5

Classification of isohedral tilings

5.1 Introduction

The concepts involved in classifying discrete patterns described in the previous chapter may be adapted and developed to form a classification system for tilings. So far, as discussed in Chapter 2, both pattern and tiling structures have been analysed in the same way and subsequently divided into symmetry groups. Following this, in Chapters 3 and 4 greater attention was paid to the finer detail of the symmetrical properties of the motifs. The tiling designs and classification system discussed in this chapter are closely related to the discrete patterns in Chapter 4. For example, because the motifs in a discrete pattern are separate from each other, a tiling design may be incorporated around them to form a patterned tiling. By doing this in a particular way (described in Section 5.3) a special form of tiling is produced which is referred to as a 'Dirichlet tiling' or 'Dirichlet domain'. The discrete pattern may be removed from the design and then the structure of the remaining tiling may be analysed and classified as a particular class of 'isohedral' tiling. (Dirichlet domains/tilings were named after the mathematician Peter Gustav Lejeune Dirichlet.[1] They are also referred to as 'Voronoi cells' or 'Voronoi regions', 'Brillouin zones' or 'Wigner-Seitz cells'[2] and also 'domain of influence' or 'plesiohedron'[1].)

In connection with ditranslational designs, since there are 51 different discrete pattern types, it would be assumed that each of these may be enclosed within one Dirichlet tiling, implying that there are 51 different ditranslational isohedral tiling types. However, this is not the case. The motifs in a discrete pattern type may be surrounded by more than one form of isohedral tiling and conversely, an isohedral tiling may form a Dirichlet domain for more than one type of discrete pattern. For finite and monotranslational discrete patterns, an associated Dirichlet tiling would be unbounded (i.e. each tile would extend infinitely) thus not satisfying the conditions of a 'normal' tiling (see Section 5.2.1). Consequently, in the following discussions, restrictions will be imposed on the extremities of the tiles for these types of design.

With reference to finite and monotranslational tilings, as stated in Chapter 2 (Sections 2.7.5 and 2.7.6), it may be difficult to categorise a design as either a pattern or tiling. A tiling is usually thought of as a type of design made up of shapes that interlock or neatly join to each other, leaving no gaps, and which covers an entire plane. However, in the context of this book (where finite and monotranslational tilings are considered), an appropriate definition for a tiling T is given by Lenart as a set of m-dimensional entities, called tiles, T = {t1, t2 . . .} that covers an area of an m-dimensional space without gaps or overlaps. This area can be the entire space.[3] In the context of surface design, the 'space' to be covered is two-dimensional, that is $m = 2$, as the design will initially be covering a flat surface. If the decorated area covers the entire space (with translational symmetry in at least two non-parallel directions) then the resulting design will be referred to as a ditranslational tiling design. If the area is enclosed within a strip (with translational symmetry in one direction) then it becomes a monotranslational tiling design. If it is enclosed within a circle (with no translational symmetry) it will be referred to as a finite tiling design.

This definition appears to be straightforward but ambiguities may still arise at limiting cases. For example, each tile has a boundary and when placed next to each other these boundaries form lines which, depending on their thickness, may be regarded as merely a source for division of the plane or, when thicker, as a background for a pattern. Similarly, designs with many different shaped and/or complicated tiles may appear more pattern-like than tiling-like and, for example, some two-coloured designs may either be regarded as a two-coloured tiling or as a black pattern on a white background or vice versa. Additionally, for finite and monotranslational tiling designs, their outside boundaries may veer towards their centres or longitudinal axes, respectively, thus making it difficult to determine whether it is a pattern or tiling design (see Fig. 5.1).

There is often a grey area within which pattern and tilings may coexist, and it is difficult (particularly in the context of finite and monotranslational tiling designs) to arrive at a precise definition which is appropriate in every context. Consequently, to avoid any ambiguities, the illustrative examples presented in this chapter have obvious tiling characteristics.

Compared to the classification of finite and monotranslational tilings, the classification of ditranslational isohedral tilings is more complicated, and requires further geometrical parameters such as the topology and the relationships between the edges and adjacent tiles to be taken into account. Considering topological variation, in comparison to the limited variety discussed in Chapter 2 (as described in Section 2.13 and illustrated in Fig. 2.31), provides further scope for the interlocking nature of fundamental regions and consequently allows greater freedom in design construction. Thus through these tiling designs a wide variety of patterned tilings and interlocking patterns may be produced by a similar method to one of those described for design types (iv) and (v) in Chapter 2. Although the classification and construction methods discussed in this chapter are, in general, illustrated with tiles and motifs which have very formal and rigid graphic qualities these are merely to present a clear insight into design structure upon which surface-pattern designers may build or use as a basis for more free-flowing creative designs.

5.2 Isohedral tiling

The classification system in this chapter is only applicable to a particular range of tilings which have the characteristics of being ‘normal’, ‘monohedral’ and ‘isohedral’.

5.2.1 Normal tiling

Grünbaum and Shephard[4] define a tiling T as ‘normal’ if it satisfies the conditions N.1, N.2 and N.3 below:

> N.1 Every tile of T is a topological disk. . .
> N.2 The intersection of every two tiles of T is a connected set, that is, it does not consist of two (or more) distinct and disjoint parts . . .
> N.3 The tiles of T are uniformly bounded.

These conditions N.1 to N.3 may be thought of as follows:

- N.1′ Every tile has a boundary edge which joins up with itself and has no breaks in it.
- N.2′ If one tile is adjacent to another, they have line segment(s) in common in the form of *one* edge only.
- N.3′ A tile is uniformly bounded if it is small enough to have a circle drawn round it and yet large enough to have a circle drawn inside it, i.e. this condition prevents tiles being either too long or too thin. The exact, permissible conditions are hard to define but the dimensions of each tile, in this context, will be taken to be of ‘sensible’ proportions.

Examples of tilings which do not satisfy these characteristics, N.1′, N.2′ and N.3′, are given in Figure 5.2(a), (b) and (c), respectively.

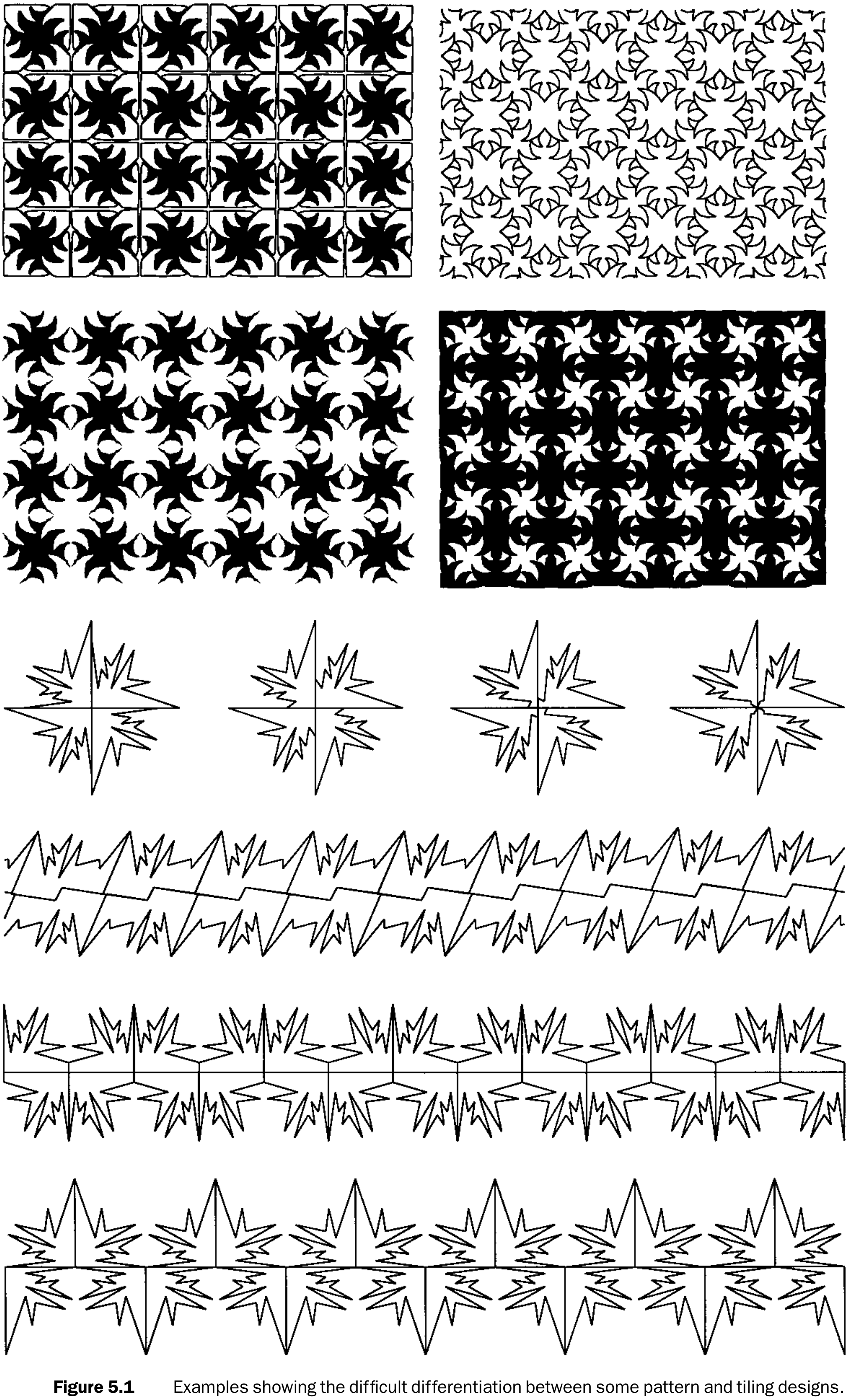


Figure 5.1 Examples showing the difficult differentiation between some pattern and tiling designs.

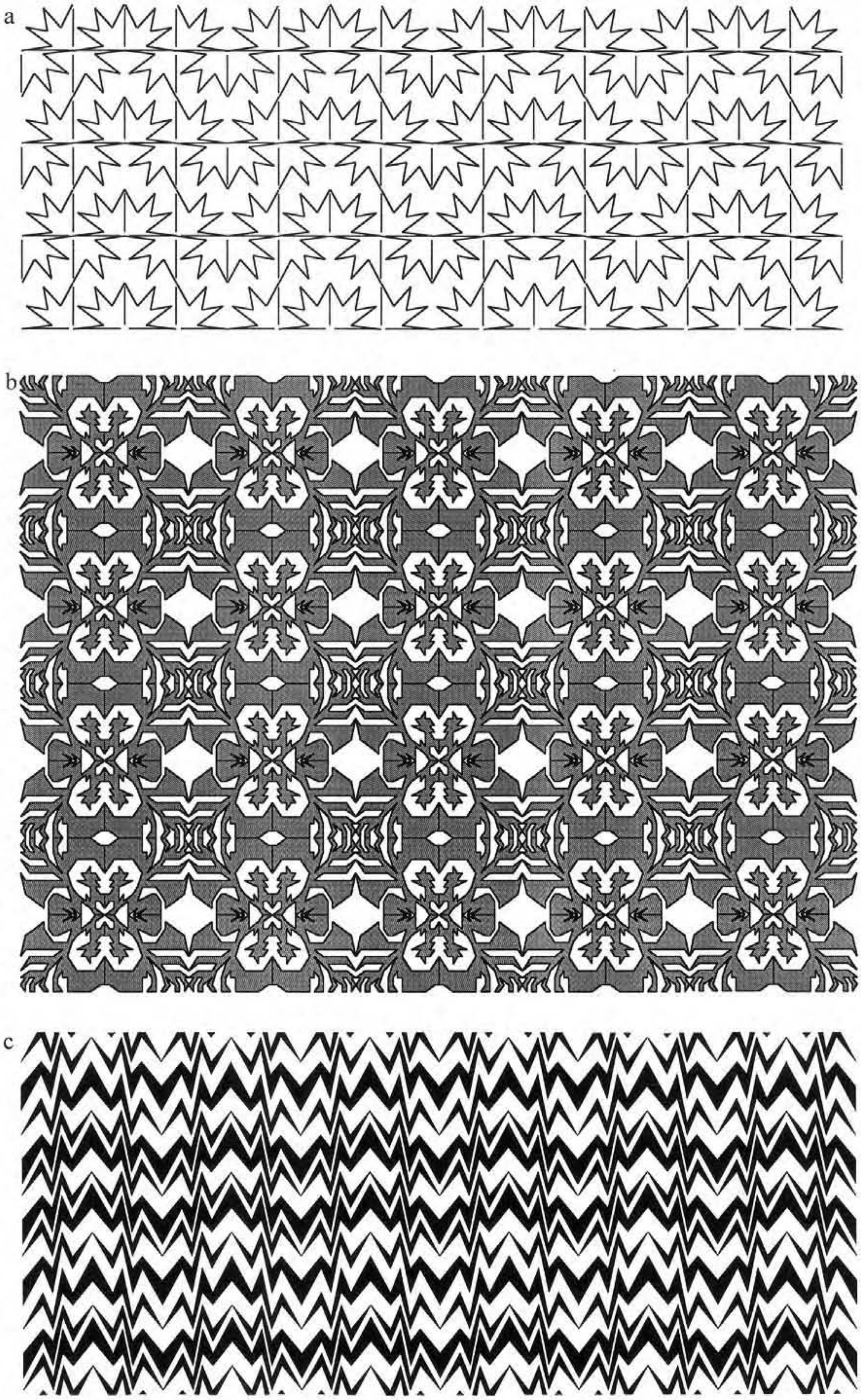


Figure 5.2 Examples of tilings which are not 'normal'.

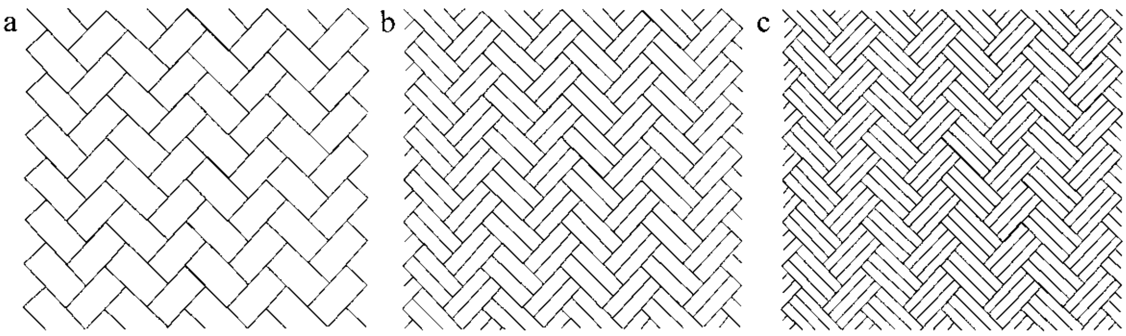


Figure 5.3 Examples of (a) and (b) isohedral and (c) non-isohedral tilings.

A monohedral tiling has a similar description to that of condition P.2′ of a monomotif pattern in Section 4.2 in that one tile is congruent to all the others, that is, all the tiles are the same size and shape. An isohedral tiling, which is a special form of monohedral tiling, is formally defined by Grünbaum and Shephard[4] as follows:

> Two tiles T_1, T_2 of a tiling T are said to be *equivalent* if the symmetry group $S(T)$ contains a transformation that maps T_1 onto T_2; the collection of all tiles of T that are equivalent to T_1 is called the *transitivity class* of T_1. If all tiles of T form one transitivity class we say that T is *tile transitive* or *isohedral*.

This definition is comparable to condition P.3′ of a monomotif pattern, that is, if each tile can be mapped onto any other tile by a symmetry of the tiling then the tiling is isohedral. Lenart defines an isohedral tiling more simply by saying that a monohedral tiling T is called *isohedral* if, given two tiles t_i and t_j, there is a symmetry transformation of the entire tiling which maps t_i onto t_j.[3]

Again, the simplest way to assess whether a translational tiling is isohedral is to look at a translation unit. If, inside one translation unit, each tile can be mapped onto any other by an isometry of the tiling, then by subsequent unit translations, any tile can be mapped onto any other in the whole tiling. Illustrations of monohedral tilings, which are either isohedral or non-isohedral, are given in Fig. 5.3 with finer details of their characteristics shown in Fig. 5.4.

Figure 5.4(a(i)) shows the incorporation of the group diagram into the first design (Fig. 5.3(a)) displaying the symmetries present in its structure. Figure 5.4(a(ii)) illustrates one way of dividing the design into translation units. By analysing the tiles and symmetries which occur in just one translation unit – (Fig. 5.4a(iii)) – note that one tile, for example T1, can be mapped onto the other, T2, by either horizontal or vertical glide–reflectional symmetry. This implies that, since tile T1 may be mapped onto the other tile in the translation unit, it is possible to map it onto any other tile in the remainder of the tiling by applying glide–reflectional and translational symmetries. Hence, the tiling is isohedral.

Similarly, Fig. 5.4(b(i)), (b(ii)) and (b(iii)) represents equivalent characteristics for the tiling in Fig. 5.3(b). In this example, each translation unit contains four tiles: T1, T2, T3 and T4. Tile T1 may be mapped onto tile T2 by two-fold rotational symmetry about a centre of rotation half way along one edge. It may be mapped onto tile T3 by vertical glide–reflectional symmetry and T4 by horizontal glide–reflectional symmetry. Therefore, since T1 may be mapped onto each of the other tiles in the translation unit, it is possible to map it onto any other tile in the whole tiling and consequently the tiling in Fig. 5.3(b) is isohedral.

Figure 5.4(c) represents equivalent characteristics for the tiling in Fig. 5.3(c). Each translation unit contains six tiles: T1 to T6. Tile T1 may be mapped onto T3 by two-fold rotational symmetry; onto T4 by vertical glide–reflectional symmetry; and onto T6 by horizontal glide–reflectional symmetry. However, there is no symmetry in the tiling which will allow tile T1 to be mapped onto either T2 or T4. Therefore the tiling in Fig. 5.3(c) is non-isohedral.

Although a finite tiling design does obviously not have translational symme-

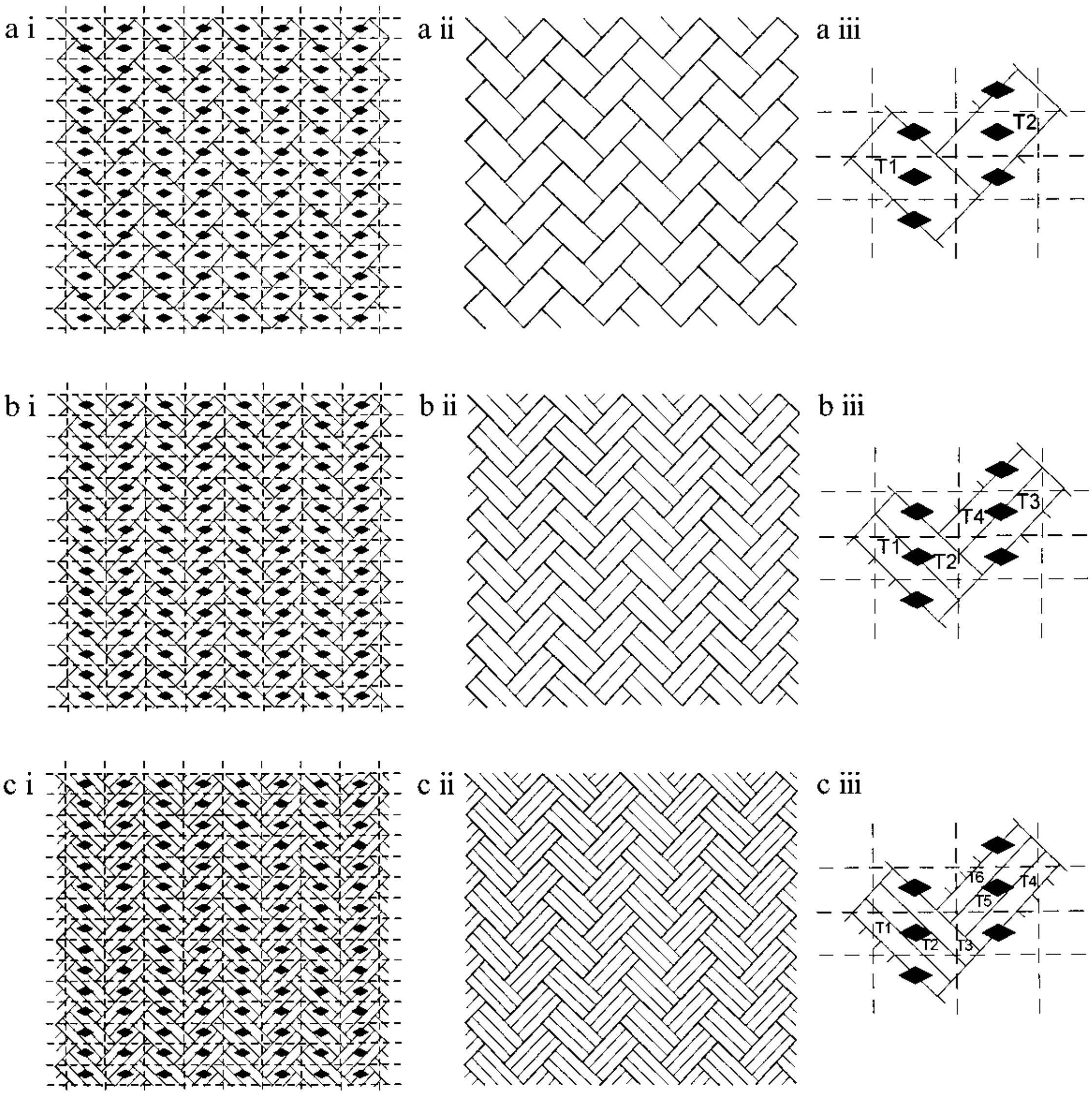


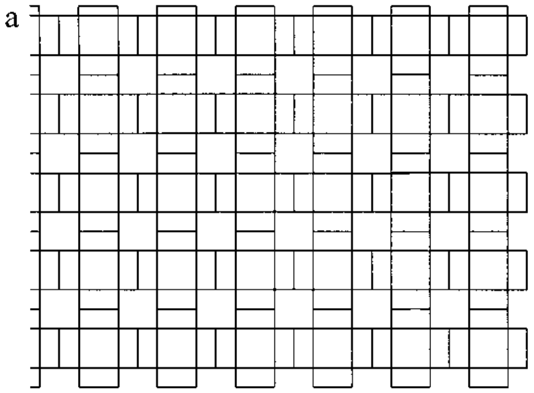
Figure 5.4 Analysis of the tilings in Fig. 5.3.

try, it may be analysed in a similar way to determine whether it is isohedral. Provided that any tile in the design can be mapped onto every other one then the tiling is isohedral.

5.2.2 *k-isohedral*

A tiling may be non-isohedral but if T is a tiling with precisely k transitivity classes then T is called k-isohedral.[4]

In the previous example, illustrated in Fig. 5.4(c), tiles T1, T3, T4 and T6 are in equivalent positions since each one can be mapped onto any of the others in this set of tiles. Thus, if one of these tiles was labelled A, and then copies of this letter were mapped to all equivalent positions in the tiling, then four out of six of all the tiles would be labelled A. If one of the remaining unlabelled tiles in the translation unit was labelled B and then mapped onto all the other possible equivalent positions, first in the translation unit and then in the remainder of the tiling, then all the other tiles would be labelled B. Hence, each of the tiles would have had a tile mapped onto itself (since none of them would be left unlabelled). Consequently, the tiles would have been divided into two sets: those labelled A and those labelled B, in other words there are two sets of tiles (two transitivity classes)



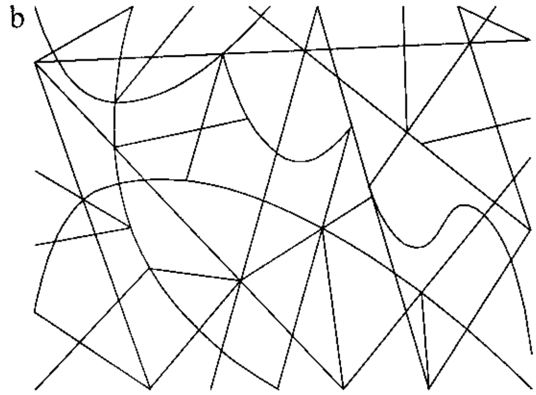


Figure 5.5 Examples of *k*-isohedral tilings.

in this tiling: those equivalent to the position of tile T1 and those equivalent to the position of tile T2. Therefore the tiling in Fig. 5.3(c) is two-isohedral.

Grünbaum and Shephard[4] state that generally if the tiles of a tiling are of *n* different shapes then there will be at least *n* transitivity classes. They go on to say that in the case of a tiling which is not symmetric, every tile is a transitivity class on its own. For example, if a tiling consists of, say, two different shaped tiles, there will be at least two transitivity classes, that is, it will be at least two-isohedral since, obviously, only tiles of the same size and shape could possibly be mapped onto each other. The tiling in Fig. 5.5(a) is composed of square and rectangular tiles. In this case, all the squares form one transitivity class and the rectangles form another; hence this tiling is two-isohedral.

If a tiling is not symmetric, the only symmetry it possesses is the identity symmetry (e.g. see the tiling in Fig. 5.5b). No tile can be mapped onto any other even if they are congruent. Thus each of the *n* tiles has to be put in a different set forming *n* different transitivity classes, that is an *n*-isohedral tiling.

However, since the classification system used in this chapter only deals with tilings which are isohedral and hence monohedral, situations where tilings have characteristics such as those illustrated in Fig. 5.5 do not arise.

5.2.3 Induced tile groups

An additional distinguishing feature of an isohedral tiling is its 'induced tile group'. This is analogous to the induced motif group of a discrete pattern type. For an isohedral tiling, the induced (tile) group or induced group is taken to be the finite symmetry group of the tile whose symmetries coincide with that of the design structure. For example, the isohedral tilings in Fig. 5.6(a), (b), (c) and (d) have induced groups *c*1, *d*2, *d*1 and *d*1, respectively.

In some cases, there may be more than one possibility for the positioning of reflection axes of an induced group. This can only occur when each tile has an even number of edges. Adopting the notation given by Grünbaum and Shephard, *d*1(l) and *d*1(s) for instance, are used to denote the different positions of reflection axes of induced group *d*1.[4] Here the '(l)' stands for 'long' and indicates that the reflection axis of the induced group passes through opposite vertices of a tile. '(s)' stands for 'short' and indicates that the reflection axis of the induced group passes through opposite edges (sides) of a tile (see Fig. 5.6(c) and (d)). (The basic features of tilings are discussed in detail in Section 5.3.) A similar analogy is used to differentiate between the positioning of reflection axes for tilings with induced groups *d*2 and *d*3. The only exception is tiling Dt(T)35 where the induced tile group *d*1(s) represents a reflection axis coinciding with the short bisector as opposed to the long bisector of each tile.

As mentioned in the introduction Section 5.1, isohedral tiling classification is

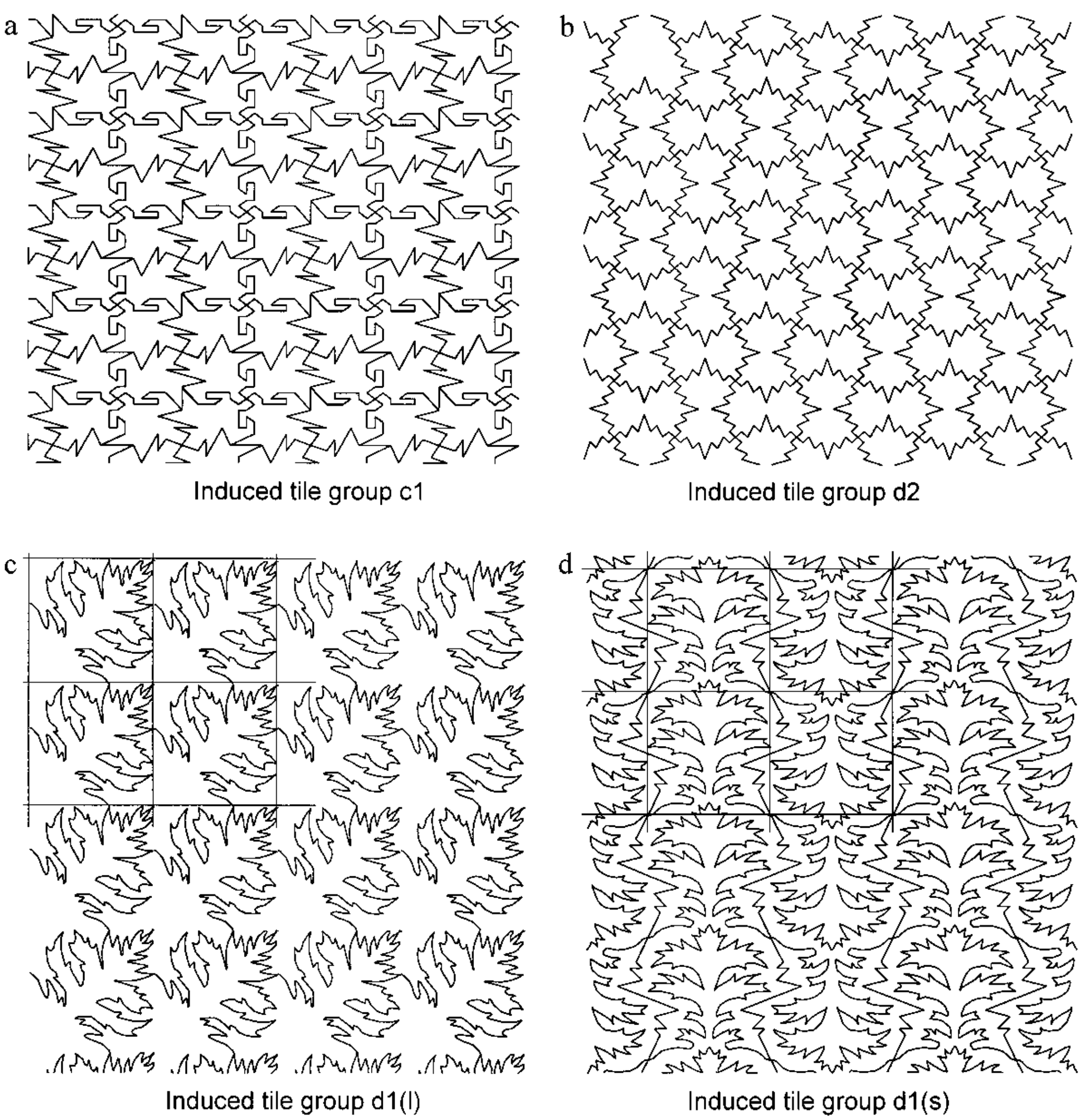


Figure 5.6 Examples of induced tile groups.

closely related to the classification of discrete patterns. As stated by Grünbaum and Shephard 'To every discrete periodic pattern *P* corresponds an isohedral tiling *D*(*P*) . . .'.[4] Here, 'periodic pattern' is analogous to a regularly repeating ditranslational pattern and the corresponding tiling, '*D*(*P*)', is an isohedral 'Dirichlet' tiling. In a similar way, in subsequent discussions and illustrations in this chapter, this analogy has been applied and adapted to incorporate the analysis and classification of monotranslational and finite tilings.

5.3 Dirichlet tiling

Grünbaum and Shephard[4] formally define a Dirichlet tiling as follows:

> Let $F = \{F_i \mid (i \in I\}$ be any non-empty family of pair-wise disjoint sets in the plane; with each F_i we associate a tile $T(F_i)$ consisting of all the points P of the plane for which the distance from P to F_i is less than or equal to the distance from P to each F_j with $j \neq i$. Then $\{T(F_i) \mid i \in I\}$ is a tiling called the *Dirichlet tiling associated with F*, which we denote by *D*(*F*).

