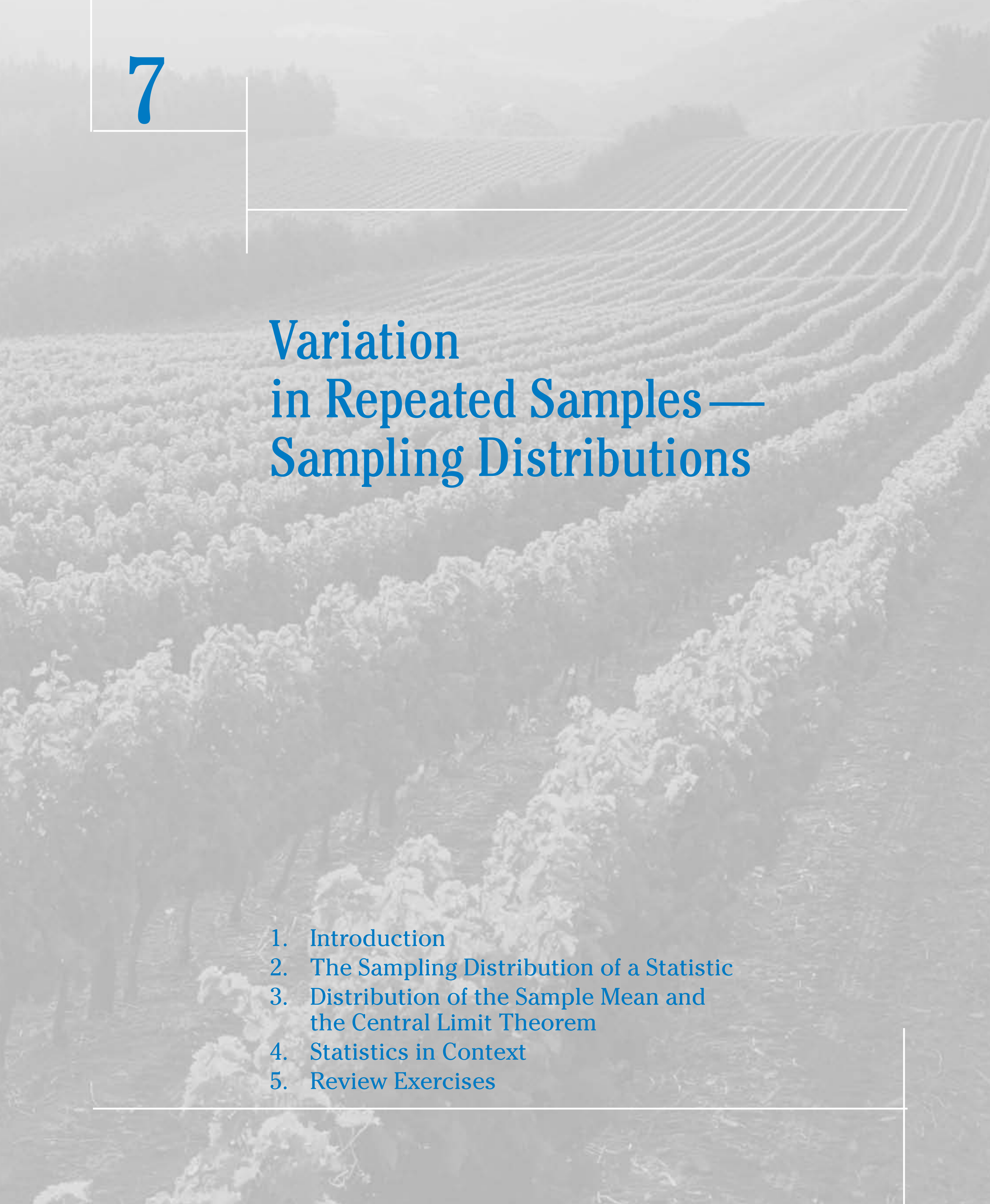
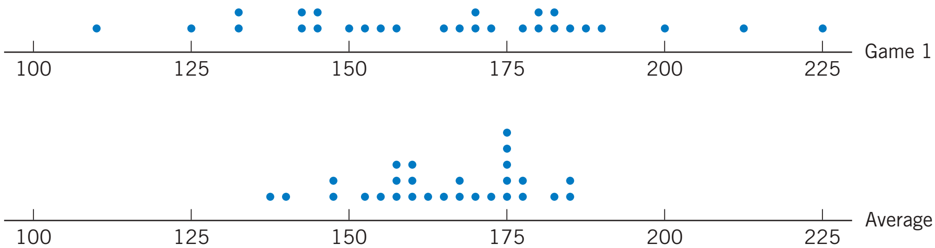


7

Variation in Repeated Samples — Sampling Distributions

Bowling Averages

A bowler records individual game scores and the average for a three-game series.



Bowlers are well aware that their three-game averages are less variable than their single-game scores. Sample means always have less variability than individual observations. Spike Mafford/Alamy.

1. INTRODUCTION

At the heart of statistics lie the ideas of inference. They enable the investigator to argue from the particular observations in a sample to the general case. These generalizations are founded on an understanding of the manner in which variation in the population is transmitted, by sampling, to variation in statistics like the sample mean. This key concept is the subject of this chapter.

Typically, we are interested in learning about some numerical feature of the population, such as the proportion possessing a stated characteristic, the mean and standard deviation of the population, or some other numerical measure of center or variability.

A numerical feature of a population is called a **parameter.**

The true value of a population parameter is an unknown constant. It can be correctly determined only by a complete study of the population. The concepts of statistical inference come into play whenever this is impossible or not practically feasible.

If we only have access to a sample from the population, our inferences about a parameter must then rest on an appropriate sample-based quantity. Whereas a parameter refers to some numerical characteristic of the population, a sample-based quantity is called a **statistic.**

A **statistic** is a numerical valued function of the sample observations.

For example, the sample mean

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n}$$

is a statistic because its numerical value can be computed once the sample data, consisting of the values of $X_1, \ldots, X_n$, are available. Likewise, the sample median and the sample standard deviation are also sample-based quantities so each is a statistic. Note that every statistic is a random variable. A sample-based quantity (statistic) must serve as our source of information about the value of a parameter. Three points are crucial:

1. Because a sample is only a part of the population, the numerical value of a statistic cannot be expected to give us the exact value of the parameter.

2. The observed value of a statistic depends on the particular sample that happens to be selected.
3. There will be some variability in the values of a statistic over different occasions of sampling.

A brief example will help illustrate these important points. Suppose an urban planner wishes to study the average commuting distance of workers from their home to their principal place of work. Here the statistical population consists of the commuting distances of all the workers in the city. The mean of this finite but vast and unrecorded set of numbers is called the population mean, which we denote by μ. We want to learn about the parameter μ by collecting data from a sample of workers. Suppose 80 workers are randomly selected and the (sample) mean of their commuting distances is found to be $\bar{x} = 8.3$ miles. Evidently, the population mean μ cannot be claimed to be exactly 8.3 miles. If one were to observe another random sample of 80 workers, would the sample mean again be 8.3 miles? Obviously, we do not expect the two results to be identical. Because the commuting distances do vary in the population of workers, the sample mean would also vary on different occasions of sampling. In practice, we observe only one sample and correspondingly a single value of the sample mean such as $\bar{x} = 8.3$. However, it is the idea of the variability of the $\bar{x}$ values in repeated sampling that contains the clue to determining how precisely we can hope to determine μ from the information on $\bar{x}$.

2. THE SAMPLING DISTRIBUTION OF A STATISTIC

The fact that the value of the sample mean, or any other statistic, will vary as the sampling process is repeated is a key concept. Because any statistic, the sample mean in particular, varies from sample to sample, it is a random variable and has its own probability distribution. The variability of the statistic, in repeated sampling, is described by this probability distribution.

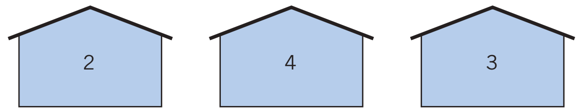
> The probability distribution of a statistic is called its **sampling distribution.**

The qualifier "sampling" indicates that the distribution is conceived in the context of repeated sampling from a population. We often drop the qualifier and simply say the **distribution of a statistic.**

Although in any given situation, we are limited to one sample and the corresponding single value for a statistic, over repeated samples from a population the statistic varies and has a sampling distribution. The sampling distribution of a statistic is determined from the distribution $f(x)$ that governs the population, and it also depends on the sample size n. Let us see how the distribution of $\overline{X}$ can be determined in a simple situation where the sample size is 2 and the population consists of 3 units.

Example 1 Illustration of a Sampling Distribution

A population consists of three housing units, where the value of X, the number of rooms for rent in each unit, is shown in the illustration.



Consider drawing a random sample of size 2 with replacement. That is, we select a unit at random, put it back, and then select another unit at random. Denote by X_1 and X_2 the observation of X obtained in the first and second drawing, respectively. Find the sampling distribution of $\overline{X} = (X_1 + X_2)/2$.

SOLUTION The population distribution of X is given in Table 1, which simply formalizes the fact that each of the X values 2, 3, and 4 occurs in $\frac{1}{3}$ of the population of the housing units.

TABLE 1 The Population Distribution

x	$f(x)$
2	$\frac{1}{3}$
3	$\frac{1}{3}$
4	$\frac{1}{3}$

Because each unit is equally likely to be selected, the observation X_1 from the first drawing has the same distribution as given in Table 1. Since the sampling is with replacement, the second observation X_2 also has this same distribution.

The possible samples (x_1, x_2) of size 2 and the corresponding values of $\overline{X}$ are

(x_1, x_2)	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(3, 2)$	$(3, 3)$	$(3, 4)$	$(4, 2)$	$(4, 3)$	$(4, 4)$
$\bar{x} = \dfrac{x_1 + x_2}{2}$	2	2.5	3	2.5	3	3.5	3	3.5	4

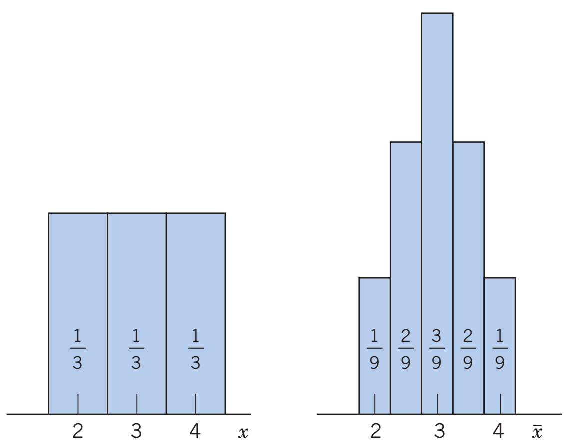
The nine possible samples are equally likely so, for instance, $P[\overline{X} = 2.5] = \frac{2}{9}$. Continuing in this manner, we obtain the distribution of $\overline{X}$, which is given in Table 2.

This sampling distribution pertains to repeated selection of random samples of size 2 with replacement. It tells us that if the random sampling is repeated a large number of times, then in about $\frac{1}{9}$, or 11%, of the cases, the sample mean would be 2, and in $\frac{2}{9}$, or 22%, of the cases, it would be 2.5, and so on.

TABLE 2 The Probability Distribution of $\overline{X} = (X_1 + X_2)/2$

Value of $\overline{X}$	Probability
2	$\frac{1}{9}$
2.5	$\frac{2}{9}$
3	$\frac{3}{9}$
3.5	$\frac{2}{9}$
4	$\frac{1}{9}$

Figure 1 shows the probability histograms of the distributions in Tables 1 and 2.



(*a*) Population distribution.

(*b*) Sampling distribution of $\overline{X} = (X_1 + X_2)/2$.

Figure 1 Idea of a sampling distribution.

In the context of Example 1, suppose instead the population consists of 300 housing units, of which 100 units have 2 rooms, 100 units have 3 rooms, and 100 units have 4 rooms for rent. When we sample two units from this large population, it would make little difference whether or not we replace the unit after the first selection. Each observation would still have the same probability distribution—namely, $P[X = 2] = P[X = 3] = P[X = 4] = \frac{1}{3}$, which characterizes the population.

When the population is very large and the sample size relatively small, it is inconsequential whether or not a unit is replaced before the next unit is selected. Under these conditions, too, we refer to the observations as a random sample. What are the key conditions required for a sample to be random?

> The observations $X_1, X_2, \ldots, X_n$ are a **random sample of size *n* from the population distribution** if they result from independent selections and each observation has the same distribution as the population.

More concisely, under the independence and same distribution conditions, we refer to the observations as a **random sample.**

Because of variation in the population, the random sample will vary and so will $\overline{X}$, the sample median, or any other statistic.

The next example further explores sampling distributions by focusing first on the role of independence in constructing a sampling distribution and then emphasizing that this distribution too has a mean and variance.

Example 2 **The Sample Mean and Median Each Have a Sampling Distribution**

A large population is described by the probability distribution

x	$f(x)$
0	.2
3	.3
12	.5

Let X_1, X_2, X_3 be a random sample of size 3 from this distribution.

(a) List all the possible samples and determine their probabilities.

(b) Determine the sampling distribution of the sample mean.

(c) Determine the sampling distribution of the sample median.

SOLUTION

(a) Because we have a random sample, each of the three observations X_1, X_2, X_3 has the same distribution as the population and they are independent. So, the sample 0, 3, 0 has probability $(.2) \times (.3) \times (.2) = 0.12$. The calculations for all $3 \times 3 \times 3 = 27$ possible samples are given in Table 3, page 270.

(b) The probabilities of all samples giving the same value $\bar{x}$ are added to obtain the sampling distribution on the second page of Table 3.

(c) The calculations and sampling distribution of the median are also given on the second page of Table 3.

TABLE 3 Sampling Distributions

Population Distribution

x	$f(x)$
0	.2
3	.3
12	.5

$$\text{Population mean: } E(X) = 0(.2) + 3(.3) + 12(.5) = 6.9 = \mu$$

$$\text{Population variance: } \text{Var}(X) = 0^2(.2) + 3^2(.3) + 12^2(.5) - 6.9^2 = 27.09 = \sigma^2$$

	Possible Samples x_1	x_2	x_3	Sample Mean $\bar{x}$	Sample Median m	Probability
1	0	0	0	0	0	(.2)(.2)(.2) = .008
2	0	0	3	1	0	(.2)(.2)(.3) = .012
3	0	0	12	4	0	(.2)(.2)(.5) = .020
4	0	3	0	1	0	(.2)(.3)(.2) = .012
5	0	3	3	2	3	(.2)(.3)(.3) = .018
6	0	3	12	5	3	(.2)(.3)(.5) = .030
7	0	12	0	4	0	(.2)(.5)(.2) = .020
8	0	12	3	5	3	(.2)(.5)(.3) = .030
9	0	12	12	8	12	(.2)(.5)(.5) = .050
10	3	0	0	1	0	(.3)(.2)(.2) = .012
11	3	0	3	2	3	(.3)(.2)(.3) = .018
12	3	0	12	5	3	(.3)(.2)(.5) = .030
13	3	3	0	2	3	(.3)(.3)(.2) = .018
14	3	3	3	3	3	(.3)(.3)(.3) = .027
15	3	3	12	6	3	(.3)(.3)(.5) = .045
16	3	12	0	5	3	(.3)(.5)(.2) = .030
17	3	12	3	6	3	(.3)(.5)(.3) = .045
18	3	12	12	9	12	(.3)(.5)(.5) = .075
19	12	0	0	4	0	(.5)(.2)(.2) = .020
20	12	0	3	5	3	(.5)(.2)(.3) = .030
21	12	0	12	8	12	(.5)(.2)(.5) = .050
22	12	3	0	5	3	(.5)(.3)(.2) = .030
23	12	3	3	6	3	(.5)(.3)(.3) = .045
24	12	3	12	9	12	(.5)(.3)(.5) = .075
25	12	12	0	8	12	(.5)(.5)(.2) = .050
26	12	12	3	9	12	(.5)(.5)(.3) = .075
27	12	12	12	12	12	(.5)(.5)(.5) = .125
						Total = 1.000

(Continued)

TABLE 3 *(Cont.)*

Sampling Distribution of $\overline{X}$

$\bar{x}$	$f(\bar{x})$
0	.008
1	.036 = .012 + .012 + .012
2	.054 = .018 + .018 + .018
3	.027
4	.060 = .020 + .020 + .020
5	.180 = .030 + .030 + .030 + .030 + .030 + .030
6	.135 = .045 + .045 + .045
8	.150 = .050 + .050 + .050
9	.225 = .075 + .075 + .075
12	.125 = .125

$$\begin{aligned} E(\overline{X}) = \sum \bar{x} f(\bar{x}) &= 0(.008) + 1(.036) + 2(.054) + 3(.027) \\ &\quad + 4(.060) + 5(.180) + 6(.135) \\ &\quad + 8(.150) + 9(.225) + 12(.125) \\ &= 6.9 \text{ same as } E(X)\text{, pop. mean} \end{aligned}$$

$$\begin{aligned} \operatorname{Var}(\overline{X}) = \sum \bar{x}^2 f(\bar{x}) - \mu^2 &= 0^2(.008) + 1^2(.036) + 2^2(.054) + 3^2(.027) \\ &\quad + 4^2(.060) + 5^2(.180) + 6^2(.135) \\ &\quad + 8^2(.150) + 9^2(.225) + 12^2(.125) - (6.9)^2 \\ &= 9.03 = \frac{27.09}{3} = \frac{\sigma^2}{3} \end{aligned}$$

$\operatorname{Var}(\overline{X})$ is one-third of the population variance.

Sampling Distribution of the Median m

m	$f(m)$
0	.104 = .008 + .012 + .020 + .012 + .020 + .012 + .020
3	.396 = .018 + .030 + .030 + .018 + .030 + .018 + .027 + .045 + .030 + .045 + .030 + .030 + .045
12	.500 = .050 + .075 + .050 + .075 + .050 + .075 + .125

Mean of the distribution of sample median

$= 0(.104) + 3(.396) + 12(.500) = 7.188 \neq 6.9 = \mu$

Different from the mean of the population distribution

Variance of the distribution of sample median

$= 0^2(.104) + 3^2(.396) + 12^2(.500) - (7.188)^2 = 23.897$

[*not* one-third of the population variance 27.09]

To illustrate the idea of a sampling distribution, we considered simple populations with only three possible values and small sample sizes $n = 2$ and $n = 3$. The calculation gets more tedious and extensive when a population has many values of X and n is large. However, the procedure remains the same. Once the population and sample size are specified:

1. List all possible samples of size n.
2. Calculate the value of the statistic for each sample.
3. List the distinct values of the statistic obtained in step 2. Calculate the corresponding probabilities by identifying all the samples that yield the same value of the statistic.

We leave the more complicated cases to statisticians who can sometimes use additional mathematical methods to derive exact sampling distributions.

Instead of a precise determination, one can turn to the computer in order to approximate a sampling distribution. The idea is to program the computer to actually draw a random sample and calculate the statistic. This procedure is then repeated a large number of times and a relative frequency histogram constructed from the values of the statistic. The resulting histogram will be an approximation to the sampling distribution. This approximation will be used in Example 4.

Exercises

7.1 Identify each of the following as either a parameter or a statistic.

(a) Sample standard deviation.

(b) Sample interquartile range.

(c) Population 20th percentile.

(d) Population first quartile.

(e) Sample median.

7.2 Identify the parameter, statistic, and population when they appear in each of the following statements.

(a) During 2008, forty-one different movies received the distinction of generating the most box office revenue for a weekend.

(b) A survey of 400 minority persons living in Chicago revealed that 41 were out of work.

(c) Out of a sample of 100 dog owners who applied for dog licenses in northern Wisconsin, 18 had a Labrador retriever.

7.3 Data obtained from asking the wrong questions at the wrong time or in the wrong place can lead to misleading summary statistics. Explain why the following collection procedures are likely to produce useless data.

(a) To evaluate the number of students who are employed at least part time, the investigator interviews students who are taking an evening class.

(b) To study the pattern of spending of persons earning the minimum wage, a survey is taken during the first three weeks of December.

7.4 Explain why the following collection procedures are likely to produce data that fail to yield the desired information.

(a) To evaluate public opinion about a new global trade agreement, an interviewer asks persons, “Do you feel that this unfair trade agreement should be canceled?”

(b) To determine how eighth-grade girls feel about having boys in the classroom, a random sample from a private girls' school is polled.

7.5 From the set of numbers $\{3, 5, 7\}$, a random sample of size 2 will be selected with replacement.

(a) List all possible samples and evaluate $\bar{x}$ for each.

(b) Determine the sampling distribution of $\overline{X}$.

7.6 A random sample of size 2 will be selected, with replacement, from the set of numbers $\{0, 2, 4\}$.

(a) List all possible samples and evaluate $\bar{x}$ and s^2 for each.

(b) Determine the sampling distribution of $\overline{X}$.

(c) Determine the sampling distribution of S^2.

7.7 A bride-to-be asks a prospective wedding photographer to show a sample of her work. She provides ten pictures. Should the bride-to-be consider this a random sample of the quality of pictures she will get? Comment.

7.8 To determine the time a cashier spends on a customer in the express lane, the manager decides to record the time to check-out for the customer who is being served at 10 past the hour, 20 past the hour, and so on. Will measurements collected in this manner be a random sample of the times a cashier spends on a customer?

7.9 ***Using a physical device to generate random samples.*** Using a die, generate a sample and evaluate the statistic. Then repeat many times and obtain an estimate of the sampling distribution. In particular, investigate the sampling distribution of the median for a sample of size 3 from the population distribution.

x	$f(x)$
1	2/6
2	3/6
4	1/6

(a) Roll the die. Assign $X = 1$ if 1 or 2 dots show. Complete the assignment of values so that X has the population distribution when the die is fair.

(b) Roll the die two more times and obtain the median of the three observed values of X.

(c) Repeat to obtain a total of 25 samples of size 3. Calculate the relative frequencies, among the 75 values, of 1, 2, and 4. Compare with the population probabilities and explain why they should be close.

(d) Obtain the 25 values of the sample median and create a frequency table. Explain how the distribution in this table approximates the actual sampling distribution. It is easy to see how this approach extends to any sample size.

7.10 Referring to Exercise 7.9, use a die to generate samples of size 3. Investigate the sampling distribution of the number of times a value 1 occurs in a sample of size 3.

(a) Roll the die and assign $X = 1$ if 1 dot shows and $X = 0$, otherwise. Repeat until you obtain a total of 25 samples of size 3. Calculate the relative frequencies, among the 75 values, of 1. Compare with the population probabilities and explain why they should be close.

(b) Let a random variable Y equal 1 if at least one value of X in the sample is 1. Set Y equal to 0 otherwise. Then the relative frequency of $[Y = 1]$ is an estimate of $P[Y = 1]$, the probability of at least one value 1 in the sample of size 3. Give your estimate.

3. DISTRIBUTION OF THE SAMPLE MEAN AND THE CENTRAL LIMIT THEOREM

Statistical inference about the population mean is of prime practical importance. Inferences about this parameter are based on the sample mean

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

and its sampling distribution. Consequently, we now explore the basic properties of the sampling distribution of $\overline{X}$ and explain the role of the normal distribution as a useful approximation.

In particular, we want to relate the sampling distribution of $\overline{X}$ to the population from which the random sample was selected. We denote the parameters of the population by

$$\text{Population mean} = \mu$$
$$\text{Population standard deviation} = \sigma$$

The sampling distribution of $\overline{X}$ also has a mean $E(\overline{X})$ and a standard deviation $\text{sd}(\overline{X})$. These can be expressed in terms of the population mean μ and standard deviation σ. (The interested reader can consult Appendix A.4 for details.)

