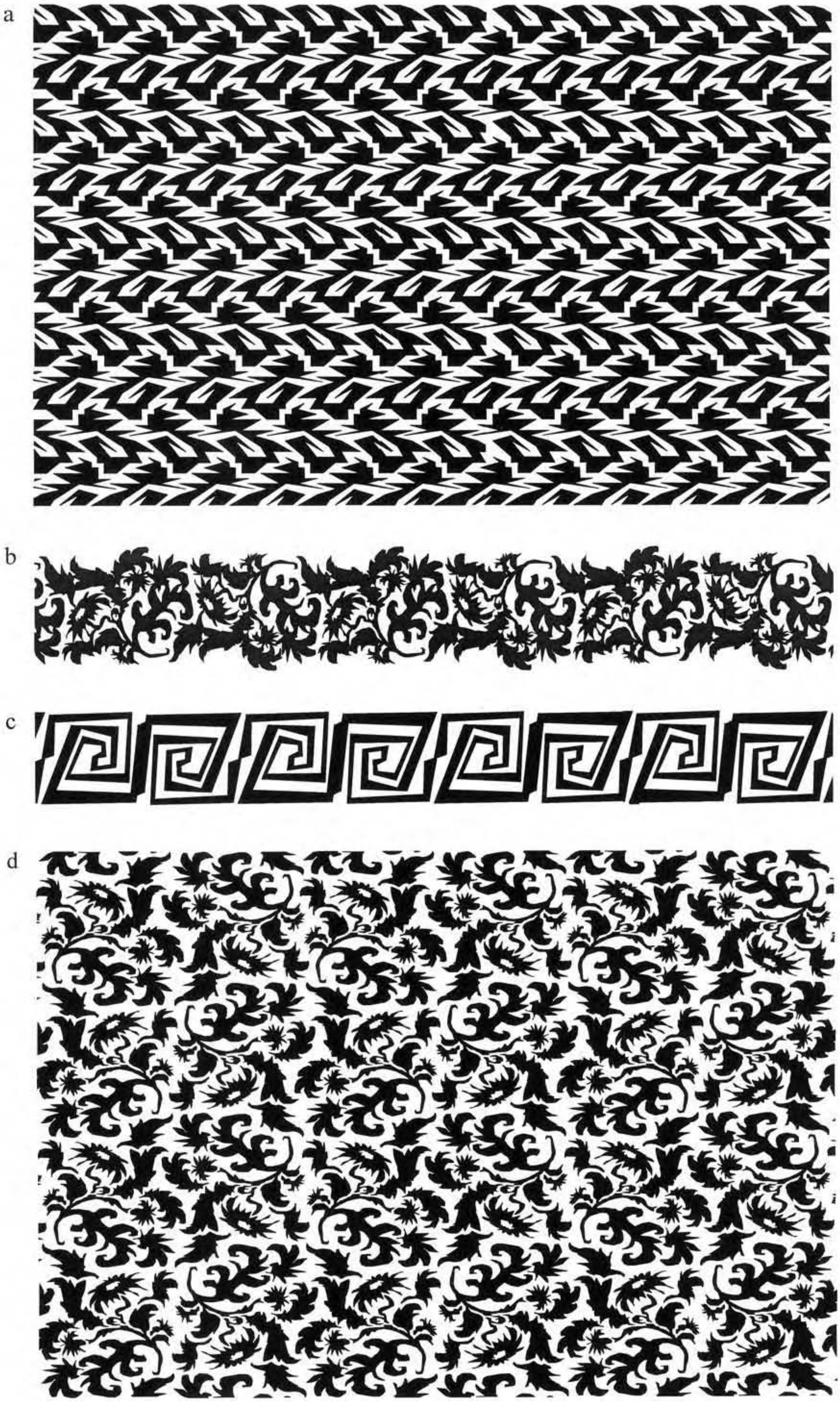


Figure 4.2 Examples of (a) and (b) discrete and (c) and (d) non-discrete patterns.

pattern is only applicable to those patterns which are known to be monomotif. On checking the monomotif conditions for the patterns in Fig. 4.2(a), (b), (c) and (d), it is found that:

- P.1′ none of the motifs overlap or touch any other motif
- P.2′ each motif is congruent to every other motif and
- P.3′ each motif can be mapped onto any other motif by a symmetry of the pattern.

Thus, since all three conditions hold for each example, they are all monomotif. Each pattern may now be analysed in turn to test whether its characteristics fit the criteria for a discrete pattern.

In both Fig. 4.2(a) and Fig. 4.2(b):

- DP.1′ (i) each motif is finite and so bounded
- DP.1′ (ii) each motif does consist of one piece only
- DP.2′ the motifs are separate from each other and so, since all three conditions are satisfied, the pattern is discrete.

In Fig. 4.2(c):

- DP.1′ the motif, of which there is only one, continues endlessly and so is not bounded, hence this pattern is not discrete.

In Fig. 4.2(d):

- DP.1′ (i) each motif is finite and so bounded
- DP.1′ (ii) each motif consists of more than one piece (separate flowers, leaves and stalks), hence this pattern is not discrete.

These examples clearly illustrate that only a proportion of the group of monomotif patterns is also discrete. This proportion of monomotif discrete patterns forms the subgroup of designs which are classified later in this chapter.

4.3.1 Non-trivial discrete pattern

An additional condition imposed on the subgroup of monomotif, discrete patterns is that they are also non-trivial. This simply means that there is more than one copy of the motif in each pattern. Examples of trivial and non-trivial, monomotif discrete patterns are given in Fig. 4.3. The following explanations discuss whether the non-trivial condition holds for each design.

Figure 4.3(a) illustrates a finite pattern, with a $d1$ motif, which satisfies all the criteria for a monomotif discrete pattern. It also has more than one copy of the motif therefore it is non-trivial. Figure 4.3(b) shows a finite pattern with two joined, reflectionally symmetric elements as the motif. It satisfies all the criteria for a monomotif discrete pattern but there is only one copy of the motif, so it is trivial. If the motif was regarded as being a single element (symmetry group $d1$), with the pattern consisting of two copies of the motif, the condition of non-triviality is not even considered because, in this case, the finite pattern is not monomotif as condition P.1′ is not satisfied. In Fig. 4.3(c) the finite pattern is monomotif and discrete but, as there is only one copy of the motif, it is trivial.

Another feature of a subgroup of the group of non-trivial monomotif discrete patterns is the characteristic of being ‘primitive’.

4.4 Primitive pattern

A pattern is described as being primitive if the only symmetry of each motif, which coincides with one of the structures of the whole pattern, is the identity symmetry. A motif may be symmetrical, but if none of its symmetries coincide (by superimposition) with those of the pattern structure then it is primitive.

The following examples, in Fig. 4.4, illustrate primitive and non-primitive, discrete patterns. (Note that throughout the remainder of this book, to reduce unnecessary complication, when referring to a discrete pattern, it will be assumed that it is also monomotif and non-trivial).

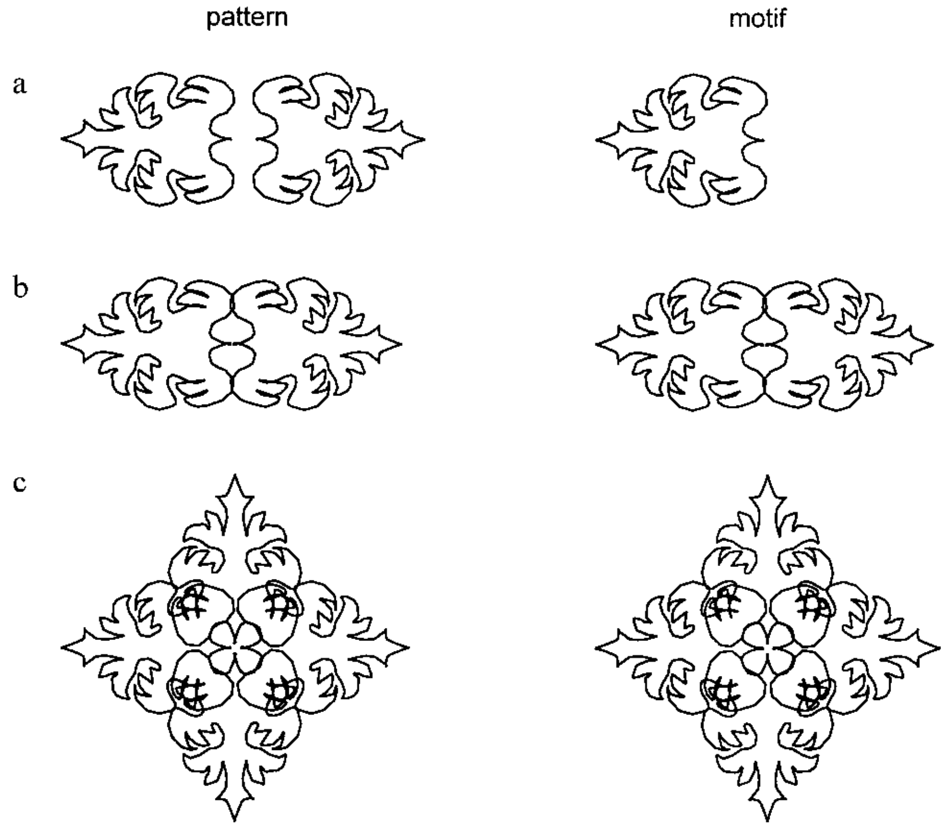


Figure 4.3 Examples of (b) and (c) trivial and (a) non-trivial monomotif discrete patterns.

Figure 4.4(a(i)) (which is represented schematically in Fig. 4.4a(ii)) illustrates a ditranslational discrete pattern composed of individual motifs, each of finite symmetry group *d*1. However, none of the vertical reflection axes passing through the motifs coincide with ones in the design structure. In fact, the only symmetries of the design structure are translational symmetries and the identity symmetry. Hence, since there is only the identity symmetry in common with both the pattern structure and each motif, the pattern is primitive.

Figure 4.4(b) illustrates a monotranslational pattern, symmetry group *p*112. Again each motif has bilateral symmetry but since their symmetry axes do not coincide with any symmetries in the design structure, the pattern is primitive.

Figure 4.4(c) illustrates a ditranslational discrete pattern composed of individual motifs, each of finite symmetry group *c*4. However, in this instance each centre of rotation passing through a motif coincides with one in the design structure. Hence, since the identity symmetry and centres of four-fold rotational symmetry coincide with both the pattern structure and each motif, the pattern is non-primitive.

The monotranslational pattern in Fig. 4.4(e) has been derived from the primitive pattern in Fig. 4.4(d) by joining adjacent asymmetric motifs (half butterflies), in other words each pair of motifs has been transformed to make one motif (a whole butterfly). Therefore, since in Fig. 4.4(e) reflection axes of the design structure now pass through each motif, the pattern is non-primitive.

The finite patterns in Fig. 4.4(f(i), (ii) and (iii)) illustrate non-primitive, non-primitive and primitive patterns, respectively.

Note that for symmetry groups *p*1*a*1 and *p*111 the only symmetries in the patterns' structures, other than the identity symmetry, are glide–reflectional and/or translational symmetries respectively, neither of which can coincide with one individual motif of a discrete pattern. Thus, in these two cases and the two equivalent cases for ditranslational discrete patterns (symmetry groups *p*1 and *pg*) the primitive condition always holds. However, as described in Chapter 3, this does not imply that each inividual motif is necessarily asymmetric (for example see Fig. 4.4(a)).

The previous illustrations show that although a pattern may be discrete, it is not necessarily primitive. Only a proportion of the discrete patterns are primitive, which leaves the remaining non-primitive discrete patterns to be differentiated from each other by their 'induced motif groups' or 'induced groups'.

4.5 Induced motif groups

The induced (motif) group (or induced group) of a discrete pattern relates to the symmetry of each motif which coincides with one or more of the symmetries in structure of the whole pattern. It is taken to be the finite symmetry group of the motif, the symmetries of which coincide with those of the structure. For example, if each of the motifs of a discrete pattern fall on centres of two-fold rotation of the pattern structure but do not intersect any reflectional axes, the motifs will each have at least two-fold rotational symmetry and therefore, the induced group of the discrete pattern will be *c*2. All primitive discrete patterns have induced group *c*1 since each motif has only the identity symmetry coinciding with the design structure. Figure 4.5 shows some examples which illustrate the concept of induced groups for finite, monotranslational and ditranslational discrete patterns.

Figure 4.5(a(i)) shows a finite discrete pattern whose symmetry group is *d*3. Each motif has no symmetries which coincide with the reflection axes of the underlying structure. Therefore, the pattern is primitive and hence has induced group *c*1. Figure 4.5(a(ii)) illustrates a finite, discrete pattern whose symmetry group is *d*4. Each motif has two reflection axes but only one which coincides with one in the underlying structure. Therefore, the induced group is *d*1 as this is the symmetry group corresponding to a finite design with one reflection axis. Similarly, Fig. 4.5(a(iii)) shows a finite design with symmetry group *d*4 and induced group *d*1.

Figure 4.5(b) shows a monotranslational discrete pattern whose symmetry group is *pma*2. Each motif has one reflection axis which coincides with that of the underlying structure; therefore the induced group is *d*1 as this is the symmetry group corresponding to a finite design with one reflection axis.

Figure 4.5(c) illustrates a monotranslational discrete pattern whose symmetry group is *pma*2. Although each motif has two reflection axes, only their centres of two-fold rotation coincide with ones in the underlying structure. Therefore, the induced group is *c*2 as this is the symmetry group corresponding to a finite design with two-fold rotational symmetry only.

Figure 4.5(d(i) to (vi)) illustrates six ditranslational discrete patterns whose symmetry groups are *p*31*m*, *cmm*, *p*4*g*, *p*6*m*, *p*6*m* and *p*3*m*1, respectively. Their corresponding induced groups are *c*3, *c*2, *c*4, *d*6, *d*2 and *d*3.

Figure 4.5(e) shows a ditranslational discrete pattern whose symmetry group is *p*4*m*. Each motif has two reflection axes which coincide with ones in the underlying structure; therefore, the induced group is *d*2.

In Fig. 4.5(f) a *pmm*2 monotranslational discrete pattern has been constructed from *d*4 motifs. Each of these motifs has a centre of two-fold rotation and two perpendicular reflection axes which coincide with ones in the underlying structure; therefore the induced group is *d*2.

Figure 4.5(g) illustrates a ditranslational discrete pattern whose symmetry group is *p*4*m*. Each motif has four reflection axes which coincide with those of the underlying structure; therefore the induced group is *d*4. Further examples of induced groups may be derived from referring back to the illustrations in Fig. 4.4(a), (b), (c), (d), (e), (f(i)), (f(ii)) and (f(iii)). These patterns have induced groups *c*1, *c*1, *c*4, *c*1, *d*1, *d*1, *d*1 and *c*1, respectively.

a i

a ii

b

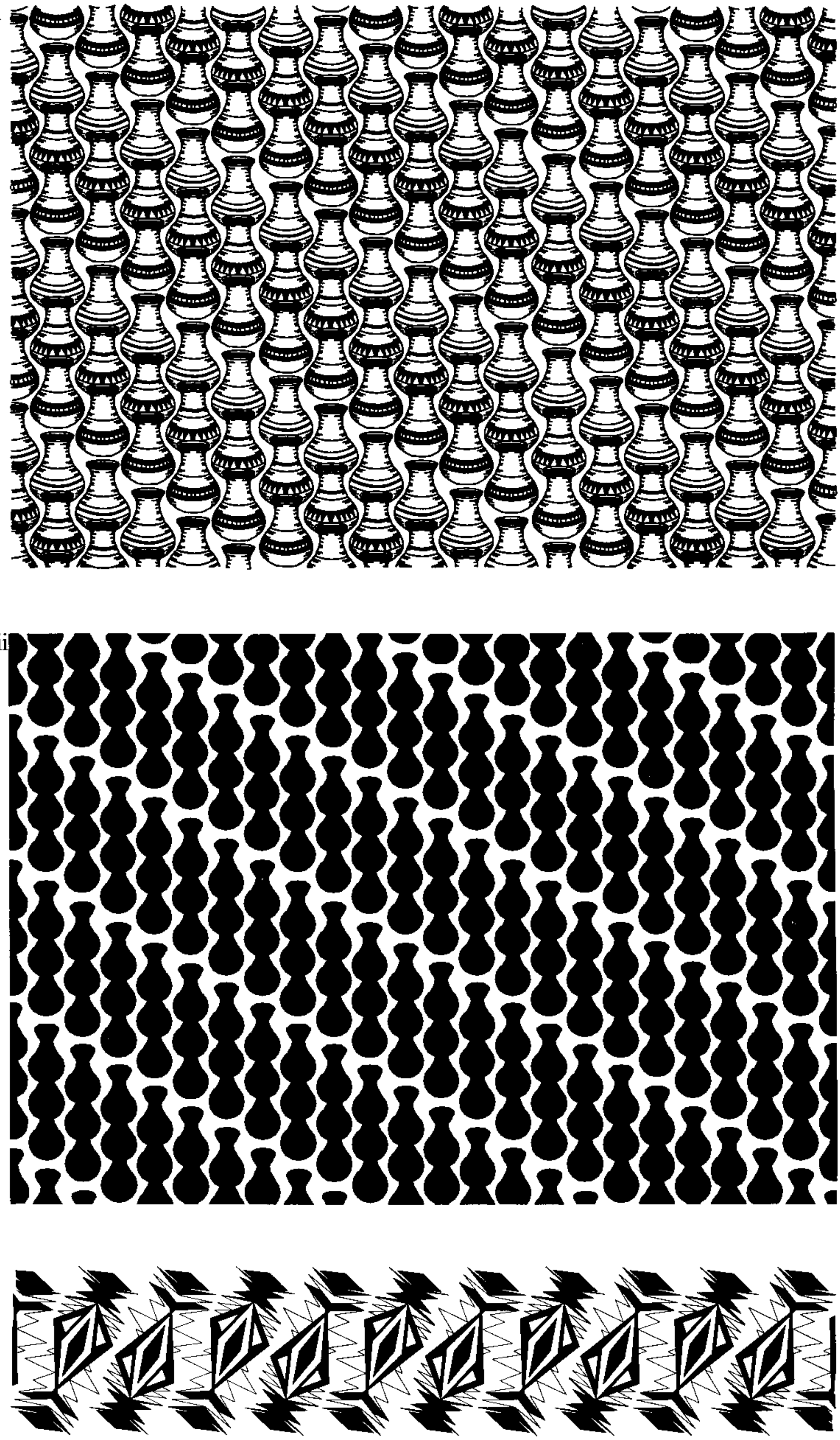


Figure 4.4 Examples of (a), (b), (d) and (f(iii)) primitive and (c), (e), (f(i)) and (f(ii)) non-primitive discrete patterns.