Alternatively, the theory of Dirichlet domains is explained by Kappraff with reference to schools and the districts to which they are allocated. He explains this by

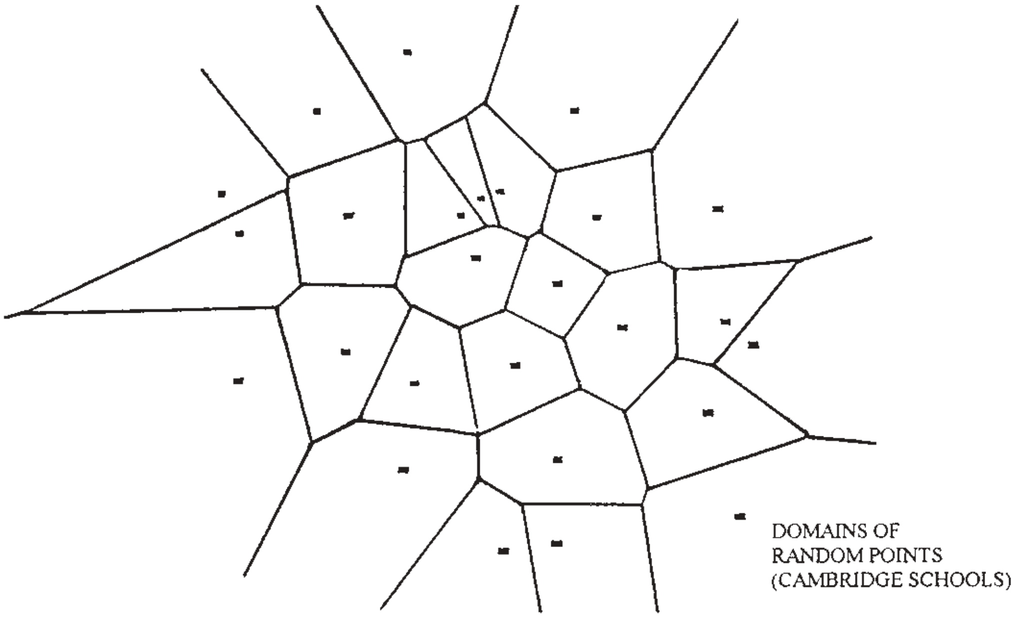


Figure 5.7 An example of a Dirichlet domain. Source: derived from Kappraff J, *Connections: The Geometric Bridge Between Art and Science*, New York, McGraw-Hill Inc., 1991, with permission.

saying that each point of a school district is nearer to the school in that district than to any other school (see Fig. 5.7).[5] In this context, each school district represents a tile and each school represents a motif. In connection with isohedral tilings, when such a tiling is placed over a discrete pattern, if every point within a tile is closer to the motif contained within it than any other motif in the pattern, then the tiling is a Dirichlet tiling for that pattern type. Notice that in the example in Fig. 5.7 the extremities of the outside districts are unbounded. Similarly for finite and monotranslational pattern types the associated Dirichlet tilings would, strictly speaking, be unbounded. However, in this book adaptations will be made to form more practical solutions by insisting on bounded tiles for these classes of tilings.

Figure 5.8 shows some examples of discrete patterns, their enclosure within Dirichlet tilings and the resulting associated isohedral tilings.

An isohedral Dirichlet tiling achieves a sense of 'fitting' with the discrete pattern enclosed within it. This does not necessarily imply that both the tiling and pattern have the same symmetry group. However, if they do, the Dirichlet tiling may still not be unique. For example, the discrete pattern in Fig. 5.9(a) may be associated with both the isohedral tilings in Fig. 5.9(b) and (c) by the Dirichlet relationship. Yet, these tilings appear, conceptually, to be very different despite the pattern and both tilings having the same symmetry group and induced group. This is due to the interlocking and joining relationship of adjacent tiles, the structure of which is described by the topology of the tiling.

Before introducing elements of topology, the following descriptions and diagrams (in Fig. 5.10) illustrate the main concepts and terminology used to define the basic features of a tiling.

- Corners of **T**: A, B, D, F, G, H, I, J, K and L in tiling *A* and A, B, C and D in tiling *B*. A corner is a point at which two lines join at an angle (≠180°).
- Vertices of **T**: A, C, E, G, I and K in tiling *A* and A, B, C and D in tiling *B*. A vertex is a point at which at least three line segments join together.
- Line segments of **T**: AB, BD, DF, FG, GH, HI, IJ, JK, KL and LA in tiling *A* and AB, BC, CD and DA in tiling *B*. The line segments correspond to the sides of a tile **T**.
- Edges of **T**: AC, CE, EG, GI, IK and KA and all equivalent lengths in tiling

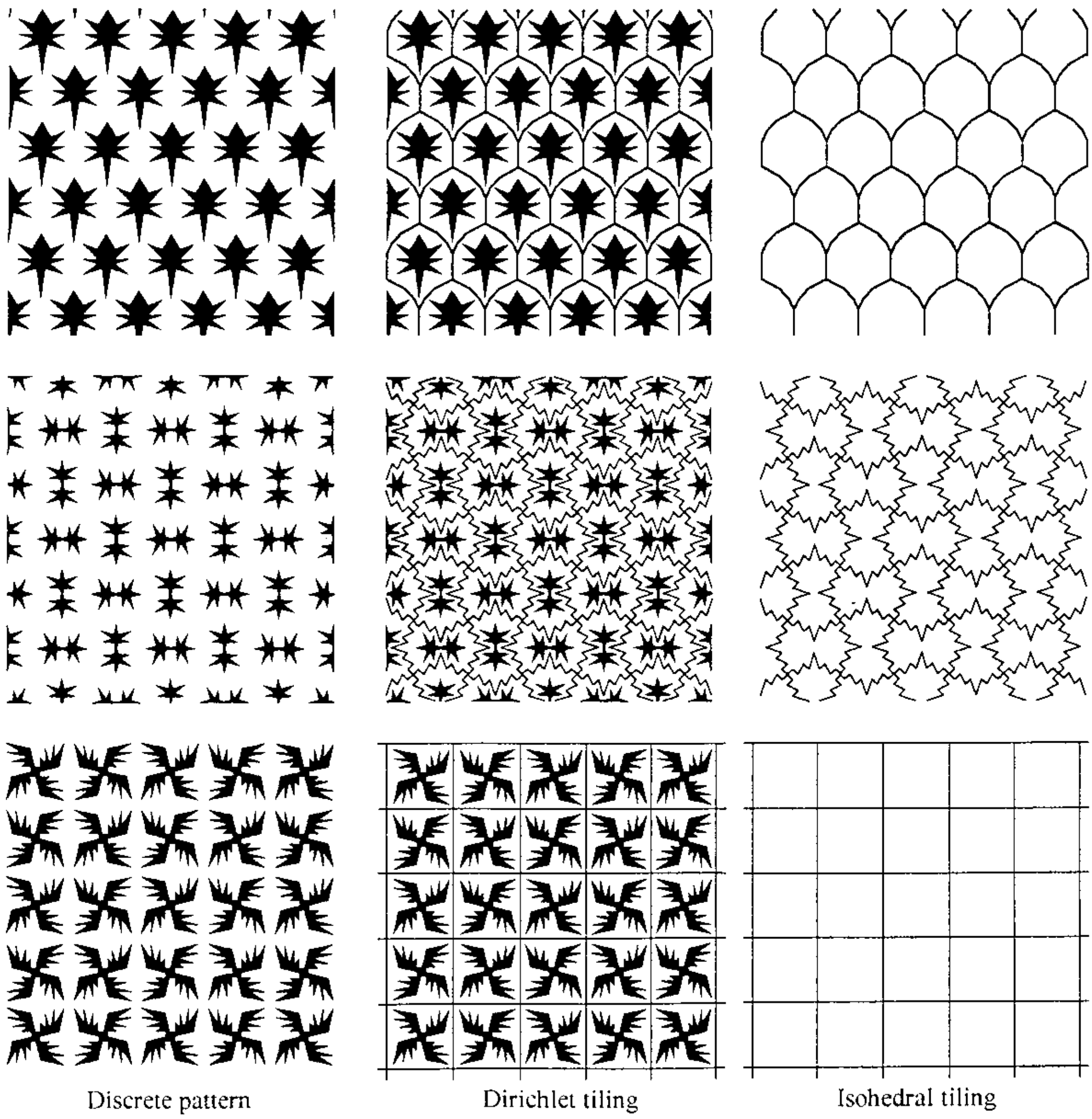


Figure 5.8 Further examples illustrating the Dirichlet relationship.

A and AB, BC, CD and DA and all equivalent positions in tiling *B*. The edges are the line segments or combination of line segments between each vertex.

- Valency: The valency of each vertex of tiling *A* is three and of each vertex of tiling *B* is four. The valency of a vertex is the number of edges that meet at that point.
- Adjacents of **T**: In tiling *A*, T_1, T_2, T_3, T_4, T_5 and T_6 are adjacents of tile **T** and in tiling *B*, T_2, T_4, T_6 and T_8 are all adjacents of tile **T**. Two tiles must have an edge in common to be adjacent to each other.
- Neighbours of **T**: In tiling *A*, T_1, T_2, T_3, T_4, T_5 and T_6 are neighbours of tile **T** and in tiling *B*, T_1, T_2, T_3, T_4, T_5, T_6, T_7 and T_8 are all neighbours of tile **T**. Two tiles are neighbours if they have at least one point in common.

5.4 Topology of tilings

The two tilings in Fig. 5.9 illustrate that the derivation of Dirichlet tilings from a discrete pattern type does not necessarily give a one-to-one correspondence and hence, does not provide sufficient information for the classification of isohedral tilings. This is observed by Grünbaum and Shephard who state that the classification of isohedral tilings by pattern type is deficient in that it takes no account of

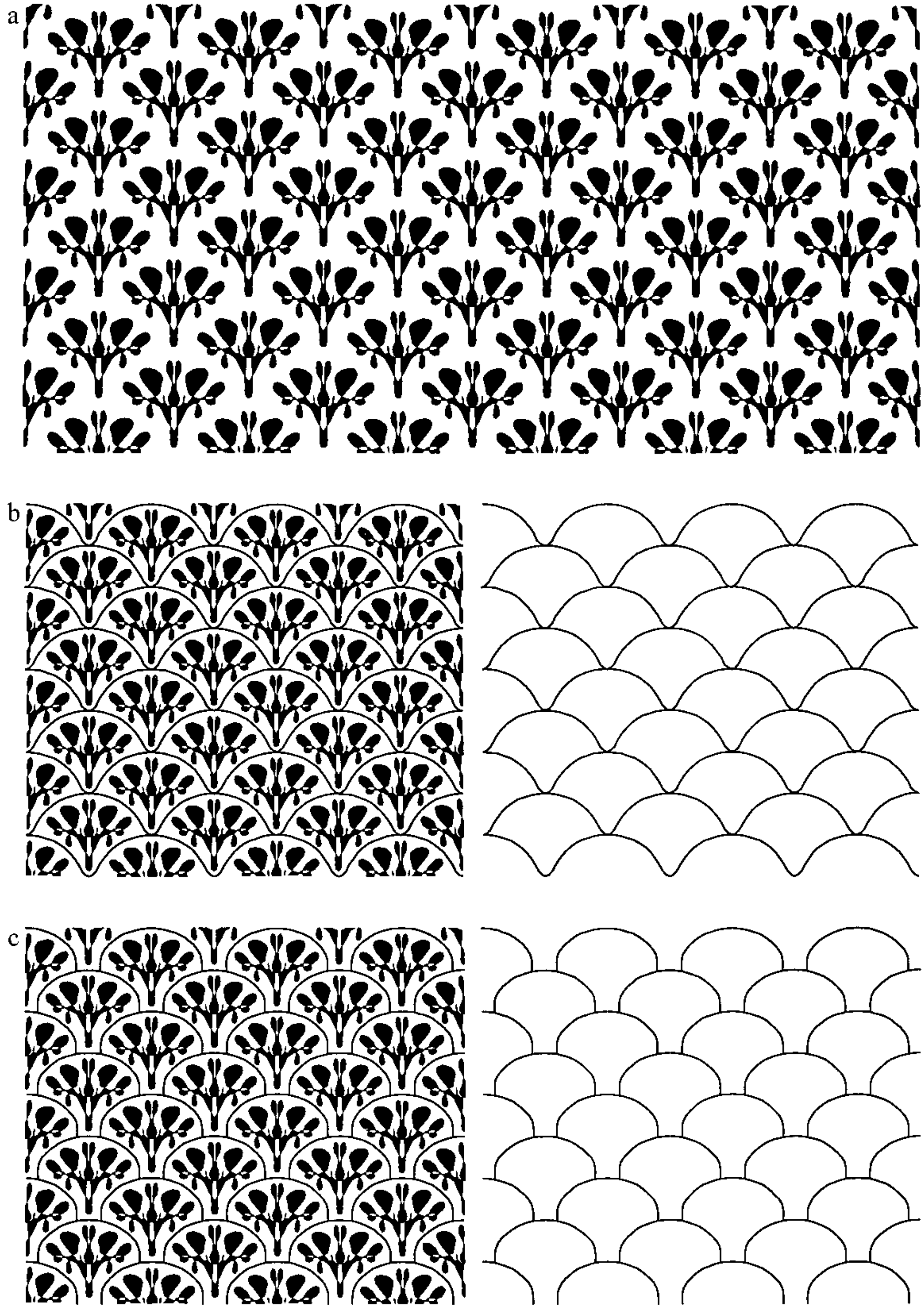
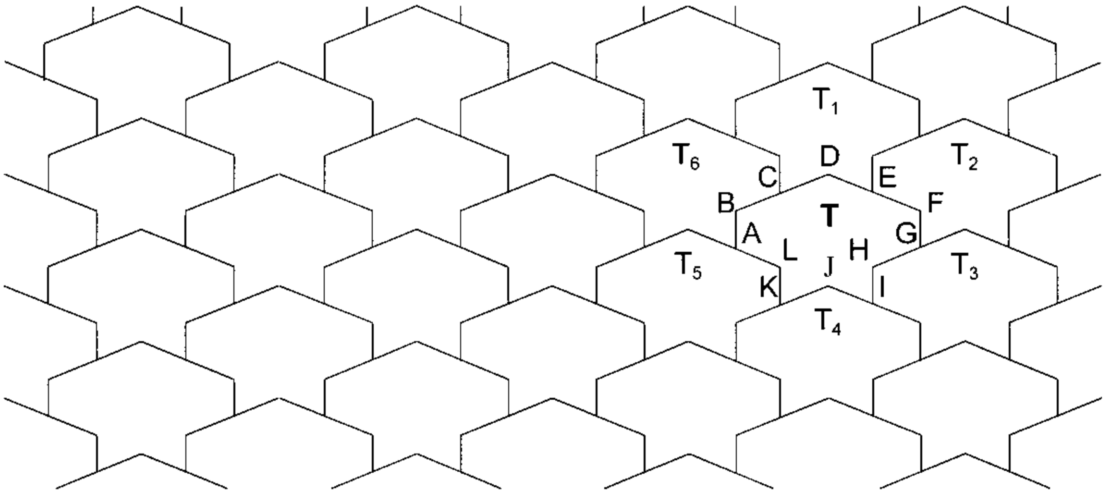


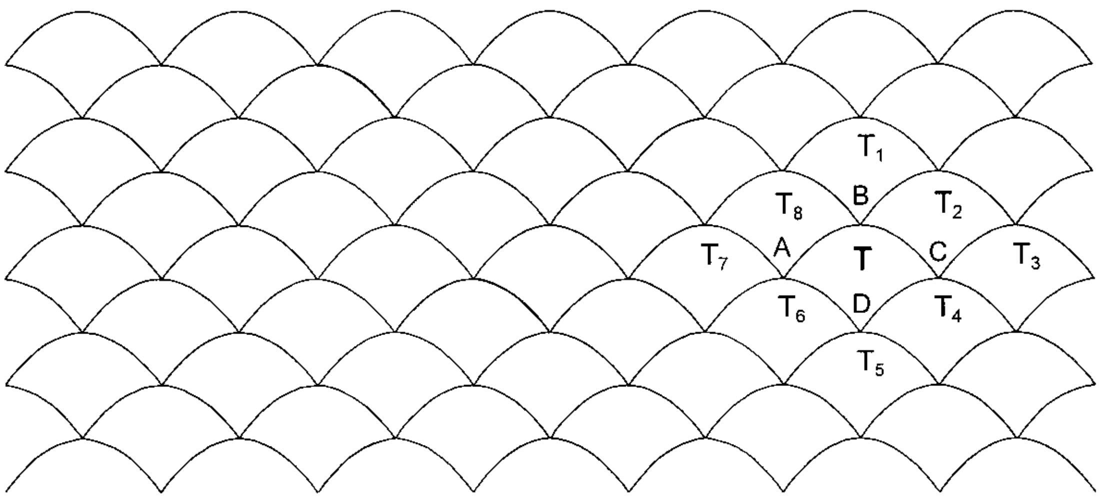
Figure 5.9 Examples of two Dirichlet tilings associated with the same pattern type.

one of the most important features of the tiling, namely its topological type.[4] In other words, since in most cases (as in Fig. 5.9) more than one Dirichlet tiling may be associated with each ditranslational discrete pattern type, additional features involved in their topology, such as their vertices and valencies, must be taken into consideration to enable one form of Dirichlet tiling to be distinguished from another. Alternatively, Bergamini described topology as a special kind of geome-



Tiling *A*

Corners of **T**:	A, B, D, F, G, H, I, J, K, L
Vertices of **T**:	A, C, E, G, I, K
Line Segments of **T**:	AB, BD, DF, FG, GH, HI, IJ, JK, KL, LA
Edges of **T**:	AC, CE, EG, GI, IK, KA
Valencies of vertices A, C, E, G, I, K:	3, 3, 3, 3, 3, 3
Adjacents of **T**:	T_1, T_2, T_3, T_4, T_5, T_6
Neighbours of **T**:	T_1, T_2, T_3, T_4, T_5, T_6



Tiling *B*

Corners of **T**:	A, B, C, D
Vertices of **T**:	A, B, C, D
Line Segments of **T**:	AB, BC, CD, DA
Edges of **T**:	AB, BC, CD, DA
Valencies of vertices A, B, C and D:	4, 4, 4, 4
Adjacents of **T**:	T_2, T_4, T_6, T_8
Neighbours of **T**:	T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8

Figure 5.10 Examples illustrating the basic features of a tiling.

try concerned with the ways in which surfaces can be twisted, bent, pulled stretched or otherwise deformed from one shape into another.[6]

5.4.1 Topological equivalence

To deduce whether Dirichlet tilings, associated with a particular pattern type, are topologically equivalent (i.e. are classed as the same 'topological type') involves a special kind of transformation or mapping referred to as a 'homeomorphism'. Grünbaum and Shephard describe two tilings to be of the same *topological type* (or to be *topologically equivalent*) if there is a homeomorphism which maps one onto the other. They go on to define a homeomorphism as follows[4]:

> A mapping Φ: $E^2 \to E^2$ of the plane onto itself is called a *homeomorphism* or *topological transformation* if it is one-to-one and bicontinuous. *One-to-one* (or *bijective*) means that for any two points P, Q in the plane, $\Phi(P) = \Phi(Q)$ if and only if $P = Q$; this implies that there exists an inverse transformation Φ^{-1} such that $\Phi^{-1}(R) = P$ if and only if $\Phi(P) = R$. . . . *Bicontinuity* means that both Φ and Φ^{-1} are continuous.

In a context more suitable for surface designers, topologically equivalent tilings may be thought of more simply as follows: if one tiling can be transformed into another by applying a special kind of mapping or transformation called a homeomorphism, which squashes, stretches or deforms tiles of the first tiling without removing or adding any edges, and hence tiles, then the two tilings are topologically equivalent.

For each of the examples (a) to (d), in Fig. 5.11, tiling *A* is topologically equivalent to tiling *B*. In the first example it is easy to see how a form of horizontal stretch produces tiling *B*. The homeomorphic transformation in the second, third and fourth examples is more difficult to visualise. The metamorphosis, for each example, is given in Fig. 5.12. In the third example, some edges, initially composed of one line segment, have been transformed to those made of two. This does not alter the topology of the tiling since the number of edges, and hence tiles, has not increased or decreased.

Similarly, the six tilings in Fig. 5.13(a) to (f) are all topologically equivalent to each other despite their tiles edges being composed of one, one, two, three, three and four line segment(s), respectively. The number of edges remains the same in each example although their differences in appearance are quite distinct.

The presence of topological equivalence is sometimes difficult to visualise through a homeomorphic transformation. An alternative method of establishing whether two tilings are topologically equivalent is to test for 'combinatorial equivalence' because, as stated by Grünbaum and Shephard, for normal tilings the concepts of topological and combinatorial equivalence coincide.[4]

5.4.2 Combinatorial equivalence

Two tilings are combinatorially equivalent if the following condition holds[4]:

> Let $\varepsilon(T)$denote the set of all *elements* of a tiling T, that is, the set whose members are the vertices, edges and tiles of T. A map Φ of $\varepsilon(T_1)$ onto $\varepsilon(T_2)$ is said to be inclusion-preserving if, whenever e_1, $e_2 \in \varepsilon(T_1)$, then $\Phi(e_1)$ includes $\Phi(e_2)$ if and only if e_1 includes e_2. If there exists an inclusion-preserving map between T_1 and T_2, then T_1 and T_2 are said to be combinatorially isomorphic or combinatorially equivalent. If V is any n-valent vertex of T_1, then $\Phi(V)$ will be an n-valent vertex of the combinatorially equivalent tiling T_2. Similarly, if a tile T of T_1 has n adjacents, then so does the corresponding tile $\Phi(T)$ of T_2.

In other words, given two tilings A and B, if each tile in A can be mapped onto a tile in B such that, for example, a tile a_1 in A corresponds to a tile b_1 in B, and the number of edges, vertices and valencies of a_1 are the same as those of b_1 and they have the same number of adjacents, then they are combinatorially equivalent. These conditions must apply to every single tile in A and their corresponding tiles in B. The relationship between the tiles in A and the tiles in B is one-to-one, that is

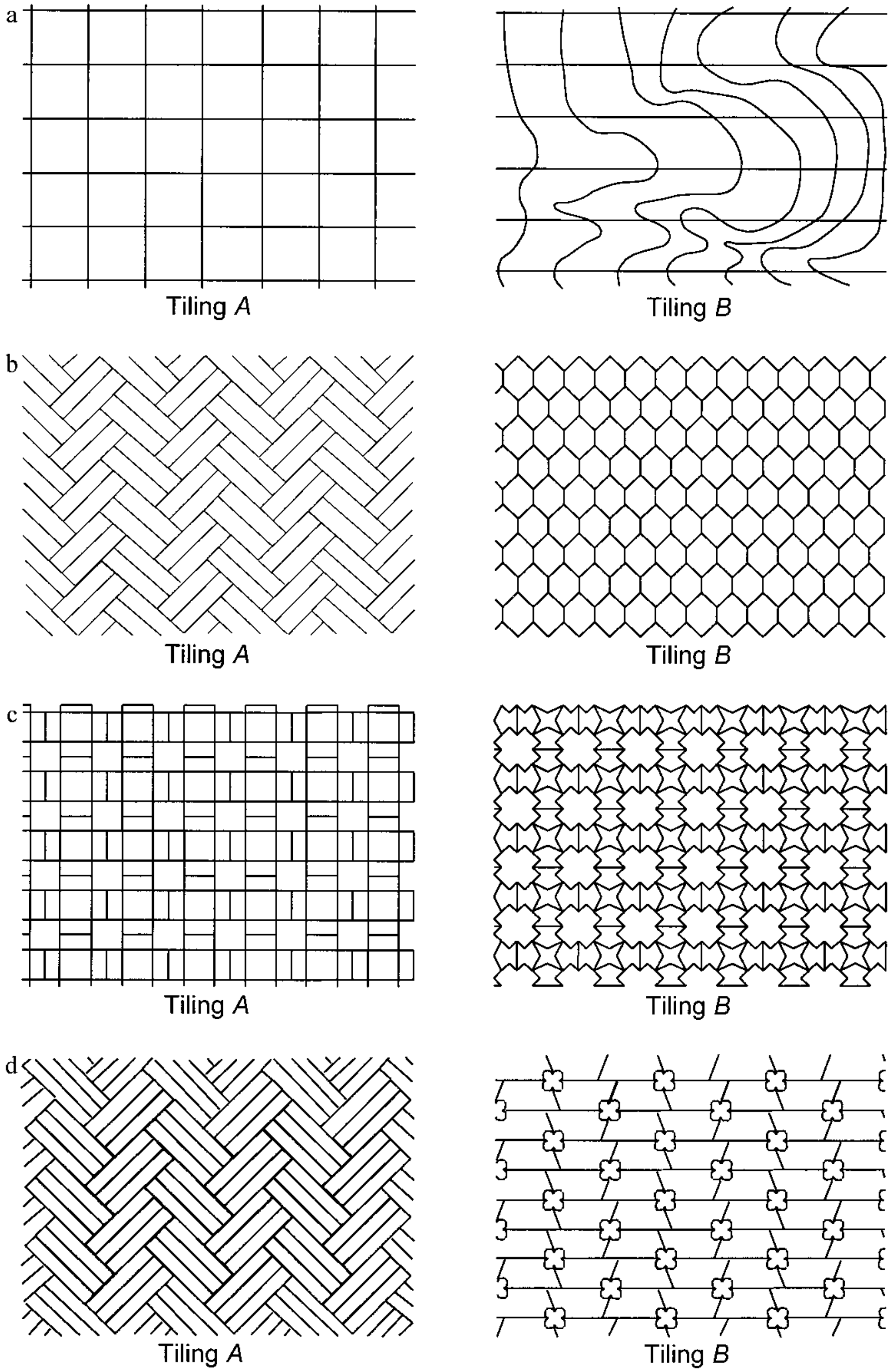


Figure 5.11 Examples of topologically equivalent tilings.

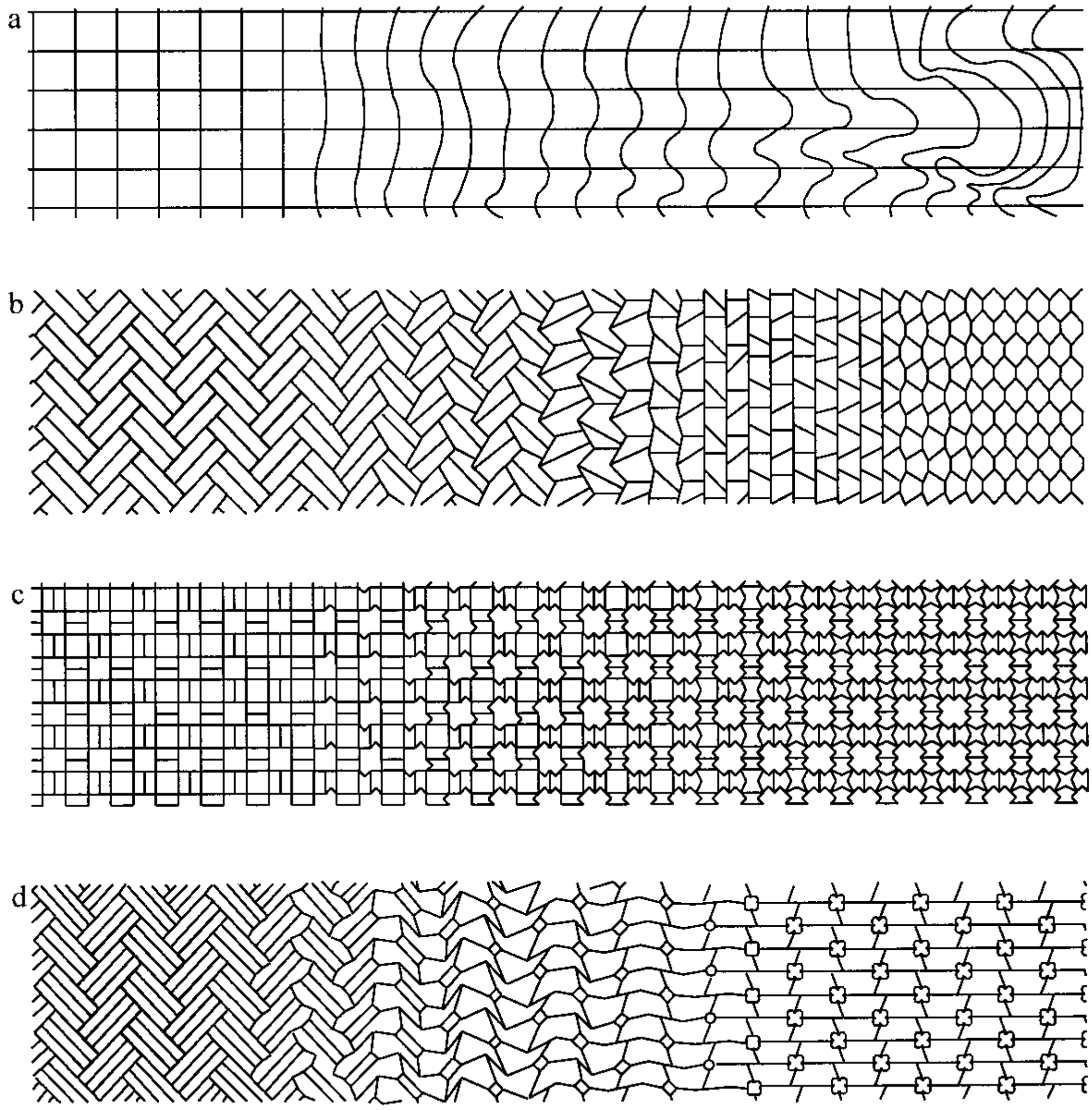


Figure 5.12 Metamorphosis of topologically equivalent tilings.

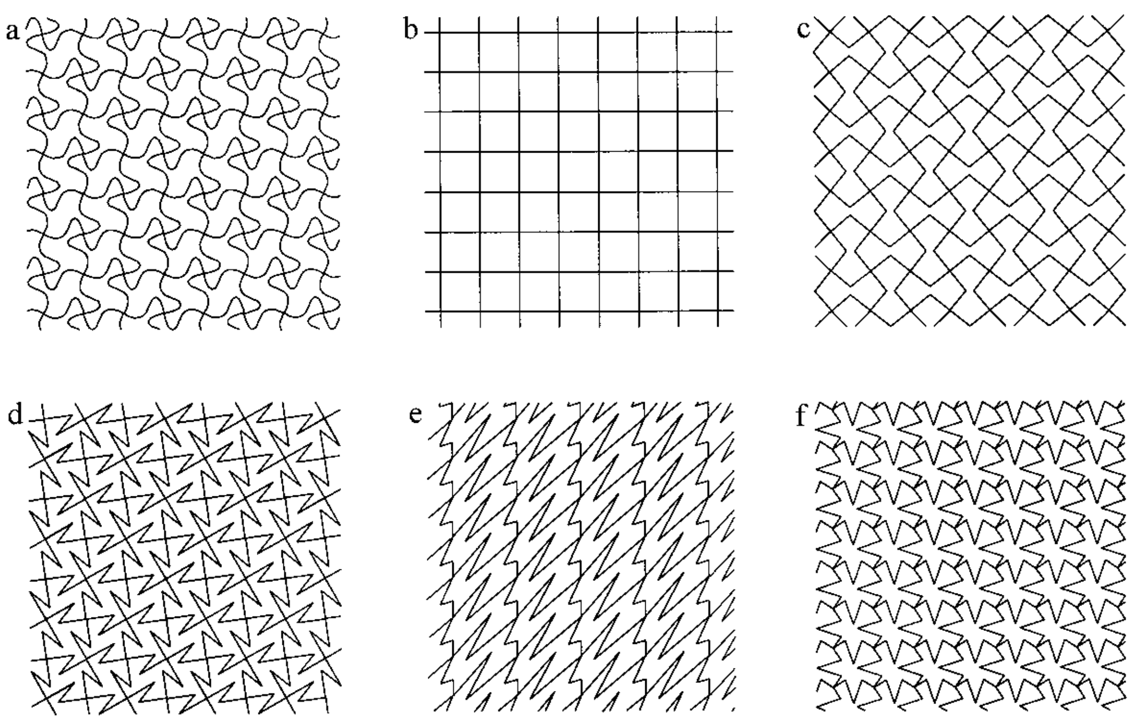


Figure 5.13 Further illustrations of topologically equivalent tilings.

one tile in A is mapped onto only one tile in B and that same tile in B may only be mapped back onto the same particular tile in A.

The conditions of combinatorial equivalence may be applied to the tilings in Fig. 5.11 to confirm their topological equivalence. Examples illustrating their combinatorial equivalence are given in Fig. 5.14.

For example, in Fig. 5.14(a(i)) each tile has four edges, four vertices (each with valency four) and four adjacents. Although the shapes of the tiles have been altered for the tiling in Fig. 5.14(a(ii)), each still retains these characteristics. For example, tile a_{11} may be mapped onto b_{11}, a_{12} onto b_{12}, a_{13} onto b_{13}, a_{14} onto b_{14} . . . a_{1n} onto b_{1n}. Similarly a_{21} may be mapped onto b_{21}, a_{22} onto b_{22}, a_{23} onto b_{23} . . . a_{2n} onto b_{2n} . . . a_{nn} onto b_{nn} and so on until each of the tiles of tiling (a(i)) has been mapped onto one in tiling (a(ii)). Throughout the mapping there has been no alteration of the tilings' elements or number of adjacents; therefore they are combinatorially and hence topologically equivalent.