Mean and Standard Deviation of $\overline{X}$

The distribution of the sample mean, based on a random sample of size n, has

$$E(\overline{X}) = \mu \qquad (= \text{Population mean})$$

$$\text{Var}(\overline{X}) = \frac{\sigma^2}{n} \qquad \left(= \frac{\text{Population variance}}{\text{Sample size}} \right)$$

$$\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}} \qquad \left(= \frac{\text{Population standard deviation}}{\sqrt{\text{Sample size}}} \right)$$

The first result shows that the distribution of $\overline{X}$ is centered at the population mean μ in the sense that expectation serves as a measure of center of a distribution. The last result states that the standard deviation of $\overline{X}$ equals the population standard deviation divided by the square root of the sample size. That is, the variability of the sample mean is governed by the two factors: the population variability σ and the sample size n. Large variability in the population induces large variability in $\overline{X}$, thus making the sample information about μ less dependable. However, this can be countered by choosing n large. For instance, with $n = 100$, the standard deviation of $\overline{X}$ is $\sigma/\sqrt{100} = \sigma/10$, a tenth of the population standard deviation. With increasing sample size, the standard deviation $\sigma/\sqrt{n}$ decreases and the distribution of $\overline{X}$ tends to become more concentrated around the population mean μ.

Example 3 The Mean and Variance of the Sampling Distribution of $\overline{X}$

Calculate the mean and standard deviation for the population distribution given in Table 1 and for the distribution of $\overline{X}$ given in Table 2. Verify the relations $E(\overline{X}) = \mu$ and $\text{sd}(\overline{X}) = \sigma/\sqrt{n}$.

SOLUTION The calculations are performed in Table 4.

TABLE 4 Mean and Variance of $\overline{X} = (X_1 + X_2)/2$

Population Distribution

x	$f(x)$	$x f(x)$	$x^2 f(x)$
2	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
3	$\frac{1}{3}$	$\frac{3}{3}$	$\frac{9}{3}$
4	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{16}{3}$
Total	1	3	$\frac{29}{3}$

$\mu = 3$

$\sigma^2 = \frac{29}{3} - (3)^2 = \frac{2}{3}$

Distribution of $\overline{X} = (X_1 + X_2)/2$

$\bar{x}$	$f(\bar{x})$	$\bar{x} f(\bar{x})$	$\bar{x}^2 f(\bar{x})$
2	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$
2.5	$\frac{2}{9}$	$\frac{5}{9}$	$\frac{12.5}{9}$
3	$\frac{3}{9}$	$\frac{9}{9}$	$\frac{27}{9}$
3.5	$\frac{2}{9}$	$\frac{7}{9}$	$\frac{24.5}{9}$
4	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
Total	1	3	$\frac{84}{9}$

$$E(\overline{X}) = 3 = \mu$$

$$\text{Var}(\overline{X}) = \frac{84}{9} - (3)^2 = \frac{1}{3}$$

By direct calculation, $\text{sd}(\overline{X}) = 1/\sqrt{3}$. This is confirmed by the relation

$$\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{2}{3}}\Big/\sqrt{2} = \frac{1}{\sqrt{3}}$$

We now state two important results concerning the shape of the sampling distribution of $\overline{X}$. The first result gives the exact form of the distribution of $\overline{X}$ when the population distribution is normal:

$\overline{X}$ Is Normal When Sampling from a Normal Population

In random sampling from a **normal** population with mean μ and standard deviation σ, the sample mean $\overline{X}$ has the normal distribution with mean μ and standard deviation $\sigma / \sqrt{n}$.

Example 4 Determining Probabilities Concerning $\overline{X}$—Normal Populations

The weight of a pepperoni and cheese pizza from a local provider is a random variable whose distribution is normal with mean 16 ounces and standard deviation 1 ounce. You intend to purchase four pepperoni and cheese pizzas. What is the probability that:

(a) The average weight of the four pizzas will be greater than 17.1 ounces?

(b) The total weight of the four pizzas will not exceed 61.0 ounces?

SOLUTION Because the population is normal, the distribution of the sample mean $\overline{X} = (X_1 + X_2 + X_3 + X_4)/4$ is exactly normal with mean 16 ounces and standard deviation $1/\sqrt{4} = .5$ ounce.

(a) Since $\overline{X}$ is $N(16, .5)$

$$P[\overline{X} > 17.1] = P\left[\frac{\overline{X} - 16}{.5} > \frac{17.1 - 16}{.5}\right]$$
$$= P[Z > 2.20] = 1 - .9861 = .0139$$

Only rarely, just over one time in a hundred purchases of four pizzas, would the average weight exceed 17.1 ounces.

(b) The event that the total weight $X_1 + X_2 + X_3 + X_4 = 4\overline{X}$ does not exceed 61.0 ounces is the same event that the average weight $\overline{X}$ is less than or equal to $61.0/4 = 15.25$. Consequently

$$P[X_1 + X_2 + X_3 + X_4 \le 61.0] = P[\overline{X} \le 15.25]$$
$$= P\left[\frac{\overline{X} - 16}{.5} \le \frac{15.25 - 16}{.5}\right]$$
$$= P[Z \le -1.50] = .0668$$

Only about seven times in one hundred purchases would the total weight be less than 61.0 ounces.

When sampling from a nonnormal population, the distribution of $\overline{X}$ depends on the particular form of the population distribution that prevails. A surprising result, known as the **central limit theorem,** states that when the sample size n is large, the distribution $\overline{X}$ is approximately normal, regardless of the shape of the population distribution. In practice, the normal approximation is usually adequate when n is greater than 30.

Central Limit Theorem

Whatever the population, the distribution of $\overline{X}$ is approximately normal when n is large.

In random sampling from an arbitrary population with mean μ and standard deviation σ, when n is large, the distribution of $\overline{X}$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$. Consequently,

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \quad \text{is approximately } N(0, 1)$$

Whether the population distribution is continuous, discrete, symmetric, or asymmetric, the central limit theorem asserts that as long as the population variance is finite, the distribution of the sample mean $\overline{X}$ is nearly normal if the sample size is large. In this sense, the normal distribution plays a central role in the development of statistical procedures.

Example 5 Probability Calculations for $\overline{X}$—Based on a Large Sample of Activities

Extensive data, including that in Exercise 2.3, suggest that the number of extracurricular activities per week can be modeled as distribution with mean 1.9 and standard deviation 1.6.

(a) If a random sample of size 41 is selected, what is the probability that the sample mean will lie between 1.6 and 2.1?

(b) With a sample size of 100, what is the probability that the sample mean will lie between 1.6 and 2.1?

SOLUTION

(a) We have $\mu = 1.9$ and $\sigma = 1.6$. Since $n = 41$ is large, the central limit theorem tells us that the distribution of $\overline{X}$ is approximately normal with

$$\text{Mean} = \mu = 1.9$$

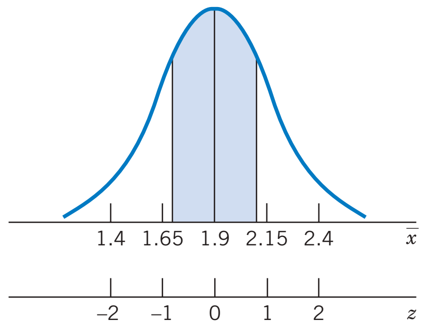
$$\text{Standard deviation} = \frac{\sigma}{\sqrt{n}} = \frac{1.6}{\sqrt{41}} = .250$$

To calculate $P[1.7 < \overline{X} < 2.1]$, we convert to the standardized variable

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{\overline{X} - 1.9}{.250}$$

The z values corresponding to 1.7 and 2.1 are

$$\frac{1.7 - 1.9}{.250} = -.8 \qquad \text{and} \qquad \frac{2.1 - 1.9}{.250} = .8$$



Consequently,

$$\begin{aligned} P[1.7 < \overline{X} < 2.1] &= P[-.8 < Z < .8] \\ &= .7881 - .2119 \quad \text{(using the normal table)} \\ &= .5762 \end{aligned}$$

(b) We now have $n = 100$, so $\sigma/\sqrt{n} = 1.6/\sqrt{100} = .16$, and

$$Z = \frac{\overline{X} - 1.9}{.16}$$

Therefore,

$$\begin{aligned} P[1.7 < \overline{X} < 2.1] &= P\left[\frac{1.7 - 1.9}{.16} < Z < \frac{2.1 - 1.9}{.16}\right] \\ &= P[-1.25 < Z < 1.25] \\ &= .8944 - .1056 \\ &= .7888 \end{aligned}$$

Note that the interval $(1.7, 2.1)$ is centered at $\mu = 1.9$. The probability that $\overline{X}$ will lie in this interval is larger for $n = 100$ than for $n = 41$.

Although a proof of the central limit theorem requires higher mathematics, we can empirically demonstrate how this result works.

Example 6 Demonstrating the Central Limit Theorem

Consider a population having a discrete uniform distribution that places a probability of .1 on each of the integers 0, 1, . . . , 9. This may be an appropriate model for the distribution of the last digit in telephone numbers or the first overflow digit in computer calculations. The line diagram of this distribution appears in Figure 2. The population has $\mu = 4.5$ and $\sigma = 2.872$.

Take 100 samples of size 5, calculate $\bar{x}$ for each, and make a histogram to approximate the sampling distribution of $\overline{X}$.

TABLE 5 Samples of Size 5 from a Discrete Uniform Distribution

Sample Number	Observations	Sum	Mean $\bar{x}$	Sample Number	Observations	Sum	Mean $\bar{x}$
1	4, 7, 9, 0, 6	26	5.2	51	4, 7, 3, 8, 8	30	6.0
2	7, 3, 7, 7, 4	28	5.6	52	2, 0, 3, 3, 2	10	2.0
3	0, 4, 6, 9, 2	21	4.2	53	4, 4, 2, 6, 3	19	3.8
4	7, 6, 1, 9, 1	24	4.8	54	1, 6, 4, 0, 6	17	3.4
5	9, 0, 2, 9, 4	24	4.8	55	2, 4, 5, 8, 9	28	5.6
6	9, 4, 9, 4, 2	28	5.6	56	1, 5, 5, 4, 0	15	3.0
7	7, 4, 2, 1, 6	20	4.0	57	3, 7, 5, 4, 3	22	4.4
8	4, 4, 7, 7, 9	31	6.2	58	3, 7, 0, 7, 6	23	4.6
9	8, 7, 6, 0, 5	26	5.2	59	4, 8, 9, 5, 9	35	7.0
10	7, 9, 1, 0, 6	23	4.6	60	6, 7, 8, 2, 9	32	6.4
11	1, 3, 6, 5, 7	22	4.4	61	7, 3, 6, 3, 6	25	5.0
12	3, 7, 5, 3, 2	20	4.0	62	7, 4, 6, 0, 1	18	3.6
13	5, 6, 6, 5, 0	22	4.4	63	7, 9, 9, 7, 5	37	7.4
14	9, 9, 6, 4, 1	29	5.8	64	8, 0, 6, 2, 7	23	4.6
15	0, 0, 9, 5, 7	21	4.2	65	6, 5, 3, 6, 2	22	4.4
16	4, 9, 1, 1, 6	21	4.2	66	5, 0, 5, 2, 9	21	4.2
17	9, 4, 1, 1, 4	19	3.8	67	2, 9, 4, 9, 1	25	5.0
18	6, 4, 2, 7, 3	22	4.4	68	9, 5, 2, 2, 6	24	4.8
19	9, 4, 4, 1, 8	26	5.2	69	0, 1, 4, 4, 4	13	2.6
20	8, 4, 6, 8, 3	29	5.8	70	5, 4, 0, 5, 2	16	3.2
21	5, 2, 2, 6, 1	16	3.2	71	1, 1, 4, 2, 0	8	1.6
22	2, 2, 9, 1, 0	14	2.8	72	9, 5, 4, 5, 9	32	6.4
23	1, 4, 5, 8, 8	26	5.2	73	7, 1, 6, 6, 9	29	5.8
24	8, 1, 6, 3, 7	25	5.0	74	3, 5, 0, 0, 5	13	2.6
25	1, 2, 0, 9, 6	18	3.6	75	3, 7, 7, 3, 5	25	5.0
26	8, 5, 3, 0, 0	16	3.2	76	7, 4, 7, 6, 2	26	5.2
27	9, 5, 8, 5, 0	27	5.4	77	8, 1, 0, 9, 1	19	3.8
28	8, 9, 1, 1, 8	27	5.4	78	6, 4, 7, 9, 3	29	5.8
29	8, 0, 7, 4, 0	19	3.8	79	7, 7, 6, 9, 7	36	7.2
30	6, 5, 5, 3, 0	19	3.8	80	9, 4, 2, 9, 9	33	6.6
31	4, 6, 4, 2, 1	17	3.4	81	3, 3, 3, 3, 3	15	3.0
32	7, 8, 3, 6, 5	29	5.8	82	8, 7, 7, 0, 3	25	5.0
33	4, 2, 8, 5, 2	21	4.2	83	5, 3, 2, 1, 1	12	2.4
34	7, 1, 9, 0, 9	26	5.2	84	0, 4, 5, 2, 6	17	3.4
35	5, 8, 4, 1, 4	22	4.4	85	3, 7, 5, 4, 1	20	4.0
36	6, 4, 4, 5, 1	20	4.0	86	7, 4, 5, 9, 8	33	6.6
37	4, 2, 1, 1, 6	14	2.8	87	3, 2, 9, 0, 5	19	3.8
38	4, 7, 5, 5, 7	28	5.6	88	4, 6, 6, 3, 3	22	4.4
39	9, 0, 5, 9, 2	25	5.0	89	1, 0, 9, 3, 7	20	4.0
40	3, 1, 5, 4, 5	18	3.6	90	2, 9, 6, 8, 5	30	6.0
41	9, 8, 6, 3, 2	28	5.6	91	4, 8, 0, 7, 6	25	5.0
42	9, 4, 2, 2, 8	25	5.0	92	5, 6, 7, 6, 3	27	5.4
43	8, 4, 7, 2, 2	23	4.6	93	3, 6, 2, 5, 6	22	4.4
44	0, 7, 3, 4, 9	23	4.6	94	0, 1, 1, 8, 4	14	2.8
45	0, 2, 7, 5, 2	16	3.2	95	3, 6, 6, 4, 5	24	4.8
46	7, 1, 9, 9, 9	35	7.0	96	9, 2, 9, 8, 6	34	6.8
47	4, 0, 5, 9, 4	22	4.4	97	2, 0, 0, 6, 8	16	3.2
48	5, 8, 6, 3, 3	25	5.0	98	0, 4, 5, 0, 5	14	2.8
49	4, 5, 0, 5, 3	17	3.4	99	0, 3, 7, 3, 9	22	4.4
50	7, 7, 2, 0, 1	17	3.4	100	2, 5, 0, 0, 7	14	2.8

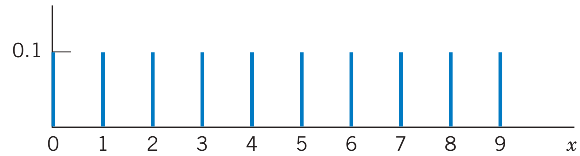


Figure 2 Uniform distribution on the integers 0, 1, . . . , 9.

SOLUTION By means of a computer, 100 random samples of size 5 were generated from this distribution, and $\bar{x}$ was computed for each sample. The results of this repeated random sampling are presented in Table 5. The relative frequency histogram in Figure 3 is constructed from the 100 observed values of $\bar{x}$. Although the population distribution (Figure 2) is far from normal, the top of the histogram of the $\bar{x}$ values (Figure 3) has the appearance of a bell-shaped curve, even for the small sample size of 5. For larger sample sizes, the normal distribution would give an even closer approximation.

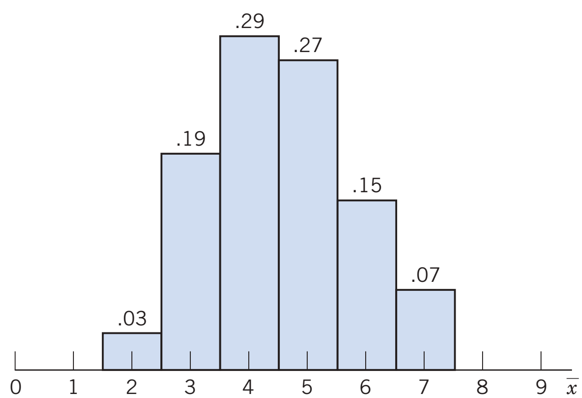


Figure 3 Relative frequency histogram of the $\bar{x}$ values recorded in Table 5.

Calculating from the 100 simulated $\bar{x}$ values in Table 5, we find the sample mean and standard deviation to be 4.54 and 1.215, respectively. These are in close agreement with the theoretical values for the mean and standard deviation of $\bar{X}$: $\mu = 4.5$ and $\sigma/\sqrt{n} = 2.872/\sqrt{5} = 1.284$.

It might be interesting for the reader to collect similar samples by reading the last digits of numbers from a telephone directory and then to construct a histogram of the $\bar{x}$ values.

Another graphic example of the central limit theorem appears in Figure 4, where the population distribution represented by the solid curve is a continuous asymmetric distribution with $\mu = 2$ and $\sigma = 1.41$. The distributions of the sample mean $\bar{X}$ for sample sizes $n = 3$ and $n = 10$ are plotted as dashed and dotted

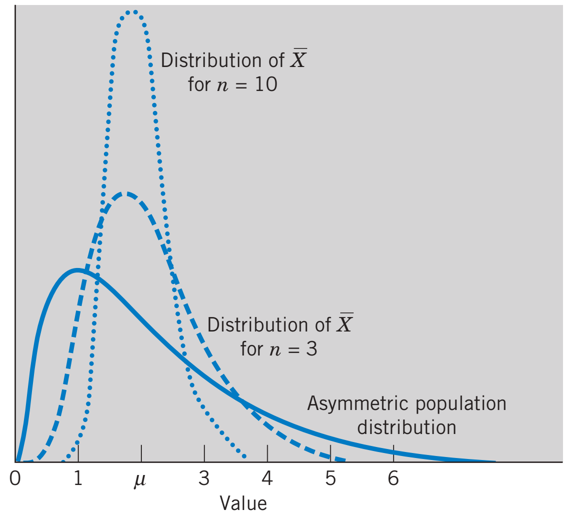


Figure 4 Distributions of $\overline{X}$ for $n = 3$ and $n = 10$ in sampling from an asymmetric population.

curves on the graph. These indicate that with increasing n, the distributions become more concentrated around μ and look more like the normal distribution.

Example 7 More Probability Calculations for $\overline{X}$, Number of Items Returned

Retail stores experience their heaviest returns on December 26 and December 27 each year. A small sample of number of items returned is given in Example 3, Chapter 2, but a much larger sample size is required to approximate the probability distribution. Suppose the relative frequencies, determined from a sample of size 300, suggest the probability distribution in Table 6.

TABLE 6 Number X of Items Returned

Number Items Returned (x)	Probability
1	.25
2	.28
3	.20
4	.17
5	.08
6	.02

This distribution for number of gifts returned has mean 2.61 and standard deviation 1.34. Assume the probability distribution in Table 6 still holds for this year.

(a) If this year, a random sample of size 45 is selected, what is the probability that the sample mean will be greater than 2.9 items?

(b) Find an upper bound b such that the total number of items returned by 45 customers will be less b with probability .95.

SOLUTION Here the population mean and the standard deviation are $\mu = 2.61$ and $\sigma = 1.34$, respectively. The sample size $n = 45$ is large, so the central limit theorem ensures that the distribution of $\overline{X}$ is approximately normal with

$$\text{Mean} = 2.61$$
$$\text{Standard deviation} = \frac{\sigma}{\sqrt{n}} = \frac{1.34}{\sqrt{45}} = .200$$

(a) To find $P[\overline{X} > 2.9]$, we convert to the standardized variable

$$Z = \frac{\overline{X} - 2.61}{.200}$$

and calculate the z value

$$\frac{2.9 - 2.61}{.200} = 1.45$$

The required probability is

$$\begin{aligned} P[\overline{X} > 2.9] &= P[Z > 1.45] \\ &= 1 - P[Z \leq 1.45] \\ &= 1 - .9265 = .0735 \end{aligned}$$

(b) Let X_i denote the number of items returned by the i–th person. Then we recognize that the event the total number of items returned is less than or equal b, $[X_1 + X_2 + \cdots + X_{45} \leq b]$ is the same event as $[\overline{X} < b/45]$.

By the central limit theorem, since $Z_{.05} = 1.645$, b must satisfy

$$.95 = P[Z \leq 1.645] = P\left[\frac{\overline{X} - 2.61}{.200} \leq \frac{b/45 - 2.61}{.200}\right]$$

It follows that

$$b = 45 \cdot (1.645 \times .200 + 2.61) = 132.3$$

In the context of this problem, the total must be an integer so, conservatively, we may take 133 as the bound.

Exercises

7.11 Refer to the lightning data in Exercise 2.121 of Chapter 2. One plausible model for the population distribution has mean $\mu = 83$ and standard deviation $\sigma = 38$ deaths per year. Calculate the mean and standard deviation of $\overline{X}$ for a random sample of size (a) 4 and (b) 25.

7.12 Refer to the data on earthquakes of magnitude greater than 6.0 in Exercise 2.20. The data suggests that one plausible model, for X = magnitude, is a population distribution having mean $\mu = 6.7$ and standard deviation sigma $\sigma = .47$. Calculate the expected value and standard deviation of $\overline{X}$ for a random sample of size (a) 9 and (b) 16.

7.13 Refer to the monthly intersection accident data in Exercise 5.92. The data suggests that one plausible model, for X = the number of accidents in one month, is a population distribution having mean $\mu = 2.6$ and variance $\sigma^2 = 2.4$. Determine the standard deviation of $\overline{X}$ for a random sample of size (a) 25 and (b) 100 and (c) 400. (d) How does quadrupling the sample size change the standard deviation of $\overline{X}$?

7.14 Refer to Exercise 7.11. Determine the standard deviation of $\overline{X}$ for a random sample of size (a) 9 (b) 36 and (c) 144. (d) How does quadrupling the sample size change the standard deviation of $\overline{X}$?

7.15 Using the sampling distribution determined for $\overline{X} = (X_1 + X_2)/2$ in Exercise 7.5, verify that $E(\overline{X}) = \mu$ and $\text{sd}(\overline{X}) = \sigma/\sqrt{2}$.

7.16 Using the sampling distribution determined for $\overline{X} = (X_1 + X_2)/2$ in Exercise 7.6, verify that $E(\overline{X}) = \mu$ and $\text{sd}(\overline{X}) = \sigma/\sqrt{2}$.

7.17 Suppose the number of different computers used by a student last week has distribution

Value	Probability
0	.3
1	.4
2	.3

Let X_1 and X_2 be independent and each have the same distribution as the population.

(a) Determine the missing elements in the table for the sampling distribution of $\overline{X} = (X_1 + X_2)/2$.

$\bar{x}$	Probability
0.0	
0.5	
1.0	.34
1.5	.24
2.0	

(b) Find the expected value of $\overline{X}$.

(c) If the sample size is increased to 36, give the mean and variance of $\overline{X}$.

7.18 As suggested in Example 8, Chapter 6, the population of hours of sleep can be modeled as a normal distribution with mean 7.2 hours and standard deviation 1.3 hours. For a sample of size 4, determine the

(a) mean of $\overline{X}$.

(b) standard deviation of $\overline{X}$.

(c) distribution of $\overline{X}$.

7.19 According to Example 12, Chapter 6, a normal distribution with mean 115 and standard deviation 22 hundredths of an inch describes the variation in female salmon growth in fresh water. For a sample of size 6, determine the

(a) mean of $\overline{X}$.

(b) standard deviation of $\overline{X}$.

(c) distribution of $\overline{X}$.

7.20 A population has distribution

Value	Probability
0	.7
2	.1
4	.2

Let X_1 and X_2 be independent and each have the same distribution as the population.

(a) Determine the missing elements in the table for the sampling distribution of $\overline{X} = (X_1 + X_2)/2$.

$\bar{x}$	Probability
0	
1	
2	.29
3	
4	.04

(b) Find the expected value of $\bar{X}$.

(c) If the sample size is increased to 25, give the mean and variance of $\bar{X}$.