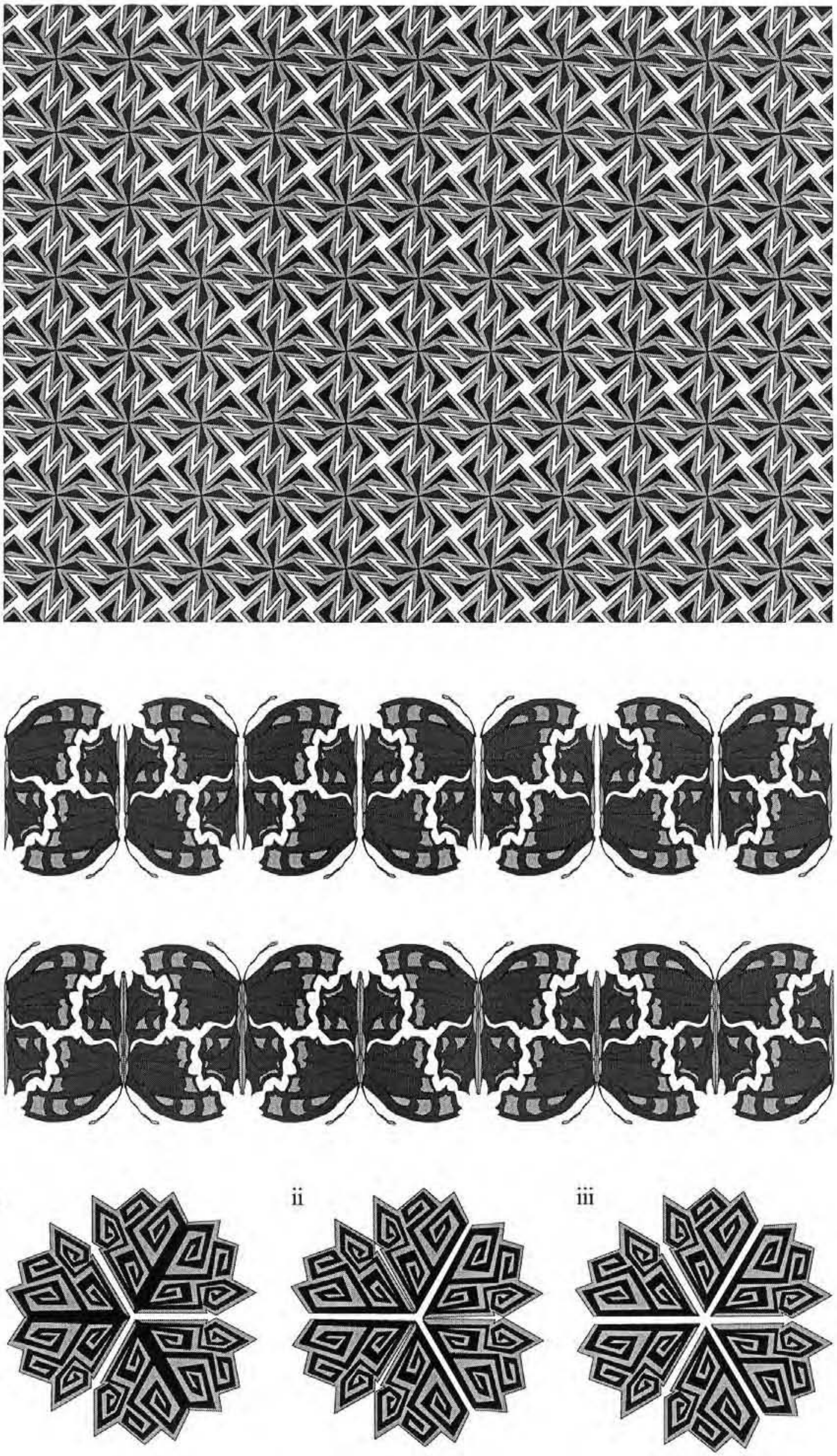


Figure 4.4 (cont.)

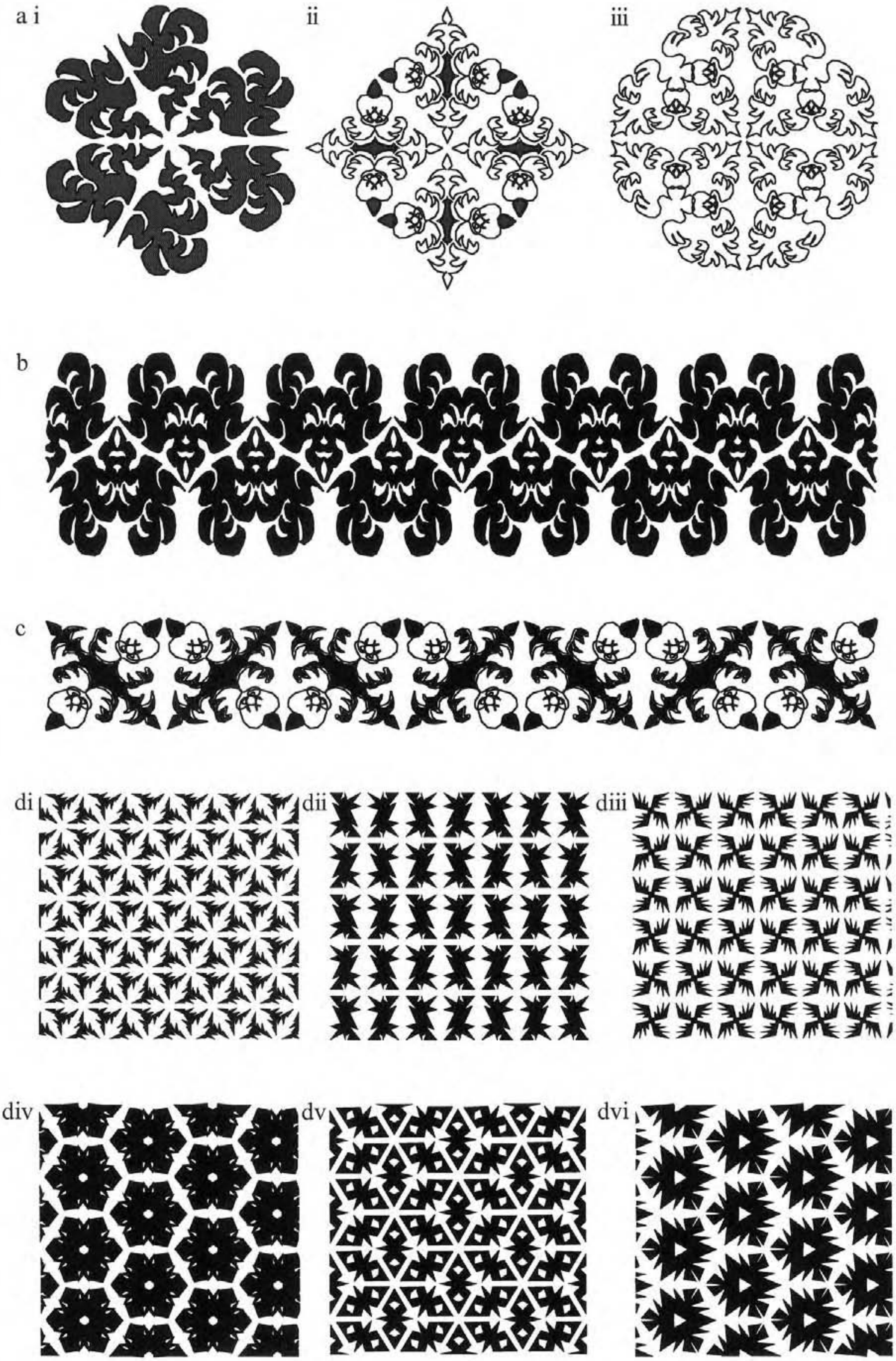
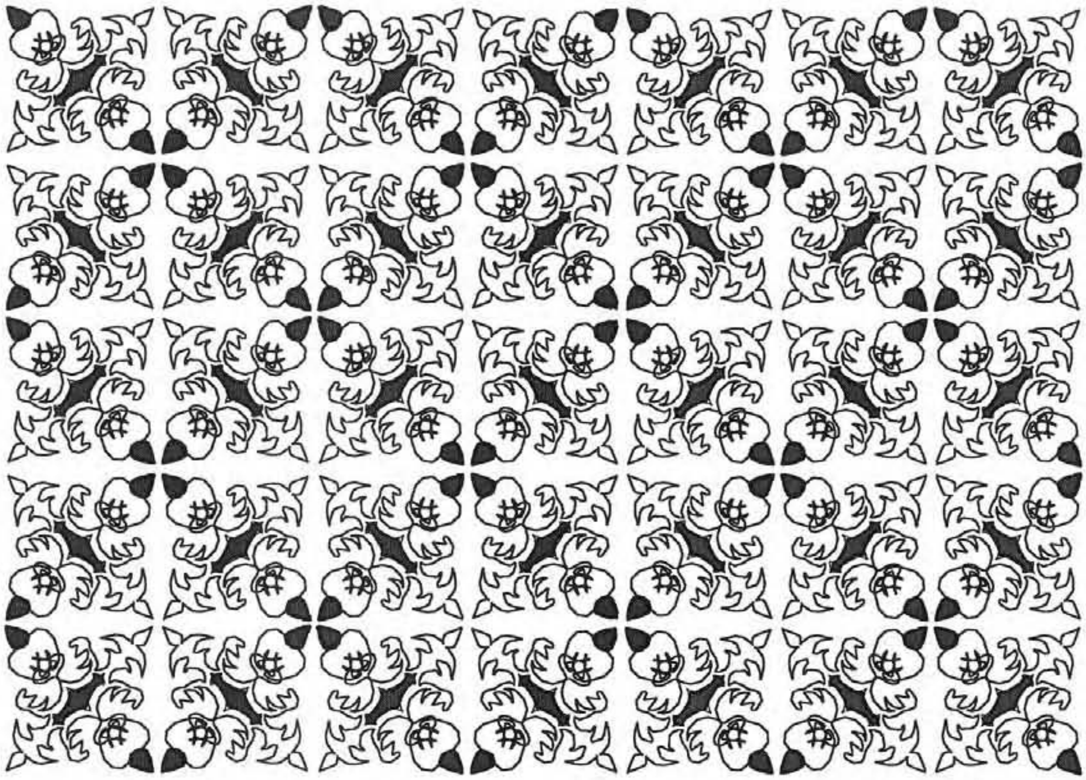


Figure 4.5 Examples illustrating induced motif groups of discrete patterns.

e



f

g

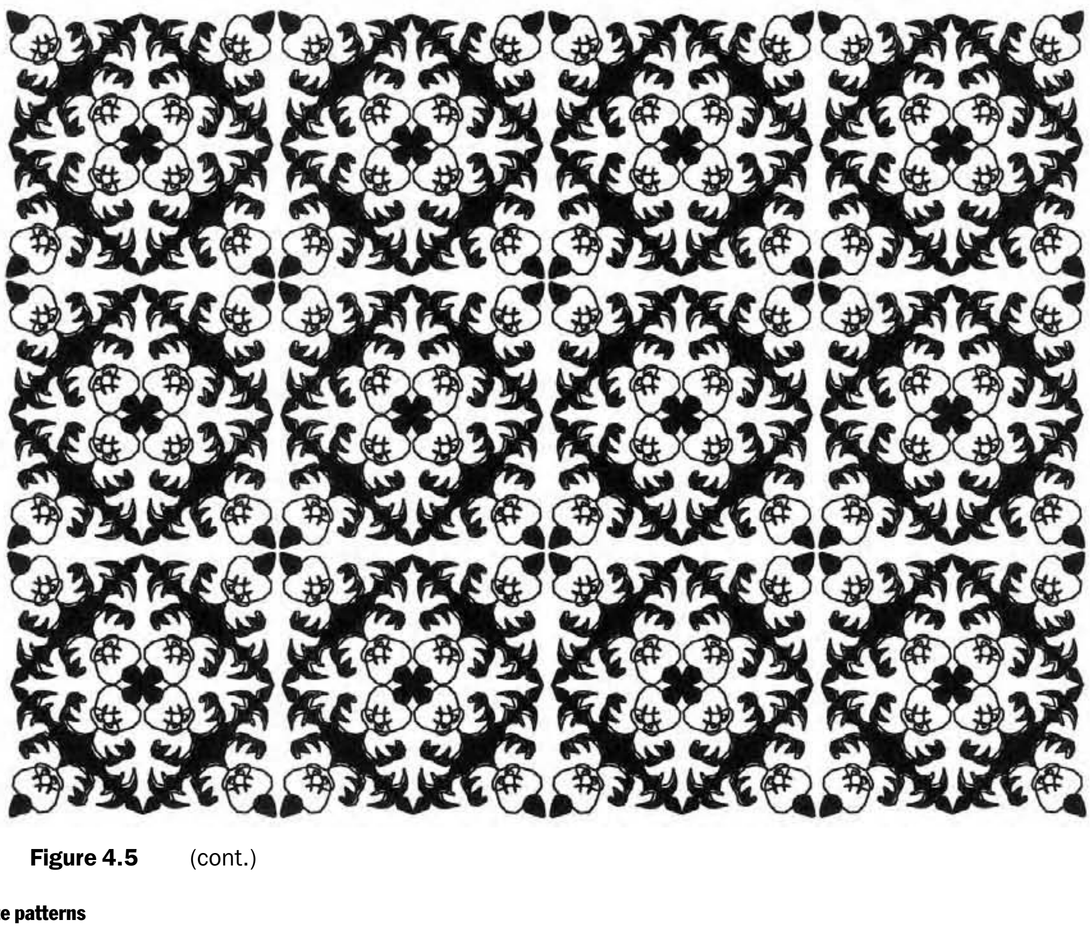


Figure 4.5 (cont.)

4.6 Motif-transitive subgroups

Each of the finite, monotranslational and ditranslational symmetry groups may be divided into 'pattern types' by their induced groups. However, there are three exceptions where these criteria do not provide sufficient information for discrete pattern classification. For example, the monotranslational symmetry group *pmm*2 is divided into three pattern types, two of which have the same induced group, *d*1 (as shown in Fig. 4.6(a)). Similarly, the ditranslational symmetry group *p*4*m* is divided into three pattern types, two of which have the same induced group *d*1 (these are shown in Fig. 4.6(b)). Also, two of the six pattern types of symmetry group p6 have the same induced group *d*1 (see Fig. 4.6(c)). Unlike the remaining discrete pattern types, the structures and relationships between adjacent motifs in these patterns, with the same symmetry group and induced group, appear to be different. To differentiate between them requires further geometrical analysis. The mathematical theory for this process requires that a distinction be made between their 'motif-transitive subgroups'.

A subgroup of symmetries of a symmetry group may be thought of as a proportion of the symmetries contained within the symmetry group. The proportion may include the identity, all the symmetries or a selection of the symmetries in the symmetry group. A subgroup of symmetries in the symmetry group is defined as being 'motif transitive' if it satisfies the following condition according to Grünbaum and Shephard[1]:

> Let $T(P)$ be a subgroup of the symmetry group $S(P)$ of a given discrete pattern P. Then $T(P)$ is called *motif transitive* if it contains isometries that map any copy M_o of the motif of P onto any other copy M_j.

In other words, a subgroup of the symmetries in a symmetry group is motif transitive if symmetries in it are able to map any one motif onto any of the others in the pattern.

An alternative way of explaining this theory is to imagine generating the design by mapping a single motif onto its equivalent positions. For example (as shown in Fig. 4.7(a)) a *pm*11 monotranslational design, with induced group *d*1, may be generated in two different ways: either by translating a *d*1 motif at unit intervals in the direction of the longitudinal axis or by continually reflecting a motif about reflection axes positioned at unit intervals, between adjacent motifs, perpendicular to the longitudinal axis of the strip. These two sets of symmetries used to generate the design in this way form subgroups of the symmetry group *pm*11 and since each can map one motif onto the remaining ones, they are both motif transitive. In the first instance, translational symmetry alone, besides the identity, is also used to represent the monotranslational design symmetry group *p*111, and second, the parallel axes of reflectional symmetry described above represent the group *pm*11. Thus, these two symmetry groups form motif-transitive subgroups of the discrete pattern with symmetry group *pm*11 and induced group *d*1.

Similarly, one individual motif in the monotranslational discrete pattern type, with symmetry group *p*1*m*1 and induced group *d*1 (Fig. 4.7(b)), may be mapped onto the remaining ones either by translational symmetry or glide–reflectional symmetry about an axis coinciding with the longitudinal axis of the strip. Thus, these two symmetries form the motif-transitive subgroups *p*111 and *p*1*a*1, respectively.

By analysing the geometry of the pattern type in Fig. 4.7(c), with symmetry group *pmm*2 and induced group *d*1, it will be noticed that one motif cannot be mapped onto the remaining ones by translational symmetry alone, therefore *p*111 is not a motif-transitive subgroup of this design. However, one motif can be mapped onto the remaining ones by two-fold rotational symmetries only; by reflection about the longitudinal axis and translations; by alternating two-fold rotations and transverse reflections; and/or by transverse and longitudinal reflectional symmetries. These different sets of symmetries represent the symmetry groups *p*112, *pm*11, *pma*2 and *pmm*2, respectively and form motif-transitive subgroups of this monotranslational discrete pattern.

