CHAPTER

2

Electrical and Thermal Conduction in Solids

Electrical conduction involves the motion of charges in a material under the influence of an applied electric field. A material can generally be classified as a conductor if it contains a large number of "free" or mobile charge carriers. In metals, due to the nature of metallic bonding, the valence electrons from the atoms form a sea of electrons that are free to move within the metal and are therefore called conduction electrons. In this chapter, we will treat the conduction electrons in metal as "free charges" that can be accelerated by an applied electric field. In the presence of an electric field, the conduction electrons attain an average velocity, called the drift velocity, that depends on the field. By applying Newton's second law to electron motion and using such concepts as mean free time between electron collisions with lattice vibrations, crystal defects, impurities, etc., we will derive the fundamental equations that govern electrical conduction in solids. A key concept will be the drift mobility, which is a measure of the ease with which charge carriers in the solid drift under the influence of an external electric field.

Good electrical conductors, such as metals, are also known to be good thermal conductors. The conduction of thermal energy from higher to lower temperature regions in a metal involves the conduction electrons carrying the energy. Consequently, there is an innate relationship between the electrical and thermal conductivities, which is supported by theory and experiments.

2.1 CLASSICAL THEORY: THE DRUDE MODEL

2.1.1 METALS AND CONDUCTION BY ELECTRONS

The electric current density J is defined as the net amount of charge flowing across a unit area per unit time, that is,

Current density definition

$$J = \frac{\Delta q}{A\,\Delta t}$$

where Δq is the net quantity of charge flowing through an area A in time Δt. Figure 2.1 shows the net flow of electrons in a conductor section of cross-sectional area A in the presence of an applied field $\mathcal{E}_x$. Notice that the direction of electron motion is opposite to that of the electric field $\mathcal{E}_x$ and of conventional current, because the electrons experience a Coulombic force $e\mathcal{E}_x$ in the x direction, due to their negative charge.

We know that the conduction electrons are actually moving around randomly[1] in the metal, but we will assume that as a result of the application of the electric field $\mathcal{E}_x$, they all acquire a net velocity in the x direction. Otherwise, there would be no net flow of charge through area A.

The average velocity of the electrons in the x direction at time t is denoted $v_{dx}(t)$. This is called the **drift velocity,** which is the instantaneous velocity v_x in the x direction averaged over many electrons (perhaps, $\sim 10^{28}$ m^{-3}); that is

Definition of drift velocity

$$v_{dx} = \frac{1}{N}[v_{x1} + v_{x2} + v_{x3} + \cdots + v_{xN}] \qquad [2.1]$$

where v_{xi} is the x direction velocity of the ith electron, and N is the number of conduction electrons in the metal. Suppose that n is the number of electrons per unit volume in the conductor $(n = N/V)$. In time Δt, electrons move a distance $\Delta x = v_{dx}\,\Delta t$, so the total charge Δq crossing the area A is $enA\,\Delta x$. This is valid because all the electrons within distance Δx pass through A; thus, $n(A\,\Delta x)$ is the total number of electrons crossing A in time Δt.

The current density in the x direction is

$$J_x = \frac{\Delta q}{A\,\Delta t} = \frac{enAv_{dx}\,\Delta t}{A\,\Delta t} = env_{dx}$$

This general equation relates J_x to the average velocity v_{dx} of the electrons. It must be appreciated that the average velocity at one time may not be the same as at another time, because the applied field, for example, may be changing: $\mathcal{E}_x = \mathcal{E}_x(t)$. We therefore allow for a time-dependent current by writing

Current density and drift velocity

$$J_x(t) = env_{dx}(t) \qquad [2.2]$$

To relate the current density J_x to the electric field $\mathcal{E}_x$, we must examine the effect of the electric field on the motion of the electrons in the conductor. To do so, we will consider the copper crystal.

---

[1] All the conduction electrons are "free" within the metal and move around randomly, being scattered from vibrating metal ions, as we discuss in this chapter.

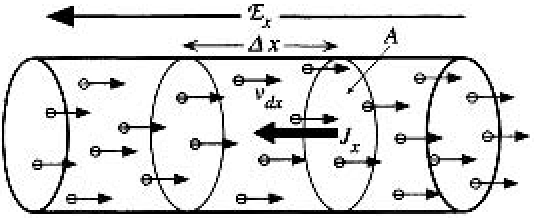


Figure 2.1 Drift of electrons in a conductor in the presence of an applied electric field.
Electrons drift with an average velocity v_{dx} in the x direction.

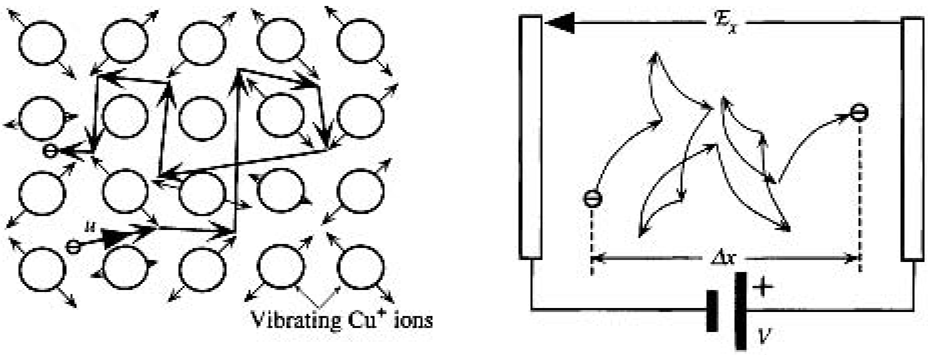
The copper atom has a single valence electron in its 4*s* subshell, and this electron is loosely bound. The solid metal consists of positive ion cores, Cu^+, at regular sites, in the face-centered cubic (FCC) crystal structure. The valence electrons detach themselves from their parents and wander around freely in the solid, forming a kind of electron cloud or gas. These mobile electrons are free to respond to an applied field, creating a current density J_x. The valence electrons in the electron gas are therefore **conduction electrons.**

The attractive forces between the negative electron cloud and the Cu^+ ions are responsible for metallic bonding and the existence of the solid metal. (This simplistic view of metal was depicted in Figure 1.7 for copper.) The electrostatic attraction between the conduction electrons and the positive metal ions, like the electrostatic attraction between the electron and the proton in the hydrogen atom, results in the conduction electron having both potential energy *PE* and kinetic energy *KE*. The conduction electrons move about the crystal lattice in the same way that gas atoms move randomly in a cylinder. Although the average *KE* for gas atoms is $\frac{3}{2}kT$; this is not the case for electrons in a metal, because these electrons strongly interact with the metal ions and with each other as a result of electrostatic interactions.

The mean *KE* of the conduction electrons in a metal is primarily determined by the electrostatic interaction of these electrons with the positive metal ions and also with each other. For most practical purposes, we will therefore neglect the temperature dependence of the mean *KE* compared with other factors that control the behavior of the conduction electrons in the metal crystal. We can speculate from Example 1.1, that the magnitude of mean *KE* must be comparable to the magnitude of the mean *PE* of electrostatic interaction[2] or, stated differently, to the metal bond energy which is several electron volts per atom. If u is the **mean speed** of the conduction electrons, then, from electrostatic interactions alone, we expect $\frac{1}{2}m_e u^2$ to be several electron volts which means that u is typically $\sim 10^6$ m s^{-1}. This purely classical and intuitive reasoning is not sufficient, however, to show that the mean speed u is relatively temperature insensitive and much greater than that expected from kinetic molecular theory. The true reasons are quantum mechanical and are discussed in Chapter 4. (They arise from what is called the Pauli exclusion principle.)

---

[2] There is a theorem in classical mechanics called the **virial theorem**, which states that for a collection of particles, the mean *KE* has half the magnitude of the mean *PE* if the only forces acting on the particles are such that they follow an inverse square law dependence on the particle–particle separation (as in Coulombic and gravitational forces).



(a) A conduction electron in the electron gas moves about randomly in a metal (with a mean speed u) being frequently and randomly scattered by thermal vibrations of the atoms. In the absence of an applied field there is no net drift in any direction.

(b) In the presence of an applied field, $\mathcal{E}_x$, there is a net drift along the x direction. This net drift along the force of the field is superimposed on the random motion of the electron. After many scattering events the electron has been displaced by a net distance, Δx, from its initial position toward the positive terminal.

Figure 2.2 Motion of a conduction electron in a metal.

In general, the copper crystal will not be perfect and the atoms will not be stationary. There will be crystal defects, vacancies, dislocations, impurities, etc., which will scatter the conduction electrons. More importantly, due to their thermal energy, the atoms will vibrate about their lattice sites (equilibrium positions), as depicted in Figure 2.2a. An electron will not be able to avoid collisions with vibrating atoms; consequently, it will be "scattered" from one atom to another. In the absence of an applied field, the path of an electron may be visualized as illustrated in Figure 2.2a, where scattering from lattice vibrations causes the electron to move randomly in the lattice. On those occasions when the electron reaches a crystal surface, it becomes "deflected" (or "bounced") back into the crystal. Therefore, in the absence of a field, after some duration of time, the electron crosses its initial x plane position again. Over a long time, the electrons therefore show no net displacement in any one direction.

When the conductor is connected to a battery and an electric field is applied to the crystal, as shown in Figure 2.2b, the electron experiences an acceleration in the x direction in addition to its random motion, so after some time, it will drift a finite distance in the x direction. The electron accelerates along the x direction under the action of the force $e\mathcal{E}_x$, and then it suddenly collides with a vibrating atom and loses the gained velocity. Therefore, there is an average velocity in the x direction, which, if calculated, determines the current via Equation 2.2. Note that since the electron experiences an acceleration in the x direction, its trajectory between collisions is a parabola, like the trajectory of a golf ball experiencing acceleration due to gravity.

To calculate the drift velocity v_{dx} of the electrons due to applied field $\mathcal{E}_x$, we first consider the velocity v_{xi} of the ith electron in the x direction at time t. Suppose its last collision was at time t_i; therefore, for time $(t - t_i)$, it accelerated *free of collisions,* as

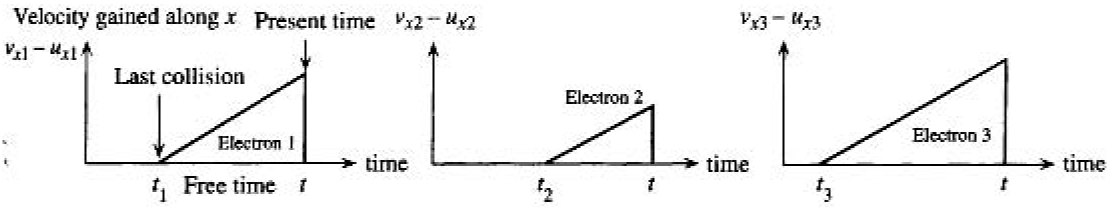


Figure 2.3 Velocity gained in the x direction at time t from the electric field ($\mathcal{E}_x$) for three electrons. There will be N electrons to consider in the metal.

indicated in Figure 2.3. Let u_{xi} be the velocity of electron i in the x direction just after the collision. We will call this the initial velocity. Since $e\mathcal{E}_x/m_e$ is the acceleration of the electron, the velocity v_{xi} in the x direction at time t will be

$$v_{xi} = u_{xi} + \frac{e\mathcal{E}_x}{m_e}(t - t_i)$$

However, this is only for the ith electron. We need the average velocity v_{dx} for all such electrons along x. We average the expression for $i = 1$ to N electrons, as in Equation 2.1. We assume that immediately after a collision with a vibrating ion, the electron may move in any random direction; that is, it can just as likely move along the negative or positive x, so that u_{xi} averaged over many electrons is zero. Thus,

$$v_{dx} = \frac{1}{N}[v_{x1} + v_{x2} + \cdots + v_{xN}] = \frac{e\mathcal{E}_x}{m_e}\overline{(t - t_i)}$$

Drift velocity

where $\overline{(t - t_i)}$ is the **average free time** for N electrons between collisions.

Suppose that τ is the mean free time, or the **mean time between collisions** (also known as the **mean scattering time**). For some electrons, $(t - t_i)$ will be greater than τ, and for others, it will be shorter, as shown in Figure 2.3. Averaging $(t - t_i)$ for N electrons will be the same as τ. Thus, we can substitute τ for $\overline{(t - t_i)}$ in the previous expression to obtain

$$v_{dx} = \frac{e\tau}{m_e}\mathcal{E}_x \qquad [2.3]$$

Equation 2.3 shows that the drift velocity increases linearly with the applied field. The constant of proportionality $e\tau/m_e$ has been given a special name and symbol. It is called the **drift mobility** μ_d, which is defined as

$$v_{dx} = \mu_d\mathcal{E}_x \qquad [2.4]$$

Definition of drift mobility

where

$$\mu_d = \frac{e\tau}{m_e} \qquad [2.5]$$

Drift mobility and mean free time

Equation 2.5 relates the drift mobility of the electrons to their mean scattering time τ. To reiterate, τ, which is also called the **relaxation time**, is directly related to the microscopic processes that cause the scattering of the electrons in the metal; that is, lattice vibrations, crystal imperfections, and impurities, to name a few.

From the expression for the drift velocity v_{dx}, the current density J_x follows immediately by substituting Equation 2.4 into 2.2, that is,

Ohm's law

$$J_x = en\mu_d \mathcal{E}_x \tag{2.6}$$

Therefore, the current density is proportional to the electric field and the conductivity σ is the term multiplying $\mathcal{E}_x$, that is,

Unipolar conductivity

$$\sigma = en\mu_d \tag{2.7}$$

It is gratifying that by treating the electron as a particle and applying classical mechanics ($F = ma$), we are able to derive Ohm's law. We should note, however, that we assumed τ to be independent of the field.

Drift mobility is important because it is a widely used electronic parameter in semiconductor device physics. The drift mobility gauges how fast electrons will drift when driven by an applied field. If the electron is not highly scattered, then the mean free time between collisions will be long, τ will be large, and by Equation 2.5, the drift mobility will also be large; the electrons will therefore be highly mobile and be able to "respond" to the field. However, a large drift mobility does not necessarily imply high conductivity, because σ also depends on the concentration of conduction electrons n.

The mean time between collisions τ has further significance. Its reciprocal $1/\tau$ represents the **mean frequency of collisions** or **scattering events;** that is, $1/\tau$ is the mean probability per unit time that the electron will be scattered (see Example 2.1). Therefore, during a small time interval δt, the probability of scattering will be $\delta t/\tau$. The probability of scattering per unit time $1/\tau$ is time independent and depends only on the nature of the electron scattering mechanism.

There is one important assumption in the derivation of the drift velocity v_{dx} in Equation 2.3. We obtained v_{dx} by averaging the velocities v_{xi} of N electrons along x at one instant, as defined in Equation 2.1. The drift velocity therefore represents the average velocity of *all* the electrons along x at one instant; that is, v_{dx} is a number average at one instant. Figure 2.2b shows that after many collisions, after a time interval $\Delta t \gg \tau$, an electron would have been displaced by a net distance Δx along x. The term $\Delta x/\Delta t$ represents the effective velocity with which the electron drifts along x. It is an average velocity for one electron over many collisions, that is, over a long time (hence, $\Delta t \gg \tau$), so $\Delta x/\Delta t$ is a time average. Provided that Δt contains many collisions, it is reasonable to expect that the drift velocity $\Delta x/\Delta t$ from the time average for one electron is the same as the drift velocity v_{dx} per electron from averaging for all electrons at one instant, as in Equation 2.1, or

Drift velocity

$$\frac{\Delta x}{\Delta t} = v_{dx}$$

The two velocities are the same only under steady-state conditions ($\Delta t \gg \tau$). The proof may be found in more advanced texts.

EXAMPLE 2.1

PROBABILITY OF SCATTERING PER UNIT TIME AND THE MEAN FREE TIME If $1/\tau$ is defined as the mean probability per unit time that an electron is scattered, show that the mean time between collisions is τ.

SOLUTION

Consider an infinitesimally small time interval dt at time t. Let N be the number of unscattered electrons at time t. The probability of scattering during dt is $(1/\tau)\,dt$, and the number of scattered electrons during dt is $N(1/\tau)\,dt$. The change dN in N is thus

$$dN = -N\left(\frac{1}{\tau}\right)dt$$

The negative sign indicates a reduction in N because, as electrons become scattered, N decreases. Integrating this equation, we can find N at any time t, given that at time $t = 0$, N_0 is the total number of unscattered electrons. Therefore,

$$N = N_0 \exp\left(-\frac{t}{\tau}\right)$$

Unscattered electron concentration

This equation represents the number of unscattered electrons at time t. It reflects an exponential decay law for the number of unscattered electrons. The **mean free time** $\bar{t}$ can be calculated from the mathematical definition of $\bar{t}$,

$$\bar{t} = \frac{\int_0^\infty tN\,dt}{\int_0^\infty N\,dt} = \tau$$

Mean free time

where we have used $N = N_0 \exp(-t/\tau)$. Clearly, $1/\tau$ is the mean probability of scattering per unit time.

EXAMPLE 2.2

ELECTRON DRIFT MOBILITY IN METALS Calculate the drift mobility and the mean scattering time of conduction electrons in copper at room temperature, given that the conductivity of copper is $5.9 \times 10^5\ \Omega^{-1}\ \text{cm}^{-1}$. The density of copper is 8.96 g cm^{-3} and its atomic mass is 63.5 g mol^{-1}.

SOLUTION

We can calculate μ_d from $\sigma = en\mu_d$ because we already know the conductivity σ. The number of free electrons n per unit volume can be taken as equal to the number of Cu atoms per unit volume, if we assume that each Cu atom donates one electron to the conduction electron gas in the metal. One mole of copper has N_A (6.02×10^{23}) atoms and a mass of 63.5 g. Therefore, the number of copper atoms per unit volume is

$$n = \frac{dN_A}{M_{at}}$$

where d = density = 8.96 g cm^{-3}, and M_{at} = atomic mass = 63.5 (g mol^{-1}). Substituting for d, N_A, and M_{at}, we find $n = 8.5 \times 10^{22}$ electrons cm^{-3}.

The electron drift mobility is therefore

$$\mu_d = \frac{\sigma}{en} = \frac{5.9 \times 10^5\ \Omega^{-1}\ \text{cm}^{-1}}{[(1.6 \times 10^{-19}\ \text{C})(8.5 \times 10^{22}\ \text{cm}^{-3})]}$$
$$= 43.4\ \text{cm}^2\ \text{V}^{-1}\ \text{s}^{-1}$$

From the drift mobility we can calculate the mean free time τ between collisions by using Equation 2.5,

$$\tau = \frac{\mu_d m_e}{e} = \frac{(43.4 \times 10^{-4}\ \text{m}^2\ \text{V}^{-1}\ \text{s}^{-1})(9.1 \times 10^{-31}\ \text{kg})}{1.6 \times 10^{-19}\ \text{C}} = 2.5 \times 10^{-14}\ \text{s}$$

Note that the mean speed u of the conduction electrons is about 1.5×10^6 m s^{-1}, so that their mean free path is about 37 nm.

EXAMPLE 2.3

DRIFT VELOCITY AND MEAN SPEED What is the applied electric field that will impose a drift velocity equal to 0.1 percent of the mean speed $u(\sim 10^6\ \text{m s}^{-1})$ of conduction electrons in copper? What is the corresponding current density and current through a Cu wire of diameter 1 mm?

SOLUTION

The drift velocity of the conduction electrons is $v_{dx} = \mu_d \mathcal{E}_x$, where μ_d is the drift mobility, which for copper is 43.4 cm^2 V^{-1} s^{-1} (see Example 2.2). With $v_{dx} = 0.001u = 10^3$ m s^{-1}, we have

$$\mathcal{E}_x = \frac{v_{dx}}{\mu_d} = \frac{10^3\ \text{m s}^{-1}}{43.4 \times 10^{-4}\ \text{m}^2\ \text{V}^{-1}\ \text{s}^{-1}} = 2.3 \times 10^5\ \text{V m}^{-1} \quad \text{or} \quad 230\ \text{kV m}^{-1}$$

This is an unattainably large electric field in a metal. Given the conductivity σ of copper, the equivalent current density is

$$J_x = \sigma\mathcal{E}_x = (5.9 \times 10^7\ \Omega^{-1}\ \text{m}^{-1})(2.3 \times 10^5\ \text{V m}^{-1})$$
$$= 1.4 \times 10^{13}\ \text{A m}^{-2} \quad \text{or} \quad 1.4 \times 10^7\ \text{A mm}^{-2}$$

This means a current of 1.1×10^7 A through a 1 mm diameter wire! It is clear from this example that for all practical purposes, even under the highest working currents and voltages, the drift velocity is much smaller than the mean speed of the electrons. Consequently, when an electric field is applied to a conductor, for all practical purposes, the mean speed is unaffected.

EXAMPLE 2.4

DRIFT VELOCITY IN A FIELD: A CLOSER LOOK There is another way to explain the observed dependence of the drift velocity on the field, and Equation 2.3. Consider the path of a conduction electron in an applied field $\mathcal{E}$ as shown in Figure 2.4. Suppose that at time $t = 0$ the electron has just been scattered from a lattice vibration. Let u_{x1} be the initial velocity in the x direction just after this initial collision (to which we assign a collision number of zero). We will assume that immediately after a collision, the velocity of the electron is in a random *direction*. Suppose that the first collision occurs at time t_1. Since $e\mathcal{E}_x/m_e$ is the acceleration, the distance s_1 covered in the x direction during the free time t_1 will be

Distance traversed along x before collision

$$s_1 = u_{x1}t_1 + \frac{1}{2}\left(\frac{e\mathcal{E}_x}{m_e}\right)t_1^2$$

At time t_1, the electron collides with a lattice vibration (its first collision), and the velocity is randomized again to become u_{x2}. The whole process is then repeated during the next interval which lasts for a free time t_2, and the electron traverses a distance s_2 along x, and so on. To find the overall distance traversed by the electron after p such scattering events, we sum all the

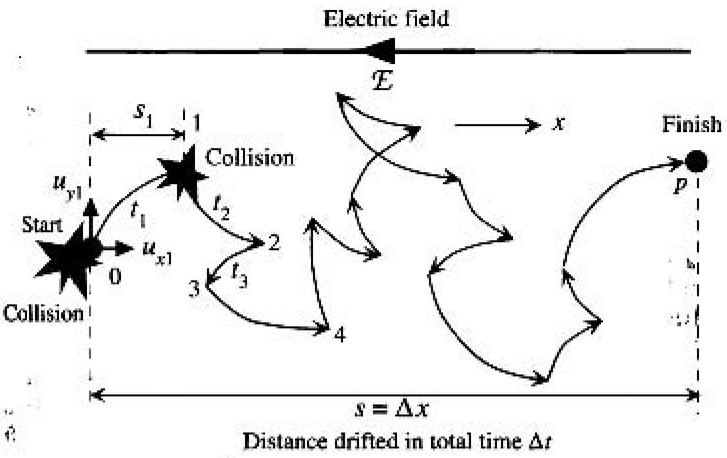


Figure 2.4 The motion of a single electron in the presence of an electric field $\mathcal{E}$. During a time interval t_i, the electron traverses a distance s_i along x. After p collisions, it has drifted a distance $s = \Delta x$.

above distances $s_1, s_2, \ldots$ for p free time intervals,

$$s = s_1 + s_2 + \cdots + s_p = [u_{x1}t_1 + u_{x2}t_2 + \cdots + u_{xp}t_p] + \frac{1}{2}\left(\frac{e\mathcal{E}_x}{m_e}\right)[t_1^2 + t_1^2 + \cdots + t_p^2] \quad [2.8]$$

Since after a collision the "initial" velocity u_x is always random, the first term has u_x values that are randomly negative and positive, so for many collisions (large p) the first term on the right-hand side of Equation 2.8 is nearly zero and can certainly be neglected compared with the second term. Thus, after many collisions, the net distance $s = \Delta x$ traversed in the x direction is given by the second term in Equation 2.8, which is the electric field induced displacement term. If $\overline{t^2}$ is the **mean square free time,** then

Distance drifted after p scattering events

$$s = \frac{1}{2}\left(\frac{e\mathcal{E}_x}{m_e}\right)p\overline{t^2}$$

where

$$\overline{t^2} = \frac{1}{p}[t_1^2 + t_1^2 + \cdots + t_p^2]$$

Mean square free time definition

Suppose that τ is the **mean free time between collisions,** where $\tau = (t_1 + t_2 + \cdots + t_p)/p$. Then from straightforward elementary statistics it can be shown that $\overline{t^2} = 2(\overline{t})^2 = 2\tau^2$. So in terms of the mean free time τ between collisions, the overall distance $s = \Delta x$ drifted in the x direction after p collisions is

$$s = \frac{e\mathcal{E}_x}{m_e}(p\tau^2)$$

Further, since the total time Δt taken for these p scattering events is simply $p\tau$, the drift velocity v_{dx} is given by $\Delta x/\Delta t$ or $s/(p\tau)$, that is,

Drift velocity and mean free time

$$v_{dx} = \frac{e\tau}{m_e}\mathcal{E}_x \quad [2.9]$$

This is the same expression as Equation 2.3, except that τ is defined here as the average free time for a single electron over a long time, that is, over many collisions, whereas previously it was the mean free time averaged over many electrons. Further, in Equation 2.9 v_{dx} is an average drift for an electron over a long time, over many collisions. In Equation 2.1 v_{dx} is the average velocity averaged over all electrons at one instant. For all practical purposes, the two are equivalent. (The equivalence breaks down when we are interested in events over a time scale that is comparable to one scattering, $\sim 10^{-14}$ second.)

The drift mobility μ_d from Equation 2.9 is identical to that of Equation 2.5, $\mu_d = e\tau/m_e$. Suppose that the mean speed of the electrons (not the drift velocity) is u. Then an electron moves a distance $\ell = u\tau$ in mean free time τ, which is called the **mean free path.** The drift mobility and conductivity become,

Drift mobility and conductivity and mean free path

$$\mu_d = \frac{e\ell}{m_e u} \qquad \text{and} \qquad \sigma = en\mu_d = \frac{e^2 n\ell}{m_e u} \tag{2.10}$$

Equations 2.3 and 2.10 both assume that after each collision the velocity is randomized. The scattering process, lattice scattering, is able to randomize the velocity in one single scattering. In general not all electron scattering processes can randomize the velocity in one scattering process. If it takes more than one collision to randomize the velocity, then the electron is able to carry with it some velocity gained from a previous collision and hence possesses a higher drift mobility. In such cases one needs to consider the effective mean free path a carrier has to move to eventually randomize the velocity gained; this is a point considered in Chapter 4 when we calculate the resistivity at low temperatures.

2.2 TEMPERATURE DEPENDENCE OF RESISTIVITY: IDEAL PURE METALS

When the conduction electrons are only scattered by thermal vibrations of the metal ions, then τ in the mobility expression $\mu_d = e\tau/m_e$ refers to the mean time between scattering events by this process. The resulting conductivity and resistivity are denoted by σ_T and ρ_T, where the subscript T represents "thermal vibration scattering."

To find the temperature dependence of σ, we first consider the temperature dependence of the mean free time τ, since this determines the drift mobility. An electron moving with a mean speed u is scattered when its path crosses the cross-sectional area S of a scattering center, as depicted in Figure 2.5. The scattering center

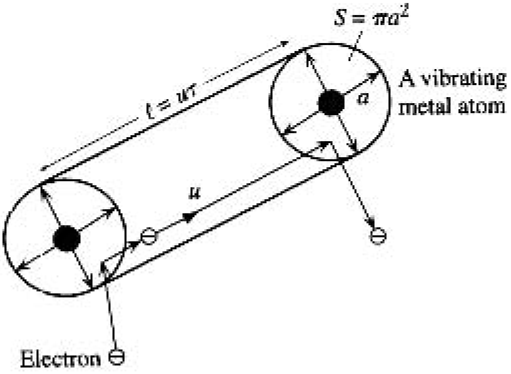


Figure 2.5 Scattering of an electron from the thermal vibrations of the atoms.

The electron travels a mean distance $\ell = u\tau$ between collisions. Since the scattering cross-sectional area is S, in the volume $S\ell$ there must be at least one scatterer, $N_s\,[Su\tau] = 1$.

may be a vibrating atom, impurity, vacancy, or some other crystal defect. Since τ is the mean time taken for one scattering process, the **mean free path** ℓ of the electron between scattering processes is $u\tau$. If N_s is the concentration of scattering centers, then in the volume $S\ell$, there is one scattering center, that is, $(Su\tau)N_s = 1$. Thus, the mean free time is given by

Mean free time between collisions

$$\tau = \frac{1}{SuN_s} \qquad [2.11]$$

The mean speed u of conduction electrons in a metal can be shown to be only slightly temperature dependent.[3] In fact, electrons wander randomly around in the metal crystal with an almost constant mean speed that depends largely on their concentration and hence on the crystal material. Taking the number of scattering centers per unit volume to be the atomic concentration, the temperature dependence of τ then arises essentially from that of the cross-sectional area S. Consider what a free electron "sees" as it approaches a vibrating crystal atom as in Figure 2.5. Because the atomic vibrations are random, the atom covers a cross-sectional area πa^2, where a is the amplitude of the vibrations. If the electron's path crosses πa^2, it gets scattered. Therefore, the mean time between scattering events τ is inversely proportional to the area πa^2 that scatters the electron, that is, $\tau \propto 1/\pi a^2$.

The thermal vibrations of the atom can be considered to be simple harmonic motion, much the same way as that of a mass M attached to a spring. The average kinetic energy of the oscillations is $\frac{1}{4}Ma^2\omega^2$, where ω is the oscillation frequency. From the kinetic theory of matter, this average kinetic energy must be on the order of $\frac{1}{2}kT$. Therefore,

$$\tfrac{1}{4}Ma^2\omega^2 \approx \tfrac{1}{2}kT$$

so $a^2 \propto T$. Intuitively, this is correct because raising the temperature increases the amplitude of the atomic vibrations. Thus,

$$\tau \propto \frac{1}{\pi a^2} \propto \frac{1}{T} \qquad \text{or} \qquad \tau = \frac{C}{T}$$

where C is a temperature-independent constant. Substituting for τ in $\mu_d = e\tau/m_e$, we obtain

$$\mu_d = \frac{eC}{m_e T}$$

So, the resistivity of a metal is

$$\rho_T = \frac{1}{\sigma_T} = \frac{1}{en\mu_d} = \frac{m_e T}{e^2 n C}$$

---

[3] The fact that the mean speed of electrons in a metal is only weakly temperature dependent can be proved from what it called the Fermi–Dirac statistics for the collection of electrons in a metal (see Chapter 4). This result contrasts sharply with the kinetic molecular theory of gases (Chapter 1), which predicts that the mean speed of molecules is proportional to $\sqrt{T}$. For the time being, we simply use a constant mean speed u for the conduction electrons in a metal.