7.21 Suppose the weights of the contents of cans of mixed nuts have a normal distribution with mean 32.4 ounces and standard deviation .4 ounce.

(a) If every can is labeled 32 ounces, what proportion of the cans have contents that weigh less than than the labeled amount?

(b) If two packages are randomly selected, specify the mean, standard deviation, and distribution of the average weight of the contents.

(c) If two packages are randomly selected, what is the probability that the average weight is less than 32 ounces?

7.22 Suppose the amount of a popular sport drink in bottles leaving the filling machine has a normal distribution with mean 101.5 milliliters (ml) and standard deviation 1.6 ml.

(a) If the bottles are labeled 100 ml, what proportion of the bottles contain less than the labeled amount?

(b) If four bottles are randomly selected, find the mean and standard deviation of the average content.

(c) What is the probability that the average content of four bottles is less than 100 ml?

7.23 The distribution of personal income of full-time retail clerks working in a large eastern city has $\mu = \$41,000$ and $\sigma = \$5000$.

(a) What is the approximate distribution for $\bar{X}$ based on a random sample of 100 persons?

(b) Evaluate $P[\bar{X} > 41,500]$.

7.24 The result of a survey[1] suggests that one plausible population distribution, for X = number of persons with whom an adult discusses important matters, can be modeled as a population having mean $\mu = 2.0$ and standard deviation $\sigma = 2.0$. A random sample of size 100 will be obtained.

(a) What can you say about the probability distribution of the sample mean $\bar{X}$?

(b) Find the probability that $\bar{X}$ will exceed 2.3.

7.25 The lengths of the trout fry in a pond at the fish hatchery are approximately normally distributed with mean 3.4 inches and standard deviation .8 inch. Three dozen fry will be netted and their lengths measured.

(a) What is the probability that the sample mean length of the 36 netted trout fry will be less than 3.2 inches?

(b) Why might the fish in the net not represent a random sample of trout fry in the pond?

7.26 The heights of male students at a university have a nearly normal distribution with mean 70 inches and standard deviation 2.8 inches. If 5 male students are randomly selected to make up an intramural basketball team, what is the probability that the heights of the team will average over 72.0 inches?

7.27 According to the growth chart that doctors use as a reference, the heights of two-year-old boys are normally distributed with mean 34.5 inches and standard deviation 1.3 inches. For a random sample of 6 two-year-old boys, find the probability that the sample mean will be between 34.1 and 35.2 inches.

[1] M. McPherson, L. Smith-Lovin, and M. Brashears. "Social Isolation in America: Changes in Core Discussion Networks Over Two Decades," *American Sociological Review*, 71 (3) (2006), pp. 353–375.

7.28 The weight of an almond is normally distributed with mean .05 ounce and standard deviation .015 ounce. Find the probability that a package of 100 almonds will weigh between 4.8 and 5.3 ounces. That is, find the probability that $\overline{X}$ will be between .048 and .053 ounce.

*7.29 Refer to Table 5 on page 279.

(a) Calculate the sample median for each sample.

(b) Construct a frequency table and make a histogram.

(c) Compare the histogram for the median with that given in Figure 3 for the sample mean. Does your comparison suggest that the sampling distribution of the mean or median has the smaller variance?

7.30 The number of days, X, that it takes the post office to deliver a letter between City A and City B has the probability distribution

x	3	4	5
$f(x)$	.5	.3	.2

(a) Find the expected number of days and the standard deviation of the number of days.

(b) A company in City A sends a letter to a company in City B with a return receipt request that is to be mailed immediately upon receiving the letter. Find the probability distribution of total number of days from the time the letter is mailed until the return receipt arrives back at the company in City A. Assume the two delivery times are independent.

(c) A single letter will be sent from City A on each of 100 different days. What is the approximate probability that more than 25 of the letters will take 5 days to reach City B?

7.31 The number of complaints per day, X, received by a cable TV distributor has the probability distribution

x	0	1	2	3
$f(x)$	.4	.3	.1	.2

(a) Find the expected number of complaints per day.

(b) Find the standard deviation of the number of complaints.

(c) What is the probability distribution of total number of complaints received in two days? Assume the numbers of complaints on different days are independent.

(d) What is the approximate probability that the distributor will receive more than 125 complaints in 90 days?

4. STATISTICS IN CONTEXT

Troy, a Canadian importer of cut flowers, must effectively deal with uncertainty every day that he is in business. For instance, he must order enough of each kind of flower to supply his regular wholesale customers and yet not have too many left each day. Fresh flowers are no longer fresh on the day after arrival.

Troy purchases his fresh flowers from growers in the United States, Mexico, and Central and South America. Because most of the growers purchase their growing stock and chemicals from the United States, all of the selling prices are quoted in U.S. dollars. On a typical day, he purchases tens of thousands of cut flowers. Troy knows their price in U.S. dollars, but this is not his ultimate cost. Because of a fluctuating exchange rate, he does not know his ultimate cost at the time of purchase.

As with most businesses, Troy takes about a month to pay his bills. He must pay in Canadian dollars, so fluctuations in the Canadian/U.S. exchange rate from the time of purchase to the time the invoice is paid are a major source of uncertainty. Can this uncertainty be quantified and modeled?

The Canadian dollar to U.S. dollar exchange rate equals the number of Canadian dollars which must be paid for each U.S. dollar. Data from several years will provide the basis for a model. As given in Table 7, the exchange rate was 1.3669 on January 1, 1996 and 1.2337 on December 1, 2008 (reading across each row). It peaked at 1.5997 in January 2002.

If the exchange rate is 1.2000, one dollar and twenty cents Canadian is required to pay for each single U.S. dollar. An invoice for 1000 U.S. dollars would cost Troy 1200 Canadian dollars while it would cost 1210 dollars if the exchange rate were 1.2100. It is the change in the exchange rate from time of purchase to payment that creates uncertainty.

Although the exchange rate changes every day, we consider the monthly rates. The value of the difference

$$\text{Current month exchange rate} - \text{Previous month exchange rate}$$

TABLE 7 Monthly Canadian to U.S. Dollar Exchange Rates 1/1/1996–12/1/2008

1.3669	1.3752	1.3656	1.3592	1.3693	1.3658	1.3697	1.3722
1.3694	1.3508	1.3381	1.3622	1.3494	1.3556	1.3725	1.3942
1.3804	1.3843	1.3775	1.3905	1.3872	1.3869	1.4128	1.4271
1.4409	1.4334	1.4166	1.4298	1.4452	1.4655	1.4869	1.5346
1.5218	1.5452	1.5404	1.5433	1.5194	1.4977	1.5176	1.4881
1.4611	1.4695	1.4890	1.4932	1.4771	1.4776	1.4674	1.4722
1.4486	1.4512	1.4608	1.4689	1.4957	1.4770	1.4778	1.4828
1.4864	1.5125	1.5426	1.5219	1.5032	1.5216	1.5587	1.5578
1.5411	1.5245	1.5308	1.5399	1.5679	1.5717	1.5922	1.5788
1.5997	1.5964	1.5877	1.5815	1.5502	1.5318	1.5456	1.5694
1.5761	1.5780	1.5715	1.5592	1.5414	1.5121	1.4761	1.4582
1.3840	1.3525	1.3821	1.3963	1.3634	1.3221	1.3130	1.3128
1.2958	1.3299	1.3286	1.3420	1.3789	1.3578	1.3225	1.3127
1.2881	1.2469	1.1968	1.2189	1.2248	1.2401	1.2160	1.2359
1.2555	1.2402	1.2229	1.2043	1.1777	1.1774	1.1815	1.1615
1.1572	1.1489	1.1573	1.1441	1.1100	1.1137	1.1294	1.1182
1.1161	1.1285	1.1359	1.1532	1.1763	1.1710	1.1682	1.1350
1.0951	1.0651	1.0502	1.0579	1.0267	0.9754	0.9672	1.0021
1.0099	0.9986	1.0029	1.0137	0.9993	1.0166	1.0130	1.0535
1.0582	1.1847	1.2171	1.2337				

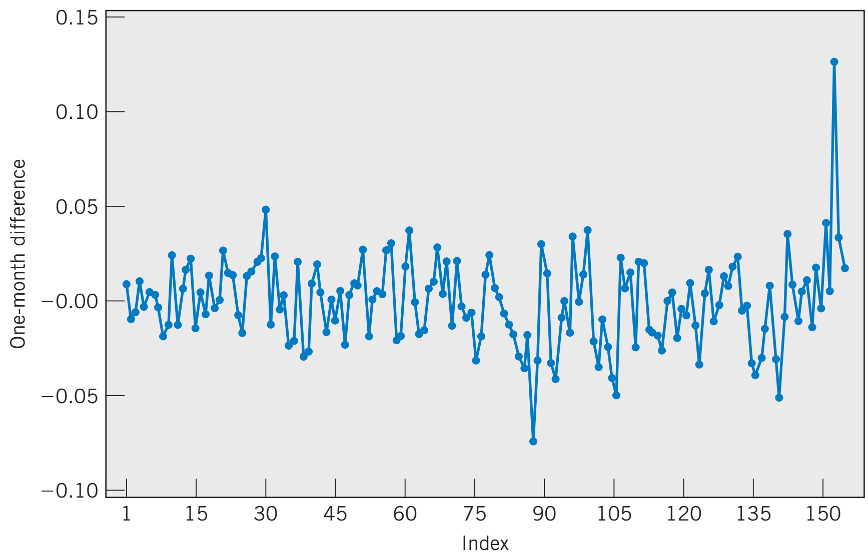


Figure 5 Time plot of one-month differences in the Canadian/U. S. exchange rate.

would describe the change in cost, per dollar invoiced, resulting from the one-month delay between purchasing and paying for a shipment of cut flowers. If the rate goes down, Troy makes money. If the rate goes up, he loses money. Figure 5 gives a time plot of these differences for the period 1/1/1996 to 12/1/2008. The 155 differences appear to be stable except for the one very large change of .1265 which is the difference in the exchange rate from September to October 2008. The one-month differences have $\bar{x} = -.00086$ and $s = .02303$.

The single value .1265 is very profitable to Troy but, according to a long historical record, large changes are very rare. To model future changes, we choose to ignore this difference. Doing so, the $n = 154$ one-month differences have mean $\bar{x} = -.00169$ and $s = .02067$. The mean changes substantially and the standard deviation somewhat. Figure 6 presents a histogram and normal scores plot. There is one slightly small value but the mean and variance do not change much if this is dropped.

According to the methods developed in the next chapter, 0 is a plausible value for the mean. Our approximating normal distribution has $\mu = 0$ and $\sigma = .0207$. We have successfully modeled the uncertainty in the exchange rate over a one-month period. If Troy paid all of his invoices in exactly one month, this then is the variability that he would face.

Over a three-month period, Troy would pay three times and the total uncertainty would be the sum of three independent mean 0 normal random variables. The variance of the sum is $3\sigma^2 = 3\,(.0207)^2 = .00129$, so the standard deviation is

$$\sqrt{3\ (.0207)^2} = .0359$$

Although the variance is 3 times as large as that of a single difference, the standard deviation does not increase that fast. The standard deviation is $\sqrt{3}\ (.0207)$.

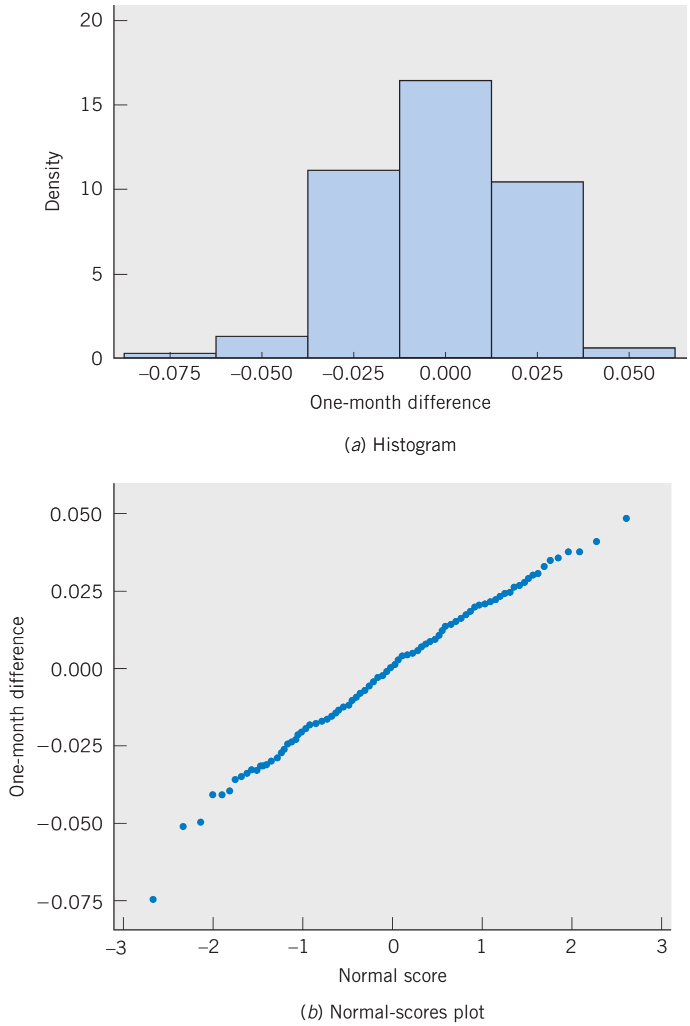


Figure 6 (*a*) Histogram and (*b*) normal-scores plot of differences.

Exercises

7.32 Refer to the model for monthly differences in the exchange rate. Find the mean and standard deviation for the sampling distribution of the sample mean of three monthly differences. Relate these quantities to the population mean and standard deviation for the sum of three differences given in the Statistics in Context section.

7.33 Refer to the Statistics in Context section concerning the flower importer.

(a) Suppose it takes the importer two months to pay his invoices. Proceed by taking the differences of the one-month differences after deleting the outlier .1265. Make a histogram of the differences two months apart. (The two-month differences may not be independent but the histogram is the correct summary of uncertainty for some two-month period in the future.)

(b) Compare your histogram in part (a) with that for the one-month changes. Which is more variable?

USING STATISTICS WISELY

1. Understand the concept of a sampling distribution. Each observation is the value of a random variable so a sample of n observations varies from one possible sample to another. Consequently, a statistic such as a sample mean varies from one possible sample to another. The probability distribution which describes the chance behavior of the sample mean is called its sampling distribution.
2. When the underlying distribution has mean μ and variance σ^2, remember that the sampling distribution of $\overline{X}$ has

$$\text{Mean of } \overline{X} = \mu = \text{Population mean}$$
$$\text{Variance of } \overline{X} = \frac{\sigma^2}{n} = \frac{\text{Population variance}}{n}$$

3. When the underlying distribution is normal with mean μ and variance σ^2, calculate exact probabilities for $\overline{X}$ using the normal distribution with mean μ and variance $\dfrac{\sigma^2}{n}$.

$$P[\overline{X} \le b] = P\left[Z \le \frac{b - \mu}{\sigma/\sqrt{n}}\right]$$

4. Apply the central limit theorem, when the sample size is large, to approximate the sampling distribution of $\overline{X}$ by a normal distribution with mean μ and variance σ^2/n. The probability $P[\overline{X} \le b]$ is approximately equal to the standard normal probability $P\left[Z \le \dfrac{b - \mu}{\sigma/\sqrt{n}}\right]$.
5. Do not confuse the population distribution, which describes the variation for a single random variable, with the sampling distribution of a statistic.
6. When the population distribution is noticeably nonnormal, do not try to conclude that the sampling distribution of $\overline{X}$ is normal unless the sample size is at least moderately large, 30 or more.

KEY IDEAS

The observations $X_1, X_2, \ldots, X_n$ are a **random sample of size n from the population distribution** if they result from independent selections and each observation has the same distribution as the population. Under these conditions, we refer to the observations as a **random sample.**

A **parameter** is a numerical characteristic of the population. It is a constant, although its value is typically unknown to us. The object of a statistical analysis of sample data is to learn about the parameter.

A numerical characteristic of a sample is called a **statistic.** The value of a statistic varies in repeated sampling.

When random sampling from a population, a statistic is a random variable. The probability distribution of a statistic is called its **sampling distribution.**

The sampling distribution of $\overline{X}$ has mean μ and standard deviation $\sigma/\sqrt{n}$, where μ = population mean, σ = population standard deviation, and n = sample size.

With increasing n, the distribution of $\overline{X}$ is more concentrated around μ.

If the population distribution is normal, $N(\mu, \sigma)$, the distribution of $\overline{X}$ is $N(\mu, \sigma/\sqrt{n})$.

Regardless of the shape of the population distribution, the distribution of $\overline{X}$ is approximately $N(\mu, \sigma/\sqrt{n})$, provided that n is large. This result is called the **central limit theorem.**

5. REVIEW EXERCISES

7.34 A population consists of the four numbers {0, 2, 4, 6}. Consider drawing a random sample of size 2 with replacement.

(a) List all possible samples and evaluate $\overline{x}$ for each.

(b) Determine the sampling distribution of $\overline{X}$.

(c) Write down the population distribution and calculate its mean μ and standard deviation σ.

(d) Calculate the mean and standard deviation of the sampling distribution of $\overline{X}$ obtained in part (b), and verify that these agree with μ and $\sigma/\sqrt{2}$, respectively.

7.35 Refer to Exercise 7.34 and, instead of $\overline{X}$, consider the statistic

$$\text{Sample range } R = \text{Largest observation} - \text{Smallest observation}$$

For instance, if the sample observations are (2, 6), the range is $6 - 2 = 4$.

(a) Calculate the sample range for all possible samples.

(b) Determine the sampling distribution of R.

7.36 What sample size is required in order that the standard deviation of $\overline{X}$ be:

(a) $\frac{1}{4}$ of the population standard deviation?

(b) $\frac{1}{7}$ of the population standard deviation?

(c) 12% of the population standard deviation?

7.37 A population has distribution

Value	Probability
1	.2
2	.6
3	.2

Let X_1 and X_2 be independent and each have the same distribution as the population.

(a) Determine the missing elements in the table for the sampling distribution of $\overline{X} = (X_1 + X_2)/2$.

$\overline{x}$	Probability
1.0	
1.5	
2.0	.44
2.5	.24
3.0	

(b) Find the expected value of $\overline{X}$.

(c) If the sample size is increased to 81, give the mean and variance of $\overline{X}$.

7.38 A population has distribution

Value	Probability
1	.6
3	.3
5	.1

Let X_1 and X_2 be independent and each have the same distribution as the population.

(a) Determine the missing elements in the table for the sampling distribution of $\overline{X} = (X_1 + X_2)/2$.

$\bar{x}$	Probability
1	
2	.36
3	.21
4	
5	

(b) Find the expected value of $\overline{X}$.

(c) If the sample size is increased to 25, give the mean and variance of $\overline{X}$.

7.39 Suppose the weights of the contents of cans of mixed nuts have a normal distribution with mean 32.4 ounces and standard deviation .4 ounce. For a random sample of size $n = 9$

(a) What are the mean and standard deviation of $\overline{X}$?

(b) What is the distribution of $\overline{X}$? Is this distribution exact or approximate?

(c) Find the probability that $\overline{X}$ lies between 32.3 and 32.6.

7.40 The weights of pears in an orchard are normally distributed with mean .32 pound and standard deviation .08 pound.

(a) If one pear is selected at random, what is the probability that its weight will be between .28 and .34 pound?

(b) If $\overline{X}$ denotes the average weight of a random sample of four pears, what is the probability that $\overline{X}$ will be between .28 and .34 pound?

7.41 Suppose that the size of pebbles in a river bed is normally distributed with mean 12.1 mm and standard deviation 3.2 mm. A random sample of 9 pebbles will be measured. Let $\overline{X}$ denote the average size of the sampled pebbles.

(a) What is the distribution of $\overline{X}$?

(b) What is the probability that $\overline{X}$ is smaller than 10 mm?

(c) What percentage of the pebbles in the river bed are of size smaller than 10 mm?

7.42 A random sample of size 150 is taken from the population of the ages of juniors enrolled at a large university during one semester. This population has mean 21.1 years and standard deviation 2.6. The population distribution is not normal.

(a) Is it reasonable to assume a normal distribution for the sample mean $\overline{X}$? Why or why not?

(b) Find the probability that $\overline{X}$ lies between 17.85 and 25.65 years.

(c) Find the probability that $\overline{X}$ exceeds 25.91 years.

7.43 A company that manufactures car mufflers finds that the labor to set up and run a nearly automatic machine has mean $\mu = 1.9$ hours and $\sigma = 1.2$ hours. For a random sample of 36 runs,

(a) determine the mean and standard deviation of $\overline{X}$.

(b) What can you say about the distribution of $\overline{X}$?

7.44 Refer to Exercise 7.43. Evaluate

(a) $P[\,\overline{X} > 2.2\,]$

(b) $P[\,1.65 < \overline{X} < 2.25\,]$

7.45 Visitors to a popular Internet site rated the newest gaming console on a scale of 1 to 5 stars. The following probability distribution is proposed based on over 1400 individual ratings.

$\bar{x}$	$f(x)$
1	.02
2	.02
3	.04
4	.12
5	.80

(a) For a future random sample of 40 ratings, what are the mean and standard deviation of $\overline{X}$?

(b) What is the distribution of $\overline{X}$? Is this distribution exact or approximate?

(c) Find the probability that $\overline{X}$ lies between 4.6 and 4.8 stars.

7.46 A special purpose coating must have the proper abrasion. The standard deviation is known to be

21. Consider a random sample of 49 abrasion measurements.

(a) Find the probability that the sample mean $\overline{X}$ will lie within 2 units of the population mean—that is, $P[-2 \le \overline{X} - \mu \le 2]$.

(b) Find the number k so that $P[-k \le \overline{X} - \mu \le k] = .90$.

(c) What is the probability that $\overline{X}$ will differ from μ by more than 4 units?

7.47 The daily number of kayaks sold, X, at a water sports store has the probability distribution

x	0	1	2
$f(x)$	.5	.3	.2

(a) Find the expected number of kayaks sold in a day.

(b) Find the standard deviation of the number of kayaks sold in a day.

(c) Find the probability distribution of the total number of kayaks sold in the next two days. Suppose that the number of sales on different days are independent.

(d) Over the next 64 days, what is the approximate probability that at least 53 kayaks will be sold?

(e) How many kayaks should the store order to have approximate probability .95 of meeting the total demand in the next 64 days?

7.48 Suppose packages of cream cheese coming from an automated processor have weights that are normally distributed. For one day's production run, the mean is 8.2 ounces and the standard deviation is 0.1 ounce.

(a) If the packages of cream cheese are labeled 8 ounces, what proportion of the packages weigh less than the labeled amount?

(b) If only 5% of the packages exceed a specified weight w, what is the value of w?

(c) Suppose two packages are selected at random from the day's production. What is the probability that the average weight of the two packages is less than 8.3 ounces?

(d) Suppose 5 packages are selected at random from the day's production. What is the probability that at most one package weighs at least 8.3 ounces?

7.49 Suppose the amount of sun block lotion in plastic bottles leaving a filling machine has a normal distribution. The bottles are labeled 300 milliliters (ml) but the actual mean is 302 ml and the standard deviation is 2 ml.

(a) What is the probability that an individual bottle will contain less than 299 ml?

(b) If only 5% of the bottles have contents that exceed a specified amount v, what is the value of v?

(c) Two bottles can be purchased together in a twin-pack. What is the probability that the mean content of bottles in a twin-pack is less than 299 ml? Assume the contents of the two bottles are independent.

(d) If you purchase two twin-packs of the lotion, what is the probability that only one of the twin-packs has a mean bottle content less than 299 ml?

CLASS PROJECTS

1. (a) Count the number of occupants X including the driver in each of 20 passing cars. Calculate the mean $\overline{x}$ of your sample.

(b) Repeat part (a) 10 times.

(c) Collect the data sets of the individual car counts x from the entire class and construct a relative frequency histogram.