The tilings in Fig. 5.11(c) are not monohedral. However, since both designs are periodic, that is, regularly repeating, the combinatorial condition may be assessed for a translation unit of each tiling. A translation unit for tiling A is composed of six tiles (see Fig. 5.14c(i)). Four tiles each has four edges and four vertices with valencies 3, 3, 4 and 4 (ordered by following the boundary of the tile starting with the lowest numbered). Each of these tiles also has four adjacents. One of the square tiles in the translation unit has eight edges and vertices with valencies 3, 4, 3, 4, 3, 4, 3 and 4 and eight adjacents. The other square tile has vertices with valencies 4, 4, 4 and 4 and four adjacents. These conditions coincide with the characteristics of the tiles in the translation unit of tiling B (see Fig. 5.14c(ii)). Therefore these two tilings are combinatorially and hence topologically equivalent.

A translation unit of tiling A, in Fig. 5.11(d), is composed of six tiles (see Fig. 5.14 d(i)). Two tiles each has four edges and four vertices each with valencies 3, 3, 3, 3 and four adjacents; and four tiles each of which has seven edges and seven vertices each with valencies 3, 3, 3, 3, 3, 3, 3 and seven adjacents. These characteristics coincide with those of a translation unit of tiling B in Fig. 5.11(d) (see Fig. 5.14d(ii)) therefore these tilings A and B are combinatorially and hence topologically equivalent.

Both the tilings in Fig. 5.11(b) are isohedral (unlike the other three examples). Therefore, instead of analysing the characteristics of a translation unit, it is only necessary to look at one tile of each tiling (since each tile is equivalent to any other). Figure 5.14(b) shows that each of the tiles in tilings A and B, in Fig. 5.11(b), has the same number of edges and vertices with the same valencies. Also, they each have the same number of adjacents. Therefore the two corresponding tilings are combinatorially and hence topologically equivalent.

The principle of combinatorial equivalence is evident in a number of metamorphic drawings by M.C. Escher. For example in his XXXIV Emblata, Padlock design, a chequered black and white parallelogram tiling is transformed into tiles shaped as bird-like images. Throughout the metamorphic transformation the elements, valencies and adjacents of each tile remain the same from beginning to end.[7]

5.4.2.1 Notation

The notation used to classify tilings by topological type involves vertex valencies. Since this chapter is concerned with the classification of isohedral tilings, the classification by topological type will also be limited to this group of tilings. Consequently, as one tile, in an isohedral tiling, is equivalent to each of the others, it is only necessary to look at the characteristics of a single tile. Figure 5.15 gives some examples to show how the topological type of an isohedral tiling is derived.

A single tile, in the first example (Fig. 5.15(a)), has three vertices with valencies 4, 8 and 8. To find the topological type, the smallest vertex valency is noted, and then the valencies of the other vertices, whilst following the boundary of a tile in

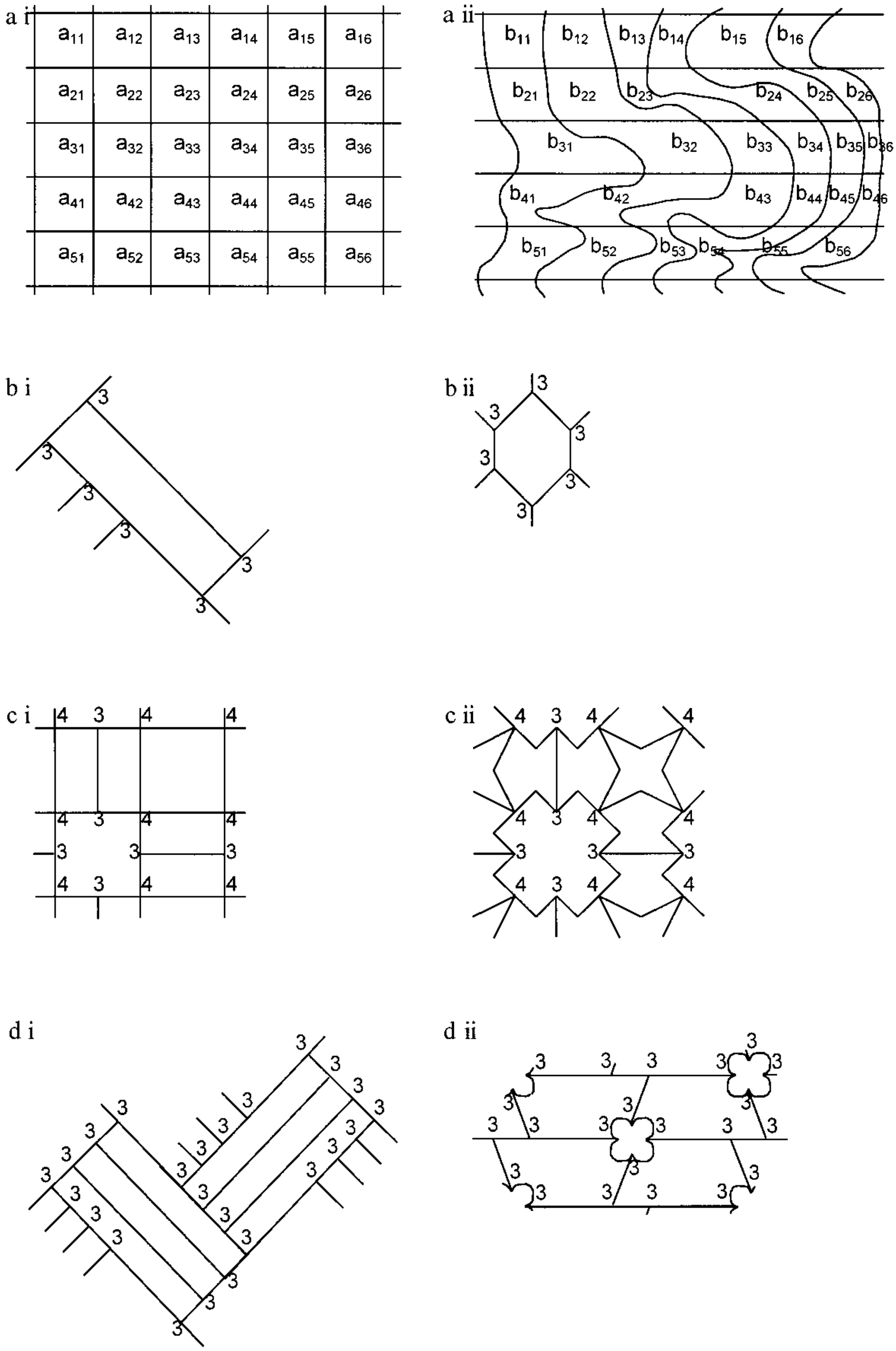


Figure 5.14 Examples of combinatorially equivalent tilings.

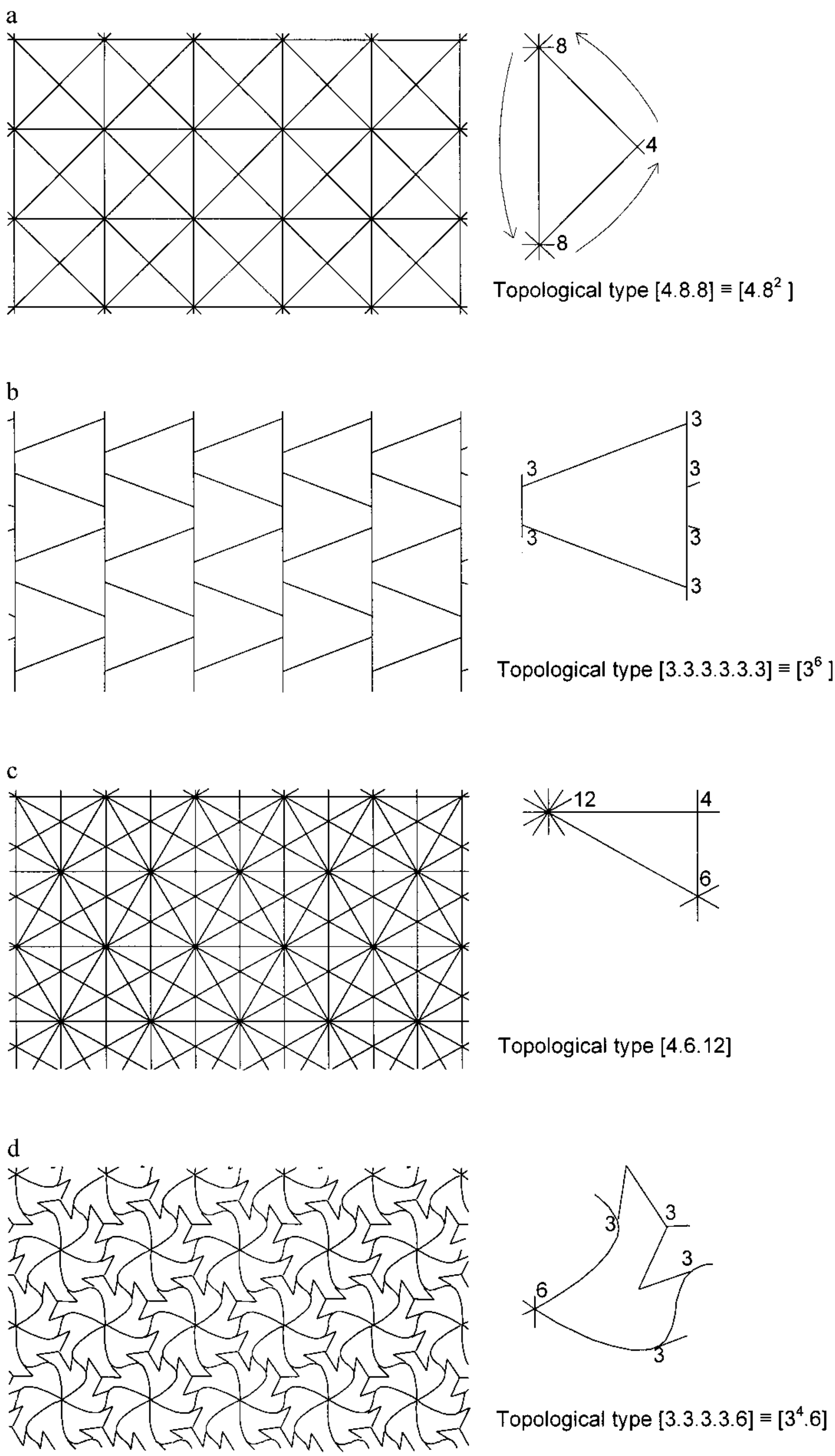


Figure 5.15 Derivation of the topological type of an isohedral tiling.

one direction. The direction in which to follow the boundary of the tile is determined by numerical order, that is, by the next lowest valency of an adjacent vertex. The sides of the tile are then followed by continuing in that same direction until one circuit of the boundary is completed. Consequently, for the tiling in Fig. 5.15(a), this gives topological type [4. 8. 8] which is reduced to [4.8^2] in shorthand. The topological types of the tilings in Fig. 5.15(b), (c) and (d) are derived in the same way.

5.5 Incidence symbols

Amalgamating the topological classification system with an isohedral tiling's associated discrete pattern type still does not result in differentiation between all possible classes of isohedral tiling. Two tilings may be classed in the same symmetry group, be of the same topological type and even be associated with the same discrete pattern type but still appear to be quite different. This is due to the relationship between a tile and its adjacents.

For example, the two tilings illustrated in Fig. 5.16(a(i)) and (b(i)) (which are schematically illustrated in Fig. 5.16a(ii) and b(ii), respectively) are both in the same symmetry group, *pg*, are of the same topological type, 4^4, and are associated with the same discrete pattern type, Dt(P)2, but are classed as different isohedral tiling types.

In Fig. 5.16(a(ii)), a tile T is mapped onto its adjacents t1, t2, t3 and t4 by either glide–reflection or translation, whereas in Fig. 5.16(b(ii)), a tile P is mapped onto its adjacents, p1, p2, p3 and p4 by glide–reflectional symmetry only. This difference in the relationship a tile has with its adjacents, may be analysed and incorporated into a term referred to as the tiling's 'incidence symbol'. The incidence symbol, together with the topological type, distinguishes one isohedral tiling type from another. Two tilings may be topologically equivalent but have different incidence symbols (for example, the tilings in Fig. 5.16) or conversely they may have the same incidence symbol and be unlike topologically. Either way, they would differ under the classification by isohedral tiling type.

Grünbaum and Shephard stated that two tilings are the same isohedral tiling type if and only if they are of the same topological type and their incidence symbols [L;A] differ trivially.[2] Here the two letters, L and A, in the incidence symbol, [L;A], represent the tile symbol and the adjacency symbol, respectively.

5.5.1 Tile symbol

The tile symbol, L, consists of a sequence of letters with superscripts which labels the edges of each tile in a particular order. Figure 5.17 gives some examples to show how the tile symbol is derived.

An edge of a single tile, in Fig. 5.17(a), is allocated the letter 'a' and is orientated by adding an arrow to it to indicate the direction which will be followed around the boundary of the tile (Fig. 5.17a(i)). (The direction of the first labelled edge does not matter.) This letter and arrow are then mapped onto equivalent 'inside' edges of the tile by using an isometry of the tiling (which in this case is reflectional symmetry) (see Fig. 5.17a(ii)). The letter 'b' and an arrow is then assigned to an edge following on from an edge labelled 'a' (Fig. 5.17a(iii)) which, again, is mapped onto any other equivalent positions inside the tile. The tile symbol, L, is determined by following the boundary of the tile in one direction and noting down the letters in order and adding a superscript '+' if the arrows point in the same direction as that which is being followed or '–' if the arrow points in the opposite direction to that being followed. This gives tile symbol $a^+ b^+ b^- a^-$ for the tiling in Fig. 5.17(a). (It is conventional and simplest to begin by labelling consecutive edges alphabetically and to start with the letter 'a' when deriving the tile symbol.)

The same edge-labelling procedure, for the second example, gives the result in Fig. 5.17(b). As the top and bottom edges of the tile can be mapped onto themselves by reflectional symmetry of the tiling, these two are assigned double-

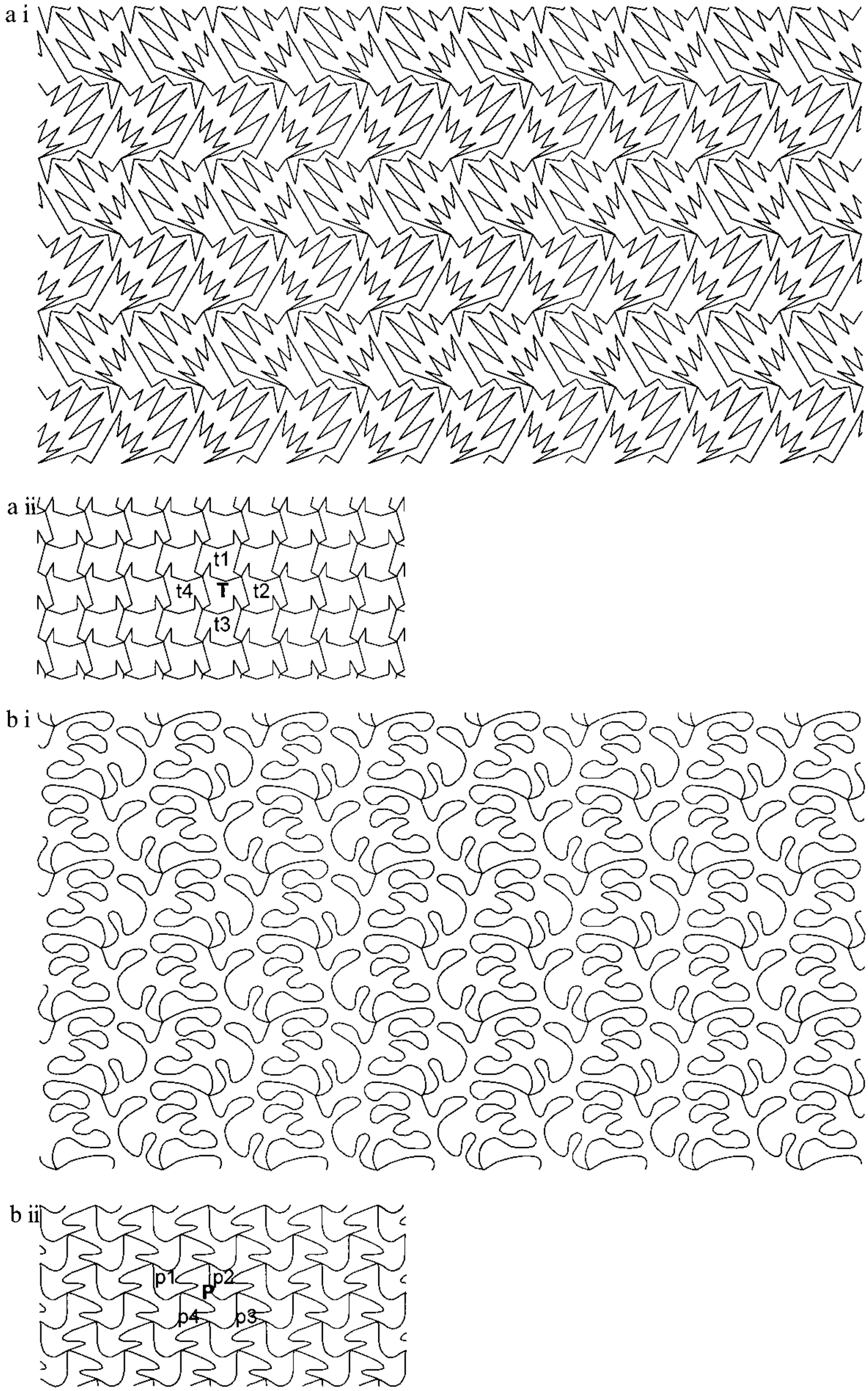


Figure 5.16 Examples of different isohedral tiling types with the same topological type, symmetry group and induced tile group.

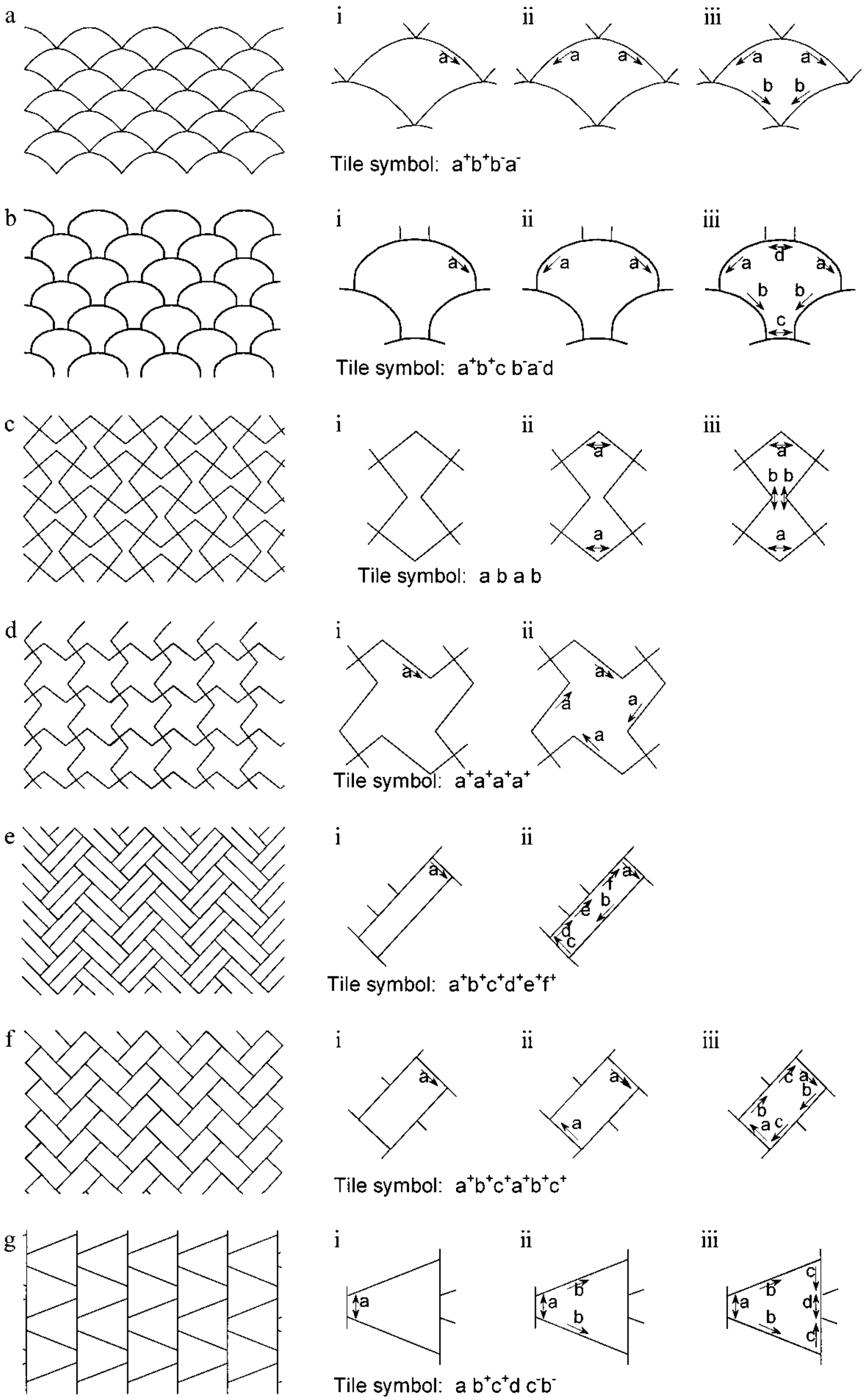


Figure 5.17 Derivation of the tile symbol.

headed arrows. In these cases, there is no superscript added to the letters in the tile symbol which, for this example, is written $a^+ b^+ c b^- a^- d$.

The tile symbol, for the example in Fig. 5.17(c), is derived in the same way as for the tiling in Fig. 5.17(d). These two tilings are of the same topological type but have different tile symbols: a b a b and $a^+ a^+ a^+ a^+$ respectively. In Fig. 5.17(d), each edge can be mapped onto every other by four-fold rotational symmetry of the tiling which is why they are all assigned the same letter.

In Fig. 5.17(e), no edge can be mapped onto any other so therefore each edge is allocated a different letter.

In Fig. 5.17(f), two-fold rotational symmetry allows some edges to be allocated the same letter resulting in the tile symbol $a^+ b^+ c^+ a^+ b^+ c^+$. By similar analysis, but involving reflectional symmetry, the tiling in Fig. 5.17(g) is assigned the tile symbol $a b^+ c^+ d c^- b^-$.

The tile symbol of a tiling contributes to only half of the incidence symbol. The remaining component is referred to as the 'adjacency symbol'.

5.5.2 Adjacency symbol

The adjacency symbol, A, is also a sequence of letters with superscripts. It relates to the letters contained within the tile symbol. It is derived by first mapping the labelled inside edges of a tile, used to determine the tile symbol, onto every other tile in the tiling by using symmetries of the tiling structure. This results in each edge being allocated two letters and two arrows. The examples in Fig. 5.18 illustrate the results of this operation with respect to one tile in each of the tilings in Fig. 5.17(a) to (g) and the derivation of the resulting adjacency and incidence symbols.

Figure 5.18(b(ii)) shows a tile which has had its edges labelled with letters and arrows by the procedure described above. The adjacency symbol is determined by beginning at the same point as that for the tile symbol and continuing in the same direction along the same edges whilst noting the edge labels of adjacent tiles in order. If the parallel arrow of an adjacent tile points in the same direction, a negative superscript is added to the adjacent letter. If the arrow points in the opposite direction, a positive superscript is attached. However, in the adjacency symbol (unlike the tile symbol) if a letter has been noted down once, and whilst following the boundary of the tile the same letter appears again, it is not repeated a second time, that is, if the tile symbol consists of four distinct letters (ignoring their superscripts and repetitions) then the adjacency symbol will consist of four letters only with their appropriate superscripts. Thus, following this system of letter allocation, the tiling in Fig. 5.18(b), with tile symbol $a^+ b^+ c b^- a^- d$, is given the adjacency symbol $b^- a^- d c$ implying that all tile edges labelled 'a' abut edges labelled 'b' with equally orientated arrows; and all edges labelled 'c' abut edges labelled 'd' neither of which is orientated, that is, they have double-headed arrows. The combination of the tile symbol, L, and adjacency symbol, A, gives the incidence symbol, [L;A], of the tiling which, for the example in Fig. 5.18(b) is $[a^+ b^+ c b^- a^- d; b^- a^- d c]$.

The adjacency symbol, for the tiling in Fig. 5.18(a), is found by following the boundary of a tile from the same starting point that was used to derive the tile symbol and in the same direction as described above. The initial letter of the adjacency symbol is 'b' as this lies next to the first edge in the tile symbol. The superscript is negative as the arrows are equally orientated. The next letter is 'a' as this lies next to the second letter in the tile symbol. Again, the superscript is negative. The third letter in the adjacency symbol would be 'a' but since this letter has been used previously in the adjacency symbol, it is not repeated a second time. Thus, the tile symbol $a^+ b^+ b^- a^-$ leads to the adjacency symbol $b^- a^-$ which gives the incidence symbol $[a^+ b^+ b^- a^-; b^- a^-]$

The same principle has been used to determine the adjacency and incidence symbols for the remaining tilings in Fig. 5.18(c) to (g).

Of course, the incidence symbol may vary according to how the letters and arrows were initially allocated to the tiling when first deriving the tile symbol. For

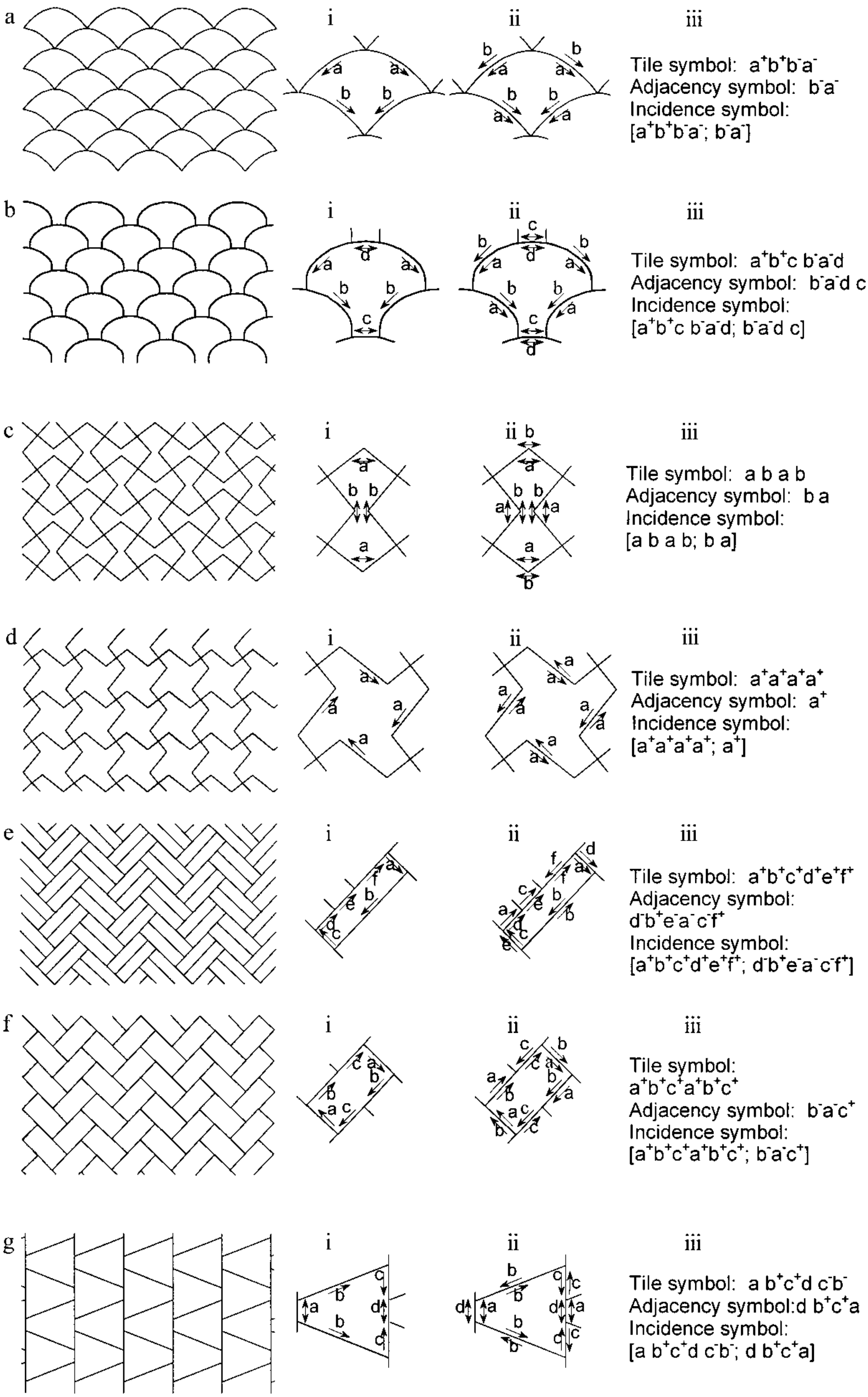
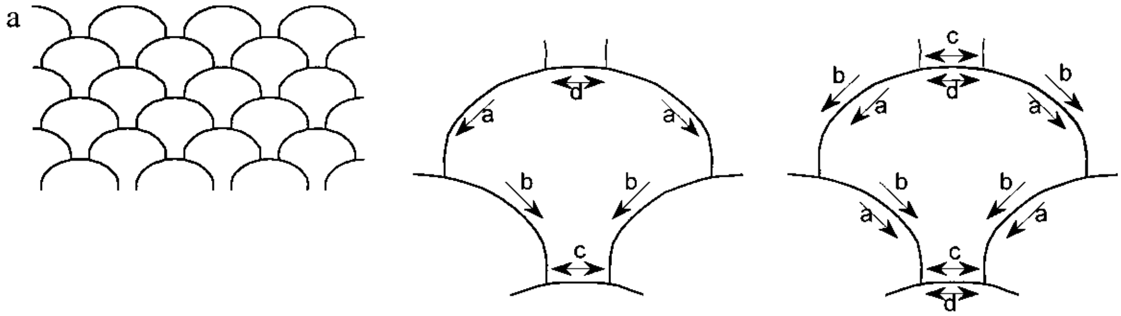
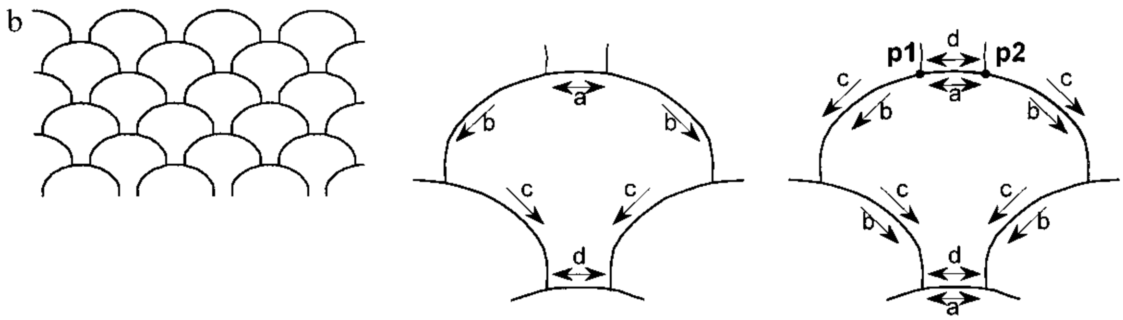


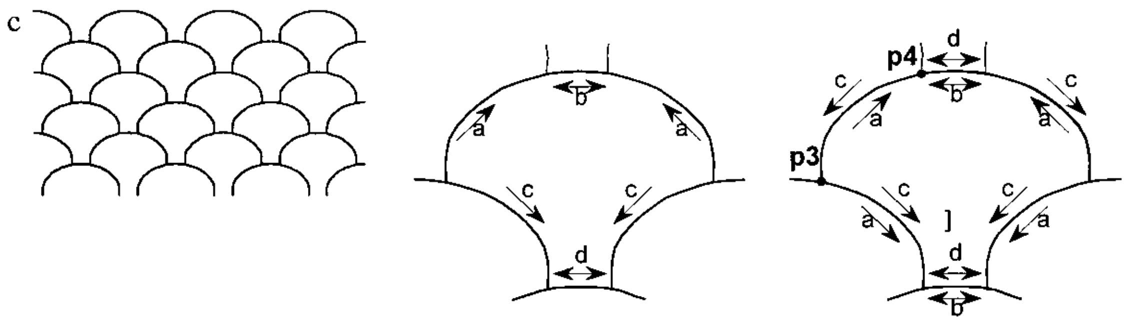
Figure 5.18 Derivation of the adjacency and incidence symbols.