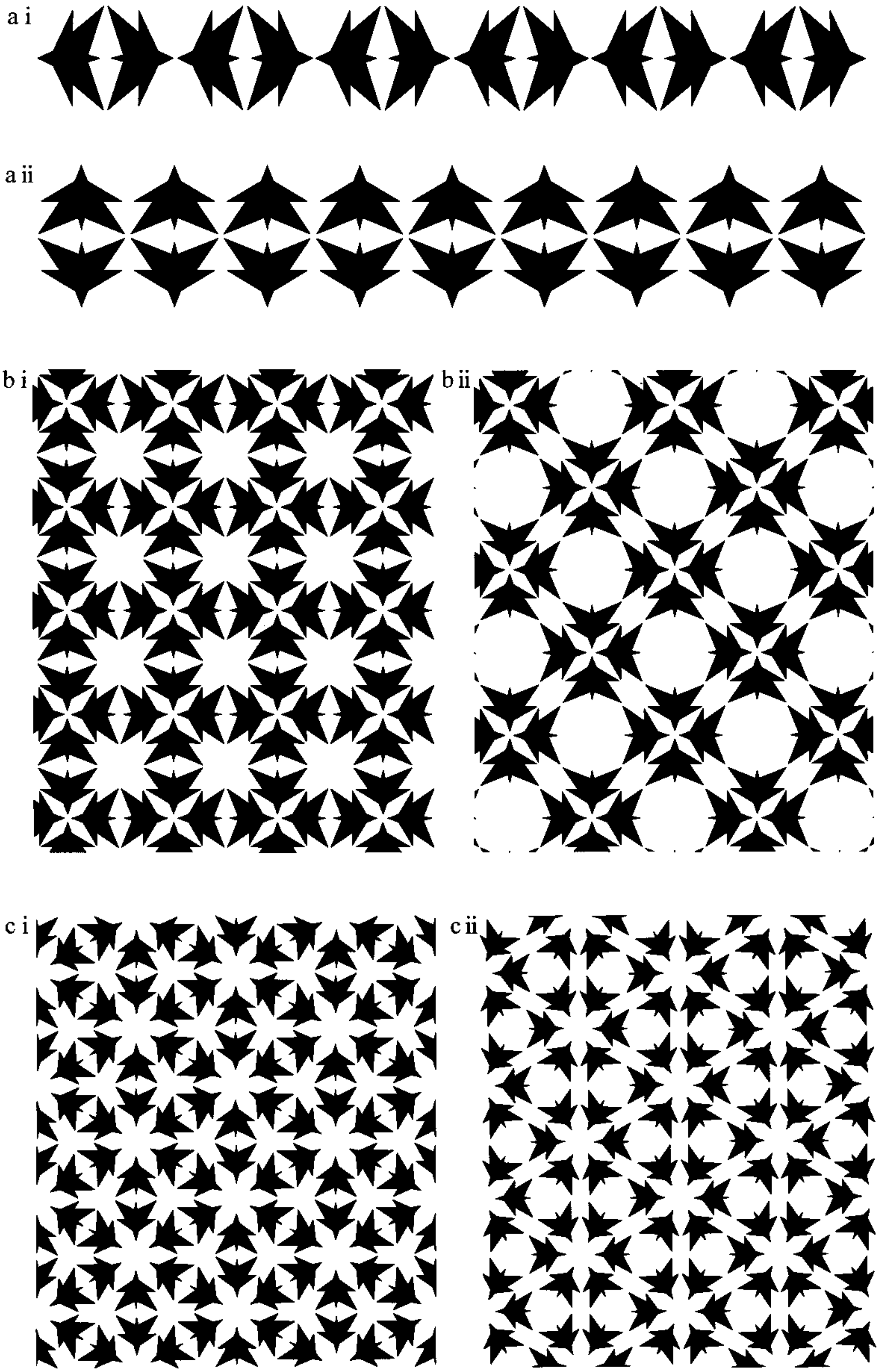


Figure 4.6 Examples of discrete pattern types with the same symmetry groups and induced motif groups.

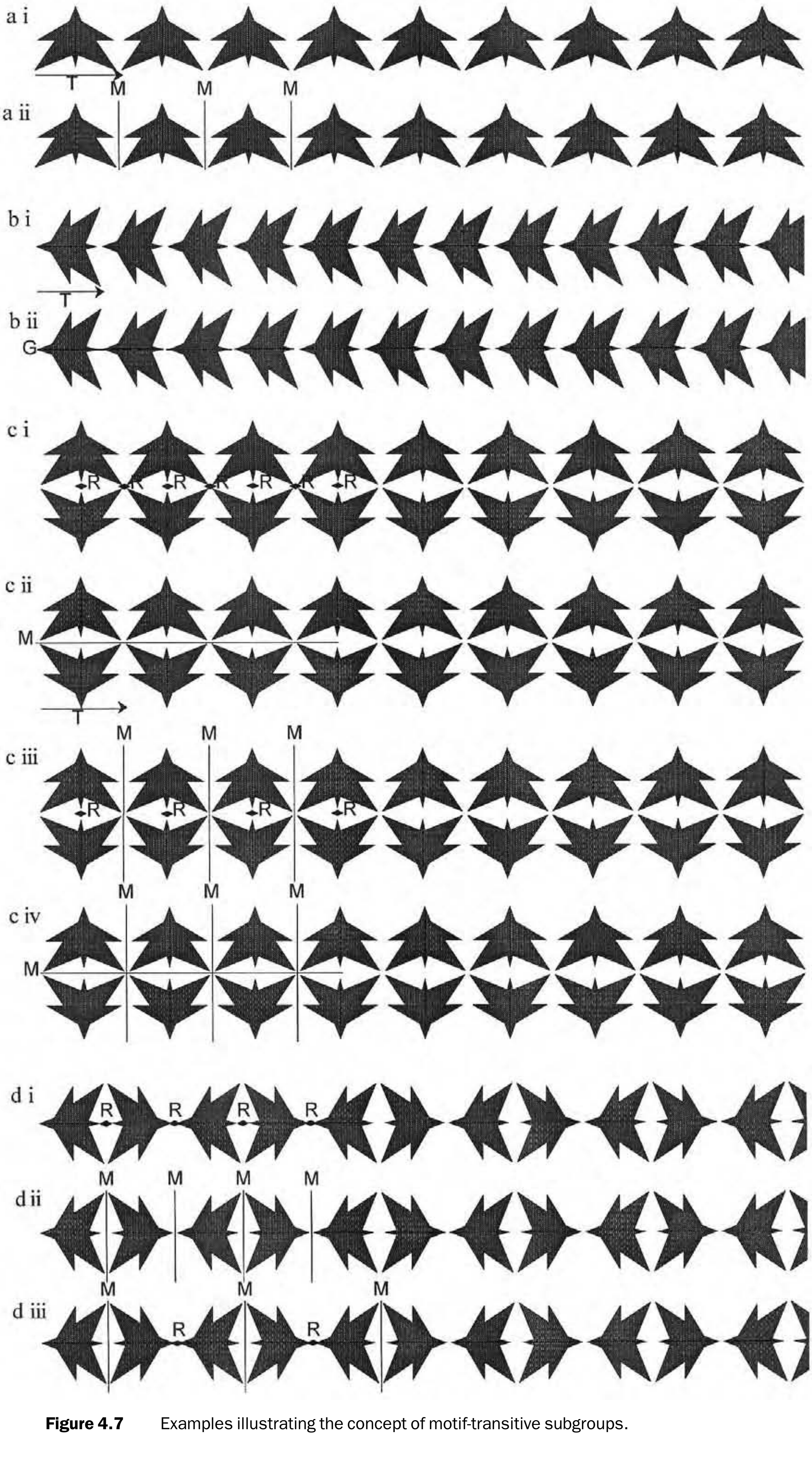


Figure 4.7 Examples illustrating the concept of motif-transitive subgroups.

By analysing the geometry of the pattern type in Fig. 4.7(d), with symmetry group *pmm*2 and induced group *d*1, again it will be noticed that one motif cannot be mapped onto the remaining ones by translational symmetry alone, therefore *p*111 is not a motif-transitive subgroup of this design. However, one motif can be mapped onto the remaining ones by two-fold rotational symmetries only; by reflections about transverse axes; by alternating two-fold rotations and transverse reflections. These different sets of symmetries represent the symmetry groups *p*112, *p*1*m*1, and *pma*2, respectively and form motif-transitive subgroups of this discrete pattern.

From the analysis of the two pattern types in Fig. 4.7(c) and (d), it is noticed that although they have the same symmetry group and induced group, they have different motif-transitive subgroups which, therefore, characterises them differently. Thus, in order to class two patterns as the same type, as described by Grünbaum and Shephard they must have the same symmetry group, induced group and the same set of motif-transitive subgroups.[1]

To generate a primitive discrete pattern, all the symmetries in the pattern structure are required. This implies that only the whole symmetry group itself forms a motif-transitive subgroup.

In some cases there is more than one form of a motif-transitive subgroup. For example, Fig. 4.8(a) illustrates the 16 different motif-transitive subgroups of the discrete pattern with symmetry group *pmm* and induced group *d*2. Note that there are at least two inequivalent motif-transitive subgroups of *pm*, *p*2, *pmg* and *cmm*. For each of these motif-transitive subgroups, the number or fraction of motifs contained within a unit cell is different, for example for motif-transitive subgroup *cmm*, there are two, four and eight motifs contained within a *cmm* unit cell. This implies that these subgroups are inequivalent and must be regarded as being different from each other. Where the symmetries of the same subgroup are represented in different positions but may be superimposed on each other by a translation (e.g. for subgroup *p*2 in Fig. 4.8(b)) the motif-transitive subgroups are considered to be equivalent and not counted separately.

The motif-transitive subgroups represented by an asterisk in Tables 4.2 and 4.3 (see Sections 4.8 and 4.9, respectively) indicate a subgroup equivalent to the symmetry group of the overall design. Again, in these cases, the motif-transitive subgroup contains equivalent symmetries as the overall symmetry group but a unit cell contains a larger number or fraction of motifs because not all the symmetries of the symmetry group are included (for example see Fig. 4.7a(ii)). Where a motif-transitive subgroup is followed by a number in parentheses, the number indicates how many inequivalent motif-transitive subgroups there are for that particular subgroup.

This theoretical perspective resolves the problem of distinguishing between the three cases where symmetry groups and induced groups coincide. However, on further analysis it is found that there are two ditranslational pattern types in which symmetry groups, induced groups and motif-transitive subgroups coincide and yet the structures and relationships between motifs still appear to be different. The theory for distinguishing between these two pattern structures will not be described here because there is only one possible pattern type bearing these characteristics. The analytic differentiation between these patterns is described in detail by Grünbaum and Shephard[1] but in the context of this book they are merely represented by pattern types Dt(P)48A and Dt(P)48B (see Section 4.12).

4.7 Classification of finite discrete pattern types

The two finite symmetry groups may be divided into three discrete pattern types as shown in Table 4.1. Symmetry group *cn* has one associated discrete pattern type with induced group *c*1 (i.e. primitive) and symmetry group *dn* has two associated discrete pattern types with induced groups *c*1 and *d*1, respectively. Figure

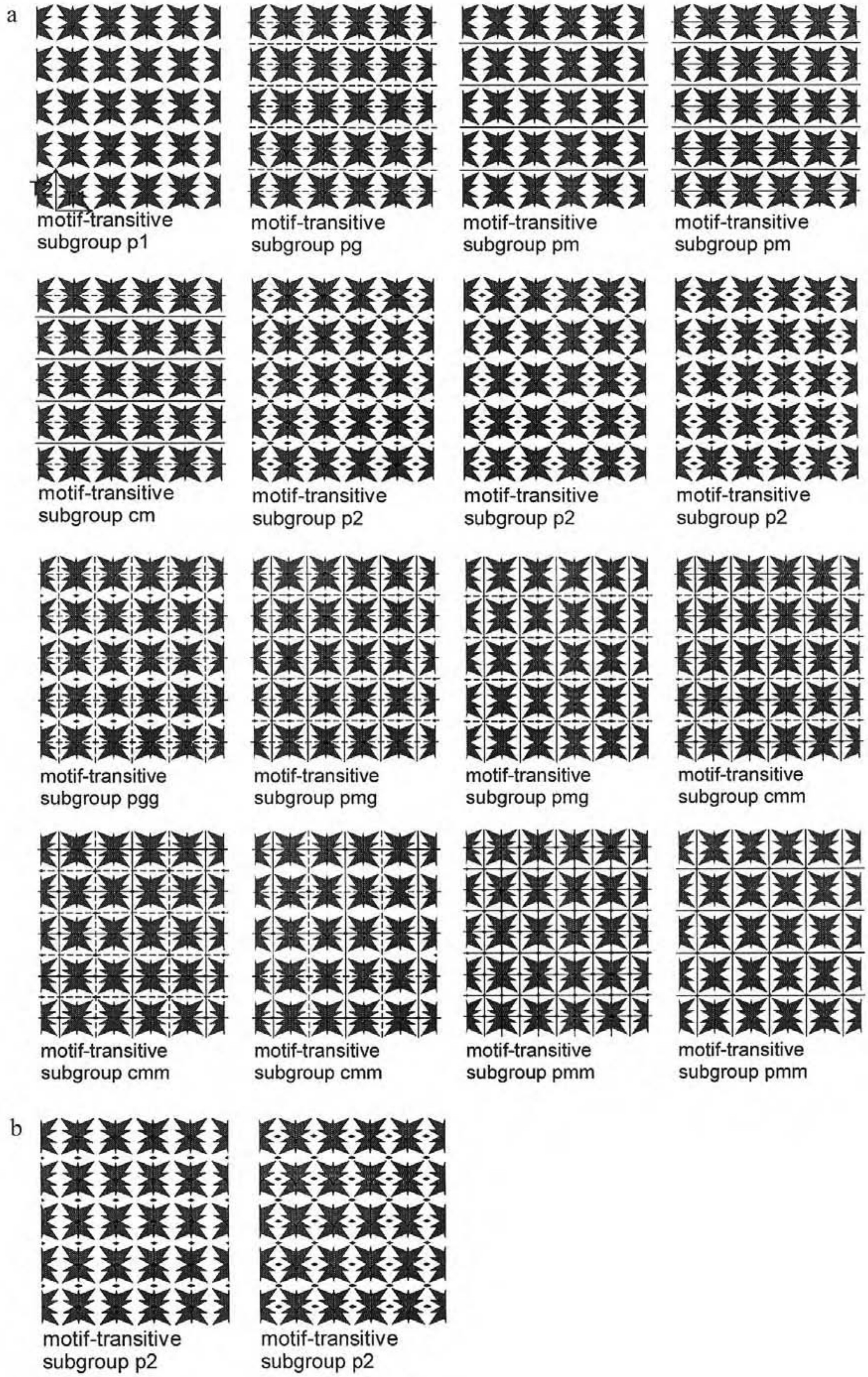


Figure 4.8 Examples illustrating the 16 distinct motif-transitive subgroups of pattern type Dt(P)16.

Table 4.1 The three finite discrete pattern types

Pattern type	Symmetry group	Induced group	Motif-transitive subgroups
$F(P)1_n$	*cn*	*c*1	primitive
$F(P)2_n$	*dn*	*c*1	primitive
$F(P)3_n$	*dn*	*d*1	*cn* for all *n* *dn*/2 for even *n*

Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

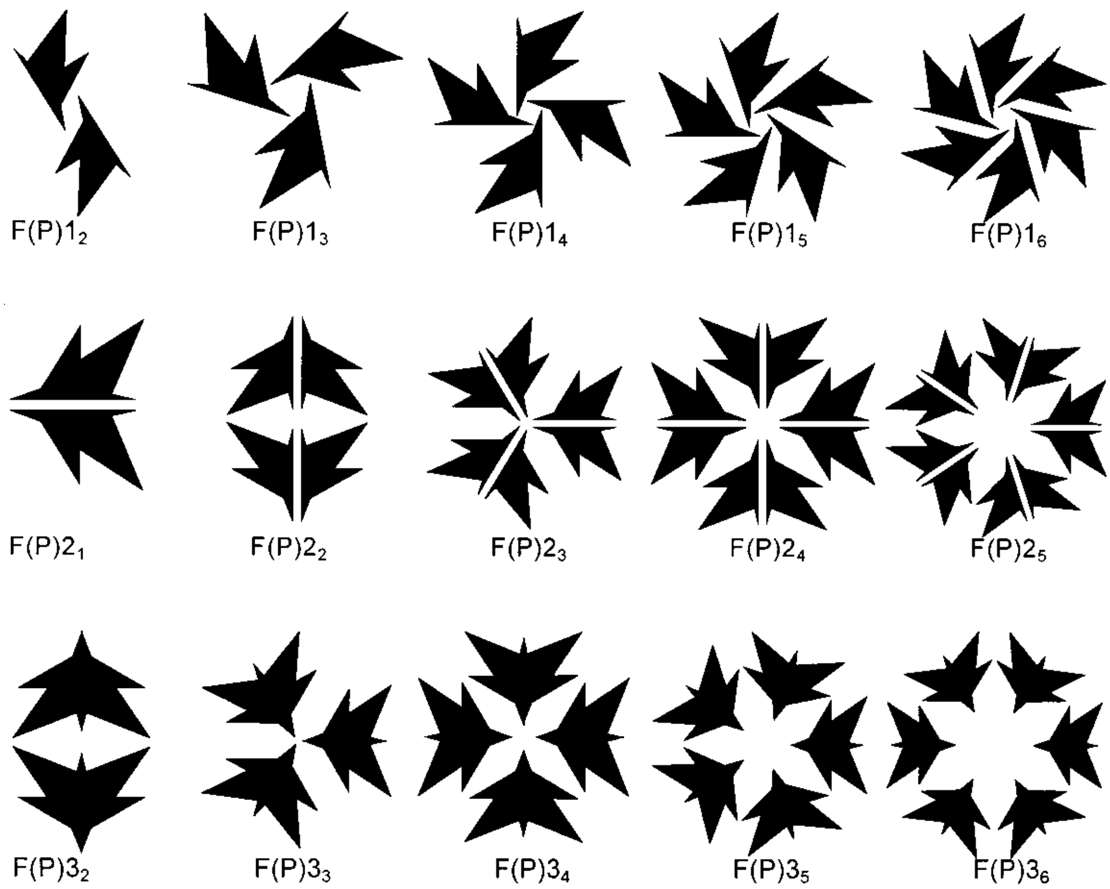


Figure 4.9 Schematic illustrations of the three finite discrete pattern types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

4.9 shows schematic illustrations of these pattern types and further illustrations are given in Fig. 4.10.

4.7.1 Notation

The notation used to represent the finite discrete pattern types has been derived from that given by Grünbaum and Shephard who denote the three types by $PF1_n$, $PF2_n$ and $PF3_n$.[1] However, in the context of this book, the analogous notation $F(P)1_n$, $F(P)2_n$ and $F(P)3_n$ is used where n represents the number of reflection axes and/or order of rotation of the overall design structure.