(d) Collect the $\overline{x}$ values from the entire class (10 from each student) and construct a relative frequency histogram for $\overline{x}$, choosing appropriate class intervals.

(e) Plot the two relative frequency histograms and comment on the closeness of their shapes to the normal distribution.

2. (a) Collect a sample of size 7 and compute $\bar{x}$ and the sample median.
   (b) Repeat part (a) 30 times.
   (c) Plot dot diagrams for the values of the two statistics in part (a). These plots reflect the individual sampling distributions.
   (d) Compare the amount of variation in $\overline{X}$ and the median.

   In this exercise, you might record weekly soft-drink consumptions, sentence lengths, or hours of sleep for different students.

COMPUTER PROJECT

1. Conduct a simulation experiment on the computer to verify the central limit theorem. Generate $n = 6$ observations from the continuous distribution that is uniform on 0 to 1. Calculate $\overline{X}$. Repeat 150 times. Make a histogram of the $\overline{X}$ values and a normal-scores plot. Does the distribution of $\overline{X}$ appear to be normal for $n = 6$? You may wish to repeat with $n = 20$.

   If MINITAB is available, you could use the commands

> **Calc > Random Data > Uniform.**
> Type *150* after **Generate** and *C1 – C6* in **Store in Column(s).** Click **OK.**
> **Calc > Row Statistics.** Click **Mean** and type *C1 – C6* in **Input Variables.**
> Type C7 in **Store.** Click **OK.**

The 150 means in C7 can then be summarized using the dialog box sequence described in Chapter 2.

> **Stat > Basic Statistics > Graphical summary.**
> Type C7 in **Variables.** Click **OK.**

8

Drawing Inferences from Large Samples

Building Strong Evidence from Diverse Individual Cases

One of the major contributions of statistics to modern thinking is the understanding that information on single, highly variable observations can be combined in great numbers to obtain very precise information about a population.

Although each individual is satisfied or not satisfied with his or her job, a sample survey can obtain accurate information about the population proportion that are satisfied.

About 50% are satisfied with their present job.

1. INTRODUCTION

Inferences are generalizations about a population that are made on the basis of a sample collected from the population. For instance, a researcher interested in the growth of pine trees plants 40 seedlings. The heights of these 40 plants would be a sample that is hopefully representative of the population consisting of all current and future seedlings that could be planted.

More specifically, we begin by modeling the population by a probability distribution which has a numerical feature of interest called a **parameter.** A random sample from the population distribution will provide information about the parameter.

The problem of statistical inference arises when we wish to make generalizations about a population when only a sample will be available. Once a sample is observed, its main features can be determined by the methods of descriptive summary discussed in Chapters 2 and 3. However, more often than not, our principal concern is with not just the particular data set, but what can be said about the population based on the information extracted from analyzing the sample data. We call these generalizations **statistical inferences** or just **inferences.**

Consider a study on the effectiveness of a diet program in which 30 participants report their weight loss. We have on hand a sample of 30 measurements of weight loss. But is the goal of the study confined to this particular group of 30 persons? No, it is not. We need to evaluate the effectiveness of the diet program for the population of potential users. The sample measurements must, of course, provide the basis for any conclusions.

> **Statistical inference** deals with drawing conclusions about population parameters from an analysis of the sample data.

Any inference about a population parameter will involve some uncertainty because it is based on a sample rather than the entire population. To be meaningful, a statistical inference must include a specification of the uncertainty that is determined using the ideas of probability and the sampling distribution of the statistic.

The purpose of an investigation, spelled out in a clear statement of purpose as described in Chapter 1, can reveal the nature of the inference required to answer important questions.

The two most important types of inferences are (1) **estimation of parameter(s)** and (2) **testing of statistical hypotheses.** The true value of a parameter is an unknown constant that can be correctly ascertained only by an exhaustive study of the population, if indeed that were possible. Our objective may be to obtain a guess or an estimate of the unknown true value along with a determination of its accuracy. This type of inference is called **estimation of parameters.** An alternative objective may be to examine whether the sample data support or contradict the investigator's conjecture about the true value of the parameter. This latter type of inference is called **testing of statistical hypotheses.**

Example 1 Types of Inference: Point Estimation, Interval Estimation, and Testing Hypotheses

What is the degree of participation in community service? A student at a large midwestern university questioned $n = 40$ students in her dorm concerning the amount of time they spent doing community service during the past month. The data on times, in hours, are presented in Table 1.

TABLE 1 Community Service (hours) in Month

0	0	0	0	0	0	0	1	1	1
2	2	2	2	2	3	3	3	3	4
4	4	4	5	5	5	5	5	5	5
5	5	6	6	6	8	10	15	20	25

Employing the ideas of Chapter 2, we can calculate a descriptive summary for this set of measurements.

Sample mean $\overline{x} = 4.55$ Sample standard deviation $s = 5.17$
Sample median $= 4.00$ First quartile $= 1.50$ Third quartile $= 5.00$

However the target of our investigation is not just the particular set of measurements recorded, but also concerns the vast population of hours of community service for all students at this university or even similar universities. The population distribution of time is unknown to us. Consequently, parameters such as the population mean μ and population standard deviation σ are also unknown. If we take the view that the 40 observations represent a random sample from the population distribution of times, one goal of this study may be to "learn about μ." More specifically, depending on the purpose of the study, we may wish to do one, two, or all three of the following:

1. Estimate a single value for the unknown μ (**point estimation**).
2. Determine an interval of plausible values for μ (**interval estimation**).
3. Decide whether or not the mean time μ is 2.6 hours, which was found to be the mean time in an earlier study (**testing statistical hypotheses**)

Example 2 Inferences about an Unknown Proportion

A market researcher wishes to determine what proportion of new-car buyers in her city are satisfied with their new car one year after the purchase. She feels, correctly, that this assessment could be made quickly and effectively by sampling a small fraction of the new-car buyers. The persons selected will be asked if they are satisfied as opposed to not satisfied. Suppose that a sample of 200 randomly selected new-car purchasers are interviewed and 168 say they are satisfied.

A descriptive summary of this finding is provided by

$$\text{Sample proportion satisfied} = \frac{168}{200} = .84$$

Here the target of our investigation is the proportion of new-car purchasers who are satisfied, p, in the entire collection of new-car buyers in the city. The value of p is unknown. The sample proportion $= .84$ sheds some light on p, but it is subject to some error since it draws only on a part of the population. The investigator would like to evaluate its margin of error and provide an interval of plausible values of p.

Also, she may wish to test the hypothesis that proportion satisfied, p, for her city is not lower than the value given by a nationwide vehicle satisfaction study.

2. POINT ESTIMATION OF A POPULATION MEAN

The object of point estimation is to calculate, from the sample data, a single number that is likely to be close to the unknown value of the parameter. The available information is assumed to be in the form of a random sample $X_1, X_2, \ldots, X_n$ of size n taken from the population. We wish to formulate a statistic such that its value computed from the sample data would reflect the value of the population parameter as closely as possible.

A statistic intended for estimating a parameter is called a **point estimator,** or simply an **estimator.**
The standard deviation of an estimator is called its **standard error:** S.E.

When we estimate a population mean from a random sample, perhaps the most intuitive estimator is the sample mean,

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

For instance, to estimate the mean hours of community service in Example 1, we would naturally compute the mean of the sample measurements. Employing the estimator $\overline{X}$, with the data of Table 1, we get the result $\overline{x} = 4.55$ hours, which we call a **point estimate,** or simply an **estimate of μ.**

Without an assessment of accuracy, a single number quoted as an estimate may not serve a very useful purpose. We must indicate the extent of variability in the distribution of the estimator. The standard deviation, alternatively called the **standard error** of the estimator, provides information about its variability.

In order to study the properties of the sample mean $\overline{X}$ as an estimator of the population mean μ, let us review the results from Chapter 7.

1. $E(\overline{X}) = \mu$.

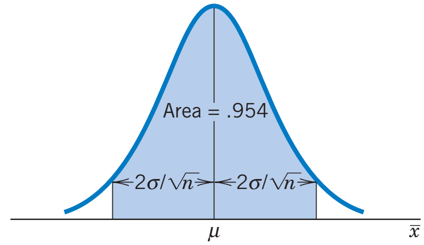


Figure 1 Approximate normal distribution of $\overline{X}$.

2. $\text{sd}(\overline{X}) = \dfrac{\sigma}{\sqrt{n}}$ so $\text{S.E.}(\overline{X}) = \dfrac{\sigma}{\sqrt{n}}$.
3. With large n, $\overline{X}$ is nearly normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$.

The first two results show that the distribution of $\overline{X}$ is centered around μ and its standard error is $\sigma/\sqrt{n}$, where σ is the population standard deviation and n the sample size.

To understand how closely $\overline{X}$ is expected to estimate μ, we now examine the third result, which is depicted in Figure 1. Recall that, in a normal distribution, the interval running two standard deviations on either side of the mean contains probability .954. Thus, prior to sampling, the probability is .954 that the estimator $\overline{X}$ will be at most a distance $2\sigma/\sqrt{n}$ from the true parameter value μ. This probability statement can be rephrased by saying that **when we are estimating μ by $\overline{X}$, the 95.4% error margin is $2\sigma/\sqrt{n}$.**

Use of the probability .954, which corresponds to the multiplier 2 of the standard error, is by no means universal. The following notation will facilitate our writing of an expression for the $100(1 - \alpha)\%$ error margin, where $1 - \alpha$ denotes the desired high probability such as .95 or .90.

Notation

$$z_{\alpha/2} = \text{Upper } \alpha/2 \text{ point of standard normal distribution}$$

That is, the area to the right of $z_{\alpha/2}$ is $\alpha/2$, and the area between $-z_{\alpha/2}$ and $z_{\alpha/2}$ is $1 - \alpha$ (see Figure 2).

A few values of $z_{\alpha/2}$ obtained from the normal table appear in Table 2 for easy reference. To illustrate the notation, suppose we want to determine the 90% error margin. We then set $1 - \alpha = .90$ so $\alpha/2 = .05$ and, from Table 2, we have $z_{.05} = 1.645$. Therefore, when estimating μ by $\overline{X}$, the 90% error margin is $1.645\,\sigma/\sqrt{n}$.

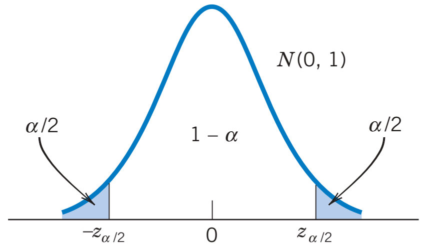


Figure 2 The notation $z_{\alpha/2}$.

TABLE 2 Values of $z_{\alpha/2}$

$1 - \alpha$	.80	.85	.90	.95	.99
$z_{\alpha/2}$	1.28	1.44	1.645	1.96	2.58

A minor difficulty remains in computing the standard error of $\overline{X}$. The expression involves the unknown population standard deviation σ, but we can estimate σ by the sample standard deviation.

$$S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n - 1}}$$

When n is large, the effect of estimating the standard error $\sigma/\sqrt{n}$ by $S/\sqrt{n}$ can be neglected. We now summarize.

Point Estimation of the Mean

Parameter: Population mean μ.

Data: $X_1, \ldots, X_n$ (a random sample of size n)

Estimator: $\overline{X}$ (sample mean)

$$\text{S.E.}(\overline{X}) = \frac{\sigma}{\sqrt{n}} \qquad \text{Estimated S.E.}(\overline{X}) = \frac{S}{\sqrt{n}}$$

For large n, the $100(1 - \alpha)\%$ error margin is $z_{\alpha/2}\,\sigma/\sqrt{n}$.
(If σ is unknown, use S in place of σ.)

Example 3 Point Estimation of the Mean Time of Community Service

From the data of Example 1, consisting of 40 measurements of the time devoted to community service the past month, give a point estimate of the population mean amount of time and state a 95% error margin.

SOLUTION The sample mean and the standard deviation computed from the 40 measurements in Table 1 are

$$\bar{x} = \frac{\sum x_i}{40} = 4.55$$

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{39}} = \sqrt{26.715} = 5.17$$

To calculate the 95% error margin, we set $1 - \alpha = .95$ so that $\alpha/2 = .025$ and $z_{\alpha/2} = 1.96$. Therefore, the 95% error margin is

$$\frac{1.96\, s}{\sqrt{n}} = \frac{1.96 \times 5.17}{\sqrt{40}} = 1.60 \text{ hours}$$

Our estimate of the population mean time is 4.55 hours per month. We do not expect the population mean to be exactly this value and we attach a 95% error margin of plus and minus 1.60 hours.

Caution: (a) Standard error should not be interpreted as the "typical" error in a problem of estimation as the word "standard" may suggest. For instance, when $\text{S.E.}(\bar{X}) = .3$, we should not think that the error $(\bar{X} - \mu)$ is likely to be .3, but rather, prior to observing the data, the probability is approximately .954 that the error will be within $\pm 2(\text{S.E.}) = \pm .6$.

(b) An estimate and its variability are often reported in either of the forms: estimate $\pm$ S.E. or estimate $\pm 2(\text{S.E.})$. In reporting a numerical result such as 53.4 ± 4.6, we must specify whether 4.6 represents S.E., 2(S.E.), or some other multiple of the standard error.

DETERMINING THE SAMPLE SIZE

During the planning stage of an investigation, it is important to address the question of sample size. Because sampling is costly and time-consuming, the investigator needs to know beforehand the sample size required to give the desired precision.

In order to determine how large a sample is needed for estimating a population mean, we must specify

$$d = \text{Desired error margin}$$

and

$$1 - \alpha = \text{Probability associated with error margin}$$

Referring to the expression for a $100(1 - \alpha)\%$ error margin, we then equate:

$$z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = d$$

This gives an equation in which n is unknown. Solving for n, we obtain

$$n = \left[\frac{z_{\alpha/2}\,\sigma}{d} \right]^2$$

which determines the required sample size. Of course, the solution is rounded to the next higher integer, because a sample size cannot be fractional.

This determination of sample size is valid provided $n > 30$, so that the normal approximation to $\overline{X}$ is satisfactory.

To be $100(1 - \alpha)\%$ sure that the error of estimation $|\overline{X} - \mu|$ does not exceed d, the **required sample size** is

$$n = \left[\frac{z_{\alpha/2}\,\sigma}{d} \right]^2$$

If σ is completely unknown, a small-scale preliminary sampling is necessary to obtain an estimate of σ to be used in the formula to compute n.

Example 4 Determining a Sample Size for Collecting Water Samples

A limnologist wishes to estimate the mean phosphate content per unit volume of lake water. It is known from studies in previous years that the standard deviation has a fairly stable value of $\sigma = 4$. How many water samples must the limnologist analyze to be 90% certain that the error of estimation does not exceed 0.8 milligrams?

SOLUTION Here $\sigma = 4$ and $1 - \alpha = .90$, so $\alpha/2 = .05$. The upper .05 point of the $N(0, 1)$ distribution is $z_{.05} = 1.645$. The tolerable error is $d = .8$. Computing

$$n = \left[\frac{1.645 \times 4}{.8} \right]^2 = 67.65$$

we round up to determine that the required sample size is $n = 68$.

Exercises

8.1 A researcher wants to estimate μ, the mean number of minutes before a person scores over 200 on a new computer game. She wishes to get estimates, separately, for each of the groups (a) novices, (b) occasional game players, and (c) expert game players. When using $\overline{X}$ to estimate μ find the (i) standard error and the (ii) $100(1 - \alpha)\%$ error margin for each group.

(a) Novice $n = 125$ $\sigma = 65$ 98% error margin 4
(b) Occasonal $n = 47$ $\sigma = 38$ 95% error margin 8
(c) Expert $n = 6$ $\sigma = 20$ 90% error margin 10

8.2 The same owners operate two coffee shops in a large building. One is (a) small and the other (b) large. On any day, the number of customers is only observed for one shop. Determine the point estimate of μ, the mean number of persons served during a weekday morning, and the $100(1 - \alpha)\%$ error margin, separately, for each shop when

(a) Small $n = 65$ $\bar{x} = 103$
$s = 15$ $1 - \alpha = .95$

(b) Large $n = 45$ $\overline{x} = 260$
$s = 40$ $1 - \alpha = .975$

8.3 Consider the problem of estimating μ, the mean time per day surfing the Internet, for each of (a) business, (b) physical, and (c) social science majors. Obtain a point estimate of μ and the estimated standard error separately for each of three majors when

(a) Business $n = 30$ $\sum x_i = 185$ $\sum (x_i - \overline{x})^2 = 116$
(b) Phy. Sci. $n = 25$ $\sum x_i = 145$ $\sum (x_i - \overline{x})^2 = 103$
(c) Social Sci. $n = 45$ $\sum x_i = 297$ $\sum (x_i - \overline{x})^2 = 194$

8.4 To study the growth of pine trees at an early stage, a nursery worker records 40 measurements of the heights (cm) of one-year-old red pine seedlings.

2.6	1.9	1.8	1.6	1.4	2.2	1.2	1.6
1.6	1.5	1.4	1.6	2.3	1.5	1.1	1.6
2.0	1.5	1.7	1.5	1.6	2.1	2.8	1.0
1.2	1.2	1.8	1.7	0.8	1.5	2.0	2.2
1.5	1.6	2.2	2.1	3.7	1.7	1.7	1.2

Courtesy of Professor Alan Ek.

The summary Statistics are

$$n = 40 \quad \overline{x} = 1.715 \quad s = .475 \text{ centimeter}$$

Find the (a) point estimate of the population mean height μ, (b) standard error, and (c) 98% error margin.

8.5 A credit company randomly selected 50 contested items and recorded the dollar amount being contested. These contested items had sample mean $\overline{x} = 95.74$ dollars and $s = 24.63$ dollars. Construct a point estimate for the population mean contested amount, μ, and give its 90% error margin.

8.6 A manager at a power company monitored the employee time required to process high-efficiency lamp bulb rebates. A random sample of 40 applications gave a sample mean time of 3.8 minutes and a standard deviation of 1.2 minutes. Construct a point estimate for the population mean time to process, μ, and give its 90% error margin.

8.7 When estimating μ from a large sample, suppose that one has found the 95% error margin of $\overline{X}$ to be 4.2. From this information, determine:

(a) The estimated S.E. of $\overline{X}$.

(b) The 90% error margin.

8.8 A small business owner wants to estimate the value of her inventory and she needs to estimate, separately, the mean value of (a) expensive items and (b) other items. She needs the most accuracy for expensive items. Determine the sample size n that is required for estimating the population mean of each class of items. The population standard deviation σ and the desired error margin are specified.

(a) Expensive $\sigma = 135$ 98% error margin 2
(b) Other $\sigma = 18$ 90% error margin 5

8.9 When estimating the mean of a population, how large a sample is required in order that the 95% error margin be:

(a) $\frac{1}{8}$ of the population standard deviation?

(b) 15% of the population standard deviation?

8.10 Assume that the standard deviation of the number of violent incidents in one hour of children's shows on television is 3.2. An investigator would like to be 99% sure that the true mean number of violent incidents per hour is estimated within 1.4 incidents. For how many randomly selected hours does she need to count the number of violent incidents?

8.11 Referring to Exercise 8.5, suppose that the survey of 50 contested items was, in fact, a pilot study intended to give an idea of the population standard deviation. Assuming $\sigma = \$25$, determine the sample size that is needed for estimating the population mean amount contested with a 98% error margin of \$5.00.

8.12 Assume that the standard deviation of the heights of five-year-old boys is 3.5 inches. How many five-year-old boys need to be sampled if we want to be 90% sure that the population mean height is estimated within .5 inch?

8.13 Let the abbreviation PSLT stand for the percent of the gross family income that goes into paying state and local taxes. Suppose one wants to estimate the mean PSLT for the population of all families in New York City with gross incomes in the range \$50,000 to \$60,000. If $\sigma = 2.5$, how many such families should be surveyed if one wants to be 90% sure of being able to estimate the true mean PSLT within .5?

8.14 To estimate μ with a 90% error margin of 2.9 units, one has determined that the required sample size is 108. What then is the required sample size if one wants the 95% error margin to be 1.8 units?

3. CONFIDENCE INTERVAL FOR A POPULATION MEAN

For point estimation, a single number lies in the forefront even though a standard error is attached. Instead, it is often more desirable to produce an interval of values that is likely to contain the true value of the parameter.

Ideally, we would like to be able to collect a sample and then use it to calculate an interval that would definitely contain the true value of the parameter. This goal, however, is not achievable because of sample-to-sample variation. Instead, we insist that before sampling the proposed interval will contain the true value with a specified high probability. This probability, called the **level of confidence,** is typically taken as .90, .95, or .99.

To develop this concept, we first confine our attention to the construction of a confidence interval for a population mean μ, assuming that the population is normal and the standard deviation σ is *known*. This restriction helps to simplify the initial presentation of the concept of a confidence interval. Later on, we will treat the more realistic case where σ is also unknown.

A probability statement about $\overline{X}$ based on the normal distribution provides the cornerstone for the development of a confidence interval. From Chapter 7, recall that when the population is normal, the distribution of $\overline{X}$ is also normal. It has mean μ and standard deviation $\sigma/\sqrt{n}$. Here μ is unknown, but $\sigma/\sqrt{n}$ is a known number because the sample size n is known and we have assumed that σ is known.

The normal table shows that the probability is .95 that a normal random variable will lie within 1.96 standard deviations from its mean. For $\overline{X}$, we then have

$$P\left[\mu - 1.96\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + 1.96\frac{\sigma}{\sqrt{n}}\right] = .95$$

as shown in Figure 3.

Now, the relation

$$\mu - 1.96\frac{\sigma}{\sqrt{n}} < \overline{X} \qquad \text{is the same as} \qquad \mu < \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}$$

and

$$\overline{X} < \mu + 1.96\frac{\sigma}{\sqrt{n}} \qquad \text{is the same as} \qquad \overline{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu$$

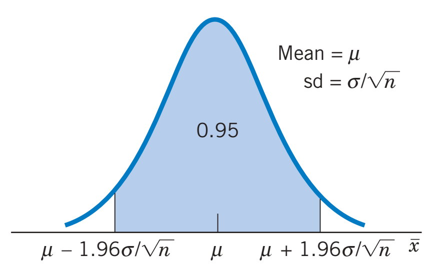


Figure 3 Normal distribution of $\overline{X}$.

as we can see by transposing $1.96\sigma/\sqrt{n}$ from one side of an inequality to the other. Therefore, the event

$$\left[\mu - 1.96\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

is equivalent to

$$\left[\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

In essence, both events state that the difference $\overline{X} - \mu$ lies between $-1.96\sigma/\sqrt{n}$ and $1.96\sigma/\sqrt{n}$. Thus, the probability statement

$$P\left[\mu - 1.96\frac{\sigma}{\sqrt{n}} < \overline{X} < \mu + 1.96\frac{\sigma}{\sqrt{n}}\right] = .95$$

can also be expressed as

$$P\left[\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right] = .95$$

This second form tells us that, before we sample, the random interval from $\overline{X} - 1.96\sigma/\sqrt{n}$ to $\overline{X} + 1.96\sigma/\sqrt{n}$ will include the unknown parameter μ with a probability of .95. Because σ is assumed to be known, both the upper and lower endpoints can be computed as soon as the sample data are available. Guided by the above reasonings, we say that the interval

$$\left(\overline{X} - 1.96\frac{\sigma}{\sqrt{n}}, \quad \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

or its realization $(\overline{x} - 1.96\,\sigma/\sqrt{n},\ \overline{x} + 1.96\,\sigma/\sqrt{n})$ is a **95% confidence interval for μ** when the population is normal and σ known.

Example 5 Calculating a Confidence Interval—Normal Population σ Known

The daily carbon monoxide (CO) emission from a large production plant will be measured on 25 randomly selected weekdays. The production process is always being modified and the current mean value of daily CO emissions μ is unknown. Data collected over several years confirm that, for each year, the distribution of CO emission is normal with a standard deviation of .8 ton.