Tile symbol: $a^+b^+c\ b^-a^-d$
Adjacency symbol: $b^-a^-d\ c$
Incidence symbol: $[a^+b^+c\ b^-a^-d;\ b^-a^-d\ c]$



Tile symbol: $a\ b^+c^+d\ c^-b^-$
Adjacency symbol: $d\ c^-b^-\ a$
Incidence symbol: $[a\ b^+c^+d\ c^-b^-;\ d\ c^-b^-\ a]$



Tile symbol: $a^+b\ a^-c^+d\ c^-$
Adjacency symbol: $c^+d\ a^+b$
Incidence symbol: $[a^+b\ a^-c^+d\ c^-;\ c^+d\ a^+b]$

Figure 5.19 Examples of isohedral tilings with equivalent incidence symbols.

example, referring to the tiling in Fig. 5.19(a), the tile symbol could have been derived from a different lettering system, shown in Fig. 5.19(b) or 5.19(c), to give the corresponding tile, adjacency and incidence symbols. These incidence symbols must be equivalent since they are taken from the same isohedral tiling. In essence, they differ 'trivially'.

The symbols contained within two different, but equivalent, incidence symbols may be made to coincide by the reallocation of letters to one of the tilings. After all, each letter represents a particular edge but its name is not significant as long as it is allocated correctly. For simplicity, though, it is most logical to label the edges in alphabetical order.

The incidence symbols, derived from the labelled tilings in Fig. 5.19(a) to (c) are listed below.

- Figure 5.19(a) [$a^+ b^+ c\, b^- a^- d$; $b^- a^- d\, c$] (i)
- Figure 5.19(b) [$a\, b^+ c^+ d\, c^- b^-$; $d\, c^- b^- a$] (ii)
- Figure 5.19(c) [$a^+ b\, a^- c^+ d\, c^-$; $c^+ d\, a^+ b$] (iii)

Suppose, in (i), the letters were cyclically permuted one step, that is, 'a' is replaced by 'b', 'b' by 'c', 'c' by 'd' and 'd' by 'a'. Then the incidence symbol becomes [$b^+ c^+ d\, c^- b^- a$; $c^- b^- a\, d$] which coincides with (ii) except that the starting point for the tile symbol is $p2$ instead of $p1$ (see Fig. 5.19(b)). This confirms that the first and second incidence symbols differ trivially and so are equivalent.

Alternatively, differences in equivalent incidence symbols may occur due to arrow orientation instead of, or as well as, edge lettering.

For incidence symbols (ii) and (iii), unorientated edges in these tile symbols must coincide if the incidence symbols differ trivially. If the edges labelled 'd' in the tile symbols of (ii) and (iii) coincide, then adjacent edges labelled 'c' do also since 'c' occurs next in the sequence in the tile symbol of (ii) and (iii). The other unorientated edges are labelled 'a' in (ii), and 'b' in (iii). Suppose, in (iii), letters a and b are interchanged (remembering that the edge letter does not matter, but the order and occurrence of superscripts and equally labelled edges do matter). Then this transforms (iii) to incidence symbol [$b^+ a\, b^- c^+ d\, c^-$; $c^+ d\, b^+ a$]. By cyclically permuting the terms in this symbol by one step, which is equivalent to starting the tile symbol at an adjacent vertex ($p4$ instead of $p3$), then this symbol becomes [$a\, b^- c^+ d\, c^- b^+$; $d\, b^+ a\, c^+$]. This is the same as (i) except for the orientation of the edge labelled 'b' (see Fig. 5.19(b)). If this arrow is reorientated, all the superscripts of the edges labelled 'b' in the tile symbol will be reversed as will the superscripts in the edges labelled 'c' in the associated adjacency symbol. This results in the transformation of the incidence symbol such that it coincides exactly with that of Fig. 5.19(a). Thus, with some simple reallocation and manipulation of letters, it has been shown that the incidence symbols, in Fig. 5.19(a) to (c), differ trivially.

This may prove to be a time-consuming procedure when determining an isohedral tiling type. However, after initially deducing the topological type, the group of possible incidence symbols may be reduced to a minimum by checking certain characteristics of the tiling. For example, the number of different letters in the tile symbol restricts the range of possible incidence symbols. Then, by checking the symmetry group and induced tile group of the tiling the range is reduced further and, in most cases, enables a ditranslational isohedral tiling to be classified. However, for some tilings with topological types [3^6] or [4^4], the number of different letters in the symbol, the symmetry group and tile induced group coincide. These are listed in Table 5.1.

If the adjacency symbol contains superscripts which are all the same, the classification is straightforward because, if they are all negative, each edge is a reflection or glide–reflection of another edge and if they are all positive, each edge is a two-fold rotation or translation of another edge. If, in addition, each letter in the tile symbol corresponds to the same one in the adjacency symbol, then every edge is mapped onto itself by two-fold rotation.

If there is still doubt with regard to classifying the type of an isohedral tiling, edge lettering reallocation may be necessary and/or more detailed investigation of edge characteristics. A visual comparison with the uncomplicated illustrations of the 81 distinct isohedral tilings in Fig. 5.25 may also aid classification. These examples clearly show the properties of the edges in the tilings and the relationships between them. A tiling which may be classed as one listed in Table 5.1 (and whose type may be difficult to distinguish) may be more easily identified by observing its visual characteristics in comparison with those tilings in Fig. 5.25.

Table 5.1 Isohedral tilings with the same topological type and similar incidence symbols

Topological type	Isohedral tiling type	Incidence symbol	Symmetry group	Induced group
$[3^6]$	Dt(T)2	$[a^+ b^+ c^+ d^+ e^+ f^+; b^- a^- f^+ e^- d^- c^+]$	*pg*	*c*1
	Dt(T)3	$[a^+ b^+ c^+ d^+ e^+ f^+; c^- e^+ a^- f^- b^+ d^-]$	*pg*	*c*1
	Dt(T)5	$[a^+ b^+ c^+ d^+ e^+ f^+; a^+ e^+ d^- c^- b^- f^+]$	*pgg*	*c*1
	Dt(T)6	$[a^+ b^+ c^+ d^+ e^+ f^+; a^+ e^- c^+ f^- b^- d^-]$	*pgg*	*c*1
$[4^4]$	Dt(T)43	$[a^+ b^+ c^+ d^+; c^- d^+ a^- b^+]$	*pg*	*c*1
	Dt(T)44	$[a^+ b^+ c^+ d^+; b^- a^- d^- c^-]$	*pg*	*c*1
	Dt(T)46	$[a^+ b^+ c^+ d^+; a^+ b^+ c^+ d^+]$	*p*2	*c*1
	Dt(T)47	$[a^+ b^+ c^+ d^+; c^+ b^+ a^+ d^+]$	*p*2	*c*1
	Dt(T)49	$[a^+ b^+ c^+ d^+; a^- b^+ c^- d^+]$	*pmg*	*c*1
	Dt(T)50	$[a^+ b^+ c^+ d^+; c^+ b^- a^+ d^+]$	*pmg*	*c*1
	Dt(T)51	$[a^+ b^+ c^+ d^+; c^- b^+ a^- d^+]$	*pgg*	*c*1
	Dt(T)52	$[a^+ b^+ c^+ d^+; c^- d^- a^- b^-]$	*pgg*	*c*1
	Dt(T)53	$[a^+ b^+ c^+ d^+; b^- a^- c^+ d^+]$	*pgg*	*c*1

5.6 Marked isohedral tilings

In the previous discussion in Section 5.4, it was noted that a Dirichlet tiling, associated with a discrete pattern, may not necessarily be unique, in other words more than one type of isohedral tiling may be derived from a ditranslational discrete pattern (see Fig. 5.9). This results in 39 of the 51 ditranslational discrete pattern types forming a basis for 81 different types of isohedral tiling by the Dirichlet relationship, that is, as stated by Grünbaum and Shephard there exist precisely 81 distinct types of isohedral tilings.[4]

Conversely, one isohedral tiling may be associated with more than one discrete pattern type. For example, each of the pattern types Dt(P)24, Dt(P)27 and Dt(P)47 is associated with the same Dirichlet tiling – an equilateral triangle tiling (as shown in Fig. 5.20(a)). Similarly, pattern types Dt(P)14, Dt(P)18 and Dt(P)15 may only be enclosed in either a rectangular or square tiling (depending in the lattice structure of the pattern) and pattern types Dt(P)39, Dt(P)34 and Dt(P)40 may only be enclosed in a square Dirichlet tiling (see Fig. 5.20(b) and (c)). Likewise, pattern types Dt(P)28, Dt(P)29 and Dt(P)37 are associated with the tilings shown in Fig. 5.20(d). Yet, each of these tilings is already a Dirichlet tiling for an associated discrete pattern of its own where the pattern and tiling have the same symmetry group and induced group and the tiling forms one of the distinct isohedral tiling types (see Fig. 5.21). To differentiate between the distinct isohedral tilings and the ones associated with a pattern type with a different symmetry group and induced motif group, the pattern type is incorporated into the isohedral Dirichlet tiling to form a 'marked isohedral tiling' (as shown in Fig. 5.20). A marked tiling is defined as one in which there is a *marking* or *motif* on each tile where a symmetry of the marked tiling is an isometry which not only maps the tiles of T onto tiles of T, but also maps each marking on a tile of T onto a marking on the image tile.[4] In other words the symmetries of the marked isohedral tiling must not only map the tiles onto each other but also the motifs positioned on the tiles onto each other.

Unlike the unmarked tilings, where the symmetries of the discrete pattern and Dirichlet tiling coincide, the symmetries of a pattern of a marked tiling form a subgroup of the symmetries of the tiling enclosing them, that is, by incorporating a pattern into an unmarked tiling the order of symmetry of the tiling is reduced. The 12 marked isohedral tilings, combined with the 81 distinct ones, form the 93 different types of ditranslational isohedral tiling. The identifi-

Table 5.2 Three finite isohedral tiling types

Isohedral tiling type	Symmetry group	Induced tile group	Pattern type
$F(T)1_n$	cn ($n \geq 2$)	$c1$	$F(P)1_n$
$F(T)2_n$	dn ($n \geq 1$)	$c1$	$F(P)2_n$
$F(T)3_n$	dn ($n \geq 2$)	$d1$	$F(P)3_n$

Table 5.3 The 15 monotranslational isohedral tiling types

Isohedral tiling type	Symmetry group	Induced tile group	Pattern type
Mt(T)1	*p*111	*c*1	Mt(T)1
Mt(T)2	*p*1*a*1	*c*1	Mt(T)2
Mt(T)3	*p*1*m*1	*c*1	Mt(T)3
Mt(T)4	*p*1*m*1	*d*1	Mt(T)4
Mt(T)5	*pm*11	*c*1	Mt(T)5
Mt(T)6	*pm*11	*d*1	Mt(T)6
Mt(T)7	*p*112	*c*1	Mt(T)7
Mt(T)8	*p*112	*c*2	Mt(T)8
Mt(T)9	*pma*2	*c*1	Mt(T)9
Mt(T)10	*pma*2	*c*2	Mt(T)10
Mt(T)11	*pma*2	*d*1	Mt(T)11
Mt(T)12	*pmm*2	*c*1	Mt(T)12
Mt(T)13	*pmm*2	*d*1	Mt(T)13
Mt(T)14	*pmm*2	*d*1	Mt(T)14
Mt(T)15	*pmm*2	*d*2	Mt(T)15

cation and classification of the marked isohedral tilings is determined in the same way as the distinct isohedral tilings, whilst accounting for the reduction in the order of symmetry of each tiling caused by marking the superimposed discrete pattern.

The identification and classification of finite and monotranslational isohedral tilings involves a much simpler process since they form a one-to-one correspondence with their associated discrete pattern types.

5.7 Classification of finite isohedral tiling types

Each of the three finite discrete pattern types is associated with one isohedral tiling type. These tilings are listed in Table 5.2 together with their symmetry groups, induced tile groups and associated discrete pattern types. Illustrations of each type are given in Fig. 5.22(a), (b) and (c).

5.7.1 Notation

The notation used for finite isohedral tilings has been derived from that of the finite discrete pattern types. The three types are denoted by $F(T)1_n$, $F(T)2_n$ and $F(T)3_n$ where n represents the number of reflection axes and/or the order of rotation of the overall design structure. Because each of these tilings is associated with a discrete pattern type, which must satisfy the non-trivial condition, the same restrictions apply to the limitations on the values of n (see Table 5.2).

5.8 Classification of monotranslational isohedral tiling types

Each of the 15 monotranslational discrete pattern types is associated with one isohedral tiling type. These tilings are listed in Table 5.3 together with their symmetry groups, induced tile groups and associated discrete pattern types. Figure 5.23 shows an illustration of each type.

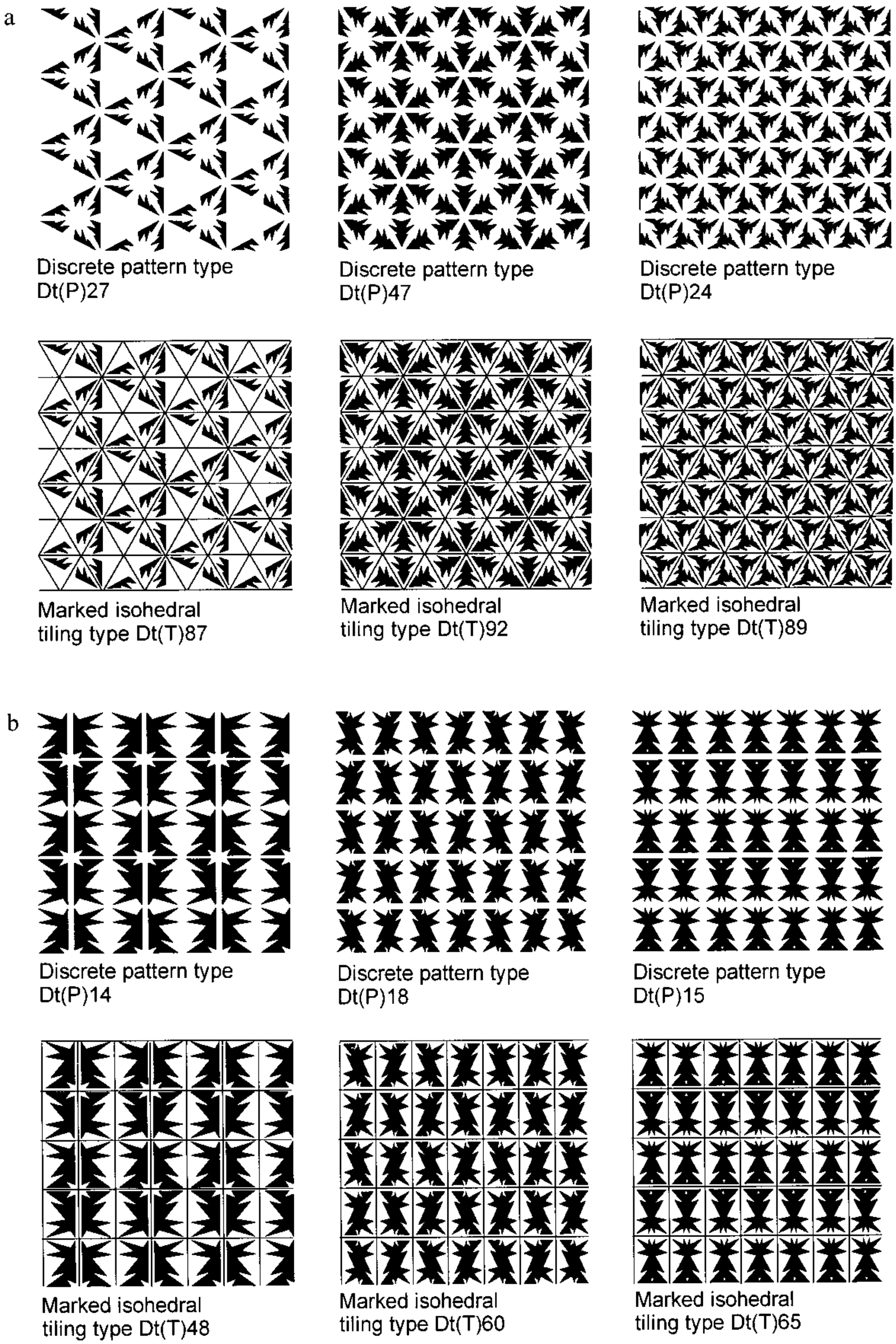
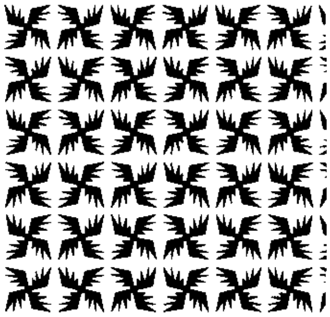
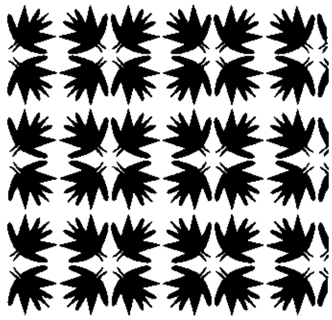


Figure 5.20 Examples of different pattern types associated with the same Dirichlet tiling.

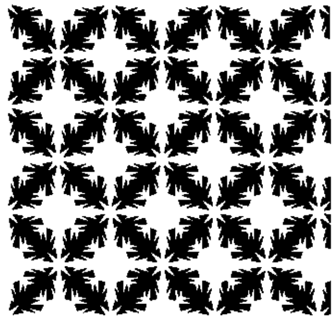
c



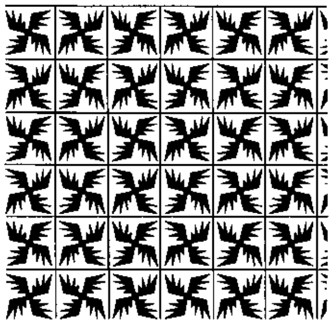
Discrete pattern type Dt(P)34



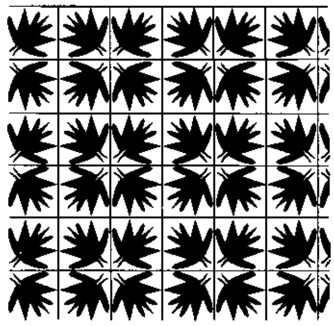
Discrete pattern type Dt(P)39



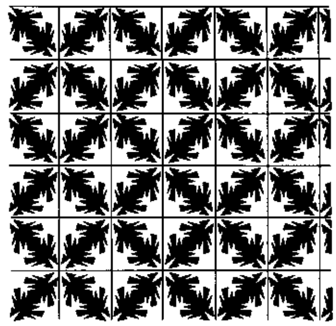
Discrete pattern type Dt(P)40



Marked isohedral tiling type Dt(T)63

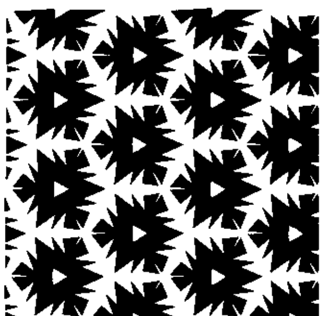


Marked isohedral tiling type Dt(T)70

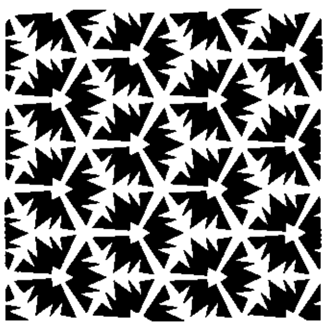


Marked isohedral tiling type Dt(T)75

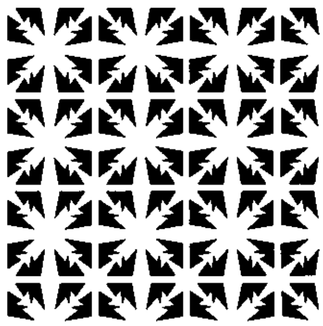
d



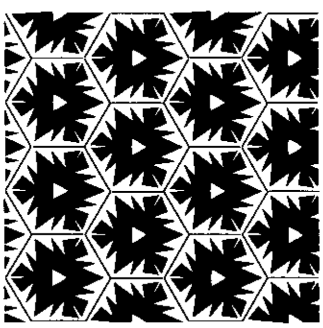
Discrete pattern type Dt(P)28



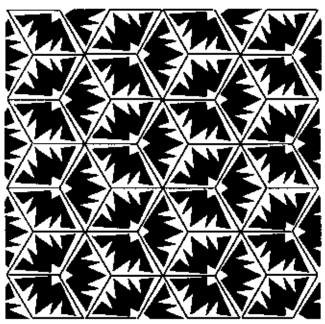
Discrete pattern type Dt(P)29



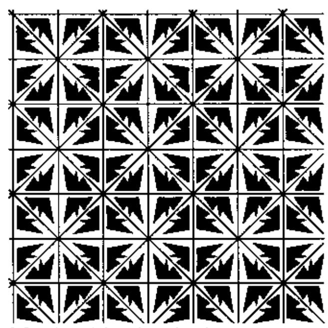
Discrete pattern type Dt(P)37



Marked isohedral tiling type Dt(T)19

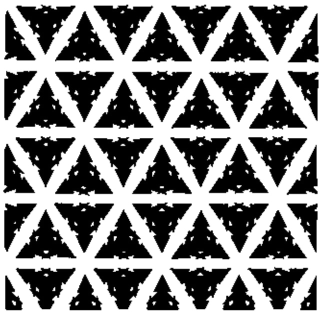


Marked isohedral tiling type Dt(T)35

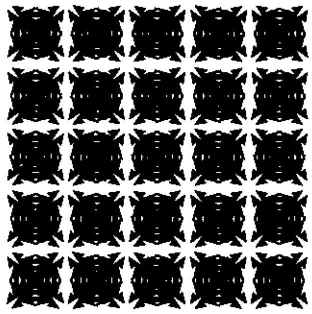


Marked isohedral tiling type Dt(T)80

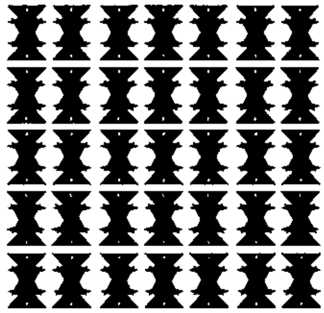
Figure 5.20 (cont.)



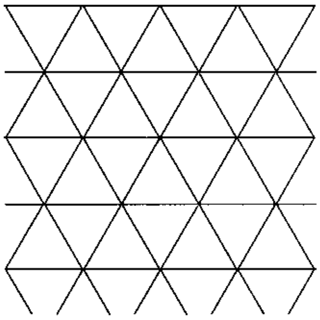
Discrete pattern type
Dt(P)50



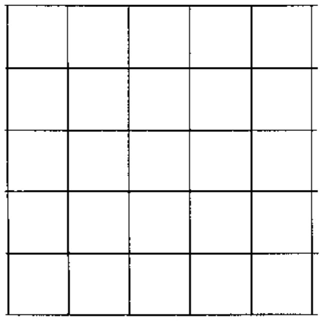
Discrete pattern type
Dt(P)41



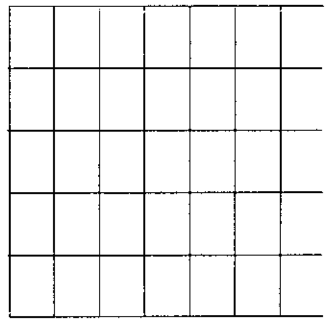
Discrete pattern type
Dt(P)16



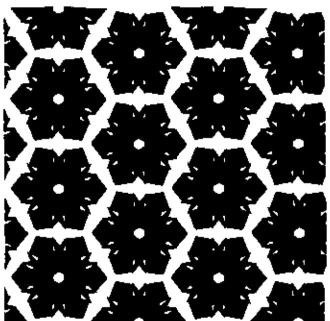
Isohedral tiling type
Dt(T)93



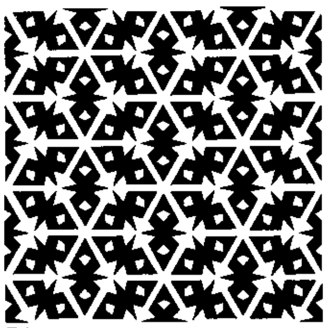
Isohedral tiling type
Dt(T)76



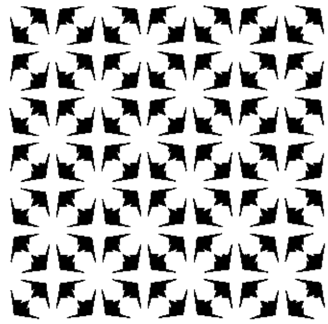
Isohedral tiling type
Dt(T)72



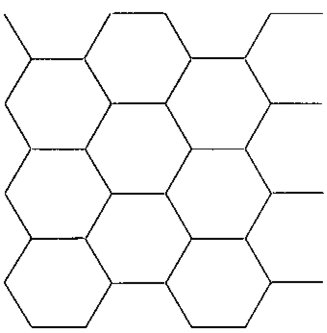
Discrete pattern type
Dt(P)51



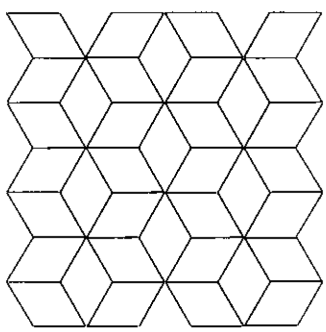
Discrete pattern type
Dt(P)49



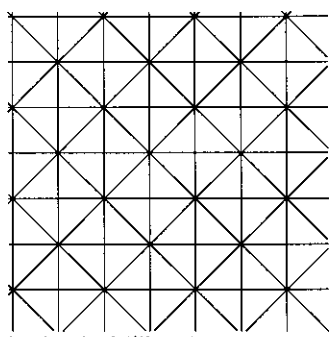
Discrete pattern type
Dt(P)38



Isohedral tiling type
Dt(T)20



Isohedral tiling type
Dt(T)37



Isohedral tiling type
Dt(T)82

Figure 5.21 Distinct isohedral tilings (and their associated pattern types) which are used to form the marked isohedral tilings.

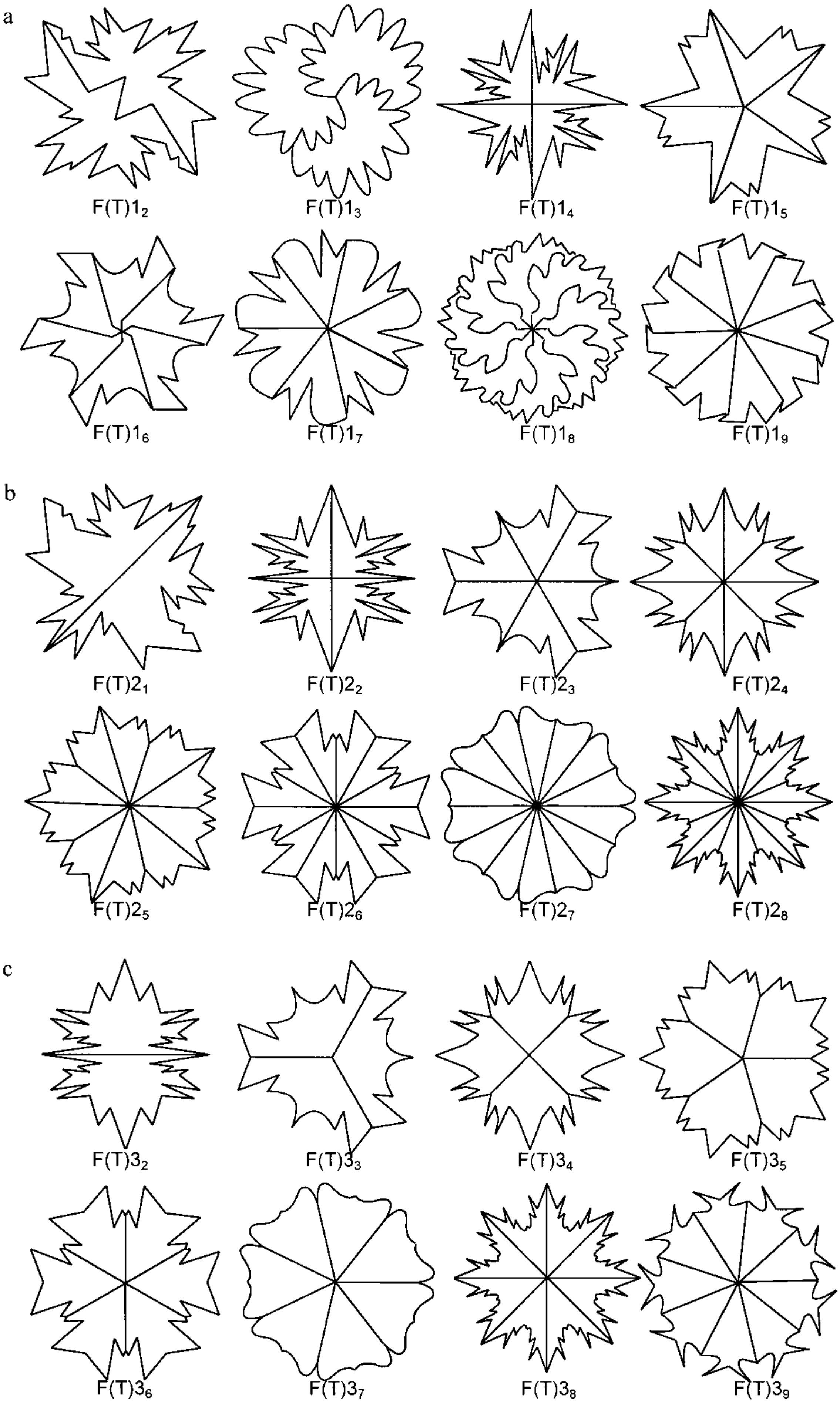


Figure 5.22 Illustrations of finite isohedral tiling types.