Pure metal resistivity due to thermal vibrations of the crystal

that is,

$$\rho_T = AT \tag{2.12}$$

where A is a temperature-independent constant. This shows that the resistivity of a pure metal wire increases linearly with the temperature, and that the resistivity is due simply to the scattering of conduction electrons by the thermal vibrations of the atoms. We term this conductivity **lattice-scattering-limited conductivity.**

EXAMPLE 2.5 **TEMPERATURE DEPENDENCE OF RESISTIVITY** What is the percentage change in the resistance of a pure metal wire from Saskatchewan's summer to winter, neglecting the changes in the dimensions of the wire?

SOLUTION

Assuming 20 °C for the summer and perhaps −30 °C for the winter, from $R \propto \rho = AT$, we have

$$\frac{R_{\text{summer}} - R_{\text{winter}}}{R_{\text{summer}}} = \frac{T_{\text{summer}} - T_{\text{winter}}}{T_{\text{summer}}} = \frac{(20+273)-(-30+273)}{(20+273)}$$

$$= 0.171 \qquad \text{or} \qquad 17\%$$

Notice that we have used the absolute temperature for T. How will the outdoor cable power losses be affected?

EXAMPLE 2.6 **DRIFT MOBILITY AND RESISTIVITY DUE TO LATTICE VIBRATIONS** Given that the mean speed of conduction electrons in copper is 1.5×10^6 m s^{-1} and the frequency of vibration of the copper atoms at room temperature is about 4×10^{12} s^{-1}, estimate the drift mobility of electrons and the conductivity of copper. The density d of copper is 8.96 g cm^{-3} and the atomic mass M_{at} is 63.56 g mol^{-1}.

SOLUTION

The method for calculating the drift mobility and hence the conductivity is based on evaluating the mean free time τ via Equation 2.11, that is, $\tau = 1/SuN_s$. Since τ is due to scattering from atomic vibrations, N_s is the atomic concentration,

$$N_s = \frac{dN_A}{M_{at}} = \frac{(8.96 \times 10^3 \text{ kg m}^{-3})(6.02 \times 10^{23} \text{ mol}^{-1})}{63.56 \times 10^{-3} \text{ kg mol}^{-1}}$$

$$= 8.5 \times 10^{28} \text{ m}^{-3}$$

The cross-sectional area $S = \pi a^2$ depends on the amplitude a of the thermal vibrations as shown in Figure 2.5. The average kinetic energy KE_{av} associated with a vibrating mass M attached to a spring is given by $KE_{av} = \frac{1}{4}Ma^2\omega^2$, where ω is the angular frequency of the vibration ($\omega = 2\pi 4 \times 10^{12}$ rad s^{-1}). Applying this equation to the vibrating atom and equating the average kinetic energy KE_{av} to $\frac{1}{2}kT$, by virtue of equipartition of energy theorem, we have $a^2 = 2kT/M\omega^2$ and thus

$$S = \pi a^2 = \frac{2\pi kT}{M\omega^2} = \frac{2\pi(1.38 \times 10^{-23} \text{ J K}^{-1})(300 \text{ K})}{\left(\dfrac{63.56 \times 10^{-3} \text{ kg mol}^{-1}}{6.022 \times 10^{23} \text{ mol}^{-1}}\right)(2\pi \times 4 \times 10^{12} \text{ rad s}^{-1})^2}$$

$$= 3.9 \times 10^{-22} \text{ m}^2$$

Therefore,

$$\tau = \frac{1}{SuN_s} = \frac{1}{(3.9 \times 10^{-22}\ \mathrm{m^2})(1.5 \times 10^6\ \mathrm{m\ s^{-1}})(8.5 \times 10^{28}\ \mathrm{m^{-3}})}$$

$$= 2.0 \times 10^{-14}\ \mathrm{s}$$

The drift mobility is

$$\mu_d = \frac{e\tau}{m_e} = \frac{(1.6 \times 10^{-19}\ \mathrm{C})(2.0 \times 10^{-14}\ \mathrm{s})}{(9.1 \times 10^{-31}\ \mathrm{kg})}$$

$$= 3.5 \times 10^{-3}\ \mathrm{m^2\ V^{-1}\ s^{-1}} = 35\ \mathrm{cm^2\ V^{-1}\ s^{-1}}$$

The conductivity is then

$$\sigma = en\mu_d = (1.6 \times 10^{-19}\ \mathrm{C})(8.5 \times 10^{22}\ \mathrm{cm^{-3}})(35\ \mathrm{cm^2\ V^{-1}\ s^{-1}})$$

$$= 4.8 \times 10^5\ \Omega^{-1}\ \mathrm{cm^{-1}}$$

The experimentally measured value for the conductivity is $5.9 \times 10^5\ \Omega^{-1}\ \mathrm{cm^{-1}}$, so our crude calculation based on Equation 2.11 is actually only 18 percent lower, which is not bad for an estimate. (As we might have surmised, the agreement is brought about by using reasonable values for the mean speed u and the atomic vibrational frequency ω. These values were taken from quantum mechanical calculations, so our evaluation for τ was not truly based on classical concepts.)

2.3 MATTHIESSEN'S AND NORDHEIM'S RULES

2.3.1 MATTHIESSEN'S RULE AND THE TEMPERATURE COEFFICIENT OF RESISTIVITY (α)

The theory of conduction that considers scattering from lattice vibrations only works well with pure metals; unfortunately, it fails for metallic alloys. Their resistivities are only weakly temperature dependent. We must therefore search for a different type of scattering mechanism.

Consider a metal alloy that has randomly distributed impurity atoms. An electron can now be scattered by the impurity atoms because they are not identical to the host atoms, as illustrated in Figure 2.6. The impurity atom need not be larger than the host atom; it can be smaller. As long as the impurity atom results in a local distortion of the crystal lattice, it will be effective in scattering. One way of looking at the scattering process from an impurity is to consider the scattering cross section. What actually scatters the electron is a local, unexpected change in the potential energy *PE* of the electron as it approaches the impurity, because the force experienced by the electron is given by

$$F = -\frac{d(PE)}{dx}$$

For example, when an impurity atom of a different size compared to the host atom is placed into the crystal lattice, the impurity atom distorts the region around it, either by

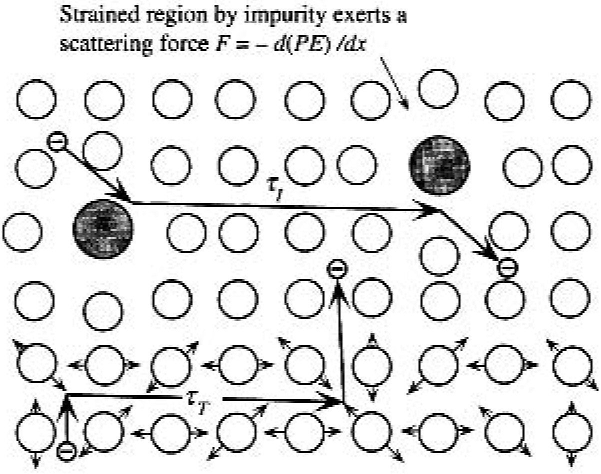


Figure 2.6 Two different types of scattering processes involving scattering from impurities alone and from thermal vibrations alone.

pushing the host atoms farther away, or by pulling them in, as depicted in Figure 2.6. The cross section that scatters the electron is the lattice region that has been elastically distorted by the impurity (the impurity atom itself and its neighboring host atoms), so that in this zone, the electron suddenly experiences a force $F = -d(PE)/dx$ due to a sudden change in the PE. This region has a large scattering cross section, since the distortion induced by the impurity may extend a number of atomic distances. These impurity atoms will therefore hinder the motion of the electrons, thereby increasing the resistance.

We now effectively have two types of mean free times between collisions: one, τ_T, for scattering from thermal vibrations only, and the other, τ_I, for scattering from impurities only. We define τ_T as the mean time between scattering events arising from thermal vibrations alone and τ_I as the mean time between scattering events arising from collisions with impurities alone. Both are illustrated in Figure 2.6.

In general, an electron may be scattered by both processes, so the effective mean free time τ between any two scattering events will be less than the individual scattering times τ_T and τ_I. The electron will therefore be scattered when it collides with either an atomic vibration or an impurity atom. Since in unit time, $1/\tau$ is the net probability of scattering, $1/\tau_T$ is the probability of scattering from lattice vibrations alone, and $1/\tau_I$ is the probability of scattering from impurities alone, then within the realm of elementary probability theory for independent events, we have

Overall frequency of scattering

$$\frac{1}{\tau} = \frac{1}{\tau_T} + \frac{1}{\tau_I} \qquad [2.13]$$

In writing Equation 2.13 for the various probabilities, we make the reasonable assumption that, to a greater extent, the two scattering mechanisms are essentially independent. Here, the effective mean scattering time τ is clearly smaller than both τ_T and τ_I. We can also interpret Equation 2.13 as follows: In unit time, the overall number of collisions $(1/\tau)$ is the sum of the number of collisions with thermal vibrations alone $(1/\tau_T)$ and the number of collisions with impurities alone $(1/\tau_I)$.

The drift mobility μ_d depends on the effective scattering time τ via $\mu_d = e\tau/m_e$, so Equation 2.13 can also be written in terms of the drift mobilities determined by the various scattering mechanisms. In other words,

Effective drift mobility

$$\frac{1}{\mu_d} = \frac{1}{\mu_L} + \frac{1}{\mu_I} \qquad [2.14]$$

where μ_L is the **lattice-scattering-limited drift mobility,** and μ_I is the **impurity-scattering-limited drift mobility.** By definition, $\mu_L = e\tau_T/m_e$ and $\mu_I = e\tau_I/m_e$. The effective (or overall) resistivity ρ of the material is simply $1/en\mu_d$, or

$$\rho = \frac{1}{en\mu_d} = \frac{1}{en\mu_L} + \frac{1}{en\mu_I}$$

which can be written

Matthiessen's rule

$$\rho = \rho_T + \rho_I \qquad [2.15]$$

where $1/en\mu_L$ is defined as the resistivity due to scattering from thermal vibrations, and $1/en\mu_I$ is the resistivity due to scattering from impurities, or

Resistivities due to lattice and impurity scattering

$$\rho_T = \frac{1}{en\mu_L} \qquad \text{and} \qquad \rho_I = \frac{1}{en\mu_I}$$

The final result in Equation 2.15 simply states that the effective resistivity ρ is the sum of two contributions. First, $\rho_T = 1/en\mu_L$ is the resistivity due to scattering by thermal vibrations of the host atoms. For those near-perfect pure metal crystals, this is the dominating contribution. As soon as we add impurities, however, there is an additional resistivity, $\rho_I = 1/en\mu_I$, which arises from the scattering of the electrons from the impurities. The first term is temperature dependent because $\tau_T \propto T^{-1}$ (see Section 2.2), but the second term is not.

The mean time τ_I between scattering events involving electron collisions with impurity atoms depends on the separation between the impurity atoms and therefore on the concentration of those atoms (see Figure 2.6). If ℓ_I is the mean separation between the impurities, then the mean free time between collisions with impurities alone will be ℓ_I/u, which is temperature independent because ℓ_I is determined by the impurity concentration N_I (*i.e.*, $\ell_I = N_I^{-1/3}$), and the mean speed of the electrons u is nearly constant in a metal. In the absence of impurities, τ_I is infinitely long, and thus $\rho_I = 0$. The summation rule of resistivities from different scattering mechanisms, as shown by Equation 2.15, is called **Matthiessen's rule.**

There may also be electrons scattering from dislocations and other crystal defects, as well as from grain boundaries. All of these scattering processes add to the resistivity of a metal, just as the scattering process from impurities. We can therefore write the effective resistivity of a metal as

Matthiessen's rule

$$\rho = \rho_T + \rho_R \qquad [2.16]$$

where ρ_R is called the **residual resistivity** and is due to the scattering of electrons by impurities, dislocations, interstitial atoms, vacancies, grain boundaries, etc. (which means that ρ_R also includes ρ_I). The residual resistivity shows very little temperature dependence, whereas $\rho_T = AT$, so the effective resistivity ρ is given by

$$\rho \approx AT + B \tag{2.17}$$

where A and B are temperature-independent constants.

Equation 2.17 indicates that the resistivity of a metal varies almost linearly with the temperature, with A and B depending on the material. Instead of listing A and B in resistivity tables, we prefer to use a temperature coefficient that refers to small, normalized changes around a reference temperature. The **temperature coefficient of resistivity (TCR)** α_0 is defined as the fractional change in the resistivity per unit temperature increase at the reference temperature T_0, that is,

Definition of temperature coefficient of resistivity

$$\alpha_0 = \frac{1}{\rho_0}\left[\frac{\delta\rho}{\delta T}\right]_{T=T_0} \tag{2.18}$$

where ρ_0 is the resistivity at the reference temperature T_0, usually 273 K (0 °C) or 293 K (20 °C), and $\delta\rho = \rho - \rho_0$ is the change in the resistivity due to a small increase in temperature, $\delta T = T - T_0$.

When the resistivity follows the behavior $\rho \approx AT + B$ in Equation 2.17, then according to Equation 2.18, α_0 is constant over a temperature range T_0 to T, and Equation 2.18 leads to the well-known equation,

Temperature dependence of resistivity

$$\rho = \rho_0[1 + \alpha_0(T - T_0)] \tag{2.19}$$

Equation 2.19 is actually only valid when α_0 is constant over the temperature range of interest, which requires Equation 2.17 to hold. Over a limited temperature range, this will usually be the case. Although it is not obvious from Equation 2.19, we should note that α_0 depends on the reference temperature T_0, by virtue of ρ_0 depending on T_0.

The equation $\rho = AT$, which we used for pure-metal crystals to find the change in the resistance with temperature, is only approximate; nonetheless, for pure metals, it is useful to recall in the absence of tabulated data. To determine how good the formula $\rho = AT$ is, put it in Equation 2.19, which leads to $\alpha_0 = T_0^{-1}$. If we take the reference temperature T_0 as 273 K (0 °C), then α_0 is simply 1/273 K; stated differently, Equation 2.19 is then equivalent to $\rho = AT$.

Table 2.1 shows that $\rho \propto T$ is not a bad approximation for some of the familiar pure metals used as conductors (Cu, Al, Au, etc.), but it fails badly for others, such as indium, antimony, and, in particular, the magnetic metals, iron and nickel.

The temperature dependence of the resistivity of various metals is shown in Figure 2.7, where it is apparent that except for the magnetic materials, such as iron and nickel, the linear relationship $\rho \propto T$ seems to be approximately obeyed almost all the way to the melting temperature for many pure metals. It should also be noted that for the alloys, such as nichrome (Ni–Cr), the resistivity is essentially dominated by the residual resistivity, so the resistivity is relatively temperature insensitive, with a very small TCR.

Table 2.1 Resistivity, thermal coefficient of resistivity α_0 at 273 K (0 °C) for various metals. The resistivity index n in $\rho \propto T^n$ for some of the metals is also shown.

Metal	ρ_0 (nΩ m)	$\alpha_0\left(\frac{1}{\text{K}}\right)$	n	Comment
Aluminum, Al	25.0	$\frac{1}{233}$	1.20	
Antimony, Sb	38	$\frac{1}{196}$	1.40	
Copper, Cu	15.7	$\frac{1}{232}$	1.15	
Gold, Au	22.8	$\frac{1}{251}$	1.11	
Indium, In	78.0	$\frac{1}{196}$	1.40	
Platinum, Pt	98	$\frac{1}{255}$	0.94	
Silver, Ag	14.6	$\frac{1}{244}$	1.11	
Tantalum, Ta	117	$\frac{1}{294}$	0.93	
Tin, Sn	110	$\frac{1}{217}$	1.11	
Tungsten, W	50	$\frac{1}{220}$	1.20	
Iron, Fe	84.0	$\frac{1}{152}$	1.80	Magnetic metal; $273 < T < 1043$ K
Nickel, Ni	59.0	$\frac{1}{125}$	1.72	Magnetic metal; $273 < T < 627$ K

SOURCE: Data were extracted and combined from several sources. Typical values.

Frequently, the resistivity versus temperature behavior of pure metals can be empirically represented by a power law of the form

$$\rho = \rho_0\left[\frac{T}{T_0}\right]^n \qquad [2.20]$$

Resistivity of pure metals

where ρ_0 is the resistivity at the reference temperature T_0, and n is a characteristic index that best fits the data. Table 2.1 lists some typical n values for various pure metals above 0 °C. It is apparent that for the nonmagnetic metals, n is close to unity, whereas it is closer to 2 than 1 for the magnetic metals Fe and Ni. In iron, for example, the conduction electron is not scattered simply by atomic vibrations, as in copper, but is affected by its magnetic interaction with the Fe ions in the lattice. This leads to a complicated temperature dependence.

Although our oversimplified theoretical analysis predicts a linear $\rho = AT + B$ behavior for the resistivity down to the lowest temperatures, this is not true in reality,

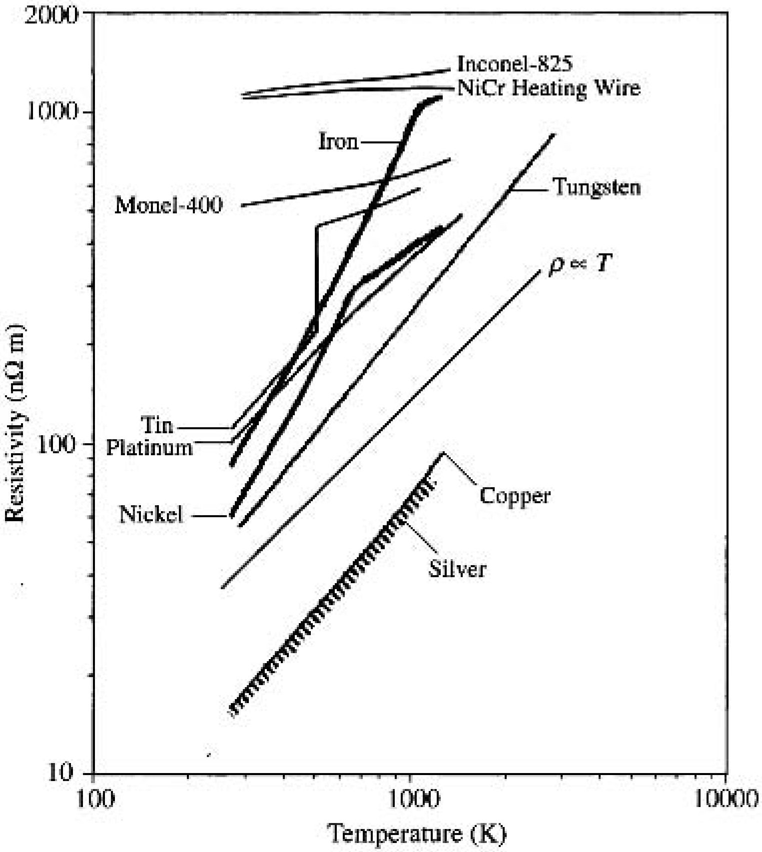


Figure 2.7 The resistivity of various metals as a function of temperature above 0 °C.

Tin melts at 505 K, whereas nickel and iron go through a magnetic-to-nonmagnetic (Curie) transformation at about 627 K and 1043 K, respectively. The theoretical behavior ($\rho \sim T$) is shown for reference.

SOURCE: Data selectively extracted from various sources, including sections in *Metals Handbook*, 10th ed., 2 and 3. Metals Park, Ohio: ASM, 1991.

as depicted for copper in Figure 2.8. As the temperature decreases, typically below ~100 K for many metals, our simple and gross assumption that all the atoms are vibrating with a constant frequency fails. Indeed, the number of atoms that are vibrating with sufficient energy to scatter the conduction electrons starts to decrease rapidly with decreasing temperature, so the resistivity due to scattering from thermal vibrations becomes more strongly temperature dependent. The mean free time $\tau = 1/SuN_s$ becomes longer and strongly temperature dependent, leading to a smaller resistivity than the $\rho \propto T$ behavior. A full theoretical analysis, which is beyond the scope of this chapter, shows that $\rho \propto T^5$. Thus, at the lowest temperature, from Matthiessen's rule, the resistivity becomes $\rho = DT^5 + \rho_R$, where D is a constant. Since the slope of ρ versus T is $d\rho/dT = 5DT^4$, which tends to zero as T becomes small, we have ρ curving

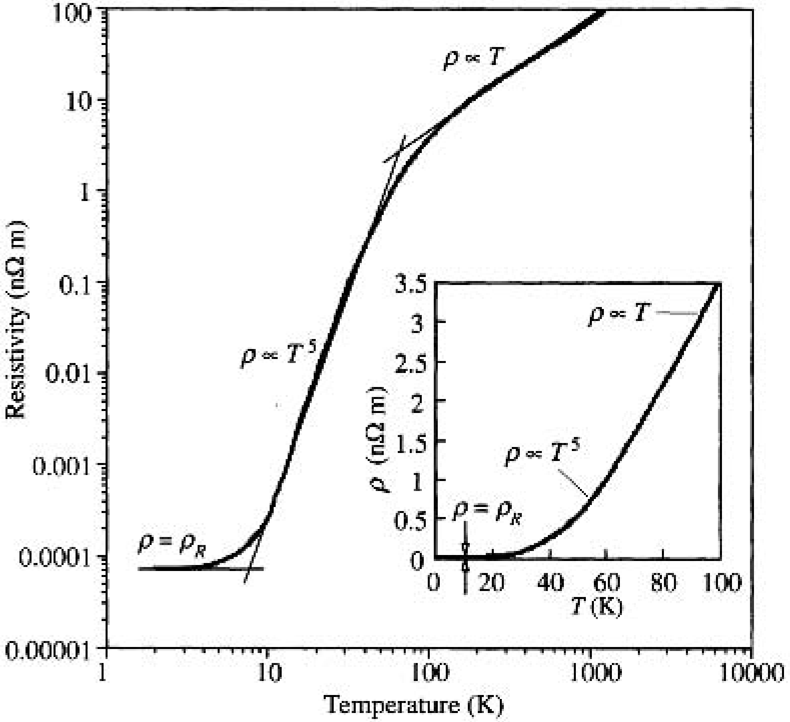


Figure 2.8 The resistivity of copper from lowest to highest temperatures (near melting temperature, 1358 K) on a log-log plot.

Above about 100 K, $\rho \propto T$, whereas at low temperatures, $\rho \propto T^5$, and at the lowest temperatures ρ approaches the residual resistivity ρ_R. The inset shows the ρ vs. T behavior below 100 K on a linear plot. (ρ_R is too small on this scale.)

toward ρ_R as T decreases toward 0 K. This is borne out by experiments, as shown in Figure 2.8 for copper. Therefore, at the lowest temperatures of interest, the resistivity is limited by scattering from impurities and crystal defects.[4]

MATTHIESSEN'S RULE Explain the typical resistivity versus temperature behavior of annealed and cold-worked (deformed) copper containing various amounts of Ni as shown in Figure 2.9.

EXAMPLE 2.7

SOLUTION

When small amounts of nickel are added to copper, the resistivity increases by virtue of Matthiessen's rule, $\rho = \rho_T + \rho_R + \rho_I$, where ρ_T is the resistivity due to scattering from thermal vibrations; ρ_R is the residual resistivity of the copper crystal due to scattering from crystal defects, dislocations, trace impurities, etc.; and ρ_I is the resistivity arising from Ni addition

---

[4] At sufficiently low temperatures (typically, below 10–20 K for many metals and below ~135 K for certain ceramics) certain materials exhibit superconductivity in which the resistivity vanishes ($\rho = 0$), even in the presence of impurities and crystal defects. Superconductivity and its quantum mechanical origin will be explained in Chapter 8.

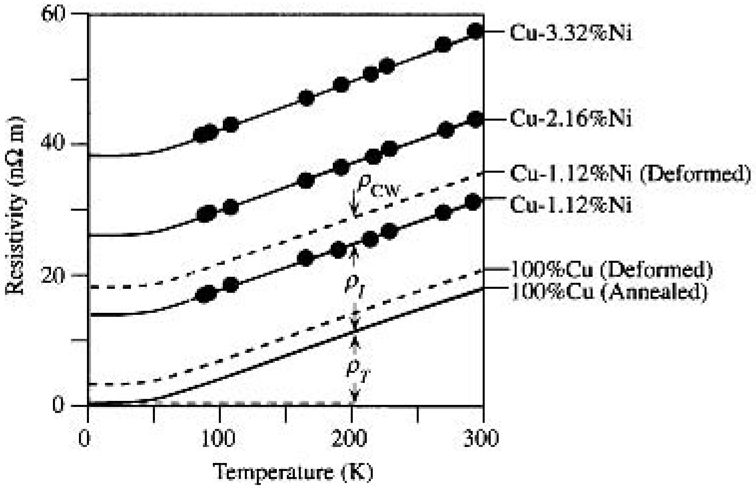


Figure 2.9 Typical temperature dependence of the resistivity of annealed and cold-worked (deformed) copper containing various amounts of Ni in atomic percentage.

SOURCE: Data adapted from J.O. Linde, *Ann Pkysik*, 5, 219 (Germany, 1932).

alone (scattering from Ni impurity regions). Since ρ_I is temperature independent, for small amounts of Ni addition, ρ_I will simply shift up the ρ versus T curve for copper, by an amount proportional to the Ni content, $\rho_I \propto N_{Ni}$, where N_{Ni} is the Ni impurity concentration. This is apparent in Figure 2.9, where the resistivity of Cu–2.16% Ni is almost twice that of Cu–1.12% Ni. Cold working (CW) or deforming a metal results in a higher concentration of dislocations and therefore increases the residual resistivity ρ_R by ρ_{CW}. Thus, cold-worked samples have a resistivity curve that is shifted up by an additional amount ρ_{CW} that depends on the extent of cold working.

EXAMPLE 2.8 **TEMPERATURE COEFFICIENT OF RESISTIVITY α AND RESISTIVITY INDEX *n*** If α_0 is the temperature coefficient of resistivity (TCR) at temperature T_0 and the resistivity obeys the equation

$$\rho = \rho_0 \left[\frac{T}{T_0}\right]^n$$

show that

$$\alpha_0 = \frac{n}{T_0}\left[\frac{T}{T_0}\right]^{n-1}$$

What is your conclusion?

Experiments indicate that $n = 1.2$ for W. What is its α_0 at 20 °C? Given that, experimentally, $\alpha_0 = 0.00393$ for Cu at 20 °C, what is n?

SOLUTION

Since the resistivity obeys $\rho = \rho_0(T/T_0)^n$, we substitute this equation into the definition of TCR,

$$\alpha_0 = \frac{1}{\rho_0}\left[\frac{d\rho}{dT}\right] = \frac{n}{T_0}\left[\frac{T}{T_0}\right]^{n-1}$$

It is clear that, in general, α_0 depends on the temperature T, as well as on the reference temperature T_0. The TCR is only independent of T when $n = 1$.

At $T = T_0$, we have

$$\frac{\alpha_0 T_0}{n} = 1 \qquad \text{or} \qquad n = \alpha_0 T_0$$

For W, $n = 1.2$, so at $T = T_0 = 293$ K, we have $\alpha_{293\text{ K}} = 0.0041$, which agrees reasonably well with $\alpha_{293\text{ K}} = 0.0045$, frequently found in data books.

For Cu, $\alpha_{293\text{ K}} = 0.00393$, so that $n = 1.15$, which agrees with the experimental value of n.

TCR AT DIFFERENT REFERENCE TEMPERATURES If α_1 is the temperature coefficient of resistivity (TCR) at temperature T_1 and α_0 is the TCR at T_0, show that

EXAMPLE 2.9

$$\alpha_1 = \frac{\alpha_0}{1 + \alpha_0(T_1 - T_0)}$$

SOLUTION

Consider the resistivity at temperature T in terms of α_0 and α_1:

$$\rho = \rho_0[1 + \alpha_0(T - T_0)] \qquad \text{and} \qquad \rho = \rho_1[1 + \alpha_1(T - T_1)]$$

These equations are expected to hold at any temperature T, so the first and second equations at T_1 and T_0, respectively, give

$$\rho_1 = \rho_0[1 + \alpha_0(T_1 - T_0)] \qquad \text{and} \qquad \rho_0 = \rho_1[1 + \alpha_1(T_0 - T_1)]$$

These two equations can be readily solved to eliminate ρ_0 and ρ_1 to obtain

$$\alpha_1 = \frac{\alpha_0}{1 + \alpha_0(T_1 - T_0)}$$

TEMPERATURE OF THE FILAMENT OF A LIGHT BULB

EXAMPLE 2.10

a. Consider a 40 W, 120 V incandescent light bulb. The tungsten filament is 0.381 m long and has a diameter of 33 μm. Its resistivity at room temperature is 5.51×10^{-8} Ω m. Given that the resistivity of the tungsten filament varies at $T^{1.2}$, estimate the temperature of the bulb when it is operated at the rated voltage, that is, when it is lit directly from a power outlet, as shown schematically in Figure 2.10. Note that the bulb dissipates 40 W at 120 V.

b. Assume that the electrical power dissipated in the tungsten wire is radiated from the surface of the filament. The radiated electromagnetic power at the absolute temperature T can

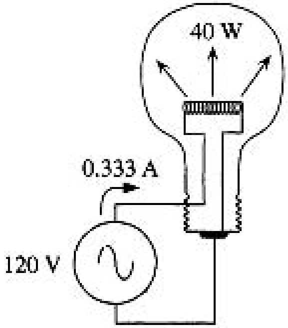


Figure 2.10 Power radiated from a light bulb is equal to the electrical power dissipated in the filament.

be described by **Stefan's law,** as follows:

$$P_{\text{radiated}} = \epsilon\sigma_S A(T^4 - T_0^4)$$

where σ_S is Stefan's constant (5.67×10^{-8} W m^{-2} K^{-4}), ϵ is the emissivity of the surface (0.35 for tungsten), A is the surface area of the tungsten filament, and T_0 is the room temperature (293 K). For $T^4 \gg T_0^4$, the equation becomes

$$P_{\text{radiated}} = \epsilon\sigma_S A T^4$$

Assuming that all the electrical power is radiated as electromagnetic waves from the surface, estimate the temperature of the filament and compare it with your answer in part (a).