Suppose the sample mean is found to be $\overline{x} = 2.7$ tons. Construct a 95% confidence interval for the current daily mean emission μ.

SOLUTION The population is normal, and the observed value $\overline{x} = 2.7$.

$$\left(2.7 - 1.96\frac{.8}{\sqrt{25}}, \quad 2.7 + 1.96\frac{.8}{\sqrt{25}}\right) = (2.39, 3.01) \text{ tons}$$

is a 95% confidence interval for μ. Since μ is unknown, we cannot determine whether or not μ lies in this interval.

Referring to the confidence interval obtained in Example 5, we must **not** speak of the probability of the fixed interval (2.39, 3.01) covering the true mean μ. The particular interval (2.39, 3.01) either does or does not cover μ, and we will never know which is the case.

We need not always tie our discussion of confidence intervals to the choice of a 95% level of confidence. An investigator may wish to specify a different high probability. We denote this probability by $1 - \alpha$ and speak of a $100(1 - \alpha)\%$ **confidence interval.** The only change is to replace 1.96 with $z_{\alpha/2}$, where $z_{\alpha/2}$ denotes the upper $\alpha/2$ point of the standard normal distribution (i.e., the area to the right of $z_{\alpha/2}$ is $\alpha/2$, as shown in Figure 2).

In summary, when the population is normal and σ is known, a $100(1 - \alpha)\%$ confidence interval for μ is given by

$$\left(\overline{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \qquad \overline{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right)$$

INTERPRETATION OF CONFIDENCE INTERVALS

To better understand the meaning of a confidence statement, we use the computer to perform repeated samplings from a normal distribution with $\mu = 100$ and $\sigma = 10$. Ten samples of size 7 are selected, and a 95% confidence interval $\overline{x} \pm 1.96 \times 10/\sqrt{7}$ is computed from each. For the first sample, $\overline{x} = 104.3$ and the interval is 104.3 ± 7.4, or 96.9 to 111.7. This and the other intervals are illustrated in Figure 4, where each vertical line segment represents one confidence

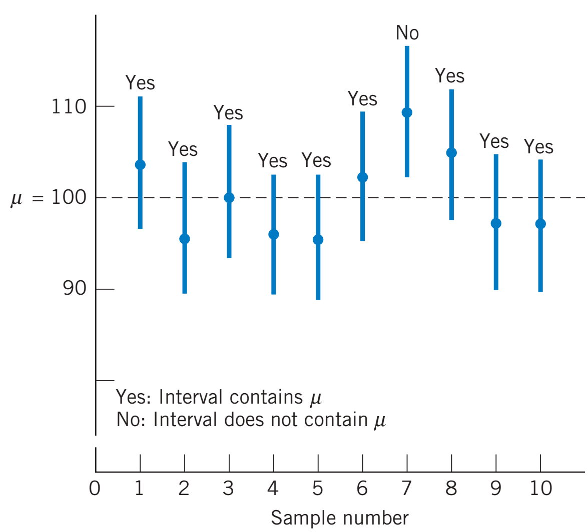


Figure 4 Interpretation of the confidence interval for μ.

interval. The midpoint of a line is the observed value of $\overline{X}$ for that particular sample. Also note that all the intervals are of the same length $2 \times 1.96\sigma/\sqrt{n} = 14.8$. Of the 10 intervals shown, 9 cover the true value of μ. This is not surprising, because the specified probability .95 represents the long-run relative frequency of these intervals covering the true $\mu = 100$.

Because confidence interval statements are the most useful way to communicate information obtained from a sample, certain aspects of their formulation merit special emphasis. Stated in terms of a 95% confidence interval for μ, these are:

1. Before we sample, a confidence interval $(\overline{X} - 1.96\sigma/\sqrt{n}, \overline{X} + 1.96\sigma/\sqrt{n})$ is a random interval that attempts to cover the true value of the parameter μ.
2. The probability

$$P\left[\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right] = .95$$

   interpreted as the long-run relative frequency over many repetitions of sampling asserts that about 95% of the intervals will cover μ.
3. Once $\overline{x}$ is calculated from an observed sample, the interval

$$\left(\overline{x} - 1.96\frac{\sigma}{\sqrt{n}}, \quad \overline{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

   which is a realization of the random interval, is presented as a 95% confidence interval for μ. A numerical interval having been determined, it is no longer sensible to speak about the probability of its covering a fixed quantity μ.
4. In any application we never know if the 95% confidence interval covers the unknown mean μ. Relying on the long-run relative frequency of coverage in property 2, we adopt the terminology **confidence** once the interval is calculated.

At this point, one might protest, "I have only one sample and I am not really interested in repeated sampling." But if the confidence estimation techniques presented in this text are mastered and followed each time a problem of interval estimation arises, then over a lifetime approximately 95% of the intervals will cover the true parameter. Of course, this is contingent on the validity of the assumptions underlying the techniques—independent normal observations here.

LARGE SAMPLE CONFIDENCE INTERVALS FOR μ

Having established the basic concepts underlying confidence interval statements, we now turn to the more realistic situation for which the population standard deviation σ is unknown. We require the sample size n to be large in

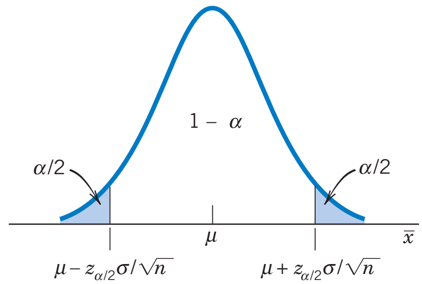


Figure 5 Normal distribution of $\overline{X}$.

order to dispense with the assumption of a normal population. The central limit theorem then tells us that $\overline{X}$ is nearly normal whatever the form of the population. Referring to the normal distribution of $\overline{X}$ in Figure 5 and the discussion accompanying Figure 2, we again have the probability statement

$$P\left[\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1 - \alpha$$

(Strictly speaking, this probability is approximately $1 - \alpha$ for a nonnormal population.) Even though the interval

$$\left(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \qquad \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

will include μ with the probability $1 - \alpha$, it does not serve as a confidence interval because it involves the unknown quantity σ. However, because n is large, replacing $\sigma/\sqrt{n}$ with its estimator $S/\sqrt{n}$ does not appreciably affect the probability statement. Summarizing, we find that the large sample confidence interval for μ has the form

$$\text{Estimate} \pm (z\ \text{Value})(\text{Estimated standard error})$$

Large Sample Confidence Interval for μ

When n is large, a $100(1 - \alpha)\%$ confidence interval for μ is given by

$$\left(\overline{X} - z_{\alpha/2}\frac{S}{\sqrt{n}}, \qquad \overline{X} + z_{\alpha/2}\frac{S}{\sqrt{n}}\right)$$

where S is the sample standard deviation.

Example 6 A Confidence Interval for the Mean Time of Community Service

Refer to the data in Example 1, consisting of 40 measurements of the time spent on community service during the past month. The summary statistics are

$$n = 40 \qquad \bar{x} = 4.55 \qquad s = 5.17$$

Compute (a) 90% and (b) 80% confidence intervals for the mean number of hours per month.

SOLUTION The sample size $n = 40$ is large, so a normal approximation for the distribution of the sample mean $\overline{X}$ is appropriate. From the sample data, we know that

$$\bar{x} = 4.55 \text{ hours} \qquad \text{and} \qquad s = 5.17 \text{ hours}$$

(a) With $1 - \alpha = .90$, we have $\alpha/2 = .05$, and $z_{.05} = 1.645$,

$$1.645 \frac{s}{\sqrt{n}} = \frac{1.645 \times 5.17}{\sqrt{40}} = 1.34$$

The 90% confidence interval for the population mean of number of hours worked μ becomes

$$\left(\bar{x} - 1.645 \frac{s}{\sqrt{n}},\ \bar{x} + 1.645 \frac{s}{\sqrt{n}} \right) = (4.55 - 1.34, 4.55 + 1.34)$$

or approximately (3.2, 5.9) hours per month. This means that we can be 90% confident that the mean hours per month μ is in the interval 3.2 to 5.9 hours. We have this confidence because about 90% of the random samples of 40 students would produce intervals $\bar{x} \pm 1.645\, s/\sqrt{n}$ that contain μ.

(b) With $1 - \alpha = .80$, we have $\alpha/2 = .10$, and $z_{.10} = 1.28$, so

$$1.28 \frac{s}{\sqrt{n}} = \frac{1.28 \times 5.17}{\sqrt{40}} = 1.05$$

The 80% confidence interval for μ becomes

$$(4.55 - 1.05, 4.55 + 1.05) \qquad \text{or} \qquad (3.5, 5.6) \text{ hours per month.}$$

Comparing the two confidence intervals, we note that the 80% confidence interval is shorter than the 90% interval. A shorter interval seems to give a more precise location for μ but suffers from a lower long-run frequency of being correct.

CONFIDENCE INTERVAL FOR A PARAMETER

The concept of a confidence interval applies to any parameter, not just the mean. It requires that a lower limit L and an upper limit U be computed from the sample data. Then the random interval from L to U must have the specified probability of covering the true value of the parameter. The large sample $100(1 - \alpha)\%$ confidence interval for μ has

$$L = \overline{X} - z_{\alpha/2} \frac{S}{\sqrt{n}} \qquad U = \overline{X} + z_{\alpha/2} \frac{S}{\sqrt{n}}$$

Definition of a Confidence Interval for a Parameter

An interval (L, U) is a $100(1 - \alpha)\%$ confidence interval for a parameter if

$$P[L < \text{Parameter} < U] = 1 - \alpha$$

and the endpoints L and U are computable from the sample.

Example 7 A Confidence Interval for the Mean Time to Complete a Test

Madison recruits for the fire department need to complete a timed test that simulates working conditions. It includes placing a ladder against a building, pulling out a section of fire hose, dragging a weighted object, and crawling in a simulated attic environment. The times, in seconds, for recruits to complete the test for Madison firefighter are

425	389	380	421	438	331	368	417	403	416	385	315
427	417	386	386	378	300	321	286	269	225	268	317
287	256	334	342	269	226	291	280	221	283	302	308
296	266	238	286	317	276	254	278	247	336	296	259
270	302	281	228	317	312	327	288	395	240	264	246
294	254	222	285	254	264	277	266	228	347	322	232
365	356	261	293	354	236	285	303	275	403	268	250
279	400	370	399	438	287	363	350	278	278	234	266
319	276	291	352	313	262	289	273	317	328	292	279
289	312	334	294	297	304	240	303	255	305	252	286
297	353	350	276	333	285	317	296	276	247	339	328
267	305	291	269	386	264	299	261	284	302	342	304
336	291	294	323	320	289	339	292	373	410	257	406
374	268										

Obtain a 95% confidence interval for the mean time of recruits who complete the test.

SOLUTION A computer calculation gives

```
SAMPLE SIZE   158
MEAN          307.77
STD DEV       51.852
```

Since $1 - \alpha = .95$, $\alpha/2 = .025$, and $z_{.025} = 1.96$, the large sample 95% confidence interval for μ becomes

$$\left(\overline{x} - z_{\alpha/2}\frac{s}{\sqrt{n}},\quad \overline{x} + z_{\alpha/2}\frac{s}{\sqrt{n}} \right)$$

$$= \left(307.77 - 1.96\frac{51.852}{\sqrt{158}},\quad 307.77 + 1.96\frac{51.852}{\sqrt{158}} \right)$$

or

$$(299.68, 315.86) \text{ seconds}$$

When the sample size is large, the sample also contains information on the shape of the distribution that can be elicited by graphical displays. Figure 6 gives the stem-and-leaf display, with the data rounded to two places, accompanied by the boxplot. The confidence interval pertains to the mean of a population with a long right-hand tail.

```
STEM-AND-LEAF of TIME N   158
LEAF UNIT = 10
        2 | 2222223333
        2 | 44444555555555
        2 | 66666666666666667777777777777
        2 | 88888888888888889999999999999999
        3 | 000000000001111111111
        3 | 222222233333333
        3 | 444555555
        3 | 6667777
        3 | 88888899
        3 | 00001111
        4 | 22233
```

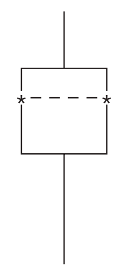


Figure 6 A stem-and-leaf display and boxplot give more information about the form of the population.

Exercises

8.15 Refer to Example 2, Chapter 2, where Table 2 records the number of items returned by 30 persons to a large discount department store in late December. The summary statistics are $n = 30 \quad \bar{x} = 2.538 \; s = 1.303.$
Obtain a 98% confidence interval for μ, the population mean number of items returned.

8.16 A company wants to check the consistency of electronic copies of signatures for consumer credit purchases. A sample of 49 electronic signatures are available from the same customer. One measure of consistency in signing is the total length that the script is outside the signature box. The sample of 49 signatures yielded a mean length of .21 with a standard deviation of .19 centimeter.
Obtain a 99% confidence interval for this customer's population mean length outside the box.

8.17 Each day of the year, a large sample of cellular phone calls is selected and a 95% confidence interval is calculated for the mean length of all cellular phone calls on that day. Of these 365 confidence intervals, one for each day of the year, approximately how many will cover their respective population means? Explain your reasoning.

8.18 A forester measures 100 needles off a pine tree and finds $\bar{x} = 3.1$ centimeters and $s = 0.7$ centimeter. She reports that a 95% confidence interval for the mean needle length is

$$3.1 - 1.96 \frac{0.7}{\sqrt{100}} \quad \text{to} \quad 3.1 + 1.96 \frac{0.7}{\sqrt{100}}$$

$$\text{or} \quad (2.96, 3.24)$$

(a) Is the statement correct?

(b) Does the interval (2.96, 3.24) cover the true mean? Explain.

8.19 In a study on the nutritional qualities of fast foods, the amount of fat was measured for a random sample of 35 hamburgers of a particular restaurant chain. The sample mean and standard deviation were found to be 30.2 and 3.8 grams, respectively. Use these data to construct a 95% confidence interval for the mean fat content in hamburgers served in these restaurants.

8.20 In the same study described in Exercise 8.19, the sodium content was also measured for the sampled hamburgers, and the sample mean and standard deviation were 658 and 47 milligrams, respectively. Determine a 98% confidence interval for the true mean sodium content.

8.21 An entomologist sprayed 120 adult Melon flies with a specific low concentration of malathion and observed their survival times. The mean and standard deviation were found to be 18.3 and 5.2 days, respectively. Use these data to construct a 99% confidence interval for the true mean survival time.

8.22 Students are asked about the number of songs they downloaded from a pay-for-songs Web site the last month. From a random sample of 39 students, the sample mean was 4.7 with a standard deviation of 3.2.

(a) Obtain a 95% confidence interval for μ, the mean number of songs downloaded by the population of all students.

(b) Does μ lie in your interval obtained in Part(a)?

(c) In a long series of repeated experiments, with new samples of 39 students collected for each experiment, what proportion of the resulting confidence intervals will contain the true population mean? Explain your reasoning.

8.23 The freshness of produce at a super-store is rated on a scale of 1 to 5 with 5 being very fresh. From a random sample of 49 customers, the average score was 3.8 with a standard deviation of .7.

(a) Obtain a 95% confidence interval for the population mean, μ, the mean score for the distribution of all possible customers.

(b) Does μ lie in your interval obtained in Part(a)? Explain.

(c) In a long series of repeated experiments, with new random samples of 49 customers each day, what proportion of the resulting confidence intervals will contain the true population mean? Explain your reasoning.

8.24 Referring to Example 7, where the 158 times to complete the firefighter test have mean 307.77 and standard deviation 51.852, obtain a 99% confidence interval for the mean time of all possible recruits who would complete the test.

8.25 Based on a survey of 140 employed persons in a city, the mean and standard deviation of the commuting distances between home and the principal place of business are found to be 8.6 and 4.3 miles, respectively. Determine a 90% confidence interval for the mean commuting distance for the population of all employed persons in the city.

8.26 A manager at a power company monitored the employee time required to process high-efficiency lamp bulb rebates. A random sample of 40 applications gave a sample mean time of 3.8 minutes and a standard deviation of 1.2 minutes. Construct a 90% confidence interval for the mean time to process μ.

8.27 A credit company randomly selected 50 contested items and recorded the dollar amount being contested. These contested items had a sample mean $\bar{x} = 95.74$ dollars and $s = 24.63$ dollars. Construct a 95% confidence interval for the mean amount contested, μ.

8.28 In a study to determine whether a certain stimulant produces hyperactivity, 55 mice were injected with 10 micrograms of the stimulant. Afterward, each mouse was given a hyperactivity rating score. The mean score was $\bar{x} = 14.9$ and $s = 2.8$. Give a 95% confidence interval for the population mean score μ.

8.29 Refer to the Statistics in Context section of Chapter 7 concerning monthly changes in the Canadian to U.S. exchange rate. A computer calculation gives $\bar{x} = -.0017$ and $s = .0207$ for the $n = 154$ monthly changes. Find a 95% confidence interval for the mean monthly change.

8.30 An employee of an on-campus copy center wants to determine the mean number of copies before a cartridge needs to be replaced. She records the life length in thousands of copies for 43 cartridges and obtains

$$n = 43 \quad \bar{x} = 8.12 \quad s = 1.78 \text{ thousand copies}$$

Obtain a 90% confidence interval for the population mean, μ, number of copies in thousands before a cartridge should be replaced.

8.31 Refer to the 40 height measurements given in Exercise 8.4, which have

$$n = 40 \quad \bar{x} = 1.715 \quad s = .475 \text{ centimeter}$$

Calculate a 99% confidence interval for the population mean height.

8.32 Radiation measurements on a sample of 65 microwave ovens produced $\bar{x} = .11$ and $s = .06$. Determine a 95% confidence interval for the mean radiation.

8.33 Refer to the data on the growth of female salmon in the marine environment in Table D.7 of the Data Bank. A computer calculation gives a 95% confidence interval.

```
One-Sample Z: Fmarine

Variable   N    Mean  StDev       95.0% CI
Fmarine   40  429.15  41.05  ( 416.43, 441.87)
```

(a) Does the 95% confidence interval cover the true mean growth of all female salmon in that marine environment?

(b) Why are you 95% confident that the interval (416.43, 441.87) covers the true mean?

8.34 Refer to the data on the girth, in centimeters, of grizzly bears in Table D.8 of the Data Bank. A computer calculation gives

```
One-Sample Z: Girth

Variable   N    Mean  StDev       95.0% CI
Girth     61   93.39  21.79   ( 87.93, 98.86)
```

(a) Does the 95% confidence interval cover the true mean girth of all grizzly bears in the area of the study? Explain.

(b) Why are you 95% confident that the interval (87.93, 98.86) covers the true mean?

8.35 The amount of PCBs (polychlorinated biphenyls) was measured in 40 samples of soil that were treated with contaminated sludge. The following summary statistics were obtained.

$$\bar{x} = 3.56 \qquad s = .5 \text{ ppm}$$

(a) Obtain a 95% confidence interval for the population mean μ, amount of PCBs in the soil.

Answer parts (b), (c), and (d) Yes, No, or Cannot tell. Explain your answer.

(b) Does the sample mean PCB content lie in your interval obtained in part (a)?

(c) Does the population mean PCB content lie in your interval obtained in part (a)?

(d) It is likely that 95% of the data lie in your interval obtained in part (a)?

8.36 A national fast food chain, with thousands of franchise locations, needed to audit the books at each location. They first selected a sample of 50 locations and performed the audit. They determined that a 95% confidence interval for the mean time to complete an audit is

$$(28.4 \text{ hours}, 52.7 \text{ hours})$$

Answer the following questions "Yes," "No," or "Cannot tell" and justify your answer.

(a) Does the population mean lie in the interval (28.4, 52.7)?

(b) Does the sample mean lie in the interval (28.4, 52.7)?

(c) For a future sample of 50 franchise locations, will the sample mean fall in the interval (28.4, 52.7)?

(d) Does 95% of the sample data lie in the interval (28.4, 52.7)?

4. TESTING HYPOTHESES ABOUT A POPULATION MEAN

Broadly speaking, the goal of testing statistical hypotheses is to determine if a claim or conjecture about some feature of the population, a parameter, is strongly supported by the information obtained from the sample data. Here we illustrate the testing of hypotheses concerning a population mean μ. The available data will be assumed to be a random sample of size n from the population of interest. Further, the sample size n will be large ($n > 30$ for a rule of thumb).

The formulation of a hypotheses testing problem and then the steps for solving it require a number of definitions and concepts. We will introduce these key statistical concepts

Null hypothesis and the **alternative hypothesis**

Type I and **Type II errors**

Level of significance

Rejection region

***P*–value**

in the context of a specific problem to help integrate them with intuitive reasoning.

PROBLEM: Can an upgrade reduce the mean transaction time at automated teller machines? At peak periods, customers are subject to unreasonably long waits before receiving cash. To help alleviate this difficulty, the bank wants to reduce the time it takes a customer to complete a transaction. From extensive records, it is found that the transaction times have a distribution with mean 270 and standard deviation 24 seconds. The teller machine vendor suggests that a new software and hardware upgrade will reduce the mean time for a customer to complete a transaction. For experimental verification, a random sample of 38 transaction times will be taken at a machine with the upgrade and the sample mean $\overline{X}$ calculated. How should the result be used toward a statistical validation of the claim that the true (population) mean transaction time is less than 270 seconds?

Whenever we seek to establish a claim or conjecture on the basis of strong support from sample data, the problem is called one of hypothesis testing.

FORMULATING THE HYPOTHESES

In the language of statistics, the claim or the research hypothesis that we wish to establish is called the **alternative hypothesis** H_1. The opposite statement, one that nullifies the research hypothesis, is called the **null hypothesis** H_0. The word "null" in this context means that the assertion we are seeking to establish is actually void.

Formulation of H_0 and H_1

When our goal is to establish an assertion with substantive support obtained from the sample, the negation of the assertion is taken to be the null hypothesis H_0 and the assertion itself is taken to be the alternative hypothesis H_1.

Our initial question, "Is there strong evidence in support of the claim?" now translates to "Is there strong evidence for rejecting H_0?" The first version typically appears in the statement of a practical problem, whereas the second version is ingrained in the conduct of a statistical test. It is crucial to understand the correspondence between the two formulations of a question.

Before claiming that a statement is established statistically, adequate evidence from data must be produced to support it. A close analogy can be made to a criminal court trial where the jury clings to the null hypothesis of "not guilty" unless there is convincing evidence of guilt. The intent of the hearings is to establish the assertion that the accused is guilty, rather than to prove that he or she is innocent.

	Court Trial	Testing Statistical Hypothesis
Requires strong evidence to establish:	Guilt.	Conjecture (research hypothesis).
Null hypothesis (H_0):	Not guilty.	Conjecture is false.
Alternative hypothesis (H_1):	Guilty.	Conjecture is true.
Attitude:	Uphold "not guilty" unless there is a strong evidence of guilt.	Retain the null hypothesis unless it makes the sample data very unlikely to happen.

False rejection of H_0 is a more serious error than failing to reject H_0 when H_1 is true.

Once H_0 and H_1 are formulated, our goal is to analyze the sample data in order to choose between them.

A **decision rule,** or a **test of the null hypothesis,** specifies a course of action by stating what sample information is to be used and how it is to be used in making a decision. Bear in mind that we are to make one of the following two decisions:

Decisions

Either

Reject H_0 and conclude that H_1 is substantiated

or

Retain H_0 and conclude that H_1 fails to be substantiated

Rejection of H_0 amounts to saying that H_1 is substantiated, whereas nonrejection or retention of H_0 means that H_1 fails to be substantiated. A key point is that a decision to reject H_0 must be based on strong evidence. Otherwise, the claim H_1 could not be established beyond a reasonable doubt.