The definition of a non-trivial discrete pattern, given in Section 4.3.1, states that each pattern must have more than one motif. For $F(P)1_n$, this implies that n

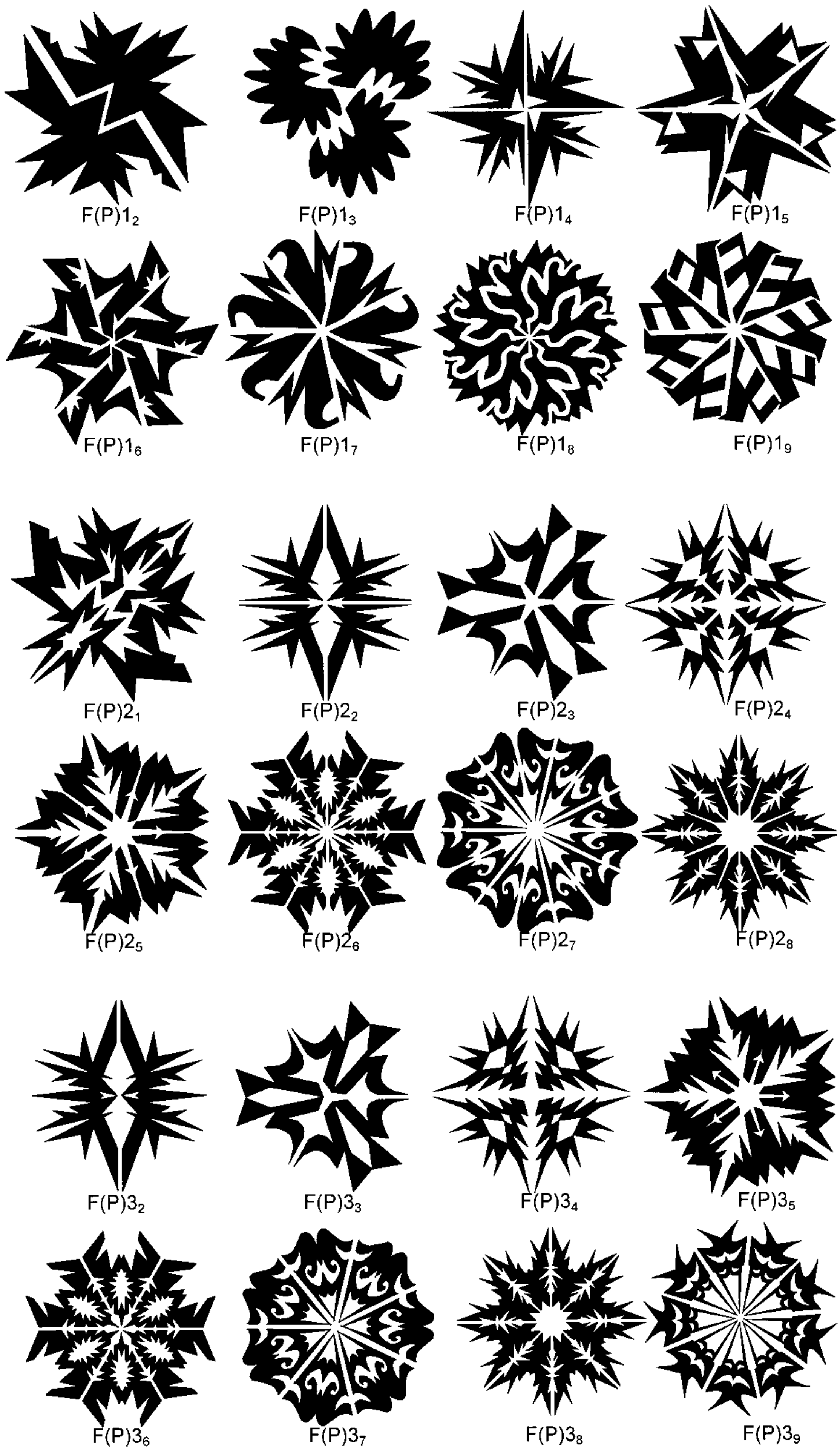


Figure 4.10 Further illustrations of finite discrete pattern types.

Table 4.2 The 15 monotranslational discrete pattern types

Pattern type	Symmetry group	Induced group	Motif-transitive subgroups
Mt(P)1	*p*111	*c*1	primitive
Mt(P)2	*p*1*a*1	*c*1	primitive
Mt(P)3	*p*1*m*1	*c*1	primitive
Mt(P)4	*p*1*m*1	*d*1	*p*111, *p*1*a*1
Mt(P)5	*pm*11	*c*1	primitive
Mt(P)6	*pm*11	*d*1	*p*111, *
Mt(P)7	*p*112	*c*1	primitive
Mt(P)8	*p*112	*c*2	*p*111, *
Mt(P)9	*pma*2	*c*1	primitive
Mt(P)10	*pma*2	*c*2	*pm*11
Mt(P)11	*pma*2	*d*1	*p*112, *p*1*a*1
Mt(P)12	*pmm*2	*c*1	primitive
Mt(P)13	*pmm*2	*d*1	*p*112, *p*1*m*1, *pma*2, *
Mt(P)14	*pmm*2	*d*1	*p*112, *pm*11, *pma*2(2)
Mt(P)15	*pmm*2	*d*2	*p*111, *p*112(2), *p*1*a*1, *p*1*m*1, *pm*11(2), *pma*2(3), *

Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

must be greater or equal to 2 (i.e. $n \geq 2$), since if $n = 1$ the design consists of one asymmetric motif of symmetry group $c1$. For finite pattern type $F(P)2_n$, $n \geq 1$ since for the minim condition, when $n = 1$, there are two motifs. However if $n = 1$ for finite pattern type $F(P)3_n$, there is just one motif as the one reflection axis passes through the centre of the motif; therefore $n \geq 2$.

4.8 Classification of monotranslational discrete pattern types

The seven monotranslational symmetry groups are divided into 15 discrete pattern types. These are listed in Table 4.2 together with their symmetry groups, induced groups and motif-transitive subgroups. Schematic illustrations of the fifteen monotranslational pattern types and further illustrations are given in Figs. 4.11 and 4.12.

4.8.1 Notation

The notation used to represent these pattern types has been derived from that given by Grünbaum and Shephard who denote the 15 monotranslational pattern types by PS1 to PS15 (where PS stands for ‘strip pattern’). However, in this book, the analogous notation Mt(P)1 to Mt(P)15 is used where Mt(P) stands for ‘monotranslational pattern type’.

4.9 Classification of ditranslational discrete pattern types

The 17 ditranslational symmetry groups are divided into 51 discrete pattern types. These are listed in Table 4.3 together with their symmetry groups, induced groups and motif-transitive subgroups. Schematic illustrations of the 51 ditranslational pattern types and further illustrations are given in Figs. 4.13 and 4.14.

4.9.1 Notation

The notation used to represent these pattern types has been derived from that given by Grünbaum and Shephard who denote the 51 monotranslational pattern

Pattern type	Symmetry group	Induced group
Mt(P)1	*p*111	c1
Mt(P)2	p1a1	c1
Mt(P)3	p1m1	c1
Mt(P)4		d1
Mt(P)5	pm11	c1
Mt(P)6		d1
Mt(P)7	p112	c1
Mt(P)8		c2
Mt(P)9	pma2	c1
Mt(P)10		c2
Mt(P)11		d1
Mt(P)12	pmm2	c1
Mt(P)13		d1
Mt(P)14		d1
Mt(P)15		d2

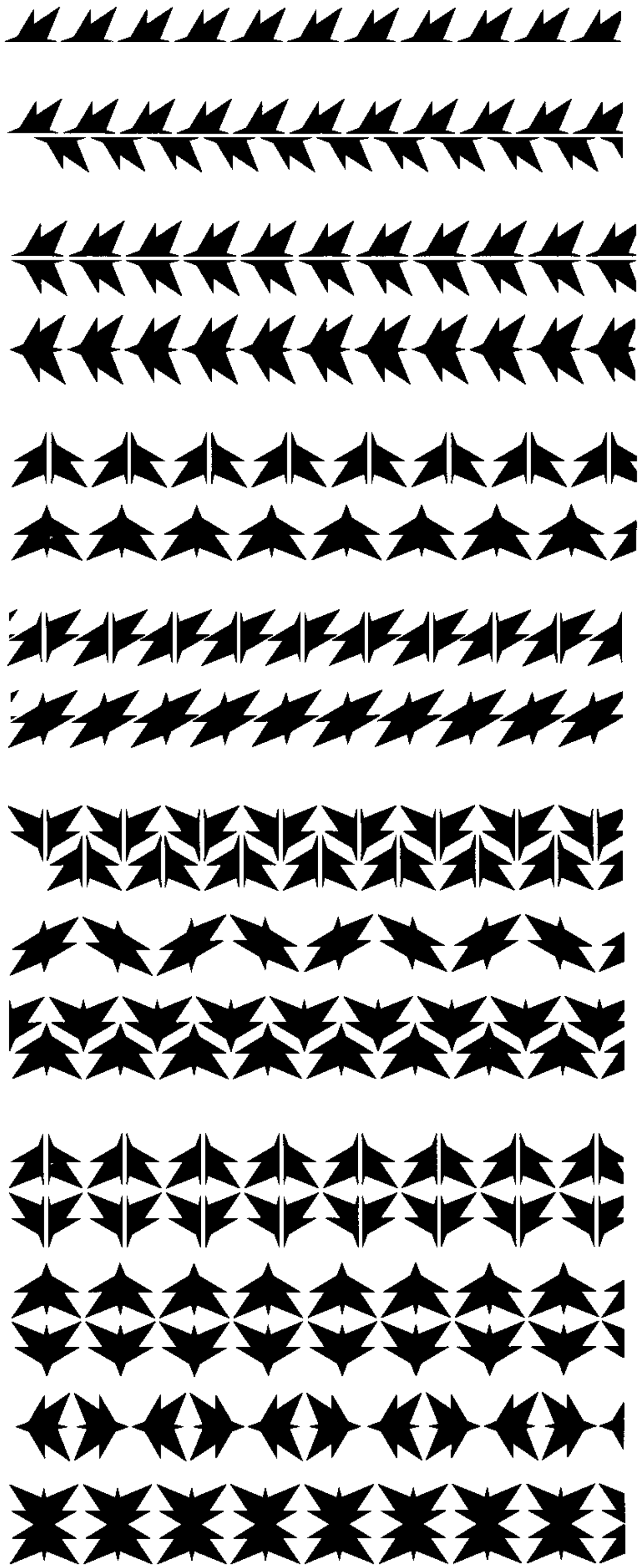


Figure 4.11 Schematic illustrations of the 15 monotranslational discrete pattern types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

types by PP1 to PP51. Here 'PP' stands for 'periodic pattern'.[1] However Senechal states that the points of a lattice are related by shifts called *translations*. She goes on to say that a pattern whose symmetry includes translation is said to be *periodic*.[3] This suggests the inclusion of monotranslational patterns in the group of 'periodic patterns' which may cause confusion if the 'PP' notation is adopted. Therefore, in the context of this book, the PP1 to PP51 notation is replaced by Dt(P)1 to Dt(P)51, where Dt(P) stands for 'ditranslational pattern type'.

4.10 Construction of finite discrete pattern types

The techniques used to construct finite pattern types F(P)1_n to F(P)3_n are similar to those described in Section 2.11 but with additional restrictions imposed on the initial motif. In each case, the structure is based on the division of a circle into n or $2n$ equal sectors depending on the symmetries in the symmetry group, as described in Chapter 2, Section 2.11. However, in this instance the shaded area (in the illustrations given in Fig. 4.15) represents a fundamental region or group of fundamental regions containing the motif.

4.10.1 Finite pattern types, induced group c1

Symmetry groups *cn* and *dn* each have one associated primitive pattern type (i.e. with induced group *c*1): F(P)1_n and F(P)2_n, respectively.

To construct F(P)1_n and F(P)2_n pattern types, the same rules are followed as those described for the first methods in Sections 2.11.1 and 2.11.2, respectively. However, the initial design unit added to a fundamental region must be made of one piece (condition DP′.1(ii)) and, on application of the generating symmetries, be separate from the others (condition DP′.2). This second condition is satisfied by ensuring that the initial design unit only touches the boundary of the fundamental region which coincides with the circumference of the circle and not those radiating from the circle centre. After applying the generating symmetries to map this design unit to all equivalent positions in the design, the boundaries of the fundamental regions are removed to give F(P)1_n and F(P)2_n pattern types. Examples are given for $n = 8$ and $n = 4$ in Fig. 4.15(a) and (b), respectively.

4.10.2 Finite pattern types, induced group d1

Symmetry group *dn* is the only finite symmetry group with an associated pattern type having induced group *d*1. To construct this finite pattern type, F(P)3_n, a *dn* motif (made of one piece) is placed in two sectors of a circle, divided into $2n$ equal sectors, such that one of its reflection axes bisects the two sectors. The motif must not touch the circle centre or any other boundary of this 'double sector' except the portion on the circle circumference. As described in Section 4.6, a discrete pattern may be generated by applying a motif-transitive subgroup of symmetries, of the symmetry group, to a motif. An F(P)3_n pattern has motif-transitive subgroups *cn*, if n is odd, and *cn* and *dn*/2, if n is even (see Table 4.1), that is, if n is odd, n-fold rotational symmetry may be applied to the *dn* motif about the circle centre to complete the design. If n is even, the same rotation may be applied or reflectional symmetry about axes coinciding with sector boundaries, unoccupied by the initial motif, and intersecting at angles of 360°/n at the circle centre. The sector boundaries, dividing the circle into fundamental regions, are then removed to give an F(P)3_n discrete pattern as shown in Fig. 4.15(c) for $n = 4$.

4.11 Construction of monotranslational discrete pattern types

The construction of monotranslational pattern types Mt(P)1 to Mt(P)15 employs similar techniques to those discussed in Section 2.12. The structure of each pattern type is based on the division of a strip into fundamental regions as described in Chapter 2 (design type (iii)). As described previously, the initial design unit added to a fundamental region must have no symmetries in common with the strip. The

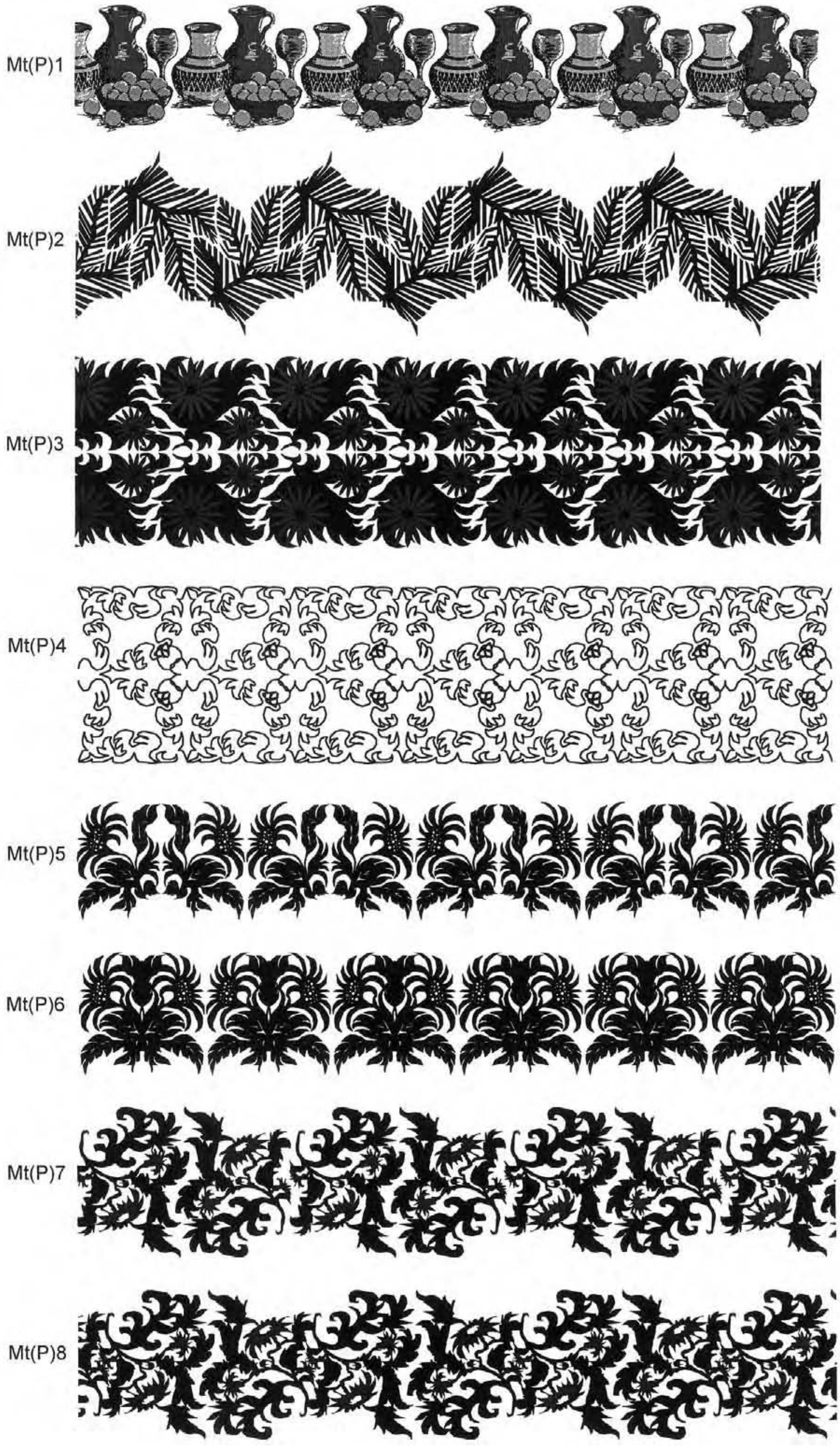


Figure 4.12 Further illustrations of monotranslational discrete pattern types.

Mt(P)9

Mt(P)10

Mt(P)11

Mt(P)12

Mt(P)13

Mt(P)14

Mt(P)15

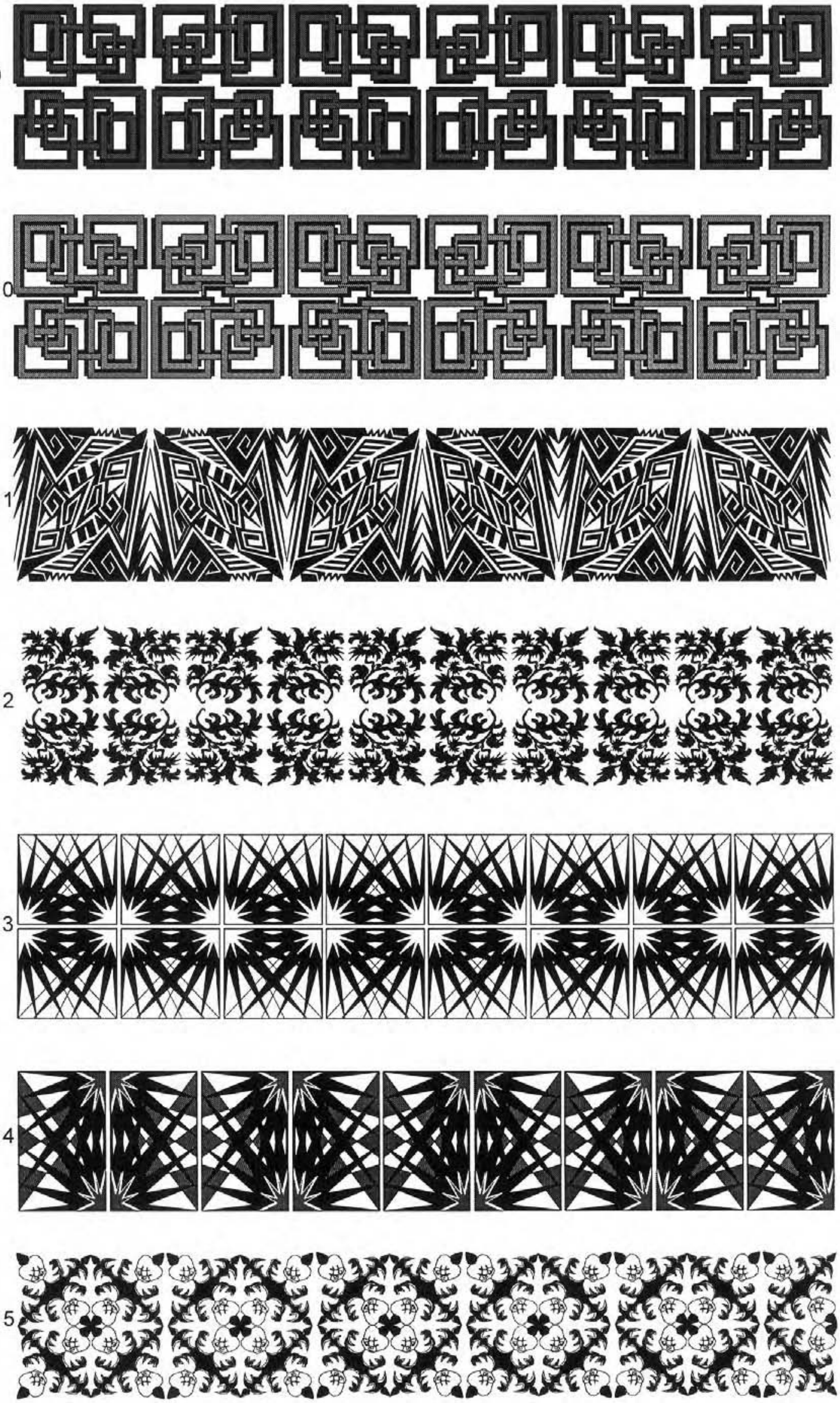


Figure 4.12 (cont.)