SOLUTION

a. When the bulb is operating at 120 V, it is dissipating 40 W, which means that the current is

$$I = \frac{P}{V} = \frac{40 \text{ W}}{120 \text{ V}} = 0.333 \text{ A}$$

The resistance of the filament at the operating temperature T must be

$$R = \frac{V}{I} = \frac{120}{0.333} = 360\ \Omega$$

Since $R = \rho L/A$, the resistivity of tungsten at the operating temperature T must be

$$\rho(T) = \frac{R(\pi D^2/4)}{L} = \frac{360\ \Omega\,\pi(33 \times 10^{-6}\text{ m})^2}{4(0.381\text{ m})} = 8.08 \times 10^{-7}\ \Omega\text{ m}$$

But, $\rho(T) = \rho_0(T/T_0)^{1.2}$, so that

$$T = T_0\left(\frac{80.8 \times 10^{-8}}{5.51 \times 10^{-8}}\right)^{1/1.2}$$

$$= 2746 \text{ K} \qquad \text{or} \qquad 2473\ ^\circ\text{C} \qquad \text{(melting temperature of W is about 3680, K)}$$

b. To calculate T from the radiation law, we note that $T = [P_{\text{radiated}}/\epsilon\sigma_S A]^{1/4}$.

The surface area is

$$A = L(\pi D) = (0.381)(\pi 33 \times 10^{-6}) = 3.95 \times 10^{-5}\text{ m}^2$$

Then,

$$T = \left[\frac{P_{\text{radiated}}}{\epsilon\sigma_S A}\right]^{1/4} = \left[\frac{40 \text{ W}}{(0.35)(5.67 \times 10^{-8}\text{ W m}^{-2}\text{ K}^{-4})(3.95 \times 10^{-5}\text{ m}^2)}\right]^{1/4}$$

$$= [5.103 \times 10^{13}]^{1/4} = 2673 \text{ K} \qquad \text{or} \qquad 2400\ ^\circ\text{C}$$

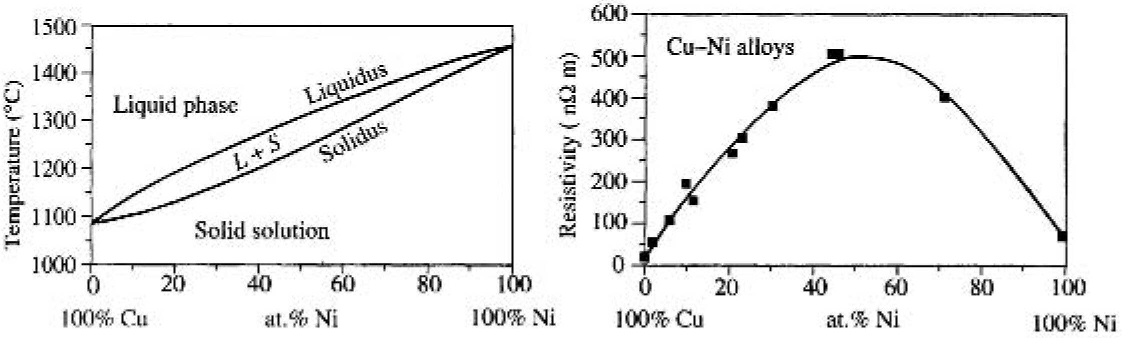
The difference between the two methods is less than 3 percent.

2.3.2 SOLID SOLUTIONS AND NORDHEIM'S RULE

In an isomorphous alloy of two metals, that is, a binary alloy that forms a solid solution, we would expect Equation 2.15 to apply, with the temperature-independent impurity contribution ρ_I increasing with the concentration of solute atoms. This means that as the alloy concentration increases, the resistivity ρ increases and becomes less temperature

Table 2.2 The effect of alloying on the resistivity

Material	Resistivity at 20 °C (nΩ m)	α at 20 °C (1/K)
Nickel	69	0.006
Chrome	129	0.003
Nichrome	1120	0.0003



(a) Phase diagram of the Cu–Ni alloy system. Above the liquidus line only the liquid phase exists. In the *L* + *S* region, the liquid (*L*) and solid (*S*) phases coexist whereas below the solidus line, only the solid phase (a solid solution) exists.

(b) The resistivity of the Cu–Ni alloy as a function of Ni content (at.%) at room temperature.

Figure 2.11 The Cu–Ni alloy system.

SOURCE: Data extracted from *Metals Handbook*, 10th ed., 2 and 3, Metals Park, Ohio: ASM, 1991, and M. Hansen and K. Anderko, *Constitution of Binary Alloys*, New York: McGraw-Hill, 1958.

dependent as ρ_I overwhelms ρ_T, leading to $\alpha \ll 1/273$. This is the advantage of alloys in resistive components. Table 2.2 shows that when 80% nickel is alloyed with 20% chromium, the resistivity of Ni increases almost 16 times. In fact, the alloy is called **nichrome** and is widely used as a heater wire in household appliances and industrial furnaces.

As a further example of the resistivity of a solid solution, consider the copper– nickel alloy. The phase diagram for this alloy system is shown in Figure 2.11a. It is clear that the alloy forms a one-phase solid solution for all compositions. Both Cu and Ni have the same FCC crystal structure, and since the Cu atom is only slightly larger than the Ni atom by about ~3 percent (easily checked on the Periodic Table), the Cu–Ni alloy will therefore still be FCC, but with Cu and Ni atoms randomly mixed, resulting in a solid solution. When Ni is added to copper, the impurity resistivity ρ_I in Equation 2.15 will increase with the Ni concentration. Experimental results for this alloy system are shown in Figure 2.11b. It should be apparent that when we reach 100% Ni, we again have a pure metal whose resistivity must be small. Therefore, ρ versus Ni concentration must pass through a maximum, which for the Cu–Ni alloy seems to be at around ~50% Ni.

There are other binary solid solutions that reflect similar behavior to that depicted in Figure 2.11, such as Cu–Au, Ag–Au, Pt–Pd, Cu–Pd, to name a few. Quite often, the use of an alloy for a particular application is necessitated by the mechanical properties, rather than the desired electrical resistivity alone. For example, brass, which is 70% Cu–30% Zn in solid solution, has a higher strength compared to pure copper; as such, it is a suitable metal for the prongs of an electrical plug.

An important semiempirical equation that can be used to predict the resistivity of an alloy is **Nordheim's rule** which relates the impurity resistivity ρ_I to the atomic fraction X of solute atoms in a solid solution, as follows:

Nordheim's rule for solid solutions

$$\rho_I = CX(1 - X) \tag{2.21}$$

where C is the constant termed the **Nordheim coefficient,** which represents the effectiveness of the solute atom in increasing the resistivity. Nordheim's rule assumes that the solid solution has the solute atoms randomly distributed in the lattice, and these random distributions of impurities cause the electrons to become scattered as they whiz around the crystal. For sufficiently small amounts of impurity, experiments show that the increase in the resistivity ρ_I is nearly always simply proportional to the impurity concentration X, that is, $\rho_I \propto X$, which explains the initial approximately equal increments of rise in the resistivity of copper with 1.11% Ni and 2.16% Ni additions as shown in Figure 2.9. For dilute solutions, Nordheim's rule predicts the same linear behavior, that is, $\rho_I = CX$ for $X \ll 1$.

Table 2.3 lists some typical Nordheim coefficients for various additions to copper and gold. The value of the Nordheim coefficient depends on the type of solute and the solvent. A solute atom that is drastically different in size to the solvent atom will result in a bigger increase in ρ_I and will therefore lead to a larger C. An important assumption

Table 2.3 Nordheim coefficient C (at 20 °C) for dilute alloys obtained from $\rho_I = CX$ and $X < 1$ at.%*

Solute in Solvent (element in matrix)	C (nΩ m)	Maximum Solubility at 25 °C (at.%)
Au in Cu matrix	5500	100
Mn in Cu matrix	2900	24
Ni in Cu matrix	1200	100
Sn in Cu matrix	2900	0.6
Zn in Cu matrix	300	30
Cu in Au matrix	450	100
Mn in Au matrix	2410	25
Ni in Au matrix	790	100
Sn in Au matrix	3360	5
Zn in Au matrix	950	15

*NOTE: For many isomorphous alloys C may be different at higher concentrations; that is, it may depend on the composition of the alloy.

SOURCES: D.G. Fink and D. Christiansen, eds., *Electronics Engineers' Handbook*, 2nd ed., New York, McGraw-Hill, 1982. J. K. Stanley, *Electrical and Magnetic Properties of Metals*, Metals Park, OH, American Society for Metals, 1963. Solubility data from M. Hansen and K. Anderko, *Constitution of Binary Alloys*, 2nd ed., New York, McGraw-Hill, 1985.

in Nordheim's rule in Equation 2.21 is that the alloying does not significantly vary the number of conduction electrons per atom in the alloy. Although this will be true for alloys with the same valency, that is, from the same column in the Periodic Table (*e.g.*, Cu–Au, Ag–Au), it will not be true for alloys of different valency, such as Cu and Zn. In pure copper, there is just one conduction electron per atom, whereas each Zn atom can donate two conduction electrons. As the Zn content in brass is increased, more conduction electrons become available per atom. Consequently, the resistivity predicted by Equation 2.21 at high Zn contents is greater than the actual value because C refers to dilute alloys. To get the correct resistivity from Equation 2.21 we have to lower C, which is equivalent to using an effective Nordheim coefficient C_{eff} that decreases as the Zn content increases. In other cases, for example, in Cu–Ni alloys, we have to increase C at high Ni concentrations to account for additional electron scattering mechanisms that develop with Ni addition. Nonetheless, the Nordheim rule is still useful for predicting the resistivities of dilute alloys, particularly in the low-concentration region.

With Nordheim's rule in Equation 2.21, the resistivity of an alloy of composition X is

Combined Matthiessen and Nordheim rules

$$\rho = \rho_{\text{matrix}} + CX(1 - X) \qquad [2.22]$$

where $\rho_{\text{matrix}} = \rho_T + \rho_R$ is the resistivity of the matrix due to scattering from thermal vibrations and from other defects, in the absence of alloying elements. To reiterate, the value of C depends on the alloying element and the matrix. For example, C for gold in copper would be different than C for copper in gold, as shown in Table 2.3.

In solid solutions, at some concentrations of certain binary alloys, such as 75% Cu–25% Au and 50% Cu–50% Au, the annealed solid has an orderly structure; that is, the Cu and Au atoms are not randomly mixed, but occupy regular sites. In fact, these compositions can be viewed as pure compound—like the solids Cu_3Au and CuAu. The resistivities of Cu_3Au and CuAu will therefore be less than the same composition random alloy that has been quenched from the melt. As a consequence, the resistivity ρ versus composition X curve does not follow the dashed parabolic curve throughout; rather, it exhibits sharp falls at these special compositions, as illustrated in Figure 2.12.

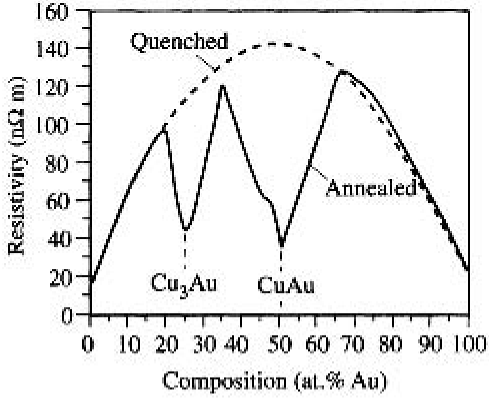


Figure 2.12 Electrical resistivity vs. composition at room temperature in Cu–Au alloys.

The quenched sample (dashed curve) is obtained by quenching the liquid, and the Cu and Au atoms are randomly mixed. The resistivity obeys the Nordheim rule. When the quenched sample is annealed or the liquid is slowly cooled (solid curve), certain compositions (Cu_3Au and CuAu) result in an ordered crystalline structure in which the Cu and Au atoms are positioned in an ordered fashion in the crystal and the scattering effect is reduced.

EXAMPLE 2.11

NORDHEIM'S RULE The alloy 90 wt.% Au–10 wt.% Cu is sometimes used in low-voltage dc electrical contacts, because pure gold is mechanically soft and the addition of copper increases the hardness of the metal without sacrificing the corrosion resistance. Predict the resistivity of the alloy and compare it with the experimental value of 108 nΩ m.

SOLUTION

We apply Equation 2.22, $\rho(X) = \rho_{\text{Au}} + CX(1 - X)$ but with 10 wt.% Cu converted to the atomic fraction for X. If w is the weight fraction of Cu, $w = 0.1$, and if M_{Au} and M_{Cu} are the atomic masses of Au and Cu, then the atomic fraction X of Cu is given by (see Example 1.2).

$$X = \frac{w/M_{\text{Cu}}}{w/M_{\text{Cu}} + (1 - w)/M_{\text{Au}}} = \frac{0.1/63.55}{(0.1/63.55) + (0.90/197)} = 0.256$$

Given that $\rho_{\text{Au}} = 22.8$ nΩ m and $C = 450$ nΩ m,

$$\rho = \rho_{\text{Au}} + CX(1 - X) = (22.8\ \text{n}\Omega\ \text{m}) + (450\ \text{n}\Omega\ \text{m})(0.256)(1 - 0.256)$$
$$= 108.5\ \text{n}\Omega\ \text{m}$$

This value is only 0.5% different from the experimental value.

EXAMPLE 2.12

RESISTIVITY DUE TO IMPURITIES The mean speed of conduction electrons in copper is about 1.5×10^6 m s^{-1}. Its room temperature resistivity is 17 nΩ m, and the atomic concentration N_{at} in the crystal is 8.5×10^{22} cm^{-3}. Suppose that we add 1 at.% Au to form a solid solution. What is the resistivity of the alloy, the effective mean free path, and the mean free path due to collisions with Au atoms only?

SOLUTION

According to Table 2.3, the Nordheim coefficient C of Au in Cu is 5500 nΩ m. With $X = 0.01$ (1 at.%), the overall resistivity from Equation 2.22 is

$$\rho = \rho_{\text{matrix}} + CX(1 - X) = 17\ \text{n}\Omega\ \text{m} + (5500\ \text{n}\Omega\ \text{m})(0.01)(1 - 0.01)$$
$$= 17\ \text{n}\Omega\ \text{m} + 54.45\ \text{n}\Omega\ \text{m} = 71.45\ \text{n}\Omega\ \text{m}$$

Suppose that ℓ is the overall or effective mean free path and τ is the effective mean free time between scattering events (includes both scattering from lattice vibrations and impurities). Since $\ell = u\tau$, and the effective drift mobility $\mu_d = e\tau/m_e$, the expression for the conductivity becomes

Conductivity and mean free path

$$\sigma = en\mu_d = \frac{e^2 n\tau}{m_e} = \frac{e^2 n\ell}{m_e u}$$

We can now calculate the effective mean free path ℓ in the alloy given that copper has a valency of I and the electron concentration $n = N_{\text{at}}$,

$$\frac{1}{71.5 \times 10^{-9}\ \Omega\ \text{m}} = \frac{(1.6 \times 10^{-19}\text{C})^2(8.5 \times 10^{28}\ \text{m}^{-3})\ell}{(9.1 \times 10^{-31}\text{kg})(1.5 \times 10^6\ \text{m s}^{-1})}$$

which gives $\ell = 8.8$ nm. We can repeat the calculation for pure copper using $\sigma = 1/\rho_{\text{matrix}} = 1/(17 \times 10^{-9}\ \Omega\ \text{m})$ to find $\ell_{\text{Cu}} = 37$ nm. The mean free path is reduced approximately by 4 times by adding only 1 at.% Au. The mean free path ℓ_I due to scattering from impurities only can be found from Equation 2.13 multiplied through by $1/u$, or by using

Matthiessen's rule in Equation 2.14:

$$\frac{1}{\ell} = \frac{1}{\ell_{\text{Cu}}} + \frac{1}{\ell_I}$$

Substituting $\ell_{\text{Cu}} = 37$ nm and $\ell = 8.8$ nm, we find $\ell_I = 11.5$ nm.

We can take these calculations one step further. If N_I is the impurity concentration in the alloy, then $N_I = 0.01N_{\text{at}} = 0.01(8.5 \times 10^{28}\ \text{m}^{-3}) = 8.5 \times 10^{26}\ \text{m}^{-3}$. The mean separation d_I between the impurities can be estimated roughly from $d_I \approx 1/N_I^{1/3}$, which gives $d_I \approx 1.0$ nm. It is clear that not all Au atoms can be involved in scattering the electrons since ℓ_I is much longer than d_I. (Another way to look at it is to say that it takes more than just one collision with an impurity to randomize the velocity of the electron.)

2.4 RESISTIVITY OF MIXTURES AND POROUS MATERIALS

2.4.1 HETEROGENEOUS MIXTURES

Nordheim's rule only applies to solid solutions that are single-phase solids. In other words, it is valid for homogeneous mixtures in which the atoms are mixed at the atomic level throughout the solid, as in the Cu–Ni alloy. The classic problem of determining the effective resistivity of a multiphase solid is closely related to the evaluation of the effective dielectric constant, effective thermal conductivity, effective elastic modulus, effective Poisson's ratio, etc., for a variety of mixtures, including such composite materials as fiberglass. Indeed, many of the mixture rules are identical.

Consider a material with two distinct phases α and β, which are stacked in layers as illustrated in Figure 2.13a. Let us evaluate the effective resistivity for current flow

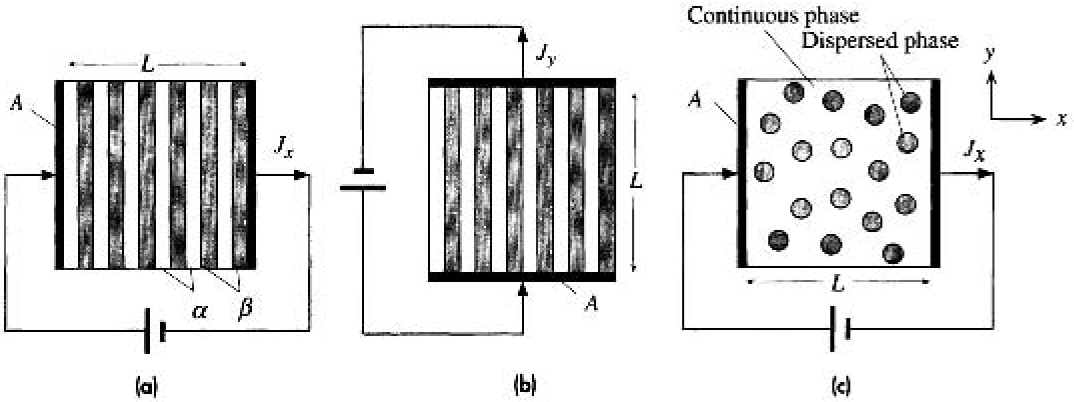


Figure 2.13 The effective resistivity of a material with a layered structure.
(a) Along a direction perpendicular to the layers.
(b) Along a direction parallel to the plane of the layers.
(c) Materials with a dispersed phase in a continuous matrix.

in the x direction. Since the layers are in series, the effective resistance R_{eff} for the whole material is

Effective resistance

$$R_{\text{eff}} = \frac{L_\alpha \rho_\alpha}{A} + \frac{L_\beta \rho_\beta}{A} \tag{2.23}$$

where L_α is the total length (thickness) of the α-phase layers, and L_β is the total length of the β-phase layers, $L_\alpha + L_\beta = L$ is the length of the sample, and A is the cross-sectional area. Let χ_α and χ_β be the volume fractions of the α and β phases. The effective resistance is defined by

$$R_{\text{eff}} = \frac{L\rho_{\text{eff}}}{A}$$

where ρ_{eff} is **the effective resistivity.** Using $\chi_\alpha = L_\alpha/L$ and $\chi_\beta = L_\beta/L$ in Equation 2.23, we find

Resistivity–mixture rule

$$\rho_{\text{eff}} = \chi_\alpha \rho_\alpha + \chi_\beta \rho_\beta \tag{2.24}$$

which is called the **resistivity–mixture rule** (or the **series rule of mixtures**).

If we are interested in the effective resistivity in the y direction, as shown in Figure 2.13b, obviously the α and β layers are in parallel, so an effective conductivity could be calculated in the same way as we did for the series case to find the **parallel rule of mixtures,** that is,

Conductivity–mixture rule

$$\sigma_{\text{eff}} = \chi_\alpha \sigma_\alpha + \chi_\beta \sigma_\beta \tag{2.25}$$

where σ is the electrical conductivity of those phases identified by the subscript. Notice that the parallel rule uses the conductivity, and the series rule uses the resistivity. Equation 2.25 is often referred to as the **conductivity–mixture rule.**

Although these two rules refer to special cases, in general, for a random mixture of phase α and phase β, we would not expect either equation to apply rigorously. When the resistivities of two randomly mixed phases are not markedly different, the series mixture rule can be applied at least approximately, as we will show in Example 2.13.

However, if the resistivity of one phase is appreciably different than the other, there are two semiempirical rules that are quite useful in materials engineering.[5] Consider a heterogeneous material that has a dispersed phase (labeled d), in the form of particles, in a continuous phase (labeled c) that acts as a matrix, as depicted in Figure 2.13c. Assume that ρ_c and ρ_d are the resistivities of the continuous and dispersed phases, and χ_c and χ_d are their volume fractions. If the dispersed phase is much more resistive with respect to the matrix, that is, $\rho_d > 10\rho_c$, then

Mixture rule

$$\rho_{\text{eff}} = \rho_c \frac{\left(1 + \frac{1}{2}\chi_d\right)}{(1 - \chi_d)} \qquad (\rho_d > 10\rho_c) \tag{2.26}$$

---

[5] Over the years, the task of predicting the resistivity of a mixture has challenged many theorists and experimentalists, including Lord Rayleigh who, in 1892, published an excellent exposition on the subject in the Philosophical Magazine. An extensive treatment of mixtures can be found in a paper by J. A. Reynolds and J. M. Hough published in 1957 (Proceedings of the Physical Society, 70, no. 769, London), which contains nearly all the mixture rules for the resistivity.

On the other hand, if $\rho_d < (\rho_c/10)$, then

$$\rho_{\text{eff}} = \rho_c \frac{(1 - \chi_d)}{(1 + 2\chi_d)} \qquad (\rho_d < 0.1\rho_c) \qquad [2.27]$$

Mixture rule

We therefore have at least four mixture rules at our disposal, the uses of which depend on the mixture geometry and the resistivities of the various phases. The problem is identifying which one to use for a given material, which in turn requires a knowledge of the microstructure and properties of the constituents. It should be emphasized that, at best, Equations 2.24 to 2.27 provide only a reasonable estimate of the effective resistivity of the mixture.[6]

Equations 2.26 and 2.27 are simplified special cases of a more general mixture rule due to Reynolds and Hough (1957). Consider a mixture that consists of a continuous conducting phase with a conductivity σ_c that has dispersed spheres of another phase of conductivity σ_d and of volume fraction χ, similar to Figure 2.13c. The effective conductivity of the mixture is given by

Reynolds and Hough rule for mixture of dispersed phases

$$\frac{\sigma - \sigma_c}{\sigma + 2\sigma_c} = \chi \frac{\sigma_d - \sigma_c}{\sigma_d + 2\sigma_c} \qquad [2.28]$$

It is assumed that the spheres are randomly dispersed in the material. It is left as an exercise to show that if $\sigma_d \ll \sigma_c$, then Equation 2.28 reduces to Equation 2.26. A good application would be the calculation of the effective resistivity of porous carbon electrodes, which can be 50–100 percent higher than the resistivity of bulk polycrystalline carbon (graphite). If, on the other hand, $\sigma_d \gg \sigma_c$, the dispersed phase is very conducting, for example, silver particles mixed into a graphite paste to increase the conductivity of the paste, then Equation 2.28 reduces to Equation 2.27. The usefulness of Equation 2.28 cannot be underestimated inasmuch as there are many types of materials in engineering that are mixtures of one type or another.

EXAMPLE 2.13

THE RESISTIVITY-MIXTURE RULE Consider a two-phase alloy consisting of phase α and phase β randomly mixed as shown in Figure 2.14a. The solid consists of a random mixture of two types of resistivities, ρ_α of α and ρ_β of β. We can divide the solid into a bundle of N parallel fibers of length L and cross-sectional area A/N, as shown in Figure 2.14b. In this fiber (infinitesimally thin), the α and β phases are in series, so if $\chi_\alpha = V_\alpha/V$ is the volume fraction of phase α and χ_β is that of β, then the total length of all α regions present in the fiber is $\chi_\alpha L$, and the total length of β regions is $\chi_\beta L$. The two resistances are in series, so the fiber resistance is

$$R_{\text{fiber}} = \frac{\rho_\alpha(\chi_\alpha L)}{(A/N)} + \frac{\rho_\beta(\chi_\beta L)}{(A/N)}$$

But the resistance of the solid is made up of N such fibers in parallel, that is,

$$R_{\text{solid}} = \frac{R_{\text{fiber}}}{N} = \frac{\rho_\alpha \chi_\alpha L}{A} + \frac{\rho_\beta \chi_\beta L}{A}$$

---

[6] More accurate mixture rules have been established for various types of mixtures with components possessing widely different properties, which the keen reader can find in P. L. Rossiter, *The Electrical Resistivity of Metals and Alloys* (Cambridge University Press, Cambridge, 1987).

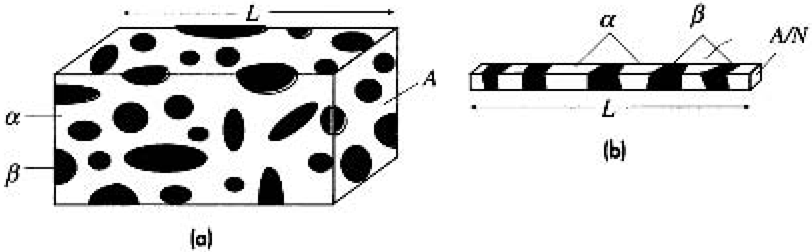


Figure 2.14
(a) A two-phase solid.
(b) A thin fiber cut out from the solid.

By definition, $R_{\text{solid}} = \rho_{\text{eff}} L/A$, where ρ_{eff} is the effective resistivity of the material, so

$$\frac{\rho_{\text{eff}} L}{A} = \frac{\rho_\alpha \chi_\alpha L}{A} + \frac{\rho_\beta \chi_\beta L}{A}$$

Thus, for a two-phase solid, the effective resistivity will be

Resistivity mixture rule

$$\rho_{\text{eff}} = \chi_\alpha \rho_\alpha + \chi_\beta \rho_\beta$$

If the densities of the two phases are not too different, we can use weight fractions instead of volume fractions. The series rule fails when the resistivities of the phases are vastly different. A major (and critical) tacit assumption here is that the current flow lines are all parallel, so that no current crosses from one fiber to another. Only then can we say that the effective resistance is R_{fiber}/N.

EXAMPLE 2.14

A COMPONENT WITH DISPERSED AIR PORES What is the effective resistivity of 95/5 (95% Cu–5% Sn) bronze, which is made from powdered metal containing dispersed pores at 15% (volume percent, vol.%). The resistivity of 95/5 bronze is 1×10^{-7} Ω m.

SOLUTION

Pores are infinitely more resistive ($\rho_d = \infty$) than the bronze matrix, so we use Equation 2.26,

$$\rho_{\text{eff}} = \rho_c \frac{1 + \frac{1}{2}\chi_d}{1 - \chi_d} = (1 \times 10^{-7}\ \Omega\ \text{m}) \frac{1 + \frac{1}{2}(0.15)}{1 - 0.15} = 1.27 \times 10^{-7}\ \Omega\ \text{m}$$

EXAMPLE 2.15

COMBINED NORDHEIM AND MIXTURE RULES Brass is an alloy composed of Cu and Zn. The alloy is a solid solution for Zn content less than 30 wt.%. Consider a brass component made from sintering 90 at.% Cu and 10 at.% Zn brass powder. The component contains dispersed air pores at 15% (vol.%). The Nordheim coefficient C of Zn in Cu is 300 nΩ m, under very dilute conditions. Each Zn atom donates two, whereas each Cu atom of the matrix donates one conduction electron, so that the Cu–Zn alloy has a higher electron concentration than in the Cu crystal itself. Predict the effective resistivity of this brass component.

SOLUTION

We first calculate the resistivity of the alloy without the pores, which forms the continuous phase in the powdered material. The simple Nordheim's rule predicts that

$$\rho_{\text{brass}} = \rho_{\text{copper}} + CX(1 - X) = 17\ \text{n}\Omega\ \text{m} + 300(0.1)(1 - 0.1) = 44\ \text{n}\Omega\ \text{m}$$

The experimental value, about 40 nΩ m, is actually less because Zn has a valency of 2, and when a Zn atom replaces a host Cu atom, it donates two electrons instead of one. We can very roughly adjust the calculated resistivity by noting that a 10 at.% Zn addition increases the conduction electron concentration by 10% and hence reduces the resistivity ρ_{brass} by 10% to 40 nΩ m.

The powdered metal has $\chi_d = 0.15$, which is the volume fraction of the dispersed phase, that is, the air pores, and $\rho_c = \rho_{\text{brass}} = 40$ nΩ m is the resistivity of the continuous matrix. The effective resistivity of the powdered metal is given by

$$\rho_{\text{eff}} = \rho_c \frac{1 + \frac{1}{2}\chi_d}{1 - \chi_d} = (40\text{ n}\Omega\text{ m})\frac{1 + \frac{1}{2}(0.15)}{1 - (0.15)} = 50.6\text{ n}\Omega\text{ m}$$

If we use the simple conductivity mixture rule, ρ_{eff} is 47.1 nΩ m, and it is underestimated.

The *effective* Nordheim coefficient C_{eff} at the composition of interest is about 255 nΩ m, which would give $\rho_{\text{brass}} = \rho_o + C_{\text{eff}}X(1 - X) = 40$ nΩ m. It is left as an exercise to show that the effective number of conduction electrons per atom in the alloy is $1 + X$ so that we must divide the ρ_{brass} calculated above by $(1 + X)$ to obtain the correct resistivity of brass if we use the listed value of C under dilute conditions. (See Question 2.8.)