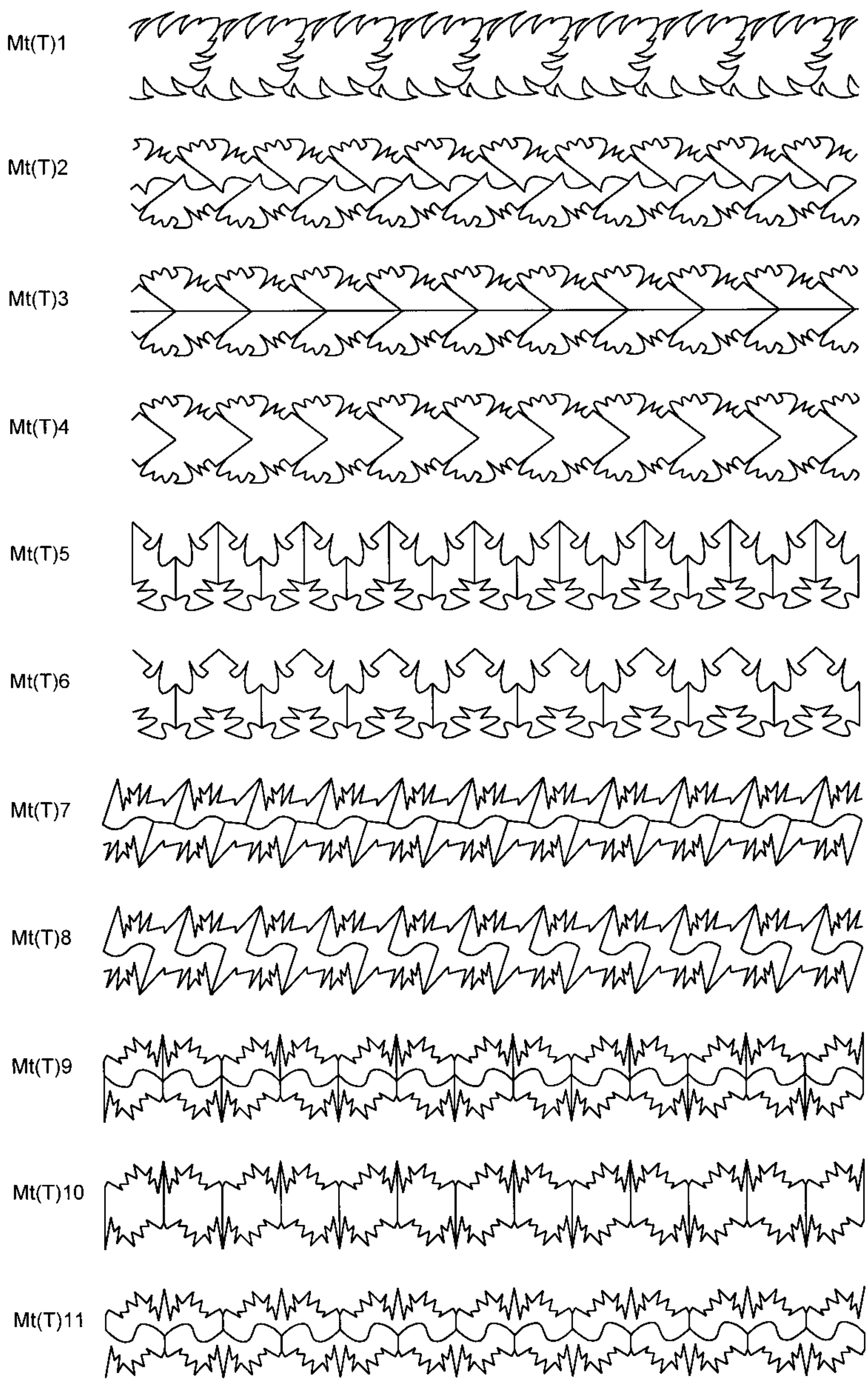
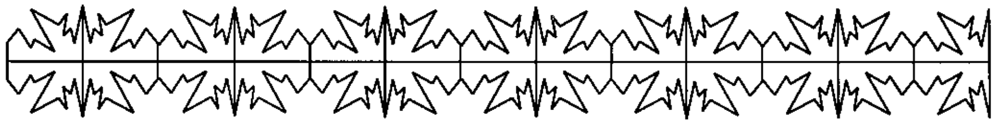


Figure 5.23 Illustrations of the 15 monotranslational isohedral tiling types.

Mt(T)12



Mt(T)13

Mt(T)14

Mt(T)15

Figure 5.23 (cont.)

5.8.1 Notation

The notation used for monotranslational isohedral tilings has been derived from that of the monotranslational discrete pattern types. The 15 types are denoted by Mt(T)1 to Mt(T)15.

5.9 Classification of ditranslational isohedral tiling types

Each of the 51 ditranslational discrete pattern types is associated with one or more isohedral tiling types which results in 12 marked and 81 distinct isohedral tiling types. These tilings are listed in Table 5.4 together with their topological types, incidence symbols, symmetry groups, induced tile groups and associated discrete pattern types. Figure 5.24 shows an illustration of each marked type and Fig. 5.25 shows an example of each distinct type.

5.9.1 Notation

The notation used for ditranslational isohedral tilings has been derived from that of the ditranslational discrete pattern types. The 93 types are denoted by Dt(T)1 to Dt(T)93.

5.10 Construction of finite isohedral tiling types

The techniques used to construct finite isohedral tilings F(T)1_n to F(T)3_n, are adapted from those described in Section 2.11. The circular area enclosing the design will be divided into fundamental regions (some or all of whose boundaries are retained) and then the circumference of the circle may be suitably adapted to produce a finite tiling design.

5.10.1 Finite isohedral tilings, induced group c1

There are two types of finite isohedral tiling design with induced group $c1$: F(T)1_n (symmetry group cn) and F(T)2_n (symmetry group dn).

The simplest method of constructing an F(T)1_n design is to begin with a circle, radius R, and add a line (which is not straight) joining its centre to the boundary. This line is then rotated, $n - 1$ times, at consecutive intervals of $360°/n$ after, if

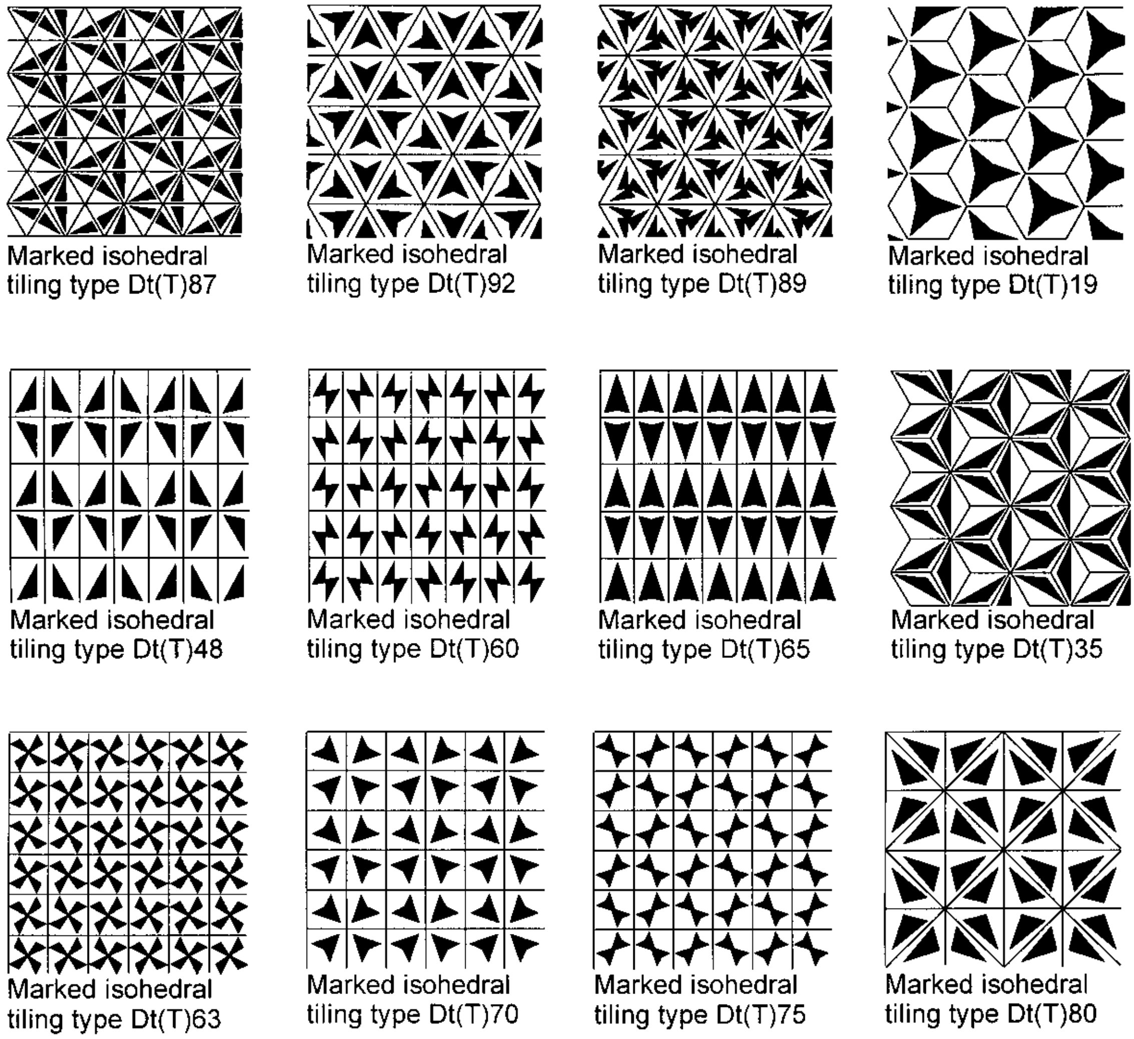


Figure 5.24 Illustrations of the 12 marked isohedral tiling types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

necessary, adapting the initial line to ensure that it does not overlap with adjacent copies of itself (see Fig. 5.26a(i) and a(ii)). The circle may be incorporated as part of the finished design or the design may be enhanced by joining the point at which one of these lines touches the circular boundary to an adjacent line segment at distance r from the circle centre (where r is a proportion of R) (see Fig. 5.26a(iii)). This line (the shape of which, in this instance, is not important) is also rotated $n - 1$ times through $360°/n$. The circular boundary may then be removed to complete the tiling design (Fig. 5.26a(iv)). The initial line may be chosen to be straight, in which case the secondary joining line must not have both end points on the circular boundary and have reflectional symmetry passing through the centre of the circle. The same procedure as above is used to complete the remainder of the design (see Fig. 5.26b).

To construct an $F(T)2_n$ tiling design, a circle of radius R is divided into $2n$ equal sectors. A line (which does not have reflectional symmetry passing through the circle centre) is used to join one straight edge of a sector to an adjacent one. It must touch at least one point on the circumference of the circle. This line is then reflected about axes coinciding with the sector boundaries. The straight sector edges inside these resulting lines are incorporated in the design whilst the proportions outside them, and the circular boundary are removed (Fig. 5.26(c)).

5.10.2 *Finite isohedral tilings, induced group d*1

There is one type of finite isohedral tiling design with induced group $d1$: $F(T)3_n$, in symmetry group dn. It is constructed by the same method as that for group $F(T)2_n$ but in the final stages only alternate straight sector edges inside the boundary of the tiling are incorporated in the design (see Fig. 5.26(d)). Different design effects may be created depending upon which of the two sets of alternate straight sector edges is removed.

5.11 Construction of monotranslational isohedral tiling types

The technique used to construct the majority of the monotranslational tiling types will initially follow the stages described in Section 2.12 for design type (iii), of dividing a strip, width W, into interlocking fundamental regions. (For symmetry groups *pm*11 and *pmm*2 recall that design type (iii) was not constructable so the initial design structures for tiling types in these symmetry groups will be based on rectangular (or square) fundamental regions described for design type (i).)

The design may then be further improved by replacing the straight edges of the strip with irregular ones. This may be achieved by adding a line (or two lines where the two opposite edges of a fundamental region coincide with the edges of the strip) which joins a vertex on the edge of the strip to an adjacent fundamental region edge in the longitudinal direction. It is then mapped onto all equivalent positions in the strip by applying the generating symmetries. In some cases more than one edge of a fundamental region may initially be replaced by a new edge (as shown for Mt(T)9, symmetry group *pma*2, in Fig. 5.27f(ii)). However, this/these new edge(s) must reach at least one point on the edge of the strip. The initial straight edges of the strip and any boundaries of the fundamental regions exceeding the tiles in the tiling are then removed to complete the design. This procedure is illustrated in the examples given throughout the remainder of this section. The non-primitive tiling types are derived from the primitive tilings by removing some of the boundaries of the fundamental regions at the end of the construction procedure.

The symmetric tiles used for the construction of design type (vi) may also be used as a basis for the construction of monotranslational tilings. However, when replacing the straight edges of the strip, the new lines added to complete the tiling must reduce the order of symmetry to the correct tiling type. These forms of tiling are not discussed in any further detail in this chapter.

5.11.1 *Monotranslational isohedral tilings, induced group c*1

Each of the seven primitive pattern types has one associated isohedral tiling type with induced group $c1$. A strip is divided into fundamental regions by the methods described for design type (iii) (or type (i) for Mt(T)5 and Mt(T)12) in Section 2.12. The procedure described above is carried out to produce the tilings given in Fig. 5.27(a)–(g). These show the isohedral tilings associated with the primitive pattern types of symmetry groups p111, p1a1, p1m1, pm11, p112, pma2 and pmm2, respectively. For Mt(T)2, symmetry group *p*1*a*1, there are two methods of construction. In one case the right hand side of a fundamental region is a glide–reflection of the left hand side and in the other case it is a translation of the left hand side. In the second case the straight longitudinal axis of the strip may also be replaced by an alternative one which has glide–reflectional symmetry (as shown in Fig. 5.27b(ii)). Three methods are given for the construction of tiling type Mt(T)7, symmetry group *p*112, which are illustrated in Fig. 5.27(e(i)), (e(ii)) and (e(iii)).

5.11.2 *Monotranslational isohedral tilings, induced group c*2

Each of the pattern types Mt(P)8 and Mt(P)10 (in symmetry groups *p*112 and *pma*2, respectively) has one associated isohedral tiling type, Mt(T)8 and

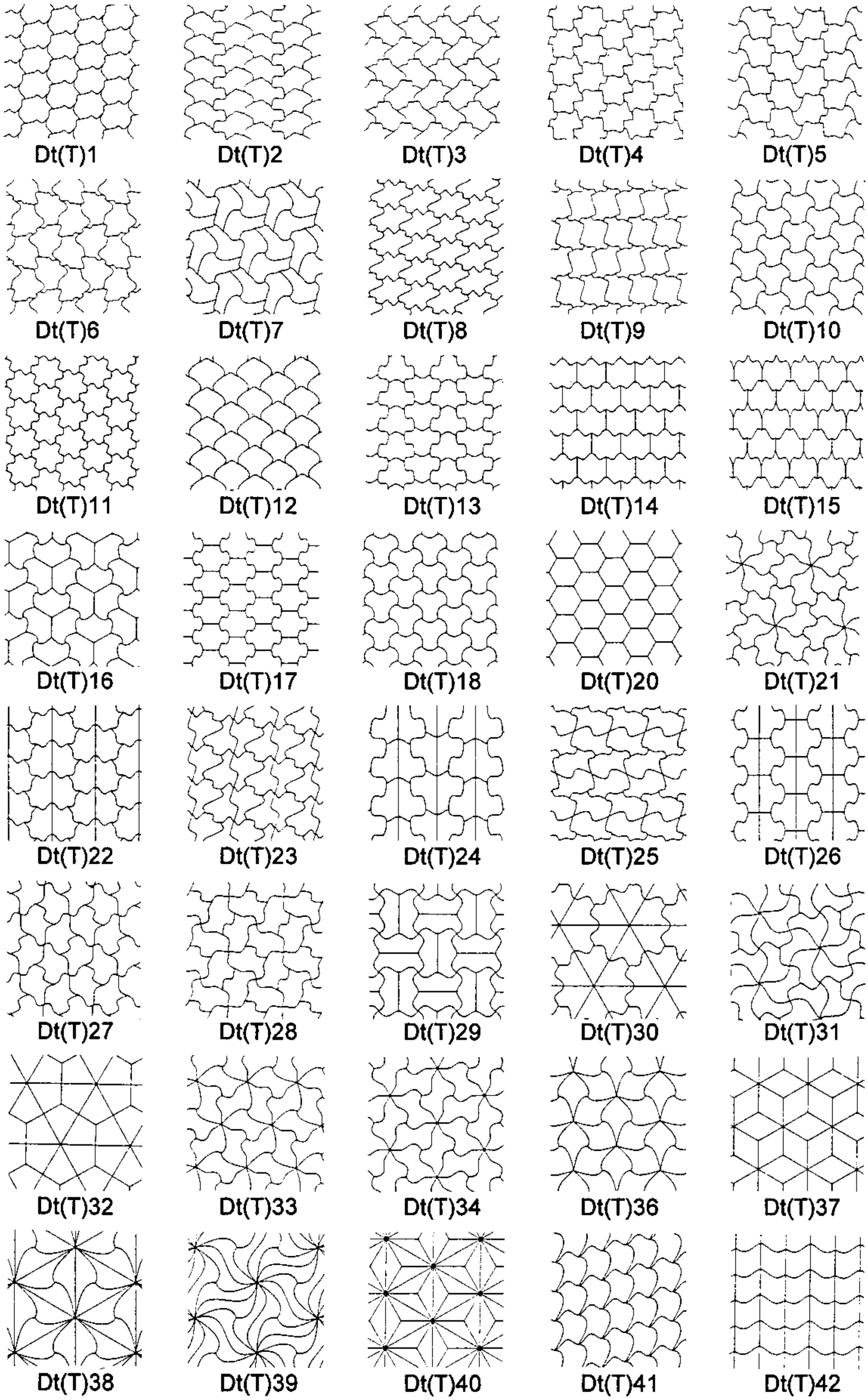


Figure 5.25 Illustrations of the 81 distinct ditranslational isohedral tiling types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

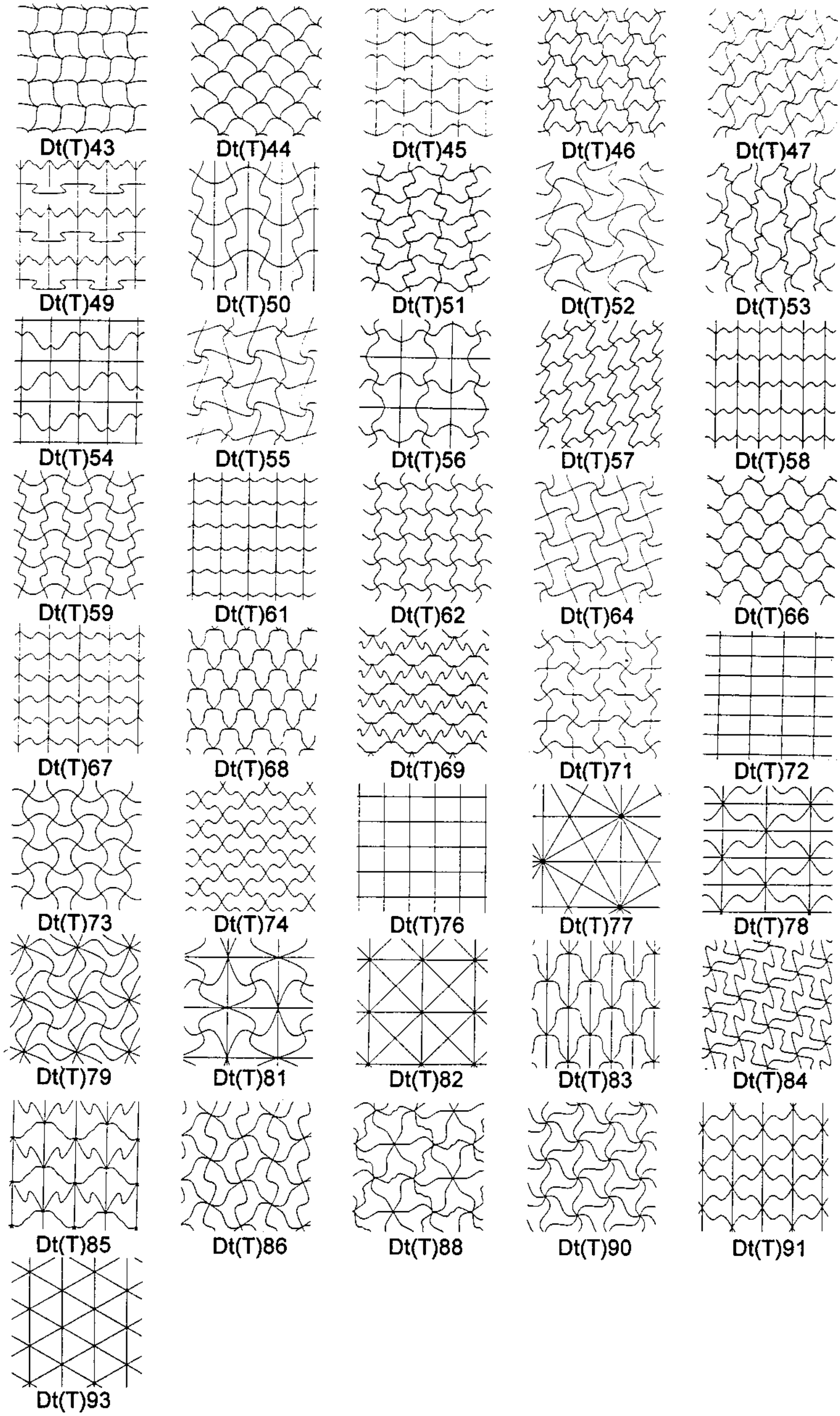


Figure 5.25 (cont.)

Table 5.4 The 93 ditranslational isohedral tiling types

Isohedral tiling type	Topological type	Incidence symbol	Symmetry group	Induced tile group	Pattern type
Dt(T)1	$[3^6]$	$[a^+ b^+ c^+ d^+ e^+ f^+; d^+ e^+ f^+ a^+ b^+ c^+]$	*p*1	*c*1	Dt(P)1
Dt(T)2		$[a^+ b^+ c^+ d^+ e^+ f^+; b^- a^- f^+ e^- d^- c^+]$	*pg*	*c*1	Dt(P)2
Dt(T)3		$[a^+ b^+ c^+ d^+ e^+ f^+; c^- e^+ a^- f^- b^+ d^-]$	*pg*	*c*1	Dt(P)2
Dt(T)4		$[a^+ b^+ c^+ d^+ e^+ f^+; a^+ e^+ c^+ d^+ b^+ f^+]$	*p*2	*c*1	Dt(P)7
Dt(T)5		$[a^+ b^+ c^+ d^+ e^+ f^+; a^+ e^+ d^- c^- b^- f^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)6		$[a^+ b^+ c^+ d^+ e^+ f^+; a^+ e^- c^+ f^- b^- d^-]$	*pgg*	*c*1	Dt(P)9
Dt(T)7		$[a^+ b^+ c^+ d^+ e^+ f^+; b^+ a^+ d^+ c^+ f^+ e^+]$	*p*3	*c*1	Dt(P)21
Dt(T)8		$[a^+ b^+ c^+ a^+ b^+ c^+; a^+ b^+ c^+]$	*p*2	*c*2	Dt(P)8
Dt(T)9		$[a^+ b^+ c^+ a^+ b^+ c^+; a^+ c^- b^-]$	*pgg*	*c*2	Dt(P)10
Dt(T)10		$[a^+ b^+ a^+ b^+ a^+ b^+; b^+ a^+]$	*p*3	*c*3	Dt(P)22
Dt(T)11		$[a^+ a^+ a^+ a^+ a^+ a^+; a^+]$	*p*6	*c*6	Dt(P)45
Dt(T)12		$[a\, b^+ c^+ d\, c^- b^-; d\, c^- b^- a]$	*cm*	*d*1(s)	Dt(P)6
Dt(T)13		$[a\, b^+ c^+ d\, c^- b^-; d\, b^+ c^+ a]$	*pmg*	*d*1(s)	Dt(P)13
Dt(T)14		$[a^+ b^+ c^+ c^- b^- a; c^- b^- a^-]$	*cm*	*d*1(1)	Dt(P)6
Dt(T)15		$[a^+ b^+ c^+ c^- b^- a^-; a^+ b^- c^+]$	*pmg*	*d*1(1)	Dt(P)13
Dt(T)16		$[a^+ b^+ c^+ c^- b^- a^-; a^- c^+ b^+]$	*p*31*m*	*d*1(1)	Dt(P)25
Dt(T)17		$[a\, b^+ b^- a\, b^+ b^-; a\, b^+]$	*cmm*	*d*2	Dt(P)20
Dt(T)18		$[a\, b\, a\, b\, a\, b; b\, a]$	*p*31*m*	*d*3(s)	Dt(P)26
Dt(T)19		$[a^+ a^- a^+ a^- a^+ a^-; a^-]$	*p*3*m*1	*d*3(1)	Dt(P)29*
Dt(T)20		$[a\, a\, a\, a\, a\, a; a]$	*p*6*m*	*d*6	Dt(P)51
Dt(T)21	$[3^4.6]$	$[a^+ b^+ c^+ d^+ e^+; e^+ c^+ b^+ d^+ a^+]$	*p*6	*c*1	Dt(P)42
Dt(T)22	$[3^3.4^2]$	$[a^+ b^+ c^+ d^+ e^+; a^- e^+ d^- c^- b^+]$	*cm*	*c*1	Dt(P)5
Dt(T)23		$[a^+ b^+ c^+ d^+ e^+; a^+ e^+ c^+ d^+ b^+]$	*p*2	*c*1	Dt(P)7
Dt(T)24		$[a^+ b^+ c^+ d^+ e^+; a^- e^+ c^+ d^+ b^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)25		$[a^+ b^+ c^+ d^+ e^+; a^+ e^+ d^- c^- d^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)26		$[a\, b^+ c^+ c^- b^-; a\, b^- c^+]$	*cmm*	*d*1	Dt(P)19
Dt(T)27	$[3^2.4.3.4]$	$[a^+ b^+ c^+ d^+ e^+; a^+ d^- e^- b^- c^-]$	*pgg*	*c*1	Dt(P)9
Dt(T)28		$[a^+ b^+ c^+ d^+ e^+; a^+ c^+ b^+ e^+ d^+]$	*p*4	*c*1	Dt(P)30
Dt(T)29		$[a\, b^+ c^+ c^- b^-; a\, c^+ b^+]$	*p*4*g*	*d*1	Dt(P)35
Dt(T)30	[3.4.6.4]	$[a^+ b^+ c^+ d^+; a^- b^- d^+ c^+]$	*p*31*m*	*c*1	Dt(P)23
Dt(T)31		$[a^+ b^+ c^+ d^+; b^+ a^+ d^+ c^+]$	*p*6	*c*1	Dt(P)42
Dt(T)32		$[a^+ a^- b^+ b^-; a^- b^-]$	*p*6*m*	*d*1	Dt(P)48
Dt(T)33	[3.6.3.6]	$[a^+ b^+ c^+ d^+; d^+ c^+ b^+ a^+]$	*p*3	*c*1	Dt(P)21
Dt(T)34		$[a^+ b^+ a^+ b^+; b^+ a^+]$	*p*6	*c*2	Dt(P)43
Dt(T)35		$[a^+ b^+ b^- a^-; a^- b^-]$	*p*3*m*1	*d*1(s)	Dt(P)28*
Dt(T)36		$[a^+ a^- b^+ b^-; b^- a^-]$	*p*31*m*	*d*1(1)	Dt(P)25
Dt(T)37		$[a^+ a^- a^+ a^-; a^-]$	*p*6*m*	*d*2	Dt(P)49
Dt(T)38	$[3.12^2]$	$[a^+ b^+ c^+; a^- c^+ b^+]$	*p*31*m*	*c*1	Dt(P)23
Dt(T)39		$[a^+ b^+ c^+; a^+ c^+ b^+]$	*p*6	*c*1	Dt(P)42
Dt(T)40		$[a\, b^+ b^-; a\, b^-]$	*p*6*m*	*d*1	Dt(P)48
Dt(T)41	$[4^4]$	$[a^+ b^+ c^+ d^+; c^+ d^+ a^+ b^+]$	*p*1	*c*1	Dt(P)1
Dt(T)42		$[a^+ b^+ c^+ d^+; c^+ b^- a^+ d^-]$	*pm*	*c*1	Dt(P)3
Dt(T)43		$[a^+ b^+ c^+ d^+; c^- d^+ a^- b^+]$	*pg*	*c*1	Dt(P)2
Dt(T)44		$[a^+ b^+ c^+ d^+; b^- a^- d^- c^-]$	*pg*	*c*1	Dt(P)2
Dt(T)45		$[a^+ b^+ c^+ d^+; c^- b^- a^- d^-]$	*cm*	*c*1	Dt(P)5
Dt(T)46		$[a^+ b^+ c^+ d^+; a^+ b^+ c^+ d^+]$	*p*2	*c*1	Dt(P)7
Dt(T)47		$[a^+ b^+ c^+ d^+; c^+ b^+ a^+ d^+]$	*p*2	*c*1	Dt(P)7
Dt(T)48		$[a^+ b^+ c^+ d^+; a^- b^- c^- d^-]$	*pmm*	*c*1	Dt(P)14*

Table 5.4 (cont.)

Isohedral tiling type	Topological type	Incidence symbol	Symmetry group	Induced tile group	Pattern type
Dt(T)49		$[a^+ b^+ c^+ d^+; a^- b^+ c^- d^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)50	$[4^4]$	$[a^+ b^+ c^+ d^+; c^+ b^- a^+ d^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)51		$[a^+ b^+ c^+ d^+; c^- b^+ a^- d^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)52		$[a^+ b^+ c^+ d^+; c^- d^- a^- b^-]$	*pgg*	*c*1	Dt(P)9
Dt(T)53		$[a^+ b^+ c^+ d^+; b^- a^- c^+ d^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)54		$[a^+ b^+ c^+ d^+; a^- b^- c^- d^+]$	*cmm*	*c*1	Dt(P)17
Dt(T)55		$[a^+ b^+ c^+ d^+; b^+ a^+ d^+ c^+]$	*p*4	*c*1	Dt(P)30
Dt(T)56		$[a^+ b^+ c^+ d^+; b^+ a^+ c^- d^-]$	*p*4*g*	*c*1	Dt(P)33
Dt(T)57		$[a^+ b^+ a^+ b^+; a^+ b^+]$	*p*2	*c*2	Dt(P)8
Dt(T)58		$[a^+ b^+ a^+ b^+; a^- b^+]$	*pmg*	*c*2	Dt(P)12
Dt(T)59		$[a^+ b^+ a^+ b^+; b^- a^-]$	*pgg*	*c*2	Dt(P)10
Dt(T)60		$[a^+ b^+ a^+ b^+; a^- b^-]$	*cmm*	*c*2	Dt(P)18*
Dt(T)61		$[a^+ b^+ a^+ b^+; b^+ a^+]$	*p*4	*c*2	Dt(P)31
Dt(T)62		$[a^+ a^+ a^+ a^+; a^+]$	*p*4	*c*4	Dt(P)32
Dt(T)63		$[a^+ a^+ a^+ a^+; a^-]$	*p*4*g*	*c*4	Dt(P)34*
Dt(T)64		$[a\, b^+ c^+ c\, b^-; c\, b^- a]$	*pm*	*d*1(s)	Dt(P)4
Dt(T)65		$[a\, b^+ c^+ c\, b^-; a\, b^- c]$	*pmm*	*d*1(s)	Dt(P)15*
Dt(T)66		$[a\, b^+ c^+ c\, b^-; c\, b^+ a]$	*pmg*	*d*1(s)	Dt(P)13
Dt(T)67		$[a\, b^+ c^+ c\, b^-; a\, b^+ c]$	*cmm*	*d*1(s)	Dt(P)19
Dt(T)68		$[a^+ b^+ b^- a^-; b^- a^-]$	*cm*	*d*1(1)	Dt(P)6
Dt(T)69		$[a^+ b^+ b^- a^-; a^+ b^+]$	*pmg*	*d*1(1)	Dt(P)13
Dt(T)70		$[a^+ b^+ b^- a^-; a^- b^-]$	*p*4*m*	*d*1(1)	Dt(P)39*
Dt(T)71		$[a^+ b^+ b^- a^-; b^+ a^+]$	*p*4*g*	*d*1(1)	Dt(P)35
Dt(T)72		$[a\, b\, a\, b; a\, b]$	*pmm*	*d*1(s)	Dt(P)16
Dt(T)73		$[a\, b\, a\, b; b\, a]$	*p*4*g*	*d*1(s)	Dt(P)36
Dt(T)74		$[a^+ a^- a^+ a^-; a^+]$	*cmm*	*d*1(1)	Dt(P)20
Dt(T)75		$[a^+ a^- a^+ a^-; a^-]$	*p*4*m*	*d*1(1)	Dt(P)40*
Dt(T)76		$[a\, a\, a\, a; a]$	*p*4*m*	*d*4	Dt(P)41
Dt(T)77	[4.6.12]	$[a^+ b^+ c^+; a^- b^- c^-]$	*p*6*m*	*c*1	Dt(P)46
Dt(T)78	$[4.8^2]$	$[a^+ b^+ c^+; a^+ b^- c^-]$	*cmm*	*c*1	Dt(P)17
Dt(T)79		$[a^+ b^+ c^+; a^+ c^+ b^+]$	*p*4	*c*1	Dt(P)30
Dt(T)80		$[a^+ b^+ c^+; a^- b^- c^-]$	*p*4*m*	*c*1	Dt(P)37*
Dt(T)81		$[a^+ b^+ c^+; a^- c^+ b^+]$	*p*4*g*	*c*1	Dt(P)33
Dt(T)82		$[a\, b^+ b^-; a\, b^-]$	*p*4*m*	*d*1	Dt(P)38
Dt(T)83	$[6^3]$	$[a^+ b^+ c^+; b^- a^- c^-]$	*cm*	*c*1	Dt(P)5
Dt(T)84		$[a^+ b^+ c^+; a^+ b^+ c^+]$	*p*2	*c*1	Dt(P)7
Dt(T)85		$[a^+ b^+ c^+; a^- b^+ c^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)86		$[a^+ b^+ c^+; b^- a^- c^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)87		$[a^+ b^+ c^+; a^- b^- c^-]$	*p*3*m*1	*c*1	Dt(P)27*
Dt(T)88		$[a^+ b^+ c^+; b^+ a^+ c^+]$	*p*6	*c*1	Dt(P)42
Dt(T)89		$[a^+ a^+ a^+; a^-]$	*p*31*m*	*c*3	Dt(P)24*
Dt(T)90		$[a^+ a^+ a^+; a^+]$	*p*6	*c*3	Dt(P)44
Dt(T)91		$[a\, b^+ b^-; a\, b^+]$	*cmm*	*d*1	Dt(P)19
Dt(T)92		$[a\, b^+ b^-; a\, b^-]$	*p*6*m*	*d*1	Dt(P)47*
Dt(T)93		$[a\, a\, a; a]$	*p*6*m*	*d*3	Dt(P)50

* Indicates that the tiling is one of the marked isohedral tiling types.
Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

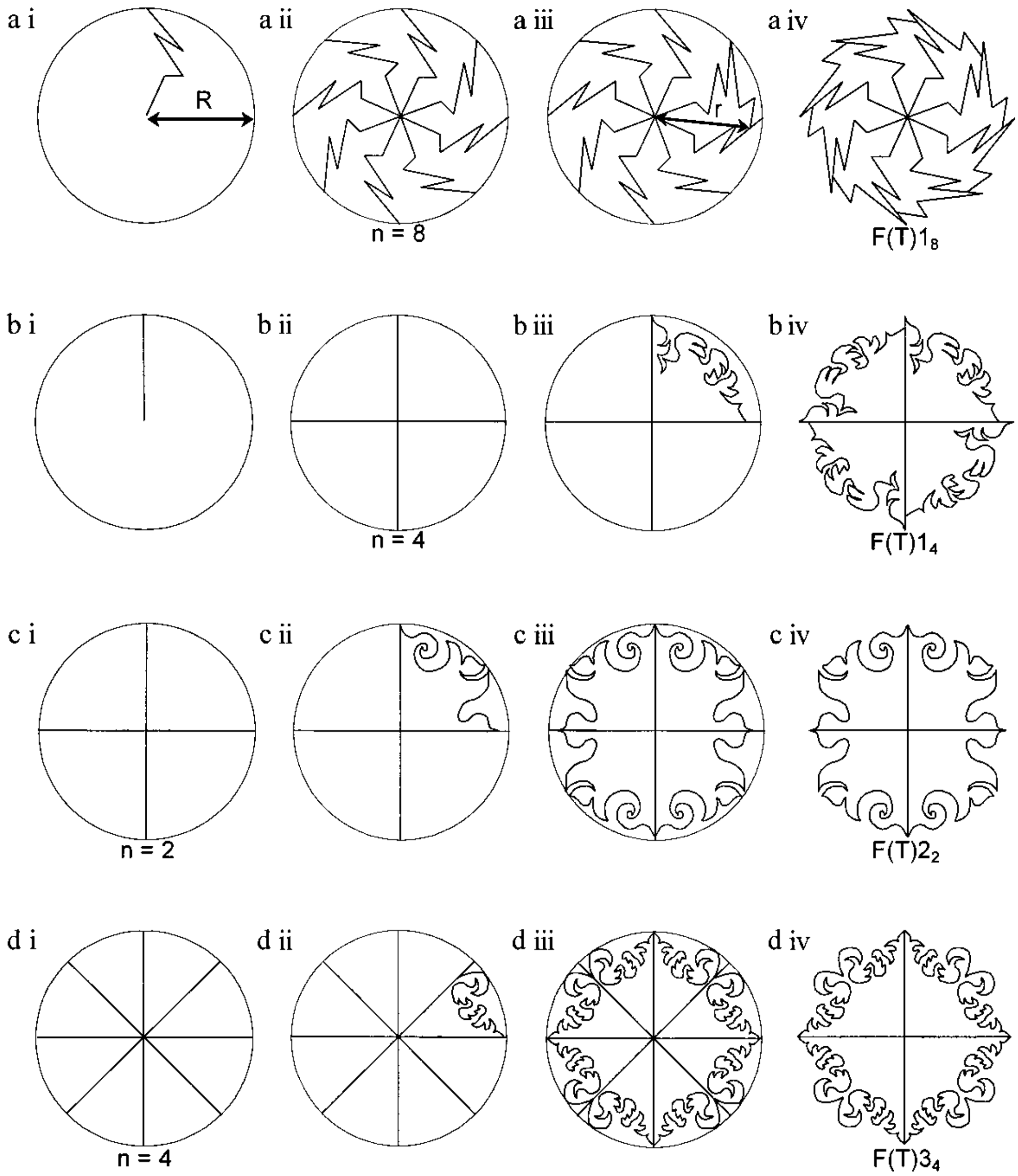


Figure 5.26 Construction of finite tiling types $F(T)1_n$, $F(T)2_n$ and $F(T)3_n$.