Table 4.3 The 51 ditranslational discrete pattern types

Pattern type	Symmetry group	Induced group	Motif-transitive subgroups
Dt(P)1	*p*1	*c*1	primitive
Dt(P)2	*pg*	*c*1	primitive
Dt(P)3	*pm*	*c*1	primitive
Dt(P)4	*pm*	*d*1	*p*1, *pg*, *cm*, *
Dt(P)5	*cm*	*c*1	primitive
Dt(P)6	*cm*	*d*1	*p*1, *pg*
Dt(P)7	*p*2	*c*1	primitive
Dt(P)8	*p*2	*c*2	*p*1, *
Dt(P)9	*pgg*	*c*1	primitive
Dt(P)10	*pgg*	*c*2	*pg*
Dt(P)11	*pmg*	*c*1	primitive
Dt(P)12	*pmg*	*c*2	*pg*, *pm*, *pgg*, *
Dt(P)13	*pmg*	*d*1	*pg*, *p*2, *pgg*
Dt(P)14	*pmm*	*c*1	primitive
Dt(P)15	*pmm*	*d*1	*pm*, *p*2, *pmg*(2), *cmm*
Dt(P)16	*pmm*	*d*2	*p*1, *pg*, *pm*(2), *cm*, *p*2(3), *pgg*, *pmg*(2), *cmm*(3), *(2)
Dt(P)17	*cmm*	*c*1	primitive
Dt(P)18	*cmm*	*c*2	*cm*, *pgg*, *pmm*
Dt(P)19	*cmm*	*d*1	*cm*, *p*2, *pgg*, *pmg*
Dt(P)20	*cmm*	*d*2	*p*1, *pg*, *cm*, *p*2(2), *pgg*(2), *pmg*
Dt(P)21	*p*3	*c*1	primitive
Dt(P)22	*p*3	*c*3	*p*1, *
Dt(P)23	*p*31*m*	*c*1	primitive
Dt(P)24	*p*31*m*	*c*3	*cm*, *p*3*m*1
Dt(P)25	*p*31*m*	*d*1	*p*3
Dt(P)26	*p*31*m*	*d*3	*p*1, *pg*, *cm*, *p*3(2)
Dt(P)27	*p*3*m*1	*c*1	primitive
Dt(P)28	*p*3*m*1	*d*1	*p*3
Dt(P)29	*p*3*m*1	*d*3	*p*1, *pg*, *cm*, *p*3(2), *p*31*m*
Dt(P)30	*p*4	*c*1	primitive
Dt(P)31	*p*4	*c*2	*
Dt(P)32	*p*4	*c*4	*p*1, *p*2(3), *(2)
Dt(P)33	*p*4*g*	*c*1	primitive
Dt(P)34	*p*4*g*	*c*4	*pg*, *cm*, *pgg*(2), *pmm*, *cmm*
Dt(P)35	*p*4*g*	*d*1	*pgg*, *p*4
Dt(P)36	*p*4*g*	*d*2	*pg*, *pgg*, *p*4(2)
Dt(P)37	*p*4*m*	*c*1	primitive
Dt(P)38	*p*4*m*	*d*1	*cmm*, *p*4, *p*4*g*, *
Dt(P)39	*p*4*m*	*d*1	*pmm*, *p*4, *p*4*g*
Dt(P)40	*p*4*m*	*d*2	*cm*, *pgg*, *pmm*, *cmm*, *p*4(2), *p*4*g*(2), *
Dt(P)41	*p*4*m*	*d*4	*p*1, *pg*(2), *pm*(2), *cm*(2), *p*2(3), *pgg*(3), *pmg*(3), *pmm*(3), *cmm*(4), *p*4(3), *p*4*g*(3), *(2)
Dt(P)42	*p*6	*c*1	primitive
Dt(P)43	*p*6	*c*2	*p*3
Dt(P)44	*p*6	*c*3	*p*2, *
Dt(P)45	*p*6	*c*6	*p*1, *p*2(2), *p*3(2)
Dt(P)46	*p*6*m*	*c*1	primitive
Dt(P)47	*p*6*m*	*d*1	*p*3*m*1, *p*6
Dt(P)48	*p*6*m*	*d*1	*p*31*m*, *p*6
Dt(P)49	*p*6*m*	*d*2	*p*3, *p*31*m*, *p*3*m*1, *p*6
Dt(P)50	*p*6*m*	*d*3	*cm*, *pgg*, *pmg*, *cmm*, *p*2, *p*31*m*, *p*3*m*1, *p*6(2)
Dt(P)51	*p*6*m*	*d*6	*p*1, *pg*(2), *cm*(2), *p*2(2), *pgg*(3), *pmg*(2), *cmm*, *p*3(2), *p*31*m*(2), *p*3*m*1, *p*6

Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

simplest way of illustrating this condition is to use an asymmetric design unit although, as described in Chapter 3, this is not the only possibility. In each case fundamental region boundaries are used as a guide for incorporating the design units. They are not included in the overall design and must be removed after the initial motif has been mapped to all its equivalent positions in the strip. Again, in the illustrations in Fig. 4.16, each shaded area represents a fundamental region or group of fundamental regions containing the motif.

Only a limited number of illustrations are given showing the construction of monotranslational discrete pattern types since they may be derived simply by following the construction techniques discussed in Chapter 2 together with the additional criteria given above.

4.11.1 Monotranslational pattern types, induced group *c*1

Each of the seven symmetry groups of monotranslational designs has one associated primitive discrete pattern type. These are constructed by dividing a strip into fundamental regions of the required symmetry group. A design unit, with no symmetries in common with the strip, is then added to one region such that the only point at which it meets a boundary is at the exterior straight edge(s) of the strip. It is then mapped onto all the remaining regions, by applying the symmetries of the design structure, to complete the pattern type. Figure 4.16(a) shows an example of this construction for pattern type Mt(P)2 (symmetry group *p*1*a*1).

4.11.2 Monotranslational pattern types, induced group *c*2

Symmetry groups *p*112 and *pma*2 each have one associated discrete pattern type with induced group *c*2. To construct these types of design, a strip is divided into appropriately shaped fundamental regions. A *cn* motif (where *n* is even) is added to the strip such that it is contained within two fundamental regions and its centre of rotation coincides with one featured in the design structure. It only intersects the edges of the fundamental regions which join at the centre of rotation and it also touches the edges of the fundamental regions which coincide with the edges of the strip. To map this motif to all its equivalent positions, a motif-transitive subgroup of pattern type Mt(P)8 or Mt(P)10 may be applied to complete each of the pattern types, respectively. Figure 4.16(b) shows an example for the construction of pattern type Mt(P)8, symmetry group *p*112.

4.11.3 Monotranslational pattern types, induced group *d*1

Symmetry groups *p*1*m*1, *pm*11 and *pma*2 each have one associated discrete pattern type with induced group *d*1, and *pmm*2 has two. Again, for each symmetry group, a strip is divided into fundamental regions and the symmetries of the group may be incorporated into the design structure. A *dn* motif (where *n* is odd) is added to the strip such that it falls into two fundamental regions and one of its reflection axes coincides with one featured in the monotranslational design structure. It does not intersect any boundaries of the two fundamental regions other than the one which bisects it and the ones which coincide with the boundaries of the strip. In the case of *pmm*2, there are two possibilities for the position of the motif for these characteristics to be satisfied. A motif-transitive subgroup of the required pattern type is applied to complete the Mt(P)4, Mt(P)6, Mt(P)11, Mt(P)13 or Mt(P)14 monotranslational design. An example is given in Fig. 4.16(c), for pattern type Mt(P)11, symmetry group *pma*2.

4.11.4 Monotranslational pattern types, induced group *d*2

Group *pmm*2 is the only monotranslational symmetry group with an associated pattern type having induced group *d*2. A strip is divided into fundamental regions and the symmetries of *pmm*2 may be incorporated into its structure. A *dn* motif

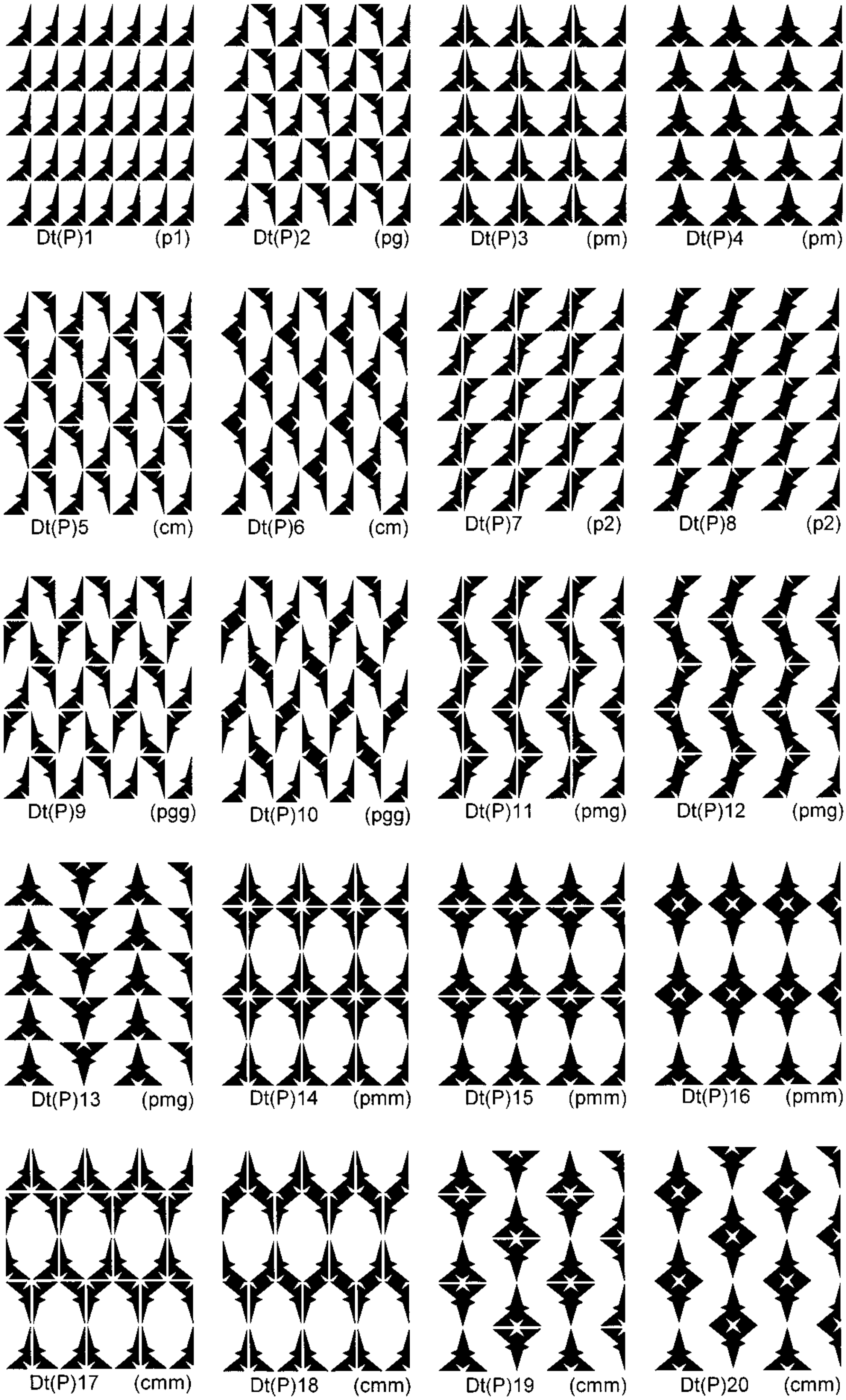


Figure 4.13 Schematic illustrations of the 51 ditranslational discrete pattern types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

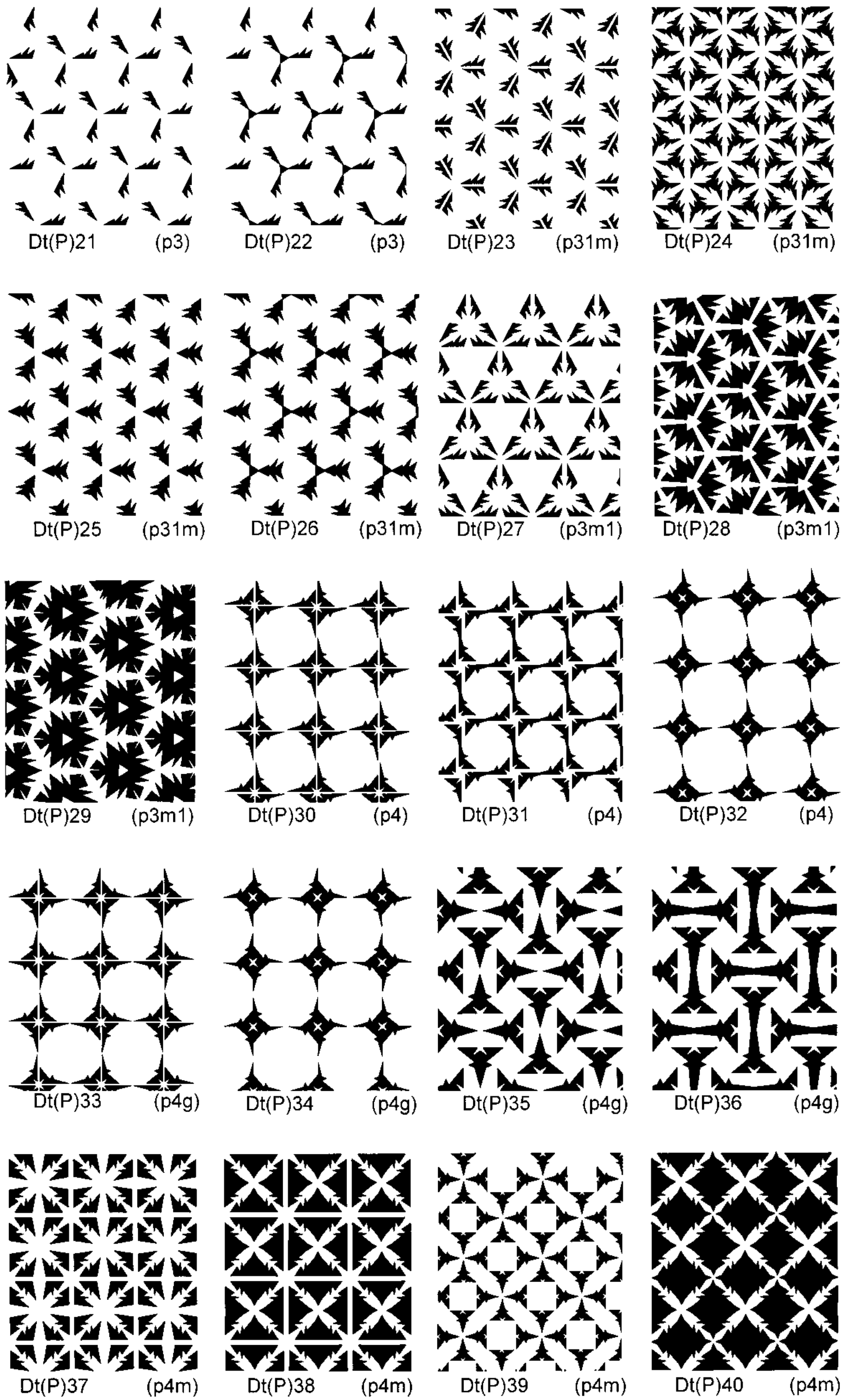


Figure 4.13 (cont.)

