4

North Wind/North Wind Picture Archive

We see that the theory of probabilities is at bottom only common sense reduced to calculation; it makes us appreciate with exactitude what reasonable minds feel by a sort of instinct, often without being able to account for it.

—Pierre-Simon Laplace

Nathan Griffith/Comet/Corbis

This quote explains how the great mathematician Pierre-Simon Laplace (1749–1827) described the theory of mathematical probability. The discovery of the mathematical theory of probability was shared by two Frenchmen: Blaise Pascal and Pierre Fermat. These 17th-century scholars were attracted to the subject by the inquiries of the Chevalier de Méré, a gentleman gambler.

Although the first applications of probability were to games of chance and gambling, today the subject seems to pervade almost every aspect of modern life. Everything from the orbits of spacecraft to the social behavior of woodchucks is described in terms of probabilities.

Elementary Probability Theory

David Young-Wolff/PhotoEdit

FOCUS PROBLEM

How Often Do Lie Detectors Lie?

James Burke is an educator who is known for his interesting science-related radio and television shows aired by the British Broadcasting Corporation. His book *Chances: Risk and Odds in Everyday Life* (Virginia Books, London) contains a great wealth of fascinating information about probabilities. The following quote is from Professor Burke's book:

> *If I take a polygraph test and lie, what is the risk I will be detected?* According to some studies, there's about a 72 percent chance you will be caught by the machine.

> *What is the risk that if I take a polygraph test it will incorrectly say that I lied?* At least 1 in 15 will be thus falsely accused.

Both of these statements contain conditional probabilities, which we will study in Section 4.2. Information from that section will enable us to answer the following:

Suppose a person answers 10% of a long battery of questions with lies. Assume that the remaining 90% of the questions are answered truthfully.

1. Estimate the percentage of answers the polygraph will *wrongly* indicate as lies.

2. Estimate the percentage of answers the polygraph will *correctly* indicate as lies.

If the polygraph indicated that 30% of the questions were answered as lies, what would you estimate for the *actual* percentage of questions the person answered as lies? (See Problems 25 and 26 in Section 4.2.)

John Neubauer/PhotoEdit

SECTION 4.1

What Is Probability?

FOCUS POINTS

- Assign probabilities to events.
- Explain how the law of large numbers relates to relative frequencies.
- Apply basic rules of probability in everyday life.
- Explain the relationship between statistics and probability.

We encounter statements given in terms of probability all the time. An excited sports announcer claims that Sheila has a 90% chance of breaking the world record in the upcoming 100-yard dash. Henry figures that if he guesses on a true–false question, the probability of getting it right is 1/2. The Right to Health lobby claims that the probability is 0.40 of getting an erroneous report from a medical laboratory at a low-cost health center. It is consequently lobbying for a federal agency to license and monitor all medical laboratories.

When we use probability in a statement, we're using a *number between 0 and 1* to indicate the likelihood of an event.

> **Probability** is a numerical measure between 0 and 1 that describes the likelihood that an event will occur. Probabilities closer to 1 indicate that the event is more likely to occur. Probabilities closer to 0 indicate that the event is less likely to occur.

Probability of an event A, P(A)

> $P(A)$, read "P of A," denotes the **probability of event A.**
>
> If $P(A) = 1$, event A is certain to occur.
>
> If $P(A) = 0$, event A is certain not to occur.

It is important to know what probability statements mean and how to determine probabilities of events, because probability is the language of inferential statistics.

> **Probability assignments**
>
> *Intuition*
>
> 1. A probability assignment based on **intuition** incorporates past experience, judgment, or opinion to estimate the likelihood of an event.
>
> *Relative frequency*
>
> 2. A probability assignment based on **relative frequency** uses the formula
>
> $$\text{Probability of event} = \text{relative frequency} = \frac{f}{n} \qquad (1)$$
>
> where f is the frequency of the event occurrence in a sample of n observations.
>
> *Equally likely outcomes*
>
> 3. A probability assignment based on **equally likely outcomes** uses the formula
>
> $$\text{Probability of event} = \frac{\text{Number of outcomes favorable to event}}{\text{Total number of outcomes}} \qquad (2)$$

EXAMPLE 1   PROBABILITY ASSIGNMENT

Consider each of the following events, and determine how the probability is assigned.

(a) A sports announcer claims that Sheila has a 90% chance of breaking the world record in the 100-yard dash.

   SOLUTION: It is likely the sports announcer used intuition based on Sheila's past performance.

(b) Henry figures that if he guesses on a true–false question, the probability of getting it right is 0.50.

   SOLUTION: In this case there are two possible outcomes: Henry's answer is either correct or incorrect. Since Henry is guessing, we assume the outcomes are equally likely. There are $n = 2$ equally likely outcomes, and only one is correct. By formula (2),

$$P(\text{correct answer}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{1}{2} = 0.50$$

(c) The Right to Health lobby claims that the probability of getting an erroneous medical laboratory report is 0.40, based on a random sample of 200 laboratory reports, of which 80 were erroneous.

   SOLUTION: Formula (1) for relative frequency gives the probability, with sample size $n = 200$ and number of errors $f = 80$.

$$P(\text{error}) = \text{relative frequency} = \frac{f}{n} = \frac{80}{200} = 0.40$$

We've seen three ways to assign probabilities: intuition, relative frequency, and—when outcomes are equally likely—a formula. Which do we use? Most of the time it depends on the information that is at hand or that can be feasibly obtained. Our choice of methods also depends on the particular problem. In Guided Exercise 1, you will see three different situations, and you will decide the best way to assign the probabilities. *Remember, probabilities are numbers between 0 and 1, so don't assign probabilities outside this range.*

GUIDED EXERCISE 1 | **Determine a probability**

Assign a probability to the indicated event on the basis of the information provided. Indicate the technique you used: intuition, relative frequency, or the formula for equally likely outcomes.

(a) A random sample of 500 students at Hudson College were surveyed and it was determined that 375 wear glasses or contact lenses. Estimate the probability that a Hudson College student selected at random wears corrective lenses.

In this case we are given a sample size of 500, and we are told that 375 of these students wear corrective lenses. It is appropriate to use a relative frequency for the desired probability:

$$P(\text{student needs corrective lenses}) = \frac{f}{n} = \frac{375}{500} = 0.75$$

Continued

(b) The Friends of the Library hosts a fundraising barbecue. George is on the cleanup committee. There are four members on this committee, and they draw lots to see who will clean the grills. Assuming that each member is equally likely to be drawn, what is the probability that George will be assigned the grill-cleaning job?

⟹   There are four people on the committee, and each is equally likely to be drawn. It is appropriate to use the formula for equally likely events. George can be drawn in only one way, so there is only one outcome favorable to the event.

$$P(\text{George}) = \frac{\text{No. of favorable outcomes}}{\text{Total no. of outcomes}} = \frac{1}{4} = 0.25$$

(c) Joanna photographs whales for Sea Life Adventure Films. On her next expedition, she is to film blue whales feeding. Based on her knowledge of the habits of blue whales, she is almost certain she will be successful. What specific number do you suppose she estimates for the probability of success?

⟹   Since Joanna is almost certain of success, she should make the probability close to 1. We could say $P(\text{success})$ is above 0.90 but less than 1. This probability assignment is based on intuition.

The technique of using the relative frequency of an event as the probability of that event is a common way of assigning probabilities and will be used a great deal in later chapters. The underlying assumption we make is that if events occurred a certain percentage of times in the past, they will occur about the same percentage of times in the future. In fact, this assumption can be strengthened to a very general statement called the *law of large numbers*.

Law of large numbers

Law of large numbers

In the long run, as the sample size increases and increases, the relative frequencies of outcomes get closer and closer to the theoretical (or actual) probability value.

The law of large numbers is the reason businesses such as health insurance, automobile insurance, and gambling casinos can exist and make a profit.

No matter how we compute probabilities, it is useful to know what outcomes are possible in a given setting. For instance, if you are going to decide the probability that Hardscrabble will win the Kentucky Derby, you need to know which other horses will be running.

To determine the possible outcomes for a given setting, we need to define a *statistical experiment*.

Statistical experiment

Event

Simple event

Sample space

A **statistical experiment** or **statistical observation** can be thought of as any random activity that results in a definite outcome.

An **event** is a collection of one or more outcomes of a statistical experiment or observation.

A **simple event** is one particular outcome of a statistical experiment.

The set of all simple events constitutes the **sample space** of an experiment.

EXAMPLE 2   USING A SAMPLE SPACE

Human eye color is controlled by a single pair of genes (one from the father and one from the mother) called a *genotype*. Brown eye color, B, is dominant over blue eye color, ℓ. Therefore, in the genotype Bℓ, consisting of one brown gene B

and one blue gene ℓ, the brown gene dominates. A person with a Bℓ genotype has brown eyes.

If both parents have brown eyes and have genotype Bℓ, what is the probability that their child will have blue eyes? What is the probability the child will have brown eyes?

SOLUTION: To answer these questions, we need to look at the sample space of all possible eye-color genotypes for the child. They are given in Table 4-1.

TABLE 4-1   **Eye Color Genotypes for Child**

	Mother	
Father	**B**	**ℓ**
B	BB	Bℓ
ℓ	ℓB	$\ell\ell$

According to genetics theory, the four possible genotypes for the child are equally likely. Therefore, we can use Formula (2) to compute probabilities. Blue eyes can occur only with the $\ell\ell$ genotype, so there is only one outcome favorable to blue eyes. By formula (2),

$$P(\text{blue eyes}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{1}{4}$$

Brown eyes occur with the three remaining genotypes: BB, Bℓ, and ℓB. By formula (2),

$$P(\text{brown eyes}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{3}{4}$$

GUIDED EXERCISE 2   *Using a sample space*

Professor Gutierrez is making up a final exam for a course in literature of the southwest. He wants the last three questions to be of the true–false type. To guarantee that the answers do not follow his favorite pattern, he lists all possible true–false combinations for three questions on slips of paper and then picks one at random from a hat.

(a) Finish listing the outcomes in the given sample space.

| TTT | FTT | TFT | _____ |
| TTF | FTF | TFF | _____ |

➡ The missing outcomes are FFT and FFF.

(b) What is the probability that all three items will be false? Use the formula

$$P(\text{all F}) = \frac{\text{No. of favorable outcomes}}{\text{Total no. of outcomes}}$$

➡ There is only one outcome, FFF, favorable to all false, so

$$P(\text{all F}) = \frac{1}{8}$$

(c) What is the probability that exactly two items will be true?

➡ There are three outcomes that have exactly two true items: TTF, TFT, and FTT. Thus,

$$P(\text{two T}) = \frac{\text{No. of favorable outcomes}}{\text{Total no. of outcomes}} = \frac{3}{8}$$

There is another important point about probability assignments of simple events.

> The **sum** of the probabilities of all simple events in a sample space must equal 1.

We can use this fact to determine the probability that an event will not occur. For instance, if you think the probability is 0.65 that you will win a tennis match, you assume the probability is 0.35 that your opponent will win.

The *complement* of an event A is the event that A *does not occur*. We use the notation A^c to designate the complement of event A. Figure 4-1 shows the event A and its complement A^c.

Notice that the two distinct events A and A^c make up the entire sample space. Therefore, the sum of their probabilities is 1.

Complement of an event A^c

> The **complement of event A** is the event that A *does not occur*. A^c designates the complement of event A. Furthermore,
>
> 1. $P(A) + P(A^c) = 1$
> 2. $P(\text{event } A \text{ does } not \text{ occur}) = P(A^c) = 1 - P(A)$ (3)
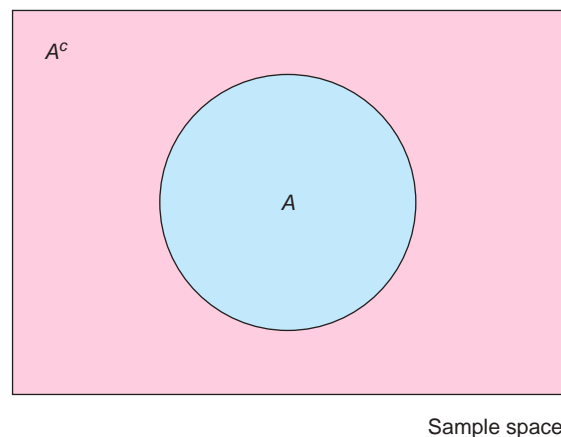
EXAMPLE 3   COMPLEMENT OF AN EVENT

The probability that a college student who has not received a flu shot will get the flu is 0.45. What is the probability that a college student will *not* get the flu if the student has not had the flu shot?

SOLUTION: In this case, we have

$P(\text{will get flu}) = 0.45$

$P(\text{will } not \text{ get flu}) = 1 - P(\text{will get flu}) = 1 - 0.45 = 0.55$

FIGURE 4-1

The Event A and Its Complement A^c

Sample space

GUIDED EXERCISE 3 | ### Complement of an event

A veterinarian tells you that if you breed two cream-colored guinea pigs, the probability that an off-spring will be pure white is 0.25. What is the probability that an offspring will not be pure white?

(a) P(pure white) + P(*not* pure white) = _____   ⟹   1

(b) P(*not* pure white) = _____   ⟹ $1 - 0.25$, or 0.75

LOOKING FORWARD

The complement rule has special application in Chapter 5. There we study probabilities associated with binomial experiments. These are experiments that have only two outcomes, "success" and "failure." One feature of these experiments is that the outcomes can be categorized according to the number of "successes" out of n independent trials.

> Simple Events of Sample Space of a Binomial Experiment
> 0 successes, 1 success, 2 successes, 3 successes, . . . , n successes

A convenient way to compute the probability of "at least one" success is to observe that the complement of "at least one success" is the event "0 successes." Then, by the complement rule

$$P(\text{at least one success}) = 1 - P(0 \text{ successes})$$

Problems 8, 9, and 10 of this section show a similar use of the complement rule.

Sometimes, instead of being told the probability of an event, we are given the odds for the event. Problems 19 and 20 discuss odds in favor and odds against an event.

Summary: Some important facts about probability

1. A **statistical experiment** or **statistical observation** is any random activity that results in a definite outcome. A **simple event** consists of one and only one outcome of the experiment. The **sample space** is the set of all simple events. An **event A** is any subset of the sample space.

2. The probability of an event A is denoted by $P(A)$.

3. The probability of an event is a number between 0 and 1. The closer to 1 the probability is, the more likely it is the event will occur. The closer to 0 the probability is, the less likely it is the event will occur.

4. The sum of the probabilities of all simple events in a sample space is 1.

5. Probabilities can be assigned by using intuition, relative frequencies, or the formula for equally likely outcomes. Additional ways to assign probabilities will be introduced in later chapters.