2.4.2 TWO-PHASE ALLOY (Ag–Ni) RESISTIVITY AND ELECTRICAL CONTACTS

Certain binary alloys, such as Pb–Sn and Cu–Ag, only exhibit a single-phase alloy structure over very small composition ranges. For most compositions, these alloys form a two-phase heterogeneous mixture of phases α and β. A typical phase diagram for such a eutectic binary alloy system is shown in Figure 2.15a, which could be a

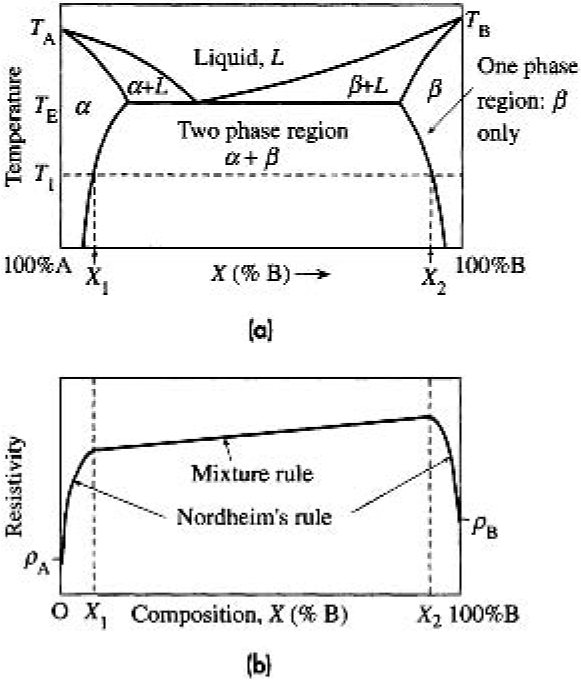


Figure 2.15 Eutectic-forming alloys, e.g., Cu–Ag.
(a) The phase diagram for a binary, eutectic-forming alloy.
(b) The resistivity versus composition for the binary alloy.

schematic scheme for the Cu–Ag system or the Pb–Sn system. The phase diagram identifies the phases existing in the alloy at a given temperature and composition. If the overall composition X is less than X_1, then at T_1, the alloy will consist of phase α only. This phase is Cu rich. When the composition X is between X_1 and X_2, then the alloy will consist of the two phases α and β randomly mixed. The phase α is Cu rich (that is, it has composition X_1) and the phase β is Ag rich (composition X_2). The relative amounts of each phase are determined by the well-known **lever rule,** which means that we can determine the volume fractions of α and β, χ_α and χ_β, as the alloy composition is changed from X_1 to X_2.

For this alloy system, the dependence of the resistivity on the alloy composition is shown in Figure 2.15b. Between O and X_1 (% Ag), the solid is one phase (isomorphous); therefore, in this region, ρ increases with the concentration of Ag by virtue of Nordheim's rule. At X_1, the solubility limit of Ag in Cu is reached, and after X_1, a second phase, which is β rich, is formed. Thus, in the composition range X_1 to X_2, we have a mixture of α and β phases, so ρ is given by Equation 2.24 for mixtures and is therefore less than that for a single-phase alloy of the same composition. Similarly, at the Ag end ($X_2 < X < 100\%$), as Cu is added to Ag, between 100% Ag and the solubility limit at X_2, the resistivity is determined by Nordheim's rule. The expected behavior of the resistivity of an eutectic binary alloy over the whole composition range is therefore as depicted in Figure 2.15b.

Electrical, thermal, and other physical properties make copper the most widely used metallic conductor. For many electrical applications, high-conductivity copper, having extremely low oxygen and other impurity contents, is produced. Although aluminum has a conductivity of only about half that of copper, it is also frequently used as an electrical conductor. On the other hand, silver has a higher conductivity than copper, but its cost prevents its use, except in specialized applications. Switches often have silver contact specifications, though it is likely that the contact metal is actually a silver alloy. In fact, silver has the highest electrical and thermal conductivity and is consequently the natural choice for use in electrical contacts. In the form of alloys with various other metals, it is used extensively in make-and-break switching applications for currents of up to about 600 A. The precious metals, gold, platinum, and palladium, are extremely resistant to corrosion; consequently, in the form of various alloys, particularly with Ag, they are widely used in electrical contacts. For example, Ag–Ni alloys are common electrical contact materials for the switches in many household appliances.

It is frequently necessary to improve the mechanical properties of a metal alloy without significantly impairing its electrical conductivity. Solid-solution alloying improves mechanical strength, but at the expense of conductivity. A compromise must often be found between electrical and mechanical properties. Most often, strength is enhanced by introducing a second phase that does not have such an adverse effect on the conductivity. For example, Ag–Pd alloys form a solid solution such that the resistivity increases appreciably due to Nordheim's rule. The resistivity of Ag–Pd is mainly controlled by the scattering of electrons from Pd atoms randomly mixed in the Ag matrix. In contrast, Ag and Ni form a two-phase alloy, a mixture of Ag-rich and Ni-rich phases. The Ag–Ni alloy is almost as strong as the Ag–Pd alloy, but it has a lower resistivity because the mixture rule volume averages the two resistivities.

2.5 THE HALL EFFECT AND HALL DEVICES

An important phenomenon that we can comfortably explain using the "electron as a particle" concept is the Hall effect, which is illustrated in Figure 2.16. When we apply a magnetic field in a perpendicular direction to the applied field (which is driving the current), we find there is a transverse field in the sample that is perpendicular to the direction of both the applied field $\mathcal{E}_x$ and the magnetic field B_z, that is, in the y direction. Putting a voltmeter across the sample, as in Figure 2.16, gives a voltage reading V_H. The applied field $\mathcal{E}_x$ drives a current J_x in the sample. The electrons move in the $-x$ direction, with a drift velocity v_{dx}. Because of the magnetic field, there is a force (called the **Lorentz force**) acting on each electron and given by $F_y = -ev_{dx}B_z$. The direction of this Lorentz force is the $-y$ direction, which we can show by applying the corkscrew rule, because, in vector notation, the force **F** acting on a charge q moving with a velocity **v** in a magnetic field **B** is given through the vector product

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad [2.29]$$

Lorentz force

All moving charges experience the Lorentz force in Equation 2.29 as shown schematically in Figure 2.17. In our example of a metal in Figure 2.16, this Lorentz force is the $-y$ direction, so it pushes the electrons downward, as a result of which there is a negative charge accumulation near the bottom of the sample and a positive charge near the top of the sample, due to exposed metal ions (*e.g.*, Cu$^+$).

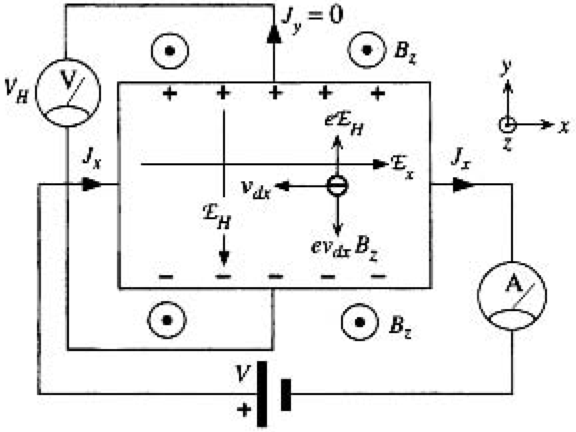


Figure 2.16 Illustration of the Hall effect.
The z direction is out of the plane of the paper. The externally applied magnetic field is along the z direction.

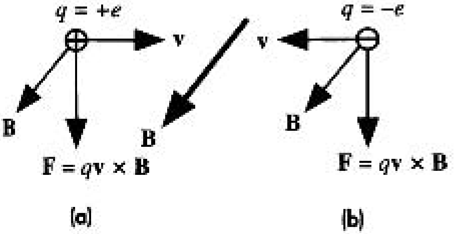


Figure 2.17 A moving charge experiences a Lorentz force in a magnetic field.
(a) A positive charge moving in the x direction experiences a force downward.
(b) A negative charge moving in the −x direction also experiences a force downward.

The accumulation of electrons near the bottom results in an internal electric field $\mathcal{E}_H$ in the $-y$ direction. This is called the **Hall field** and gives rise to a Hall voltage V_H between the top and bottom of the sample. Electron accumulation continues until the increase in $\mathcal{E}_H$ is sufficient to stop the further accumulation of electrons. When this happens, the magnetic-field force $ev_{dx}B_z$ that pushes the electrons down just balances the force $e\mathcal{E}_H$ that prevents further accumulation. Therefore, in the steady state,

$$e\mathcal{E}_H = ev_{dx}B_z$$

However, $J_x = env_{dx}$. Therefore, we can substitute for v_{dx} to obtain $e\mathcal{E}_H = J_xB_z/n$ or

$$\mathcal{E}_H = \left(\frac{1}{en}\right)J_xB_z \qquad [2.30]$$

A useful parameter called the **Hall coefficient** R_H is defined as

Definition of Hall coefficient

$$R_H = \frac{\mathcal{E}_y}{J_xB_z} \qquad [2.31]$$

The quantity R_H measures the resulting Hall field, along y, per unit transverse applied current and magnetic field. The larger R_H, the greater $\mathcal{E}_y$ for a given J_x and B_z. Therefore, R_H is a gauge of the magnitude of the Hall effect. A comparison of Equations 2.30 and 2.31 shows that for metals,

Hall coefficient for electron conduction

$$R_H = -\frac{1}{en} \qquad [2.32]$$

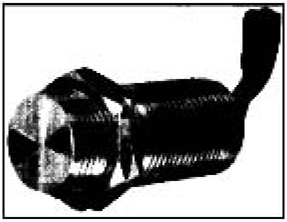
The reason for the negative sign is that $\mathcal{E}_H = -\mathcal{E}_y$, which means that $\mathcal{E}_H$ is in the $-y$ direction.

Inasmuch as R_H depends inversely on the free electron concentration, its value in metals is much less than that in semiconductors. In fact, Hall-effect devices (such as magnetometers) always employ a semiconductor material, simply because the R_H is larger. Table 2.4 lists the Hall coefficients of various metals. Note that this is negative

Table 2.4 Hall coefficient and Hall mobility ($\mu_H = |\sigma R_H|$) of selected metals

Metal	n $[\text{m}^{-3}]$ $(\times 10^{28})$	R_H (Experimental) $[\text{m}^3\,\text{A}^{-1}\,\text{s}^{-1}]$ $(\times 10^{-11})$	$\mu_H = \lvert\sigma R_H\rvert$ $[\text{m}^2\,\text{V}^{-1}\,\text{s}^{-1}]$ $(\times 10^{-4})$
Ag	5.85	−9.0	57
Al	18.06	−3.5	13
Au	5.90	−7.2	31
Be	24.2	+3.4	?
Cu	8.45	−5.5	32
Ga	15.3	−6.3	3.6
In	11.49	−2.4	2.9
Mg	8.60	−9.4	22
Na	2.56	−25	53

SOURCES: Data from various sources, including C. Nording and J. Osterman, *Physics Handbook*, Bromley, England: Chartwell-Bratt Ltd., 1982.



Magnetically operated Hall-effect position sensor as available from Micro Switch.

for most metals, although a few metals exhibit a positive Hall coefficient (see Be in Table 2.4). The reasons for the latter involve the band theory of solids, which we will discuss in Chapter 4.

Since the Hall voltage depends on the product of two quantities, the current density J_x and the transverse applied magnetic field B_z, we see that the effect naturally multiplies two independently variable quantities. Therefore, it provides a means of carrying out a multiplication process. One obvious application is measuring the power dissipated in a load, where the load current and voltage are multiplied. There are many instances when it is necessary to measure magnetic fields, and the Hall effect is ideally suited to such applications. Commercial Hall-effect magnetometers can measure magnetic fields as low as 10 nT, which should be compared to the earth's magnetic field of ~50 μT. Depending on the application, manufacturers use different semiconductors to obtain the desired sensitivity. Hall-effect semiconductor devices are generally inexpensive, small, and reliable. Typical commercial, linear Hall-effect sensor devices are capable of providing a Hall voltage of ~10 mV per mT of applied magnetic field.

The Hall effect is also widely used in magnetically actuated electronic switches. The application of a magnetic field, say from a magnet, results in a Hall voltage that is amplified to trigger an electronic switch. The switches invariably use Si and are readily available from various companies. Hall-effect electronic switches are used as noncontacting keyboard and panel switches that last almost forever, as they have no mechanical contact assembly. Another advantage is that the electrical contact is "bounce" free. There are a variety of interesting applications for Hall-effect switches, ranging from ignition systems, to speed controls, position detectors, alignment controls, brushless dc motor commutators, etc.

EXAMPLE 2.16

HALL-EFFECT WATTMETER The Hall effect can be used to implement a wattmeter to measure electrical power dissipated in a load. The schematic sketch of the Hall-effect wattmeter is shown in Figure 2.18, where the Hall-effect sample is typically a semiconductor material (usually Si). The load current I_L passes through two coils, which are called current coils and are shown as C in Figure 2.18. These coils set up a magnetic field B_z such that $B_z \propto I_L$. The Hall-effect sample is positioned in this field between the coils. The voltage V_L across the load drives a current

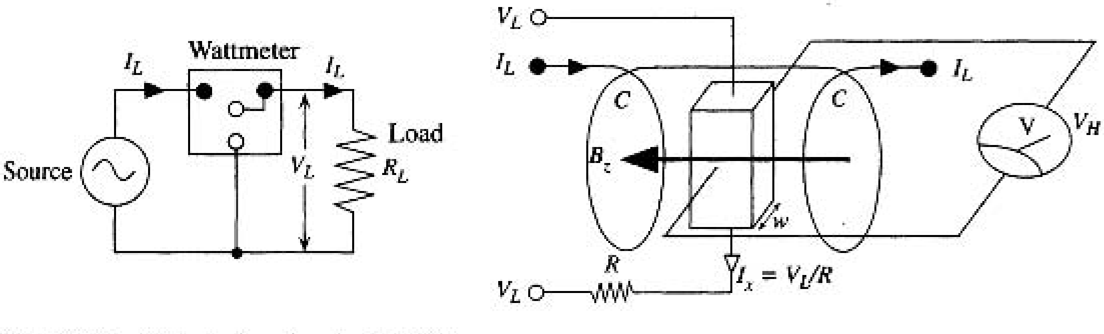


Figure 2.18 Wattmeter based on the Hall effect.

Load voltage and load current have L as subscript; C denotes the current coils for setting up a magnetic field through the Hall-effect sample (semiconductor).

$I_x = V_L/R$ through the sample, where R is a series resistance that is much larger than the resistance of the sample and that of the load. Normally, the current I_x is very small and negligible compared to the load current. If w is the width of the sample, then the measured Hall voltage is

$$V_H = w\mathcal{E}_H = wR_H J_x B_z \propto I_x B_z \propto V_L I_L$$

which is the electrical power dissipated in the load. The voltmeter that measures V_H can now be calibrated to read directly the power dissipated in the load.

EXAMPLE 2.17 **HALL MOBILITY** Show that if R_H is the Hall coefficient and σ is the conductivity of a metal, then the drift mobility of the conduction electrons is given by

$$\mu_d = |\sigma R_H| \qquad [2.33]$$

The Hall coefficient and conductivity of copper at 300 K have been measured to be -0.55×10^{-10} m^3 A^{-1} s^{-1} and 5.9×10^7 Ω^{-1} m^{-1}, respectively. Calculate the drift mobility of electrons in copper.

SOLUTION

Consider the expression for

$$R_H = \frac{-1}{en}$$

Since the conductivity is given by $\sigma = en\mu_d$, we can substitute for en to obtain

$$R_H = \frac{-\mu_d}{\sigma} \qquad \text{or} \qquad \mu_d = -R_H\sigma$$

which is Equation 2.33. The drift mobility can thus be determined from R_H and σ.

The product of σ and R_H is called the **Hall mobility** μ_H. Some values for the Hall mobility of electrons in various metals are listed in Table 2.4. From the expression in Equation 2.33, we get

$$\mu_d = -(-0.55 \times 10^{-10}\ \text{m}^3\ \text{A}^{-1}\ \text{s}^{-1})(5.9 \times 10^7\ \Omega^{-1}\ \text{m}^{-1}) = 3.2 \times 10^{-3}\ \text{m}^2\ \text{V}^{-1}\ \text{s}^{-1}$$

It should be mentioned that Equation 2.33 is an oversimplification.The actual relationship involves a numerical factor that multiplies the right term in Equation 2.33. The factor depends on the charge carrier scattering mechanism that controls the drift mobility.

EXAMPLE 2.18 **CONDUCTION ELECTRON CONCENTRATION FROM THE HALL EFFECT** Using the electron drift mobility from Hall-effect measurements (Table 2.4), calculate the concentration of conduction electrons in copper, and then determine the average number of electrons contributed to the free electron gas per copper atom in the solid.

SOLUTION

The number of conduction electrons is given by $n = \sigma/e\mu_d$. The conductivity of copper is 5.9×10^7 Ω^{-1} m^{-1}, whereas from Table 2.4, the electron drift mobility is 3.2×10^{-3} m^2 V^{-1} s^{-1}. So,

$$n = \frac{(5.9 \times 10^7\ \Omega^{-1}\ \text{m}^{-1})}{[(1.6 \times 10^{-19}\ \text{C})(3.2 \times 10^{-3}\ \text{m}^2\ \text{V}^{-1}\ \text{s}^{-1})]} = 1.15 \times 10^{29}\ \text{m}^{-3}$$

Since the concentration of copper atoms is 8.5×10^{28} m^{-3}, the average number of electrons contributed per atom is $(1.15 \times 10^{29}\ \text{m}^{-3})/(8.5 \times 10^{28}\ \text{m}^{-3}) \approx 1.36$.

2.6 THERMAL CONDUCTION

2.6.1 THERMAL CONDUCTIVITY

Experience tells us that metals are both good electrical and good thermal conductors. We may therefore surmise that the free conduction electrons in a metal must also play a role in heat conduction. Our conjecture is correct for metals, but not for other materials. The transport of heat in a metal is accomplished by the electron gas (conduction electrons), whereas in nonmetals, the conduction is due to lattice vibrations.

When a metal piece is heated at one end, the amplitude of the atomic vibrations, and thus the average kinetic energy of the electrons, in this region increases, as depicted in Figure 2.19. Electrons gain energy from energetic atomic vibrations when the two collide. By virtue of their increased random motion, these energetic electrons then transfer the extra energy to the colder regions by colliding with the atomic vibrations there. Thus, electrons act as "energy carriers."

The thermal conductivity of a material, as its name implies, measures the ease with which heat, that is, thermal energy, can be transported through the medium. Consider the metal rod shown in Figure 2.20, which is heated at one end. Heat will flow from the hot end to the cold end. Experiments show that the rate of heat flow, $Q' = dQ/dt$, through a thin section of thickness δx is proportional to the temperature gradient $\delta T/\delta x$ and the cross-sectional area A, so

$$Q' = -A\kappa\frac{\delta T}{\delta x} \qquad [2.34]$$

Fourier's law of thermal conduction

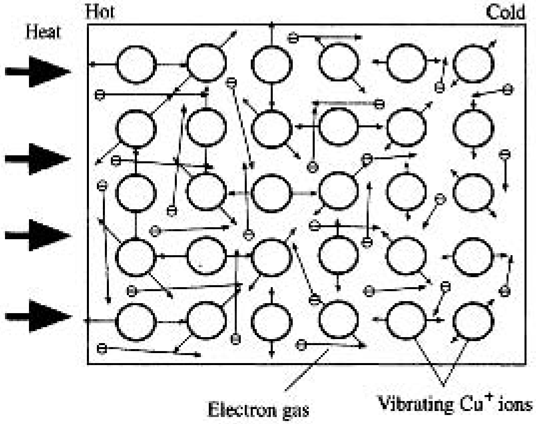


Figure 2.19 Thermal conduction in a metal involves transferring energy from the hot region to the cold region by conduction electrons.

More energetic electrons (shown with longer velocity vectors) from the hotter regions arrive at cooler regions, collide with lattice vibrations, and transfer their energy. Lengths of arrowed lines on atoms represent the magnitudes of atomic vibrations.

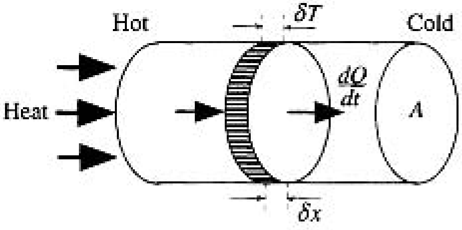


Figure 2.20 Heat flow in a metal rod heated at one end.

Consider the rate of heat flow, dQ/dt, across a thin section δx of the rod. The rate of heat flow is proportional to the temperature gradient $\delta T/\delta x$ and the cross-sectional area A.

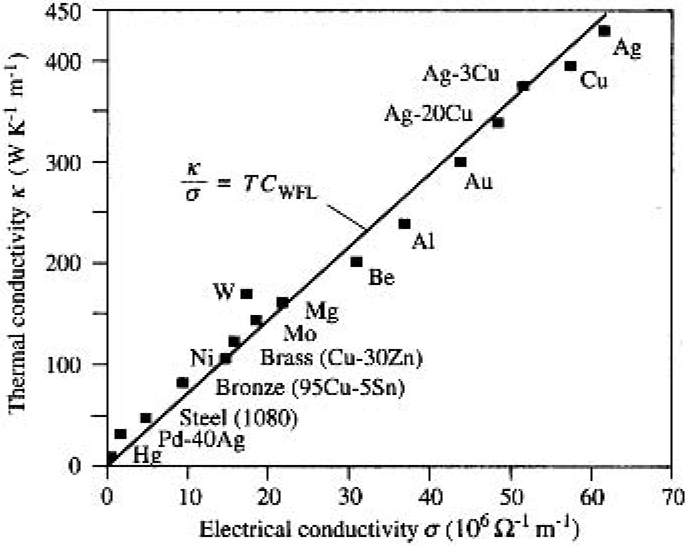


Figure 2.21 Thermal conductivity κ versus electrical conductivity σ for various metals (elements and alloys) at 20 °C.
The solid line represents the WFL law with $C_{WFL} \approx 2.44 \times 10^{8}$ W Ω K^{-2}.

where κ is a material-dependent **constant of proportionality** that we call the **thermal conductivity.** The negative sign indicates that the heat flow direction is that of decreasing temperature. Equation 2.34 is often referred to as **Fourier's law** of heat conduction and is a defining equation for κ. The driving force for the heat flow is the temperature gradient $\delta T/\delta x$. If we compare Equation 2.34 with Ohm's law for the electric current I, we see that

Ohm's law of electrical conduction

$$I = -A\sigma\frac{\delta V}{\delta x} \tag{2.35}$$

which shows that in this case, the driving force is the potential gradient, that is, the electric field.[7] In metals, electrons participate in the processes of charge and heat transport, which are characterized by σ and κ, respectively. Therefore, it is not surprising to find that the two coefficients are related by the **Wiedemann–Franz–Lorenz law,**[8] which is

Wiedemann–Franz–Lorenz law

$$\frac{\kappa}{\sigma T} = C_{WFL} \tag{2.36}$$

where $C_{WFL} = \pi^2 k^2/3e^2 = 2.44 \times 10^{-8}$ W Ω K^{-2} is a constant called the **Lorenz number** (or the Wiedemann–Franz–Lorenz coefficient).

Experiments on a wide variety of metals, ranging from pure metals to various alloys, show that Equation 2.36 is reasonably well obeyed at close to room temperature and above, as illustrated in Figure 2.21. Since the electrical conductivity of pure metals is inversely proportional to the temperature, we can immediately conclude that the thermal conductivity of these metals must be relatively temperature independent at room temperature and above.

---

[7] Recall that $J = \sigma E$ which is equivalent to Equation 2.35.

[8] Historically, Wiedemann and Franz noted in 1853 that κ/σ is the same for all metals at the same temperature. Lorenz in 1881 showed that κ/σ is proportional to the temperature with a proportionality constant that is nearly the same for many metals. The law stated in Equation 2.36 reflects both observations. By the way, Lorenz, who was a Dane, should not be confused with Lorentz, who was Dutch.

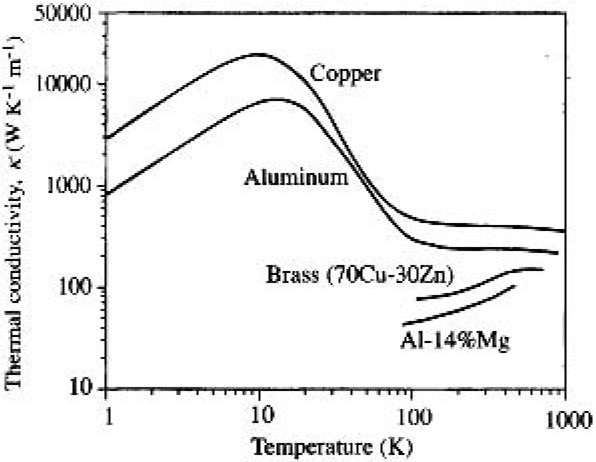


Figure 2.22 Thermal conductivity versus temperature for two pure metals (Cu and Al) and two alloys (brass and Al–14% Mg).

SOURCE: Data extracted from Y. S. Touloukian, *et al.*, *Thermophysical Properties of Matter*, vol. 1: "Thermal Conductivity, Metallic Elements and Alloys," New York: Plenum, 1970.

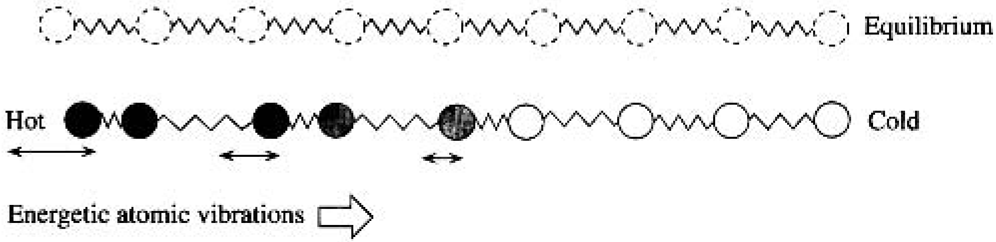


Figure 2.23 Conduction of heat in insulators involves the generation and propagation of atomic vibrations through the bonds that couple the atoms (an intuitive figure).

Figure 2.22 shows the temperature dependence of κ for copper and aluminum down to the lowest temperatures. It can be seen that for these two metals, above ~100 K, the thermal conductivity becomes temperature independent, in agreement with Equation 2.36. Qualitatively, above ~100 K, κ is constant, because heat conduction depends essentially on the rate at which the electron transfers energy from one atomic vibration to another as it collides with them (Figure 2.19). This rate of energy transfer depends on the mean speed of the electron u, which increases only fractionally with the temperature. In fact, the fractionally small increase in u is more than sufficient to carry the energy from one collision to another and thereby excite more energetic lattice vibrations in the colder regions.

Nonmetals do not have any free conduction electrons inside the crystal to transfer thermal energy from hot to cold regions of the material. In nonmetals, the energy transfer involves lattice vibrations, that is, atomic vibrations of the crystal. We know that we can view the atoms and bonds in a crystal as balls connected together through springs as shown for one chain of atoms in Figure 2.23. As we know from the kinetic molecular theory, all the atoms would be vibrating and the average vibrational kinetic energy would be proportional to the temperature. Intuitively, as depicted in Figure 2.23, when we heat one end of a crystal, we set up large-amplitude atomic vibrations at this hot end. The springs *couple* the vibrations to neighboring atoms and thus allow the large-amplitude vibrations to propagate, as a **vibrational wave,** to the cooler regions of the crystal. If we were to grab the left-end atom in Figure 2.23 and vibrate it violently, we would be sending vibrational waves down the ball-spring-ball chain. The efficiency of heat transfer depends not only on the efficiency of coupling between the atoms, and hence on the nature of interatomic bonding, but also on how the vibrational waves propogate in the crystal and how they are scattered by crystal imperfections and by their interactions with other vibrational waves; this topic is discussed in Chapter 4. The stronger the coupling, the greater will be the thermal conductivity, a trend that is intuitive but also borne out by experiments. Diamond has an exceptionally strong covalent bond and also has a very high thermal conductivity; $\kappa \approx 1000$ W m^{-1} K^{-1}. On the other hand, polymers have weak secondary bonding between the polymer chains and their thermal conductivities are very poor; $\kappa < 1$ W m^{-1} K^{-1}.

The thermal conductivity, in general, depends on the temperature. Different classes of materials exhibit different κ values and also different κ versus T behavior. Table 2.5

Table 2.5 Typical thermal conductivities of various classes of materials at 25 °C

Material	κ (W m^{-1} K^{-1})
Pure metal	
Nb	52
Fe	80
Zn	113
W	178
Al	250
Cu	390
Ag	420
Metal alloys	
Stainless steel	12–16
55% Cu–45% Ni	19.5
70% Ni–30% Cu	25
1080 steel	50
Bronze (95% Cu–5% Sn)	80
Brass (63% Cu–37% Zn)	125
Dural (95% Al–4% Cu–1% Mg)	147
Ceramics and glasses	
Glass-borosilicate	0.75
Silica-fused (SiO_2)	1.5
S_3N_4	20
Alumina (Al_2O_3)	30
Sapphire (Al_2O_3)	37
Beryllium (BeO)	260
Diamond	~1000
Polymers	
Polypropylene	0.12
PVC	0.17
Polycarbonate	0.22
Nylon 6.6	0.24
Teflon	0.25
Polyethylene, low density	0.3
Polyethylene, high density	0.5

summarizes κ at room temperature for various classes of materials. Notice how ceramics have a very large range of κ values.

EXAMPLE 2.19

THERMAL CONDUCTIVITY A 95/5 (95% Cu–5% Sn) bronze bearing made of powdered metal contains 15% (vol.%) porosity. Calculate its thermal conductivity at 300 K, given that the electrical conductivity of 95/5 bronze is $10^7\ \Omega^{-1}\ \mathrm{m}^{-1}$.

SOLUTION

Recall that in Example 2.14, we found the electrical resistivity of the same bronze by using the mixture rule in Equation 2.26 in Section 2.4. We can use the same mixture rule again here, but we need the thermal conductivity of 95/5 bronze. From $\kappa/\sigma T = C_{\mathrm{WFL}}$, we have

$$\kappa = \sigma T C_{\mathrm{WFL}} = (1 \times 10^7)(300)(2.44 \times 10^{-8}) = 73.2\ \mathrm{W\ m^{-1}\ K^{-1}}$$

Thus, the effective thermal conductivity is

$$\frac{1}{\kappa_{\mathrm{eff}}} = \frac{1}{\kappa_c}\left[\frac{1+\frac{1}{2}\chi_d}{1-\chi_d}\right] = \frac{1}{(73.2\ \mathrm{W\ m^{-1}\ K^{-1}})}\left[\frac{1+\frac{1}{2}(0.15)}{1-0.15}\right]$$

so that

$$\kappa_{\mathrm{eff}} = 57.9\ \mathrm{W\ m^{-1}\ K^{-1}}$$

2.6.2 THERMAL RESISTANCE

Consider a component of length L that has a temperature difference ΔT between its ends as in Figure 2.24a. The temperature gradient is $\Delta T/L$. Thus, the rate of heat flow, or the **heat current,** is

$$Q' = A\kappa\frac{\Delta T}{L} = \frac{\Delta T}{(L/\kappa A)} \qquad [2.37]$$

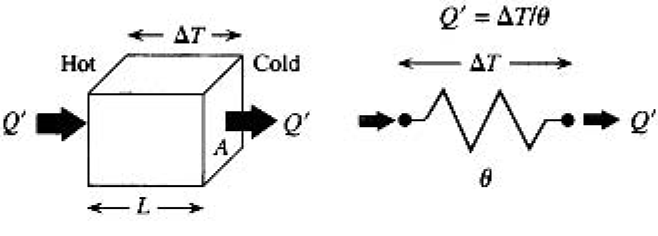
Fourier's law

This should be compared with Ohm's law in electric circuits,

$$I = \frac{\Delta V}{R} = \frac{\Delta V}{(L/\sigma A)} \qquad [2.38]$$

Ohm's law

where ΔV is the voltage difference across a conductor of resistance R, and I is the electric current.