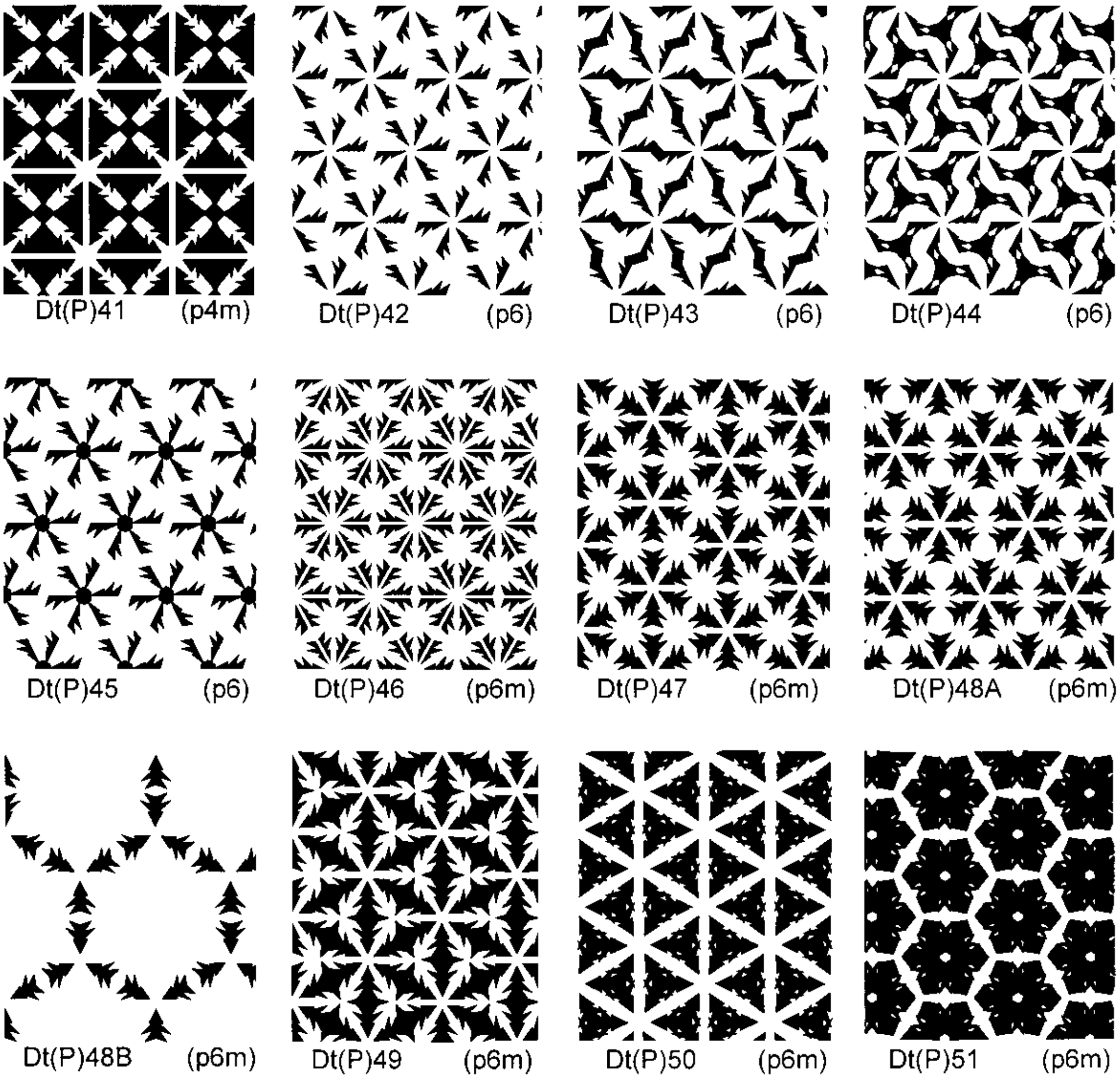


Figure 4.13 (cont.)

(where n is even) is added to the strip such that it falls into four fundamental regions and two of its perpendicular reflection axes coincide with ones featured in the monotranslational design structure. It does not intersect any fundamental region boundaries other than the ones which meet at its centre of rotation and the edges which coincide with the boundaries of the strip. A motif-transitive subgroup of Mt(P)15 is applied to complete the monotranslational design. An example is given in Fig. 4.16(d).

4.12 Construction of ditranslational discrete pattern types

The construction of ditranslational pattern types Dt(P)1 to Dt(P)51 follows similar techniques to those discussed in Section 2.13. Again, in each case, the boundaries of the fundamental regions are used as a guide for incorporating the design units. They are not included in the overall design and must be removed after the initial motif has been mapped to all its equivalent positions in the strip. The motif is incorporated in one, two, three, four or six fundamental regions for cyclic induced groups $c1$, $c2$, $c3$, $c4$ or $c6$ and two, four, six, eight or twelve fundamental regions for induced dihedral groups $d1$, $d2$, $d3$, $d4$ or $d6$, respectively. In each case, the cn ($n \geq 2$) or dn motif only intersects the boundaries of the fundamental regions which meet at the centre of the group of fundamental regions. The motif does not join, at any point, the boundary enclosing the group of fundamental regions containing the motif (except when fundamental region boundaries meet at a centre of rotation). Thus, when constructing ditranslational pattern types by methods described in Chapter 2 (i.e. by placing strips of width W next to each other) the initial design unit must not touch the edges of the strip. For some non-primitive pattern types it is not always possible to construct the same strip of fundamental regions described for the associated symmetry groups in Chapter 2 without splitting the motifs. In these cases, it is more suitable to construct a strip or double strip of whole motifs before consecutively applying the translations T_2 or T_3, respectively. These situations may be observed in the illustrations in the following sections.

As described previously, the design unit added to the fundamental region must have no symmetries in common with the design structure. For simplicity, this condition is most easily satisfied by ensuring that the design unit is asymmetric, as in the schematic illustrations in Fig. 4.13. As described in Chapter 3, additional symmetries are possible as characteristics of the design unit. However, to take all the values of N (in connection with the order of symmetry of the design unit) and induced symmetries into consideration for each pattern type would add further complication. Hence for simplicity, in the following construction methods the symmetry of design unit is taken to be asymmetric and consequently the induced group is the same symmetry group as that of the motif.

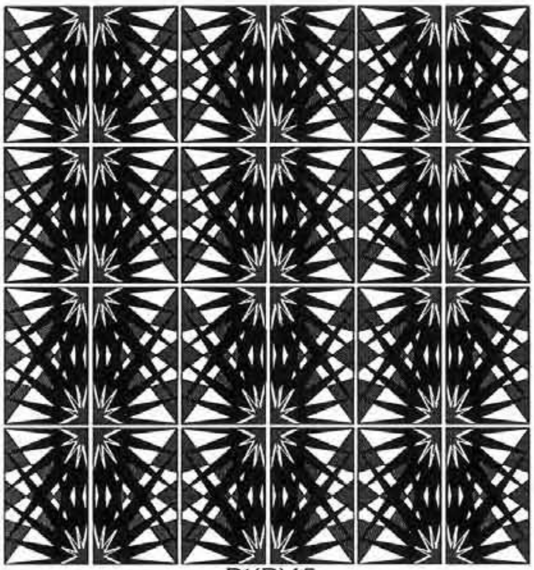
Only a limited number of illustrations are given showing the construction of ditranslational discrete pattern types because they may be derived simply by following the construction techniques discussed in Chapter 2 together with the additional criteria given above. In the first illustration in each of the Figs. 4.17 to 4.26 the dark shaded area represents a fundamental region or group of fundamental regions containing the motif and the light shaded area represents an appropriate strip to which translations T_2 or T_3 may be applied.

4.12.1 Ditranslational pattern types, induced group *c*1

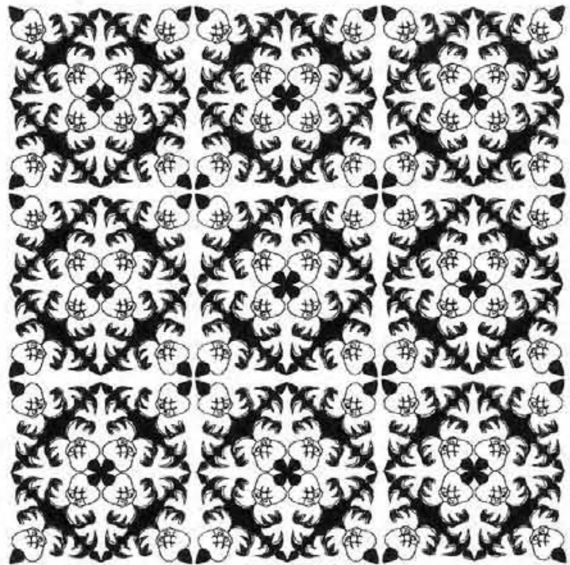
Each of the 17 symmetry groups of ditranslational design has one associated primitive discrete pattern type. These are derived by following exactly the same construction methods as those described for each symmetry group in Section 2.13 design types (ii) and (v) with the only difference being that the design unit consists of one piece and must not touch any fundamental region boundaries. An example is given for Dt(P)2, symmetry group *pg*, in Fig. 4.17.

4.12.2 Ditranslational pattern types, induced group *c*2

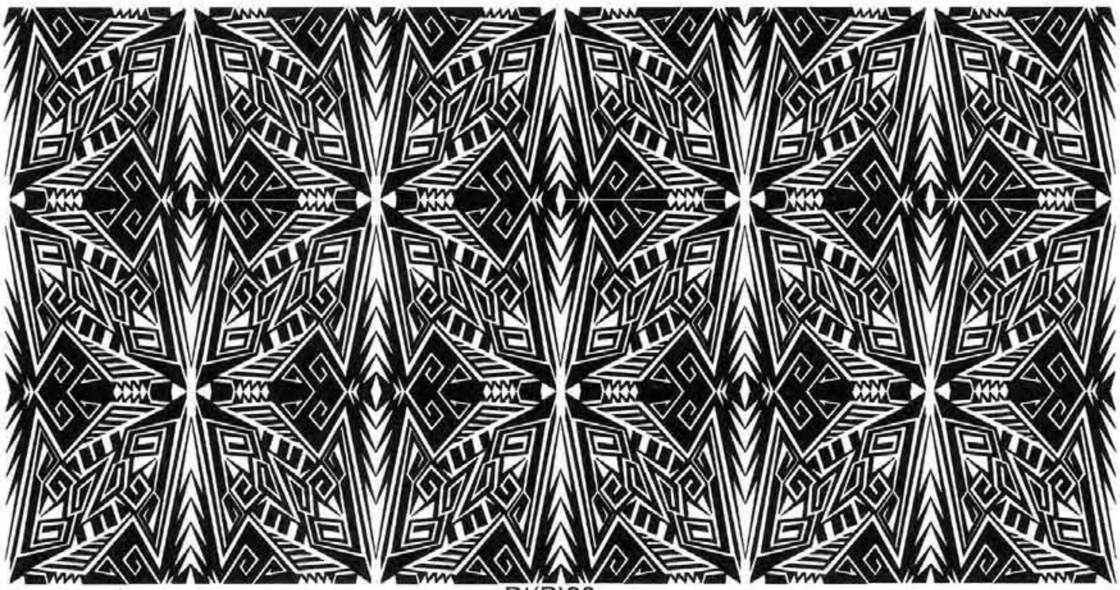
Symmetry groups *p*2, *pgg*, *pmg*, *cmm*, *p*4 and *p*6 each have one associated discrete pattern type with induced group *c*2. To construct these types of design, similar methods to those described for design types (ii) or (v) in Chapter 2 are followed but instead of the initial design unit being a *c*1 motif added to one fundamental region, a *c*2 motif is added to two fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure. The motif must not touch any other edges of the fundamental regions other than the ones joining at the point of its centre of two-fold rotation. Although the motif may touch these edges which join at this point, it must not meet any other adjacent centres of rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*2. Examples are given in Fig. 4.18 for pattern types Dt(P)31 (symmetry group *p*4) and Dt(P)8 (symmetry group *p*2). In the first example, the strip of translation units (derived from Fig. 2.42(iii)) has been altered to accommodate the *c*2 motifs. In the second example the lattice strucure is not rectangular and so the strip would have to be modified if it was used as the initial band for flat screen printing.



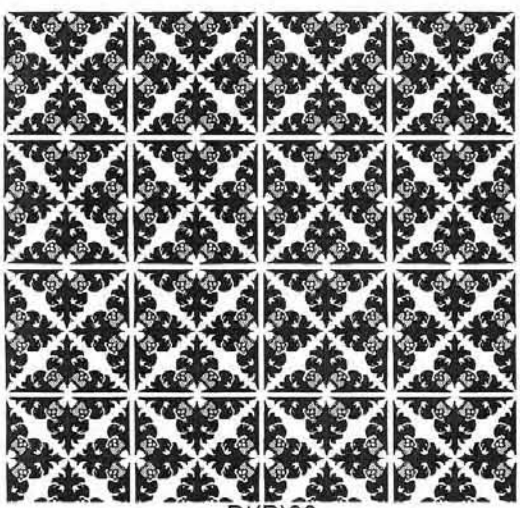
Dt(P)15



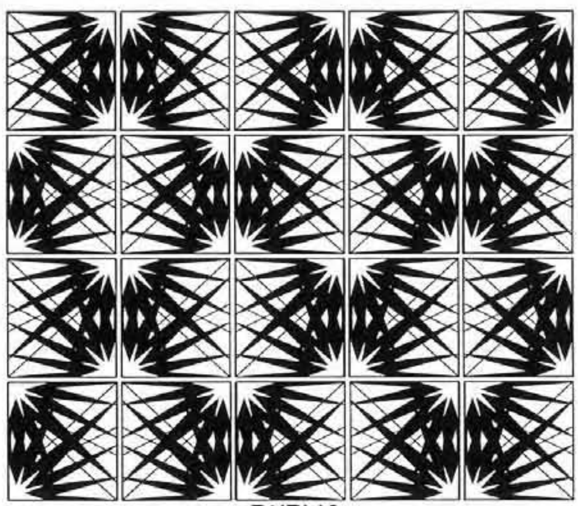
Dt(P)41



Dt(P)20



Dt(P)38



Dt(P)19

Figure 4.14 Further illustrations of ditranslational discrete pattern types.

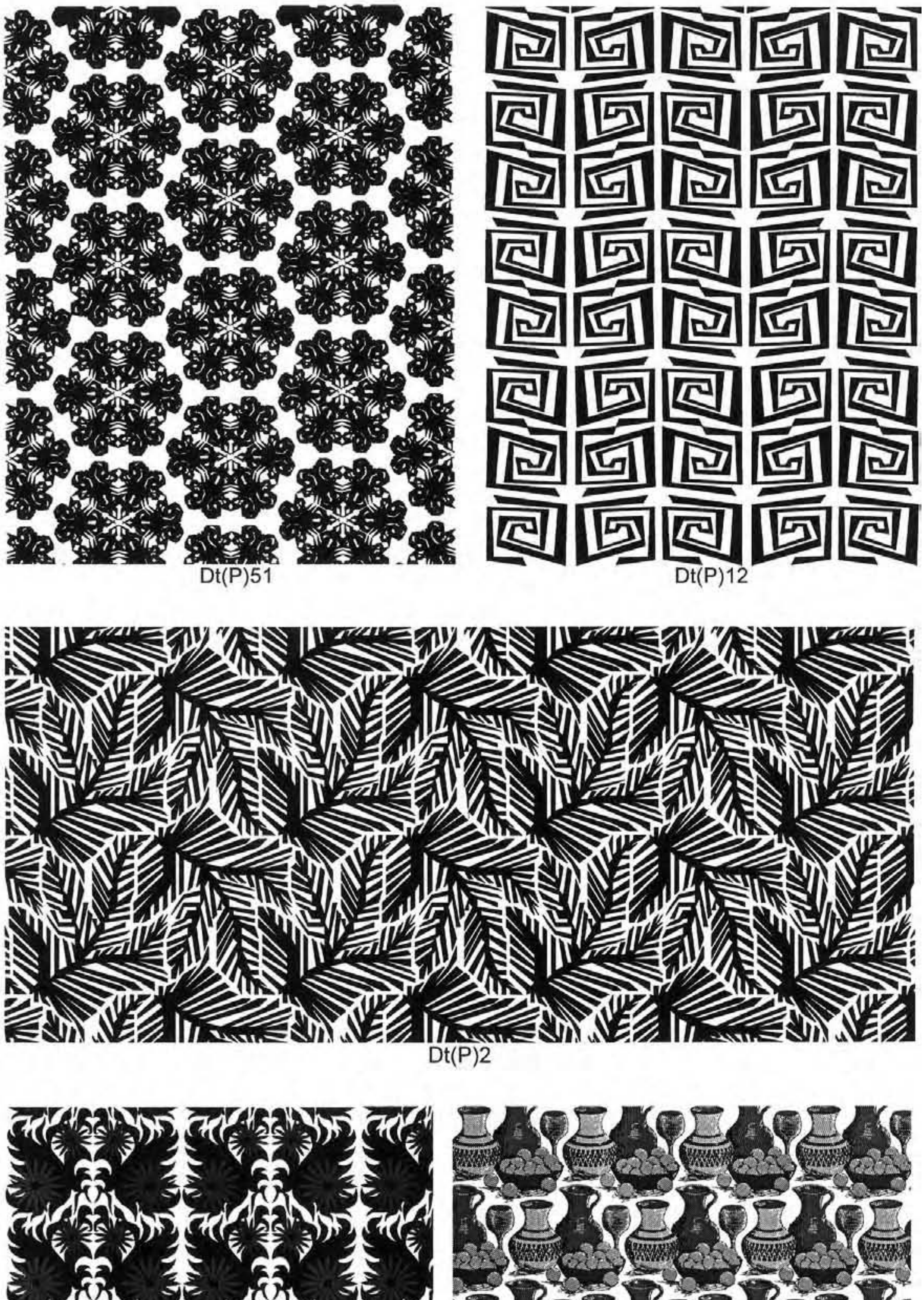


Figure 4.14 (cont.)

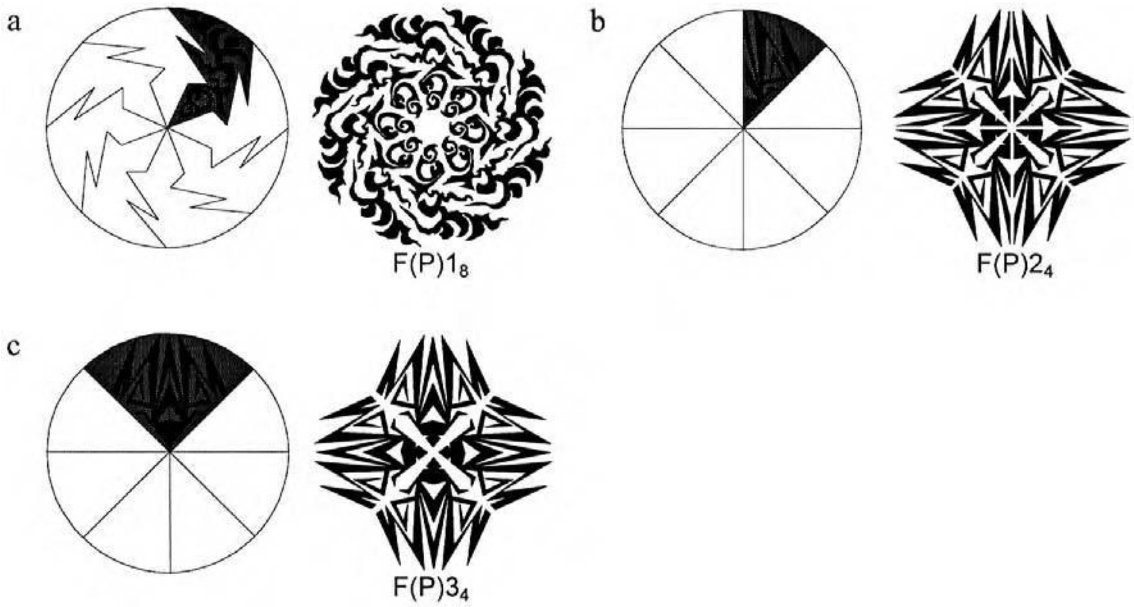


Figure 4.15 Construction of finite pattern types (a) F(P)1_n, (b) F(P)2_n and (c) F(P)3_n.