Mt(T)10, with induced group *c*2. These may be derived from the primitive isohedral tiling types Mt(T)7 and Mt(T)9, respectively.

Tiling type Mt(T)8 is constructed from Mt(T)7 by removing each edge in common with two tiles that passes through alternate centres of two-fold rotation which occur along the longitudinal axis of the strip. Tiling type Mt(T)10 is constructed from Mt(T)9 by removing every edge in common with two tiles that passes through a centre of two-fold rotation along the longitudinal axis of the strip. Examples showing the construction of Mt(T)8 and Mt(T)10 are given in Fig. 5.28(a) and (b), respectively.

*5.11.3 Monotranslational isohedral tilings, induced group d*1

Each of the pattern types Mt(P)4, Mt(P)6, Mt(P)11, Mt(P)13 and Mt(P)14 (in symmetry groups *p*1*m*1, *pm*11, *pma*2, *pmm*2 and *pmm*2, respectively) has one associated isohedral tiling type: Mt(T)4, Mt(T)6, Mt(T)11, Mt(T)13 and

Mt(T)14 with induced group *d*1. These may be derived from the primitive isohedral tiling types Mt(T)3, Mt(T)5, Mt(T)9, Mt(T)12 and Mt(T)12, respectively.

Mt(T)4 is constructed from Mt(T)3 by removing each edge in common with two tiles that coincides with the longitudinal reflection axis of the strip (see Fig. 5.29(a)). Mt(T)6 is constructed from Mt(T)5 by removing each edge in common with two tiles that coincides with each alternate transverse reflection axis (see Fig. 5.29(b)). Mt(T)11 is constructed from Mt(T)9 by removing each edge in common with two tiles that coincides with a transverse reflection axis (see Fig. 5.29(c)). Mt(T)13 is constructed from Mt(T)12 type by removing each edge in common with two tiles that coincides with each alternate transverse reflection axis (see Fig. 5.29(d)). Mt(T)14 is constructed from Mt(T)12 by removing each edge in common with two tiles that coincides with the longitudinal axis of the strip (see Fig. 5.29(e)).

5.11.4 Monotranslational isohedral tilings, induced group d2

There is one pattern type Mt(P)15 (in symmetry group *pmm*2) which has one associated isohedral tiling type Mt(T)15 with induced group *d*2. It is constructed from Mt(T)12 by removing each edge in common with two tiles that coincides with the longitudinal reflection axis of the strip and each edge in common with two tiles that coincides with each alternate transverse reflection axis. Examples are given in Fig. 5.30.

5.12 Construction of ditranslational isohedral tiling types

The techniques used to construct ditranslational isohedral tiling designs will differ from those described in Section 2.13 because the primary concern in this classification and construction involves establishing and building upon the topological characteristics of the design. Hence, the following methods will be divided into 11 sections to coincide with the 11 different topological types of ditranslational isohedral tiling: [3^6], [$3^4.6$], [$3^3.4^2$], [$3^2.4.3.4$], [3.4.6.4], [3.6.3.6], [3.12^2], [4^4], [4.6.12], [4.8^2] and [6^3].

Having chosen which particular tiling type to construct and established its topological type (from Table 5.4), a framework is required upon which to build it. Since its topology is most important, the clearest possible representation of its topological form seems the most logical basis. A tiling with this characteristic may not be of the desired symmetry group, induced tile group or have the correct incidence symbol. However, the required isohedral tiling type may be derived from its gradual metamorphosis, by the application of topological and geometric transformations interpreted from the analysis of the incidence symbol.

5.12.1 Regular tiling

The clearest way to illustrate each of the 11 topological types is through a 'regular tiling'. A regular tiling is defined by the properties at its vertices as follows: if v edges meet at a vertex of a tiling (that is, if the valence of the vertex is v) then the vertex is called *regular* if the angle between each consecutive pair of edges is $2\pi/v$ (Grünbaum and Shephard).[4] In other words, if the angle between each adjacent pair of edges joining at a vertex is the same (and this is a characteristic of every vertex in the tiling) then the tiling is regular.

It has been proved that, for monohedral tilings, the number of possible tiling structures satisfying this criteria is 11. They may be represented by what are referred to as the 'Laves tilings' which are illustrated in Fig. 5.31 (and named after the crystallographer Fritz Laves (see Grünbaum and Shephard,[4] and Engel[1]). There are two 'enantiomorphic' forms of [$3^4.6$], that is one is a reflection of the other in which, consequently, centres of rotation appear to be left and right orientated. In this context, and in general, they are regarded as being equivalent. This phenomenon does not occur in the other ten tilings because reflectional symmetry is present in their structures.

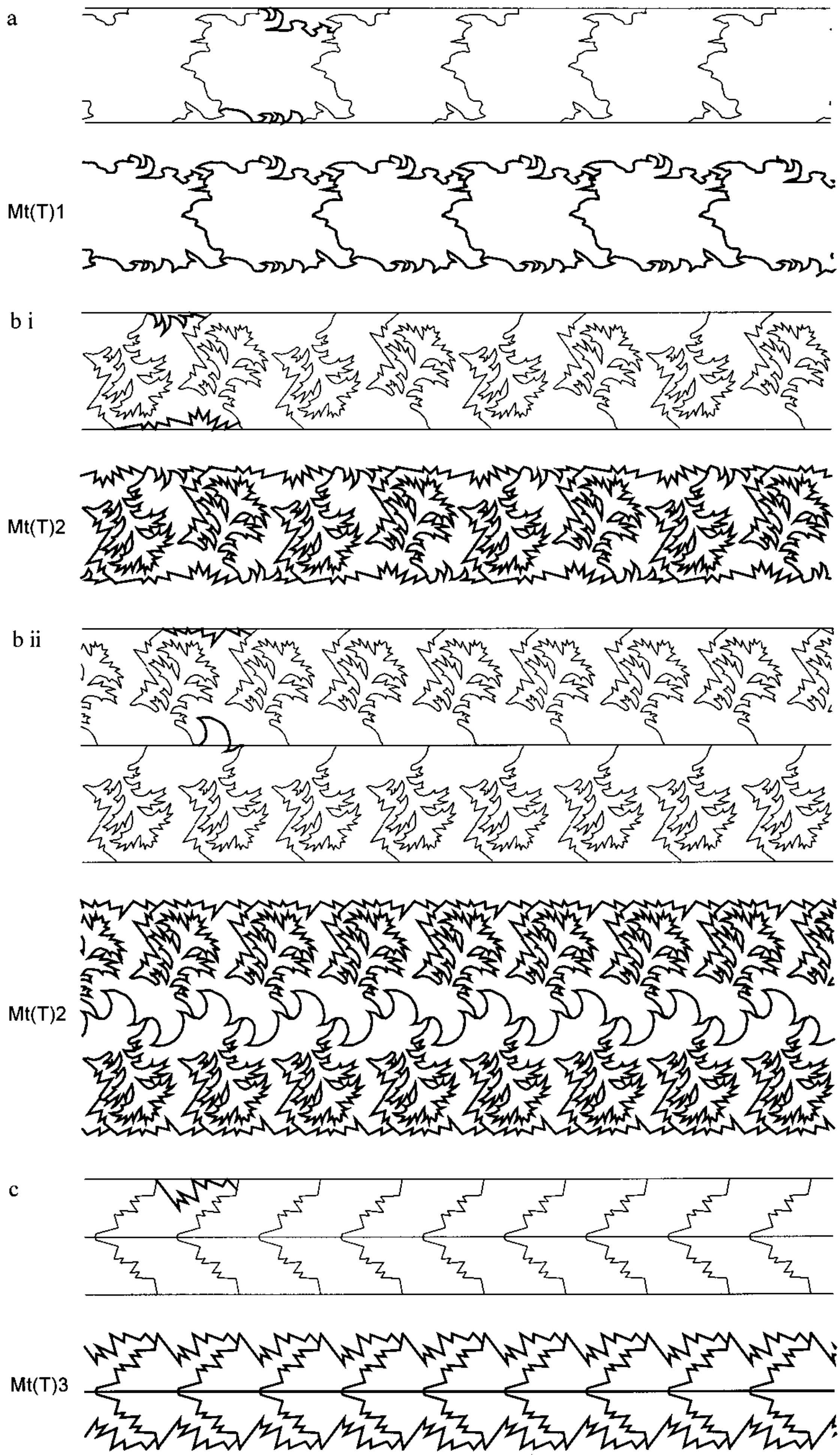


Figure 5.27 Construction of monotranslational tilings, induced group *c*1.

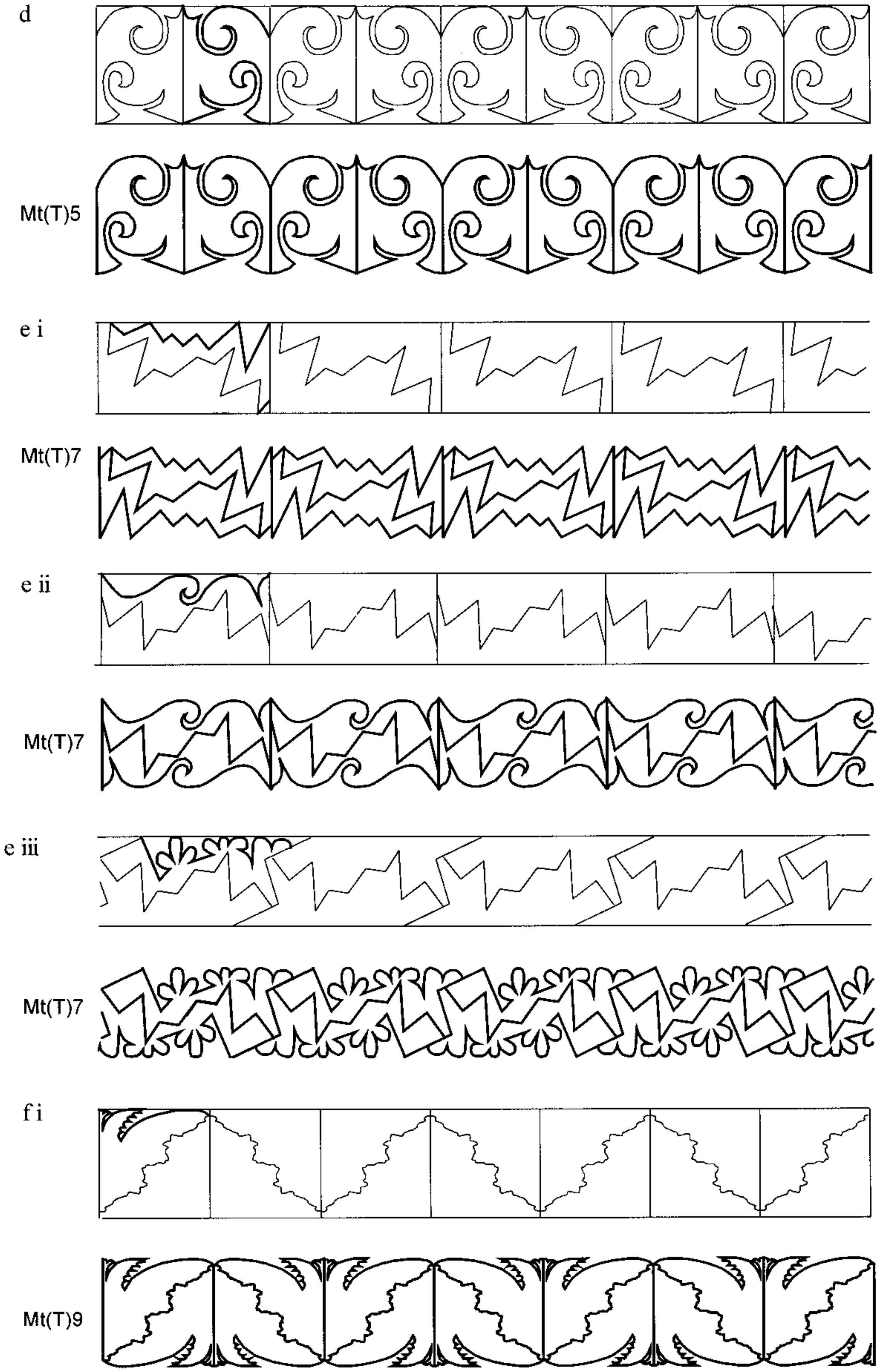


Figure 5.27 (cont.)

f ii

Mt(T)9

g i

Mt(T)12

g ii

Mt(T)12

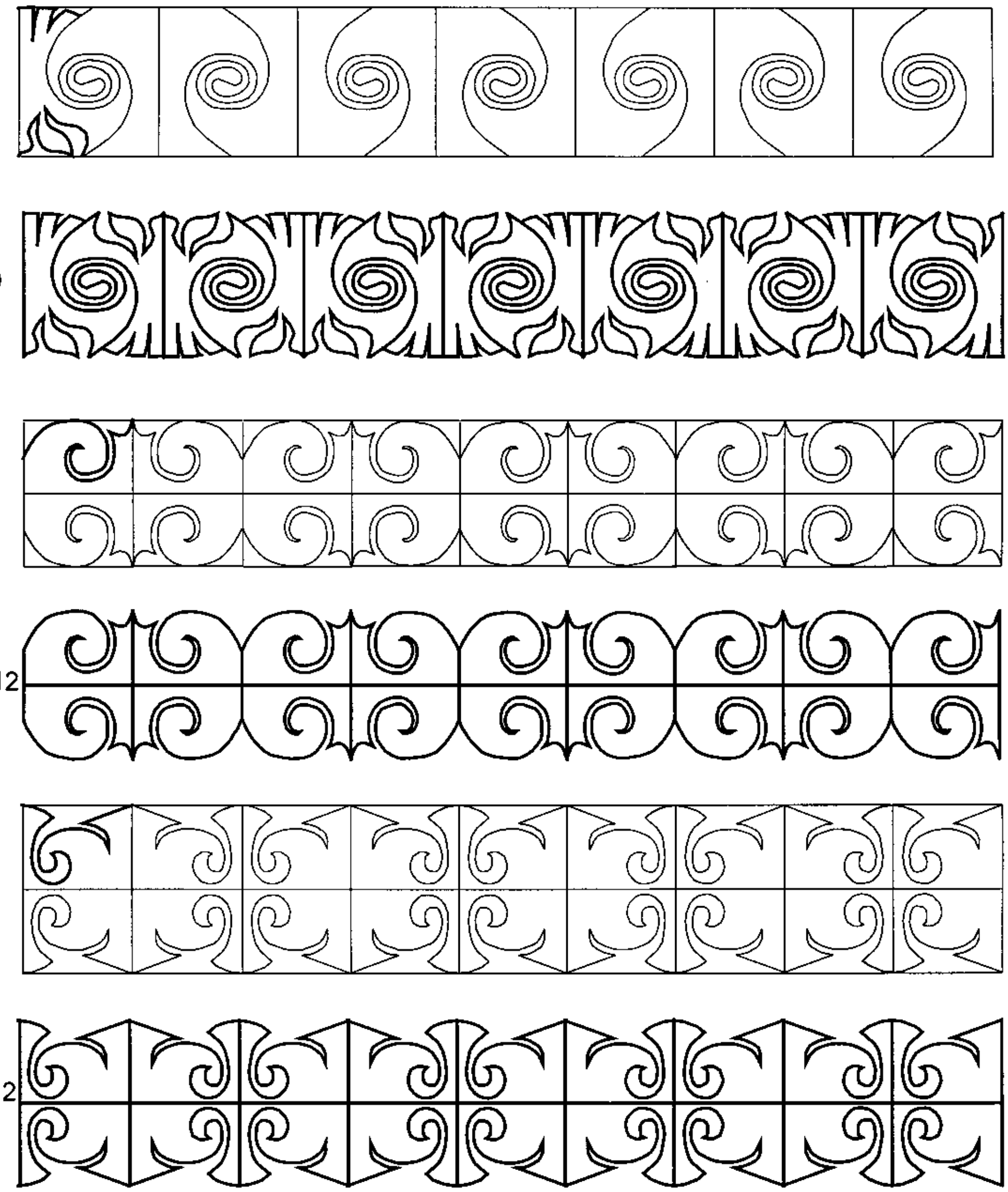


Figure 5.27 (cont.)

To aid the initial stages of metamorphosis of a Laves tiling into one of the required isohedral tiling types, the group diagram of the isohedral tiling under construction may be incorporated into its associated Laves tiling structure. In some instances there may be a number of options for the initial positioning of the group diagram since the symmetry group of the isohedral tiling being constructed usually forms a subgroup of the symmetry group of the Laves tiling upon which it is being superimposed. However, after analysing the incidence symbol, as shown in the examples below, it becomes evident how the edges relate to each other and consequently where the symmetries are positioned in the tiling structure. The induced group may also help to give an insight into the appearance of the final design.

This leaves the analysis and interpretation of the incidence symbol to determine the precise characteristics of the tiling. Some significant features of the incidence symbol were noted in Section 5.5.2 in connection with the classification of isohedral tilings. In the context of this book, it has been found that the most logical steps to follow in constructing these tilings are: first to establish the

Table 5.5 Implications of tile and adjacency letters and superscripts

Tile symbol letter and superscript	Adjacency symbol letter and superscript and relationship to tile symbol entry		Implication
x^+ or x^-	x^+	Same letter, positive superscript	The edge is mapped onto itself by two-fold rotational symmetry
x^+ or x^-	x^-	Same letter, negative superscript	The edge is mapped onto itself by reflectional symmetry
x^+ or x^-	y^+	Different letter, positive superscript	The edge x is mapped onto an edge y by rotational symmetry if x is next to y in the adjacency symbol and by translational symmetry it is not
x^+ or x^-	y^-	Different letter, negative superscript	The edge x is mapped onto edge y by glide–reflectional symmetry
x (no superscript)	x (no superscript	Same letter, no superscript	The edge x is mapped onto itself by two different perpendicular reflection axes (i.e. it is a straight line)
x (no superscript)	y (no superscript)	Different letter, no superscript	The edge x is mapped onto itself by reflectional symmetry and onto edge y by translational symmetry

number of possible different shaped edges by finding the number of different distinct mappings between tile and adjacency symbol; then to transform and label an edge of the Laves tiling which is mapped onto itself or a copy of itself, either by rotation or reflection, respectively (a letter in the tile symbol corresponding to the same letter, with a positive or negative superscript, in the adjacency symbol, respectively). This edge is then superimposed on the Laves tiling in all equivalent positions in the tiling by applying symmetries in the group diagram. (Of course, an edge may remain a straight line provided that it can only be mapped onto itself, or other edges inside a tile, by the symmetries implied by the incidence symbol and does not induce any extra symmetries into the design structure.) From this point, the relationships between edges adjacent to these edges, which are not mapped onto themselves, will result. For example, an edge in the tile symbol mapped onto a letter with a positive superscript in the adjacency symbol implies that either one edge is a translation of another or at one end of this edge there is a centre of n-fold rotation, depending on the symmetry group of the tiling structure. The value of n can be deduced from a unit cell incorporated into the Laves tiling. An edge in the tile symbol mapped onto a letter with a negative superscript in the adjacency symbol implies that this edge is a glide–reflection of another edge. Unless an edge is mapped onto itself by either rotation or reflection, the new edge, superimposed onto the Laves tiling, will be represented by an asymmetric line.

This analysis of incidence symbols, in association with the following techniques used to construct isohedral tilings, is summarised in Table 5.5. Construction methods and illustrations are described in detail for one example of each topological type. In each case, the incidence symbol has been displayed in a vertical format to aid the recognition of the relationships between edges.

5.12.2 *Topological type* $[3^6]$

There are 20 isohedral tiling types with topological type $[3^6]$: Dt(T)1 to Dt(T)20. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged for alternative ones because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)8 which has the following properties:

- Symmetry group: $p2$ $\quad b^+ \rightarrow b^+$
- Induced group: $c2$ $\quad c^+ \rightarrow c^+$

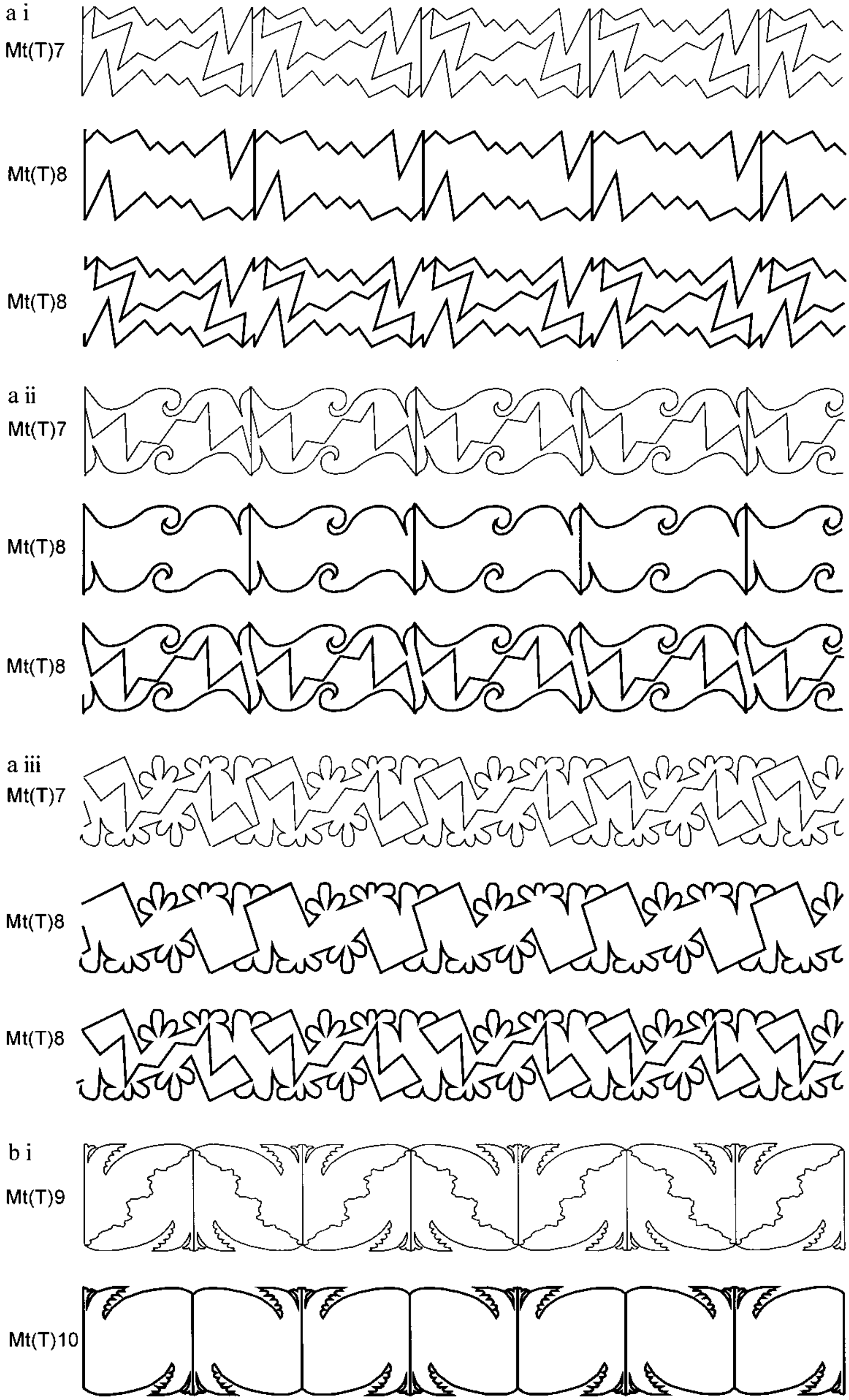
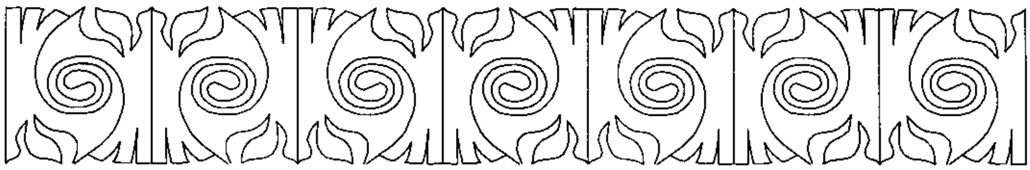


Figure 5.28 Construction of monotranslational tilings, induced group $c2$.

b ii

Mt(T)9



Mt(T)10

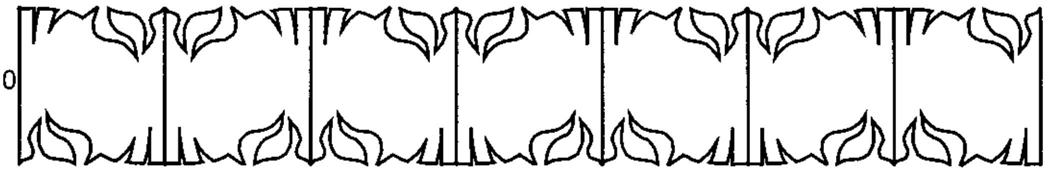


Figure 5.28 (cont.)

- Incidence symbol: [$a^+ b^+ c^+ a^+ b^+ c^+$; $a^+ b^+ c^+$] is written vertically as:

$a^+ \to a^+$
a^+
b^+
c^+

From the three distinct mappings in the incidence symbol ($a \to a$, $b \to b$ and $c \to c$) it is deduced that there may be up to three different shaped edges in the tiling. Since each of the edges 'a', 'b' and 'c' is mapped onto the same letter with a positive superscript, this implies that each one is mapped onto itself by two-fold rotational symmetry. Also, because each tile has six edges and the first and fourth, second and fifth, and third and sixth edges have the same labels, this implies that opposite edges have the same shape. By superimposing a group diagram of *p*2 onto the Laves tiling [3^6], it is obvious where centres of two-fold rotational symmetry coincide with points on the hexagonal lattice of edges (see Fig. 5.32). (Note that the symmetries of group diagram *p*2 form a subgroup of the symmetries of the Laves tiling [3^6] (symmetry group *p*6), so centres of three-fold rotation are not applicable and centres of six-fold rotation positioned at the centres of the hexagons are reduced to points of two-fold rotation.) One edge may be replaced by an alternative edge, having two-fold rotational symmetry, which is then mapped onto all equivalent positions in the tiling. One edge of a tile has this edge orientated and labelled 'a'.

The edges adjacent to the edge labelled 'a' are mapped onto themselves by two-fold rotational symmetry. One of them is replaced by another different line with two-fold rotational symmetry which is mapped to all its equivalent positions. The same operation is carried out for the remaining edge as shown in Fig. 5.32.