6. The **complement** of an event A is denoted by A^c. So, A^c is the event that A does not occur.

7. $P(A) + P(A^c) = 1$

Interpreting Probabilities

As stressed in item 3 in the summary about probabilities, the closer the probability is to 1, the more likely the event is to occur. The closer the probability is to 0, the less likely the event is to occur. However, just because the probability of an event is very high, it is not a certainty that the event will occur. Likewise, even though the probability of an event is very low, the event might still occur.

Events with low probability, but big consequences, are of special concern. Such events are sometimes referred to as "black swan events," as discussed in the very popular book *The Black Swan* by Nassim Nicholas Taleb. In his varied careers Taleb has been a stock trader, author, and professor at the University of Massachusetts. His special interests are mathematical problems of luck, uncertainty, probability, and general knowledge. In his book *The Black Swan*, Taleb discusses "the impact of the highly improbable." If an event has (a) little impact, bearing, or meaning on a person's life and (b) very low probability, then (c) it seems reasonable to ignore the event.

However, some of the really big mistakes in a person's life can result from misjudging either (a) the size of an event's impact or (b) the likelihood the event will occur. An event of great importance cannot be ignored even if it has a low probability of occurrence.

So what should a person do?

Taleb describes himself as a hyperskeptic when others are gullible, and gullible when others are skeptical. In particular, he recommends being skeptical about (mentally) confirming an event when the errors of confirmation are costly. Lots and lots of data should not be thought of as confirming an event when a simple instance can disconfirm an important and costly event. Taleb recommends that one be skeptical in the face of wild randomness (and big consequences) and gullible when the randomness (and consequences) are mild. One way to estimate the "randomness" of a situation is to estimate its probability. Estimating the "consequences" of an event in your own life must be a personal decision.

Probability Related to Statistics

We conclude this section with a few comments on the nature of statistics versus probability. Although statistics and probability are closely related fields of mathematics, they are nevertheless separate fields. It can be said that probability is the medium through which statistical work is done. In fact, if it were not for probability theory, inferential statistics would not be possible.

Put very briefly, probability is the field of study that makes statements about what will occur when samples are drawn from a *known population*. Statistics is the field of study that describes how samples are to be obtained and how inferences are to be made about *unknown populations*.

A simple but effective illustration of the difference between these two subjects can be made by considering how we treat the following examples.

Example of a probability application

Condition: We *know* the exact makeup of the *entire* population.

Example: Given 3 green marbles, 5 red marbles, and 4 white marbles in a bag, draw 6 marbles at random from the bag. What is the probability that none of the marbles is red?

Example of a statistical application

Condition: We have only *samples* from an otherwise *unknown* population.

Example: Draw a random sample of 6 marbles from the (unknown) population of all marbles in a bag and observe the colors. Based on the sample results, make a conjecture about the colors and numbers of marbles in the entire population of all marbles in the bag.

In another sense, probability and statistics are like flip sides of the same coin. On the probability side, you know the overall description of the population. The central problem is to compute the likelihood that a specific outcome will happen.

On the statistics side, you know only the results of a sample drawn from the population. The central problem is to describe the sample (descriptive statistic) and to draw conclusions about the population based on the sample results (inferential statistics).

In statistical work, the inferences we draw about an unknown population are not claimed to be absolutely correct. Since the population remains unknown (in a theoretical sense), we must accept a "best guess" for our conclusions and act using the most probable answer rather than absolute certainty.

Probability is the topic of this chapter. However, we will not study probability just for its own sake. Probability is a wonderful field of mathematics, but we will study mainly the ideas from probability that are needed for a proper understanding of statistics.

VIEWPOINT | What Makes a Good Teacher?

A survey of 735 students at nine colleges in the United States was taken to determine instructor behaviors that help students succeed. Data from this survey can be found by visiting the Brase/Brase statistics site at **http://www.cengage.com/statistics/brase** *and finding the link to DASL, the Carnegie Mellon University Data and Story Library. Once at the DASL site, select Data Subjects, then Psychology, and then Instructor Behavior. You can estimate the probability of how a student would respond (very positive, neutral, very negative) to different instructor behaviors. For example, more than 90% of the students responded "very positive" to the instructor's use of real-world examples in the classroom.*

SECTION 4.1 PROBLEMS

1. *Statistical Literacy* List three methods of assigning probabilities.

2. *Statistical Literacy* Suppose the newspaper states that the probability of rain today is 30%. What is the complement of the event "rain today"? What is the probability of the complement?

3. *Statistical Literacy* What is the probability of
   (a) an event *A* that is certain to occur?
   (b) an event *B* that is impossible?

4. *Statistical Literacy* What is the law of large numbers? If you were using the relative frequency of an event to estimate the probability of the event, would it be better to use 100 trials or 500 trials? Explain.

5. *Basic Computation: Probability as Relative Frequency* A recent Harris Poll survey of 1010 U.S. adults selected at random showed that 627 consider the occupation of firefighter to have very great prestige. Estimate the probability (to the nearest hundredth) that a U.S. adult selected at random thinks the occupation of firefighter has very great prestige.

6. *Basic Computation: Probability of Equally Likely Events* What is the probability that a day of the week selected at random will be a Wednesday?

7. *Interpretation* An investment opportunity boasts that the chance of doubling your money in 3 years is 95%. However, when you research the details of the investment, you estimate that there is a 3% chance that you could lose the entire investment. Based on this information, are you certain to make money on this investment? Are there risks in this investment opportunity?

8. *Interpretation* A sample space consists of 4 simple events: *A*, *B*, *C*, *D*. Which events comprise the complement of *A*? Can the sample space be viewed as having two events, *A* and A^c? Explain.

9. *Critical Thinking* Consider a family with 3 children. Assume the probability that one child is a boy is 0.5 and the probability that one child is a girl is also 0.5, and that the events "boy" and "girl" are independent.
   (a) List the equally likely events for the gender of the 3 children, from oldest to youngest.
   (b) What is the probability that all 3 children are male? Notice that the complement of the event "all three children are male" is "at least one of the children is female." Use this information to compute the probability that at least one child is female.

10. *Critical Thinking* Consider the experiment of tossing a fair coin 3 times. For each coin, the possible outcomes are heads or tails.
    (a) List the equally likely events of the sample space for the three tosses.
    (b) What is the probability that all three coins come up heads? Notice that the complement of the event "3 heads" is "at least one tail." Use this information to compute the probability that there will be at least one tail.

11. *Critical Thinking* On a single toss of a fair coin, the probability of heads is 0.5 and the probability of tails is 0.5. If you toss a coin twice and get heads on the first toss, are you guaranteed to get tails on the second toss? Explain.

12. *Critical Thinking*
    (a) Explain why −0.41 cannot be the probability of some event.
    (b) Explain why 1.21 cannot be the probability of some event.
    (c) Explain why 120% cannot be the probability of some event.
    (d) Can the number 0.56 be the probability of an event? Explain.

13. *Probability Estimate: Wiggle Your Ears* Can you wiggle your ears? Use the students in your statistics class (or a group of friends) to estimate the percentage of people who can wiggle their ears. How can your result be thought of as an estimate for the probability that a person chosen at random can wiggle his or her ears? *Comment*: National statistics indicate that about 13% of Americans can wiggle their ears (Source: Bernice Kanner, *Are You Normal?*, St. Martin's Press, New York).

14. *Probability Estimate: Raise One Eyebrow* Can you raise one eyebrow at a time? Use the students in your statistics class (or a group of friends) to estimate the percentage of people who can raise one eyebrow at a time. How can your result be thought of as an estimate for the probability that a person chosen at random can raise one eyebrow at a time? *Comment:* National statistics indicate that about 30% of Americans can raise one eyebrow at a time (see source in Problem 13).

15. *Myers–Briggs: Personality Types* Isabel Briggs Myers was a pioneer in the study of personality types. The personality types are broadly defined according to four main preferences. Do married couples choose similar or different personality types in their mates? The following data give an indication (Source: I. B. Myers and M. H. McCaulley, *A Guide to the Development and Use of the Myers–Briggs Type Indicators*).

Similarities and Differences in a Random Sample of 375 Married Couples

Number of Similar Preferences	Number of Married Couples
All four	34
Three	131
Two	124
One	71
None	15

Suppose that a married couple is selected at random.
(a) Use the data to estimate the probability that they will have 0, 1, 2, 3, or 4 personality preferences in common.

(b) Do the probabilities add up to 1? Why should they? What is the sample space in this problem?

16. *General: Roll a Die*
    (a) If you roll a single die and count the number of dots on top, what is the sample space of all possible outcomes? Are the outcomes equally likely?
    (b) Assign probabilities to the outcomes of the sample space of part (a). Do the probabilities add up to 1? Should they add up to 1? Explain.
    (c) What is the probability of getting a number less than 5 on a single throw?
    (d) What is the probability of getting 5 or 6 on a single throw?

17. *Psychology: Creativity* When do creative people get their *best* ideas? *USA Today* did a survey of 966 inventors (who hold U.S. patents) and obtained the following information:

Time of Day When Best Ideas Occur

Time	Number of Inventors
6 A.M.–12 noon	290
12 noon–6 P.M.	135
6 P.M.–12 midnight	319
12 midnight–6 A.M.	222

(a) Assuming that the time interval includes the left limit and all the times up to but not including the right limit, estimate the probability that an inventor has a best idea during each time interval: from 6 A.M. to 12 noon, from 12 noon to 6 P.M., from 6 P.M. to 12 midnight, from 12 midnight to 6 A.M.
(b) Do the probabilities of part (a) add up to 1? Why should they? What is the sample space in this problem?

18. *Agriculture: Cotton* A botanist has developed a new hybrid cotton plant that can withstand insects better than other cotton plants. However, there is some concern about the germination of seeds from the new plant. To estimate the probability that a seed from the new plant will germinate, a random sample of 3000 seeds was planted in warm, moist soil. Of these seeds, 2430 germinated.
    (a) Use relative frequencies to estimate the probability that a seed will germinate. What is your estimate?
    (b) Use relative frequencies to estimate the probability that a seed will *not* germinate. What is your estimate?
    (c) Either a seed germinates or it does not. What is the sample space in this problem? Do the probabilities assigned to the sample space add up to 1? Should they add up to 1? Explain.
    (d) Are the outcomes in the sample space of part (c) equally likely?

19. *Expand Your Knowledge: Odds in Favor* Sometimes probability statements are expressed in terms of odds.

    The *odds in favour* of an event A are the ratio $\dfrac{P(A)}{P(notA)} = \dfrac{P(A)}{P(A^c)}$.

    For instance, if $P(A) = 0.60$, then $P(A^c) = 0.40$ and the odds in favor of A are

    $$\frac{0.60}{0.40} = \frac{6}{4} = \frac{3}{2}, \text{written as 3 to 2 or 3:2}$$

    (a) Show that if we are given the *odds in favor* of event A as $n{:}m$, the probability of event A is given by $P(A) = \dfrac{n}{n+m}$. *Hint:* Solve the equation $\dfrac{n}{m} = \dfrac{P(A)}{1 - P(A)}$ for $P(A)$.

(b) A telemarketing supervisor tells a new worker that the odds of making a sale on a single call are 2 to 15. What is the probability of a successful call?

(c) A sports announcer says that the odds a basketball player will make a free throw shot are 3 to 5. What is the probability the player will make the shot?

20. *Expand Your Knowledge: Odds Against* Betting odds are usually stated against the event happening (against winning).

The *odds against* event W are the ratio $\dfrac{P(not\ W)}{P(W)} = \dfrac{P(W^c)}{P(W)}$.

In horse racing, the betting odds are based on the probability that the horse does *not* win.

(a) Show that if we are given the *odds against* an event W as $a{:}b$, the probability of *not* W is $P(W^c) = \dfrac{a}{a+b}$. *Hint:* Solve the equation $\dfrac{a}{b} = \dfrac{P(W^c)}{1 - P(W^c)}$ for $P(W^c)$.

(b) In a recent Kentucky Derby, the betting odds for the favorite horse, Point Given, were 9 to 5. Use these odds to compute the probability that Point Given would lose the race. What is the probability that Point Given would win the race?

(c) In the same race, the betting odds for the horse Monarchos were 6 to 1. Use these odds to estimate the probability that Monarchos would lose the race. What is the probability that Monarchos would win the race?

(d) Invisible Ink was a long shot, with betting odds of 30 to 1. Use these odds to estimate the probability that Invisible Ink would lose the race. What is the probability the horse would win the race? For further information on the Kentucky Derby, visit the Brase/Brase statistics site at **http://www.cengage.com/statistics/brase** and find the link to the Kentucky Derby.

21. *Business: Customers* John runs a computer software store. Yesterday he counted 127 people who walked by his store, 58 of whom came into the store. Of the 58, only 25 bought something in the store.

(a) Estimate the probability that a person who walks by the store will enter the store.

(b) Estimate the probability that a person who walks into the store will buy something.

(c) Estimate the probability that a person who walks by the store will come in *and* buy something.

(d) Estimate the probability that a person who comes into the store will buy nothing.

SECTION 4.2

Some Probability Rules—Compound Events

FOCUS POINTS

- Compute probabilities of general compound events.
- Compute probabilities involving independent events or mutually exclusive events.
- Use survey results to compute conditional probabilities.

Conditional Probability and Multiplication Rules

You roll two dice. What is the probability that you will get a 5 on each die? Consider a collection of 6 balls identical in every way except in color. There are 3 green balls, 2 blue balls, and 1 red ball. You draw two balls at random from the collection *without replacing* the first ball before drawing the second. What is the probability that both balls will be green?

It seems that these two problems are nearly alike. They are alike in the sense that in each case, you are to find the probability of two events occurring *together*.

In the first problem, you are to find

$P(5$ on 1st die *and* 5 on 2nd die)

In the second, you want

$P($green ball on 1st draw *and* green ball on 2nd draw)

The two problems differ in one important aspect, however. In the dice problem, the outcome of a 5 on the first die does not have any effect on the probability of getting a 5 on the second die. Because of this, the events are *independent*.

Independent events

> Two events are **independent** if the occurrence or nonoccurrence of one event does *not* change the probability that the other event will occur.

In the problem concerning a collection of colored balls, the probability that the first ball is green is 3/6, since there are 6 balls in the collection and 3 of them are green. If you get a green ball on the first draw, the probability of getting a green ball on the second draw is changed to 2/5, because one green ball has already been drawn and only 5 balls remain. Therefore, the two events in the ball-drawing problem are *not* independent. They are, in fact, *dependent*, since the outcome of the first draw changes the probability of getting a green ball on the second draw.