(a) (b)

Figure 2.24 Conduction of heat through a component in (a) can be modeled as a thermal resistance θ shown in (b) where $Q' = \Delta T/\theta$.

In analogy with electrical resistance, we may define **thermal resistance** θ by

Definition of thermal resistance

$$Q' = \frac{\Delta T}{\theta} \qquad [2.39]$$

where, in terms of thermal conductivity,

Thermal resistance

$$\theta = \frac{L}{\kappa A} \qquad [2.40]$$

The rate of heat flow Q' and the temperature difference ΔT correspond to the electric current I and potential difference ΔV, respectively. Thermal resistance is the thermal analog of electrical resistance and its thermal circuit representation is shown in Figure 2.24b.

EXAMPLE 2.20 **THERMAL RESISTANCE** A brass disk of electrical resistivity 50 nΩ m conducts heat from a heat source to a heat sink at a rate of 10 W. If its diameter is 20 mm and its thickness is 30 mm, what is the temperature drop across the disk, neglecting the heat losses from the surface?

SOLUTION

We first determine the thermal conductivity:

$$\kappa = \sigma T C_{\text{WFL}} = (5 \times 10^{-8}\ \Omega\ \text{m})^{-1}(300\ \text{K})(2.44 \times 10^{-8}\ \text{W}\ \Omega\ \text{K}^{-2})$$
$$= 146\ \text{W m}^{-1}\ \text{K}^{-1}$$

The thermal resistance is

$$\theta = \frac{L}{\kappa A} = \frac{(30 \times 10^{-3}\ \text{m})}{\pi(10 \times 10^{-3}\ \text{m})^2(146\ \text{W m}^{-1}\ \text{K}^{-1})} = 0.65\ \text{K W}^{-1}$$

Therefore, the temperature drop is

$$\Delta T = \theta Q' = (0.65\ \text{K W}^{-1})(10\ \text{W}) = 6.5\ \text{K or °C}$$

2.7 ELECTRICAL CONDUCTIVITY OF NONMETALS

All metals are good conductors because they have a very large number of conduction electrons free inside the metal. We should therefore expect solids that do not have metallic bonding to be very poor conductors, indeed insulators. Figure 2.25 shows the range of conductivities exhibited by a variety of solids. Based on typical values of the conductivity, it is possible to empirically classify various materials into conductors, semiconductors, and insulators as in Figure 2.25. It is apparent that nonmetals are not perfect insulators with zero conductivity. There is no well-defined sharp boundary between what we call insulators and semiconductors. Conductors are intimately identified with metals. It is more appropriate to view insulators as **high resistivity** (or **low conductivity**) **materials.** In general terms, current conduction is due to the drift of mobile charge carriers through a solid by the application of

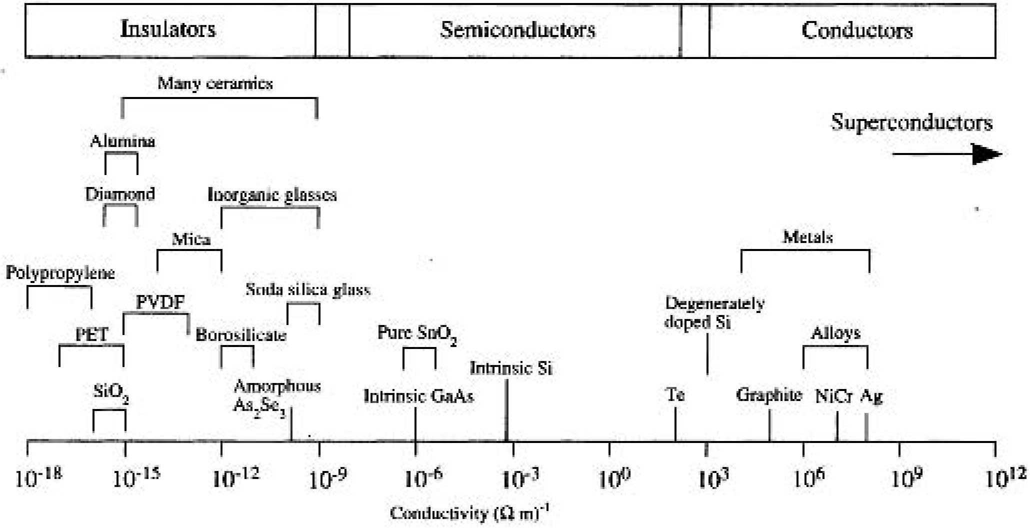


Figure 2.25 Range of conductivities exhibited by various materials.

an electric field. Each of the drifting species of charge carriers contributes to the observed current. In metals, there are only free electrons. In nonmetals there are other types of charge carriers that can drift.

2.7.1 SEMICONDUCTORS

A perfect Si crystal has each Si atom bonded to four neighbors, and each covalent bond has two shared electrons as we had shown in Figure 1.59a. We know from classical physics (the kinetic molecular theory and Boltzmann distribution) that all the atoms in the crystal are executing vibrations with a distribution of energies. As the temperature increases, the distribution spreads to higher energies. Statistically some of the atomic vibrations will be sufficiently energetic to rupture a bond as indicated in Figure 2.26a. This releases an electron from the bond which is *free* to wander inside the crystal. The free electron can drift in the presence of an applied field; it is called a **conduction electron.** As an electron has been removed from a region of the crystal that is otherwise neutral, the broken-bond region has a *net positive charge*. This broken-bond region is called a **hole** (h^+). An electron in a neighboring bond can jump and repair this bond and thereby create a hole in its original site as shown in Figure 2.26b. Effectively, the hole has been displaced in the opposite direction to the electron jump by this *bond switching*. Holes can also wander in the crystal by the repetition of bond switching. When a field is applied, both holes and electrons contribute to electrical conduction as in Figure 2.26c. For all practical purposes, these holes behave as if they were *free* positively charged particles (independent of the original electrons) inside the crystal. In the presence of an applied field, holes drift along the field direction and contribute to conduction just as the free electrons

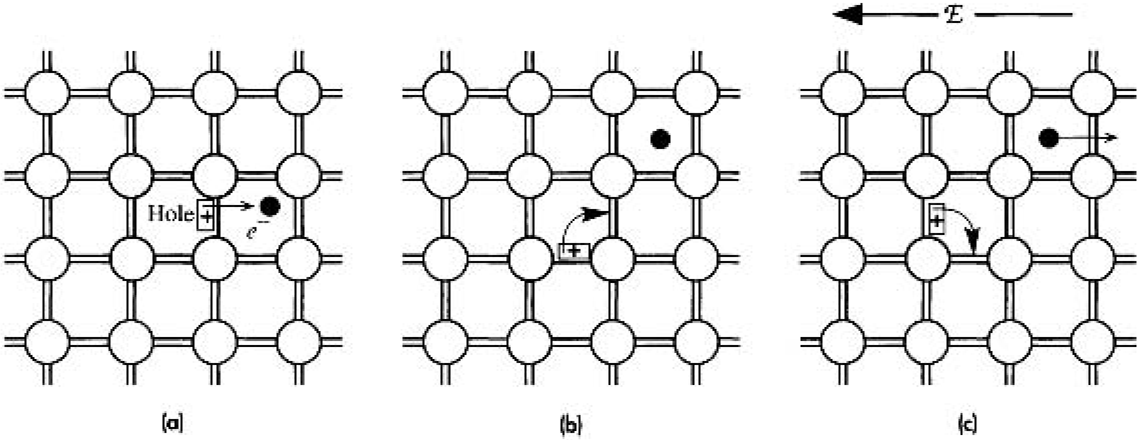


Figure 2.26

(a) Thermal vibrations of the atoms rupture a bond and release a free electron into the crystal. A hole is left in the broken bond, which has an effective positive charge.

(b) An electron in a neighboring bond can jump and repair this bond and thereby create a hole in its original site; the hole has been displaced.

(c) When a field is applied, both holes and electrons contribute to electrical conduction.

released from the broken bonds drift in the opposite direction and contribute to conduction.

It is also possible to create free electrons or holes by intentionally doping a semiconductor crystal, that is substituting impurity atoms for some of the Si atoms. Defects can also generate free carriers. The simplest example is nonstoichiometric ZnO that is shown in Figure 1.55b which has excess Zn. The electrons from the excess Zn are free to wander in the crystal and hence contribute to conduction.

Suppose that n and p are the concentrations of electrons and holes in a semiconductor crystal. If electrons and holes have drift mobilities of μ_e and μ_h, respectively, then the overall conductivity of the crystal is given by

Conductivity of a semiconductor

$$\sigma = ep\mu_h + en\mu_e \tag{2.41}$$

Unless a semiconductor has been heavily doped, the concentrations n and p are much smaller than the electron concentration in a metal. Even though carrier drift mobilities in most semiconductors are higher than electron drift mobilities in metals, semiconductors have much lower conductivities due to their lower concentration of free charge carriers.

EXAMPLE 2.21

HALL EFFECT IN SEMICONDUCTORS The Hall effect in a sample where there are both negative and positive charge carriers, for example, electrons and holes in a semiconductor, involves not only the concentrations of electrons and holes, n and p, respectively, but also the electron and hole drift mobilities, μ_e and μ_h. We first have to reinterpret the relationship between the drift velocity and the electric field $\mathcal{E}$.

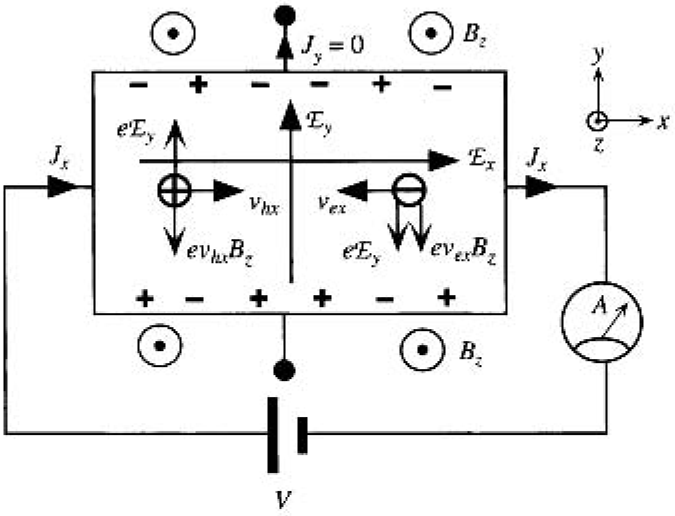


Figure 2.27 Hall effect for ambipolar conduction as in a semiconductor where there are both electrons and holes.

The magnetic field B_z is out from the plane of the paper. Both electrons and holes are deflected toward the bottom surface of the conductor and consequently the Hall voltage depends on the relative mobilities and concentrations of electrons and holes.

If μ_e is the drift mobility and v_e is the drift velocity of the electrons, then we already know that $v_e = \mu_e\mathcal{E}$. This has been derived by considering the *net electrostatic force* $e\mathcal{E}$ acting on a single electron and the imparted acceleration $a = e\mathcal{E}/m_e$. The drift is therefore due to the net force $F_{\text{net}} = e\mathcal{E}$ experienced by a conduction electron. If we were to keep $e\mathcal{E}$ as the *net force* F_{net} acting on a single electron, then we would have found

Drift velocity and net force

$$v_e = \frac{\mu_e}{e}F_{\text{net}} \qquad [2.42]$$

Equation 2.42 emphasizes the fact that drift is due to a net force F_{net} acting on an electron. A similar expression would also apply to the drift of a hole in a semiconductor.

When both electrons and holes are present in a semiconductor sample, both charge carriers experience a Lorentz force in the same direction since they would be drifting in the opposite directions as illustrated in Figure 2.27. Thus, both holes and electrons tend to pile near the bottom surface. The magnitude of the Lorentz force, however, will be different since the drift mobilities and hence drift velocities will be different in general. Once equilibrium is reached, there should be no current flowing in the y direction as we have an open circuit. Suppose that more holes have accumulated near the bottom surface so there is a built-in electric field $\mathcal{E}_y$ along y as shown in Figure 2.27. Suppose that v_{ey} and v_{hy} are the *usual* electron and hole drift velocities in the $-y$ and $+y$ directions, respectively, as if the electric field $\mathcal{E}_y$ existed alone in the $+y$ direction. The net current along y is zero, which means that

$$J_y = J_h + J_e = epv_{hy} + env_{ey} = 0 \qquad [2.43]$$

From Equation 2.43 we obtain

$$pv_{hy} = -nv_{ey} \qquad [2.44]$$

We note that either the electron or the hole drift velocity must be reversed from its usual direction; for example, holes drifting in the opposite directon to $\mathcal{E}_y$. The net force acting on the charge carriers cannot be zero. This is impossible when two types of carriers are involved and both carriers are drifting along y to give a net current J_y that is zero. This is what Equation 2.43 represents. We therefore conclude that, along y, both the electron and the hole must experience a driving force to drift them. The net force experienced by the carriers, as shown in Figure 2.27, is

$$F_{hy} = e\mathcal{E}_y - ev_{hx}B_z \quad \text{and} \quad -F_{ey} = e\mathcal{E}_y + ev_{ex}B_z \tag{2.45}$$

where v_{hx} and v_{ex} are the hole and electron drift velocities, respectively, along x. In general, the drift velocity is determined by the net force acting on a charge carrier; that is, from Equation 2.42

$$F_{hy} = \frac{ev_{hy}}{\mu_h} \quad \text{and} \quad -F_{ey} = \frac{ev_{ey}}{\mu_e}$$

so that Equation 2.45 becomes,

$$\frac{ev_{hy}}{\mu_h} = e\mathcal{E}_y - ev_{hx}B_z \quad \text{and} \quad \frac{ev_{ey}}{\mu_e} = e\mathcal{E}_y + ev_{ex}B_z$$

where v_{hy} and v_{ey} are the hole and electron drift velocities along y. Substituting $v_{hx} = \mu_h\mathcal{E}_x$ and $v_{ex} = \mu_e\mathcal{E}_x$, these become

$$\frac{v_{hy}}{\mu_h} = \mathcal{E}_y - \mu_h\mathcal{E}_xB_z \quad \text{and} \quad \frac{v_{ey}}{\mu_e} = \mathcal{E}_y + \mu_e\mathcal{E}_xB_z \tag{2.46}$$

From Equation 2.46 we can substitute for v_{hy} and v_{ey} in Equation 2.44 to obtain

$$p\mu_h\mathcal{E}_y - p\mu_h^2\mathcal{E}_xB_z = -n\mu_e\mathcal{E}_y - n\mu_e^2\mathcal{E}_xB_z$$

or

$$\mathcal{E}_y(p\mu_h + n\mu_e) = B_z\mathcal{E}_x\left(p\mu_h^2 - n\mu_e^2\right) \tag{2.47}$$

We now consider what happens along the x direction. The total current density is finite and is given by the usual expression,

Current density along x

$$J_x = epv_{hx} + env_{ex} = (p\mu_h + n\mu_e)e\mathcal{E}_x \tag{2.48}$$

We can use Equation 2.48 to substitute for $\mathcal{E}_x$ in Equation 2.47, to obtain

$$e\mathcal{E}_y(n\mu_e + p\mu_h)^2 = B_zJ_x\left(p\mu_h^2 - n\mu_e^2\right)$$

The Hall coefficient, by definition, is $R_H = \mathcal{E}_y/J_xB_z$, so

Hall effect for ambipolar conduction

$$R_H = \frac{p\mu_h^2 - n\mu_e^2}{e(p\mu_h + n\mu_e)^2} \tag{2.49}$$

or

Hall effect for ambipolar conduction

$$R_H = \frac{p - nb^2}{e(p + nb)^2} \tag{2.50}$$

where $b = \mu_e/\mu_h$. It is clear that the Hall coefficient depends on both the drift mobility ratio and the concentrations of holes and electrons. For $p > nb^2$, R_H will be positive and for $p < nb^2$, it will be negative. We should note that when only one type of carrier is involved, for example, electrons only, the $J_y = 0$ requirement means that $J_y = env_{ey} = 0$, or $v_{ey} = 0$. The drift velocity along y can only be zero, if the net driving force F_{ey} along y is zero. This occurs when $e\mathcal{E}_y - ev_{ex}B_z = 0$, that is, when the Lorentz force just balances the force due to the built-in field.

EXAMPLE 2.22 **HALL COEFFICIENT OF INTRINSIC SILICON** At room temperature, a pure silicon crystal (called **intrinsic silicon**) has electron and hole concentrations $n = p = n_i = 1.5 \times 10^{10}$ cm^{-3}, and electron and hole drift mobilities $\mu_e = 1350$ cm^2 V^{-1} s^{-1} and $\mu_h = 450$ cm^2 V^{-1} s^{-1}. Calculate the Hall coefficient and compare it with a typical metal.

SOLUTION

Given $n = p = n_i = 1.5 \times 10^{10}\ \text{cm}^{-3}$, $\mu_e = 1350\ \text{cm}^2\ \text{V}^{-1}\ \text{s}^{-1}$, and $\mu_h = 450\ \text{cm}^2\ \text{V}^{-1}\ \text{s}^{-1}$, we have

$$b = \frac{\mu_e}{\mu_h} = \frac{1350}{450} = 3$$

Then from Equation 2.50,

$$\begin{aligned} R_H &= \frac{(1.5 \times 10^{16}\ \text{m}^{-3}) - (1.5 \times 10^{16}\ \text{m}^{-3})(3)^2}{(1.6 \times 10^{-19}\ \text{C})[(1.5 \times 10^{16}\ \text{m}^{-3}) + (1.5 \times 10^{16}\ \text{m}^{-3})(3)]^2} \\ &= -208\ \text{m}^3\ \text{A}^{-1}\ \text{s}^{-1} \end{aligned}$$

which is orders of magnitude larger than that for a typical metal. All Hall-effect devices use a semiconductor rather than a metal sample.

2.7.2 IONIC CRYSTALS AND GLASSES

Figure 2.28a shows how crystal defects in an ionic crystal lead to mobile charges that can contribute to the conduction process. All crystalline solids possess vacancies and interstitial atoms as a requirement of thermal equilibrium. Many solids have interstitial impurities which are often ionized or charged. These interstitial ions can jump, *i.e.*, diffuse, from one interstitial site to another and hence drift by diffusion in the presence of a field. A positive ion at an interstitial site such as that shown in Figure 2.28a always prefers to jump into a neighboring interstitial site along the direction of the field because it experiences an effective force in this direction. When an ion with

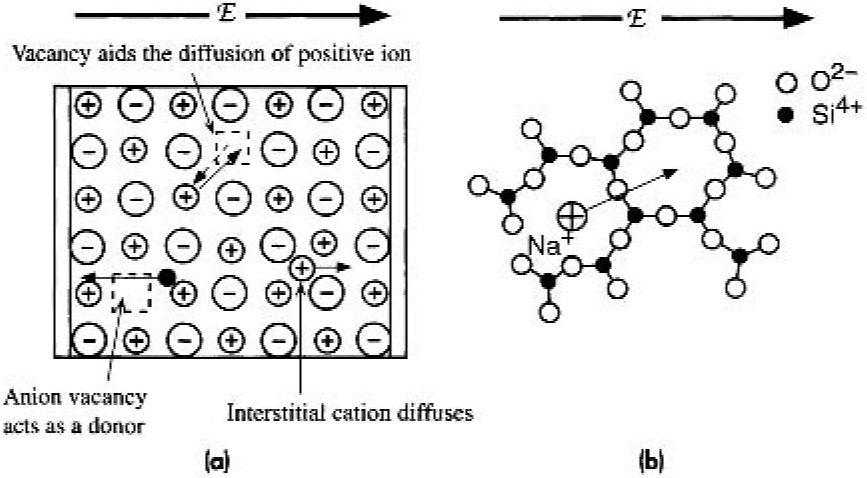
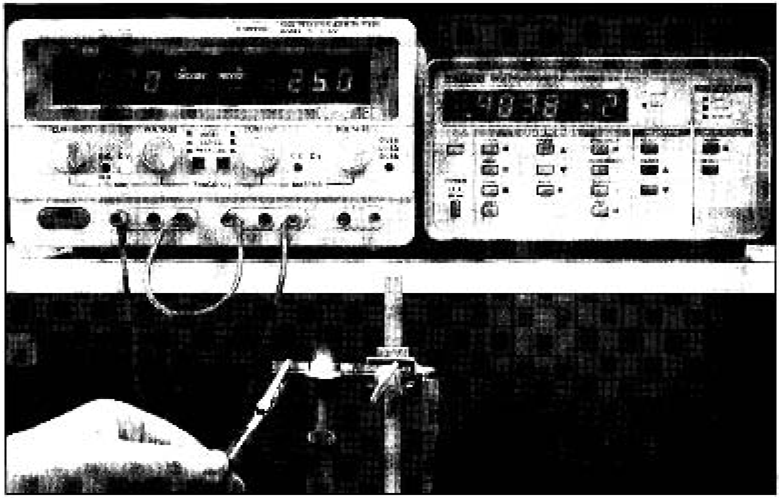


Figure 2.28 Possible contributions to the conductivity of ceramic and glass insulators.
(a) Possible mobile charges in a ceramic.
(b) An Na^+ ion in the glass structure diffuses and therefore drifts in the direction of the field.



This soda glass rod when heated under a torch becomes electrically conducting. It passes 4 mA when the voltage is 50 V (2 × 25 V); a resistance of 12.5 kΩ! Ordinary soda glass at room temperature is an insulator but can be quite conducting at sufficiently high temperatures.

charge q_{ion} jumps a distance d along the field, its potential energy decreases by $q_{\text{ion}}\mathcal{E}d$. If it tries to jump in the opposite direction, it has to do work $q_{\text{ion}}\mathcal{E}d$ against the force of the field.

Deviations from stoichiometry in compound solids often lead to the generation of mobile electrons (or holes) and point defects such as vacancies. Therefore, there are electrons, holes, and various mobile ions available for conduction under an applied field as depicted in Figure 2.28a. Many glasses and polymers contain a certain concentration of mobile ions in the structure. An example of a Na^+ ion in silica glass is shown in Figure 2.28b. Aided by the field, the Na^+ can jump from one interstice to a neighboring interstice along the field and thereby drift in the glass and contribute to current conduction. The conduction process is then essentially field-directed diffusion. Ordinary window glass, in fact, has a high concentration of Na^+ ions in the structure and becomes reasonably conducting above 300–400 °C. Some polymers may contain ions derived from the polymerization process, from the local degradation (dissociation) of the polymer itself, or from water absorption.

Conductivity σ of the material depends on all the conduction mechanisms with each species of charge carrier making a contribution, so it is given by

General conductivity

$$\sigma = \sum q_i n_i \mu_i \qquad \textbf{[2.51]}$$

where n_i is the concentration, q_i is the charge carried by the charge carrier species of type i (for electrons and holes $q_i = e$), and μ_i is the drift mobility of these carriers. The dominant conduction mechanism in Equation 2.51 is often quite difficult to uniquely identify. Further, it may change with temperature, composition, and ambient conditions such as the air pressure as in some oxide ceramics. For many insulators, whether ceramic, glass, or polymer, it has been found that, in the majority of cases, the conductivity follows an exponential or Arrhenius-type temperature dependence so that σ is

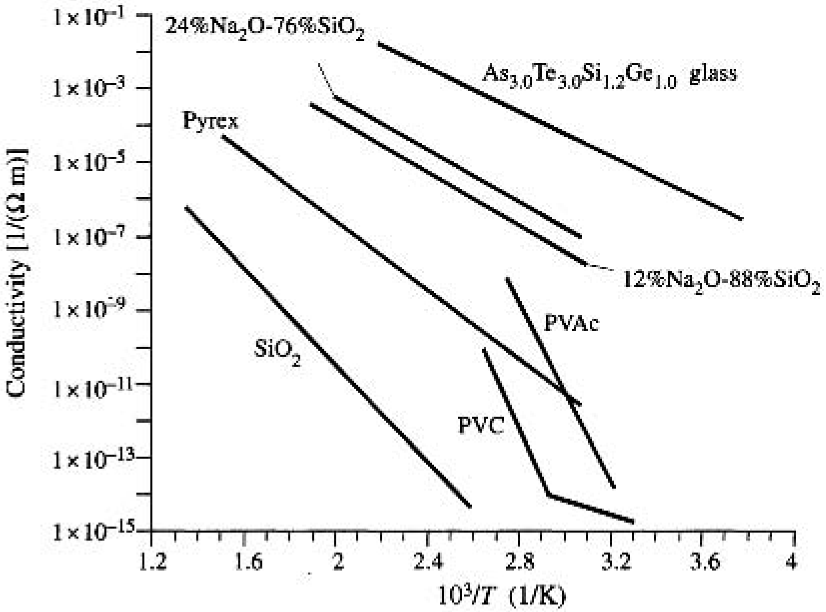


Figure 2.29 Conductivity versus reciprocal temperature for various low-conductivity solids.

| SOURCE: Data selectively combined from numerous sources.

thermally activated,

Temperature dependence of conductivity

$$\sigma = \sigma_o \exp\left(-\frac{E_\sigma}{kT}\right) \tag{2.52}$$

where E_σ is the **activation energy for conductivity.**

Figure 2.29 shows examples of the temperature dependence of conductivity for various high-resistivity solids: oxide ceramics, glasses, and polymers. When Equation 2.51 is plotted as $\log(\sigma)$ versus $1/T$, the result is a straight line with a negative slope that indicates the activation energy E_σ. Equation 2.52 is useful in predicting the conductivity at different temperatures and evaluating the temperature stability of the insulator.

EXAMPLE 2.23

CONDUCTIVITY OF A SODA–SILICATE GLASS Figure 2.29 shows the temperature dependence of 12% Na_2O–88% SiO_2, soda–silicate glass which has 12 mol% Na_2O and 88 mol% SiO_2. Calculate the activation energy of conductivity and compare this with the activation energy for the diffusion of Na^+ ions in the soda–silicate glass structure which is in the range 0.65–0.75 eV.

SOLUTION

According to Equation 2.52 when $\ln(\sigma)$ is plotted against $1/T$, the slope should be $-E_\sigma/k$. If the conductivity at temperatures T_1 and T_2 are σ_1 and σ_2, respectively, then the slope of the straight line for 12% Na_2O–88% SiO_2 in Figure 2.29 is

$$\text{Slope} = \frac{\ln(\sigma_2/\sigma_1)}{(1/T_2 - 1/T_1)} = -\frac{E_\sigma}{k}$$

Taking $\sigma_1 = 10^{-4}\ \Omega^{-1}\ \mathrm{m}^{-1}$ and $\sigma_2 = 10^{-6}\ \Omega^{-1}\ \mathrm{m}^{-1}$ in Figure 2.29, we find $1/T_1 = 0.00205$ and $1/T_2 = 0.00261$. Then, E_σ as eV is

$$E_\sigma = -\frac{\ln(\sigma_2/\sigma_1)}{(1/T_2 - 1/T_1)}\frac{k}{e} = -\frac{\ln(10^{-6}/10^{-4})}{(0.00261 - 0.00205)}\frac{1.38 \times 10^{-23}}{1.602 \times 10^{-19}} = 0.71\ \mathrm{eV}$$

A similar calculation for the 24% Na_2O–76% SiO_2 gives an activation energy of 0.69 eV.

Both of these activation energies are comparable with the activation energy for the diffusion of Na^+ ions in the structure. Thus, Na^+ diffusion is responsible for the conductivity.

EXAMPLE 2.24

DRIFT MOBILITY DUE TO IONIC CONDUCTION The soda–silicate glass of composition 20% Na_2O–80% SiO_2 and density of approximately 2.4 g cm^{-3} has a conductivity of $8.25 \times 10^{-6}\ \Omega^{-1}\ \mathrm{m}^{-1}$ at 150 °C. If conduction occurs by the diffusion of Na^+ ions, what is their drift mobility?

SOLUTION

We can calculate the drift mobility μ_i of the Na^+ ions from the conductivity expression $\sigma = q_i n_i \mu_i$ where q_i is the charge of the ion Na^+, so that it is $+e$, and n_i is the concentration of Na^+ ions in the structure. For simplicity we can take the glass to be made of $(Na_2O)_{0.2}(SiO_2)_{0.8}$ units. The atomic masses of Na, O, and Si are 23, 16, and 28.1, respectively. The atomic mass of $(Na_2O)_{0.2}(SiO_2)_{0.8}$ is

$$M_{at} = 0.2[2(23) + 1(16)] + 0.8[1(28.1) + 2(16)]$$
$$= 60.48\ \mathrm{g\ mol^{-1}}\ \text{of}\ (Na_2O)_{0.2}(SiO_2)_{0.8}$$

The number of $(Na_2O)_{0.2}(SiO_2)_{0.8}$ units per unit volume can be found from the density d by

$$n = \frac{dN_A}{M_{at}} = \frac{(2.4 \times 10^3\ \mathrm{kg\ m^{-3}})(6.02 \times 10^{23}\ \mathrm{mol^{-1}})}{(10^{-3}\ \mathrm{kg/g})(60.48\ \mathrm{g\ mol^{-1}})}$$
$$= 2.39 \times 10^{28}\ (Na_2O)_{0.2}(SiO_2)_{0.8}\ \text{units m}^{-3}$$

The concentration n_i of Na^+ ions is the concentration of Na atoms as each would be ionized. Then n_i can be expressed as $n_i = n_{Na} =$ [atomic fraction of Na in $(Na_2O)_{0.2}(SiO_2)_{0.8}$] $\times\ n$.

$$n_i = \left[\frac{0.2(2)}{0.2(2+1) + 0.8(1+2)}\right](2.39 \times 10^{28}\ \mathrm{m^{-3}}) = 3.186 \times 10^{27}\ \mathrm{m^{-3}}$$

and

$$\mu_i = \frac{\sigma}{en_i} = \frac{(8.25 \times 10^{-6}\ \Omega^{-1}\ \mathrm{m^{-1}})}{(1.60 \times 10^{-19}\ \mathrm{C})(3.186 \times 10^{27}\ \mathrm{m^{-3}})} = 1.62 \times 10^{-14}\ \mathrm{m^2\ V^{-1}\ s^{-1}}$$

This is an extremely small drift mobility, by orders of magnitude, compared with the typical electron drift mobility in metals and semiconductors. The reason is that the drift involves the Na^+ ion jumping from one site to another by a diffusion process. This diffusion requires overcoming a potential energy barrier, typically 0.5 to 1 eV, which limits drastically the rate of diffusion by virtue of the Boltzmann factor.