To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.3 *Topological type* [$3^4.6$]

Dt(T)21 is the only isohedral tiling type with topological type [$3^4.6$]. This implies that the Laves tiling with this topological type is already, in fact, Dt(T)21. However, it may still be transformed into one of the same type but having a less rigid appearance. Dt(T)21 has the following properties:

- Symmetry group: *p*6
- Induced group: *c*1
- Incidence symbol: [$a^+ b^+ c^+ d^+ e^+$; $e^+ c^+ b^+ d^+ a^+$] is written vertically as:

$b^+ \to c^+$
$c^+ \to b^+$
$a^+ \to e^+$
$d^+ \to d^+$
$e^+ \to a^+$

From the three distinct mappings in the incidence symbol ($a \to e$, $b \to c$ and $d \to d$) it is deduced that there may be up to three different shaped edges in the tiling. Since edge 'd' is mapped onto the same letter with a positive superscript, this

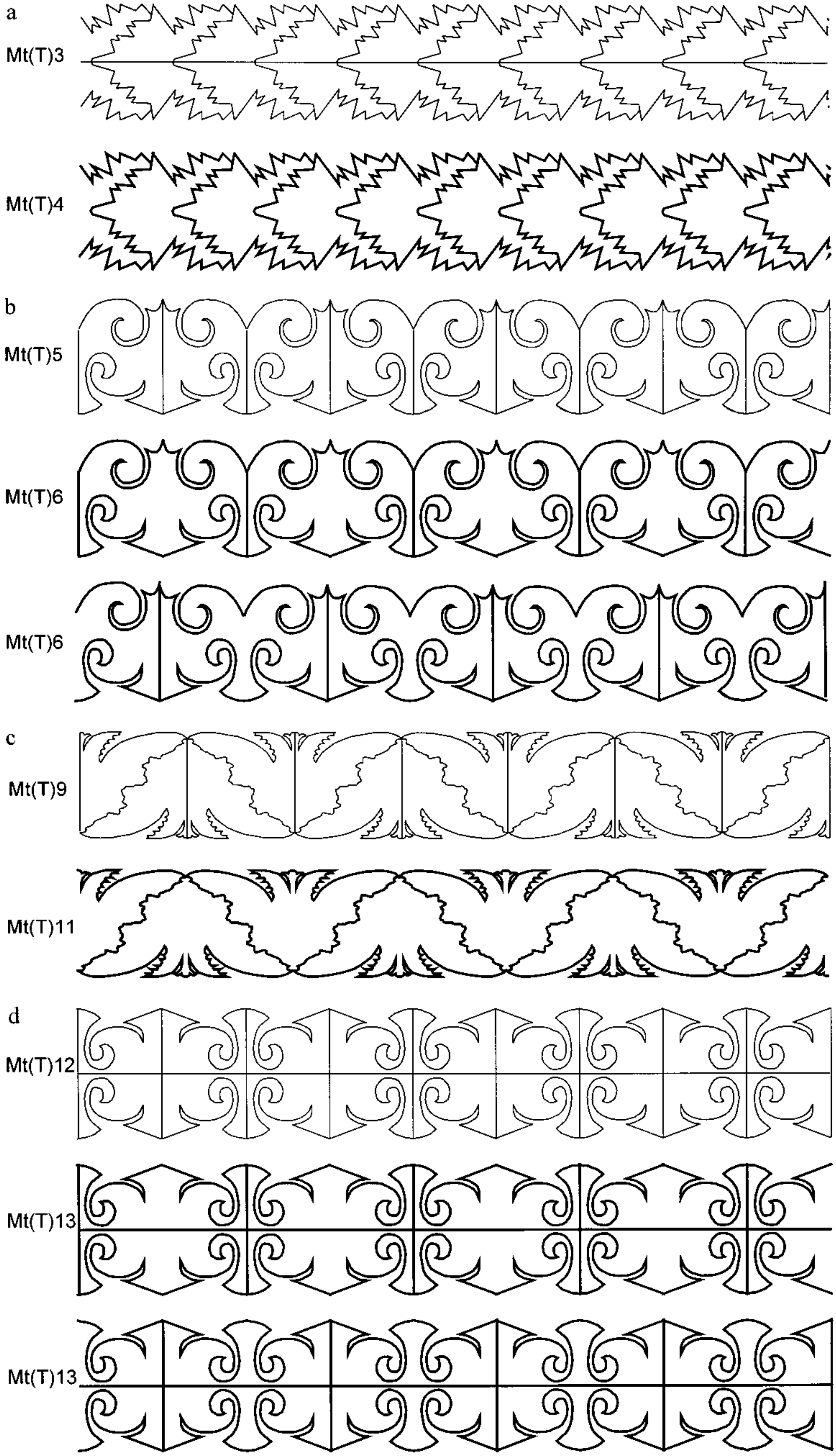


Figure 5.29 Construction of monotranslational tilings, induced group *d*1.

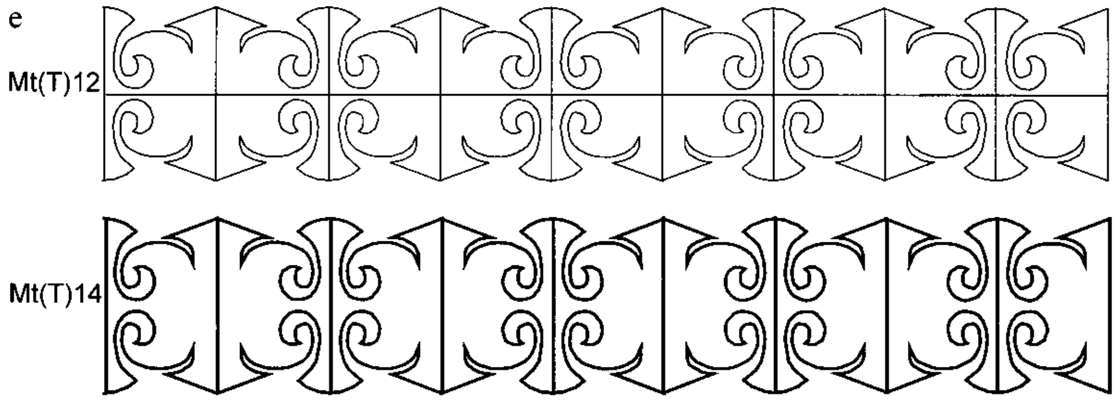


Figure 5.29 Construction of monotranslational tilings, induced group *d*1 (cont.)

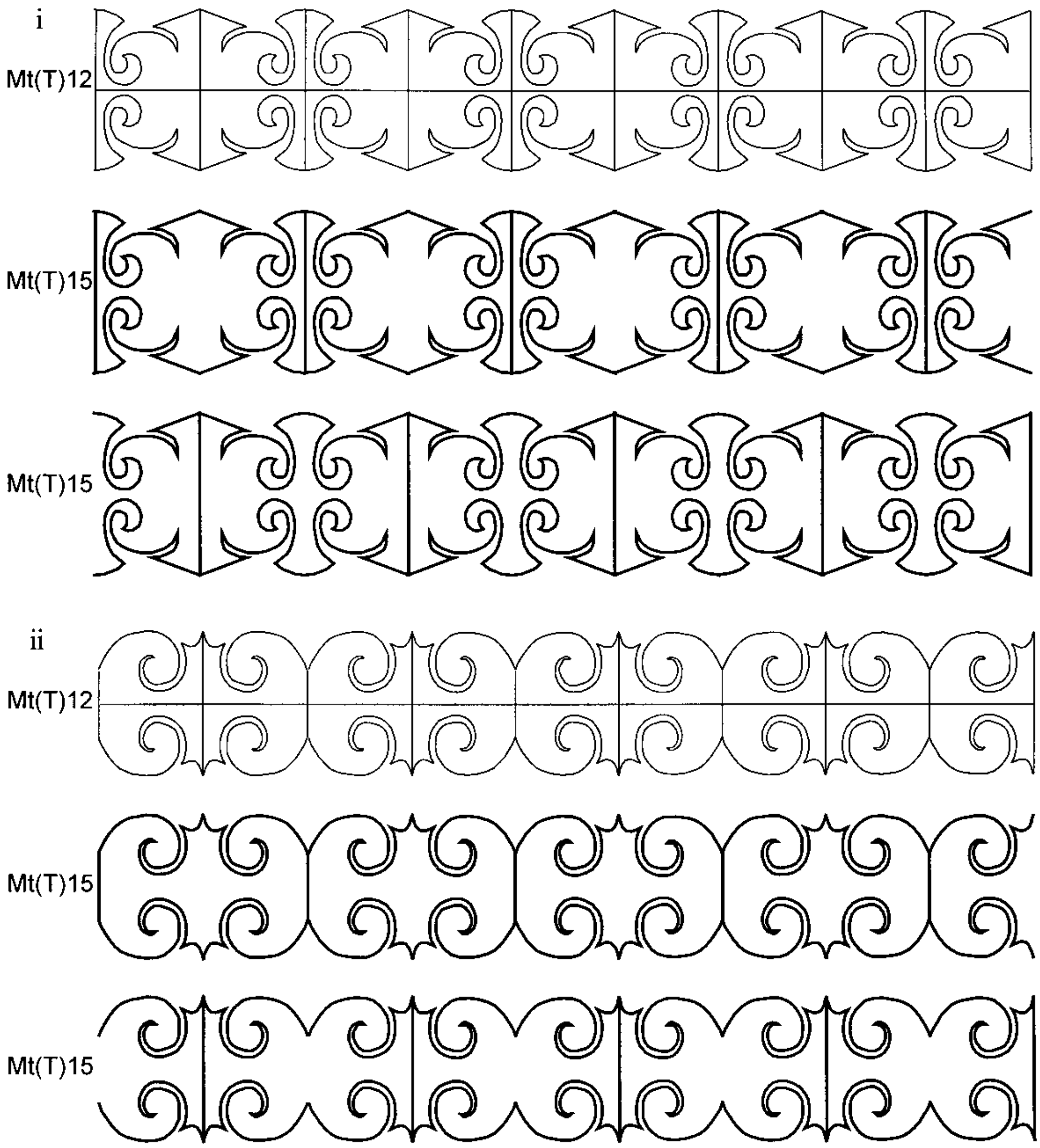


Figure 5.30 Construction of monotranslational tilings, induced group *d*2.

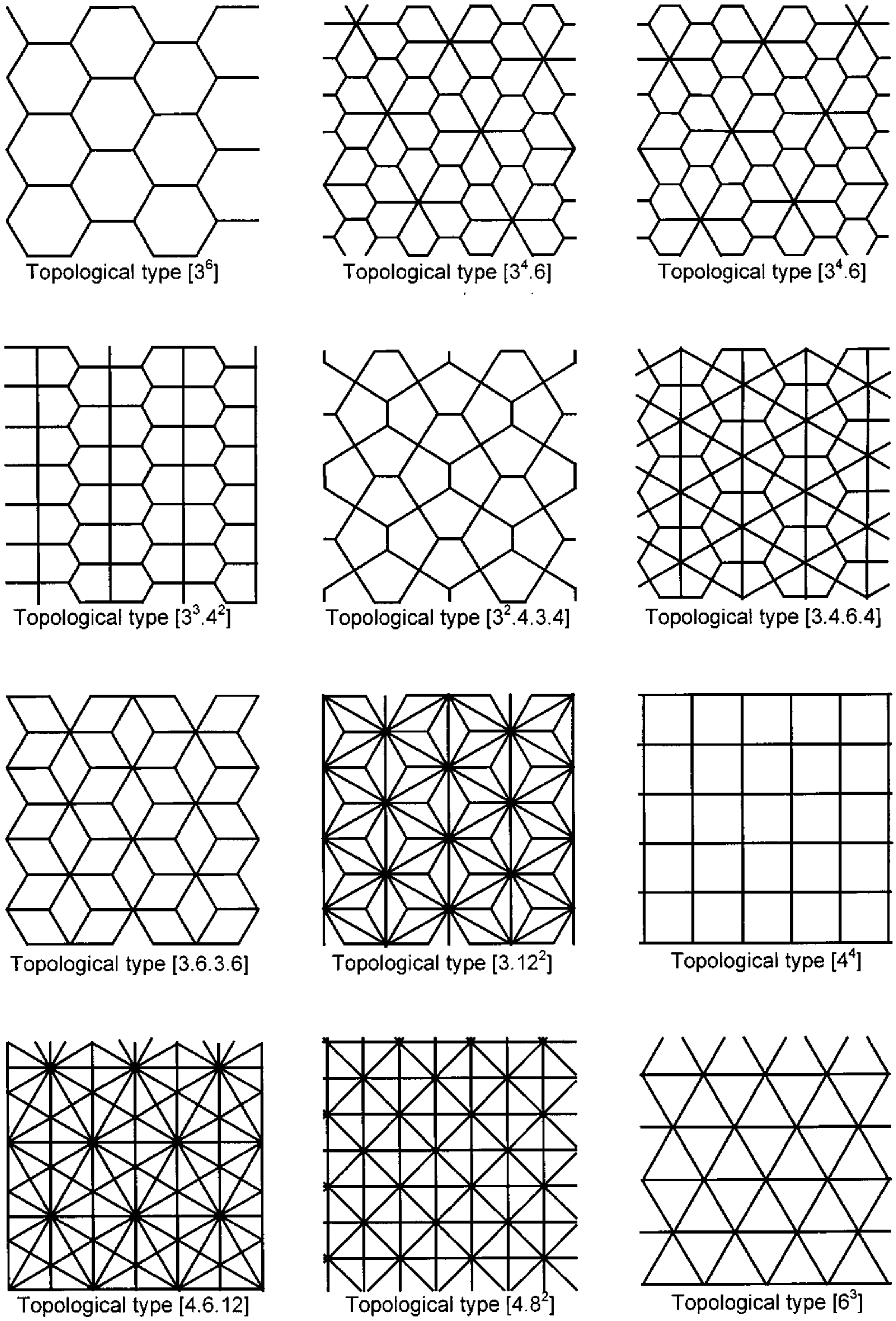


Figure 5.31 The 11 Laves tilings and their topological types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

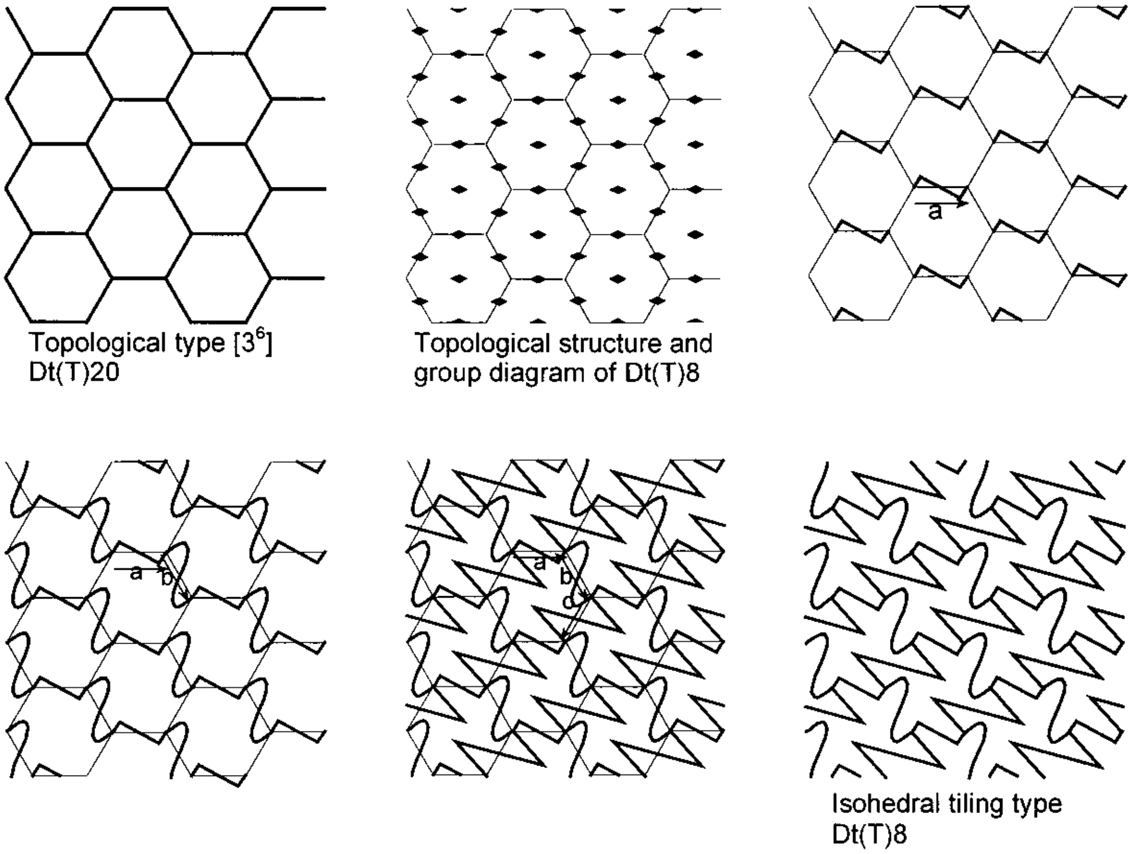


Figure 5.32 Construction of a ditranslational isohedral tiling, topological type [3^6].

implies that there is one edge which is mapped onto itself by two-fold rotational symmetry in common with the design structure. By superimposing a group diagram of *p*6 onto the Laves tiling [$3^4.6$], it is obvious which edge satisfies this criteria because there is only one edge passing through a centre of two-fold rotational symmetry of the unit cell (see Fig. 5.33). (The positioning of the symmetries of the group diagram are easily deduced by associating its six-fold centres of rotation with those occurring in the Laves tiling.) This edge may be replaced by an alternative edge, having two-fold rotational symmetry, which is then mapped onto all equivalent positions in the tiling. One edge of a tile has this edge orientated and labelled ‘d’.

The pairs of adjacent edges on either side of the edge labelled ‘d’ are mapped onto each other by rotational symmetry which may be deduced from the fact that c^+ and b^+, in the tile symbol, are mapped onto b^+ and c^+ in the adjacency symbol, and similarly for edges a^+ and e^+. Thus, the edges adjacent to the ones labelled ‘d’ may be exchanged for alternative ones which, again, are mapped onto the remainder of the tiling.

To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.4 *Topological type* [$3^3.4^2$]

There are five isohedral tiling types with topological type [$3^3.4^2$]: Dt(T)22 to Dt(T)26. The last of these gives the classification of the corresponding Laves tiling, although some of its edges may be exchanged. The discussion below gives an explanation of the construction of Dt(T)25 which has the following properties:

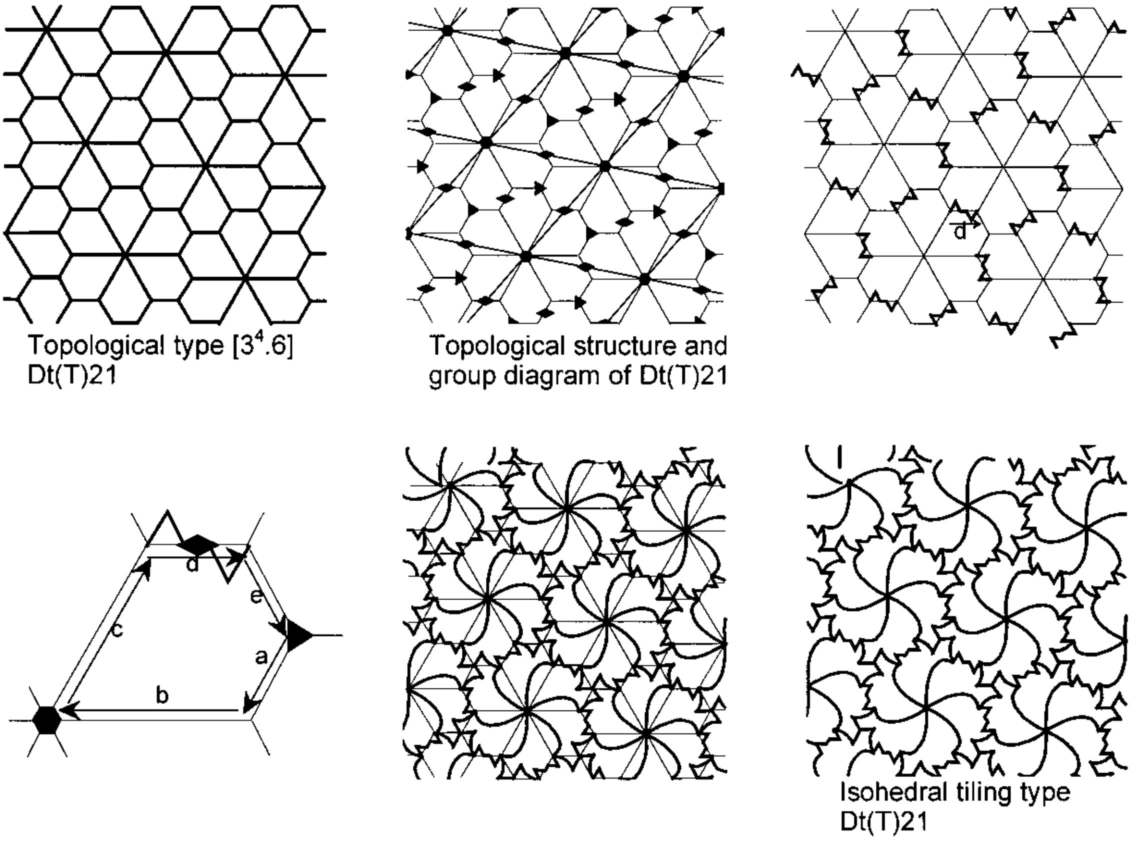


Figure 5.33 Construction of a ditranslational isohedral tiling, topological type [$3^4.6$].

- Symmetry group: *pgg*
- Induced group: *c*1
- Incidence symbol: [$a^+ b^+ c^+ d^+ e^+$; $a^+ e^+ d^- c^- b^+$] is written vertically as:

$$\begin{array}{l} b^+ \to e^+ \\ c^+ \to d^- \\ a^+ \to a^+ \\ d^+ \to c^- \\ e^+ \to b^+ \end{array}$$

The three distinct mappings in the incidence symbol (a → a, b → e and c → d) indicate that there may be up to three different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that one edge is mapped onto itself by two-fold rotational symmetry (edge 'a'), two adjacent edges are mapped onto each other by glide–reflectional symmetry (edges 'c' and 'd') and edges labelled 'b' and 'e' must be mapped onto each other by translational symmetry (rather than rotational symmetry) because they have positive superscripts in the adjacency symbol but do not follow consecutively. Illustration of the process of construction from this information is given in Fig. 5.34.

Adding the group diagram of *pgg* to the Laves tiling [$3^3.4^2$] establishes which edge is positioned on a centre of two-fold rotation. (Note that the symmetries of group diagram *pgg* form a subgroup of the symmetries of the Laves tiling [$3^3.4^2$] (symmetry group *cmm*), so centres of two-fold rotation positioned at the intersection of glide–reflection axes and reflection axes occurring in a *cmm* structure are not applicable in a *pgg* group diagram.) This edge may be exchanged for an alternative two-fold rotationally symmetric line and then mapped to all equivalent positions in the tiling. One of them is labelled 'a' and orientated. Similarly, after these mappings, the positioning of edges 'c' and 'd' becomes evident since, apart from the information displayed by the group diagram, these glide–reflectional symmetries occur on the second and third edges away from edge 'a'. This leaves the two remaining edges 'b' and 'e' which are translated onto each other (see Fig. 5.34).

To confirm that the tiling has been constructed correctly, the remaining edge

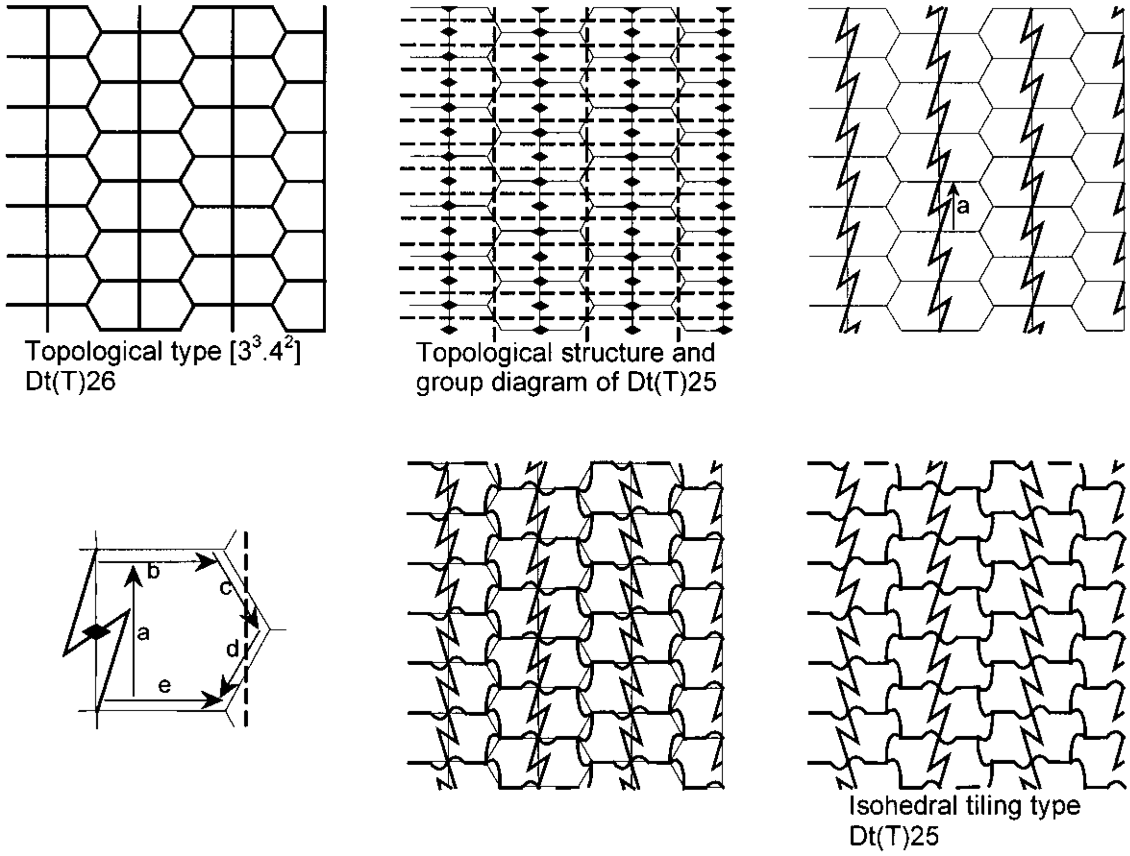


Figure 5.34 Construction of a ditranslational isohedral tiling, topological type [$3^3.4^2$].

labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.5 *Topological type* [3^2.4.3.4]

There are three isohedral tiling types with topological type [3^2.4.3.4]: Dt(T)27 to Dt(T)29. The last of these gives the classification of the corresponding Laves tiling, although some of its edges may be exchanged. The discussion below gives an explanation of the construction of Dt(T)27 which has the following properties:

- Symmetry group: *pgg*
- Induced group: *c*1
- Incidence symbol: [$a^+ b^+ c^+ d^+ e^+; a^+ d^- e^- b^- c^-$] is written vertically as:

$$\begin{array}{l} b^+ \to d^- \\ c^+ \to e^- \\ a^+ \to a^+ \\ d^+ \to b^- \\ e^+ \to c^- \end{array}$$

The three distinct mappings in the incidence symbol (a → a, b → d and c → e) indicate that there may be up to three different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that one edge is mapped onto itself by two-fold rotational symmetry (edge ‘a’), two sets of alternate edges are mapped onto themselves by glide–reflectional symmetry (edges ‘b’ and ‘d’ are mapped onto each other and edges ‘c’ and ‘e’ are mapped onto each other). Illustration of the process of construction from this information is given in Fig. 5.35.

Adding the group diagram of *pgg* to the Laves tiling [3^2.4.3.4] establishes which edge is positioned on a centre of two-fold rotation. This edge may be exchanged for an alternative two-fold rotationally symmetric line and then mapped to all equivalent positions in the tiling. One of them is labelled ‘a’ and orientated. Similarly, after these mappings, the positioning of edges ‘b’ and ‘d’

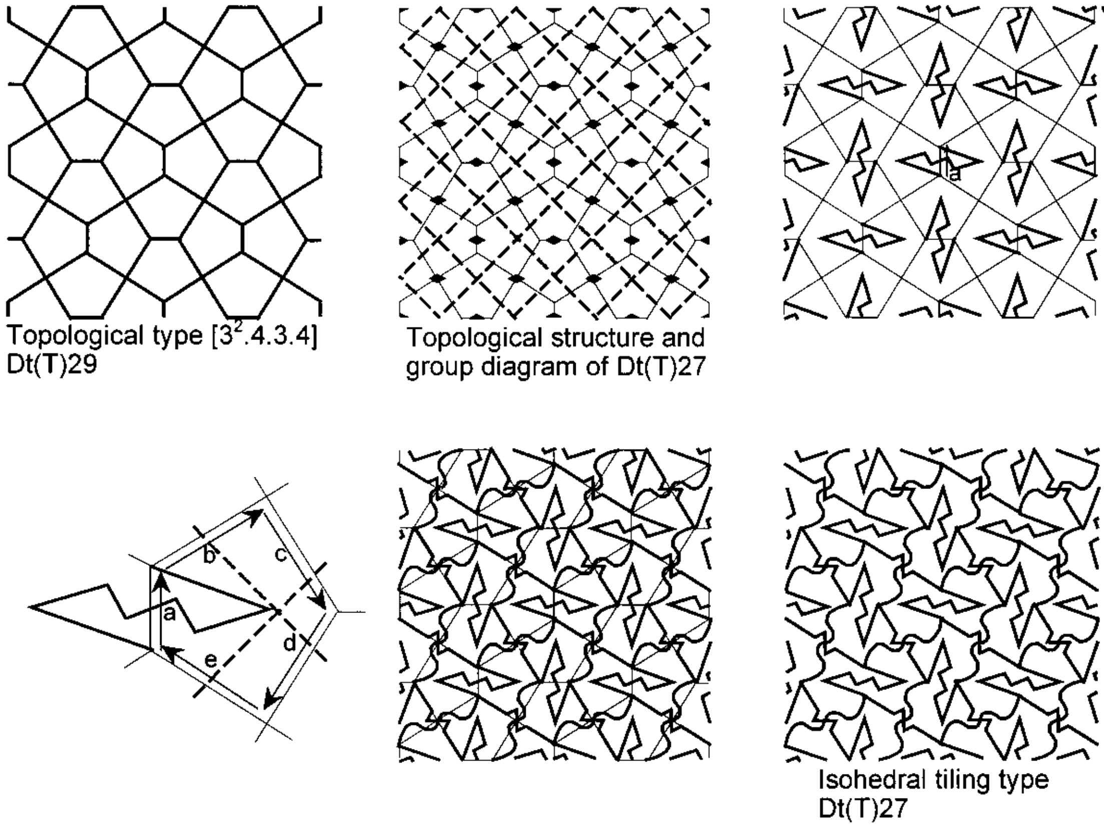


Figure 5.35 Construction of a ditranslational isohedral tiling, topological type [3^2.4.3.4].

becomes evident because, apart from the information displayed by the group diagram, these glide–reflectional symmetries occur on the first and third edges away from edge ‘a’. This leaves two remaining edges which must be glide reflected onto each other and labelled ‘c’ and ‘e’ in cyclic order (see Fig. 5.35).

To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.6 *Topological type* [3.4.6.4]

There are three isohedral tiling types with topological type [3.4.6.4]: Dt(T)30 to Dt(T)32. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)30 which has the following properties:

- Symmetry group: $p31m$
- Induced group: $c1$
- Incidence symbol: [$a^+ b^+ c^+ d^+$; $a^- b^- d^+ c^+$] is written vertically as:

$$\begin{array}{l} b^+ \to b^- \\ c^+ \to d^+ \\ a^+ \to a^- \\ d^+ \to c^+ \end{array}$$

The three distinct mappings in the incidence symbol ($a \to a$, $b \to b$ and $c \to d$) indicate that there may be up to three different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there are two adjacent edges which are mapped onto themselves by reflectional symmetry (edges ‘a’ and ‘b’). The other two adjacent edges are mapped onto each other by rotational symmetry (edges ‘c’ and ‘d’). Illustration of the process of construction from this information is given in Fig. 5.36.