Dependent events

Probability of *A* and *B*

Why does the *independence* or *dependence* of two events matter? The type of events determines the way we compute the probability of the two events happening together. If two events *A* and *B* are *independent*, then we use formula (4) to compute the probability of the event *A* and *B*:

> **Multiplication rule for independent events**
>
> $P(A \text{ } and \text{ } B) = P(A) \cdot P(B)$ (4)

If the events are *dependent*, then we must take into account the changes in the probability of one event caused by the occurrence of the other event. The notation $P(A, \text{ }given \text{ } B)$ denotes the probability that event *A* will occur *given* that event *B* has occurred. This is called a *conditional probability*. We read $P(A, \text{ }given \text{ } B)$ as "probability of *A* given *B*." If *A* and *B* are dependent events, then $P(A) \neq P(A, \text{ }given \text{ } B)$ because the occurrence of event *B* has changed the probability that event *A* will occur. A standard notation for $P(A, \text{ }given \text{ } B)$ is $P(A \mid B)$.

Conditional probability

$P(A \mid B)$

> **General multiplication rule for any events**
>
> $P(A \text{ } and \text{ } B) = P(A) \cdot P(B \mid A)$ (5)
>
> $P(A \text{ } and \text{ } B) = P(B) \cdot P(A \mid B)$ (6)

We will use either formula (5) or formula (6) according to the information available.

Multiplication rules of probability

Formulas (4), (5), and (6) constitute the *multiplication rules of probability*. They help us compute the probability of events happening together when the sample space is too large for convenient reference or when it is not completely known.

Note: For conditional probability, observe that the multiplication rule

$P(A \text{ } and \text{ } B) = P(B) \cdot P(A \mid B)$

can be solved for $P(A \mid B)$, leading to

> **Conditional probability (when $P(B) \neq 0$)**
>
> $$P(A \mid B) = \frac{P(A \ and \ B)}{P(B)}$$

We will see some applications of this formula in later chapters.

Let's use the multiplication rules to complete the dice and ball-drawing problems just discussed. We'll compare the results with those obtained by using the sample space directly.

EXAMPLE 4   MULTIPLICATION RULE, INDEPENDENT EVENTS

Suppose you are going to throw two fair dice. What is the probability of getting a 5 on each die?

SOLUTION USING THE MULTIPLICATION RULE: The two events are independent, so we should use formula (4). $P(5$ on 1st die *and* 5 on 2nd die$) = P(5$ on 1st$) \cdot P(5$ on 2nd$)$. To finish the problem, we need to compute the probability of getting a 5 when we throw one die.

There are six faces on a die, and on a fair die each is equally likely to come up when you throw the die. Only one face has five dots, so by formula (2) for equally likely outcomes,

$$P(5 \text{ on die}) = \frac{1}{6}$$

Now we can complete the calculation.

$$P(5 \text{ on 1st die } and \ 5 \text{ on 2nd die}) = P(5 \text{ on 1st}) \cdot P(5 \text{ on 2nd})$$
$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

SOLUTION USING SAMPLE SPACE: The first task is to write down the sample space. Each die has six equally likely outcomes, and each outcome of the second die can be paired with each of the first. The sample space is shown in Figure 4-2. The total number of outcomes is 36, and only one is favorable to a 5 on the first die *and* a 5 on the second. The 36 outcomes are equally likely, so by formula (2) for equally likely outcomes,

FIGURE 4-2

Sample Space for Two Dice
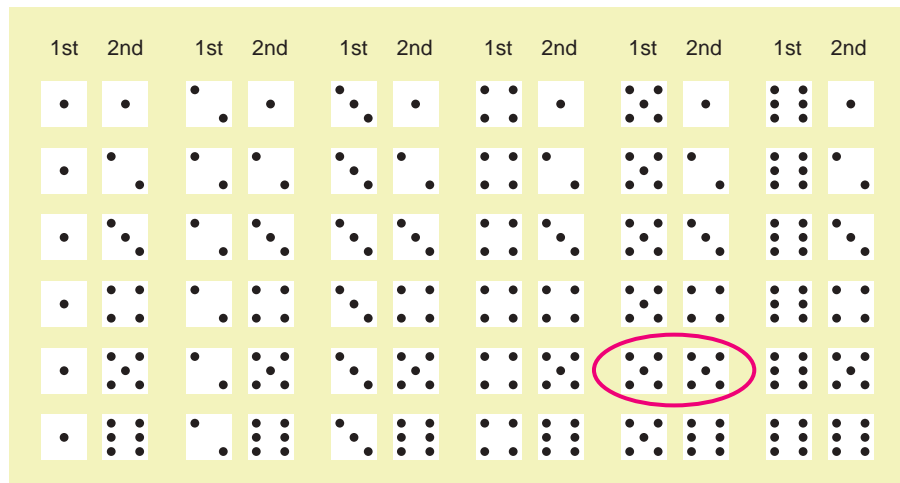
$$P(5 \text{ on 1st } and \text{ 5 on 2nd}) = \frac{1}{36}$$

The two methods yield the same result. The multiplication rule was easier to use because we did not need to look at all 36 outcomes in the sample space for tossing two dice.

EXAMPLE 5   MULTIPLICATION RULE, DEPENDENT EVENTS

Consider a collection of 6 balls that are identical except in color. There are 3 green balls, 2 blue balls, and 1 red ball. Compute the probability of drawing 2 green balls from the collection if the first ball is *not replaced* before the second ball is drawn.

MULTIPLICATION RULE METHOD: These events are *dependent*. The probability of a green ball on the first draw is 3/6, but on the second draw the probability of a green ball is only 2/5 if a green ball was removed on the first draw. By the multiplication rule for dependent events,

$$P(\text{green ball on 1st draw } and = P(\text{green on 1st}) \cdot P(\text{green on 2nd | green on 1st})$$
$$\text{green ball on 2nd draw})$$
$$= \frac{3}{2} \cdot \frac{2}{5} = \frac{1}{5} = 0.2$$

SAMPLE SPACE METHOD: Each of the 6 possible outcomes for the 1st draw must be paired with each of the 5 possible outcomes for the second draw. This means that there are a total of 30 possible pairs of balls. Figure 4-3 shows all the possible pairs of balls. In 6 of the pairs, both balls are green.

$$P(\text{green ball on 1st draw } and \text{ green ball on 2nd draw}) = \frac{6}{30} = 0.2$$

Again, the two methods agree.

FIGURE 4-3

Sample Space of Drawing Two Balls Without Replacement from a Collection of 3 Green Balls, 2 Blue Balls, and 1 Red Ball
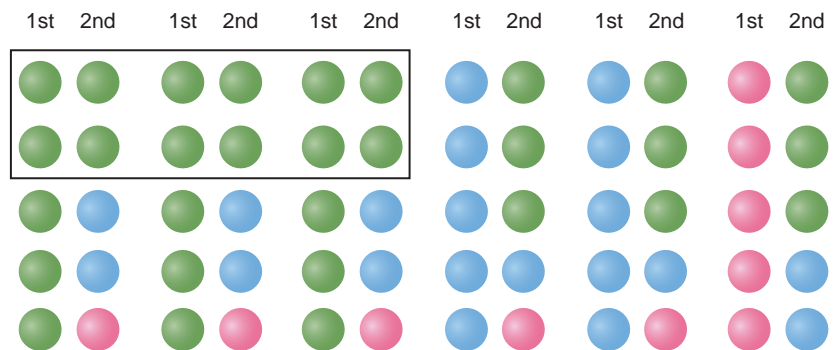
The multiplication rules apply whenever we wish to determine the probability of two events happening *together*. To indicate "together," we use *and* between the events. But before you use a multiplication rule to compute the probability of *A and B*, you must determine if *A* and *B* are independent or dependent events.

PROCEDURE

HOW TO USE THE MULTIPLICATION RULES

1. First determine whether A and B are independent events. If $P(A) = P(A \mid B)$, then the events are independent.

2. If A and B are independent events,
$$P(A \text{ and } B) = P(A) \cdot P(B) \qquad\qquad (4)$$

3. If A and B are any events,
$$P(A \text{ and } B) = P(A) \cdot P(B \mid A) \qquad\qquad (5)$$
$$\text{or} \quad P(A \text{ and } B) = P(B) \cdot P(A \mid B) \qquad\qquad (6)$$

Let's practice using the multiplication rule.

GUIDED EXERCISE 4   **Multiplication rule**

Andrew is 55, and the probability that he will be alive in 10 years is 0.72. Ellen is 35, and the probability that she will be alive in 10 years is 0.92. Assuming that the life span of one will have no effect on the life span of the other, what is the probability they will both be alive in 10 years?

(a) Are these events dependent or independent?

⟹ Since the life span of one does not affect the life span of the other, the events are independent.

(b) Use the appropriate multiplication rule to find P(Andrew alive in 10 years *and* Ellen alive in 10 years).

⟹ We use the rule for independent events:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

P(Andrew alive *and* Ellen alive)

$= P$(Andrew alive) $\cdot P$(Ellen alive)

$= (0.72)(0.92) \approx 0.66$

GUIDED EXERCISE 5   **Dependent events**

A quality-control procedure for testing Ready-Flash disposable cameras consists of drawing two cameras at random from each lot of 100 without replacing the first camera before drawing the second. If both are defective, the entire lot is rejected. Find the probability that both cameras are defective if the lot contains 10 defective cameras. Since we are drawing the cameras at random, assume that each camera in the lot has an equal chance of being drawn.

(a) What is the probability of getting a defective camera on the first draw?

⟹ The sample space consists of all 100 cameras. Since each is equally likely to be drawn and there are 10 defective ones,

$$P(\text{defective camera}) = \frac{10}{100} = \frac{1}{10}$$

Continued

(b) The first camera drawn is not replaced, so there are only 99 cameras for the second draw. What is the probability of getting a defective camera on the second draw if the first camera was defective?

If the first camera was defective, then there are only 9 defective cameras left among the 99 remaining cameras in the lot.

P(def. cam. on 2nd draw | def. cam. on 1st)

$$= \frac{9}{99} = \frac{1}{11}$$

(c) Are the probabilities computed in parts (a) and (b) different? Does drawing a defective camera on the first draw change the probability of getting a defective camera on the second draw? Are the events dependent?

The answer to all these questions is yes.

(d) Use the formula for dependent events,

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

to compute P(1st camera defective *and* 2nd camera defective).

P(1st defective *and* 2nd defective) $= \dfrac{1}{10} \cdot \dfrac{1}{11}$

$$= \frac{1}{110}$$
$$\approx 0.009$$

More than two independent events

The multiplication rule for independent events extends to *more than two independent events*. For instance, if the probability that a single seed will germinate is 0.85, and you plant 3 seeds, the probability that they will all germinate (assuming seed germinations are independent) is

P(1st germinates *and* 2nd germinates *and* 3rd germinates) $= (0.85)(0.85)(0.85)$

$$\approx 0.614$$

Addition Rules

One of the multiplication rules can be used any time we are trying to find the probability of two events happening *together*. Pictorially, we are looking for the probability of the shaded region in Figure 4-4(a).
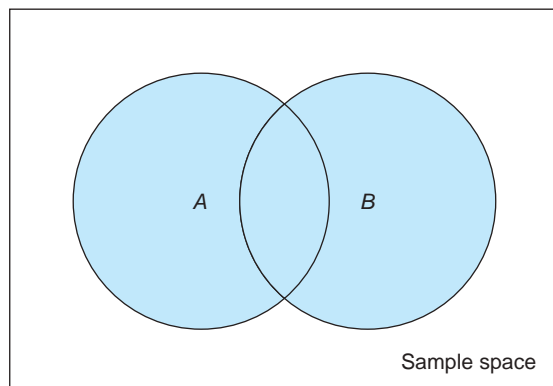
Probability of *A or B*

Another way to combine events is to consider the possibility of one event *or* another occurring. For instance, if a sports car saleswoman gets an extra bonus if she sells a convertible or a car with leather upholstery, she is interested in the

FIGURE 4-4

(a) The Event *A* and *B*
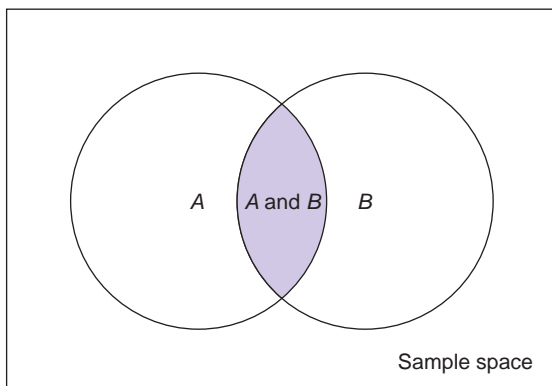
(b) The Event *A* or *B*

probability that you will buy a car that is a convertible *or* that has leather upholstery. Of course, if you bought a convertible with leather upholstery, that would be fine, too. Pictorially, the shaded portion of Figure 4-4(b) represents the outcomes satisfying the *or* condition. Notice that the condition *A or B* is satisfied by any one of the following conditions:

1. Any outcome in *A* occurs.

2. Any outcome in *B* occurs.

3. Any outcome in both *A* and *B* occurs.

It is important to distinguish between the *or* combinations and the *and* combinations because we apply different rules to compute their probabilities.

GUIDED EXERCISE 6 | **Combining events**

Indicate how each of the following pairs of events are combined. Use either the *and* combination or the *or* combination.

(a) Satisfying the humanities requirement by taking a course in the history of Japan or by taking a course in classical literature   ⟹   Use the *or* combination.

(b) Buying new tires and aligning the tires   ⟹   Use the *and* combination.

(c) Getting an A not only in psychology but also in biology   ⟹   Use the *and* combination.

(d) Having at least one of these pets: cat, dog, bird, rabbit   ⟹   Use the *or* combination.

Once you decide that you are to find the probability of an *or* combination rather than an *and* combination, what formula do you use? It depends on the situation. In particular, it depends on whether or not the events being combined share any outcomes. Example 6 illustrates two situations.

EXAMPLE 6   PROBABILITY OF EVENTS COMBINED WITH *OR*

Consider an introductory statistics class with 31 students. The students range from freshmen through seniors. Some students are male and some are female. Figure 4-5 shows the sample space of the class.

FIGURE 4-5

Sample Space for Statistics Class

F Designates Female   M Designates Male

Freshman	Sophomore	Junior	Senior
FFFFF	FFF	FFFF	F
FFFF			
MMM	MMM	MM	M
MMM	MM		
15 students	8 students	6 students	2 students

(a) Suppose we select one student at random from the class. Find the probability that the student is either a freshman or a sophomore.

Since there are 15 freshmen out of 31 students, $P(\text{freshmen}) = \dfrac{15}{31}$.

Since there are 8 sophomores out of 31 students, $P(\text{sophomore}) = \dfrac{8}{31}$.

$$P(\text{freshman } or \text{ sophomore}) = \frac{15}{31} + \frac{8}{31} = \frac{23}{31} \approx 0.742$$

Notice that we can simply add the probability of freshman to the probability of sophomore to find the probability that a student selected at random will be either a freshman or a sophomore. No student can be both a freshman and a sophomore at the same time.