In our problem of evaluating the upgraded teller machine, let μ be the population mean transaction time. Because μ is claimed to be lower than 270 seconds, we formulate the alternative hypothesis as $H_1 : \mu < 270$. According to the description of the problem, the researcher does not care to distinguish between the situations that $\mu = 270$ and $\mu > 270$ for the claim is false in either case. For this reason, it is customary to write the null hypothesis simply as a statement of no difference. Accordingly, we formulate the

Testing Problem

$$\text{Test } H_0 : \mu = 270 \qquad \text{versus} \qquad H_1 : \mu < 270$$

TEST CRITERION AND REJECTION REGION

Naturally, the sample mean $\overline{X}$, calculated from the measurements of $n = 38$ randomly selected transaction times, ought to be the basis for rejecting H_0 or not. The question now is: For what sort of values of $\overline{X}$ should we reject H_0? Because the claim states that μ is low (a left-sided alternative), only low values of $\overline{X}$ can contradict H_0 in favor of H_1. Therefore, a reasonable decision rule should be of the form

$$\begin{array}{l} \text{Reject } H_0 \text{ if } \overline{X} \leq c \\ \text{Retain } H_0 \text{ if } \overline{X} > c \end{array}$$

This decision rule is conveniently expressed as $R : \overline{X} \leq c$, where R stands for the rejection of H_0. Also, in this context, the set of outcomes $[\, \overline{X} \leq c\,]$ is called the **rejection region** or **critical region,** and the cutoff point c is called the **critical value.**

The cutoff point c must be specified in order to fully describe a decision rule. To this end, we consider the case when H_0 holds, that is, $\mu = 270$. Rejection of H_0 would then be a wrong decision, amounting to a false acceptance of the claim—a serious error. For an adequate protection against this kind of error, we must ensure that $P[\, \overline{X} \leq c\,]$ is very small when $\mu = 270$. For example, suppose that we wish to hold a low probability of $\alpha = .05$ for a wrong rejection of H_0. Then our task is to find the c that makes

$$P[\, \overline{X} \leq c\,] = .05 \qquad \text{when} \qquad \mu = 270$$

We know that, for large n, the distribution of $\overline{X}$ is approximately normal with mean μ and standard deviation $\sigma / \sqrt{n}$, whatever the form of the underlying population. Here $n = 38$ is large, and we initially assume that σ is known. Specifically, we assume that $\sigma = 24$ seconds, the same standard deviation as with the original money machines. Then, when $\mu = 270$, the distribution of $\overline{X}$ is $N(270,\ 24 / \sqrt{38}\,)$ so

$$Z = \frac{\overline{X} - 270}{24 / \sqrt{38}}$$

has the $N(0, 1)$ distribution.

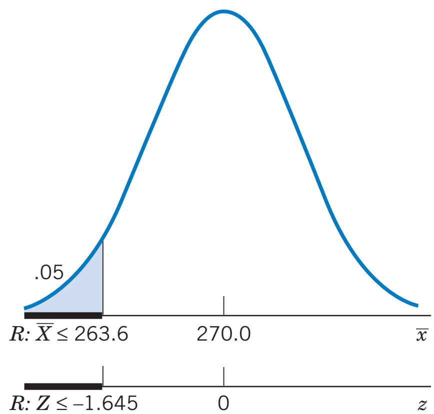


Figure 7 Rejection region with the cutoff $c = 263.6$.

Because $P[Z \leq -1.645] = .05$, the cutoff c on the $\overline{x}$ scale must be 1.645 standard deviations below $\mu_0 = 270$ or

$$c = 270 - 1.645\left(\frac{24}{\sqrt{38}}\right) = 270 - 6.40 = 263.60$$

Our decision rule is now completely specified by the rejection region (see Figure 7)

$$R: \overline{X} \leq 263.6$$

that has $\alpha = .05$ as the probability of wrongly rejecting H_0.

Instead of locating the rejection region on the scale of $\overline{X}$, we can cast the decision criterion on the standardized scale as well:

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{\overline{X} - 270}{24 / \sqrt{38}}$$

and set the rejection region as $R: Z \leq -1.645$ (see Figure 7). This form is more convenient because the cutoff -1.645 is directly read off the normal table, whereas the determination of c involves additional numerical work.

The random variable $\overline{X}$ whose value serves to determine the action is called the **test statistic.**

A **test of the null hypothesis** is a course of action specifying the set of values of a test statistic $\overline{X}$, for which H_0 is to be rejected.

This set is called the **rejection region** of the test.

A test is completely specified by a test statistic and the rejection region.

TWO TYPES OF ERROR AND THEIR PROBABILITIES

Up to this point we only considered the probability of rejecting H_0 when, in fact, H_0 is true and illustrated how a decision rule is determined by setting this probability equal to .05. The following table shows all the consequences that might arise from the use of a decision rule.

Decision Based on Sample	Unknown True Situation: H_0 True $\mu = 270$	Unknown True Situation: H_1 True $\mu < 270$
Reject H_0	Wrong rejection of H_0 (Type I error)	Correct decision
Retain H_0	Correct decision	Wrong retention of H_0 (Type II error)

In particular, when our sample-based decision is to reject H_0, either we have a correct decision (if H_1 is true) or we commit a Type I error (if H_0 is true). On the other hand, a decision to retain H_0 either constitutes a correct decision (if H_0 is true) or leads to a Type II error. To summarize:

Two Types of Error

Type I error: Rejection of H_0 when H_0 is true

Type II error: Nonrejection of H_0 when H_1 is true

α = Probability of making a Type I error (also called the **level of significance**)

β = Probability of making a Type II error

In our problem of evaluating the upgraded teller machine, the rejection region is of the form $R: \overline{X} \le c$; so that,

$$\alpha = P[\overline{X} \le c] \quad \text{when} \quad \mu = 270 \quad (H_0 \textit{ true})$$

$$\beta = P[\overline{X} > c] \quad \text{when} \quad \mu < 270 \quad (H_1 \textit{ true})$$

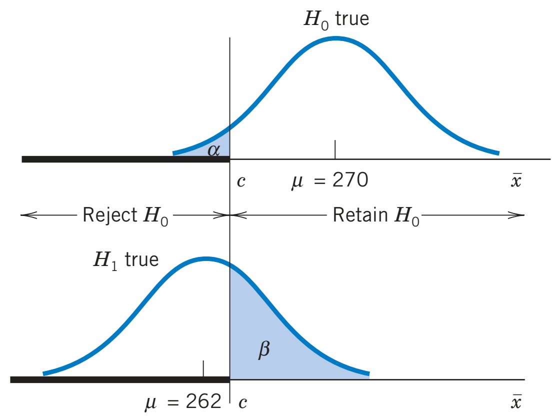


Figure 8 The error probabilities α and β.

Of course, the probability β depends on the numerical value of μ that prevails under H_1. Figure 8 shows the Type I error probability α as the shaded area under the normal curve that has $\mu = 270$ and the Type II error probability β as the shaded area under the normal curve that has $\mu = 262$, a case of H_1 being true.

From Figure 8, it is apparent that no choice of the cutoff c can minimize both the error probabilities α and β. If c is moved to the left, α gets smaller but β gets larger, and if c is moved to the right, just the opposite effects take place. In view of this dilemma and the fact that a wrong rejection of H_0 is the more serious error, we hold α at a predetermined low level such as .10, .05, or .01 when choosing a rejection region. We will not pursue the evaluation of β, but we do note that if the β turns out to be uncomfortably large, the sample size must be increased.

PERFORMING A TEST

When determining the rejection region for this example, we assumed that $\sigma = 24$ seconds, the same standard deviation as with the original money machines. Then, when $\mu = 270$, the distribution of $\overline{X}$ is $N(270, 24/\sqrt{38})$ and the rejection region $R: \overline{X} \leq 263.6$ was arrived at by fixing $\alpha = .05$ and referring

$$Z = \frac{\overline{X} - 270}{24/\sqrt{38}}$$

to the standard normal distribution.

In practice, we are usually not sure about the assumption that $\sigma = 24$, the standard deviation of the transaction times using the upgraded machine, is the same as with the original teller machine. But that does not cause any problem as long as the sample size is large. When n is large $(n > 30)$, the normal approximation for $\overline{X}$ remains valid even if σ is estimated by the sample standard deviation S. Therefore, for testing $H_0: \mu = \mu_0$ versus $H_1: \mu < \mu_0$ with level of significance α, we employ the test statistic

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

and set the rejection region $R: Z \leq -z_\alpha$. This test is commonly called a **large sample normal test** or a **Z test.**

Example 8 A One-Sided Test of Hypotheses to Establish That $\mu < 270$

Referring to the automated teller machine transaction times, suppose that, from the measurements of a random sample of 38 transaction times, the sample mean and standard deviation are found to be 261 and 22 seconds, respectively. Test the null hypothesis $H_0: \mu = 270$ versus $H_1: \mu < 270$ using a 2.5% level of significance and state whether or not the claim $\mu < 270$ is substantiated.

SOLUTION Because $n = 38$ and the null hypothesis specifies that μ has the value $\mu_0 = 270$, we employ the test statistic

$$Z = \frac{\overline{X} - 270}{S/\sqrt{38}}$$

The rejection region should consist of small values of Z because H_1 is left-sided. For a 2.5% level of significance, we take $\alpha = .025$, and since $z_{.025} = 1.96$, the rejection region is (see Figure 9) $R: Z \leq -1.96$.

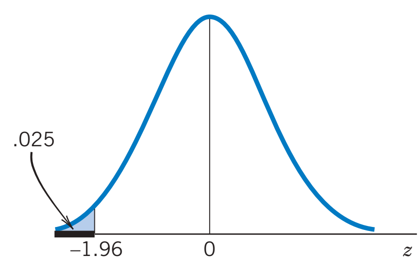


Figure 9 Rejection region for Z.

With the observed values $\overline{x} = 261$ and $s = 22$, we calculate the test statistic

$$z = \frac{261 - 270}{22/\sqrt{38}} = -2.52$$

Because this observed z is in R, the null hypothesis is rejected at the level of significance $\alpha = .025$. We conclude that the claim of a reduction in the mean transaction time is strongly supported by the data.

P–VALUE: HOW STRONG IS A REJECTION OF H_0?

Our test in Example 8 was based on the fixed level of significance $\alpha = .025$, and we rejected H_0 because the observed $z = -2.52$ fell in the rejection region $R: Z \leq -1.96$. A strong evidence against H_0 emerged due to the fact that

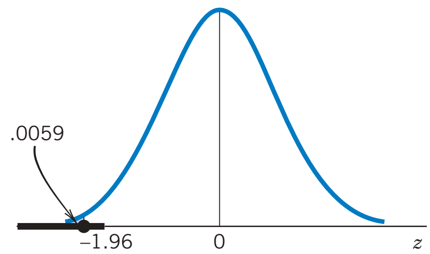


Figure 10 P−value with left-sided rejection region.

a small α was used. The natural question at this point is: How small an α could we use and still arrive at the conclusion of rejecting H_0? To answer this question, we consider the observed $z = -2.52$ itself as the cutoff point (critical value) and calculate the rejection probability

$$P[Z \leq -2.52] = .0059$$

The smallest possible α that would permit rejection of H_0, on the basis of the observed $z = -2.52$, is therefore .0059 (see Figure 10). It is called the **significance probability** or ***P*−value** of the observed z. This very small P−value, .0059, signifies a **strong rejection** of H_0 or that the result is highly statistically significant.

> The ***P*−value** is the probability, calculated under H_0, that the test statistic takes a value equal to or more extreme than the value actually observed.

The P−value serves as a measure of the strength of evidence against H_0. A **small *P*−value** means that the null hypothesis is **strongly rejected** or the result is **highly statistically significant.**

Our illustrations of the basic concepts of hypothesis tests thus far focused on a problem where the alternative hypothesis is of the form $H_1: \mu < \mu_0$, called a left-sided alternative. If the alternative hypothesis in a problem states that the true μ is larger than its null hypothesis value of μ_0, we formulate the right-sided alternative $H_1: \mu > \mu_0$ and use a right-sided rejection region $R: Z \geq z_\alpha$.

The Steps for Testing Hypotheses

1. Formulate the null hypothesis H_0 and the alternative hypothesis H_1.
2. Test criterion: State the test statistic and the form of the rejection region.
3. With a specified α, determine the rejection region.
4. Calculate the test statistic from the data.
5. Draw a conclusion: State whether or not H_0 is rejected at the specified α and interpret the conclusion in the context of the problem. Also, it is a good statistical practice to calculate the P−value and strengthen the conclusion.

We illustrate the right-sided case and the main steps for conducting a statistical test as summarized above.

Example 9 Evaluating a Weight Loss Diet—Calculation of a P–Value

A brochure inviting subscriptions for a new diet program states that the participants are expected to lose over 22 pounds in five weeks. Suppose that, from the data of the five-week weight losses of 56 participants, the sample mean and standard deviation are found to be 23.5 and 10.2 pounds, respectively. Could the statement in the brochure be substantiated on the basis of these findings? Test with $\alpha = .05$. Also calculate the P–value and interpret the result.

SOLUTION Let μ denote the population mean weight loss from five weeks of participation in the program. Because our aim is to substantiate the assertion that $\mu > 22$ pounds, we formulate the hypotheses

$$H_0 : \mu = 22 \qquad \text{versus} \qquad H_1 : \mu > 22$$

The sample size is $n = 56$. Denoting the sample mean weight loss of the 56 participants by $\overline{X}$ and the standard deviation by S, our test statistic is

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{\overline{X} - 22}{S/\sqrt{56}}$$

Because H_1 is right-sided, the rejection region should be of the form $R: Z \geq c$. Because $z_{.05} = 1.645$, the test with level of significance .05 has the rejection region (see Figure 11) $R: Z \geq 1.645$.

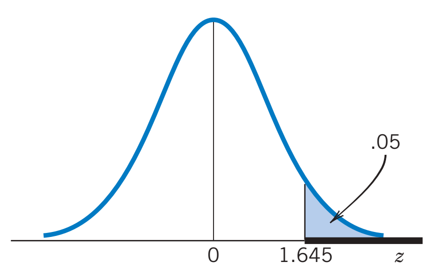


Figure 11 Right-sided rejection region with $\alpha = .05$.

With the observed values $\overline{x} = 23.5$ and $s = 10.2$, we calculate

$$z = \frac{23.5 - 22}{10.2/\sqrt{56}} = 1.10$$

Because 1.10 is not in R, we do not reject the null hypothesis. We conclude that, with level of significance $\alpha = .05$, the stated claim that $\mu > 22$ is not substantiated.

Because our observed z is 1.10 and larger values are more extreme, the significance probability of this result is

$$P\text{–value} = P[Z \geq 1.10] = .1357 \quad \text{(from the normal table)}$$

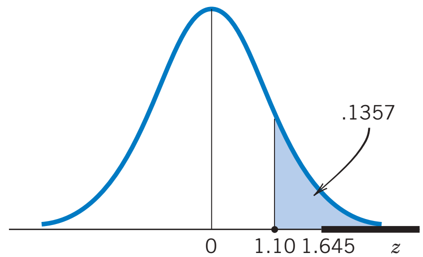


Figure 12 P−value with right-sided rejection region.

That is, .1357 is the smallest α at which H_0 could be rejected (Figure 12). This is not ordinarily considered a negligible chance so we conclude that the data do not provide a strong basis for rejection of H_0.

The preceding hypotheses are called **one-sided hypotheses,** because the values of the parameter μ under the alternative hypothesis lie on one side of those under the null hypothesis. The corresponding tests are called **one-sided tests** or **one-tailed tests.** By contrast, we can have a problem of testing the null hypothesis

$$H_0 : \mu = \mu_0$$

versus the **two-sided alternative** or **two-sided hypothesis**

$$H_1 : \mu \neq \mu_0$$

Here H_0 is to be rejected if $\overline{X}$ is too far away from μ_0 in either direction, that is, if Z is too small or too large. For a level α test, we divide the rejection probability α equally between the two tails and construct the rejection region

$$R : Z \leq -z_{\alpha/2} \quad \text{or} \quad Z \geq z_{\alpha/2}$$

which can be expressed in the more compact notation

$$R : |Z| \geq z_{\alpha/2}$$

Example 10 Testing Hypotheses about the Mean Time of Community Service

Consider the data of Example 1 concerning 40 observations on time devoted to community service in the past month. Do these data indicate that the population mean time is different from 2.6 hours?

SOLUTION We are seeking evidence in support of $\mu \neq 2.6$ so the hypotheses should be formulated as

$$H_0 : \mu = 2.6 \quad \text{versus} \quad H_1 : \mu \neq 2.6$$

The sample size $n = 40$ being large, we will employ the test statistic

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{\overline{X} - 2.6}{S/\sqrt{40}}$$

The two-sided form of H_1 dictates that the rejection region must also be two-sided.

Let us choose $\alpha = .05$, then $\alpha/2 = .025$ and $z_{.025} = 1.96$. Consequently, for $\alpha = .05$, the rejection region is (see Figure 13)

$$R: |Z| \geq 1.96$$

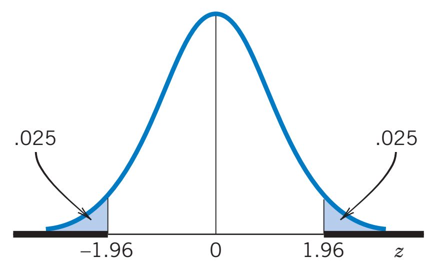


Figure 13 Two-sided rejection region with $\alpha = .05$.

From Example 1, $\overline{x} = 4.55$ and $s = 5.17$, so the observed value of the test statistic is

$$z = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} = \frac{4.55 - 2.6}{5.17/\sqrt{40}} = 2.39$$

Because $|z| = 2.39$ is larger than 1.96, we reject H_0 at $\alpha = .05$.

In fact, the large value $|z| = 2.39$ seems to indicate a much stronger rejection of H_0 than that arising from the choice of $\alpha = .05$. How small an α can we set and still reject H_0? This is precisely the idea underlying the significance probability or the P–value. We calculate (see Figure 14)

$$\begin{aligned} P\text{–value} &= P[\,|Z| \geq 2.39\,] \\ &= P[\,Z \leq -2.39\,] + P[\,Z \geq 2.39\,] \\ &= 2 \times .0084 = .0168 \end{aligned}$$

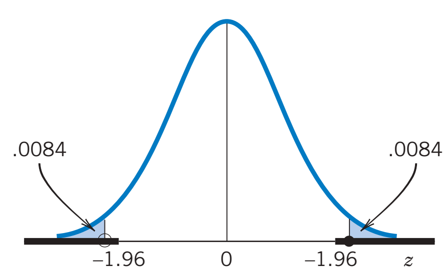


Figure 14 P–value with two-sided rejection region.

With α as small as .0138, H_0 is still rejected. This very small $P-$value gives strong support for H_1.

In summary:

Large Sample Tests for μ

When the sample size is large, a **Z test** concerning μ is based on the normal test statistic

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

The rejection region is one- or two-sided depending on the alternative hypothesis. Specifically,

$$\begin{array}{lll} H_1: \mu > \mu_0 & \text{requires} & R: Z \geq z_\alpha \\ H_1: \mu < \mu_0 & & R: Z \leq -z_\alpha \\ H_1: \mu \neq \mu_0 & & R: |Z| \geq z_{\alpha/2} \end{array}$$

Because the central limit theorem prevails for large n, no assumption is required as to the shape of the population distribution.

Exercises

8.37 Stated here are some claims or research hypotheses that are to be substantiated by sample data. In each case, identify the null hypothesis H_0 and the alternative hypothesis H_1 in terms of the population mean μ.

(a) The mean time a health insurance company takes to pay claims is less than 14 working days.

(b) The average person watching a movie at a local multiplex theater spends over \$4.50 on refreshments.

(c) The mean hospital bill for a birth in the city is less than \$5000.

(d) The mean time between purchases of a brand of mouthwash by loyal customers is different from 60 days.

8.38 From an analysis of the sample data, suppose that the decision has been to reject the null hypothesis. In the context of each part (a–d) of Exercise 8.37, answer the following questions:

In what circumstance is it a correct decision?

When is it a wrong decision, and what type of error is then made?

8.39 From an analysis of the sample data, suppose that the decision has been made to retain the null hypothesis. In the context of each part (a–d) of Exercise 8.37, answer the following questions.

In what circumstance is it a correct decision?

When is it a wrong decision, and what type of error is then made?

8.40 For each situation (a–d) in Exercise 8.37, state which of the following three forms of the rejection region is appropriate when σ is known.

$$R: \overline{X} \le c \quad \text{(left-sided)}$$
$$R: \overline{X} \ge c \quad \text{(right-sided)}$$
$$R: |\overline{X} - \mu_0| \ge c \quad \text{(two-sided)}$$

8.41 Each part (a–d) of this problem gives the population standard deviation σ, the statement of a claim about μ, the sample size n, and the desired level of significance α. Formulate (i) the hypotheses, (ii) the test statistic Z, and (iii) the rejection region. (The answers to part (a) are provided.)

(a) $\sigma = 2$ claim: $\mu > 30$, $n = 55$, $\alpha = .05$

[Answers: (i) $H_0: \mu = 30$, $H_1: \mu > 30$

(ii) $Z = \dfrac{\overline{X} - 30}{2/\sqrt{55}}$ (iii) $R: Z \ge 1.645$]

(b) $\sigma = .085$ claim: $\mu < .15$ $n = 125$ $\alpha = .025$

(c) $\sigma = 8.6$ claim: $\mu \ne 80$ $n = 38$ $\alpha = .01$

(d) $\sigma = 1.23$ claim: $\mu \ne 0$ $n = 40$ $\alpha = .06$

8.42 Suppose that the observed values of the sample mean in the contexts of parts (a–d) of Exercise 8.41 are given as follows. Calculate the test statistic Z and state the conclusion with the specified α.

(a) $\bar{x} = 30.54$ (b) $\bar{x} = .136$

(c) $\bar{x} = 77.35$ (d) $\bar{x} = -.59$

8.43 A market researcher wants to perform a test with the intent of establishing that his company's medium pump bottle of soap has a mean life greater than 40 days. The sample size is 70 and he knows that $\sigma = 5.6$.

(a) If you set the rejection region to be $R: \overline{X} \ge 41.31$, what is the level of significance of your test?

(b) Find the numerical value of c so that the test $R: \overline{X} \ge c$ has a 5% level of significance.

8.44 With reference to Exercise 8.15, perform a test with the intent of establishing that the mean number of items returned is greater than 2.0. Take $\alpha = .02$.

8.45 With reference to Exercise 8.44,

(a) test

$$H_0: \mu = 2.0 \qquad H_1: \mu \ne 2.0$$

with $\alpha = .02$.

(b) Based on your decision in Part (a), what error could you have possibly made? Explain in the context of the problem.

8.46 An investigator at a large midwestern university wants to determine the typical weekly amount of time students work on part-time jobs. More particularly, he wants to test the null hypothesis that the mean time is 15 hours versus a two-sided alternative. A sample of 39 students who hold part-time jobs is summarized by the computer output

```
Descriptive Statistics: hours

Variable   N   Mean   Median  StDev
Hours     39  16.69   15.00    7.61
```

(a) Perform the hypothesis test at the 1% level of significance.

(b) Calculate the significance probability and interpret the result.

8.47 Refer to the data on the growth of female salmon growth in the marine environment in Table D.7 of the Data Bank. A computer calculation for a test of $H_0: \mu = 411$ versus $H_1: \mu \ne 411$ is given below.

```
Test of mu = 411 vs mu not = 411

Variable   N    Mean   StDev
Fmarine   40  429.15   41.05

Variable    Z      P
Fmarine   2.80  0.005
```

(a) What is the conclusion if you test with $\alpha = .01$?

(b) What mistake could you have made in part (a)?

(c) Before you collected the data, what was the probability of making the mistake in part (a)?

(d) Give a long-run relative frequency interpretation of the probability in part (c).

(e) Give the P–value of your test with $\alpha = .01$.

8.48 Refer to the data on the girth, in centimeters, of grizzly bears in Table D.8 of the Data Bank. A computer calculation for a test of $H_0: \mu = 100$ centimeters versus $H_1: \mu \ne 100$ gives

```
One-Sample Z: Girth

Test of mu = 100 vs mu not = 100

Variable    N     Mean     StDev
Girth      61    93.39     21.79

Variable    Z       P
Girth     -2.37   0.018
```

(a) What is the conclusion if you test with $\alpha = .02$?