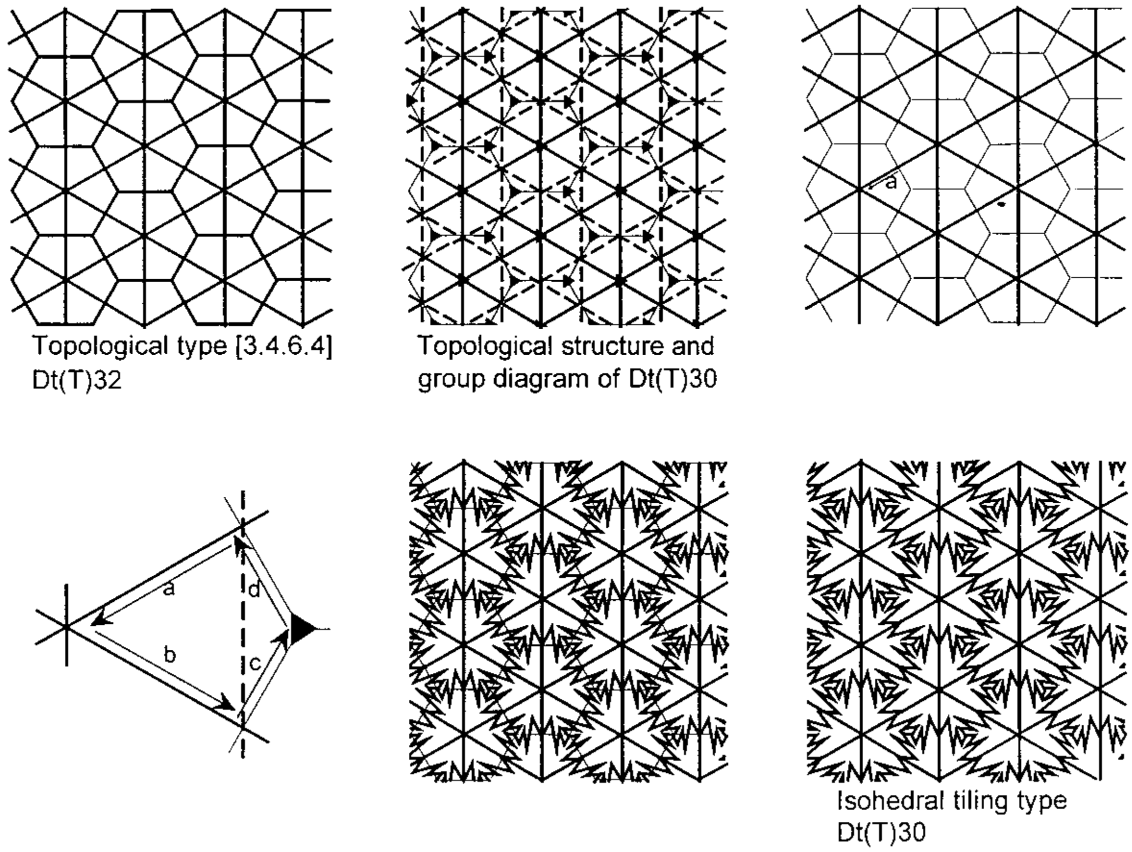


Figure 5.36 Construction of a ditranslational isohedral tiling, topological type [3.4.6.4].

5.12.7 Topological type [3.6.3.6]

There are five isohedral tiling types with topological type [3.6.3.6]: Dt(T)33 to Dt(T)37. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)33 which has the following properties:

- Symmetry group: $p3$
- Induced group: $c1$
- Incidence symbol: [$a^+ b^+ c^+ d^+$; $d^+ c^+ b^+ a^+$] is written vertically as:

$b^+ \rightarrow c^+$
$c^+ \rightarrow b^+$
$a^+ \rightarrow d^+$
$d^+ \rightarrow a^+$

The two distinct mappings in the incidence symbol (a $\rightarrow$ d and b $\rightarrow$ c) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there are two adjacent edges, 'a' and 'd', which are mapped onto each other by rotational symmetry followed by adjacent edges, 'b' and 'c', which are also mapped onto each other by rotational symmetry. The illustration of the process of construction, from this information, is given in Fig. 5.37.

5.12.8 Topological type [3.12^2]

There are three isohedral tiling types with topological type [3.12^2]: Dt(T)38 to Dt(T)40. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)38 which has the following properties:

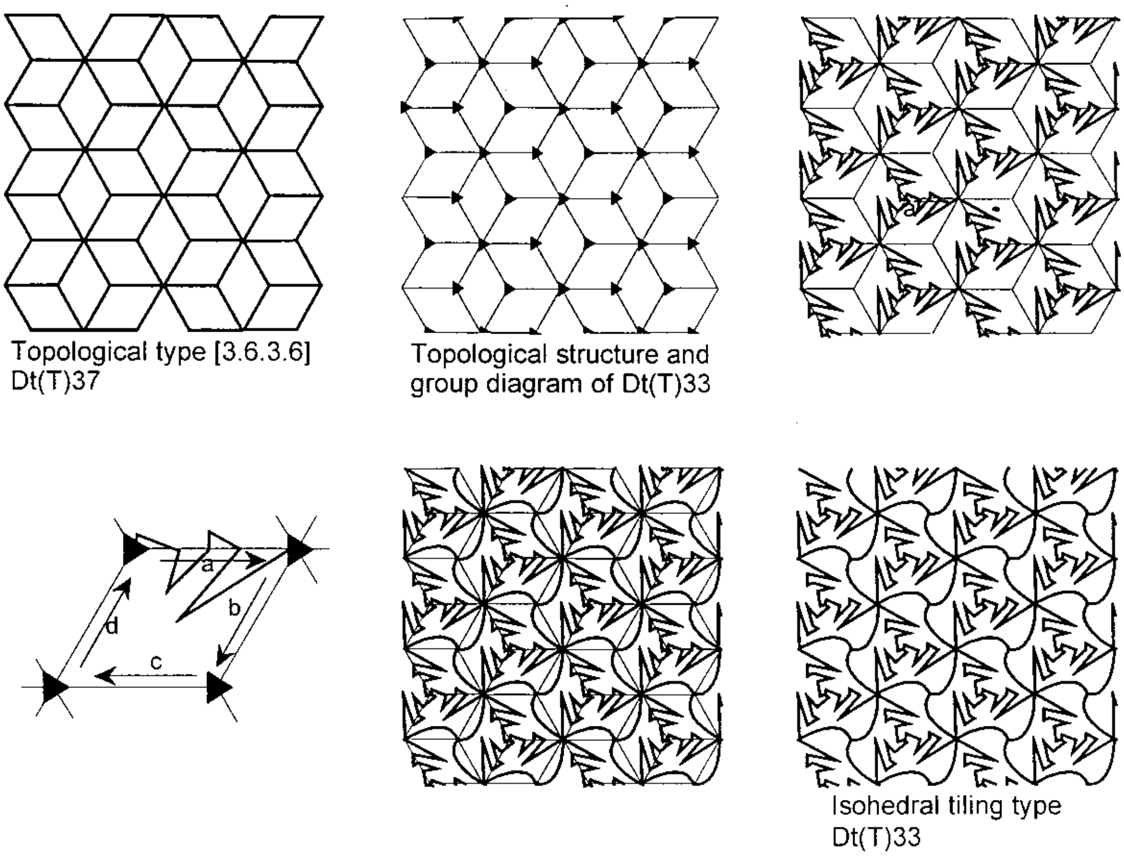


Figure 5.37 Construction of a ditranslational isohedral tiling, topological type [3.6.3.6].

- Symmetry group: $p31m$
- Induced group: $c1$
- Incidence symbol: [$a^+ b^+ c^+$; $a^- c^+ b^+$] is written vertically as:

$b^+ \to c^+$
$c^+ \to b^+$
$a^+ \to a^-$

The two distinct mappings in the incidence symbol (a → a and b → c) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there is one edge, ‘a’, which is mapped onto itself by reflectional symmetry followed by adjacent edges, ‘b’ and ‘c’, which are mapped onto each other by rotational symmetry. The illustration of the process of construction, from this information, is given in Fig. 5.38.

5.12.9 *Topological type* [4^4]

There are 36 isohedral tiling types with topological type [4^4]: Dt(T)41 to Dt(T)76. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged for alternative ones because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)71 which has the following properties:

- Symmetry group: $p4g$
- Induced group: $d1$
- Incidence symbol: [$a^+ b^+ b^- a^-$; $b^+ a^+$] is written vertically as:

$b^+ \to a^+$
b^-
$a^+ \to b^+$
a^-

From the one distinct mapping in the incidence symbol (a → b) it is deduced that each edge has the same shape. Since each edge ‘a’ is mapped onto edge ‘b’ with a positive superscript (and vice versa), this implies that one is mapped onto the other by rotational symmetry about a centre of rotation at a mutual end point of these edges. By superimposing a group diagram of $p4g$ onto the Laves tiling 4^4, it is obvious where centres of four-fold rotational symmetry coincide with points, at the ends of edges, on the square lattice (see Fig. 5.39). One edge may be replaced

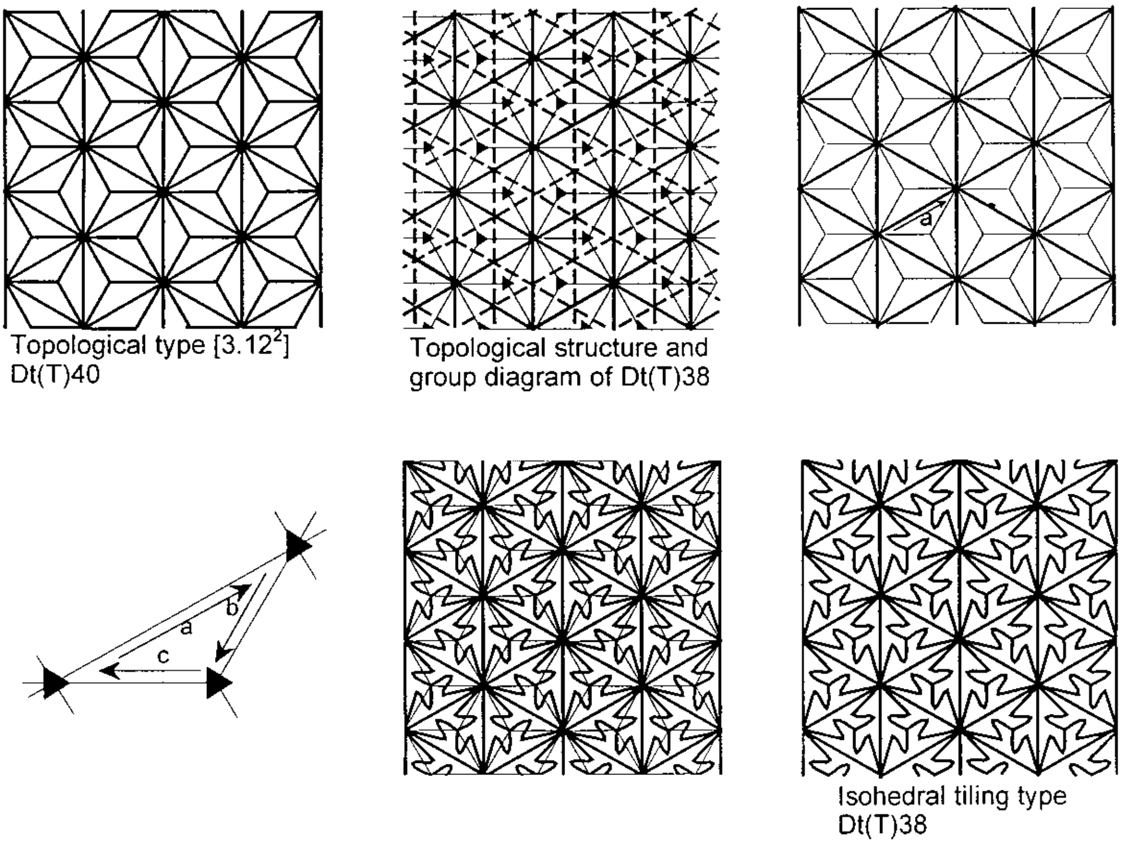


Figure 5.38 Construction of a ditranslational isohedral tiling, topological type [3.12^2].

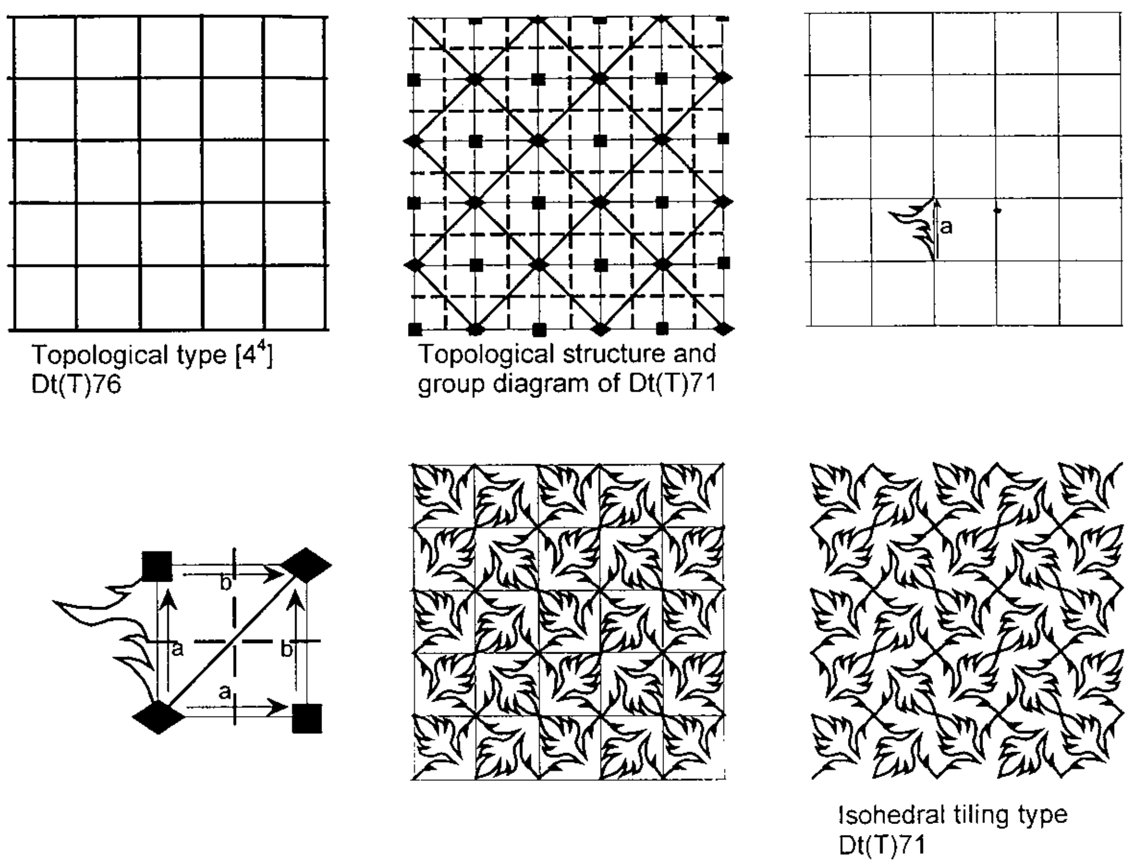
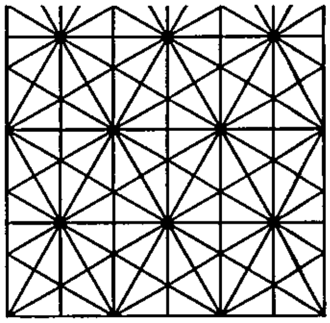


Figure 5.39 Construction of a ditranslational isohedral tiling, topological type [4^4].



Topological type [4.6.12]
Dt(T)77

Figure 5.40 Construction of a ditranslational isohedral tiling, topological type [4.6.12].

by an alternative edge which is then mapped onto all equivalent positions in the tiling by applying the symmetries of the group diagram. One edge of a tile has this edge orientated and labelled ‘a’. To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.10 Topological type [4.6.12]

Dt(T)77 is the only isohedral tiling type with topological type [4.6.12]. This implies that the Laves tiling with this topological type is already, in fact, Dt(T)77. Its edges may not be exchanged for alternative ones because each one in a tile is mapped onto itself by reflectional symmetry only. The properties and illustration of this tiling are given in Table 5.4 and Fig. 5.40, respectively.

5.12.11 Topological type [4.8^2]

There are five isohedral tiling types with topological type [4.8^2]: Dt(T)78 to Dt(T)82. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)81 which has the following properties:

• Symmetry group: *p*4*g*	$b^+ \to c^+$
• Induced group: *c*1	$c^+ \to b^+$
• Incidence symbol: [$a^+ b^+ c^+; a^- c^+ b^+$], written vertically as:	$a^+ \to a^-$

The two distinct mappings in the incidence symbol ($a \to a$ and $b \to c$) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there is one edge, ‘a’, which is mapped onto itself by reflectional symmetry followed by adjacent edges, ‘b’ and ‘c’, which are mapped onto each other by rotational symmetry. Illustration of the process of construction from this information is given in Fig. 5.41.

5.12.12 Topological type [6^3]

There are 11 isohedral tiling types with topological type [6^3]: Dt(T)83 to Dt(T)93. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)88 which has the following properties:

• Symmetry group: *p*6	$b^+ \to a^+$
• Induced group: *c*1	$c^+ \to c^+$
• Incidence symbol: [$a^+ b^+ c^+; b^+ a^+ c^+$] is written vertically as:	$a^+ \to b^+$

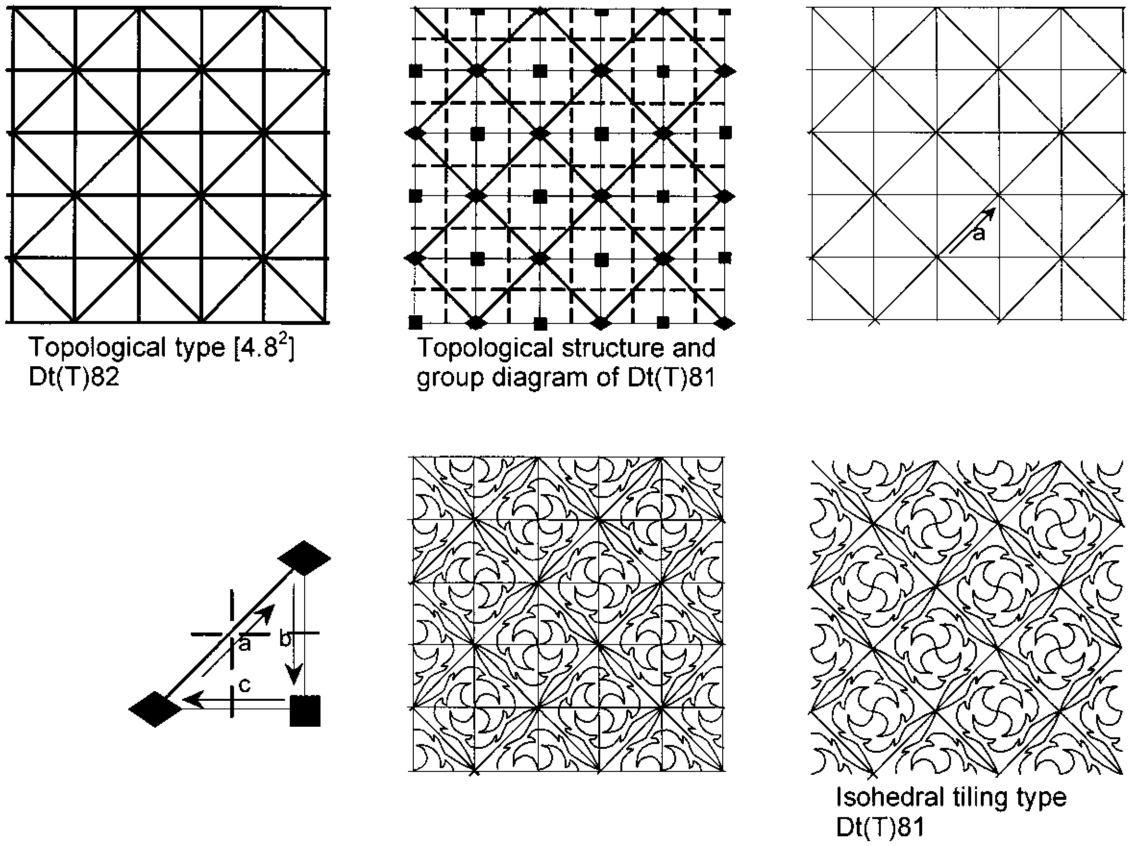


Figure 5.41 Construction of a ditranslational isohedral tiling, topological type $[4.8^2]$.

The two distinct mappings in the incidence symbol (a → b and c → c) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there is one edge, 'c', which is mapped onto itself by two-fold rotational symmetry followed by adjacent edges, 'a' and 'b', which are mapped onto each other by rotational symmetry. Illustration of the process of construction from this information is given in Fig. 5.42.

5.12.13 Marked isohedral tiling types

The techniques used to construct marked ditranslational isohedral tiling designs are similar to those for the unmarked tilings but less involved. Each of these types of tiling consists of a discrete pattern enclosed within a tiling. The construction of the tiling is straightforward because each one is an unmodified Laves tiling. The marking involves incorporating the appropriate discrete pattern into the tiling (listed in Table 5.4) such that each tile contains one motif. The positioning of the motifs within the tiling should be fairly obvious. However, if not, it may be derived by the following technique: one motif is placed within a tile such that the tile induced group is satisfied (see Table 5.4 to evaluate the tiling's induced group). All the edges of the tiles, enclosing the discrete pattern, fall on reflection axes. Provided that the initial motif is positioned correctly, it may be mapped to all its equivalent positions by applying these reflectional symmetries which coincide with the tile boundaries. For example, consider isohedral tiling type Dt(T)70 which has the following properties:

- Topological type: $[4^4]$
- Symmetry group: $p4m$
- Induced group: $d1(1)$
- Incidence symbol: $[a^+ b^+ b^- a^-; a^- b^-]$ is written vertically as:

$$\begin{array}{l} b^+ \rightarrow b^- \\ b^- \\ a^+ \rightarrow a^- \\ a^- \end{array}$$

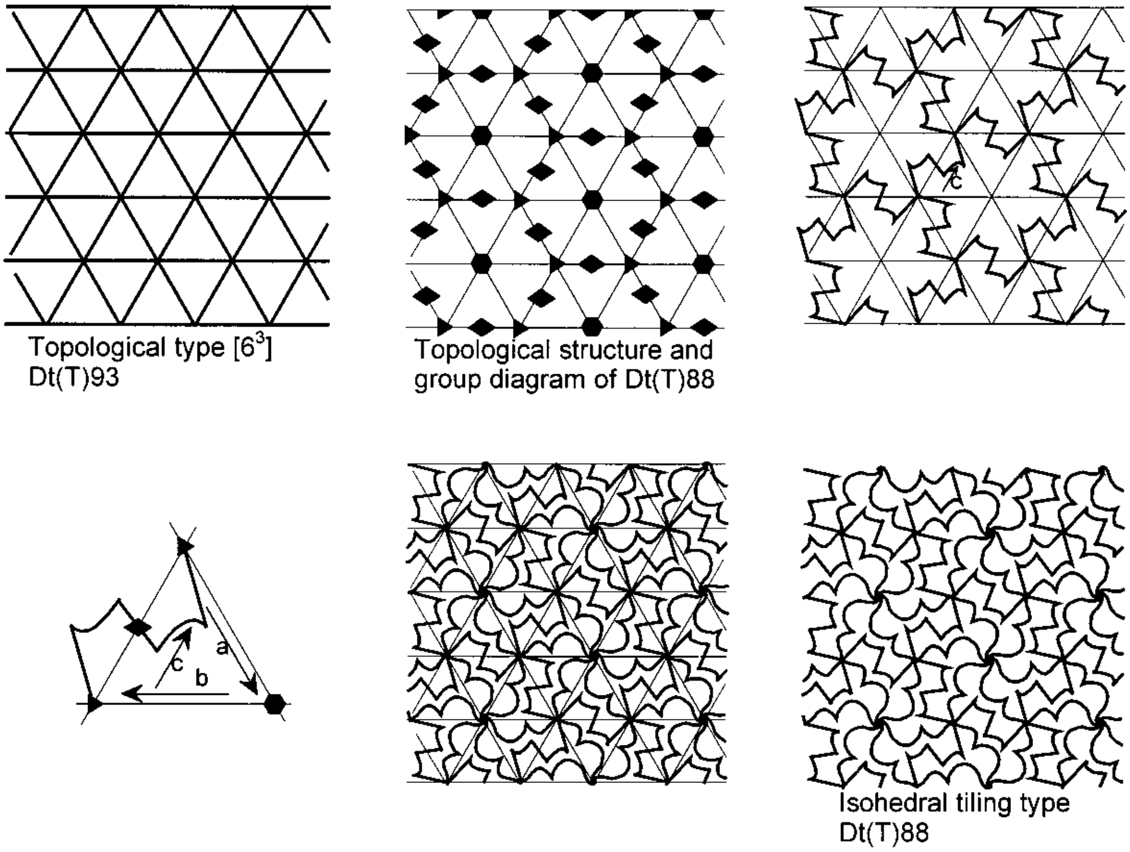


Figure 5.42 Construction of a ditranslational isohedral tiling, topological type [6^3].

The unmarked isohedral tiling associated with Dt(T)70 is Dt(T)76, the square tiling (or Laves tiling with topological type [4^4]). A motif may be added to one tile such that it reduces the tile induced group from *d*4 to *d*1. In this instance, the reflection axes of the motif must coincide with the 'longest' reflection axes inside a square tile (as opposed to the 'short' ones parallel to the sides). This motif may then be mapped onto the remaining tiles in the tiling by applying the reflectional symmetries occurring on the boundaries of the tiles (see Fig. 5.43).

5.13 Summary

Throughout this chapter a classification system has been developed which incorporates finite and monotranslational tiling designs. Notation has been devised to represent these different categories of tiling, and construction techniques have been described and illustrated. The characteristics and classification of ditranslational isohedral tilings have also been defined, explained and extensively illustrated.

The methods described for the construction of ditranslational isohedral tilings give a simple comprehensive procedure in which to create each of the 93 tiling types. However, they do not provide a generalised technique for the construction of all forms of isohedral tiling because the method has been based upon the operation of edge replacement where the vertices of the derived isohedral tiling remain in the same positions as those of the corresponding Laves tiling.

In the majority of cases, the positioning of the vertices is dictated by the symmetries of the design structure, topological type and incidence symbol. However, in some instances certain features of the associated Laves tiling may be altered to accommodate a wider variety of isohedral tilings within one isohedral tiling type. For example, Fig. 5.44(a(iii)) illustrates a ditranslational tiling, topological type [3^6], constructed by methods described previously. Figure 5.44(b(iii)) also illustrates a ditranslational tiling of exactly the same isohedral tiling type but the

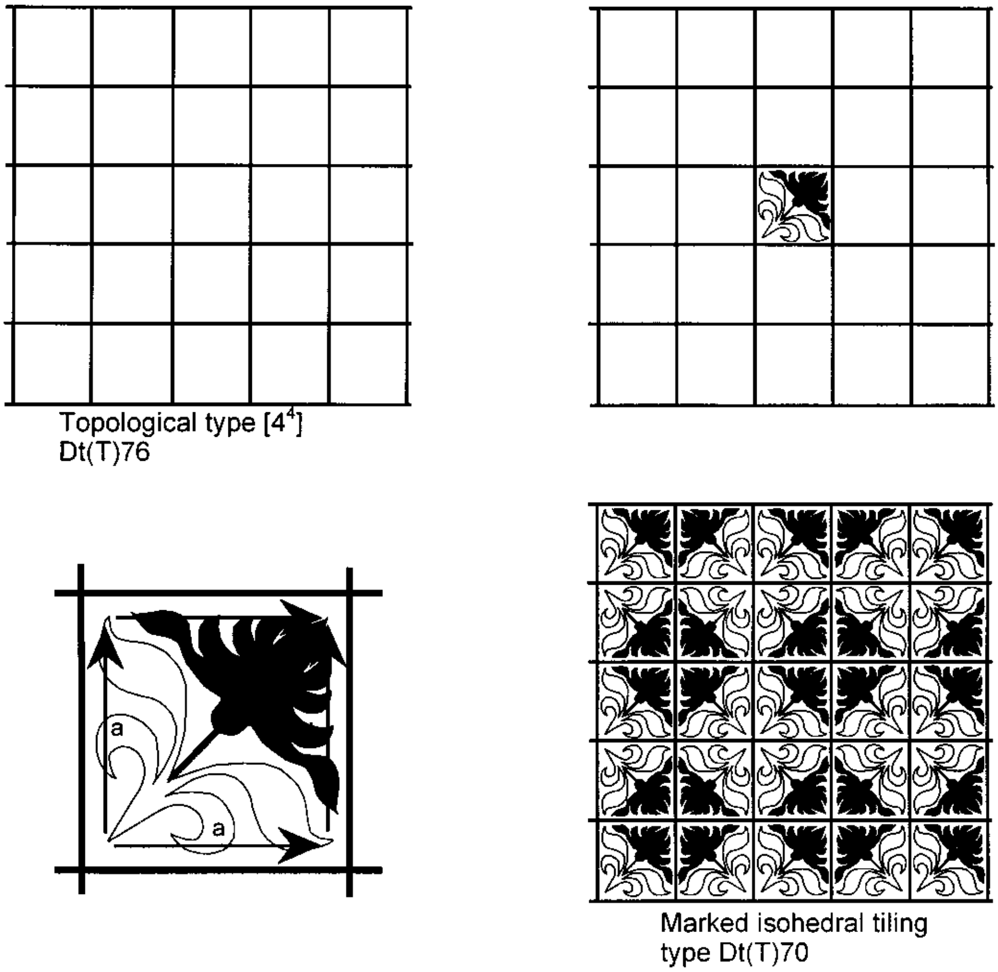


Figure 5.43 Construction of a ditranslational isohedral tiling, topological type [4^4].

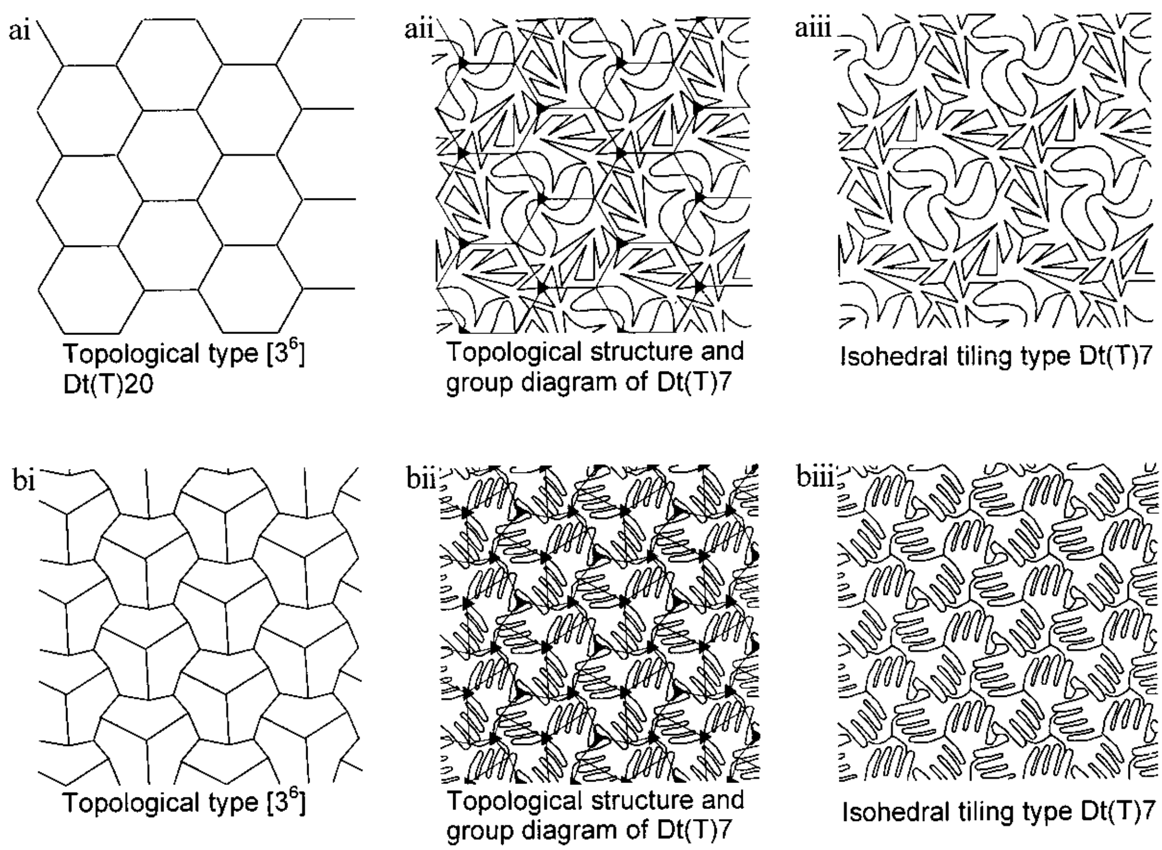


Figure 5.44 Examples of ditranslational tilings with the same isohedral tiling type but different vertex positions.

underlying simplified representation of its topological structure (derived by replacing each edge with a straight line whilst retaining the same positions for the vertices) does not correspond to a Laves tiling. Thus, the range of tilings which may be constructed within one isohedral tiling type extends beyond the methods discussed in this chapter. This extension would require further analysis and explanation of the properties of tilings. As a consequence, because there is already a vast range of tilings which may be constructed within each tiling type (owing to the variety of choice of lines used to replace the edges of the Laves tilings) further construction techniques will not be discussed in this book.

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