(b) Select one student at random from the class. What is the probability that the student is either a male or a sophomore? Here we note that

$$P(\text{sophomore}) = \frac{8}{31} \qquad P(\text{male}) = \frac{14}{31} \qquad P(\text{sophomore } and \text{ male}) = \frac{5}{31}$$

If we simply add $P(\text{sophomore})$ and $P(\text{male})$, we're including $P(\text{sophomore } and \text{ male})$ twice in the sum. To compensate for this double summing, we simply subtract $P(\text{sophomore } and \text{ male})$ from the sum. Therefore,

$$P(\text{sophomore } or \text{ male}) = P(\text{sophomore}) + P(\text{male}) - P(\text{sophomore } and \text{ male})$$

$$= \frac{8}{31} + \frac{14}{31} - \frac{5}{31} = \frac{17}{31} \approx 0.548$$

Mutually exclusive events

We say the events A and B are *mutually exclusive* or *disjoint* if they cannot occur together. This means that A and B have no outcomes in common or, put another way, that $P(A \text{ and } B) = 0$.

> Two events are **mutually exclusive** or **disjoint** if they cannot occur together. In particular, events A and B are mutually exclusive if $P(A \text{ and } B) = 0$.

Formula (7) is the *addition rule for mutually exclusive events A and B*.

Addition rules

> **Addition rule for *mutually exclusive* events *A* and *B***
>
> $P(A \text{ or } B) = P(A) + P(B)$ (7)

If the events are not mutually exclusive, we must use the more general formula (8), which is the *general addition rule for any events A and B*.

> **General addition rule for any events *A* and *B***
>
> $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ (8)

You may ask: Which formula should I use? The answer is: Use formula (7) only if you know that A and B are mutually exclusive (i.e., cannot occur together); if you do not know whether A and B are mutually exclusive, then use

formula (8). Formula (8) is valid either way. Notice that when A and B are mutually exclusive, then $P(A\ and\ B) = 0$, so formula (8) reduces to formula (7).

PROCEDURE

HOW TO USE THE ADDITION RULES

1. First determine whether A and B are mutually exclusive events.
   If $P(A\ and\ B) = 0$, then the events are mutually exclusive.

2. If A and B are mutually exclusive events,
$$P(A\ or\ B) = P(A) + P(B) \tag{7}$$

3. If A and B are any events,
$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B) \tag{8}$$

GUIDED EXERCISE 7 | **Mutually exclusive events**

The Cost Less Clothing Store carries remainder pairs of slacks. If you buy a pair of slacks in your regular waist size without trying them on, the probability that the waist will be too tight is 0.30 and the probability that it will be too loose is 0.10.

(a) Are the events "too tight" and "too loose" mutually exclusive?

⟹ The waist cannot be both too tight and too loose at the same time, so the events are mutually exclusive.

(b) If you choose a pair of slacks at random in your regular waist size, what is the probability that the waist will be too tight or too loose?

⟹ Since the events are mutually exclusive,

P(too tight *or* too loose)

$= P$(too tight) $+ P$(too loose)

$= 0.30 + 0.10$

$= 0.40$

GUIDED EXERCISE 8 | **General addition rule**

Professor Jackson is in charge of a program to prepare people for a high school equivalency exam. Records show that 80% of the students need work in math, 70% need work in English, and 55% need work in both areas.

(a) Are the events "needs math" and "needs English" mutually exclusive?

⟹ These events are not mutually exclusive, since some students need both. In fact,

P(needs math *and* needs English) $= 0.55$

(b) Use the appropriate formula to compute the probability that a student selected at random needs math *or* needs English.

⟹ Since the events are not mutually exclusive, we use formula (8):

P(needs math *or* needs English)

$= P$(needs math) $+ P$(needs English)

$\quad - P$(needs math *and* English)

$= 0.80 + 0.70 - 0.55$

$= 0.95$

The addition rule for mutually exclusive events can be extended to apply to the situation in which we have more than two events, all of which are mutually exclusive to all the other events.

EXAMPLE 7   MUTUALLY EXCLUSIVE EVENTS

Laura is playing Monopoly. On her next move she needs to throw a sum bigger than 8 on the two dice in order to land on her own property and pass Go. What is the probability that Laura will roll a sum bigger than 8?

SOLUTION: When two dice are thrown, the largest sum that can come up is 12. Consequently, the only sums larger than 8 are 9, 10, 11, and 12. These outcomes are mutually exclusive, since only one of these sums can possibly occur on one throw of the dice. The probability of throwing more than 8 is the same as

$$P(9 \text{ or } 10 \text{ or } 11 \text{ or } 12)$$

Since the events are mutually exclusive,

$$P(9 \text{ or } 10 \text{ or } 11 \text{ or } 12) = P(9) + P(10) + P(11) + P(12)$$
$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$
$$= \frac{10}{36} = \frac{5}{18}$$

To get the specific values of $P(9)$, $P(10)$, $P(11)$, and $P(12)$, we used the sample space for throwing two dice (see Figure 4-2 on page 144). There are 36 equally likely outcomes—for example, those favorable to 9 are 6, 3; 3, 6; 5, 4; and 4, 5. So $P(9) = 4/36$. The other values can be computed in a similar way.

LOOKING FORWARD

Chapters 5 and 6 involve several probability distributions. In these chapters, most of the events of interest are mutually exclusive as well as independent. This means we can compute probabilities by using the extended addition rule for mutually exclusive events and the extended multiplication rule for independent events.

Further Examples

Most of us have been asked to participate in a survey. Schools, retail stores, news media, and government offices all conduct surveys. There are many types of surveys, and it is not our intention to give a general discussion of this topic. Let us study a very popular method called the *simple tally survey*. Such a survey consists of questions to which the responses can be recorded in the rows and columns of a table called a *contingency table*. These questions are appropriate to the information you want and are designed to cover the *entire* population of interest. In addition, the questions should be designed so that you can partition the sample space of responses into distinct (that is, mutually exclusive) sectors.

If the survey includes responses from a reasonably large random sample, then the results should be representative of your population. In this case, you can estimate simple probabilities, conditional probabilities, and the probabilities of some combinations of events directly from the results of the survey.

EXAMPLE 8   SURVEY

At Hopewell Electronics, all 140 employees were asked about their political affiliations. The employees were grouped by type of work, as executives or production workers. The results with row and column totals are shown in Table 4-2.

TABLE 4-2   **Employee Type and Political Affiliation**

Employee Type	Political Affiliation			Row Total
	Democrat (*D*)	Republican (*R*)	Independent (*I*)	
Executive (*E*)	5	34	9	48
Production worker (*PW*)	63	21	8	92
Column Total	68	55	17	140 Grand Total

Suppose an employee is selected at random from the 140 Hopewell employees. Let us use the following notation to represent different events of choosing: E = executive; PW = production worker; D = Democrat; R = Republican; I = Independent.

(a) Compute $P(D)$ and $P(E)$.

SOLUTION: To find these probabilities, we look at the *entire* sample space.

$$P(D) = \frac{\text{Number of Democrats}}{\text{Number of employees}} = \frac{68}{140} \approx 0.486$$

$$P(E) = \frac{\text{Number of executives}}{\text{Number of employees}} = \frac{48}{140} \approx 0.343$$

(b) Compute $P(D \mid E)$.

SOLUTION: For the conditional probability, we restrict our attention to the portion of the sample space satisfying the condition of being an executive.

$$P(D \mid E) = \frac{\text{Number of executives who are Democrats}}{\text{Number of executives}} = \frac{5}{48} \approx 0.104$$

(c) Are the events D and E independent?

SOLUTION: One way to determine if the events D and E are independent is to see if $P(D) = P(D \mid E)$ [or equivalently, if $P(E) = P(E \mid D)$]. Since $P(D) \approx 0.486$ and $P(D \mid E) \approx 0.104$, we see that $P(D) \neq P(D \mid E)$. This means that the events D and E are *not* independent. The probability of event D "depends on" whether or not event E has occurred.

(d) Compute $P(D \text{ and } E)$.

SOLUTION: This probability is not conditional, so we must look at the entire sample space.

$$P(D \text{ and } E) = \frac{\text{Number of executives who are Democrats}}{\text{Total number of employees}} = \frac{5}{140} \approx 0.036$$

Let's recompute this probability using the rules of probability for dependent events.

$$P(D \text{ and } E) = P(E) \cdot P(D \mid E) = \frac{48}{140} \cdot \frac{5}{48} = \frac{5}{140} \approx 0.036$$

The results using the rules are consistent with those using the sample space.

(e) Compute $P(D \text{ or } E)$.

SOLUTION: From part (d), we know that the events "Democrat" and "executive" are not mutually exclusive, because $P(D \text{ and } E) \neq 0$. Therefore,

$$P(D \text{ or } E) = P(D) + P(E) - P(D \text{ and } E)$$

$$= \frac{68}{140} + \frac{48}{140} - \frac{5}{140} = \frac{111}{140} \approx 0.793$$

GUIDED EXERCISE 9 | **Survey**

Using Table 4-2 on page 152, let's consider other probabilities regarding the types of employees at Hopewell and their political affiliations. This time let's consider the production worker and the affiliation of Independent. Suppose an employee is selected at random from the group of 140.

(a) Compute $P(I)$ and $P(PW)$.

$$P(I) = \frac{\text{No. of Independents}}{\text{Total no. of employees}}$$

$$= \frac{17}{140} \approx 0.121$$

$$P(PW) = \frac{\text{No. of production workers}}{\text{Total no. of employees}}$$

$$= \frac{92}{140} \approx 0.657$$

(b) Compute $P(I \mid PW)$. This is a conditional probability. Be sure to restrict your attention to production workers, since that is the condition given.

$$P(I \mid PW) = \frac{\text{No. of Independent production workers}}{\text{No. of production workers}}$$

$$= \frac{8}{92} \approx 0.087$$

(c) Compute $P(I \text{ and } PW)$. In this case, look at the entire sample space and the number of employees who are both Independent and in production.

$$P(I \text{ and } PW) = \frac{\text{No. of Independent production workers}}{\text{Total no. employees}}$$

$$= \frac{8}{140} \approx 0.057$$

(d) Use the multiplication rule for dependent events to calculate $P(I \text{ and } PW)$. Is the result the same as that of part (c)?

By the multiplication rule,

$$P(I \text{ and } PW) = P(PW) \cdot P(I \mid PW)$$

$$= \frac{92}{140} \cdot \frac{8}{92} = \frac{8}{140} \approx 0.057$$

The results are the same.

(e) Compute $P(I \text{ or } PW)$. Are the events mutually exclusive?

Since the events are not mutually exclusive,

$$P(I \text{ or } PW) = P(I) + P(PW) - P(I \text{ and } PW)$$

$$= \frac{17}{140} + \frac{92}{140} - \frac{8}{140}$$

$$= \frac{101}{140} \approx 0.721$$

As you apply probability to various settings, keep the following rules in mind.

Basic probability rules

> **Summary of basic probability rules**
>
> A statistical experiment or statistical observation is any random activity that results in a recordable outcome. The sample space is the set of all simple events that are the outcomes of the statistical experiment and cannot be broken into other "simpler" events. A general event is any subset of the sample space. The notation $P(A)$ designates the probability of event A.
>
> 1. $P(\text{entire sample space}) = 1$
> 2. For any event A: $0 \leq P(A) \leq 1$
> 3. A^c designates the **complement** of A: $P(A^c) = 1 - P(A)$
> 4. Events A and B are **independent events** if $P(A) = P(A \mid B)$.
> 5. Multiplication Rules
>   **General:** $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$
> $\qquad\qquad\ P(A \text{ and } B) = P(B) \cdot P(A \mid B)$
>   **Independent events:** $P(A \text{ and } B) = P(A) \cdot P(B)$
> 6. Conditional Probability: $P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(B)}$
> 7. Events A and B are **mutually exclusive** if $P(A \text{ and } B) = 0$.
> 8. Addition Rules
>   **General:** $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
>   **Mutually exclusive events:** $P(A \text{ or } B) = P(A) + P(B)$

CRITICAL THINKING

Translating events described by common English phrases into events described using *and, or, complement,* or *given* takes a bit of care. Table 4-3 shows some common phrases and their corresponding translations into symbols.

TABLE 4-3   English Phrases and Corresponding Symbolic Translations

Consider the following events for a person selected at random from the general population:
$\quad A$ = person is taking college classes
$\quad B$ = person is under 30 years old

Phrase	Symbolic Expression
1. The probability that a person is under 30 years old and is taking college classes is 40%.	$P(B \text{ and } A) = 0.40$ or $P(A \text{ and } B) = 0.40$
2. The probability that a person under 30 years old is taking college classes is 45%.	$P(A \mid B) = 0.45$
3. The probability is 45% that a person is taking college classes if the person is under 30.	$P(A \mid B) = 0.45$
4. The probability that a person taking college classes is under 30 is 0.60.	$P(B \mid A) = 0.60$
5. The probability that a person is not taking college classes or is under 30 years old is 0.75.	$P(A^c \text{ or } B) = 0.75$

In this section, we have studied some important rules that are valid in all probability spaces. The rules and definitions of probability are not only interesting, but also have extensive *applications* in our everyday lives. If you are inclined to continue your study of probability a little further, we recommend reading about *Bayes's theorem* in Appendix I. The Reverend Thomas Bayes (1702–1761) was an English mathematician who discovered an important relation for conditional probabilities.

VIEWPOINT | The Psychology of Odors

The Smell and Taste Treatment Research Foundation of Chicago collected data on the time required to complete a maze while subjects were smelling different scents. Data for this survey can be found by visiting the Brase/Brase statistics site at **http://www.cengage.com/statistics/brase** *and finding the link to DASL, the Carnegie Mellon University Data and Story Library. Once at the DASL site, select Data Subjects, then Psychology, and then Scents. You can estimate conditional probabilities regarding response times for smokers, nonsmokers, and types of scents.*

**SECTION 4.2
PROBLEMS**

1. *Statistical Literacy* If two events are mutually exclusive, can they occur concurrently? Explain.

2. *Statistical Literacy* If two events A and B are independent and you know that $P(A) = 0.3$, what is the value of $P(A \mid B)$?

3. *Basic Computation: Addition Rule* Given $P(A) = 0.3$ and $P(B) = 0.4$:
   (a) If A and B are mutually exclusive events, compute $P(A \text{ or } B)$.
   (b) If $P(A \text{ and } B) = 0.1$, compute $P(A \text{ or } B)$.

4. *Basic Computation: Addition Rule* Given $P(A) = 0.7$ and $P(B) = 0.4$:
   (a) Can events A and B be mutually exclusive? Explain.
   (b) If $P(A \text{ and } B) = 0.2$, compute $P(A \text{ or } B)$.

5. *Basic Computation: Multiplication Rule* Given $P(A) = 0.2$ and $P(B) = 0.4$:
   (a) If A and B are independent events, compute $P(A \text{ and } B)$.
   (b) If $P(A \mid B) = 0.1$, compute $P(A \text{ and } B)$.