ADDITIONAL TOPICS

2.8 SKIN EFFECT: HF RESISTANCE OF A CONDUCTOR

Consider the cylindrical conductor shown in Figure 2.30a, which is carrying a current I into the paper (×). The magnetic field B of I is clockwise. Consider two magnetic field values B_1 and B_2, which are shown in Figure 2.30a. B_1 is inside the core and B_2 is just outside the conductor.

Assume that the conductor is divided into two conductors. The hypothetical cut is taken just outside of B_1. The conductor in Figure 2.30a is now cut into a hollow cylinder and a smaller solid cylinder, as shown in Figure 2.30b and c, respectively. The currents I_1 and I_2 in the solid and hollow cylinders sum to I. We can arrange things and choose B_1 such that our cut gives $I_1 = I_2 = \frac{1}{2}I$. Obviously, I_1 flowing in the inner conductor is threaded (or linked) by both B_1 and B_2. (Remember that B_1 is just inside the conductor in Figure 2.30b, so it threads at least 99% of I_1.) On the other hand, the outer conductor is only threaded by B_2, simply because I_2 flows in the hollow cylinder and there is no current in the hollow, which means that B_1 is not threaded by I_2. Clearly, I_1 threads more magnetic field than I_2 and thus conductor (c) has a higher inductance than (b). Recall that **inductance** is defined as the *total magnetic flux threaded per unit current*. Consequently, an ac current will prefer paths near the surface where the inductive impedance is smaller. As the frequency increases, the current is confined more and more to the surface region.

For a given conductor, we can assume that most of the current flows in a surface region of depth δ, called the **skin depth,** as indicated in Figure 2.31. In the central region,

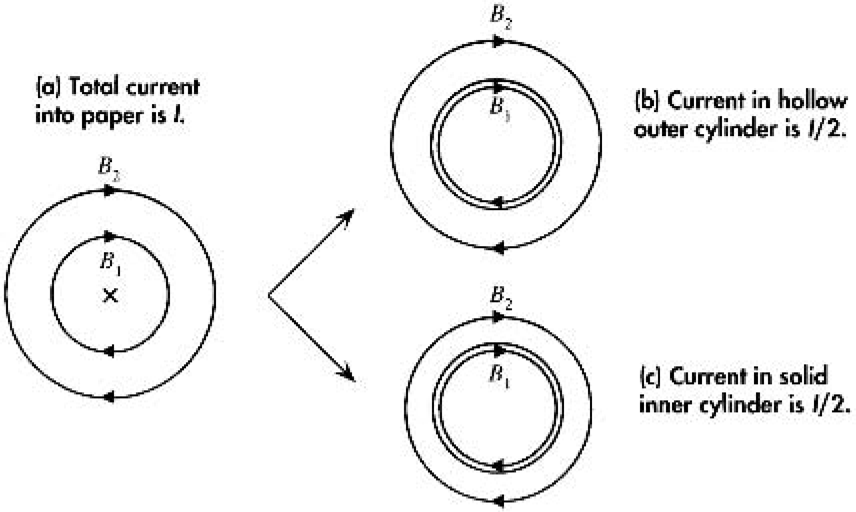


Figure 2.30 Illustration of the skin effect.
A hypothetical cut produces a hollow outer cylinder and a solid inner cylinder. Cut is placed where it would give equal current in each section. The two sections are in parallel so that the currents in (b) and (c) sum to that in (a).

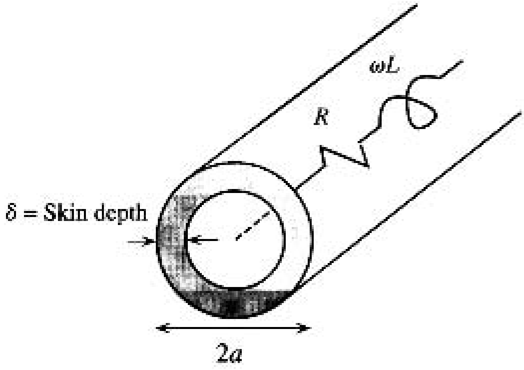


Figure 2.31 At high frequencies, the core region exhibits more inductive impedance than the surface region, and the current flows in the surface region of a conductor defined approximately by the skin depth, δ.

the current will be negligibly small. The skin depth will obviously depend on the frequency ω. To find δ, we must solve Maxwell's equations in a conductive medium, a tedious task that, fortunately, has been done by others. We can therefore simply take the result that the skin depth δ is given by

Skin depth for conduction

$$\delta = \frac{1}{\sqrt{\frac{1}{2}\omega\sigma\mu}} \tag{2.53}$$

where ω is the angular frequency of the current, σ is the conductivity (σ is constant from dc up to $\sim 10^{14}$ Hz in metals), and μ is the magnetic permeability of the medium, which is the product of the absolute (free space) permeability μ_o and the relative permeability μ_r.

We can imagine the central conductor as a resistance R in series with an inductance L. Intuitively, those factors that enhance the inductive impedance ωL over the resistance R will also tend to emphasize the skin effect and will hence tend to decrease the skin depth. For example, the greater the permeability of the conducting medium, the stronger the magnetic field inside the conductor, and hence the larger the inductance of the central region. The higher the frequency of the current, the greater the inductive impedance ωL compared with R and the more significant is the skin effect. The greater is the conductivity σ, the smaller is R compared with ωL and hence the more important is the skin effect. All these dependences are accounted for in Equation 2.53.

With the skin depth known, the effective cross-sectional area is given approximately by

$$A = \pi a^2 - \pi(a-\delta)^2 \approx 2\pi a\delta$$

where δ^2 is neglected ($\delta \ll a$). The ac resistance r_{ac} of the conductor per unit length is therefore

HF resistance per unit length due to skin effect

$$r_{ac} = \frac{\rho}{A} \approx \frac{\rho}{2\pi a\delta} \tag{2.54}$$

where ρ is the ac resistivity at the frequency of interest, which for all practical purposes is equal to the dc resistivity of the metal. Equation 2.54 clearly shows that as ω increases, δ decreases, by virtue of $\delta \propto \omega^{-1/2}$ and, as a result, r_{ac} increases.

From this discussion, it is obvious that the skin effect arises because the magnetic field of the ac current in the conductor restricts the current flow to the surface region within a depth of $\delta < a$. Since the current can only flow in the surface region, there is an effective increase in the resistance due to a decrease in the cross-sectional area for current flow. Taking this effective area for current flow as $2\pi a\delta$ leads to Equation 2.54.

The skin effect plays an important role in electronic engineering because it limits the use of solid-core conductors in high-frequency applications. As the signal frequencies reach and surpass the gigahertz (10^9 Hz) range, the transmission of the signal over a long distance becomes almost impossible through an ordinary, solid-metal conductor. We must then resort to pipes (or waveguides).

EXAMPLE 2.25

SKIN EFFECT FROM DIMENSIONAL ANALYSIS Using dimensional analysis, obtain the general form of the equation for the skin depth δ in terms of the angular frequency of the current ω, conductivity σ, and permeability μ.

SOLUTION

The skin effect depends on the angular frequency ω of the current, the conductivity σ, and the magnetic permeability μ of the conducting medium. In the most general way, we can group these effects as

$$[\delta] = [\omega]^x[\sigma]^y[\mu]^z$$

where the indices x, y, and z are to be determined. We then substitute the dimensions of each quantity in this expression. The dimensions of each, in terms of the fundamental units, are as follows:

Quantity	Units	Fundamental Units	Comment
δ	m	m	
ω	$\mathrm{s^{-1}}$	$\mathrm{s^{-1}}$	
σ	$\Omega^{-1}\,\mathrm{m^{-1}}$	$\mathrm{C^2\,s\,kg^{-1}\,m^{-3}}$	$\Omega = \mathrm{V\,A^{-1}} = (\mathrm{J\,C^{-1}})(\mathrm{C\,s^{-1}})^{-1}$ $= \mathrm{N\,m\,s\,C^{-2}} = (\mathrm{kg\,m\,s^{-2}})(\mathrm{m\,s\,C^{-2}})$
μ	$\mathrm{Wb\,A^{-1}\,m^{-1}}$	$\mathrm{kg\,m\,C^{-2}}$	$\mathrm{Wb} = \mathrm{T\,m^2} = (\mathrm{N\,A^{-1}\,m^{-1}})(\mathrm{m^2})$ $= (\mathrm{kg\,m\,s^{-2}})(\mathrm{C^{-1}\,s})(\mathrm{m})$

Therefore,

$$[\mathrm{m}] = [\mathrm{s^{-1}}]^x[\mathrm{C^2\,s\,kg^{-1}\,m^{-3}}]^y[\mathrm{kg\,m\,C^{-2}}]^z$$

Matching the dimensions of both sides, we see that $y = z$; otherwise C and kg do not cancel.

For m $\quad 1 = -3y + z$

For s $\quad 0 = -x + y$

For C or kg $\quad 0 = 2y - 2z \quad$ or $\quad 0 = -y + z$

Clearly, $x = y = z = -\frac{1}{2}$ is the only possibility. Then, $\delta \propto [\omega\sigma\mu]^{-1/2}$. It should be reemphasized that the dimensional analysis is not a proof of the skin depth expression, but a consistency check that assures confidence in the equation.

EXAMPLE 2.26

SKIN EFFECT IN AN INDUCTOR What is the change in the dc resistance of a copper wire of radius 1 mm for an ac signal at 10 MHz? What is the change in the dc resistance at 1 GHz? Copper has $\rho_{dc} = 1.70 \times 10^{-8}$ Ω m or $\sigma_{dc} = 5.9 \times 10^{7}$ Ω^{-1} m^{-1} and a relative permeability near unity.

SOLUTION

Per unit length, $r_{dc} = \rho_{dc}/\pi a^2$ and at high frequencies, from Equation 2.54, $r_{ac} = \rho_{dc}/2\pi a\delta$. Therefore, $r_{ac}/r_{dc} = a/2\delta$.

We need to find δ. From Equation 2.53, at 10 MHz we have

$$\delta = \left[\tfrac{1}{2}\omega\sigma_{dc}\mu\right]^{-1/2} = \left[\tfrac{1}{2} \times 2\pi \times 10 \times 10^{6} \times 5.9 \times 10^{7} \times 1.257 \times 10^{-6}\right]^{-1/2}$$
$$= 2.07 \times 10^{-5} \text{ m} = 20.7\ \mu\text{m}$$

Thus

$$\frac{r_{ac}}{r_{dc}} = \frac{a}{2\delta} = \frac{(10^{-3}\text{ m})}{(2 \times 2.07 \times 10^{-5}\text{ m})} = 24.13$$

The resistance has increased by 24 times. At 1 GHz, the increase is 240 times. Furthermore, the current is confined to a surface region of about $\sim 2 \times 10^{-5}$ (20 μm) at 10 MHz and $\sim 2 \times 10^{-6}$ m (2 μm) at 1 GHz, so most of the material is wasted. This is exactly the reason why solid conductors would not be used for high-frequency work. As very high frequencies, in the gigahertz range and above, are reached, the best bet would be to use pipes (waveguides).

One final comment is appropriate. An inductor wound from a copper wire would have a certain Q (quality factor) value[9] that depends inversely on its resistance. At high frequencies, Q would drop, because the current would be limited to the surface of the wire. One way to overcome this problem is to use a thick conductor that has a surface coating of higher-conductivity metal, such as silver. This is what the early radio engineers practiced. In fact, tank circuits of high-power radio transmitters often have coils made from copper tubes with a coolant flowing inside.

2.9 THIN METAL FILMS

2.9.1 CONDUCTION IN THIN METAL FILMS

The resistivity of a material, as listed in materials tables and in our analysis of conduction, refers to the resistivity of the material in bulk form; that is, any dimension of the specimen is much larger than the mean free path for electron scattering. In such cases resistivity is determined by scattering from lattice vibrations and, if significant, scattering from various impurities and defects in the crystal. In certain applications,

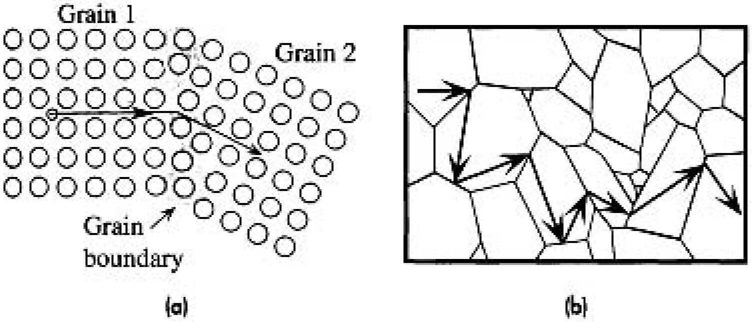
---

[9] The Q value refers to the quality factor of an inductor, which is defined by $Q = \omega_o L/R$, where ω_o is the resonant frequency, L is the inductance, and R is the resistance due to the losses in the inductor.

notably microelectronics, metal films are widely used to provide electrical conduction paths to and from the semiconductor devices. Various methods are used to deposit thin films. In many applications, the metal film is simply deposited onto a substrate, such as a semiconductor or an insulator (*e.g.*, SiO_2), by **physical vapor deposition** (PVD), that is, by **vacuum deposition,** which typically involves either evaporation or sputtering. In **thermal evaporation,** the metal is evaporated from a heated source in a vacuum chamber as depicted in Figure 1.74. As the metal atoms, evaporated from the source, impinge and adhere to the semiconductor surface, they form a metal film which is often highly polycrystalline. Stated differently, the metal atoms in the vapor condense to form a metal film on a suitably placed substrate. In **electron beam deposition,** an energetic electron beam is used to melt and evaporate the metal. **Sputtering** is a vacuum deposition process that involves bombarding a metal target material with energetic Ar ions, which dislodges the metal atoms and then condenses them onto a substrate. The use of sputtering is quite common in microelectronic fabrication. Copper metal interconnect films used in microelectronics are usually grown by **electrodeposition,** that is, using electroplating, an electrochemical process, to deposit the metal film onto the required chip areas. In many applications, especially in microelectronics, we are interested in the resistivity of a metal film in which the thickness of the film or the average size of the grains is comparable to the mean distance between scattering events ℓ_{bulk} (the mean free path) in the bulk material. In such cases, the resistivity of the metal film is greater than the corresponding resistivity of the bulk crystal. A good example is the resistivity of interconnects and various metal films used in the "shrinking" world of microelectronics, in which more and more transistors are packed into a single Si crystal, and various device dimensions are scaled down.

2.9.2 RESISTIVITY OF THIN FILMS

Polycrystalline Films and Grain Boundary Scattering In a highly polycrystalline sample the conduction electrons are more likely to be scattered by grain boundaries than by other processes as depicted in Figure 2.32a. Consider the resistivity due to scattering from grain boundaries alone as shown in Figure 2.32b. The conduction electron is free within a grain, but becomes scattered at the grain boundary. Its mean free path ℓ_{grains} is therefore roughly equal to the average grain size d. If $\lambda = \ell_{crystal}$ is



(a)

(b)

Figure 2.32

(a) Grain boundaries cause scattering of the electron and therefore add to the resistivity by Matthiessen's rule.

(b) For a very grainy solid, the electron is scattered from grain boundary to grain boundary and the mean free path is approximately equal to the mean grain diameter.

the mean free path of the conduction electrons in the *single crystal* (no grain boundaries), then

Mean free path in polycrystalline sample

$$\frac{1}{\ell} = \frac{1}{\ell_{\text{crystal}}} + \frac{1}{\ell_{\text{grains}}} = \frac{1}{\lambda} + \frac{1}{d} \qquad [2.55]$$

The resistivity is inversely proportional to the mean free path which means that the resistivity of the bulk single crystal $\rho_{\text{crystal}} \propto 1/\lambda$ and the resistivity of the polycrystalline sample $\rho \propto 1/\ell$. Thus,

Resistivity of a polycrystalline sample

$$\frac{\rho}{\rho_{\text{crystal}}} = 1 + \left(\frac{\lambda}{d}\right) \qquad [2.56]$$

Polycrystalline metal films with a smaller grain diameter d (*i.e.*, more grainy films) will have a higher resistivity.

In a more rigorous theory we have to consider a number of effects. It may take more than one scattering at a grain boundary to totally randomize the velocity, so we need to calculate the effective mean free path that accounts for how many collisions are needed to randomize the velocity. There is a possibility that the electron may be totally reflected back at a grain boundary (bounce back). Suppose that the probability of reflection at a grain boundary is R. If d is the average grain size (diameter), then the popular **Mayadas–Shatkez formula** is approximately given by[10]

Resistivity due to grain boundary scattering

$$\frac{\rho}{\rho_{\text{crystal}}} \approx 1 + 1.33\beta \qquad [2.57a]$$

where

$$\beta = \frac{\lambda}{d}\left(\frac{R}{1-R}\right) \qquad [2.57b]$$

Equation 2.57a is in the form of Matthiessen's rule and indicates that the grain boundary scattering contribution ρ_{grains} to the overall resistivity is $(1.33\beta)\rho_{\text{crystal}}$. The approximate sign in Equation 2.57 implies that Matthiessen's rule is "approximately," though reasonably well, obeyed. For copper, typical R values are 0.24 to 0.40, and R is somewhat smaller for Al. Equation 2.57 for a Cu film with $R \approx 0.3$ predicts $\rho/\rho_{\text{crystal}} \approx 1.20$ for $d \approx 3\lambda$ or a grain size $d \approx 120$ nm since the bulk crystal $\lambda \approx 40$ nm.

Surface Scattering Consider the scattering of electrons from the surfaces of a conducting film as in Figure 2.33. Take the film thickness as D. Assume that the scattering from the surface is *inelastic;* that is, the electron loses the gained velocity from the field. Put differently, the direction of the electron after the scattering process is *independent* of the direction before the scattering process. This type of scattering is called *nonspecular*. (If the electron is elastically reflected from the surface just like a rubber ball bouncing off a wall, then there is no increase in the resistivity.) It is unlikely that one surface scattering will completely randomize the electron's velocity. The mean free path ℓ_{surf} of the electron will depend on its direction right after the scattering

---

[10] This is obtained by expanding the original long expression about $\beta = 1$ to the first term. To two decimal places, the expansion is $1 + 1.33\beta$.

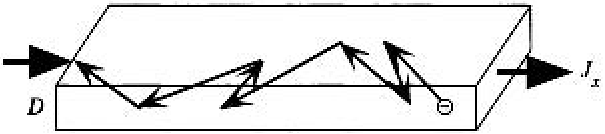


Figure 2.33 Conduction in thin films may be controlled by scattering from the surfaces.

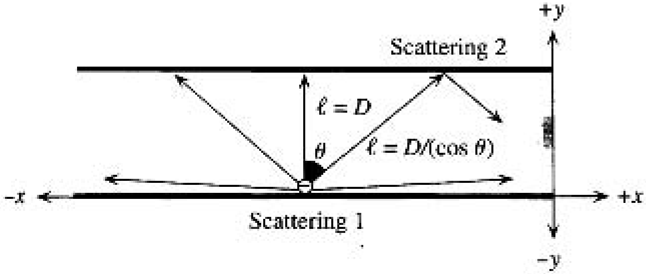


Figure 2.34 The mean free path of the electron depends on the angle θ after scattering.

process as depicted in Figure 2.34. For example, if the angle θ after surface scattering is zero, (the electron moves transversely to the film length), then $\ell_{\text{surf}} = D$. In general, the mean free path ℓ_{surf} will be $D/(\cos\theta)$ as illustrated in Figure 2.34.

Consider the surface scattering example in Figure 2.34 where the electron is scattered from the bottom surface. If the scattering of the electron were truly random, then the probability of being scattered in a direction back into the film, that is, in the $+y$ direction, would be 0.5 on average. However, the electron's direction right after the surface scattering is not totally random because we know that the electron cannot leave the film; thus θ is between $-\pi/2$ and $+\pi/2$ and cannot be between $-\pi$ and $+\pi$. The electron's velocity after the first surface scattering must have a y component along $+y$ and not along $-y$. The electron can only acquire a velocity component along $-y$ again after the second surface scattering as shown in Figure 2.34. It therefore takes two collisions to randomize the velocity, which means that the *effective mean free path* must be twice as long, that is $2D/\cos\theta$. To find the overall mean free path ℓ for calculating the resistivity we must use Matthiessen's rule. If λ is the mean free path of the conduction electrons in the *bulk crystal* (no surface scattering), then

$$\frac{1}{\ell} = \frac{1}{\lambda} + \frac{1}{\ell_{\text{surf}}} = \frac{1}{\lambda} + \frac{\cos\theta}{2D} \qquad [2.58]$$

Mean free path in a film

We have to average for all possible θ values per scattering, that is, θ from $-\pi/2$ to $+\pi/2$. Once this is done we can relate ℓ to λ as follows:

Averaged mean free path in a film

$$\frac{\lambda}{\ell} = 1 + \frac{\lambda}{\pi D}$$

The resistivity of the bulk crystal is $\rho_{\text{bulk}} \propto 1/\lambda$, and the resistivity of the film is $\rho \propto 1/\ell$. Thus,

Resistivity of a conducting thin film

$$\frac{\rho}{\rho_{\text{bulk}}} = 1 + \frac{1}{\pi}\left(\frac{\lambda}{D}\right) \qquad [2.59]$$

Table 2.6 Resistivities of some thin Cu and Au films at room temperature

Film	D (nm)	d (nm)	ρ (nΩ m)	Comment
			Cu films (Polycrystalline)	
Cu on TiN, W, and TiW [1]	>250	186	21	Chemical vapor deposition (CVD).
		45	32	Substrate temperature 200 °C. ρ depends on d not $D = 250$–900 nm.
Cu on 500 nm SiO_2 [2]	20.5		35	Thermal evaporation. Substrate at RT.
	37		27	
Cu on Si (100) [3]	52		38	Sputtered Cu films. Annealing at 150 °C has no effect. $R \approx 0.40$ and $p \approx 0$.
	100		22	
Cu on glass [4]	40		50	As deposited
	40		29	Annealed at 200 °C
	40		25	Annealed at 250 °C All thermal evaporated and PC.
			Au films	
Au epitaxial film on mica	30		25	Single crystal on mica. $p \approx 0.8$. Specular scattering.
Au PC film on mica	30		54	PC. Sputtered on mica. p is small.
Au film on glass	30		70	PC. Evaporated onto glass. p is small. Nonspecular scattering.
Au on glass [5]	40	8.5	92	PC. Sputtered films. $R = 0.27$–0.33.
	40	3.8	189	

NOTE: PC-polycrystalline film, RT-room temperature, $D =$ film thickness, $d =$ average grain size. At RT for Cu, $\lambda = 38$–40 nm, and for Au, $\lambda = 36$–38 nm.

SOURCES: Data selectively combined from various sources, including [1] S. Riedel *et al., Microelec. Engin.* **33,** 165, 1997; [2] H. D. Liu *et al., Thin Solid Films,* **34,** 151, 2001; [3] J. W. Lim *et al., Appl. Surf. Sci.* **217,** 95, 2003. [4] R. Suri *et al., J. Appl. Phys.,* **46,** 2574, 1975; [5] R. H. Carnely and T. A. Ali, *J. Appl. Phys.,* **49,** 4094, 1978.

A more rigorous calculation modifies the numerical factor $1/\pi$ and also considers what fraction p of surface collisions is specular and results in[11]

Surface scattering resistivity

$$\frac{\rho}{\rho_{\text{bulk}}} \approx 1 + \frac{3\lambda}{8D}(1-p) \qquad \frac{D}{\lambda} > 0.3 \tag{2.60}$$

which is valid down to about $D \approx 0.3\lambda$. Equation 2.60 is in Matthiessen's rule format, which means that the second term is the fractional contribution of the surfaces to the resistivity. It can be seen that for elastic or specular scattering $p = 1$ and there is no change in the resistivity. For $p = 0$, Equation 2.60 predicts $\rho/\rho_{\text{bulk}} \approx 1.20$ for roughly $D \approx 1.9\lambda$ or a thickness $D \approx 75$ nm for Cu for which $\lambda \approx 40$ nm. The value of p depends on the film preparation method (*e.g.*, sputtering, epitaxial growth) and the substrate on which the film has been deposited.

Equation 2.60 involves scattering from two surfaces, that is, from the two interfaces of the film. In general the two interfaces will not be identical and hence will have different p coefficients; p in Equation 2.60 is some mean p value. Table 2.6

---

[11] This is known as the **Fuchs–Sondheimer equation** in a simplified form.

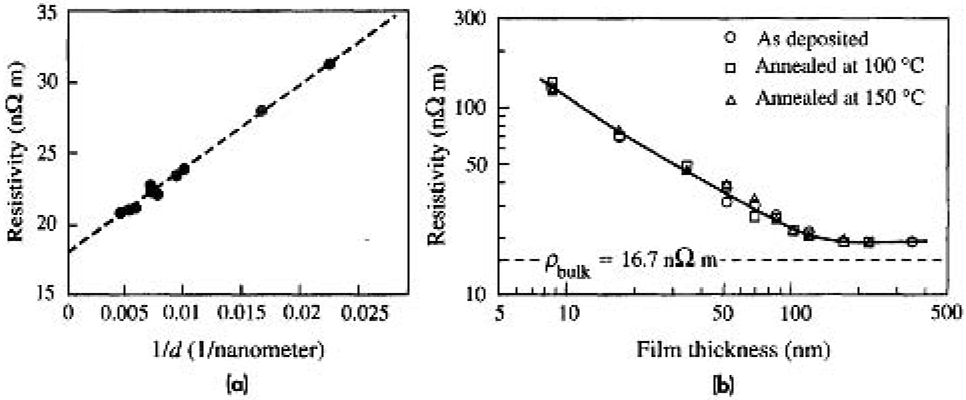


Figure 2.35

(a) ρ_{film} of Cu polycrystalline films versus reciprocal mean grain size (diameter) $1/d$. Film thickness $D = 250$–900 nm does not affect the resistivity. The straight line is $\rho_{\text{film}} = 17.8$ nΩ m + (595 nΩ m nm)(1/d).

(b) ρ_{film} of thin Cu polycrystalline films versus film thickness D. In this case, annealing (heat treating) the films to reduce the polycrystallinity does not significantly affect the resistivity because ρ_{film} is controlled mainly by surface scattering.

SOURCES: Data extracted from (a) S. Riedel *et al.*, *Microelec. Engin.* **33**, 165, 1997, and (b) W. Lim *et al.*, *Appl. Surf. Sci.*, **217**, 95, 2003.

summarizes the resistivity of thin Cu and Au gold films deposited by various preparation techniques. Notice the large difference between the Au films deposited on a noncrystalline glass substrate and on a crystalline mica substrate. Such differences between films are typically attributed to different values of p. The p value can also change (increase) when the film is annealed. Obviously, the polycrystallinity of the film will also affect the resistivity as discussed previously. Typically, most epitaxial thin films, unless very thin ($D \ll \lambda$), deposited onto heated crystalline substrates exhibit highly specular scattering with $p = 0.9$–1.

It is generally very difficult to separate the effects of surface and grain boundary scattering in thin polycrystalline films; the contribution from grain boundary scattering is likely to exceed that from the surfaces. In any event, both contributions, by Matthiessen's general rule, increase the overall resistivity. Figure 2.35a shows an example in which the resistivity ρ_{film} of thin Cu polycrystalline films is due to grain boundary scattering, and thickness has no effect (D was 250–900 nm and much greater than λ). The resistivity ρ_{film} is plotted against the reciprocal mean grain size $1/d$, which then follows the expected linear behavior in Equation 2.57a. On the other hand, Figure 2.35b shows the resistivity of Cu films as a function of film thickness D. In this case, annealing (heat treating) the films to reduce the polycrystallinity does not significantly affect the resistivity because ρ_{film} is controlled primarily by surface scattering and is given by Equation 2.60.

EXAMPLE 2.27

THIN-FILM RESISTIVITY Consider the data presented in Figure 2.35a. What can you conclude from the plot given that the mean free path $\lambda \approx 40$ nm in Cu?

SOLUTION

Consider the results in Figure 2.35a. It is stated that the film thickness $D = 250$–900 nm does not affect the resistivity, which implies that ρ_{film} is controlled only by the grain size d. From Equation 2.57a and b we expect

$$\rho_{\text{film}} \approx \rho_{\text{crystal}}(1 + 1.33\beta) \approx \rho_{\text{crystal}} + 1.33\rho_{\text{crystal}}\left(\frac{R}{1-R}\right)\frac{\lambda}{d}$$

This equation represents the observed line when ρ_{film} is plotted against $1/d$ as in Figure 2.35a. The $\rho_{\text{film}} - 1/d$ line has an intercept given by 17.8 nΩ m and a slope given by 595 (nΩ m) (nm). The intercept approximately matches the bulk resistivity ρ_{crystal} of Cu. The slope is

$$\text{Slope} \approx 1.33\rho_{\text{crystal}}\left(\frac{R}{1-R}\right)\lambda$$

or

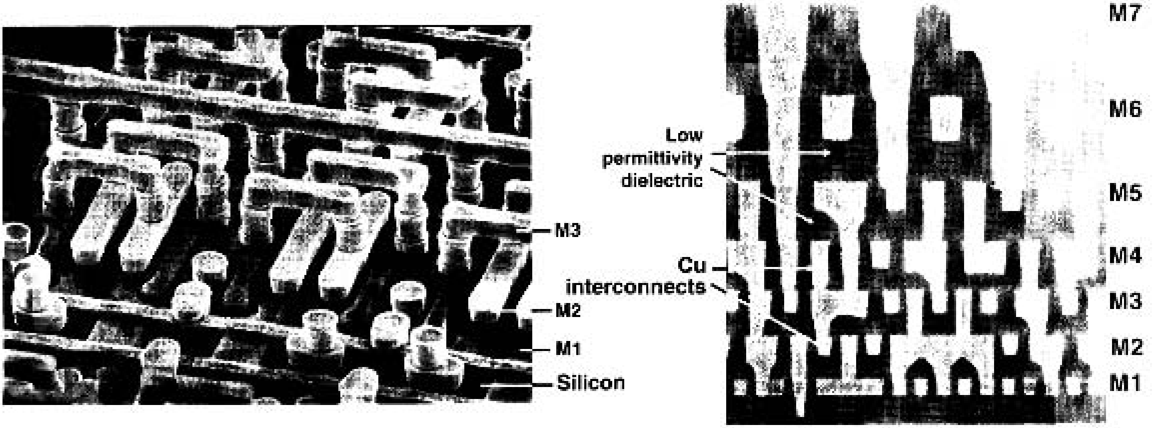
$$595(\text{n}\Omega\ \text{m})(\text{nm}) \approx 1.33(17.8\ \text{n}\Omega\ \text{m})\left(\frac{1}{R^{-1}-1}\right)(40\ \text{nm})$$

Solving this equation yields $R \approx 0.39$ for these copper films.