(b) What mistake could you have made in part (a)?

(c) Before you collected the data, what was the probability of making the mistake in part (a)?

(d) Give a long-run relative frequency interpretation of the probability in part (c).

(e) Give the P–value of your test with $\alpha = .02$.

8.49 A manager at a power company monitored the employee time required to process high-efficiency lamp bulb rebates. A random sample of 40 applications gave a sample mean time of 3.8 minutes and a standard deviation of 1.2 minutes.

(a) Is the claim that $\mu > 3.5$ minutes substantiated by these data? Test with $\alpha = .10$.

(b) Based on your decision in Part (a), what error could you have possibly made? Explain in the context of the problem.

8.50 A credit company randomly selected 50 contested items and recorded the dollar amount being contested. These contested items had sample mean $\bar{x} = 95.74$ dollars and $s = 24.63$ dollars. Is the claim "μ differs from 105 dollars" substantiated by these data? Test with $\alpha = .01$.

8.51 A company wants to establish that the mean life of its batteries, when used in a wireless mouse, is over 183 days. The data consist of the life lengths of batteries in 64 different wireless mice.

(a) Formulate the null and alternative hypotheses.

(b) What is the conclusion to your test if $\bar{x} = 190.5$ and $s = 32$ days? Take $\alpha = .05$.

(c) Based on your decision in Part (b), what error could you have possibly made?

8.52 In a given situation, suppose H_0 was not rejected at $\alpha = .02$. Answer the following questions as "yes," "no," or "can't tell" as the case may be.

(a) Would H_0 also be retained at $\alpha = .01$?

(b) Would H_0 also be retained at $\alpha = .05$?

(c) Is the $P-$value smaller than .02?

8.53 A company wishing to improve its customer service collected hold times from 75 randomly selected incoming calls to its hot line that were put on hold. These calls had sample mean hold time $\bar{x} = 3.4$ minutes and $s = 2.4$ minutes. Is the claim that $\mu > 3.0$ minutes substantiated by these data? Test with $\alpha = .05$.

8.54 A company's mixed nuts are sold in cans and the label says that 25% of the contents is cashews. Suspecting that this might be an overstatement, an inspector takes a random sample of 35 cans and measures the percent weight of cashews [i.e., 100(weight of cashews / weight of all nuts)] in each can. The mean and standard deviation of these measurements are found to be 23.5 and 3.1, respectively. Do these results constitute strong evidence in support of the inspector's belief? (Answer by calculating and interpreting the $P-$value.)

8.55 Biological oxygen demand (BOD) is an index of pollution that is monitored in the treated effluent of paper mills on a regular basis. From 43 determinations of BOD (in pounds per day) at a particular paper mill during the spring and summer months, the mean and standard deviation were found to be 3246 and 757, respectively. The company had set the target that the mean BOD should be 3000 pounds per day. Do the sample data indicate that the actual amount of BOD is significantly off the target? (Use $\alpha = .05$.)

8.56 Refer to Exercise 8.55. Along with the determinations of BOD, the discharge of suspended solids (SS) was also monitored at the same site. The mean and standard deviation of the 43 determinations of SS were found to be 5710 and 1720 pounds per day, respectively. Do these results strongly support the company's claim that the true mean SS is lower than 6000 pounds per day? (Answer by calculating and interpreting the $P-$value.)

5. INFERENCES ABOUT A POPULATION PROPORTION

The reasoning leading to estimation of a mean also applies to the problem of estimation of a population proportion. Example 2 considers sampling $n = 200$ new car buyers to infer about the proportion of the population that is satisfied after one year. When n elements are randomly sampled from the population, the data will consist of the count X of the number of sampled elements possessing the characteristic. Common sense suggests the sample proportion

$$\hat{p} = \frac{X}{n}$$

as an estimator of p. The hat notation reminds us that $\hat{p}$ is a statistic.

When the sample size n is only a small fraction of the population size, the sample count X has the binomial distribution with mean np and standard deviation $\sqrt{npq}$, where $q = 1 - p$. Recall from Chapter 6 that, when n is large, the binomial variable X is well approximated by a normal with mean np and standard deviation $\sqrt{npq}$. That is,

$$Z = \frac{X - np}{\sqrt{npq}}$$

is approximately standard normal. This statement can be converted into a statement about proportions by dividing the numerator and the denominator by n. In particular,

$$Z = \frac{(X - np)/n}{(\sqrt{npq})/n} = \frac{\hat{p} - p}{\sqrt{pq/n}}$$

This last form, illustrated in Figure 15, is crucial to all inferences about a population proportion p. It shows that $\hat{p}$ is approximately normally distributed with mean p and standard deviation $\sqrt{pq/n}$.

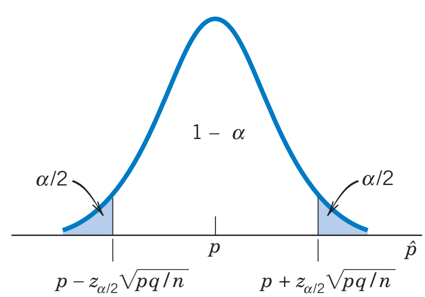


Figure 15 Approximate normal distribution of $\hat{p}$.

POINT ESTIMATION OF p

Intuitively, the sample proportion $\hat{p}$ is a reasonable estimator of the population proportion p. When the count X has a binomial distribution,

$$E(X) = np \qquad \text{sd}(X) = \sqrt{npq}$$

Since $\hat{p} = X/n$, the properties of expectation give

$$E(\hat{p}) = p$$
$$\text{sd}(\hat{p}) = \sqrt{pq/n}$$

In other words, the sampling distribution of $\hat{p}$ has a mean equal to the population proportion. The second result shows that the standard error of the estimator $\hat{p}$ is

$$\text{S.E.}(\hat{p}) = \sqrt{\frac{pq}{n}}$$

The estimated standard error can be obtained by substituting the sample estimate $\hat{p}$ for p and $\hat{q} = 1 - \hat{p}$ for q in the formula, or

$$\text{Estimated S.E.}(\hat{p}) = \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

When n is large, prior to sampling, the probability is approximately .954 that the error of estimation $|\hat{p} - p|$ will be less than $2 \times$ (estimated S.E.).

Point Estimation of a Population Proportion

Parameter: Population proportion p

Data: X = Number having the characteristic in a random sample of size n

Estimator: $\hat{p} = \dfrac{X}{n}$

$$\text{S.E.}(\hat{p}) = \sqrt{\frac{pq}{n}} \qquad \text{and} \qquad \text{estimated S.E.}(\hat{p}) = \sqrt{\frac{\hat{p}\,\hat{q}}{n}}$$

For large n, an approximate $100(1 - \alpha)\%$ error margin is $z_{\alpha/2}\sqrt{\hat{p}\,\hat{q}/n}$.

Example 11 Estimating the Proportion of Purchasers

A large mail-order club that offers monthly specials wishes to try out a new item. A trial mailing is sent to a random sample of 250 members selected from the list of over 9000 subscribers. Based on this sample mailing, 70 of the members decide to purchase the item. Give a point estimate of the proportion of club members that could be expected to purchase the item and attach a 95.4% error margin.

SOLUTION The number in the sample represents only a small fraction of the total membership, so the count can be treated as if it were a binomial variable.

Here $n = 250$ and $X = 70$, so the estimate of the population proportion is

$$\hat{p} = \frac{70}{250} = .28$$

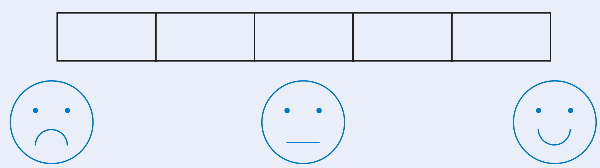
$$\text{Estimated S.E.}(\hat{p}) = \sqrt{\frac{\hat{p}\,\hat{q}}{n}} = \sqrt{\frac{.28 \times .72}{250}} = .028$$

$$95.4\% \text{ error margin} = 2 \times .028 = .056$$

Therefore, the estimated proportion is $\hat{p} = .28$, with a 95.4% error margin of .06 (rounded to two decimals).

Novel Collection of Count Data

Producers of breakfast cereals continually experiment with new products. Each promising new cereal must be market-tested on a sample of potential purchasers. An added twist here is that youngsters are a major component of the market. In order to elicit accurate information from young people, one firm developed a smiling face scale.



After tasting a new product, respondents are asked to check one box to rate the taste. A good product should have most of the youngsters responding in the top two boxes. Grouping these into a single highest category and the lower three boxes into a lower category, we are in the situation of estimating the proportion of the market population that would rate taste in the highest category.

Out of a sample of 42 youngsters, 30 rated a new cereal in the top category.

CONFIDENCE INTERVAL FOR p

A large sample confidence interval for a population proportion can be obtained from the approximate normality of the sample proportion $\hat{p}$. Since $\hat{p}$ is nearly normal with mean p and standard deviation $\sqrt{pq/n}$, the random interval

$\hat{p} \pm z_{\alpha/2}\sqrt{pq/n}$ is a candidate. However, the standard deviation involves the unknown parameter p, so we use the estimated standard deviation $\sqrt{\hat{p}\,\hat{q}\,/n}$ to set the endpoints of the confidence interval. Notice again that the common form of the confidence interval is

$$\text{Estimate} \pm (z \text{ value})(\text{estimated standard error})$$

Large Sample Confidence Interval for p

For large n, a $100(1 - \alpha)\%$ confidence interval for p is given by

$$\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\,\hat{q}}{n}}, \quad \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\,\hat{q}}{n}} \right)$$

Example 12 A Confidence Interval for the Proportion Satisfied

Consider the data in Example 2 where 168 out of a random sample of 200 new car purchasers were satisfied with their car after one year. Compute a 95% confidence interval for the population proportion of satisfied new car purchasers.

SOLUTION The sample size $n = 200$ is large so a normal approximation to the distribution of $\hat{p}$ is justified. Since $1 - \alpha = .95$, we have $\alpha/2 = .025$ and $z_{.025} = 1.96$. The observed $\hat{p} = 168/200 = .84$, and $\hat{q} = 1 - .84 = .16$. We calculate

$$z_{.025}\sqrt{\frac{\hat{p}\,\hat{q}}{n}} = 1.96\sqrt{\frac{.84 \times .16}{200}} = 1.96 \times .0259 = .051$$

Therefore, a 95% confidence interval for the population proportion of satisfied new-car buyers is $.840 \pm .051$, or $(.789, .891)$.

Because our procedure will produce true statements 95% of the time, we can be 95% confident that the proportion of satisfied new-car buyers is between .789 and .891.

DETERMINING THE SAMPLE SIZE

Note that, prior to sampling, the numerical estimate $\hat{p}$ of p is not available. Therefore, for a $100(1 - \alpha)\%$ error margin for the estimation of p, we use the expression $z_{\alpha/2}\sqrt{pq/n}$. The required sample size is obtained by equating $z_{\alpha/2}\sqrt{pq/n} = d$, where d is the specified error margin. We then obtain

$$n = pq\left[\frac{z_{\alpha/2}}{d}\right]^2$$

If the value of p is known to be roughly in the neighborhood of a value p^*, then n can be determined from

$$n = p^*(1 - p^*)\left[\frac{z_{\alpha/2}}{d}\right]^2$$

Without prior knowledge of p, pq can be replaced by its maximum possible value $\frac{1}{4}$ and n determined from the relation

$$n = \frac{1}{4}\left[\frac{z_{\alpha/2}}{d}\right]^2$$

Example 13 Selecting a Sample Size for Estimating a Proportion

A public health survey is to be designed to estimate the proportion p of a population having defective vision. How many persons should be examined if the public health commissioner wishes to be 98% certain that the error of estimation is below .05 when:

(a) There is no knowledge about the value of p?

(b) p is known to be about .3?

SOLUTION The tolerable error is $d = .05$. Also $1 - \alpha = .98$, so $\alpha/2 = .01$. From the normal table, we know that $z_{.01} = 2.33$.

(a) Since p is unknown, the conservative bound on n yields

$$\frac{1}{4}\left[\frac{2.33}{.05}\right]^2 = 543$$

A sample of size 543 would suffice.

(b) If $p^* = .3$, the required sample size is

$$n = (.3 \times .7)\left[\frac{2.33}{.05}\right]^2 = 456$$

LARGE SAMPLE TESTS ABOUT p

We consider testing H_0: $p = p_0$ versus H_1: $p \neq p_0$. With a large number of trials n, the sample proportion

$$\hat{p} = \frac{X}{n}$$

is approximately normally distributed. Under the null hypothesis, p has the specified value p_0 and the distribution of $\hat{p}$ is approximately $N(p_0, \sqrt{p_0 q_0 / n})$. Consequently, the standardized statistic

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$$

has the $N(0, 1)$ distribution. Since the alternative hypothesis is two-sided, the rejection region of a level α test is given by

$$R: |Z| \geq z_{\alpha/2}$$

For one-sided alternatives, we use a one-tailed test in exactly the same way we discussed in Section 4 in connection with tests about μ.

Example 14 **Testing for a Change in the Proportion below the Poverty Level**

A five-year-old census recorded that 20% of the families in a large community lived below the poverty level. To determine if this percentage has changed, a random sample of 400 families is studied and 70 are found to be living below the poverty level. Does this finding indicate that the current percentage of families earning incomes below the poverty level has changed from what it was five years ago?

SOLUTION Let p denote the current population proportion of families living below the poverty level. Because we are seeking evidence to determine whether p is *different* from .20, we wish to test

$$H_0: p = .20 \qquad \text{versus} \qquad H_1: p \neq .20$$

The sample size $n = 400$ being large, the Z test is appropriate. The test statistic is

$$Z = \frac{\hat{p} - .2}{\sqrt{.2 \times .8 / 400}}$$

If we set $\alpha = .05$, the rejection region is $R: |Z| \geq 1.96$. From the sample data the computed value of Z is

$$z = \frac{(70/400) - .2}{\sqrt{.2 \times .8/400}} = \frac{.175 - .2}{.020} = -1.25$$

Because $|z| = 1.25$ is smaller than 1.96, the null hypothesis is not rejected at $\alpha = .05$. We conclude that the data do not provide strong evidence that a change in the percentage of families living below the poverty level has occurred.

The significance probability of the observed value of Z is

$$\begin{aligned} P\text{-value} &= P[|Z| \geq 1.25] \\ &= P[Z \leq -1.25] + P[Z \geq 1.25] \\ &= 2 \times .1056 = .2112 \end{aligned}$$

We would have to inflate α to more than .21 in order to reject the null hypothesis. Thus, the evidence against H_0 is really weak.

Exercises

8.57 Suppose that n units are randomly sampled and x number of the sampled units are found to have the characteristic of interest. In each case, (i) define p in the context of the problem, (ii) provide a point estimate of p and (iii) determine the 95% error margin.[1]

(a) A survey is conducted of $n = 986$ adults and $x = 295$ reported that reading is a favorite leisure time activity.

(b) A survey of $n = 440$ pet owners revealed that $x = 293$ buy their pets holiday presents.

8.58 For each case in Exercise 8.57, determine the 98% error margin of the estimate.

8.59 In a psychological experiment, individuals are permitted to react to a stimulus in one of two ways, say, A or B. The experimenter wishes to estimate the proportion p of persons exhibiting reaction A. How many persons should be included in the experiment to be 90% confident that the error of estimation is within .03 if the experimenter:

(a) Knows that p is about .3?

(b) Has no idea about the value of p?

8.60 A national safety council wishes to estimate the proportion of automobile accidents that involve pedestrians. How large a sample of accident records must be examined to be 98% certain that the estimate does not differ from the true proportion by more than .03? (The council believes that the true proportion is below .25.)

8.61 An automobile club which pays for emergency road services (ERS) requested by its members wishes to estimate the proportions of the different types of ERS requests. Upon examining a sample of 2927 ERS calls, it finds that 1499 calls related to starting problems, 849 calls involved serious mechanical failures requiring towing, 498 calls involved flat tires or lockouts, and 81 calls were for other reasons.

(a) Estimate the true proportion of ERS calls that involved serious mechanical problems requiring towing and determine its 95% margin of error.

(b) Calculate a 98% confidence interval for the true proportion of ERS calls that related to starting problems.

8.62 Each year, an insurance company reviews its claim experience in order to set future rates. Regarding their damage-only automobile insurance policies, at least one claim was made on 2073 of the 12,299 policies in effect for the year. Treating these data as a random sample for the population of all possible damage-only policies that could be issued, find a 95% confidence interval for the population proportion of at least one claim.

8.63 Out of a sample of 94 purchases at the drive-up window of a fast-food establishment, 27 were made with a major credit card.

(a) Estimate the proportion of sales made with a credit card.

(b) Obtain the estimated standard error.

(c) Find a 98% confidence interval for the population proportion of purchases paid with a major credit card.

8.64 A sample of size $n = 400$ observations are made on the brand of cola purchased. Based on the sample, it is found that 249 purchases were made of Brand P.

(a) Find a 95% confidence interval for the population proportion of purchases of Brand P among cola purchases.

(b) Does p lie in your interval obtained in Part(a)?

(c) Why are you 95% confident about your interval in Part(a)?

8.65 Identify the null and the alternative hypotheses in the following situations.

(a) A university official believes that the proportion of students who currently hold part-time jobs has increased from the value .26 that prevailed four years ago.

[1]These proportions of successes are close to those obtained in 2008 Harris interactive polls.

(b) A cable company claims that, because of improved procedures, the proportion of its cable subscribers that have complaints against the cable company is now less than .13.

(c) Referring to part (b), suppose a consumer advocate feels the proportion of cable subscribers that have complaints against the cable company this year is greater than .13. She will conduct a survey to challenge the cable company's claim.

(d) An inspector wants to establish that 2 × 4 lumber at a mill does not meet a specification that requires at most 5% break under a standard load.

8.66 Given here are the descriptive statements of some claims that one intends to establish on the basis of data. In each case, identify the null and the alternative hypotheses in terms of a population proportion p.

(a) Of smokers who eventually quit smoking, less than 40% are able to do so in just one attempt.

(b) On a particular freeway, over 25% of the cars that use a lane restricted exclusively to multipassenger cars use the lane illegally.

(c) At a particular clinic, less than 20% of the patients wait over half an hour to see the doctor.

8.67 Each part of this problem specifies a claim about a population proportion, the sample size n, and the desired level of significance α. Formulate (i) the hypotheses, (ii) the test statistic, and (iii) the rejection region. (The answers to part (a) are provided for illustration.)

(a) Claim: $p < .32$ $n = 120$ $\alpha = .05$

[Answers:

(i) $H_0\colon p = .32$ $H_1\colon p < .32$

(ii) $Z = \dfrac{\hat{p} - .32}{\sqrt{.32 \times .68/120}} = \dfrac{\hat{p} - .32}{.0426}$

(iii) $R\colon Z \leq -1.645$

(b) Claim: $p > .75$ $n = 228$ $\alpha = .02$

(c) Claim: $p \neq .60$ $n = 77$ $\alpha = .02$

(d) Claim: $p < .56$ $n = .86$ $\alpha = .10$

8.68 Given here are the observed sample proportions $\hat{p}$ in the contexts of parts (a–d) of Exercise 8.67. Calculate the test statistic and draw a conclusion of the test at the specified level of significance.

(a) $\hat{p} = .233$

(b) $\hat{p} = .818$

(c) $\hat{p} = .709$

(d) $\hat{p} = .387$

8.69 A manager of a campus store that sells posters conjectures that more than 30% of all freshman dorm rooms have a poster of a rock group. From $n = 60$ rooms selected at random, an investigator will record X = number of rooms having a poster of a rock group

(a) Formulate a null and alternative hypotheses for verifying the conjecture.

(b) Select a rejection region for which $\alpha = .05$.

(c) If $p = .4$, what error can be made? Explain in words in the context of the problem

(d) What is the conclusion of your test if $X = 25$?

(e) Based on your conclusion in Part (d), what error could you have made? Explain in the context of the problem.

8.70 An educator wishes to test $H_0\colon p = .3$ against $H_1\colon p > .3$, where p = proportion of college football players who graduate in four years.

(a) State the test statistic and the rejection region for a large sample test having $\alpha = .05$.

(b) If 19 out of a random sample of 48 players graduated in four years, what does the test conclude? Calculate the P–value and interpret the result.

8.71 A concerned group of citizens wants to show that less than half the voters support the President's handling of a recent crisis. Let p = proportion of voters who support the handling of the crisis.

(a) Determine H_0 and H_1.

(b) If a random sample of 500 voters gives 228 in support, what does the test conclude? Use $\alpha = .05$. Also evaluate the P–value.

8.72 Refer to Exercise 8.61. Perform a test of hypotheses to determine whether the proportion of ERS calls involving flat tires or lockouts was significantly smaller than .19, the true proportion for previous years. (Use a 5% level of significance.)

8.73 Refer to Exercise 8.64. Perform a test with intent of showing that the proportion of persons who purchase brand P is greater than .55. Calculate the P–value and interpret the result.

8.74 An independent bank concerned about its customer base decided to conduct a survey of bank customers. Out of 505 customers who returned the survey form, 258 rated the overall bank services as excellent.

(a) Test, at level $\alpha = .10$, the null hypothesis that the proportion of customers who would rate the overall bank services as excellent is .46 versus a two-sided alternative.

(b) Calculate the $P-$value and comment on the strength of evidence.

8.75 With reference to Exercise 8.74,

(a) Find a 90% confidence interval for the proportion of customers who would rate the overall bank services as excellent.

(b) The bank has 8200 customers. Convert your confidence interval for the proportion in part (a) into a 90% confidence interval for the total number of customers who would rate the overall bank services as excellent.

8.76 From telephone interviews with 980 adults, it was found that 78% of those persons supported tougher legislation for antipollution measures. Does this poll substantiate the conjecture that more than 75% of the adult population are in favor of tougher legislation for antipollution measures? (Answer by calculating the $P-$value.)

8.77 Refer to the box with the smiling face scale for rating cereals on page 331. Using the data that 30 out of 42 youngsters in a sample rated a cereal in the highest category, find an approximate 95% confidence interval for the corresponding population proportion.

USING STATISTICS WISELY

1. Calculate the estimated standard error $s/\sqrt{n}$ to accompany the point estimate $\bar{x}$ of a population mean.
2. Understand the interpretation of a $100(1 - \alpha)\%$ confidence interval. When the sample size is large, before the data are collected

$$\left(\overline{X} - z_{\alpha/2} \frac{S}{\sqrt{n}}, \qquad \overline{X} + z_{\alpha/2} \frac{S}{\sqrt{n}} \right)$$

is a random interval that will cover the fixed unknown mean μ with probability $1 - \alpha$. The long-run frequency interpretation of the probability $1 - \alpha$ says that, in many repeated applications of this method, about proportion $1 - \alpha$ of the times the interval will cover the respective population mean.

3. When conducting a test of hypothesis, formulate the assertion that the experiment seeks to confirm as the alternative hypothesis.

4. When the sample size is large, base a test of the null hypothesis $H_0 : \mu = \mu_0$ on the test statistic

$$\frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

which has, approximately a standard normal distribution. The rejection region is one-sided or two-sided corresponding to the alternative hypothesis.

5. Understand the interpretation of a level α test. If the null hypothesis is true, before the data are collected, the probability is α that the experiment will produce observations that lead to the rejection of the null hypothesis. Consequently, after many independent experiments, the proportion that lead to rejection of the null hypothesis will be nearly α.
6. To obtain a precise estimate of a proportion usually requires a sample size of a few hundred.
7. Remember that the statistical procedures presented in this chapter will not be valid if the large sample is not randomly selected but collected from convenient units.