6. *Basic Computation: Multiplication Rule* Given $P(A) = 0.7$ and $P(B) = 0.8$:
   (a) If A and B, are independent events, compute $P(A \text{ and } B)$.
   (b) If $P(B \mid A) = 0.3$, compute $P(A \text{ and } B)$.

7. *Basic Computations: Rules of Probability* Given $P(A) = 0.2$; $P(B) = 0.5$, $P(A \mid B) = 0.3$:
   (a) Compute $P(A \text{ and } B)$.
   (b) Compute $P(A \text{ or } B)$.

8. *Basic Computation: Rules of Probability* Given $P(A^c) = 0.8$, $P(B) = 0.3$, $P(B \mid A) = 0.2$:
   (a) Compute $P(A \text{ and } B)$.
   (b) Compute $P(A \text{ or } B)$.

9. *Critical Thinking* Suppose two events A and B are mutually exclusive, with $P(A) \neq 0$ and $P(B) \neq 0$. By working through the following steps, you'll see why two mutually exclusive events are not independent.
   (a) For mutually exclusive events, can event A occur if event B has occurred? What is the value of $P(A \mid B)$?
   (b) Using the information from part (a), can you conclude that events A and B are *not* independent if they are mutually exclusive? Explain.

10. *Critical Thinking* Suppose two events A and B are independent, with $P(A) \neq 0$ and $P(B) \neq 0$. By working through the following steps, you'll see why two independent events are not mutually exclusive.
(a) What formula is used to compute $P(A \text{ and } B)$? Is $P(A \text{ and } B) \neq 0$? Explain.
(b) Using the information from part (a), can you conclude that events A and B are *not* mutually exclusive?

11. *Critical Thinking* Consider the following events for a driver selected at random from the general population:

$A =$ driver is under 25 years old
$B =$ driver has received a speeding ticket

Translate each of the following phrases into symbols.
(a) The probability the driver has received a speeding ticket and is under 25 years old
(b) The probability a driver who is under 25 years old has received a speeding ticket
(c) The probability a driver who has received a speeding ticket is 25 years old or older
(d) The probability the driver is under 25 years old or has received a speeding ticket
(e) The probability the driver has not received a speeding ticket or is under 25 years old

12. *Critical Thinking* Consider the following events for a college student selected at random:

$A =$ student is female
$B =$ student is majoring in business

Translate each of the following phrases into symbols.
(a) The probability the student is male or is majoring in business
(b) The probability a female student is majoring in business
(c) The probability a business major is female
(d) The probability the student is female and is not majoring in business
(e) The probability the student is female and is majoring in business

13. *General: Candy Colors* M&M plain candies come in various colors. According to the M&M/Mars Department of Consumer Affairs (link to the Mars company web site from the Brase/Brase statistics site at **http://www.cengage.com/statistics /brase**), the distribution of colors for plain M&M candies is

Color	Purple	Yellow	Red	Orange	Green	Blue	Brown
Percentage	20%	20%	20%	10%	10%	10%	10%

Suppose you have a large bag of plain M&M candies and you choose one candy at random. Find
(a) $P(\text{green candy } or \text{ blue candy})$. Are these outcomes mutually exclusive? Why?
(b) $P(\text{yellow candy } or \text{ red candy})$. Are these outcomes mutually exclusive? Why?
(c) $P(not \text{ purple candy})$

14. *Environmental: Land Formations* Arches National Park is located in southern Utah. The park is famous for its beautiful desert landscape and its many natural sandstone arches. Park Ranger Edward McCarrick started an inventory (not yet complete) of natural arches within the park that have an opening of at least 3 feet. The following table is based on information taken from the book *Canyon Country Arches and Bridges* by F. A. Barnes. The height of the arch opening is rounded to the nearest foot.

Height of arch, feet	3–9	10–29	30–49	50–74	75 and higher
Number of arches in park	111	96	30	33	18

For an arch chosen at random in Arches National Park, use the preceding information to estimate the probability that the height of the arch opening is
(a) 3 to 9 feet tall
(b) 30 feet or taller
(c) 3 to 49 feet tall
(d) 10 to 74 feet tall
(e) 75 feet or taller

15. *General: Roll Two Dice* You roll two fair dice, a green one and a red one.
(a) Are the outcomes on the dice independent?
(b) Find P(5 on green die *and* 3 on red die).
(c) Find P(3 on green die *and* 5 on red die).
(d) Find P((5 on green die *and* 3 on red die) *or* (3 on green die *and* 5 on red die)).

16. *General: Roll Two Dice* You roll two fair dice, a green one and a red one.
(a) Are the outcomes on the dice independent?
(b) Find P(1 on green die *and* 2 on red die).
(c) Find P(2 on green die *and* 1 on red die).
(d) Find P((1 on green die *and* 2 on red die) *or* (2 on green die *and* 1 on red die)).

17. *General: Roll Two Dice* You roll two fair dice, a green one and a red one.
(a) What is the probability of getting a sum of 6?
(b) What is the probability of getting a sum of 4?
(c) What is the probability of getting a sum of 6 *or* 4? Are these outcomes mutually exclusive?

18. *General: Roll Two Dice* You roll two fair dice, a green one and a red one.
(a) What is the probability of getting a sum of 7?
(b) What is the probability of getting a sum of 11?
(c) What is the probability of getting a sum of 7 *or* 11? Are these outcomes mutually exclusive?

Problems 19–22 involve a standard deck of 52 playing cards. In such a deck of cards there are four suits of 13 cards each. The four suits are: hearts, diamonds, clubs, and spades. The 26 cards included in hearts and diamonds are red in color. The 26 cards included in clubs and spades are black in color. The 13 cards in each suit are: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, and Ace. This means there are four Aces, four Kings, four Queens, four 10's, etc., down to four 2's in each deck.

19. *General: Deck of Cards* You draw two cards from a standard deck of 52 cards without replacing the first one before drawing the second.
(a) Are the outcomes on the two cards independent? Why?
(b) Find P(Ace on 1st card *and* King on 2nd).
(c) Find P(King on 1st card *and* Ace on 2nd).
(d) Find the probability of drawing an Ace *and* a King in either order.

20. *General: Deck of Cards* You draw two cards from a standard deck of 52 cards without replacing the first one before drawing the second.
(a) Are the outcomes on the two cards independent? Why?
(b) Find P(3 on 1st card *and* 10 on 2nd).
(c) Find P(10 on 1st card *and* 3 on 2nd).
(d) Find the probability of drawing a 10 *and* a 3 in either order.

21. *General: Deck of Cards* You draw two cards from a standard deck of 52 cards, but before you draw the second card, you put the first one back and reshuffle the deck.
(a) Are the outcomes on the two cards independent? Why?
(b) Find P(Ace on 1st card *and* King on 2nd).
(c) Find P(King on 1st card *and* Ace on 2nd).
(d) Find the probability of drawing an Ace *and* a King in either order.

22. *General: Deck of Cards* You draw two cards from a standard deck of 52 cards, but before you draw the second card, you put the first one back and reshuffle the deck.
(a) Are the outcomes on the two cards independent? Why?
(b) Find $P(3$ on 1st card *and* 10 on 2nd).
(c) Find $P(10$ on 1st card *and* 3 on 2nd).
(d) Find the probability of drawing a 10 *and* a 3 in either order.

23. *Marketing: Toys USA Today* gave the information shown in the table about ages of children receiving toys. The percentages represent all toys sold.

What is the probability that a toy is purchased for someone
(a) 6 years old or older?
(b) 12 years old or younger?
(c) between 6 and 12 years old?
(d) between 3 and 9 years old?

Age (years)	Percentage of Toys
2 and under	15%
3–5	22%
6–9	27%
10–12	14%
13 and over	22%

Interpretation A child between 10 and 12 years old looks at this probability distribution and asks, "Why are people more likely to buy toys for kids older than I am [13 and over] than for kids in my age group [10–12]?" How would you respond?

24. *Health Care: Flu* Based on data from the *Statistical Abstract of the United States*, 112th Edition, only about 14% of senior citizens (65 years old or older) get the flu each year. However, about 24% of the people under 65 years old get the flu each year. In the general population, there are 12.5% senior citizens (65 years old or older).
(a) What is the probability that a person selected at random from the general population is a senior citizen who will get the flu this year?
(b) What is the probability that a person selected at random from the general population is a person under age 65 who will get the flu this year?
(c) Answer parts (a) and (b) for a community that is 95% senior citizens.
(d) Answer parts (a) and (b) for a community that is 50% senior citizens.

25. *Focus Problem: Lie Detector Test* In this problem, you are asked to solve part of the Focus Problem at the beginning of this chapter. In his book *Chances: Risk and Odds in Everyday Life*, James Burke says that there is a 72% chance a polygraph test (lie detector test) will catch a person who is, in fact, lying. Furthermore, there is approximately a 7% chance that the polygraph will falsely accuse someone of lying.
(a) Suppose a person answers 90% of a long battery of questions truthfully. What percentage of the answers will the polygraph *wrongly* indicate are lies?
(b) Suppose a person answers 10% of a long battery of questions with lies. What percentage of the answers will the polygraph *correctly* indicate are lies?
(c) Repeat parts (a) and (b) if 50% of the questions are answered truthfully and 50% are answered with lies.
(d) Repeat parts (a) and (b) if 15% of the questions are answered truthfully and the rest are answered with lies.

26. *Focus Problem: Expand Your Knowledge* This problem continues the Focus Problem. The solution involves applying several basic probability rules and a little algebra to solve an equation.
(a) If the polygraph of Problem 25 indicated that 30% of the questions were answered with lies, what would you estimate for the actual percentage of lies in the answers? *Hint:* Let $B = $ event detector indicates a lie. We are given $P(B) = 0.30$. Let $A = $ event person is lying, so $A^c = $ event person is not lying. Then

$$P(B) = P(A \text{ and } B) + P(A^c \text{ and } B)$$
$$P(B) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c)$$

Replacing $P(A^c)$ by $1 - P(A)$ gives

$$P(B) = P(A)\cdot P(B \mid A) + [1 - P(A)]\cdot P(B \mid A^c)$$

Substitute known values for $P(B)$, $P(B \mid A)$, and $P(B \mid A^c)$ into the last equation and solve for $P(A)$.

(b) If the polygraph indicated that 70% of the questions were answered with lies, what would you estimate for the actual percentage of lies?

27. *Survey: Sales Approach* In a sales effectiveness seminar, a group of sales representatives tried two approaches to selling a customer a new automobile: the aggressive approach and the passive approach. For 1160 customers, the following record was kept:

	Sale	No Sale	Row Total
Aggressive	270	310	580
Passive	416	164	580
Column Total	686	474	1160

Suppose a customer is selected at random from the 1160 participating customers. Let us use the following notation for events: A = aggressive approach, Pa = passive approach, S = sale, N = no sale. So, $P(A)$ is the probability that an aggressive approach was used, and so on.

(a) Compute $P(S)$, $P(S \mid A)$, and $P(S \mid Pa)$.
(b) Are the events S = sale and Pa = passive approach independent? Explain.
(c) Compute $P(A \text{ and } S)$ and $P(Pa \text{ and } S)$.
(d) Compute $P(N)$ and $P(N \mid A)$.
(e) Are the events N = no sale and A = aggressive approach independent? Explain.
(f) Compute $P(A \text{ or } S)$.

28. *Survey: Medical Tests* Diagnostic tests of medical conditions can have several types of results. The test result can be positive or negative, whether or not a patient has the condition. A positive test $(+)$ indicates that the patient has the condition. A negative test $(-)$ indicates that the patient does not have the condition. Remember, a positive test does not prove that the patient has the condition. Additional medical work may be required. Consider a random sample of 200 patients, some of whom have a medical condition and some of whom do not. Results of a new diagnostic test for the condition are shown.

	Condition Present	Condition Absent	Row Total
Test Result +	110	20	130
Test Result −	20	50	70
Column Total	130	70	200

Assume the sample is representative of the entire population. For a person selected at random, compute the following probabilities:

(a) $P(+ \mid \text{condition present})$; this is known as the *sensitivity* of a test.
(b) $P(- \mid \text{condition present})$; this is known as the *false-negative rate*.
(c) $P(- \mid \text{condition absent})$; this is known as the *specificity* of a test.
(d) $P(+ \mid \text{condition absent})$; this is known as the *false-positive rate*.
(e) $P(\text{condition present and } +)$; this is the *predictive value* of the test.
(f) $P(\text{condition present and } -)$.

29. *Survey: Lung/Heart* In an article entitled "Diagnostic accuracy of fever as a measure of postoperative pulmonary complications" (*Heart Lung*, Vol. 10, No. 1, p. 61), J. Roberts and colleagues discuss using a fever of 38°C or higher as a diagnostic

indicator of postoperative atelectasis (collapse of the lung) as evidenced by x-ray observation. For fever ≥38°C as the diagnostic test, the results for postoperative patients are

	Condition Present	Condition Absent	Row Total
Test Result +	72	37	109
Test Result −	82	79	161
Column Total	154	116	270

For the meaning of + and −, see Problem 28. Complete parts (a) through (f) from Problem 28.

30. *Survey: Customer Loyalty* Are customers more loyal in the east or in the west? The following table is based on information from *Trends in the United States*, published by the Food Marketing Institute, Washington, D.C. The columns represent length of customer loyalty (in years) at a primary supermarket. The rows represent regions of the United States.

	Less Than 1 Year	1–2 Years	3–4 Years	5–9 Years	10–14 Years	15 or More Years	Row Total
East	32	54	59	112	77	118	452
Midwest	31	68	68	120	63	173	523
South	53	92	93	158	106	158	660
West	41	56	67	78	45	86	373
Column Total	157	270	287	468	291	535	2008

What is the probability that a customer chosen at random
(a) has been loyal 10 to 14 years?
(b) has been loyal 10 to 14 years, given that he or she is from the east?
(c) has been loyal *at least* 10 years?
(d) has been loyal *at least* 10 years, given that he or she is from the west?
(e) is from the west, given that he or she has been loyal less than 1 year?
(f) is from the south, given that he or she has been loyal less than 1 year?
(g) has been loyal *1 or more years*, given that he or she is from the east?
(h) has been loyal *1 or more years*, given that he or she is from the west?
(i) Are the events "from the east" and "loyal 15 or more years" independent? Explain.