2.10 INTERCONNECTS IN MICROELECTRONICS

An integrated circuit (IC) is a single crystal of Si that contains millions of transistors that have been fabricated within this one crystal. **Interconnects** are simply metal conductors that are used to wire the devices together to implement the desired overall operation of the IC; see the photographs in Figure 2.36. Aluminum and Al alloys, or Al silicides, have been the workhouse of the interconnects, but today's fast chips rely on copper interconnects, which have three distinct advantages. First, copper has a resistivity that is about 40 percent lower than that of Al. In high-transistor-density chips in which various voltages are switched on and off, what limits the speed of operation is the RC time constant, that is, the time constant that is involved in charging and discharging the capacitance between the interconnects, and the input capacitance of the transistor; usually the former dominates. The RC is substantially reduced with Cu replacing Al so that the chip speed is faster. The second advantage is that a lower overall interconnect resistance leads to a lower power consumption, lower I^2R.

The third advantage is that copper has superior resistance to **electromigration,** a process in which metal atoms are forced to migrate by a large current density. Such electromigration can eventually lead to a failure of the interconnect. The current density in interconnects with a small cross-sectional area can be very high, and hence the electron drift velocities can also be very high. As these fast electrons collide with the metal ions there is a momentum transfer that slowly drifts the metal ions. Thus, the metal ions are forced to slowly migrate as a result of being bombarded by drifting electrons; the migration is in the direction of electron flow (not current flow). This atomic migration can deplete or accumulate material in certain local regions of the interconnect structure. The result is that electromigration can lead to voids (material depletion) or hillocks (material accumulation), and eventually there may be a break or a short between interconnects (an interconnect failure). The electromigration effects are reduced



(a)

(b)

Figure 2.36

(a) Metal interconnects wiring devices on a silicon crystal. Three different metallization levels M1, M2, and M3 are used. The dielectric between the interconnects has been etched away to expose the interconnect structure.

(b) Cross section of a chip with seven levels of metallization, M1 to M7. The image is obtained with a scanning electron microscope (SEM).

| SOURCES: (a) Courtesy of IBM. (b) Courtesy of Mark Bohr, Intel.

in Cu interconnects because the Cu atoms are heavier and cannot be as easily migrated by an electric current as are Al atoms.

There is a relatively simple expression for estimating the ***RC* time constant** of multilevel interconnects that is useful in comparing various interconnect technologies and the effects of interconnect metal resistance ρ, the relative permittivity ε_r of the interlevel dielectric (insulation) between the interconnects, and the geometry of the whole interconnect wiring. First consider a simple interconnect line, as in Figure 2.37a, whose thickness is T, width is W, and length is L. Its resistance R is simply $\rho L/(TW)$. In the chip, this interconnect will have other interconnects around it as shown in a simplified way in Figure 2.37b. It will couple with all these different conductors around it and will have an overall (effective) capacitance C_{eff}. RC_{eff} is what we know as the RC time constant associated with the interconnect line in Figure 2.37b.

Suppose that the interconnect is an Mth-level metallization. It will have a series of many "horizontal" neighbors along this Mth level. Let X be the nearest edge-to-edge *separation* and P be the *pitch* of these horizontal neighbors at the Mth level. The pitch P refers to the separation from center to center, or the periodicity of interconnects; $P = W + X$. At a height H above the interconnect there will be a line running at the $(M + 1)$ level. Similarly there will be an interconnect line at a distance H below at the $(M - 1)$ level. We can identify two sets of capacitances. C_V represents the capacitance in the vertical direction, between the interconnect and its upper or lower neighbor. C_H is the lateral capacitance in the horizontal direction, between a neighbor on the right or left. Both are shown in Figure 2.37c. The interconnect therefore has two C_V and two

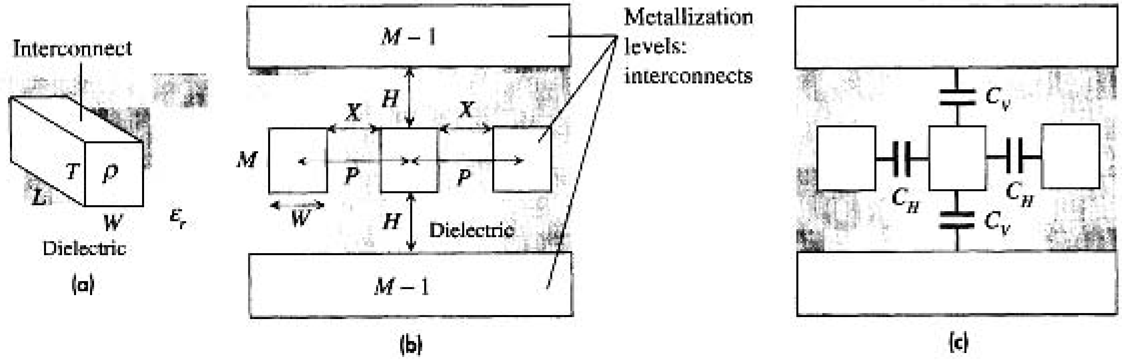


Figure 2.37

(a) A single line interconnect surrounded by dielectric insulation.

(b) Interconnects crisscross each other. There are three levels of interconnect: $M-1$, M, and $M+1$.

(c) An interconnect has vertical and horizontal capacitances C_V and C_H.

C_H, four capacitances in total, and all are in parallel as shown in Figure 2.37c. From the simple parallel plate capacitance formula we can write

$$C_H = \frac{\varepsilon_o \varepsilon_r TL}{X} \qquad \text{and} \qquad C_V = \frac{\varepsilon_o \varepsilon_r WL}{H}$$

Effective capacitance in multilevel interconnect structures

Usually C_H is greater than C_V. From Figure 2.37c, the effective capacitance $C_{\text{eff}} = 2(C_H + C_V)$,

$$C_{\text{eff}} = 2\varepsilon_o \varepsilon_r L\left(\frac{T}{X} + \frac{W}{H}\right) \qquad [2.61]$$

which is the **effective multilevel interconnect capacitance.** We now multiply this with $R = \rho L/(TW)$ to obtain the RC time constant,

RC time constant in multilevel interconnect structures

$$RC = 2\varepsilon_o \varepsilon_r \rho\left(\frac{L^2}{TW}\right)\left(\frac{T}{X} + \frac{W}{H}\right) \qquad [2.62]$$

Equation 2.62 is only an approximate first-order calculation, but, nonetheless, it turns out to be quite a useful equation for roughly predicting the RC time constant and hence the speed of multilevel interconnect based high-transistor-density chips.[12] Most significantly, it highlights the importance of *three* influencing effects: the resistivity of the interconnect metal; relative permittivity ε_r of the dielectric insulation between the conductors; and the geometry or "architecture" of the interconnects L, T, W, X, and H. Notice that L appears as L^2 in Equation 2.62 and has

---

[12] A more rigorous theory would consider the interconnect system as having a distributed resistance and a distributed capacitance, similar to a transmission line; a topical research area. The treatment here is more than sufficient to obtain approximate results and understand the factors that control the interconnect delay time.

significant control on the overall RC. Equation 2.62 does not obviously include the time it takes to turn on and off the individual transistors connected to the interconnects. In a high-transistor-density chip, the latter is smaller than the interconnect RC time constant.

The reduction in the interconnect resistivity ρ by the use of Cu instead of Al has been a commendable achievement, and cuts down RC significantly. Further reduction in ρ is limited because Cu already has a very small resistivity; the smallest ρ is for Ag which is only about 5 percent lower. Current research efforts for reducing RC further are concentrated on mainly two factors. First is the reduction of ε_r as much as possible by using dielectrics such as *fluorinated* SiO_2 (known as FSG) for which $\varepsilon_r = 3.6$, or, more importantly, using what are called **low-*k* dielectric materials** (k stands for ε_r) such as various polymers or porous dielectrics[13] that have a lower ε_r, typically 2–3, which is a substantial reduction from 3.6. The second is the development of optimized interconnect geometries that reduce L^2 in Equation 2.62. (T, W, X, and H are all of comparable size, so L^2 is the most dominant geometric factor.)

The ratio of the thickness T to width W of an interconnect is called the **aspect ratio,** $A_r = T/W$. This ratio is typically between 1 to 2. Very roughly, in many cases, X and W are the same, $X \approx W$ and $X \approx P/2$ (see Figure 2.37b). Then Equation 2.62 simplifies further,

RC time constant in multilevel interconnect structures

$$RC \approx 2\varepsilon_o\varepsilon_r\rho L^2\left(\frac{4}{P^2}+\frac{1}{T^2}\right) \qquad [2.63]$$

The signal delays between the transistors on a chip arise from the interconnect RC time constant. Equations 2.62 and 2.63 are often also used to calculate the **multilevel interconnect delay time.** Suppose that we take some typical values, $L \approx 10$ mm, $T \approx 1$ μm, $P \approx 1$ μm, $\rho = 17$ nΩ m for a Cu interconnect, and $\varepsilon_r \approx 3.6$ for FSG; then $RC \approx 0.43$ ns, not a negligible value in today's speed hungry computing.

EXAMPLE 2.28

MULTILEVEL INTERCONNECT *RC* TIME CONSTANT In a particular high-transistor-density IC where copper is used as the interconnect, one level of the multilevel interconnects has the following characteristics: pitch $P = 0.45$ μm, $T = 0.36$ μm, $A_R = 1.6$, $H = X$, and $\varepsilon_r \approx 3.6$. Find the effective capacitance per millimeter of interconnect length, and the RC delay time per L^2 as ps/mm^2 (as normally used in industry).

SOLUTION

Since $A_R = T/W$, $W = T/A_R = 0.36/1.6 = 0.225$ μm. Further, from Figure 2.37b, $P = W + X$, so that $X = P - W = 0.45 - 0.225 = 0.225$ μm. $H = X = 0.225$ μm. Thus, Equation 2.61 for $L = 1$ mm $= 10^{-3}$ m gives

$$C_{eff} = 2\varepsilon_o\varepsilon_r L\left(\frac{T}{X}+\frac{W}{H}\right) = 2(8.85\times10^{-12})(3.6)(10^{-3})\left[\frac{0.36}{0.225}+\frac{0.225}{0.225}\right] = 0.17 \text{ pF}$$

---

[13] The mixture rules mentioned in this chapter turn up again in a different but recognizable form for predicting the overall relative permittivity of porous dielectrics.

which is about 0.2 pF per millimeter of interconnect. The RC time constant per L^2 is

$$\frac{RC}{L^2} = 2\varepsilon_o\varepsilon_r\rho\left(\frac{1}{TW}\right)\left(\frac{T}{X}+\frac{W}{H}\right) = 2\varepsilon_o\varepsilon_r\rho\left(\frac{1}{WX}+\frac{1}{TH}\right)$$

$$= 2(8.85\times10^{-12})(3.6)(17\times10^{-9})$$

$$\left[\frac{1}{(0.225\times10^{-6})(0.225\times10^{-6})}+\frac{1}{(0.36\times10^{-6})(0.225\times10^{-6})}\right]$$

$$= 3.4\times10^{-5}\ \text{s m}^{-2} \qquad \text{or} \qquad 34\ \text{ps mm}^{-2}$$

2.11 ELECTROMIGRATION AND BLACK'S EQUATION

Interconnects have small cross-sectional dimensions, and consequently the current densities can be quite large. Figure 2.38a depicts how the continual bombardment of lattice atoms (metal ions) by many "fast" conduction electrons in high-current-density regions can transfer enough momentum to a host metal atom to migrate it, that is, diffuse it along a suitable path in the crystal. The bombarded metal atom has to jump to a suitable lattice location to migrate, which is usually easiest along grain boundaries or surfaces where there is sufficient space as depicted in Figure 2.38a and b. Grain boundaries that are parallel to the electron flow therefore can migrate atoms more efficiently than grain boundaries in other directions. Atomic diffusion can also occur along a surface of the interconnect, that is, along an interface between the interconnect metal and the neighboring material. The final result of atomic migration is usually either material depletion or accumulation as depicted in Figure 2.38c. The depletion of material

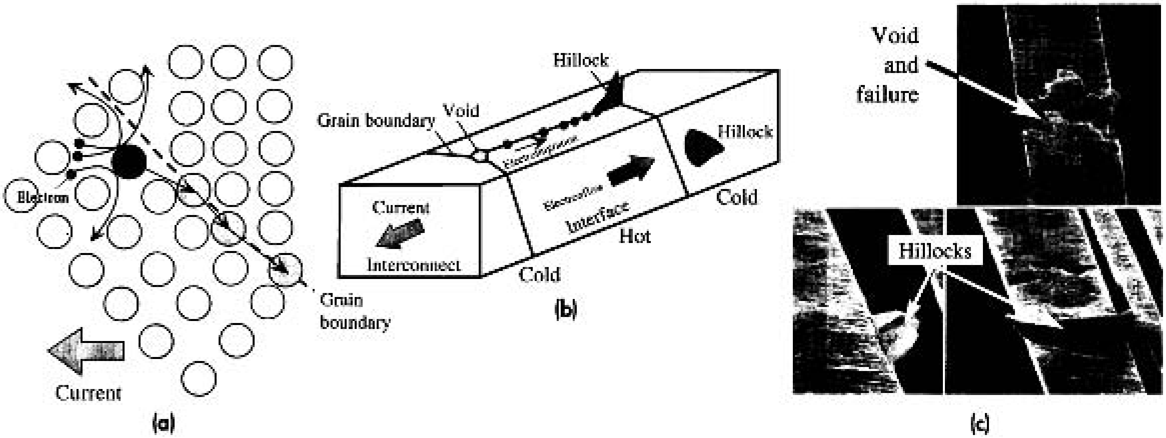


Figure 2.38

(a) Electrons bombard the metal ions and force them to slowly migrate.

(b) Formation of voids and hillocks in a polycrystalline metal interconnect by the electromigration of metal ions along grain boundaries and interfaces.

(c) Accelerated tests on a 3 μm chemical vapor deposited Cu line: $T = 200$ °C and $J = 6$ MA cm^{-2}. The photos show void formation and fatal failure (break), and hillock formation.

| SOURCE: Courtesy of L. Arnaud *et al.*, *Microelectronics Reliability*, **40,** 86, 2000.

leads to a **void** and a possible eventual break in the interconnect. The accumulation of material leads to a **hillock** and a short between lines. Interconnect failure by electromigration is measured by the **mean time to 50 percent failure** t_{MTF}. There are two factors that control the rate of electromigration R_{EM}. First is the activation energy E_A involved in migrating (diffusing) the metal atom, and the second is the rate at which the atoms are bombarded with electrons, which depends on the current density J. Thus,

Electromigration rate

$$R_{\text{EM}} \propto J^n \exp\left(-\frac{E_A}{kT}\right)$$

in which the rate is proportional to J^n, instead of just J because it is found experimentally that $n \geq 1$. From the electromigration rate we can find the average time t_{MTF} it takes for 50 percent failure of interconnects because this time is inversely proportional to the electromigration rate just given:

Black's electromigration failure equation

$$t_{\text{MTF}} = A_B J^{-n} \exp\left(\frac{E_A}{kT}\right) \qquad [2.64]$$

where A_B is a constant. Equation 2.64 is known as **Black's equation,** and it is extremely useful in extrapolating high-temperature failure tests to normal operating temperatures. Electromigration-induced interconnect failures are typically examined at elevated temperatures where the failure times are over a measurable time scale in the laboratory (perhaps several hours or a few days). These experiments are called **accelerated failure** tests because they make use of the fact that at high temperatures the electromigration failure occurs more quickly. The results are then extrapolated to room temperature using Black's equation.

Typically electromigration occurs along grain boundaries or along various interfaces that the interconnect has with its surroundings, the semiconductor, dielectric material, etc. The diffusion coefficient has a lower activation energy E_A for these migration paths than for diffusion within the volume of the crystal. The electromigration process therefore depends on the microstructure of the interconnect metal, and its interfaces. Usually another metal, called a **barrier,** is deposited to occupy the interface space between the interconnect and the semiconductor or the oxide. The barrier *passivates* the interface, rendering it relatively inactive in terms of providing an electromigration path. An interconnect can also have a temperature gradient along it. (The heat generated by I^2R may be conducted away faster at the ends of the interconnect, leaving the central region hotter.) Electromigration would be faster in the hot region and very slow (almost stationary) in the cold region since it is a thermally activated process. Consequently a pileup of electromigrated atoms can occur as atoms are migrated from hot to cold regions along the interconnect, leading to a hillock.[14]

Pure Al suffers badly from electromigration problems and is usually alloyed with small amounts of Cu, called Al(Cu), to reduce electromigration to a tolerable level. But the resistivity increases. (Why?) In recent Cu interconnects, the most important diffusion path seems to be the interface between the Cu surface and the dielectric. Surface coating of these Cu interconnects provides control over electromigration failures.

---

[14] Somewhat like a traffic accident pileup in which speeding cars run into stationary cars ahead of them.

CD Selected Topics and Solved Problems

Selected Topics

- Conduction in Metals: Electrical and Thermal Conduction
- Joule's Law
- Hall Effect
- Heat Transfer in Electrical Engineering: Fundamentals and Applications
- Thermal Conductivity

Solved Problems

- Radiation Theory of the Fuse
- The Strain Gauge

DEFINING TERMS

Alloy is a metal that contains more than one element.

Brass is a copper-rich Cu–Zn alloy.

Bronze is a copper-rich Cu–Sn alloy.

Drift mobility is the drift velocity per unit applied field. If μ_d is the drift mobility, then the defining equation is $v_d = \mu_d \mathcal{E}$, where v_d is the drift velocity and $\mathcal{E}$ is the field.

Drift velocity is the average electron velocity, over all the conduction electrons in the conductor, in the direction of an applied electrical force ($F = -e\mathcal{E}$ for electrons). In the absence of an applied field, all the electrons move around randomly, and the average velocity over all the electrons in any direction is zero. With an applied field $\mathcal{E}_x$, there is a net velocity per electron v_{dx}, in the direction opposite to the field, where v_{dx} depends on $\mathcal{E}_x$ by virtue of $v_{dx} = \mu_d \mathcal{E}_x$, where μ_d is the drift mobility.

Electrical conductivity (σ) is a property of a material that quantifies the ease with which charges flow inside the material along an applied electric field or a voltage gradient. The conductivity is the inverse of electrical resistivity ρ. Since charge flow is caused by a voltage gradient, σ is the rate of charge flow across a unit area per unit voltage gradient, $J = \sigma \mathcal{E}$.

Electromigration is current density–induced diffusion of host metal atoms due to their repeated bombardment by conduction electrons at high current densities; the metal atoms migrate in the direction of electron flow. **Black's equation** describes the mean time to failure of metal film interconnects due to electromigration failure.

Fourier's law states that the rate of heat flow Q' through a sample, due to thermal conduction, is proportional to the temperature gradient dT/dx and the cross-sectional area A, that is, $Q' = -\kappa A(dT/dx)$, where κ is the thermal conductivity.

Hall coefficient (R_H) is a parameter that gauges the magnitude of the Hall effect. If $\mathcal{E}_y$ is the electric field in the y direction, due to a current density J_x along x and a magnetic field B_z along z, then $R_H = \mathcal{E}_y / J_x B_z$.

Hall effect is a phenomenon that occurs in a conductor carrying a current when the conductor is placed in a magnetic field perpendicular to the current. The charge carriers in the conductor are deflected by the magnetic field, giving rise to an electric field (Hall field) that is perpendicular to both the current and the magnetic field. If the current density J_x is along x and the magnetic field B_z is along z, then the Hall field is along either $+y$ or $-y$, depending on the polarity of the charge carriers in the material.

Heterogeneous mixture is a mixture in which the individual components remain physically separate and possess different chemical and physical properties; that is, a mixture of different phases.

Homogeneous mixture is a mixture of two or more chemical species in which the chemical properties (*e.g.*, composition) and physical properties (*e.g.*, density, heat capacity) are uniform throughout. A homogeneous mixture is a solution.

Interconnects are various thin metal conductors in a Si integrated circuit that connect various devices to implement the required wiring of the devices. In modern ICs, these interconnects are primarily electrodeposited Cu films.

Ionic conduction is the migration of ions in the material as a result of field-directed diffusion. When a positive ion in an interstitial site jumps to a neighboring interstitial site in the direction of the field, it lowers its potential energy which is a favorable process. If it jumps in the opposite direction, then it has to do work against the force of the field which is undesirable. Thus the diffusion of the positive ion is directed along the field.

Isomorphous phase diagram is a phase diagram for an alloy that has unlimited solid solubility.

Joule's law relates the power dissipated per unit volume P_{vol} by a current-carrying conductor to the applied field $\mathcal{E}$ and the current density J, such that $P_{\text{vol}} = J\mathcal{E} = \sigma\mathcal{E}^2$.

Lorentz force is the force experienced by a moving charge in a magnetic field. When a charge q is moving with a velocity **v** in a magnetic field **B**, the charge experiences a force **F** that is proportional to the magnitude of its charge q, its velocity **v**, and the field **B**, such that $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

Magnetic field, magnetic flux density, or magnetic induction (B) is a vector field quantity that describes the magnitude and direction of the *magnetic force* exerted on a moving charge or a current-carrying conductor. The magnetic force is essentially the Lorentz force and excludes the electrostatic force $q\mathcal{E}$.

Magnetic permeability (μ) or simply permeability is a property of the medium that characterizes the effectiveness of a medium in generating as much magnetic field as possible for given external currents. It is the product of the permeability of free space (vacuum) or absolute permeability (μ_o) and relative permeability of the medium (μ_r), *i.e.*, $\mu = \mu_o\mu_r$.

Magnetometer is an instrument for measuring the magnitude of a magnetic field.

Matthiessen's rule gives the overall resistivity of a metal as the sum of individual resistivities due to scattering from thermal vibrations, impurities, and crystal defects. If the resistivity due to scattering from thermal vibrations is denoted ρ_T and the resistivities due to scattering from crystal defects and impurities can be lumped into a single resistivity term called the residual resistivity ρ_R, then $\rho = \rho_T + \rho_R$.

Mean free path is the mean distance traversed by an electron between scattering events. If τ is the mean free time between scattering events and u is the mean speed of the electron, then the mean free path is $\ell = u\tau$.

Mean free time is the average time it takes to scatter a conduction electron. If t_i is the free time between collisions (between scattering events) for an electron labeled i, then $\tau = \bar{t}_i$ averaged over all the electrons. The drift mobility is related to the mean free time by $\mu_d = e\tau/m_e$. The reciprocal of the mean free time is the mean probability per unit time that a conduction electron will be scattered; in other words, the mean frequency of scattering events.

Nordheim's rule states that the resistivity of a solid solution (an isomorphous alloy) due to impurities ρ_I is proportional to the concentrations of the solute X and the solvent $(1 - X)$.

Phase (in materials science) is a physically homogeneous portion of a materials system that has uniform physical and chemical characteristics.

Relaxation time is an equivalent term for the mean free time between scattering events.

Residual resistivity (ρ_R) is the contribution to the resistivity arising from scattering processes other than thermal vibrations of the lattice, for example, impurities, grain boundaries, dislocations, point defects.

Skin effect is an electromagnetic phenomenon that, at high frequencies, restricts ac current flow to near the surface of a conductor to reduce the energy stored in the magnetic field.

Solid solution is a crystalline material that is a homogeneous mixture of two or more chemical species. The mixing occurs at the atomic scale, as in mixing alcohol and water. Solid solutions can be substitutional (as in Cu–Ni) or interstitial (for example, C in Fe).

Stefan's law is a phenomenological description of the energy radiated (as electromagnetic waves) from a surface per second. When a surface is heated to a temperature T, it radiates net energy at a rate given by $P_{\text{radiated}} = \epsilon\sigma_S A(T^4 - T_0^4)$, where σ_S is Stefan's constant (5.67×10^{-8} W m^{-2} K^{-4}), ϵ is the emissivity of the surface, A is the surface area, and T_0 is the ambient temperature.

Temperature coefficient of resistivity (TCR) (α_0) is defined as the fractional change in the electrical resistivity of a material per unit increase in the temperature with respect to some reference temperature T_0.

Thermal conductivity (κ) is a property of a material that quantifies the ease with which heat flows along the material from higher to lower temperature regions. Since heat flow is due to a temperature gradient, κ is the rate of heat flow across a unit area per unit temperature gradient.

Thermal resistance (θ) is a measure of the difficulty with which heat conduction takes place along a material sample. The thermal resistance is defined as the temperature drop per unit heat flow, $\theta = \Delta T/Q'$. It depends on both the material and its geometry. If the heat losses from the surfaces are negligible, then $\theta = L/\kappa A$, where L is the length of the sample (along heat flow) and A is the cross-sectional area.

Thermally activated conductivity means that the conductivity increases in an exponential fashion with temperature as in $\sigma = \sigma_o \exp(-E_a/kT)$ where E_a is the activation energy.

Thin film is a conductor whose thickness is typically less than ~1 micron; the thickness is also much less than the width and length of the conductor. Typically thin films have a higher resistivity than the corresponding bulk material due to the grain boundary and surface scattering.

QUESTIONS AND PROBLEMS

2.1 **Electrical conduction** Na is a monovalent metal (BCC) with a density of 0.9712 g cm^{-3}. Its atomic mass is 22.99 g mol^{-1}. The drift mobility of electrons in Na is 53 cm^2 V^{-1} s^{-1}.

a. Consider the collection of conduction electrons in the solid. If each Na atom donates one electron to the electron sea, *estimate* the mean separation between the electrons. (Note: If n is the concentration of particles, then the particles' mean separation $d = 1/n^{1/3}$.)

b. Estimate the mean separation between an electron (e^-) and a metal ion (Na$^+$), assuming that most of the time the electron prefers to be between two neighboring Na$^+$ ions. What is the approximate Coulombic interaction energy (in eV) between an electron and an Na$^+$ ion?

c. How does this electron/metal-ion interaction energy compare with the average thermal energy per particle, according to the kinetic molecular theory of matter? Do you expect the kinetic molecular theory to be applicable to the conduction electrons in Na? If the mean electron/metal-ion interaction energy is of the same order of magnitude as the mean *KE* of the electrons, what is the mean speed of electrons in Na? Why should the mean kinetic energy be comparable to the mean electron/metal-ion interaction energy?

d. Calculate the electrical conductivity of Na and compare this with the experimental value of 2.1×10^7 Ω^{-1} m^{-1} and comment on the difference.

2.2 **Electrical conduction** The resistivity of aluminum at 25 °C has been measured to be 2.72×10^{-8} Ω m. The thermal coefficient of resistivity of aluminum at 0 °C is 4.29×10^{-3} K^{-1}. Aluminum has a valency of 3, a density of 2.70 g cm^{-3}, and an atomic mass of 27.

a. Calculate the resistivity of aluminum at −40 °C.

b. What is the thermal coefficient of resistivity at −40 °C?

c. Estimate the mean free time between collisions for the conduction electrons in aluminum at 25 °C, and hence estimate their drift mobility.

d. If the mean speed of the conduction electrons is about 2.0×10^6 m s^{-1}, calculate the mean free path and compare this with the interatomic separation in Al (Al is FCC). What should be the thickness of an Al film that is deposited on an IC chip such that its resistivity is the same as that of bulk Al?

e. What is the percentage change in the power loss due to Joule heating of the aluminum wire when the temperature drops from 25 °C to −40 °C?

2.3 **Conduction in gold** Gold is in the same group as Cu and Ag. Assuming that each Au atom donates one conduction electron, calculate the drift mobility of the electrons in gold at 22 °C. What is the mean free path of the conduction electrons if their mean speed is 1.4×10^6 m s^{-1}? (Use ρ_o and α_o in Table 2.1.)

2.4 **Effective number of conduction electrons per atom**

a. Electron drift mobility in tin (Sn) is 3.9 cm^2 V^{-1} s^{-1}. The room temperature (20 °C) resistivity of Sn is about 110 nΩ m. Atomic mass M_{at} and density of Sn are 118.69 g mol^{-1} and 7.30 g cm^{-3}, respectively. How many "free" electrons are donated by each Sn atom in the crystal? How does this compare with the position of Sn in Group IVB of the Periodic Table?

b. Consider the resistivity of few selected metals from Groups I to IV in the Periodic Table in Table 2.7. Calculate the number of conduction electrons contributed per atom and compare this with the location of the element in the Periodic Table. What is your conclusion?

Table 2.7 Selection of metals from Groups I to IV in the Periodic Table

Metal	Periodic Group	Valency	Density (g cm^{-3})	Resistivity (nΩ m)	Mobility (cm^2 $V^{-1}$$s^{-1}$)
Na	IA	1	0.97	42.0	53
Mg	IIA	2	1.74	44.5	17
Ag	IB	1	10.5	15.9	56
Zn	IIB	2	7.14	59.2	8
Al	IIIB	3	2.7	26.5	12
Sn	IVB	4	7.30	110	3.9
Pb	IVB	4	11.4	206	2.3

NOTE: Mobility from Hall-effect measurements.

2.5 **TCR and Matthiessen's rule** Determine the temperature coefficient of resistivity of pure iron and of electrotechnical steel (Fe with 4% C), which are used in various electrical machinery, at two temperatures: 0 °C and 500 °C. Comment on the similarities and differences in the resistivity versus temperature behavior shown in Figure 2.39 for the two materials.

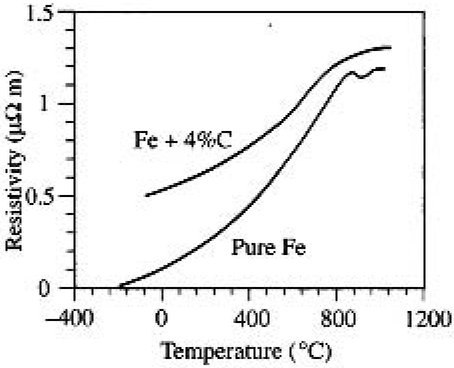


Figure 2.39 Resistivity versus temperature for pure iron and 4% C steel.

***2.6 TCR of isomorphous alloys**

a. Show that for an isomorphous alloy $A\%–B\%$ ($B\%$ solute in $A\%$ solvent), the temperature coefficient of resistivity α_{AB} is given by

$$\alpha_{AB} \approx \frac{\alpha_A \rho_A}{\rho_{AB}}$$

where ρ_{AB} is the resistivity of the alloy (AB) and ρ_A and α_A are the resistivity and TCR of pure A. What are the assumptions behind this equation?

b. Determine the composition of the Cu–Ni alloy that will have a TCR of 4×10^{-4} K^{-1}, that is, a TCR that is an order of magnitude less than that of Cu. Over the composition range of interest, the resistivity of the Cu–Ni alloy can be calculated from $\rho_{\text{CuNi}} \approx \rho_{\text{Cu}} + C_{\text{eff}} X(1 - X)$, where C_{eff}, the effective Nordheim coefficient, is about 1310 nΩ m.