4.12.3 Ditranslational pattern types, induced group c3

Symmetry groups *p*3, *p*31*m* and *p*6 each have one associated discrete pattern type with induced group *c*3. To construct these types of design, similar methods are followed to those described in Chapter 2 for design types (ii) and (v). However, instead of the initial design unit being a *c*1 motif added to one fundamental region, a *c*3 motif is added to three fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure and it must not touch any other boundaries of the fundamental regions other than the three edges joined to the centre of three-fold rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*3. An example is given for Dt(P)22, symmetry group *p*3, in Fig. 4.19.

4.12.4 Ditranslational pattern types, induced group c4

Symmetry groups *p*4 and *p*4*g* each have one associated discrete pattern type with induced group *c*4. The initial *c*4 motif is added to four fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure and it must not touch any other boundaries of the fundamental regions other than the four edges joined to the centre of four-fold rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*4. Examples are given for discrete pattern types Dt(P)32 and Dt(P)34 (symmetry groups *p*4 and *p*4*g*, respectively) in Fig. 4.20.

4.12.5 Ditranslational pattern types, induced group c6

Symmetry group *p*6 has one associated discrete pattern type with induced group *c*6. The initial *c*6 motif is added to six fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure and it must not touch any other boundaries of the fundamental regions other than the six edges joined to the centre of six-fold rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*6. An example is given for Dt(P)45, symmetry group *p*6, in Fig. 4.21.

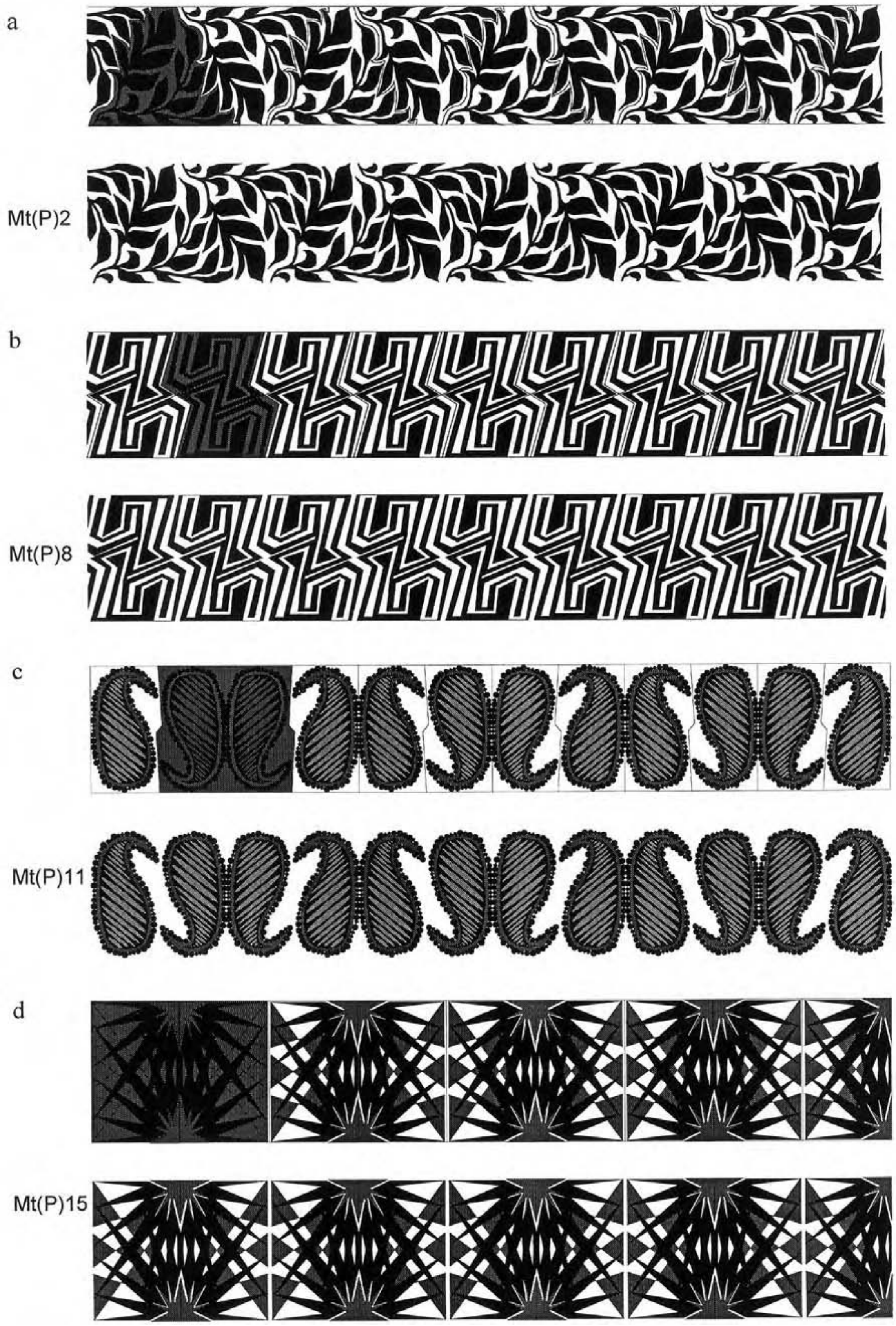


Figure 4.16 Construction of monotranslational pattern types.

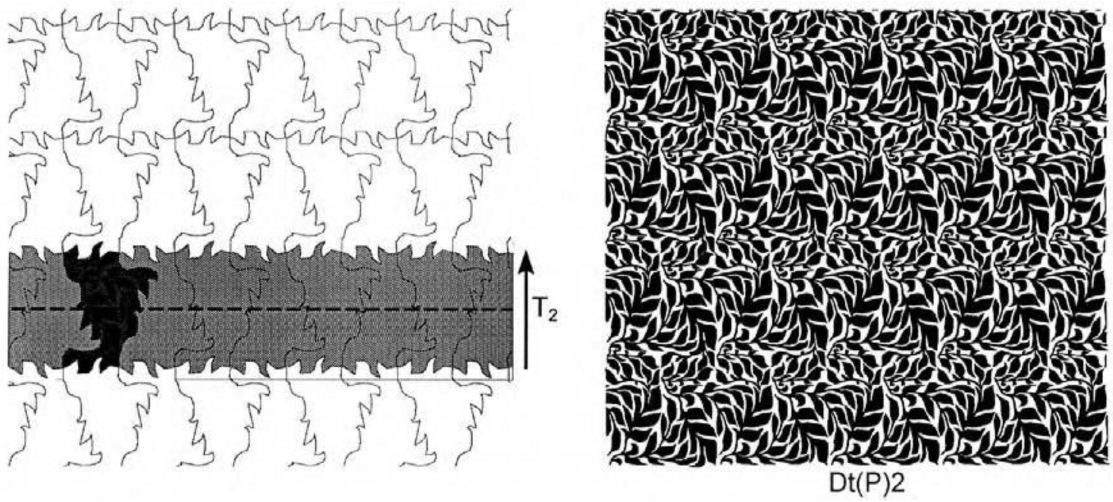


Figure 4.17 Construction of ditranslational pattern types, induced group *c*1.

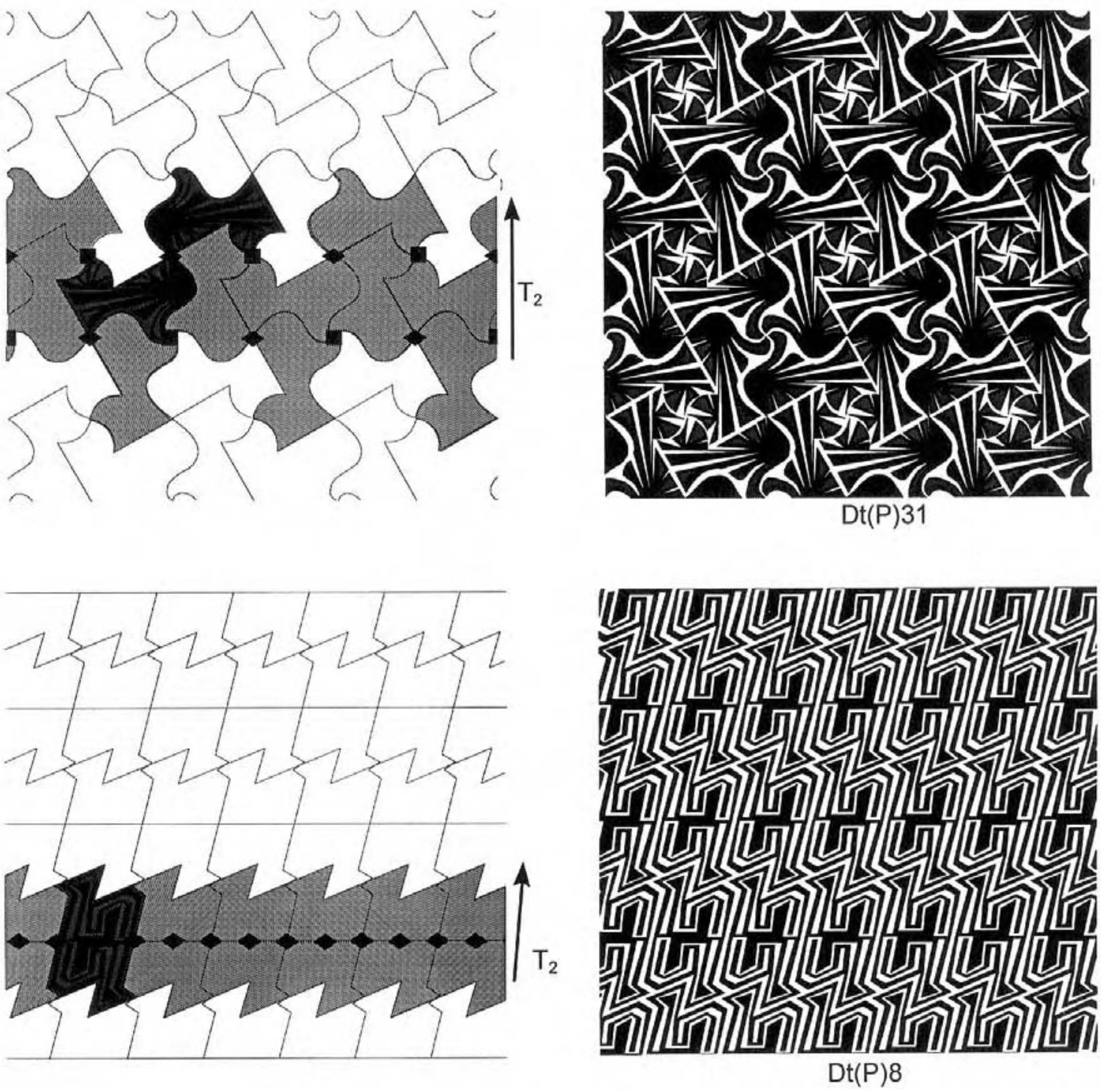


Figure 4.18 Construction of ditranslational pattern types, induced group *c*2.

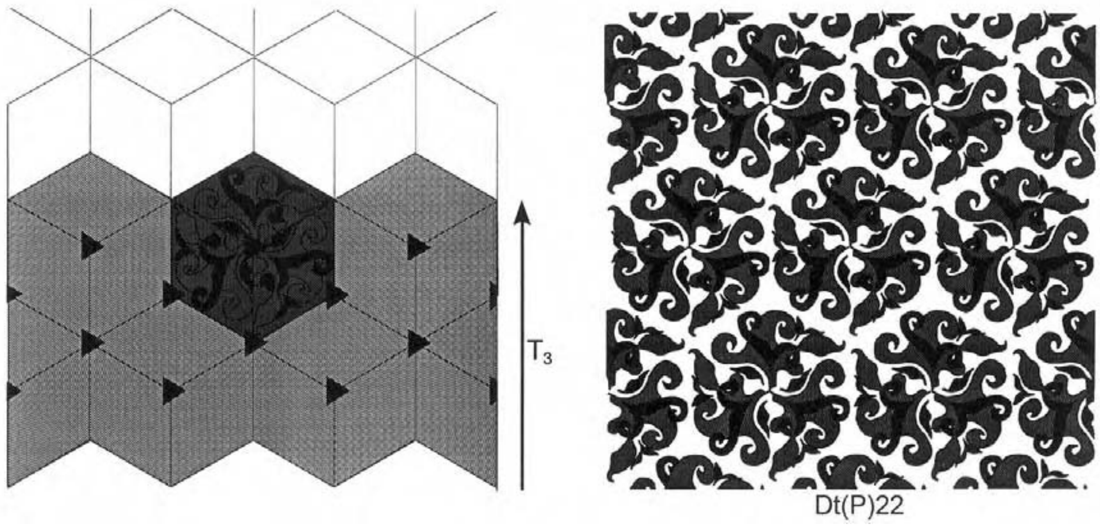


Figure 4.19 Construction of ditranslational pattern types, induced group *c*3.

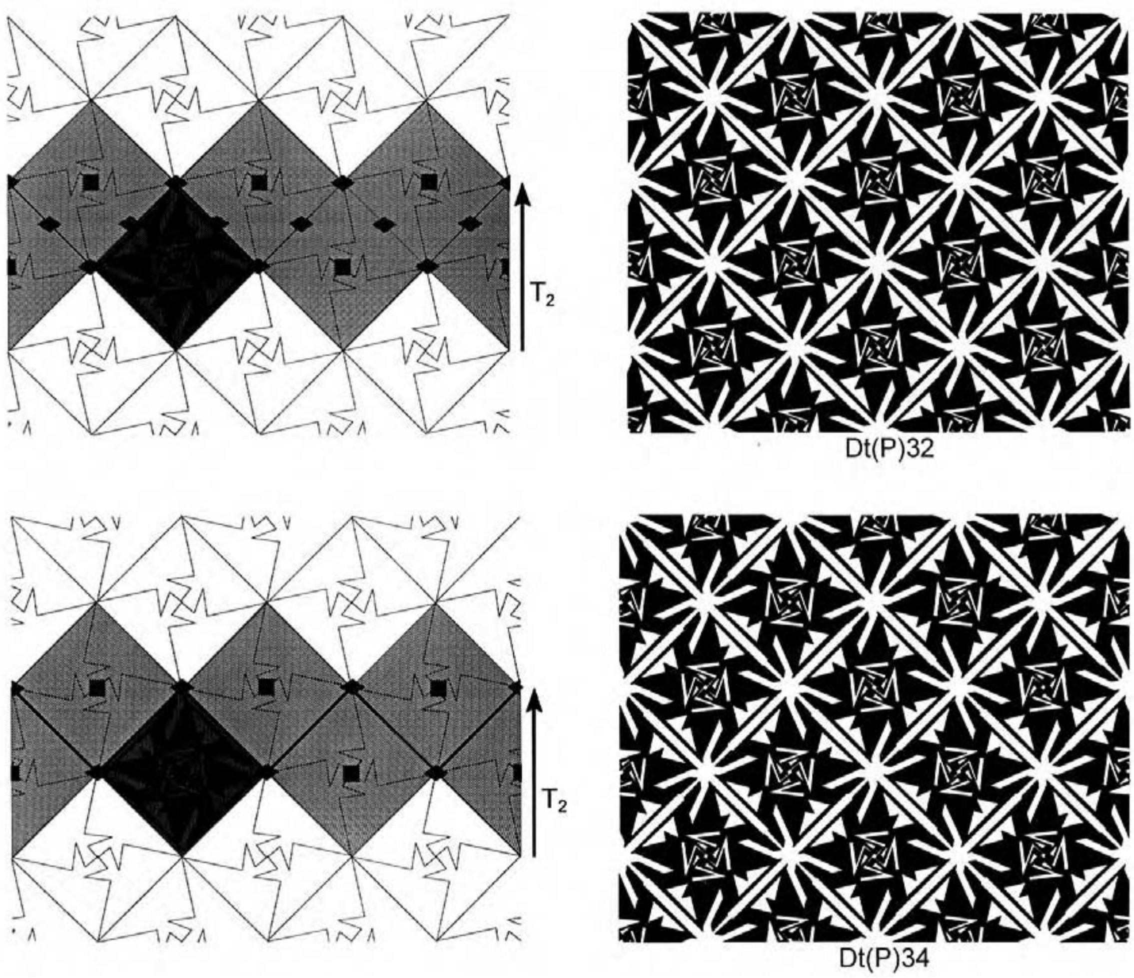


Figure 4.20 Construction of ditranslational pattern types, induced group *c*4.

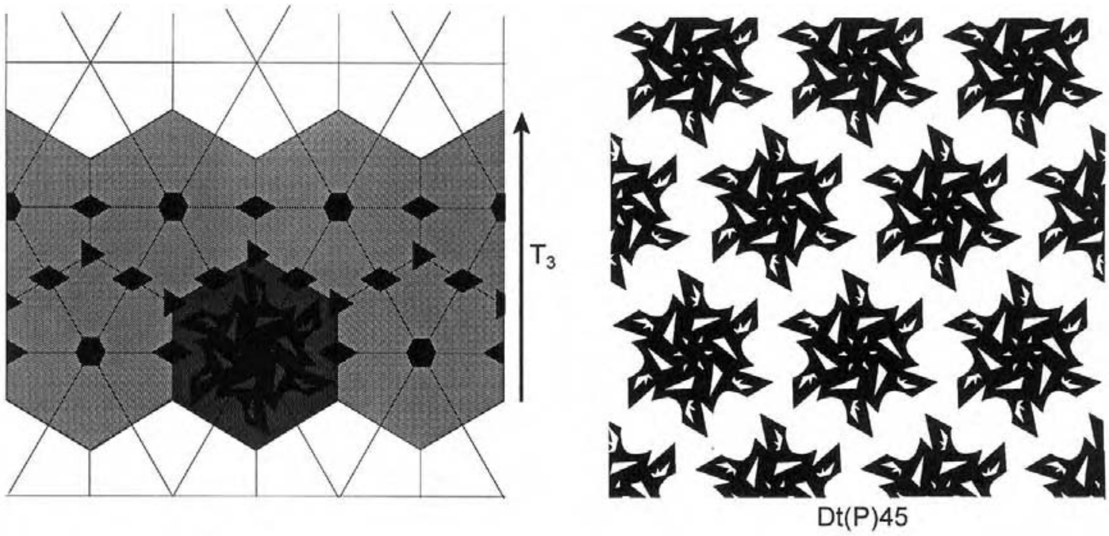


Figure 4.21 Construction of ditranslational pattern types, induced group *c*6.