31. *Franchise Stores: Profits* Wing Foot is a shoe franchise commonly found in shopping centers across the United States. Wing Foot knows that its stores will not show a profit unless they gross over $940,000 per year. Let A be the event that a new Wing Foot store grosses over $940,000 its first year. Let B be the event that a store grosses over $940,000 its second year. Wing Foot has an administrative policy of closing a new store if it does not show a profit in *either* of the first 2 years. The accounting office at Wing Foot provided the following information: 65% of *all* Wing Foot stores show a profit the first year; 71% of *all* Wing Foot stores show a profit the second year (this includes stores that did not show a profit the first year); however, 87% of Wing Foot stores that showed a profit the first year also showed a profit the second year. Compute the following:
(a) $P(A)$
(b) $P(B)$
(c) $P(B \mid A)$
(d) $P(A \text{ and } B)$
(e) $P(A \text{ or } B)$

(f) What is the probability that a new Wing Foot store will not be closed after 2 years? What is the probability that a new Wing Foot store will be closed after 2 years?

32. | *Education: College of Nursing* At Litchfield College of Nursing, 85% of incoming freshmen nursing students are female and 15% are male. Recent records indicate that 70% of the entering female students will graduate with a BSN degree, while 90% of the male students will obtain a BSN degree. If an incoming freshman nursing student is selected at random, find
(a) P(student will graduate | student is female).
(b) P(student will graduate *and* student is female).
(c) P(student will graduate | student is male).
(d) P(student will graduate *and* student is male).
(e) P(student will graduate). Note that those who will graduate are either males who will graduate or females who will graduate.
(f) The events described by the phrases "will graduate *and* is female" and "will graduate, *given* female" seem to be describing the same students. Why are the probabilities P(will graduate *and* is female) and P(will graduate | female) different?

33. | *Medical: Tuberculosis* The state medical school has discovered a new test for tuberculosis. (If the test indicates a person has tuberculosis, the test is positive.) Experimentation has shown that the probability of a positive test is 0.82, given that a person has tuberculosis. The probability is 0.09 that the test registers positive, given that the person does not have tuberculosis. Assume that in the general population, the probability that a person has tuberculosis is 0.04. What is the probability that a person chosen at random will
(a) have tuberculosis and have a positive test?
(b) not have tuberculosis?
(c) not have tuberculosis and have a positive test?

34. | *Therapy: Alcohol Recovery* The Eastmore Program is a special program to help alcoholics. In the Eastmore Program, an alcoholic lives at home but undergoes a two-phase treatment plan. Phase I is an intensive group-therapy program lasting 10 weeks. Phase II is a long-term counseling program lasting 1 year. Eastmore Programs are located in most major cities, and past data gave the following information based on percentages of success and failure collected over a long period of time: The probability that a client will have a relapse in phase I is 0.27; the probability that a client will have a relapse in phase II is 0.23. However, if a client did not have a relapse in phase I, then the probability that this client will not have a relapse in phase II is 0.95. If a client did have a relapse in phase I, then the probability that this client will have a relapse in phase II is 0.70. Let A be the event that a client has a relapse in phase I and B be the event that a client has a relapse in phase II. Let C be the event that a client has no relapse in phase I and D be the event that a client has no relapse in phase II. Compute the following:
(a) $P(A)$, $P(B)$, $P(C)$, and $P(D)$
(b) $P(B \mid A)$ and $P(D \mid C)$
(c) $P(A \text{ and } B)$ and $P(C \text{ and } D)$
(d) $P(A \text{ or } B)$
(e) What is the probability that a client will go through both phase I and phase II without a relapse?
(f) What is the probability that a client will have a relapse in both phase I and phase II?
(g) What is the probability that a client will have a relapse in either phase I or phase II?

Brain Teasers Assume A and B are events such that $0 < P(A) < 1$ and $0 < P(B) < 1$. Answer questions 35–49 true or false and give a brief explanation for each answer. *Hint:* Review the summary of basic probability rules.

35. $P(A \text{ and } A^c) = 0$

36. $P(A \text{ or } A^c) = 0$

37. $P(A \mid A^c) = 1$

38. $P(A \text{ or } B) = P(A) + P(B)$

39. $P(A \mid B) \geq P(A \text{ and } B)$

40. $P(A \text{ or } B) \geq P(A)$ if A and B are independent events

41. $P(A \text{ and } B) \leq P(A)$

42. $P(A \mid B) > P(A)$ if A and B are independent events

43. $P(A^c \text{ and } B^c) \leq 1 - P(A)$

44. $P(A^c \text{ or } B^c) \leq 2 - P(A) - P(B)$

45. If A and B are independent events, they must also be mutually exclusive events.

46. If A and B are mutually exclusive, they must also be independent.

47. If A and B are both mutually exclusive and independent, then at least one of $P(A)$ or $P(B)$ must be zero.

48. If A and B are mutually exclusive, then $P(A \mid B) = 0$.

49. $P(A \mid B) + P(A^c \mid B) = 1$

50. *Brain Teaser* The Reverend Thomas Bayes (1702–1761) was an English mathematician who discovered an important rule of probability (*see* Bayes's theorem, Appendix I part I). A key feature of Bayes's theorem is the formula

$$P(B) = P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c)$$

Explain why this formula is valid. *Hint:* See Figure Al-1 in Appendix I.

SECTION 4.3

Trees and Counting Techniques

FOCUS POINTS
- Organize outcomes in a sample space using tree diagrams.
- Compute number of ordered arrangements of outcomes using permutations.
- Compute number of (nonordered) groupings of outcomes using combinations.
- Explain how counting techniques relate to probability in everyday life.

When outcomes are equally likely, we compute the probability of an event by using the formula

$$P(A) = \frac{\text{Number of outcomes favorable to the event } A}{\text{Number of outcomes in the sample space}}$$

The probability formula requires that we be able to determine the number of outcomes in the sample space. In the problems we have done in previous sections, this task has not been difficult because the number of outcomes was small or the sample space consisted of fairly straightforward events. The tools we present in this section will help you count the number of possible outcomes in larger sample spaces or those formed by more complicated events.

When an outcome of an experiment is composed of a series of events, the multiplication rule gives us the *total number* of outcomes.

Multiplication rule of counting

> **Multiplication rule of counting**
>
> Consider the series of events E_1 through E_m, where n_1 is the number of possible outcomes for event E_1, n_2 is the number of possible outcomes for event E_2, and n_m designates the number of possible outcomes for event E_m. Then the product
>
> $$n_1 \times n_2 \times \cdots \times n_m$$
>
> gives the total number of possible outcomes for the series of events E_1, followed by E_2, up through event E_m.

EXAMPLE 9   MULTIPLICATION RULE

Jacqueline is in a nursing program and is required to take a course in psychology and one in physiology (A and P) next semester. She also wants to take Spanish II. If there are two sections of psychology, two of A and P, and three of Spanish II, how many different class schedules can Jacqueline choose from? (Assume that the times of the sections do not conflict.)

SOLUTION: Creating a class schedule can be considered an experiment with a series of three events. There are two possible outcomes for the psychology section, two for the A and P section, and three for the Spanish II section. By the multiplication rule, the total number of class schedules possible is

$$2 \times 2 \times 3 = 12$$

Tree diagram

A *tree diagram* gives a visual display of the total number of outcomes of an experiment consisting of a series of events. From a tree diagram, we can determine not only the total number of outcomes, but also the individual outcomes.

EXAMPLE 10   TREE DIAGRAM

Using the information from Example 9, let's make a tree diagram that shows all the possible course schedules for Jacqueline.

SOLUTION: Figure 4-6 shows the tree diagram. Let's study the diagram. There are two branches from Start. These branches indicate the two possible choices for psychology sections. No matter which section of psychology Jacqueline chooses, she can choose from the two available A and P sections. Therefore, we have two

FIGURE 4-6

Tree Diagram for Selecting Class Schedules
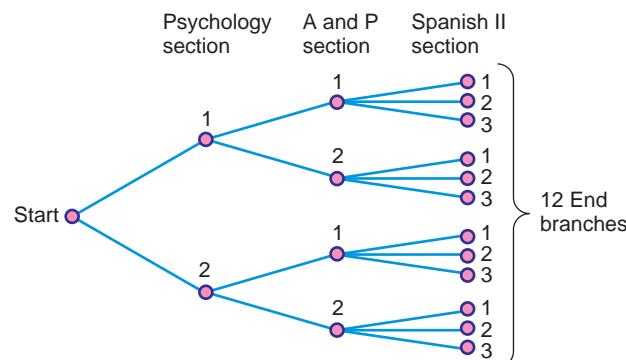
branches leading from *each* psychology branch. Finally, after the psychology and A and P sections are selected, there are three choices for Spanish II. That is why there are three branches from *each* A and P section.

The tree ends with a total of 12 branches. The number of end branches tells us the number of possible schedules. The outcomes themselves can be listed from the tree by following each series of branches from Start to End. For instance, the top branch from Start generates the schedules shown in Table 4-4. The other six schedules can be listed in a similar manner, except they begin with the second section of psychology.

TABLE 4-4   Schedules Utilizing Section 1 of Psychology

Psychology Section	A and P Section	Spanish II Section
1	1	1
1	1	2
1	1	3
1	2	1
1	2	2
1	2	3

GUIDED EXERCISE 10   Tree diagram and multiplication rule

Louis plays three tennis matches. Use a tree diagram to list the possible win and loss sequences Louis can experience for the set of three matches.

(a) On the first match Louis can win or lose. From Start, indicate these two branches.

FIGURE 4-7 W = Win, L = Lose
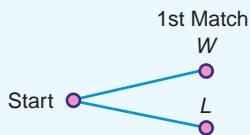
(b) Regardless of whether Louis wins or loses the first match, he plays the second and can again win or lose. Attach branches representing these two outcomes to *each* of the first match results.

FIGURE 4-8
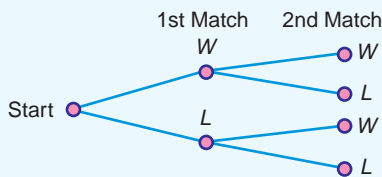
(c) Louis may win or lose the third match. Attach branches representing these two outcomes to *each* of the second match results.

FIGURE 4-9
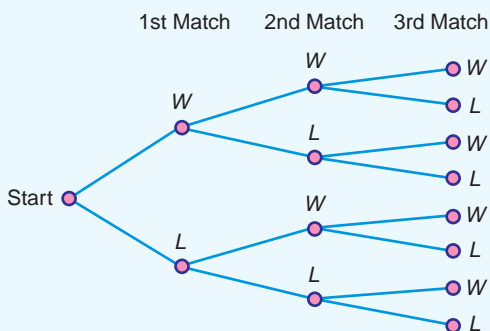
Continued

GUIDED EXERCISE 10   *continued*

(d) How many possible win–lose sequences are there for the three matches?   ➡️   Since there are eight branches at the end, there are eight sequences.

(e) Complete this list of win–lose sequences.   ➡️   The last four sequences all involve a loss on Match 1.

1st	2nd	3rd
W	W	W
W	W	L
W	L	W
W	L	L
___	___	___
___	___	___
___	___	___
___	___	___

1st	2nd	3rd
L	W	W
L	W	L
L	L	W
L	L	L

(f) Use the multiplication rule to compute the total number of outcomes for the three matches.   ➡️   The number of outcomes for a series of three events, each with two outcomes, is

$$2 \times 2 \times 2 = 8$$

Tree diagrams help us display the outcomes of an experiment involving several stages. If we label each branch of the tree with an appropriate probability, we can use the tree diagram to help us compute the probability of an outcome displayed on the tree. One of the easiest ways to illustrate this feature of tree diagrams is to use an experiment of drawing balls out of an urn. We do this in the next example.

EXAMPLE 11   TREE DIAGRAM AND PROBABILITY

Suppose there are five balls in an urn. They are identical except for color. Three of the balls are red and two are blue. You are instructed to draw out one ball, note its color, and set it aside. Then you are to draw out another ball and note its color. What are the outcomes of the experiment? What is the probability of each outcome?

SOLUTION: The tree diagram in Figure 4-10 will help us answer these questions. Notice that since you did not replace the first ball before drawing the second one, the two stages of the experiment are dependent. The probability associated with

FIGURE 4-10

Tree Diagram for Urn Experiment
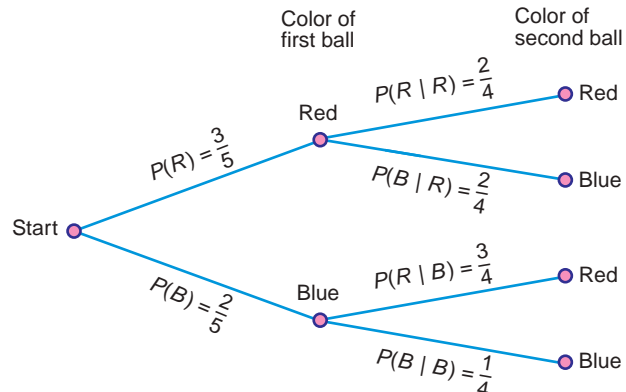
the color of the second ball depends on the color of the first ball. For instance, on the top branches, the color of the first ball drawn is red, so we compute the probabilities of the colors on the second ball accordingly. The tree diagram helps us organize the probabilities.

From the diagram, we see that there are four possible outcomes to the experiment. They are

RR = red on 1st *and* red on 2nd

RB = red on 1st *and* blue on 2nd

BR = blue on 1st *and* red on 2nd

BB = blue on 1st *and* blue on 2nd

To compute the probability of each outcome, we will use the multiplication rule for dependent events. As we follow the branches for each outcome, we will find the necessary probabilities.

$$P(R \text{ on 1st } and \ R \text{ on 2nd}) = P(R) \cdot P(R \mid R) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$$

$$P(R \text{ on 1st } and \ B \text{ and 2nd}) = P(R) \cdot P(B \mid R) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$$

$$P(B \text{ on 1st } and \ R \text{ on 2nd}) = P(B) \cdot P(R \mid B) = \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$$

$$P(B \text{ on 1st } and \ B \text{ and 2nd}) = P(B) \cdot P(B \mid B) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$$

Notice that the probabilities of the outcomes in the sample space add to 1, as they should.

Sometimes when we consider n items, we need to know the number of different *ordered arrangements* of the n items that are possible. The multiplication rules can help us find the number of possible ordered arrangements. Let's consider the classic example of determining the number of different ways in which eight people can be seated at a dinner table. For the first chair at the head of the table, there are eight choices. For the second chair, there are seven choices, since one person is already seated. For the third chair, there are six choices, since two people are already seated. By the time we get to the last chair, there is only one person left for that seat. We can view each arrangement as an outcome of a series of eight events. Event 1 is *fill the first chair*, event 2 is *fill the second chair*, and so forth. The multiplication rule will tell us the number of different outcomes.

Choices for	1st	2nd	3rd	4th	5th	6th	7th	8th	Chair position
	↓	↓	↓	↓	↓	↓	↓	↓	
	(8)	(7)	(6)	(5)	(4)	(3)	(2)	(1)	= 40,320

In all, there are 40,320 different seating arrangements for eight people. It is no wonder that it takes a little time to seat guests at a dinner table!