2.7 Resistivity of isomorphous alloys and Nordheim's rule What are the maximum atomic and weight percentages of Cu that can be added to Au without exceeding a resistivity that is twice that of pure gold? What are the maximum atomic and weight percentages of Au that can be added to pure Cu without exceeding twice the resistivity of pure copper? (Alloys are normally prepared by mixing the elements in weight.)

2.8 Nordheim's rule and brass Brass is a Cu–Zn alloy. Table 2.8 shows some typical resistivity values for various Cu–Zn compositions in which the alloy is a solid solution (up to 30% Zn).

a. Plot ρ versus $X(1 - X)$. From the slope of the best-fit line find the mean (effective) Nordheim coefficient $\overline{C}$ for Zn dissolved in Cu over this compositional range.

b. Since X is the atomic fraction of Zn in brass, for each atom in the alloy, there are X Zn atoms and $(1 - X)$ Cu atoms. The conduction electrons consist of each Zn donating two electrons and each copper donating one electron. Thus, there are $2(X) + 1(1 - X) = 1 + X$ conduction electrons per atom. Since the conductivity is proportional to the electron concentration, the combined Nordheim–Matthiessens rule must be scaled up by $(1 + X)$.

$$\rho_{\text{brass}} = \frac{\rho_o + CX(1 - X)}{(1 + X)}$$

Plot the data in Table 2.8 as $\rho(1 + X)$ versus $X(1 - X)$. From the best-fit line find C and ρ_o. What is your conclusion? (Compare the correlation coefficients of the best-fit lines in your two plots.[15])

Table 2.8 Cu–Zn brass alloys

Zn at.% in Cu–Zn	0	0.34	0.5	0.93	3.06	4.65	9.66	15.6	19.59	29.39
Resistivity nΩ m	17	18.1	18.84	20.7	26.8	29.9	39.1	49.0	54.8	63.5

SOURCE: H. A. Fairbank, *Phys. Rev.*, **66**, 274, 1944.

2.9 Resistivity of solid solution metal alloys: testing Nordheim's rule Nordheim's rule accounts for the increase in the resistivity resulting from the scattering of electrons from the random distribution of impurity (solute) atoms in the host (solvent) crystal. It can nonetheless be quite useful in approximately

---

[15] More rigorously, $\rho_{\text{brass}} = \rho_{\text{matrix}} + C_{\text{eff}} X(1-X)$, in which ρ_{matrix} is the resistivity of the perfect matrix. Accounting for the extra electrons, $\rho_{\text{matrix}} \approx \rho_n/(1+X)$, where ρ_n is the pure metal matrix resistivity and C_{eff} is the Nordheim coefficient at the composition of interest, given by $C_{\text{eff}} \approx C/(1+X)^{2/3}$. (It is assumed that the atomic concentration does not change significantly.) As always, there are also other theories; part *b* is more than sufficient for most practical purposes.

predicting the resistivity at one composition of a solid solution metal alloy, given the value at another composition. Table 2.9 lists some solid solution metal alloys and gives the resistivity ρ at one composition X and asks for a prediction ρ' based on Nordheim's rule at another composition X'. Fill in the table for ρ' and compare the predicted values with the experimental values, and comment.

Table 2.9 Resistivities of some solid solution metal alloys

	Alloy							
	Ag–Au	**Au–Ag**	**Cu–Pd**	**Ag–Pd**	**Au–Pd**	**Pd–Pt**	**Pt–Pd**	**Cu–Ni**
X (at.%)	8.8% Au	8.77% Ag	6.2% Pd	10.1% Pd	8.88% Pd	7.66% Pt	7.1% Pd	2.16% Ni
ρ_0 (nΩ m)	16.2	22.7	17	16.2	22.7	108	105.8	17
ρ at X (nΩ m)	44.2	54.1	70.8	59.8	54.1	188.2	146.8	50
C_{eff}								
X'	15.4% Au	24.4% Ag	13% Pd	15.2% Pd	17.1% Pd	15.5% Pt	13.8% Pd	23.4% Ni
ρ' at X' (nΩ m)								
ρ' at X' (nΩ m) Experimental	66.3	107.2	121.6	83.8	82.2	244	181	300

NOTE: First symbol (*e.g.*, Ag in AgAu) is the matrix (solvent) and the second (Au) is the added solute. X is in at.%, converted from traditional weight percentages reported with alloys. C_{eff} is the effective Nordheim coefficient in $\rho = \rho_0 + C_{eff}X(1-X)$.

*2.10 **TCR and alloy resistivity** Table 2.10 shows the resistivity and TCR (α) of Cu–Ni alloys. Plot TCR versus $1/\rho$, and obtain the best-fit line. What is your conclusion? Consider the Matthiessen rule, and explain why the plot should be a straight line. What is the relationship between ρ_{Cu}, α_{Cu}, ρ_{CuNi}, and α_{CuNi}? Can this be generalized?

Table 2.10 Cu–Ni alloys, resistivity, and TCR

	Ni wt.% in Cu–Ni				
	0	2	6	11	20
Resistivity (nΩ m)	17	50	100	150	300
TCR (ppm °C^{-1})	4270	1350	550	430	160

NOTE: ppm-parts per million, *i.e.*, 10^{-6}.

2.11 **Electrical and thermal conductivity of In** Electron drift mobility in indium has been measured to be 6 cm^2 V^{-1} s^{-1}. The room temperature (27 °C) resistivity of In is 8.37×10^{-8} Ω m, and its atomic mass and density are 114.82 amu or g mol^{-1} and 7.31 g cm^{-3}, respectively.

a. Based on the resistivity value, determine how many free electrons are donated by each In atom in the crystal. How does this compare with the position of In in the Periodic Table (Group IIIB)?

b. If the mean speed of conduction electrons in In is 1.74×10^8 cm s^{-1}, what is the mean free path?

c. Calculate the thermal conductivity of In. How does this compare with the experimental value of 81.6 W m^{-1} K^{-1}?

2.12 **Electrical and thermal conductivity of Ag** The electron drift mobility in silver has been measured to be 56 cm^2 V^{-1} s^{-1} at 27 °C. The atomic mass and density of Ag are given as 107.87 amu or g mol^{-1} and 10.50 g cm^{-3}, respectively.

a. Assuming that each Ag atom contributes one conduction electron, calculate the resistivity of Ag at 27 °C. Compare this value with the measured value of 1.6×10^{-8} Ωm at the same temperature and suggest reasons for the difference.

b. Calculate the thermal conductivity of silver at 27 °C and at 0 °C.

2.13 **Mixture rules** A 70% Cu–30% Zn brass electrical component has been made of powdered metal and contains 15 vol.% porosity. Assume that the pores are dispersed randomly. Given that the resistivity of 70% Cu–30% Zn brass is 62 nΩ m, calculate the effective resistivity of the brass component using the simple conductivity mixture rule, Equation 2.26, and the Reynolds and Hough rule.

2.14 **Mixture rules**

a. A certain carbon electrode used in electrical arcing applications is 47 percent porous. Given that the resistivity of graphite (in polycrystalline form) at room temperature is about 9.1 μΩ m, estimate the effective resistivity of the carbon electrode using the appropriate Reynolds and Hough rule and the simple conductivity mixture rule. Compare your estimates with the measured value of 18 μΩ m and comment on the differences.

b. Silver particles are dispersed in a graphite paste to increase the effective conductivity of the paste. If the volume fraction of dispersed silver is 30 percent, what is the effective conductivity of this paste?

2.15 **Ag–Ni alloys (contact materials) and the mixture rules** Silver alloys, particularly Ag alloys with the precious metals Pt, Pd, Ni, and Au, are extensively used as contact materials in various switches. Alloying Ag with other metals generally increases the hardness, wear resistance, and corrosion resistance at the expense of electrical and thermal conductivity. For example, Ag–Ni alloys are widely used as contact materials in switches in domestic appliances, control and selector switches, circuit breakers, and automotive switches up to several hundred amperes of current. Table 2.11 shows the resistivities of four Ag–Ni alloys used in make-and-break as well as disconnect contacts with current ratings up to ~100 A.

a. Ag–Ni is a two-phase alloy, a mixture of Ag-rich and Ni-rich phases. Using an appropriate mixture rule, predict the resistivity of the alloy and compare with the measured values in Table 2.11. Explain the difference between the predicted and experimental values.

b. Compare the resistivity of Ag–10% Ni with that of Ag–10% Pd in Table 2.9. The resistivity of the Ag–Pd alloy is almost a factor of 5 greater. Ag–Pd is an isomorphous solid solution, whereas Ag–Ni is a two-phase mixture. Explain the difference in the resistivities of Ag–Ni and Ag–Pd.

Table 2.11 Resistivity of Ag–Ni contact alloys for switches

	Ni % in Ag–Ni					
	0	10	15	20	30	100
ρ(nΩ m)	16.9	20.9	23.6	25	31.1	71.4
d(g cm^{-3})	10.5	10.3	9.76	9.4	9.47	8.9
Hardness VHN	30	50	55	60	65	80

NOTE: Compositions are in wt.%. Ag–10% Ni means 90% Ag–10% Ni. Vickers hardness number (VHN) is a measure of the hardness or strength of the alloy, and *d* is density.

2.16 **Ag–W alloys (contact materials) and the mixture rule** Silver–tungsten alloys are frequently used in heavy-duty switching applications (*e.g.*, current-carrying contacts and oil circuit breakers) and in arcing tips. Ag–W is a two-phase alloy, a mixture of Ag-rich and W-rich phases. The measured resistivity and density for various Ag–W compositions are summarized in Table 2.12.

a. Plot the resistivity and density of the Ag–W alloy against the W content (wt.%)

b. Show that the density of the mixture, d, is given by

$$d^{-1} = w_\alpha d_\alpha^{-1} + w_\beta d_\beta^{-1}$$

where w_α is the weight fraction of phase α, w_β is the weight fraction of phase β, d_α is the density of phase α, and d_β is the density of phase β.

c. Show that the resistivity mixture rule is

$$\rho = \rho_\alpha \frac{dw_\alpha}{d_\alpha} + \rho_\beta \frac{dw_\beta}{d_\beta}$$

Mixture rule and weight fractions

where ρ is the resistivity of the alloy (mixture), d is the density of the alloy (mixture), and subscripts α and β refer to phases α and β, respectively.

d. Calculate the density d and the resistivity ρ of the mixture for various values of W content (in wt.%) and plot the calculated values in the same graph as the experimental values. What is your conclusion?

Table 2.12 Dependence of resistivity in Ag–W alloy on composition as a function of wt.% W

	W(wt.%)												
	0	**10**	**15**	**20**	**30**	**40**	**65**	**70**	**75**	**80**	**85**	**90**	**100**
ρ (nΩ m)	16.2	18.6	19.7	20.9	22.7	27.6	35.5	38.3	40	46	47.9	53.9	55.6
d (g cm^{-3})	10.5	10.75	10.95	11.3	12	12.35	14.485	15.02	15.325	16.18	16.6	17.25	19.1

NOTE: ρ = resistivity and d = density.

2.17 **Thermal conduction** Consider brass alloys with an X atomic fraction of Zn. Since Zn addition increases the number of conduction electrons, we have to scale the final alloy resistivity calculated from the simple Matthiessen–Nordheim rule in Equation 2.22 down by a factor $(1 + X)$ (see Question 2.8) so that the resistivity of the alloy is $\rho \approx [\rho_o + CX(1 - X)]/(1 + X)$ in which $C = 300$ nΩ m and $\rho_o = \rho_{\text{Cu}} = 17$ nΩ m.

a. An 80 at.% Cu–20 at.% Zn brass disk of 40 mm diameter and 5 mm thickness is used to conduct heat from a heat source to a heat sink.

(1) Calculate the thermal resistance of the brass disk.

(2) If the disk is conducting heat at a rate of 100 W, calculate the temperature drop along the disk.

b. What should be the composition of brass if the temperature drop across the disk is to be halved?

2.18 **Thermal resistance** Consider a thin insulating disk made of mica to electrically insulate a semiconductor device from a conducting heat sink. Mica has $\kappa = 0.75$ W m^{-1} K^{-1}. The disk thickness is 0.1 mm, and the diameter is 10 mm. What is the thermal resistance of the disk? What is the temperature drop across the disk if the heat current through it is 25 W?

***2.19** **Thermal resistance** Consider a coaxial cable operating under steady-state conditions when the current flow through the inner conductor generates Joule heat at a rate $P = I^2R$. The heat generated per second by the core conductor flows through the dielectric; $Q' = I^2R$. The inner conductor reaches a temperature T_i, whereas the outer conductor is at T_o. Show that the thermal resistance θ of the hollow cylindrical insulation for heat flow in the radial direction is

$$\theta = \frac{(T_i - T_o)}{Q'} = \frac{\ln(b/a)}{2\pi\kappa L} \qquad [2.65]$$

Thermal resistance of hollow cylinder

where a is the inside (core conductor) radius, b is the outside radius (outer conductor), κ is the thermal conductivity of the insulation, and L is the cable length. Consider a coaxial cable that has a copper core conductor and polyethylene (PE) dielectric with the following properties: Core conductor resistivity $\rho = 19$ nΩ m, core radius $a = 4$ mm, dielectric thickness $b - a = 3.5$ mm, dielectric thermal conductivity $\kappa = 0.3$ W m^{-1} K^{-1}. The outside temperature T_o is 25 °C. The cable is carrying a current of 500 A. What is the temperature of the inner conductor?

2.20 **The Hall effect** Consider a rectangular sample, a metal or an n-type semiconductor, with a length L, width W, and thickness D. A current I is passed along L, perpendicular to the cross-sectional area WD. The face $W \times L$ is exposed to a magnetic field density B. A voltmeter is connected across the width, as shown in Figure 2.40, to read the Hall voltage V_H.

a. Show that the Hall voltage recorded by the voltmeter is

Hall voltage

$$V_H = \frac{IB}{Den}$$

b. Consider a 1-micron-thick strip of gold layer on an insulating substrate that is a candidate for a Hall probe sensor. If the current through the film is maintained at constant 100 mA, what is the magnetic field that can be recorded per μV of Hall voltage?

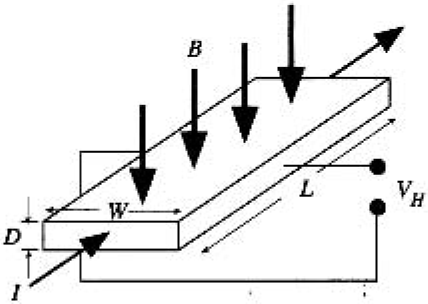


Figure 2.40 Hall effect in a rectangular material with length L, width W, and thickness D.
The voltmeter is across the width W.

2.21 **The strain gauge** A **strain gauge** is a transducer attached to a body to measure its fractional elongation $\Delta L/L$ under an applied load (force) F. The gauge is a grid of many folded runs of a thin, resistive wire glued to a flexible backing, as depicted in Figure 2.41. The gauge is attached to the body under test such that the resistive wire length is parallel to the strain.

a. Assume that the elongation does not change the resistivity and show that the change in the resistance ΔR is related to the strain $\varepsilon = \Delta L/L$ by

Strain gauge equation

$$\Delta R \approx R(1 + 2\nu)\varepsilon \qquad [2.66]$$

where ν is the **Poisson ratio**, which is defined by

Poisson ratio

$$\nu = -\frac{\text{Transverse strain}}{\text{Longitudinal strain}} = -\frac{\varepsilon_t}{\varepsilon_l} \qquad [2.67]$$

where ε_l is the strain along the applied load, that is, $\varepsilon_l = \Delta L/L = \varepsilon$, and ε_t is the strain in the transverse direction, that is, $\varepsilon_t = \Delta D/D$, where D is the diameter (thickness) of the wire.

b. Explain why a nichrome wire would be a better choice than copper for the strain gauge (consider the TCR).

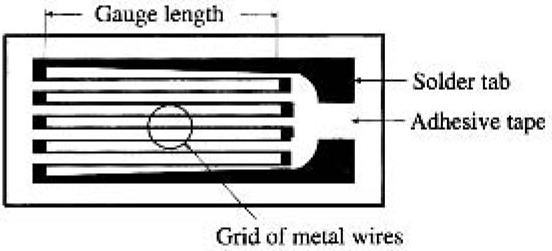


Figure 2.41 The strain gauge consists of a long, thin wire folded several times along its length to form a grid as shown and embedded in a self-adhesive tape.
The ends of the wire are attached to terminals (solder pads) for external connections. The tape is stuck on the component for which the strain is to be measured.

c. How do temperature changes affect the response of the gauge? Consider the effect of temperature on ρ. Also consider the differential expansion of the specimen with respect to the gauge wire such that even if there is no applied load, there is still strain, which is determined by the differential expansion coefficient, $\lambda_{\text{specimen}} - \lambda_{\text{gauge}}$, where λ is the thermal coefficient of linear expansion: $L = L_0[1 + \lambda(T - T_0)]$, where T_0 is the reference temperature.

d. The gauge factor for a transducer is defined as the fractional change in the measured property $\Delta R/R$ per unit input signal (ε). What is the gauge factor for a metal-wire strain gauge, given that for most metals, $\nu \approx \frac{1}{3}$?

e. Consider a strain gauge that consists of a nichrome wire of resistivity 1 μΩ m, a total length of 1 m, and a diameter of 25 μm. What is ΔR for a strain of 10^{-3}? Assume that $\nu \approx \frac{1}{3}$.

f. What will ΔR be if constantan wire with a resistivity of 500 nΩ m is used?

2.22 Thermal coefficients of expansion and resistivity

a. Consider a thin metal wire of length L and diameter D. Its resistance is $R = \rho L/A$, where $A = \pi D^2/4$. By considering the temperature dependence of L, A, and ρ individually, show that

$$\frac{1}{R}\frac{dR}{dT} = \alpha_0 - \lambda_0$$

Change in R with temperature

where α_0 is the temperature coefficient of resistivity (TCR), and λ_0 is the temperature coefficient of linear expansion (thermal expansion coefficient or expansivity), that is,

$$\lambda_0 = L_0^{-1}\left(\frac{dL}{dT}\right)_{T=T_0} \qquad \text{or} \qquad \lambda_0 = D_0^{-1}\left(\frac{dD}{dT}\right)_{T=T_0}$$

Note: Consider differentiating $R = \rho L/[(\pi D^2)/4]$ with respect to T with each parameter, ρ, L, and D, having a temperature dependence.

Given that typically, for most pure metals, $\alpha_0 \approx 1/273\ \text{K}^{-1}$ and $\lambda_0 \approx 2 \times 10^{-5}\ \text{K}^{-1}$, confirm that the temperature dependence of ρ controls R, rather than the temperature dependence of the geometry. Is it necessary to modify the given equation for a wire with a noncircular cross section?

b. Is it possible to design a resistor from a suitable alloy such that its temperature dependence is almost nil? Consider the TCR of an alloy of two metals A and B, for which $\alpha_{AB} \approx \alpha_A \rho_A/\rho_{AB}$.

2.23 Temperature of a light bulb filament

a. Consider a 100 W, 120 V incandescent bulb (lamp). The tungsten filament has a length of 0.579 m and a diameter of 63.5 μm. Its resistivity at room temperature is 56 nΩ m. Given that the resistivity of the filament can be represented as

$$\rho = \rho_0\left[\frac{T}{T_0}\right]^n \qquad [2.68]$$

Resistivity of W

where T is the temperature in K, ρ_0 is the resistance of the filament at T_0 K, and $n = 1.2$, estimate the temperature of the bulb when it is operated at the rated voltage, that is, directly from the main outlet. Note that the bulb dissipates 100 W at 120 V.

b. Suppose that the electrical power dissipated in the tungsten wire is totally radiated from the surface of the filament. The radiated power at the absolute temperature T can be described by Stefan's law

$$P_{\text{radiated}} = \epsilon\sigma_S A\left(T^4 - T_0^4\right) \qquad [2.69]$$

Radiated power

where σ_S is Stefan's constant ($5.67 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$), ϵ is the emissivity of the surface (0.35 for tungsten), A is the surface area of the tungsten filament, and T_0 is room temperature (293 K). Obviously, for $T > T_0$, $P_{\text{radiated}} = \epsilon\sigma_S A T^4$.

Assuming that all the electrical power is radiated from the surface, estimate the temperature of the filament and compare it with your answer in part (a).

c. If the melting temperature of W is 3407 °C, what is the voltage that guarantees that the light bulb will blow?

2.24 Einstein relation and ionic conductivity In the case of ionic conduction, ions have to jump from one interstice to the neighboring one. This process involves overcoming a potential energy barrier just like atomic diffusion, and drift and diffusion are related. The drift mobility μ of ions is proportional to the diffusion coefficient D because drift is limited by the atomic diffusion process. The **Einstein relation** relates the two by

Einstein relation

$$\frac{D}{\mu} = \frac{kT}{e} \qquad [2.70]$$

Diffusion coefficient of the Na^+ ion in sodium silicate (Na_2O–SiO_2) glasses at 400 °C is 3.4×10^{-9} $cm^2\ s^{-1}$. The density of such glasses is approximately 2.4 g cm^{-3}. Calculate the ionic conductivity and resistivity of (17.5 mol% Na_2O)(82.5 mol% SiO_2) sodium silicate glass at 400 °C and compare your result with the experimental values of the order of 10^4 Ω cm for the resistivity.

2.25 Skin effect

a. What is the skin depth for a copper wire carrying a current at 60 Hz? The resistivity of copper at 27 °C is 17 nΩ m. Its relative permeability $\mu_r \approx 1$. Is there any sense in using a conductor for power transmission which has a diameter more than 2 cm?

b. What is the skin depth for an iron wire carrying a current at 60 Hz? The resistivity of iron at 27 °C is 97 nΩ m. Assume that its relative permeability $\mu_r \approx 700$. How does this compare with the copper wire? Discuss why copper is preferred over iron for power transmission even though iron is nearly 100 times cheaper than copper.

2.26 Thin films

a. Consider a polycrystalline copper film that has $R = 0.40$. What is the approximate mean grain size d in terms of the mean free path λ in the bulk that would lead to the polycrystalline Cu film having a resistivity that is $1.5\rho_{bulk}$. If the mean free path in the crystal is about 40 nm at room temperature, what is d?

b. What is the thickness D of a copper film in terms of λ in which surface scattering increases the film resistivity to $1.2\rho_{bulk}$ if the specular scattering fraction p is 0.5?

c. Consider the data of Lim *et al.* (2003) presented in Table 2.13. Show that the excess resistivity, *i.e.* resistivity above that of bulk Cu, is roughly proportional to the reciprocal film thickness.

Table 2.13 Resistivity ρ_{film} of a copper film as a function of thickness D.

D (nm)	8.61	17.2	34.4	51.9	69	85.8	102.6	120.3	173.2	224.3
ρ_{film} (nΩ m)	121.8	75.3	46.1	38.5	32.1	25.2	22.0	20.5	19.9	18.8

NOTE: Film annealed at 150 °C.
SOURCE: Data extracted from J. W. Lim *et al.*, *Appl. Surf. Sci.* **217**, 95, 2003.

2.27 Interconnects Consider a high-transistor-density CMOS chip in which the interconnects are copper with a pitch P of 500 nm, interconnect thickness T of 400 nm, aspect ratio 1.4, and $H = X$. The dielectric is FSG with $\varepsilon_r = 3.6$. Consider two cases, $L = 1$ mm and $L = 10$ mm, and calculate the overall effective interconnect capacitance C_{eff} and the RC delay time. Suppose that Al, which is normally Al with about 4 wt.% Cu in the microelectronics industry with a resistivity 31 nΩ m, is used as the interconnect. What is the corresponding RC delay time?

***2.28 Thin 50 nm interconnects** Equation 2.60 is for conduction in a thin film of thickness D and assumes scattering from two surfaces, which yields an additional resistivity $\rho_2 = \rho_{bulk}\frac{3}{8}(\lambda/D)(1-p)$. An interconnect line in an IC is not quite a thin film and has four surfaces (interfaces), because the thickness T of the conductor is comparable to the width W. If we assume $T = W$, we can very roughly take $\rho_4 \approx \rho_2 + \rho_2 \approx \rho_{bulk}\frac{3}{4}(\lambda/D)(1-p)$ in which $D = T$. (The exact expression is more complicated, but the latter will suffice for this problem.) In addition there will be a contribution from grain boundary scattering, (Equation 2.57a). For simplicity assume $T \approx W \approx X \approx H \approx 60$ nm, $\lambda = 40$ nm, $p = 0.5$ and $\varepsilon_r = 3.6$. If the mean grain size d is roughly 40 nm and $R = 0.4$, estimate the resistivity of the interconnect and hence the RC delay for a 1 mm interconnect.

2.29 **TCR of thin films** Consider Matthiessen's rule applied to a thin film. Show that, very approximately, the product of the thermal coefficient of resistivity (TCR) α_{film} and the resistivity ρ_{film} is equivalent to the product of the bulk TCR and resistivity:

$$\alpha_{\text{film}}\rho_{\text{film}} \approx \alpha_{\text{bulk}}\rho_{\text{bulk}}$$

2.30 **Electromigration** Although electromigration-induced failure in Cu metallization is less severe than in Al metallization, it can still lead to interconnect failure depending on current densities and the operating temperature. In a set of experiments carried out on electroplated Cu metallization lines, failure of the Cu interconnects have been examined under accelerated tests (at elevated temperatures). The mean lifetime t_{50} (time for 50 percent of the lines to break) have been measured as a function of current density J and temperature T at a given current density. The results are summarized in Table 2.14.

a. Plot semilogarithmically t_{50} versus $1/T$ (T in Kelvins) for the first three interconnects. Al(Cu) and Cu ($1.3 \times 0.7\mu\text{m}^2$) have single activation energies E_A. Calculate E_A for these interconnects. Cu ($1.3 \times 0.7\mu\text{m}^2$) exhibits different activation energies for the high-and low-temperature regions. Estimate these E_A.

b. Plot on a log-log plot t_{50} versus J at 370 °C. Show that at low J, $n \approx 1.1$ and at high J, $n \approx 1.8$.

Table 2.14 Results of electromigration failure experiments on various Al and Cu interconnects

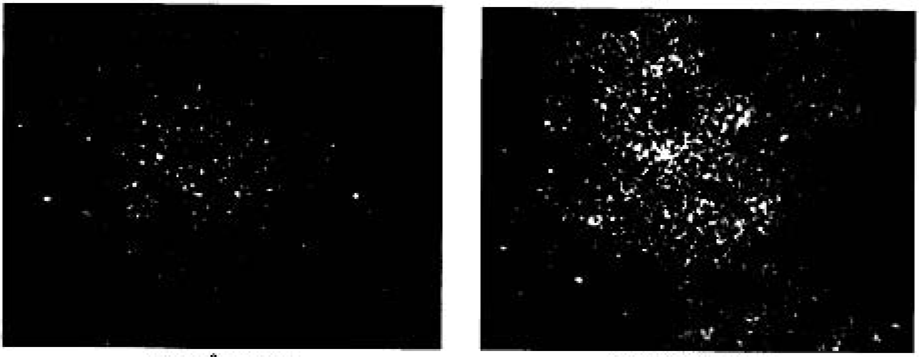
Al(Cu) [$J = 25$ mA/μm^2, $A = 0.35 \times 0.2\ (\mu\text{m})^2$]		Cu [$J = 25$ mA/μm^2, $A = 0.24 \times 0.28\ (\mu\text{m})^2$]		Cu [$J = 25$ mA/μm^2, $A = 1.3 \times 0.7\ (\mu\text{m})^2$]		Cu ($T = 370$ °C)	
T (°C)	**t_{50} (hr)**	**T (°C)**	**t_{50} (hr)**	**T (°C)**	**t_{50} (hr)**	**J mA μm^{-2}**	**t_{50} (hr)**
365	0.11	397	2.87	395	40.3	3.54	131.5
300	0.98	354	12.8	360	196	11.7	25.2
259	5.73	315	70.53	314	825	24.8	14.9
233	15.7	269	180	285	2098	49.2	4.28
		232	899			74.1	2.29
						140	0.69

NOTE: A = width × height in micron^2.

SOURCE: Data extracted from R. Rosenberg *et al.*, [IBM, T. J. Watson Research Center, *Annu. Rev. Mater. Sci.*, **30**, 229, 2000, figures 29 and 31, and subject to small extraction errors.]

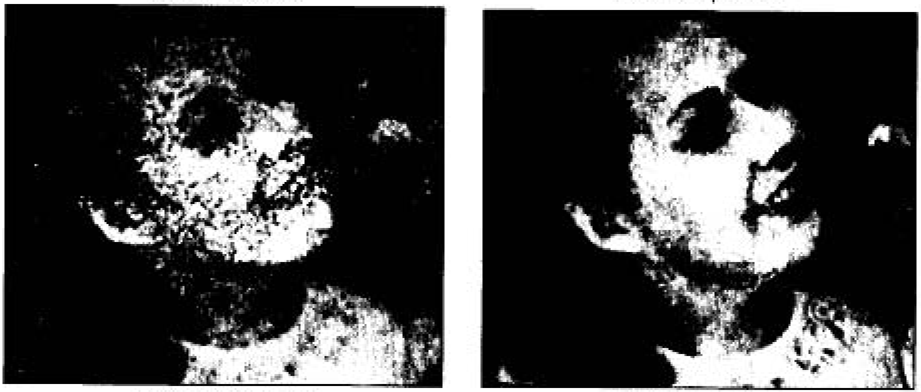
Gordon Teal (Left) and Morgan Sparks fabricated the first grown-junction Ge transistor in 1950–1951 at Bell Labs. Gordon Teal started at Bell Labs but later moved to Texas Instruments where he lead the development of the first commercial Si transistor; the first Si transistor was made at Bell Labs by Morris Tanenbaum.

SOURCE: Courtesy of Bell Laboratories, Lucent Technologies.



3×10^3 photons

1.2×10^4 photons



9.3×10^4 photons

7.6×10^5 photons

3.6×10^6 photons

2.8×10^7 photons

These electronic images were made with the number of photons indicated. The discrete nature of photons means that a large number of photons are needed to constitute an image with satisfactorily discernable details.

SOURCE: A. Rose, "Quantum and noise limitations of the visual process" *J. Opt. Soc. of America*, vol. 43, 715, 1953. (Courtesy of OSA.)