KEY IDEAS AND FORMULAS

Statistical concepts and methods provide the framework that allows us to learn about the population from a sample of observations. The process begins by modeling the population as a probability distribution, which has some numerical feature of interest called a **parameter.** Then, given a sample from this population distribution called the **data,** we make a generalization or **statistical inference** about the parameter.

Two basic forms of inference are (1) **estimation** of a population parameter and (2) **testing statistical hypotheses.**

A parameter can be estimated in two ways: by quoting (1) a single numerical value (**point estimation**) or (2) an interval of plausible values (**interval estimation**).

The statistic whose value gives a point estimate is called an **estimator.** The standard deviation of a point estimator is also called its **standard error.**

To be meaningful, a point estimate must be accompanied by an evaluation of its **error margin.**

A $100(1 - \alpha)\%$ **confidence interval** is an interval that, before sampling, will cover the true value of the parameter with probability $1 - \alpha$. The interval must be computable from the sample data.

If random samples are repeatedly drawn from a population and a $100(1 - \alpha)\%$ confidence interval is calculated from each, then about $100(1 - \alpha)\%$ of those intervals will include the true value of the parameter. We never know what happens in a single application. Our confidence draws from the success rate of $100(1 - \alpha)\%$ over many applications.

A **statistical hypothesis** is a statement about a population parameter.

A statement or claim, which is to be established with a strong support from the sample data, is formulated as the **alternative hypothesis** (H_1). The **null hypothesis** (H_0) says that the claim is void.

A **test of the null hypothesis** is a **decision rule** that tells us when to reject H_0 and when not to reject H_0. A test is specified by a **test statistic** and a **rejection region (critical region).**

A wrong decision may occur in one of the two ways:

A false rejection of H_0 (**Type I error**)

Failure to reject H_0 when H_1 is true (**Type II error**)

Errors cannot always be prevented when making a decision based on a sample. It is their probabilities that we attempt to keep small.

A Type I error is considered to be more serious. The maximum Type I error probability of a test is called its **level of significance** and is denoted by α.

The **significance probability** or ***P*–value** of an observed test statistic is the smallest α for which this observation leads to the rejection of H_0.

Main steps in testing statistical hypotheses

1. Formulate the null hypotheses H_0 and the alternative hypothesis H_1.
2. Test criterion: State the test statistic and the form of the rejection region.
3. With a specified α, determine the rejection region.
4. Calculate the test statistic from the data.
5. Draw a conclusion: State whether or not H_0 is rejected at the specified α and interpret the conclusion in the context of the problem. Also, it is a good statistical practice to calculate the P–value and strengthen the conclusion.

The Type II error probability is denoted by β.

Inferences about a Population Mean When *n* Is Large

When n is large, we need not be concerned about the shape of the population distribution. The central limit theorem tells us that the sample mean $\overline{X}$ is nearly normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$. Moreover, $\sigma/\sqrt{n}$ can be estimated by $S/\sqrt{n}$.

Parameter of interest is

$$\mu = \text{Population mean}$$

Inferences are based on

$$\overline{X} = \text{Sample mean}$$

1. A **point estimator** of μ is the sample mean $\overline{X}$.

$$\text{Estimated standard error} = \frac{S}{\sqrt{n}}$$

$$\text{Approximate } 100(1-\alpha)\% \text{ error margin} = z_{\alpha/2}\frac{S}{\sqrt{n}}$$

2. A $100(1-\alpha)\%$ **confidence interval** for μ is

$$\left(\overline{X} - z_{\alpha/2}\frac{S}{\sqrt{n}}, \quad \overline{X} + z_{\alpha/2}\frac{S}{\sqrt{n}}\right)$$

3. The **test of the null hypothesis** concerning μ, called the **large sample normal test** or ***Z* test,** uses the **test statistic**

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

where μ_0 is the value of μ that marks the boundary between H_0 and H_1. Given a level of significance α,

$$\begin{array}{llll}
\text{Reject } H_0: \mu = \mu_0 & \text{in favor of} \quad H_1: \mu > \mu_0 & \text{if} & Z \geq z_\alpha \\
\text{Reject } H_0: \mu = \mu_0 & \text{in favor of} \quad H_1: \mu < \mu_0 & \text{if} & Z \leq -z_\alpha \\
\text{Reject } H_0: \mu = \mu_0 & \text{in favor of} \quad H_1: \mu \neq \mu_0 & \text{if} & |Z| \geq z_{\alpha/2}
\end{array}$$

The first two alternative hypotheses, $H_1: \mu > \mu_0$ and $H_1: \mu < \mu_0$ are **one-sided hypothesis** and the third is a **two-sided hypothesis.** The rejection regions correspond to the alternative hypothesis so tests in the first two cases are called **one-sided tests** or **one-tailed tests.** Those in the third case are **two-tailed tests.**

Inference about a Population Proportion When *n* Is Large

Parameter of interest:

p = Population proportion of individuals possessing stated characteristic

Inferences are based on $\hat{p} = \dfrac{x}{n}$, the sample proportion.

1. A **point estimator** of p is $\hat{p}$.

$$\text{Estimated standard error} = \sqrt{\frac{\hat{p}\,\hat{q}}{n}} \qquad \text{where } \hat{q} = 1 - \hat{p}$$

$$100(1 - \alpha)\% \text{ error margin} = z_{\alpha/2}\sqrt{\frac{\hat{p}\,\hat{q}}{n}}$$

2. A $100(1 - \alpha)\%$ **confidence interval** for p is

$$\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\,\hat{q}}{n}}, \quad \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\,\hat{q}}{n}}\right)$$

3. To test hypotheses about p, the **test statistic** is

$$Z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$$

where p_0 is the value of p that marks the boundary between H_0 and H_1. The rejection region is right-sided, left-sided, or two-sided according to $H_1: p > p_0$, $H_1: p < p_0$, or $H_1: p \neq p_0$, respectively.

TECHNOLOGY

Large Sample Confidence Intervals and Tests Concerning a Mean

The software programs use a known value for the population standard deviation σ. If this is not given in your application, you need to first obtain the sample standard deviation using the technology described in Chapter 2.

MINITAB

Confidence Intervals for μ

We illustrate the calculation of a 99% confidence interval for μ when we have determined that the sample standard deviation is 8.2 (or the known population $\sigma = 8.2$).

Data: C*1*

Stat > Basic Statistics > 1-Sample Z.
Type C*1* in **Samples in columns** and *8.2* in **Standard deviation.**
Click **Options** and type *99* in **Confidence level.**
Click **OK.** Click **OK.**

Tests of Hypotheses Concerning μ

We illustrate the calculation of an $\alpha = .01$ level test of $H_0 : \mu = 32$ versus a one-sided alternative $H_1 : \mu > 32$ when we have determined that the sample standard deviation is 8.2 (or the known population $\sigma = 8.2$).

Data: C*1*

Stat > Basic Statistics > 1-Sample Z.
Type C*1* in **Samples in Columns.**
Type *8.2* in **Sigma.** Following **Test mean,** type *32*, the value of the mean under the null hypothesis.
Click **Options** and type *99* in **Confidence level.**
In the **Alternative** cell select **greater than,** the direction of the alternative hypothesis. Click **OK.** Click **OK.**

If the sample size and mean are available, instead of the second step, type these values in the corresponding cells.

EXCEL

Confidence Intervals for μ

We illustrate the calculation of a 99% confidence interval for μ when we have determined that the sample standard deviation is 8.2 (or the known population $\sigma = 8.2$) and the sample size is 100.

Select **Insert and** then **Function.**
Choose **Statistical** and then **CONFIDENCE.**
Enter *1 − .99* or *.01* for **Alpha,** *8.2* for **Standard_dev,** and *100* for **size.**

(Add and subtract this value to and from $\overline{x}$ to obtain the confidence interval.)

Tests of Hypotheses Concerning μ

We illustrate the calculation of a test of $H_0 : \mu = 32$ versus a one-sided alternative $H_1 : \mu > 32$ when we have determined that the sample standard deviation is 8.2 (or the known population $\sigma = 8.2$). Start with the data entered in column A.

> Select **Insert and** then **Function.** Choose **Statistical** and then **ZTEST.**
> Highlight the data in column A for **Array.** Enter *32* in **X** and *8.2* in **Sigma.**
> (Leave **Sigma** blank and the sample standard deviation will be used.)
> Click **OK.**

The program returns the one-sided $P-$value for right-sided alternatives. For two-sided alternatives, you need to double the $P-$value if $\overline{x}$ is above $\mu_0 = 32$.

TI-84/-83 PLUS

Confidence Intervals for μ

We illustrate the calculation of a 99% confidence interval for μ when we have determined that the sample standard deviation is 8.2 (or the known population $\sigma = 8.2$). Start with the data entered in **L1**.

> Press **STAT** and select **TESTS** and then **7: Zinterval.**
> Select **Data** with *List* set to **L1** and **FREQ** to 1.
> Following σ: enter *8.2*.
> Enter *.99* following **C-Level:** Select **Calculate.**
> Then press **ENTER.**

If instead the sample size and sample mean are available, the second step is Select **Stats** (instead of **Data**) and enter the sample size and mean for n and $\overline{x}$.

Tests of Hypotheses Concerning μ

We illustrate the calculation of an $\alpha = .01$ level test of $H_0 : \mu = 32$ versus a one-sided alternative $H_1 : \mu > 32$ when we have determined that the sample standard deviation is 8.2 (or the known population $\sigma = 8.2$). Start with the data entered in column **L1**.

> Press **STAT** and select **TESTS** and then **1: Z-Test.**
> Select **Data** with **List** set to **L1** and **Freq** to 1.
> Following σ: enter *8.2*. Enter *32* for μ_0.
> Select the direction of the alternate hypothesis. Select **Calculate.**
> Press **ENTER.**
>
> The calculator will return the $P-$value.

If instead the sample size and sample mean are available, the second step is

> Select **Stats** (instead of **Data**) and type in the sample size and mean.

6. REVIEW EXERCISES

8.78 Refer to Exercise 2.4 where the number of automobile accidents reported per month were recorded for an intersection. The sample size is $n = 59$, $\bar{x} = 1.949$, and $s = 1.558$ accidents.

(a) Give a point estimate of μ, the mean number of accidents reported per month.

(b) Determine the estimated standard error.

(c) Calculate the 98% error margin.

8.79 A student in a large lecture section asked students how much they paid for a used copy of the text. The $n = 38$ responses yielded

$$\sum x_i = 3230.84 \text{ dollars} \qquad \sum (x_i - \bar{x})^2 = 2028.35$$

(a) Give a point estimate of μ, the mean price paid.

(b) Determine the estimated standard error.

(c) Calculate the 95% error margin.

8.80 The time it takes for a taxi to drive from the office to the airport was recorded on 40 occasions. It was found that $\bar{x} = 47$ minutes and $s = 5$ minutes. Give

(a) An estimate of μ = population mean time to drive.

(b) An approximate 95.4% error margin.

8.81 By what factor should the sample size be increased to reduce the standard error of $\overline{X}$ to

(a) one-half its original value?

(b) one-fourth its original value?

8.82 A food service manager wants to be 95% certain that the error in the estimate of the mean number of sandwiches dispensed over the lunch hour is 10 or less. What sample size should be selected if a preliminary sample suggests

(a) $\sigma = 40$?

(b) $\sigma = 80$?

8.83 A zoologist wishes to estimate the mean blood sugar level of a species of animal when injected with a specified dosage of adrenaline. A sample of 55 animals of a common breed are injected with adrenaline, and their blood sugar measurements are recorded in units of milligrams per 100 milliliters of blood. The mean and standard deviation of these measurements are found to be 126.9 and 10.5, respectively.

(a) Give a point estimate of the population mean and find a 95.4% error margin.

(b) Determine a 90% confidence interval for the population mean.

8.84 Refer to Exercise 2.3 and the data on the number of extracurricular activities in which 30 students participated in the past week. These data have

$$n = 30 \qquad \bar{x} = 1.925 \qquad s = 1.607 \text{ activities}$$

(a) Obtain a 98% confidence interval for μ, the population mean number of activities.

(b) In a long series of experiments, each involving different students, what proportion of the intervals would cover μ?

8.85 After feeding a special diet to 80 mice, the scientist measures their weight in grams and obtains $\bar{x} = 35$ grams and $s = 4$ grams. He states that a 90% confidence interval for μ is given by

$$\left(35 - 1.645\frac{4}{\sqrt{80}},\quad 35 + 1.645\frac{4}{\sqrt{80}}\right)$$

$$\text{or} \qquad (34.26,\quad 35.74)$$

(a) Was the confidence interval calculated correctly? If not, provide the correct result.

(b) Does the interval (34.26, 35.74) cover the true mean? Explain your answer.

8.86 The amount of PCBs was measured in 40 samples of soil that were treated with contaminated sludge. The following summary statistics were obtained.

$$\bar{x} = 3.56 \qquad s = .5 \text{ p.p.m}$$

(a) Perform a test of hypotheses with the intent of establishing that the mean PCB contamination is less than 3.7 p.p.m. Take $\alpha = .05$.

(b) What error could you have possibly made in Part (a)?

8.87 In each case, identify the null hypothesis (H_0) and the alternative hypothesis (H_1) using the appropriate symbol for the parameter of interest.

(a) A consumer group plans to test-drive several cars of a new model in order to document that its average highway mileage is less than 50 miles per gallon.

(b) Confirm the claim that the mean number of pages per transmission sent by a campus fax station is more than 3.4.

(c) A chiropractic method will be tried on a number of persons suffering from persistent backache in order to demonstrate the claim that its success rate is higher than 50%.

(d) The setting of an automatic dispenser needs adjustment when the mean fill differs from the intended amount of 16 ounces. Several fills will be accurately measured in order to decide if there is a need for resetting.

(e) The content of fat in a gourmet chocolate ice cream is more than the amount, 4%, that is printed on the label.

8.88 A literary critic wants to establish that the mean number of words per sentence, appearing in a newly discovered short story, is different from 9.1 words.
A sample of 36 sentences provided the data

$$\bar{x} = 8.6 \text{ and } s = 1.2$$

(a) Formulate the null and alternative hypotheses. (Define any symbols you use.)

(b) Determine the test statistic.

(c) Give the form of the rejection region.

(d) What is the conclusion to your test? Take $\alpha = .10$.

(e) Calculate a P–value.

(f) Based on Part (d), what error could you have possibly made?

8.89 A test will be conducted to see how long a seven-ounce tube of toothpaste lasts. The researcher wants to establish that the mean time is greater than 30.5 days. From a random sample of size 75, an investigator obtains $\bar{x} = 32.3$ and $s = 6.2$ days.

(a) Formulate the null and alternative hypotheses. (Define any symbols you use.)

(b) Determine the test statistic.

(c) Give the form of the rejection region.

(d) What is the conclusion to your test? Take $\alpha = .10$.

(e) Calculate a P– value.

(f) Based on Part (d), what error could you have possibly made?

8.90 In a given situation, suppose H_0 was rejected at $\alpha = .05$. Answer the following questions as "yes," "no," or "can't tell" as the case may be.

(a) Would H_0 also be rejected at $\alpha = .03$?

(b) Would H_0 also be rejected at $\alpha = .10$?

(c) Is the P–value larger than .05?

8.91 Refer to the data on the amount of reflected light from urban areas in Table D.3b of the Data Bank. A computer calculation for a test of $H_0 : \mu = 84$ versus $H_1 : \mu \neq 84$ has the output

```
Test of mu = 84 vs mu not = 84

Variable     N   Mean   StDev
Lighturb    40 82.075   4.979

Variable         Z      P
Lighturb     -2.45  0.014
```

(a) What is the conclusion of the test when $\alpha = .03$?

(b) Use the value for Z to test the null hypothesis $H_0 : \mu = 84$ versus the one-sided alternative $H_0 : \mu < 84$ at the $\alpha = .01$ level of significance.

8.92 A company wishing to improve its customer service collected hold times from 75 randomly selected incoming calls to its hot line that were put on hold. These calls had sample mean hold time $\bar{x} = 3.4$ minutes and $s = 2.3$ minutes.

Obtain a 99% confidence interval for μ, the population mean hold time.

8.93 The daily number of kayaks sold, X, at a water sports store has the probability distribution

x	0	1	2
$f(x)$	.5	.3	.2

(a) Find the expected number of kayaks sold in a day.

(b) Find the standard deviation of the number of kayaks sold in a day.

(c) Suppose data from the next 64 different days give $\bar{x} = .84$ and standard deviation $s = .40$ number of kayaks sold. Can we conclude that the mean number of kayaks sold is greater than it used to be? Test with $\alpha = .05$.

8.94 In a large-scale, cost-of-living survey undertaken last January, weekly grocery expenses for families with one or two children were found to have a mean of \$148 and a standard deviation of \$25. To investigate the current situation, a random sample of families with one or two children is to be chosen and their last week's grocery expenses are to be recorded.

(a) How large a sample should be taken if one wants to be 95% sure that the error of estimation of the population mean grocery expenses per week for families with one or two children does not exceed \$2? (Use the previous s as an estimate of the current σ.)

(b) A random sample of 100 families is actually chosen, and from the data of their last week's grocery bills, the mean and the standard deviation are found to be \$155 and \$22, respectively. Construct a 98% confidence interval for the current mean grocery expense per week for the population of families with one or two children.

8.95 A random sample of 2000 persons from the labor force of a large city are interviewed, and 175 of them are found to be unemployed.

(a) Estimate the rate of unemployment based on the data.

(b) Establish a 95% error margin for your estimate.

8.96 Referring to Exercise 8.95, compute a 98% confidence interval for the rate of unemployment.

8.97 Out of a sample of $n = 625$ students interviewed, 139 had missed at least one class last week. Obtain a 95% confidence interval for p = proportion of all students that missed at least one class last week.

8.98 With reference to Exercise 8.97, conduct a test with the intent of establishing that $p > .20$.

(a) Formulate the null and alternative hypotheses.

(b) Determine the test statistic.

(c) Give the form of the rejection region.

(d) What is the conclusion to your test? Take $\alpha = .05$.

(e) Calculate a P–value.

(f) Based on Part (d), what error could you have possibly made?

8.99 From July 1 through August 15, 2008, United Airlines flew 137 flights from Chicago, Illinois, to Austin, Texas. Of these, 44 arrived late. Treat this as a random sample and conduct a test with the intent of establishing that the population proportion of late flights is greater than .25.

(a) Formulate the null and alternative hypotheses. Define any symbols.

(b) Determine the test statistic.

(c) Give the form of the rejection region.

(d) What is the conclusion to your test? Take $\alpha = .05$.

(e) Calculate a P–value.

(f) Based on Part (d), what error could you have possibly made?

8.100 Each year, an insurance company reviews its claim experience in order to set future rates. Regarding their damage-only automobile insurance policies, at least one claim was made on 2073 of the 12,299 policies in effect for the year. Treat these data as a random sample for the population of all possible damage-only policies that could be issued.

(a) Test, at level $\alpha = .05$, the null hypothesis that the probability of at least one claim is 0.16 versus a two-sided alternative.

(b) Calculate the P–value and comment on the strength of evidence.

8.101 A genetic model suggests that 80% of the plants grown from a cross between two given strains of seeds will be of the dwarf variety. After breeding 200 of these plants, 136 were observed to be of the dwarf variety.

(a) Does this observation strongly contradict the genetic model?

(b) Construct a 95% confidence interval for the true proportion of dwarf plants obtained from the given cross.

*8.102 ***Finding the power of a test.*** Consider the problem of testing $H_0 : \mu = 10$ versus $H_1 : \mu > 10$ with $n = 64$, $\sigma = 2$ (known), and $\alpha = .025$. The rejection region of this test is given by

$$R: \frac{\overline{X} - 10}{2/\sqrt{64}} \geq 1.96 \quad \text{or}$$

$$R: \overline{X} \geq 10 + 1.96 \frac{2}{\sqrt{64}} = 10.49$$

Suppose we wish to calculate the power of this test at the alternative $\mu_1 = 11$. Power = the probability of rejecting the null hypothesis when the alternative is true. Since our test rejects the null hypothesis when $\overline{X} \geq 10.49$, its power at $\mu_1 = 11$ is the probability

$$P[\overline{X} \geq 10.49 \text{ when the true mean } \mu_1 = 11]$$

If the population mean is 11, we know that $\overline{X}$ has the normal distribution with mean 11 and sd $= \sigma/\sqrt{n} = 2/\sqrt{64} = .25$. The standardized variable is

$$Z = \frac{\overline{X} - 11}{.25}$$

and we calculate

$$\begin{aligned} \text{Power} &= P[\overline{X} \geq 10.49 \text{ when } \mu_1 = 11] \\ &= P\left[Z \geq \frac{10.49 - 11}{.25}\right] \\ &= P[Z \geq -2.04] = .9793 \\ &\qquad \text{(using normal table)} \end{aligned}$$

Following the above steps, calculate the power of this test at the alternative:

(a) $\mu_1 = 10.5$

(b) $\mu_1 = 10.8$

8.103 Refer to the data on the computer attitude score (CAS) in Table D.4 of the Data Bank. A computer summary of a level $\alpha = .05$ test of H_0: $\mu = 2.6$ versus a two-sided alternative and a 95% confidence interval is given below.

```
One-Sample Z: CAS

Test of mu = 2.6 vs mu not = 2.6

Variable     N     Mean    StDev
CAS         35   2.8157   0.4840

Variable         95.0% CI           Z       P
CAS        ( 2.6554, 2.9761)  2.64  0.008
```

(a) Will the 99% confidence interval for mean CAS be smaller or larger than the one in the printout? Verify your answer by determining the 99% confidence interval.

(b) Use the value for Z to test the null hypothesis $H_0 : \mu = 2.6$ versus the one-sided alternative $H_1 : \mu > 2.6$ at the $\alpha = .05$ level of significance.

8.104 Refer to the data on percent malt extract in Table D.8 of the Data Bank. A computer summary of a level $\alpha = .05$ test of $H_0 : \mu = 77$ versus a two-sided alternative and a 95% confidence interval is given below.

```
One-Sample Z: malt extract(%)

Test of mu = 77 vs mu not = 77

Variable         N    Mean   StDev
malt extract    40  77.458   1.101

Variable            95.0% CI          Z       P
malt extract  ( 77.116, 77.799)  2.63  0.009
```

(a) Will the 98% confidence interval for mean malt extract be smaller or larger than the one in the printout? Verify your answer by determining the 98% confidence interval.

(b) Use the value for Z to test the null hypothesis $H_0 : \mu = 77.0$ versus the one-sided alternative $H_1 : \mu > 77.0$ at the $\alpha = .05$ level of significance.

The Following Exercises Require a Computer

8.105 Refer to the data on the heights of red pine seedlings in Exercise 8.4. Use MINITAB (or some other package program) to:

(a) Find a 97% percent confidence interval for the mean height.

(b) Test $H_0 : \mu = 1.9$ versus $H_1 : \mu \neq 1.9$ centimeters with $\alpha = .03$.

8.106 Referring to speedy lizard data in Exercise 2.19, page 38, obtain a 95% confidence interval for the mean speed of that genus.

8.107 Refer to the male salmon data given in Table D.7 of the Data Bank. Use MINITAB or some other package program to find a 90% large sample confidence interval for the mean freshwater growth.

8.108 Refer to the physical fitness data given in Table D.5 of the Data Bank. Use MINITAB or some other package program to:

(a) Find a 97% large sample confidence interval for the pretest number of situps.

(b) Construct a histogram to determine if the underlying distribution is symmetric or has a long tail to one side.

8.109 Refer to the sleep data given in Table D.10 of the Data Bank. Use MINITAB or some other package program to:

(a) Find a 95% large sample confidence interval for the mean number of breathing pauses per hour (BPH).

(b) Construct a histogram to determine if the underlying distribution is symmetric or has a long tail to one side.

8.110 Refer to the grizzly bear data given in Table D.8 of the Data Bank. Use MINITAB or some other package program to:

(a) Find a 95% large sample confidence interval for the mean weight in pounds of all bears living in that area.

(b) Construct a histogram to determine if the underlying distribution is symmetric or has a long tail to one side.