The multiplication pattern shown above is not unusual. In fact, it is an example of the multiplication indicated by the *factorial notation* 8!.

Factorial notation

! is read "factorial"

8! is read "8 factorial"

$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

In general, $n!$ indicates the product of n with each of the positive counting numbers less than n. *By special definition, $0! = 1$.*

Radius Images/Jupiter Images

Factorial notation

For a counting number n,

$0! = 1$

$1! = 1$

$n! = n(n - 1)(n - 2) \cdots 1$

GUIDED EXERCISE 11 | **Factorial**

(a) Evaluate 3!.   ⟹ $3! = 3 \cdot 2 \cdot 1 = 6$

(b) In how many different ways can three objects be   ⟹   You have three choices for the first position, two for
arranged in order? How many choices do you   the second position, and one for the third position.
have for the first position? for the second   By the multiplication rule, you have
position? for the third position?

$(3)(2)(1) = 3! = 6$ arrangements

We have considered the number of ordered arrangements of n objects taken as
an entire group. But what if we don't arrange the entire group? Specifically, we
considered a dinner party for eight and found the number of ordered seating
arrangements for all eight people. However, suppose you have an open house and
have only five chairs. How many ways can five of the eight people seat themselves
in the chairs? The formula we use to compute this number is called the *permuta-*
Permutations rule   *tion formula*. As we see in the next example, the *permutations rule* is really
another version of the multiplication rule.

Counting rule for permutations

The number of ways to *arrange in order n* distinct objects, taking them r at
a time, is

$$P_{n,r} = \frac{n!}{(n - r)!} \tag{9}$$

where n and r are whole numbers and $n \geq r$. Another commonly used nota-
tion for permutations is nPr.

EXAMPLE 12   PERMUTATIONS RULE

Let's compute the number of possible ordered seating arrangements for eight peo-
ple in five chairs.

SOLUTION: In this case, we are considering a total of $n = 8$ different people, and
we wish to arrange $r = 5$ of these people. Substituting into formula (9), we have

$$P_{n,r} = \frac{n!}{(n - r)!}$$

$$P_{8,5} = \frac{8!}{(8 - 5)!} = \frac{8!}{3!} = \frac{40,320}{6} = 6720$$

Using the multiplication rule, we get the same results

Chair	1		2		3		4		5		
Choices for	8	×	7	×	6	×	5	×	4	=	6720

The permutations rule has the advantage of using factorials. Most scientific calculators have a factorial key (!) as well as a permutations key (nPr) (see Tech Notes).

In each of our previous counting formulas, we have taken the *order* of the objects or people into account. However, suppose that in your political science class you are given a list of 10 books. You are to select 4 to read during the semester. The order in which you read the books is not important. We are interested in the *different groupings* or *combinations* of 4 books from among the 10 on the list. The next formula tells us how to compute the number of different combinations.

> **Counting rule for combinations**
>
> The number of *combinations* of n objects taken r at a time is
>
> $$C_{n,r} = \frac{n!}{r!(n-r)!} \tag{10}$$
>
> where n and r are whole numbers and $n \geq r$. Other commonly used notations for combinations include nCr and $\binom{n}{r}$.

Notice the difference between the concepts of permutations and combinations. When we consider permutations, we are considering groupings *and order*. When we consider combinations, we are considering only the number of different groupings. For combinations, order within the groupings is not considered. As a result, the number of combinations of n objects taken r at a time is generally smaller than the number of permutations of the same n objects taken r at a time. In fact, the combinations formula is simply the permutations formula with the number of permutations of each distinct group divided out. In the formula for combinations, notice the factor of $r!$ in the denominator.

Combinations rule

Now let's look at an example in which we use the *combinations rule* to compute the number of *combinations* of 10 books taken 4 at a time.

EXAMPLE 13   COMBINATIONS

In your political science class, you are assigned to read any 4 books from a list of 10 books. How many different groups of 4 are available from the list of 10?

SOLUTION: In this case, we are interested in *combinations*, rather than permutations, of 10 books taken 4 at a time. Using $n = 10$ and $r = 4$, we have

$$C_{n,r} = \frac{n!}{r!(n-r)!} = \frac{10!}{4!(10-4)!} = 210$$

There are 210 different groups of 4 books that can be selected from the list of 10. An alternate solution method is to use the combinations key (often nCr or $C_{n,r}$) on a calculator.

LOOKING FORWARD

We will see the combinations rule again in Section 5.2 when we discuss the formula for the binomial probability distribution.

PROCEDURE

HOW TO DETERMINE THE NUMBER OF OUTCOMES OF AN EXPERIMENT

1. If the experiment consists of a series of stages with various outcomes, use the multiplication rule or a tree diagram.

2. If the outcomes consist of ordered subgroups of r items taken from a group of n items, use the permutations rule, $P_{n,r}$.

$$P_{n,r} = \frac{n!}{(n-r)!} \tag{9}$$

3. If the outcomes consist of nonordered subgroups of r items taken from a group of n items, use the combinations rule, $C_{n,r}$.

$$C_{n,r} = \frac{n!}{r!(n-r)!} \tag{10}$$

GUIDED EXERCISE 12   **Permutations and combinations**

The board of directors at Belford Community Hospital has 12 members.

(i) Three officers—president, vice president, and treasurer—must be elected from among the members. How many different slates of officers are possible? We will view a slate of officers as a list of three people, with the president listed first, the vice president listed second, and the treasurer listed third. For instance, if Mr. Acosta, Ms. Hill, and Mr. Smith wish to be on a slate together, there are several different slates possible, depending on the person listed for each office. Not only are we asking for the number of different groups of three names for a slate, we are also concerned about order.

(a) Do we use the permutations rule or the combinations rule? What is the value of n? What is the value of r?

⟹ We use the permutations rule, since order is important. The size of the group from which the slates of officers are to be selected is n. The size of each slate is r.

$n = 12$ and $r = 3$

(b) Use the permutations rule with $n = 12$ and $r = 3$ to compute $P_{12,3}$.

⟹ $P_{n,r} = \dfrac{n!}{(n-r)!} = \dfrac{12!}{(12-3!)} = 1320$

An alternative is to use the permutations key on a calculator.

Continued

(ii) Three members from the group of 12 on the board of directors at Belford Community Hospital will be selected to go to a convention (all expenses paid) in Hawaii. How many different groups of 3 are there?

(c) Do we use the permutations rule or the combinations rule? What is the value of n? What is the value of r?

⟹ We use the combinations rule, because order is not important. The size of the board is $n = 12$ and the size of each group going to the convention is $r = 3$.

(d) Use the combinations rule with $n = 12$ and $r = 3$ to compute $C_{12,3}$.

⟹ $C_{n,r} = \dfrac{n!}{r!(n-r)!} = \dfrac{12!}{3!(12-3)!} = 220$

An alternative is to use the combinations key on a calculator.

VIEWPOINT   Powerball

Powerball is a multistate lottery game that consists of drawing five distinct whole numbers from the numbers 1 through 59 in any order. Then one more number from the numbers 1 through 39 is selected as the Powerball number (this number could be one of the original five). Powerball numbers are drawn every Wednesday and Saturday. If you match all six numbers, you win the jackpot, which is worth at least $10 million. Use methods of this section to show that there are 195,249,054 possible Powerball plays. For more information about the game of Powerball and the probability of winning different prizes, visit the Brase/Brase statistics site at **http://www.cengage.com/statistics/brase** *and find the link to the Multi-State Lottery Association. Then select Powerball.*

SECTION 4.3 PROBLEMS

1. *Statistical Literacy* What is the main difference between a situation in which the use of the permutations rule is appropriate and one in which the use of the combinations rule is appropriate?

2. *Statistical Literacy* Consider a series of events. How does a tree diagram help you list all the possible outcomes of a series of events? How can you use a tree diagram to determine the total number of outcomes of a series of events?

3. *Critical Thinking* For each of the following situations, explain why the combinations rule or the permutations rule should be used.
   (a) Determine the number of different groups of 5 items that can be selected from 12 distinct items.
   (b) Determine the number of different arrangements of 5 items that can be selected from 12 distinct items.

4. *Critical Thinking* You need to know the number of different arrangements possible for five distinct letters. You decide to use the permutations rule, but your friend tells you to use 5!. Who is correct? Explain.

5. *Tree Diagram*
   (a) Draw a tree diagram to display all the possible head–tail sequences that can occur when you flip a coin three times.
   (b) How many sequences contain exactly two heads?
   (c) *Probability Extension* Assuming the sequences are all equally likely, what is the probability that you will get exactly two heads when you toss a coin three times?

6. | *Tree Diagram*
   (a) Draw a tree diagram to display all the possible outcomes that can occur when you flip a coin and then toss a die.
   (b) How many outcomes contain a head and a number greater than 4?
   (c) *Probability Extension* Assuming the outcomes displayed in the tree diagram are all equally likely, what is the probability that you will get a head *and* a number greater than 4 when you flip a coin and toss a die?

7. | *Tree Diagram* There are six balls in an urn. They are identical except for color. Two are red, three are blue, and one is yellow. You are to draw a ball from the urn, note its color, and set it aside. Then you are to draw another ball from the urn and note its color.
   (a) Make a tree diagram to show all possible outcomes of the experiment. Label the probability associated with each stage of the experiment on the appropriate branch.
   (b) *Probability Extension* Compute the probability for each outcome of the experiment.

8. | *Tree Diagram*
   (a) Make a tree diagram to show all the possible sequences of answers for three multiple-choice questions, each with four possible responses.
   (b) *Probability Extension* Assuming that you are guessing the answers so that all outcomes listed in the tree are equally likely, what is the probability that you will guess the one sequence that contains all three correct answers?

9. | *Multiplication Rule* Four wires (red, green, blue, and yellow) need to be attached to a circuit board. A robotic device will attach the wires. The wires can be attached in any order, and the production manager wishes to determine which order would be fastest for the robot to use. Use the multiplication rule of counting to determine the number of possible sequences of assembly that must be tested. *Hint:* There are four choices for the first wire, three for the second, two for the third, and only one for the fourth.

10. | *Multiplication Rule* A sales representative must visit four cities: Omaha, Dallas, Wichita, and Oklahoma City. There are direct air connections between each of the cities. Use the multiplication rule of counting to determine the number of different choices the sales representative has for the order in which to visit the cities. How is this problem similar to Problem 9?

11. | *Counting: Agriculture* Barbara is a research biologist for Green Carpet Lawns. She is studying the effects of fertilizer type, temperature at time of application, and water treatment after application. She has four fertilizer types, three temperature zones, and three water treatments to test. Determine the number of different lawn plots she needs in order to test each fertilizer type, temperature range, and water treatment configuration.

12. | *Counting: Outcomes* You toss a pair of dice.
   (a) Determine the number of possible pairs of outcomes. (Recall that there are six possible outcomes for each die.)
   (b) There are three even numbers on each die. How many outcomes are possible with even numbers appearing on each die?
   (c) *Probability extension:* What is the probability that both dice will show an even number?

13. | Compute $P_{5,2}$.   17. | Compute $C_{5,2}$.

14. | Compute $P_{8,3}$.   18. | Compute $C_{8,3}$.

15. | Compute $P_{7,7}$.   19. | Compute $C_{7,7}$.

16. | Compute $P_{9,9}$.   20. | Compute $C_{8,8}$.

21. | *Counting: Hiring* There are three nursing positions to be filled at Lilly Hospital. Position 1 is the day nursing supervisor; position 2 is the night nursing supervisor;

and position 3 is the nursing coordinator position. There are 15 candidates qualified for all three of the positions. Determine the number of different ways the positions can be filled by these applicants.

22. *Counting: Lottery* In the Cash Now lottery game there are 10 finalists who submitted entry tickets on time. From these 10 tickets, three grand prize winners will be drawn. The first prize is $1 million, the second prize is $100,000, and the third prize is $10,000. Determine the total number of different ways in which the winners can be drawn. (Assume that the tickets are not replaced after they are drawn.)

23. *Counting: Sports* The University of Montana ski team has five entrants in a men's downhill ski event. The coach would like the first, second, and third places to go to the team members. In how many ways can the five team entrants achieve first, second, and third places?

24. *Counting: Sales* During the Computer Daze special promotion, a customer purchasing a computer and printer is given a choice of 3 free software packages. There are 10 different software packages from which to select. How many different groups of software packages can be selected?

25. *Counting: Hiring* There are 15 qualified applicants for 5 trainee positions in a fast-food management program. How many different groups of trainees can be selected?

26. *Counting: Grading* One professor grades homework by randomly choosing 5 out of 12 homework problems to grade.
    (a) How many different groups of 5 problems can be chosen from the 12 problems?
    (b) *Probability Extension* Jerry did only 5 problems of one assignment. What is the probability that the problems he did comprised the group that was selected to be graded?
    (c) Silvia did 7 problems. How many different groups of 5 did she complete? What is the probability that one of the groups of 5 she completed comprised the group selected to be graded?

27. *Counting: Hiring* The qualified applicant pool for six management trainee positions consists of seven women and five men.
    (a) How many different groups of applicants can be selected for the positions?
    (b) How many different groups of trainees would consist entirely of women?
    (c) *Probability Extension* If the applicants are equally qualified and the trainee positions are selected by drawing the names at random so that all groups of six are equally likely, what is the probability that the trainee class will consist entirely of women?

Chapter Review

SUMMARY

In this chapter we explored basic features of probability.

- The probability of an event *A* is a number between 0 and 1, inclusive. The more likely the event, the closer the probability of the event is to 1.

- Three main ways to determine the probability of an event are: the method of relative frequency, the method of equally likely outcomes, and intuition. Other important ways will be discussed later.

- The law of large numbers indicates that as the number of trials of a statistical experiment or observation increases, the relative frequency of a designated event becomes closer to the theoretical probability of that event.

- Events are mutually exclusive if they cannot occur together. Events are independent if the occurrence of one event does not change the probability of the occurrence of the other.

- Conditional probability is the probability that one event will occur, given that another event has occurred.

- The complement rule gives the probability that an event will not occur. The addition rule gives the probability that at least one of two specified events will occur. The multiplication rule gives the probability that two events will occur together.

- To determine the probability of equally likely events, we need to know how many outcomes are possible. Devices such as tree diagrams and counting rules—such as the multiplication rule of counting, the permutations rule, and the combinations rule—help us determine the total number of outcomes of a statistical experiment or observation.

In most of the statistical applications of later chapters, we will use the addition rule for mutually exclusive events and the multiplication rule for independent events.

IMPORTANT WORDS & SYMBOLS

VIEWPOINT | Deathday and Birthday

Can people really postpone death? If so, how much can the timing of death be influenced by psychological, social, or other influential factors? One special event is a birthday. Do famous people try to postpone their deaths until an important birthday? Both Thomas Jefferson and John Adams died on July 4, 1826, when the United States was celebrating its 50th birthday. Is this only a strange coincidence, or is there an unexpected connection between birthdays and deathdays? The probability associated with a death rate decline of famous people just before important birthdays has been studied by Professor D. P. Phillips of the State University of New York and is presented in the book Statistics, A Guide to the Unknown, edited by J. M. Tanur.