*4.12.6 Ditranslational pattern types, induced group d*1

Symmetry groups *pm*, *cm*, *pmg*, *pmm*, *cmm*, *p*31*m*, *p*3*m*1, *p*4*g* each have one associated discrete pattern type with induced group *d*1 and *p*4*m* and *p*6*m* each have two. The initial *d*1 motif is added to two fundamental regions. Its reflection axis must coincide with one featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the one edge bisecting it. This motif is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*1. In the case of *p*4*m* and *p*6*m* designs there are two inequivalent discrete patterns with induced group *d*1. To construct the two different types of *p*4*m* pattern either the initial motif is placed with its reflection axis perpendicular to a side of the unit cell or its reflection is placed such that it coincides with a diagonal of the unit cell. These two cases are illustrated in the second and third examples of Figure 4.22, respectively. Similarly, the two cases of pattern type *p*6*m*, with induced group *d*1, are produced by placing the reflection axis of the initial motif either parallel to or at 30° to a side of a unit cell. An illustration for the construction of Dt(P)6 (symmetry group *cm*) is given in the first example of Fig. 4.22.

*4.12.7 Ditranslational pattern types, induced group d*2

Symmetry groups *pmm*, *cmm*, *p*4*g*, *p*4*m* and *p*6*m* each have one associated discrete pattern type with induced group *d*2. The initial *d*2 motif is added to four fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of two-fold rotation, through its centre. This motif is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*2. Examples are given for Dt(P)20 and Dt(P)40 (symmetry groups *cmm* and *p*4*m*, respectively) in Fig. 4.23.

*4.12.8 Ditranslational pattern types, induced group d*3

Symmetry groups *p*31*m*, *p*3*m*1 and *p*6*m* each have one associated discrete pattern type with induced group *d*3. The initial *d*3 motif is added to six fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of three-fold rotation, through its centre. This motif is mapped to all its equivalent positions to

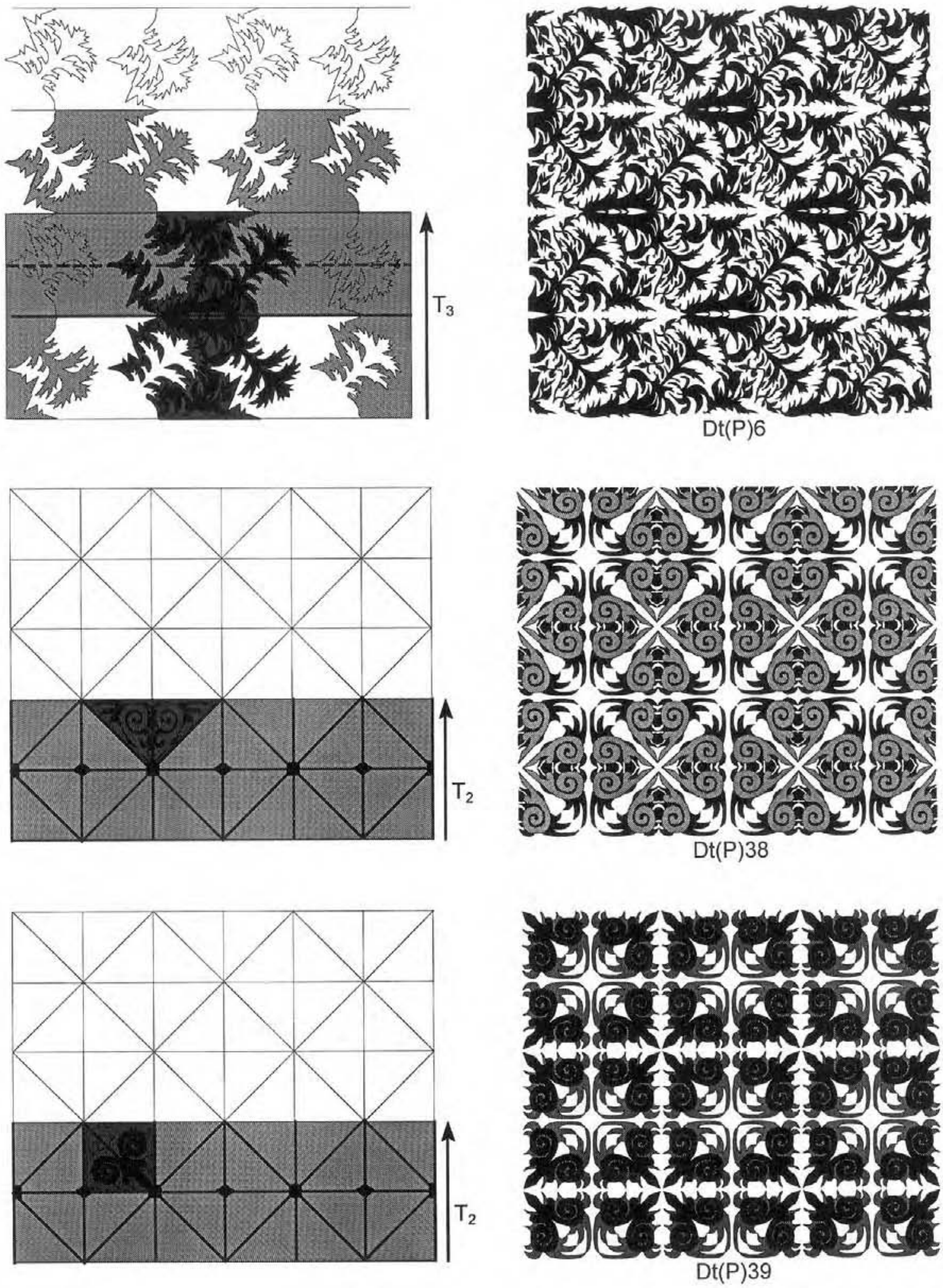


Figure 4.22 Construction of ditranslational pattern types, induced group *d*1.

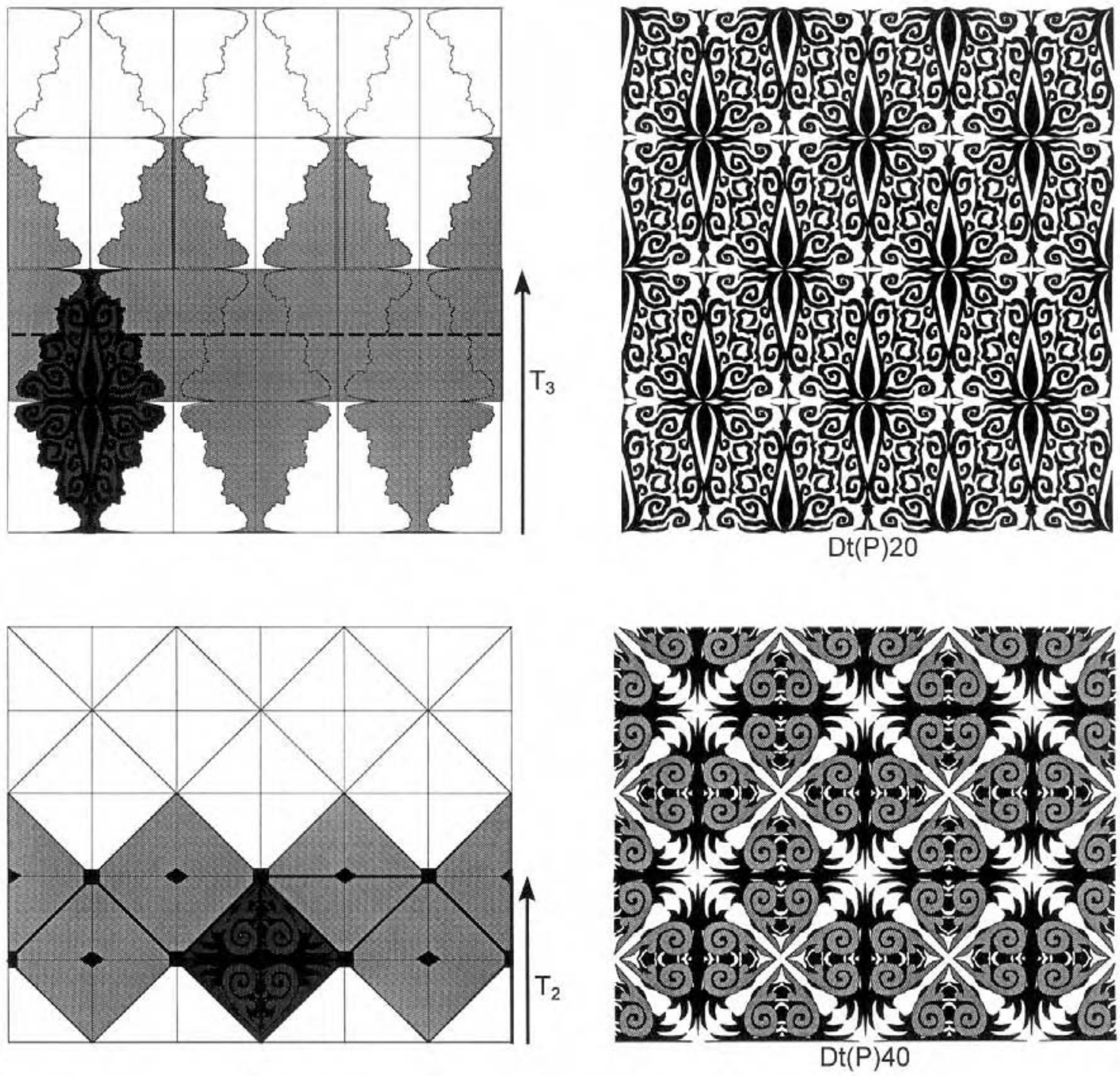


Figure 4.23 Construction of ditranslational pattern types, induced group *d*2.

complete the discrete pattern type with induced group *d*3. An example is given for Dt(P)29, symmetry group *p*3*m*1, in Fig. 4.24.

4.12.9 Ditranslational pattern types, induced group d4

Symmetry group *p*4*m* has one associated discrete pattern type with induced group *d*4. The initial *d*4 motif is added to eight fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of four-fold rotation, through its centre. This motif is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*4. An example is given for Dt(P)41, symmetry group *p*4*m*, in Fig. 4.25.

4.12.10 Ditranslational pattern types, induced group d6

Symmetry group *p*6*m* has one associated discrete pattern type with induced group *d*6. The inital *d*6 motif is added to twelve fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of six-fold rotation, through its centre. This motif

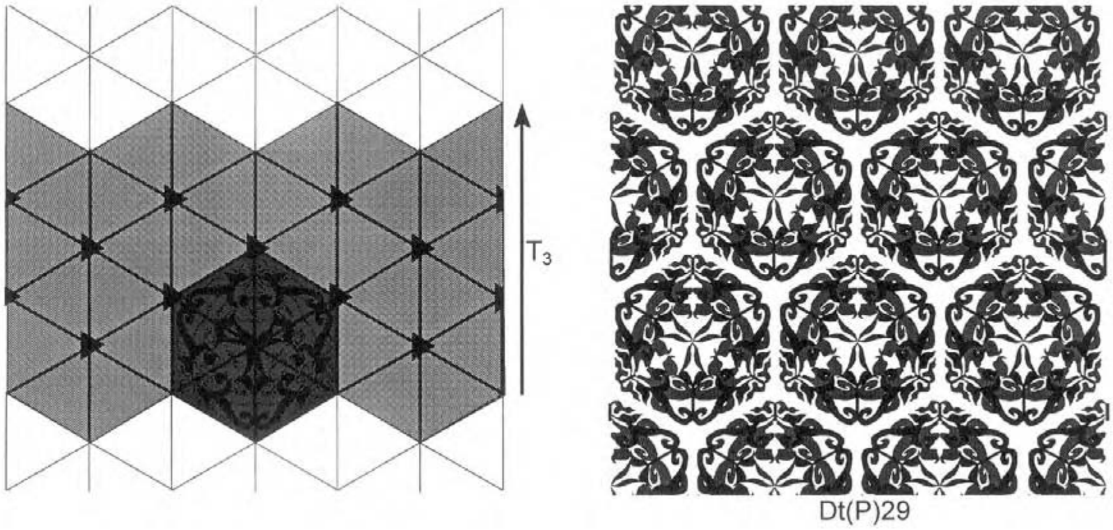


Figure 4.24 Construction of ditranslational pattern types, induced group *d*3.

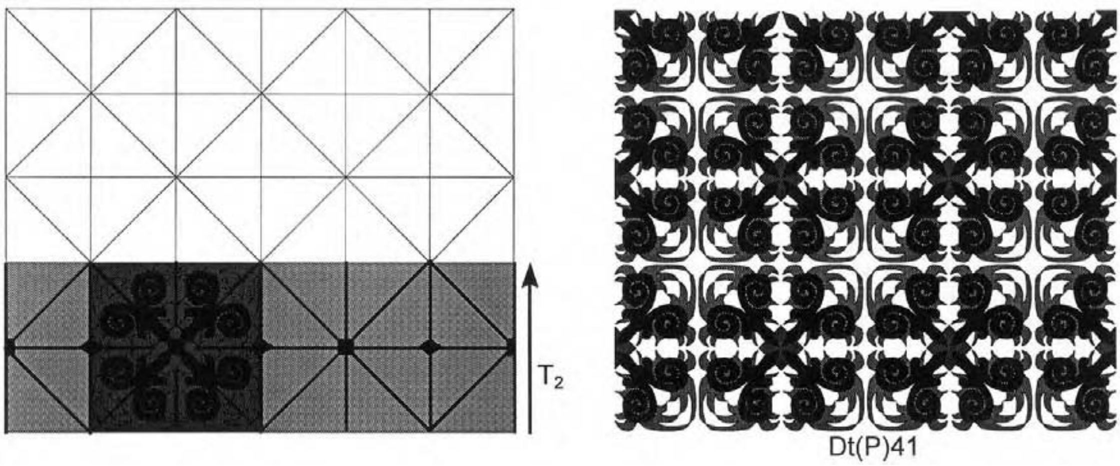


Figure 4.25 Construction of ditranslational pattern types, induced group *d*4.

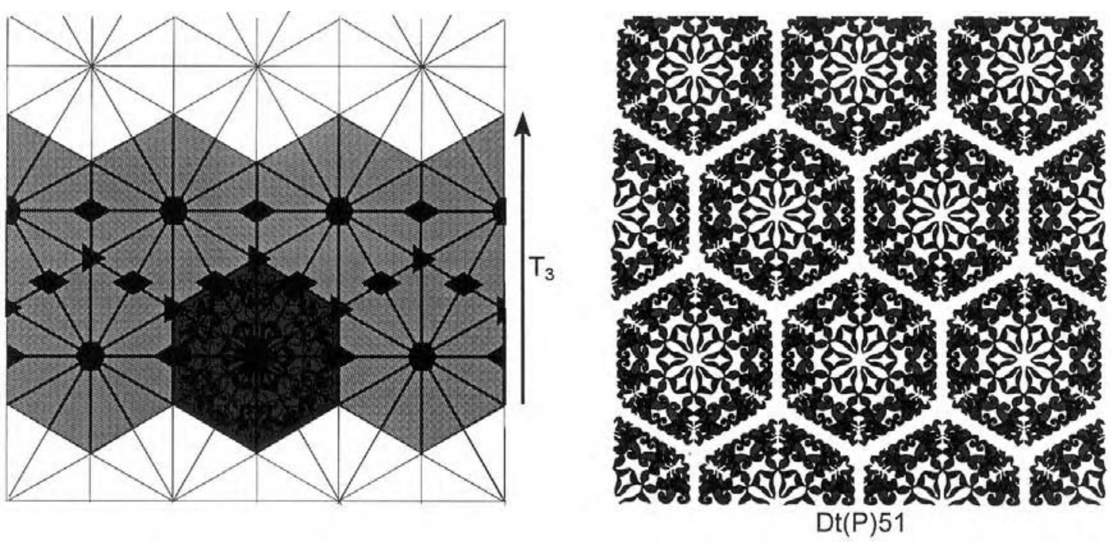


Figure 4.26 Construction of ditranslational pattern types, induced group *d*6.

is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*6. An example is given for Dt(P)51, symmetry group *p*6*m*, in Fig. 4.26.

4.13 Summary

This chapter builds on the concepts and perspectives used by Grünbaum and Shephard in their classification of discrete patterns.[1] The characteristics of discrete patterns and principles involved in categorising these types of designs are discussed in detail. The classification and construction of the three finite, 15 monotranslational and 51 ditranslational discrete pattern types have been described and illustrated with numerous examples.

The designs constructed from this classification system may have a more disjointed appearance owing to the requirement for a discrete pattern to be composed of motifs which are separate from each other. In some of the examples given in the construction of discrete patterns, although the motifs are 'pairwise disjoint' (see Section 4.2) it is sometimes difficult to visualise a motif as being able to be contained within a tile without this tile overlapping an adjacent motif (as stated in DP.2 for a discrete pattern, Section 4.3). Because, in some cases, the motifs are very close together and the scale of the patterns is small in order to exhibit a sufficient proportion of repeat, the motifs appear to be touching each other. This may contravene the precise mathematical definition given for a discrete pattern. However, with regard to the classification and construction of discrete patterns in the context of creative surface-pattern design, the less formal definitions given after the formal statements provide sufficient regulation.

As a consequence of the distinctive 'separation' characteristic of the motifs of a discrete pattern it is possible to construct a type of patterned tiling by incorporating a tiling in between, or surrounding, the motifs. A similar type of design was mentioned in Chapter 2 (as shown in the construction of design type (iv)) where the edges of the tiles corresponded to the boundaries of the fundamental regions. In this instance the design units were permitted to touch the boundaries of the tiles. Conversely, a tiling design may be derived from a discrete pattern, as described above, such that each motif is contained within one tile and the boundaries of the tiles do not touch the motifs. For ditranslational designs, the tiling may be thought of as a covering of the plane with tiles having shapes corresponding to the dark shaded areas given in the previous construction techniques for ditranslational designs (Section 4.12). However, in some of these examples the dark regions could not be regarded as tiles because each is divided into portions which meet at a point (e.g. see the first example Figure 4.18). Nevertheless, there are numerous other ways of dividing a plane into fundamental regions or surrounding a pattern by tiles, other than those discussed in Chapter 2. One particular type of tiling design which relates to a specific method of enclosing a discrete pattern is referred to as an isohedral tiling. The analysis, classification and construction of these types of tiling design are discussed in detail in Chapter 5.

References

1 Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman, 1987.
2 Shubnikov A V and Koptsik V A, *Symmetry in Science and Art*, New York, Plenum Press, 1974.
3 Senechal M, *Crystalline Symmetries: An Informal Mathematical Introduction*, Bristol, Adam Hilger IOP Publishing, 1990.