CHAPTER REVIEW PROBLEMS

1. *Statistical Literacy* Consider the following two events for an individual:

   A = owns a cell phone B = owns a laptop computer

   Translate each event into words.
   (a) A^c (d) $A \mid B$
   (b) A *and* B (e) $B \mid A$
   (c) A *or* B

2. *Statistical Literacy* If two events A and B are mutually exclusive, what is the value of $P(A$ *and* $B)$?

3. *Statistical Literacy* If two events A and B are independent, how do the probabilities $P(A)$ and $P(A \mid B)$ compare?

4. *Interpretation* You are considering 2 facial cosmetic surgeries. These are elective surgeries and their outcomes are independent. The probability of success for each surgery is 0.90. What is the probability of success for both surgeries? If the probability of success for both surgeries is less than 0.85, you will decide not to have the surgeries. Will you have the surgeries or not?

5. *Interpretation* You are applying for two jobs, and you estimate the probability of getting an offer for the first job is 0.70 while the probability of getting an offer for the second job is 0.80. Assume the job offers are independent.
   (a) Compute the probability of getting offers for both jobs. How does this probability compare to the probability of getting each individual job offer?
   (b) Compute the probability of getting an offer for either the first or the second job. How does this probability compare to the probability of getting each individual job offer? Does it seem worthwhile to apply for both jobs? Explain.

6. *Critical Thinking* You are given the information that $P(A) = 0.30$ and $P(B) = 0.40$.
   (a) Do you have enough information to compute $P(A$ *or* $B)$? Explain.
   (b) If you know that events A and B are mutually exclusive, do you have enough information to compute $P(A$ *or* $B)$? Explain.

7. *Critical Thinking* You are given the information that $P(A) = 0.30$ and $P(B) = 0.40$.
   (a) Do you have enough information to compute $P(A$ *and* $B)$? Explain.
   (b) If you know that events A and B are independent, do you have enough information to compute $P(A$ *and* $B)$? Explain.

8. *Critical Thinking* For a class activity, your group has been assigned the task of generating a quiz question that requires use of the formula for conditional probability to compute $P(B \mid A)$. Your group comes up with the following question: "If $P(A$ *and* $B) = 0.40$ and $P(A) = 0.20$, what is the value of $P(B \mid A)$?" What is wrong with this question? *Hint:* Consider the answer you get when using the correct formula, $P(B \mid A) = P(A$ *and* $B)/P(A)$.

9. *Salary Raise: Women* Does it pay to ask for a raise? A national survey of heads of households showed the percentage of those who asked for a raise and the percentage who got one (*USA Today*). According to the survey, of the women interviewed, 24% had asked for a raise, and of those women who had asked for a raise, 45% received the raise. If a woman is selected at random from the survey population of women, find the following probabilities: P(woman asked for a raise); P(woman received raise, *given* she asked for one); P(woman asked for raise *and* received raise).

10. *Salary Raise: Men* According to the same survey quoted in Problem 9, of the men interviewed, 20% had asked for a raise and 59% of the men who had asked

for a raise received the raise. If a man is selected at random from the survey population of men, find the following probabilities: *P*(man asked for a raise); *P*(man received raise, *given* he asked for one); *P*(man asked for raise *and* received raise).

11. *General: Thumbtack* Drop a thumbtack and observe how it lands.
    (a) Describe how you could use a relative frequency to estimate the probability that a thumbtack will land with its flat side down.
    (b) What is the sample space of outcomes for the thumbtack?
    (c) How would you make a probability assignment to this sample space if, when you drop 500 tacks, 340 land flat side down?

12. *Survey: Reaction to Poison Ivy* Allergic reactions to poison ivy can be miserable. Plant oils cause the reaction. Researchers at Allergy Institute did a study to determine the effects of washing the oil off within 5 minutes of exposure. A random sample of 1000 people with known allergies to poison ivy participated in the study. Oil from the poison ivy plant was rubbed on a patch of skin. For 500 of the subjects, it was washed off *within* 5 minutes. For the other 500 subjects, the oil was washed off *after* 5 minutes. The results are summarized in Table 4-5.

TABLE 4-5   **Time Within Which Oil Was Washed Off**

Reaction	Within 5 Minutes	After 5 Minutes	Row Total
None	420	50	470
Mild	60	330	390
Strong	20	120	140
Column Total	500	500	1000

Let's use the following notation for the various events: *W* = washing oil off within 5 minute, *A* = washing oil off after 5 minutes, *N* = no reaction, *M* = mild reaction, *S* = strong reaction. Find the following probabilities for a person selected at random from this sample of 1000 subjects.
(a) *P*(*N*), *P*(*M*), *P*(*S*)
(b) *P*(*N* | *W*), *P*(*S* | *W*)
(c) *P*(*N* | *A*), *P*(*S* | *A*)
(d) *P*(*N and W*), *P*(*M and W*)
(e) *P*(*N or M*). Are the events *N* = no reaction and *M* = mild reaction mutaually exclusive? Explain.
(f) Are the events *N* = no reaction and *W* = washing oil off within 5 minutes independent? Explain.

13. *General: Two Dice* In a game of craps, you roll two fair dice. Whether you win or lose depends on the sum of the numbers appearing on the tops of the dice. Let *x* be the random variable that represents the sum of the numbers on the tops of the dice.
    (a) What values can *x* take on?
    (b) What is the probability distribution of these *x* values (that is, what is the probability that *x* = 2, 3, etc.)?

14. *Academic: Passing French* Class records at Rockwood College indicate that a student selected at random has probability 0.77 of passing French 101. For the student who passes French 101, the probability is 0.90 that he or she will pass French 102. What is the probability that a student selected at random will pass both French 101 and French 102?

15. *Combination: City Council* There is money to send two of eight city council members to a conference in Honolulu. All want to go, so they decide to choose the members to go to the conference by a random process. How many different combinations of two council members can be selected from the eight who want to go to the conference?

16. | **Basic Computation** Compute. (a) $P_{7,2}$ (b) $C_{7,2}$ (c) $P_{3,3}$ (d) $C_{4,4}$

17. | **Counting: Exam Answers** There are five multiple-choice questions on an exam, each with four possible answers. Determine the number of possible answer sequences for the five questions. Only one of the sets can contain all five correct answers. If you are guessing, so that you are as likely to choose one sequence of answers as another, what is the probability of getting all five answers correct?

18. | **Scheduling: College Courses** A student must satisfy the literature, social science, and philosophy requirements this semester. There are four literature courses to select from, three social science courses, and two philosophy courses. Make a tree diagram showing all the possible sequences of literature, social science, and philosophy courses.

19. | **General: Combination Lock** To open a combination lock, you turn the dial to the right and stop at a number; then you turn it to the left and stop at a second number. Finally, you turn the dial back to the right and stop at a third number. If you used the correct sequence of numbers, the lock opens. If the dial of the lock contains 10 numbers, 0 through 9, determine the number of different combinations possible for the lock. Note: The same number can be reused.

20. | **General: Combination Lock** You have a combination lock. Again, to open it you turn the dial to the right and stop at a first number; then you turn it to the left and stop at a second number. Finally, you turn the dial to the right and stop at a third number. Suppose you remember that the three numbers for your lock are 2, 9, and 5, but you don't remember the order in which the numbers occur. How many sequences of these three numbers are possible?

**DATA HIGHLIGHTS:
GROUP PROJECTS**

Break into small groups and discuss the following topics. Organize a brief outline in which you summarize the main points of your group discussion.

1. Look at Figure 4-11, "Who's Cracking the Books?"

   (a) Does the figure show the probability distribution of grade records for male students? for female students? Describe all the grade-record probability distributions shown in Figure 4-11. Find the probability that a male student selected at random has a grade record showing mostly A's.

   (b) Is the probability distribution shown for all students making mostly A's? Explain your answer. *Hint:* Do the percentages shown for mostly A's add up

FIGURE 4-11

Who's Cracking the Books?

Undergraduate Grade Record by Student Characteristic

Student characteristic		C's and D's or lower	B's and C's	Mostly B's	A's and B's	Mostly A's
Gender:	Men	38.8 %	16.6 %	22.6 %	9.6 %	12.4 %
	Women	29.4	16.2	26.2	12.0	16.2
Class level:	Graduating senior	15.8 %	21.9 %	34.5 %	14.7 %	13.0 %
	All other class levels	35.4	15.8	23.6	10.5	14.7
Age:	18 or younger	42.6 %	14.7 %	23.4 %	9.4 %	10.0 %
	19 to 23	38.1	19.0	25.1	9.4	8.3
	24 to 29	33.3	16.9	24.7	10.3	14.9
	30 to 39	23.1	13.2	25.9	14.8	23.0
	40 and older	20.1	10.1	22.0	14.8	33.0

Source: U.S. Department of Education

to 1? Can Figure 4-11 be used to determine the probability that a student selected at random has mostly A's? Can it be used to determine the probability that a female student selected at random has mostly A's? What is the probability?

(c) Can we use the information shown in the figure to determine the probability that a graduating senior has grades consisting of mostly B's or higher? What is the probability?

(d) Does Figure 4-11 give sufficient information to determine the probability that a student selected at random is in the age range 19 to 23 *and* has grades that are mostly B's? What is the probability that a student selected at random has grades that are mostly B's, *given* that he or she is in the age range 19 to 23?

(e) Suppose that 65% of the students at State University are between 19 and 23 years of age. What is the probability that a student selected at random is in this age range *and* has grades that are mostly B's?

2. Consider the information given in Figure 4-12, "Vulnerable Knees." What is the probability that an orthopedic case selected at random involves knee problems? Of those cases, estimate the probability that the case requires full knee replacement. Compute the probability that an orthopedic case selected at random involves a knee problem *and* requires a full knee replacement. Next, look at the probability distribution for ages of patients requiring full knee replacements. Medicare insurance coverage begins when a person reaches age 65. What is the probability that the age of a person receiving a knee replacement is 65 or older?

FIGURE 4-12

Vulnerable Knees

Age of Adults Getting Knee Replacements

About 26% of orthopedic surgery involves knee problems. More than two-thirds of the surgeries involve full knee replacements.

43.3%

24.6%

26.7%

2.8%

2.9%

Age: 18–44 45–64 65–74 75–84 85–older
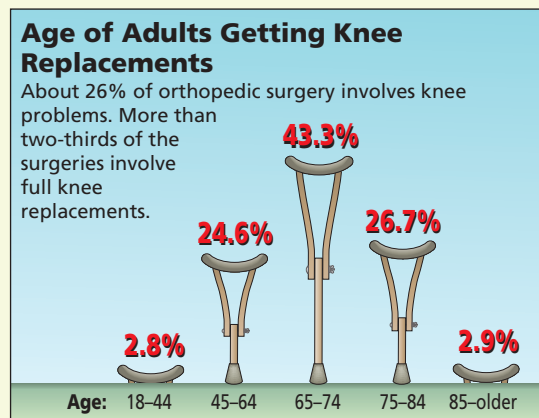
Source: American Academy of Orthopedic Surgeons

**LINKING CONCEPTS:
WRITING PROJECTS**

Discuss each of the following topics in class or review the topics on your own. Then write a brief but complete essay in which you summarize the main points. Please include formulas as appropriate.

1. Discuss the following concepts and give examples from everyday life in which you might encounter each concept. *Hint:* For instance, consider the "experiment" of arriving for class. Some possible outcomes are not arriving (that is, missing class), arriving on time, and arriving late.

   (a) Sample space.

   (b) Probability assignment to a sample space. In your discussion, be sure to include answers to the following questions.

      (i) Is there more than one valid way to assign probabilities to a sample space? Explain and give an example.

      (ii) How can probabilities be estimated by relative frequencies? How can probabilities be computed if events are equally likely?

2. Discuss the concepts of mutually exclusive events and independent events. List several examples of each type of event from everyday life.

   (a) If A and B are mutually exclusive events, does it follow that A and B *cannot* be independent events? Give an example to demonstrate your answer. *Hint:* Discuss an election where only one person can win the election. Let A be the event that party A's candidate wins, and let B be the event that party B's candidate wins. Does the outcome of one event determine the outcome of the other event? Are A and B mutually exclusive events?

   (b) Discuss the conditions under which $P(A \text{ and } B) = P(A) \cdot P(B)$ is true. Under what conditions is this not true?

   (c) Discuss the conditions under which $P(A \text{ or } B) = P(A) + P(B)$ is true. Under what conditions is this not true?

3. Although we learn a good deal about probability in this course, the main emphasis is on statistics. Write a few paragraphs in which you talk about the distinction between probability and statistics. In what types of problems would probability be the main tool? In what types of problems would statistics be the main tool? Give some examples of both types of problems. What kinds of outcomes or conclusions do we expect from each type of problem?

USING TECHNOLOGY

Demonstration of the Law of Large Numbers

Computers can be used to simulate experiments. With packages such as Excel 2007, Minitab, and SPSS, programs using random-number generators can be designed (see the *Technology Guide*) to simulate activities such as tossing a die.

The following printouts show the results of the simulations for tossing a die 6, 500, 50,000, 500,000, and 1,000,000 times. Notice how the relative frequencies of the outcomes approach the theoretical probabilities of 1/6 or 0.16667 for each outcome. Do you expect the same results every time the simulation is done? Why or why not?

Results of tossing one die 6 times

Outcome	Number of Occurrences	Relative Frequency
⚀	0	0.00000
⚁	1	0.16667
⚂	2	0.33333
⚃	0	0.00000
⚄	1	0.16667
⚅	2	0.33333

Results of tossing one die 500 times

Outcome	Number of Occurrences	Relative Frequency
⚀	87	0.17400
⚁	83	0.16600
⚂	91	0.18200
⚃	69	0.13800
⚄	87	0.17400
⚅	83	0.16600

Results of tossing one die 50,000 times

Outcome	Number of Occurrences	Relative Frequency
⚀	8528	0.17056
⚁	8354	0.16708
⚂	8246	0.16492
⚃	8414	0.16828
⚄	8178	0.16356
⚅	8280	0.16560

Results of tossing one die 500,000 times

Outcome	Number of Occurrences	Relative Frequency
⚀	83644	0.16729
⚁	83368	0.16674
⚂	83398	0.16680
⚃	83095	0.16619
⚄	83268	0.16654
⚅	83227	0.16645

Results of tossing one die 1,000,000 times

Outcome	Number of Occurrences	Relative Frequency
⚀	166643	0.16664
⚁	166168	0.16617
⚂	167391	0.16739
⚃	165790	0.16579
⚄	167243	0.16724
⚅	166765	0.16677