CHAPTER
7

Dielectric Materials and Insulation

The familiar parallel plate capacitor equation with free space as an insulator is given by

$$C = \frac{\varepsilon_o A}{d}$$

where ε_o is the absolute permittivity, A is the plate area, and d is the separation between the plates. If there is a material medium between the plates, then the capacitance, the charge storage ability per unit voltage, increases by a factor of ε_r, where ε_r is called the **dielectric constant** of the medium or its **relative permittivity.** The increase in the capacitance is due to the **polarization** of the medium in which positive and negative charges are displaced with respect to their equilibrium positions. The opposite surfaces of the dielectric medium acquire opposite surface charge densities that are related to the amount of polarization in the material. An important concept in dielectric theory is that of an **electric dipole moment** p, which is a measure of the electrostatic effects of a pair of opposite charges $+Q$ and $-Q$ separated by a finite distance a, and so is defined by

$$p = Qa$$

Although the net charge is zero, this entity still gives rise to an electric field in space and also interacts with an electric field from other sources. The relative permittivity is a material property that is frequency dependent. Some capacitors are designed to work at low frequencies, whereas others have a wide frequency range. Furthermore, even though they are regarded as energy storage devices, all practical capacitors exhibit some losses when used in an electric circuit. These losses are no different than I^2R losses in a resistor carrying a current. The power dissipation in a practical capacitor depends on the frequency, and for some applications it can be an important factor. A defining property of a dielectric medium is not only its ability to increase capacitance but also, and equally important, its insulating behavior or low conductivity so that the charges are not conducted from one plate of the capacitor to the other through the dielectric. Dielectric materials often serve to insulate current-carrying conductors or conductors at different voltages. Why can we not simply use air as insulation between high-voltage conductors? When the electric field inside an insulator exceeds a critical field called the **dielectric strength,** the medium suffers dielectric breakdown and a large discharge current flows through the dielectric. Some 40 percent of utility generator failures are linked to insulation failures in the generator. Dielectric breakdown is probably one of the oldest electrical engineering problems and that which has been most widely studied and never fully explained.

7.1 MATTER POLARIZATION AND RELATIVE PERMITTIVITY

7.1.1 RELATIVE PERMITTIVITY: DEFINITION

We first consider a parallel plate capacitor with vacuum as the dielectric medium between the plates, as shown in Figure 7.1a. The plates are connected to a constant voltage supply V. Let Q_o be the charge on the plates. This charge can be easily measured. The capacitance C_o of the parallel plate capacitor in free space, as in Figure 7.1a, is defined by

Definition of capacitance

$$C_o = \frac{Q_o}{V} \tag{7.1}$$

The electric field, directed from high to low potential, is defined by the gradient of the potential $\mathcal{E} = -dV/dx$. Thus, the electric field $\mathcal{E}$ between the plates is just V/d where d is the separation of the plates.

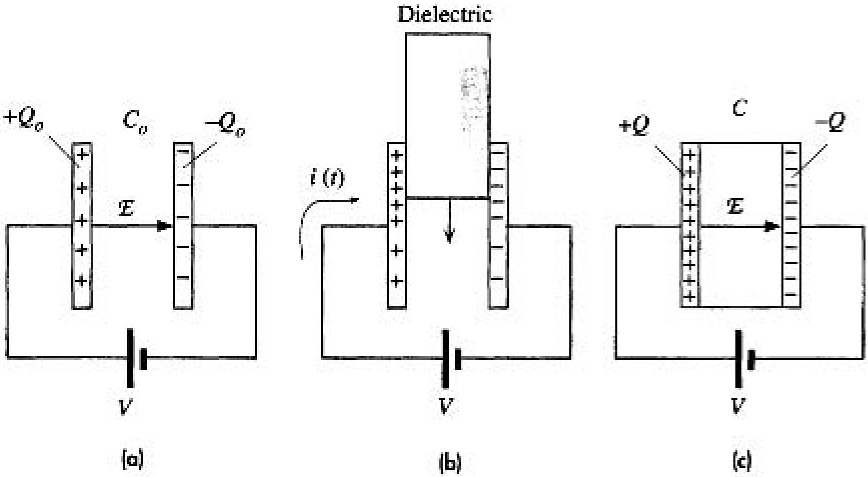

Figure 7.1

(a) Parallel plate capacitor with free space between the plates.

(b) As a slab of insulating material is inserted between the plates, there is an external current flow indicating that more charge is stored on the plates.

(c) The capacitance has been increased due to the insertion of a medium between the plates.

Consider now what happens when a dielectric slab (a slab of any nonconducting material) is inserted into this parallel plate capacitor, as shown in Figure 7.1b and c with V kept the same. During the insertion of the dielectric slab, there is an external current flow that indicates that there is additional charge being stored on the plates. The charge on the electrodes increases from Q_o to Q. We can easily measure the extra charge $Q - Q_o$ flowing from the battery to the plates by integrating the observed current in the circuit during the process of insertion, as shown in Figure 7.1b. Because there is now a greater amount of charge stored on the plates, the capacitance of the system in Figure 7.1c is larger than that in Figure 7.1a by the ratio Q to Q_o. The **relative permittivity** (or the **dielectric constant**) ε_r is defined to reflect this increase in the capacitance or the charge storage ability by virtue of having a dielectric medium. If C is the capacitance with the dielectric medium as in Figure 7.1c, then by definition

Definition of relative permittivity

$$\varepsilon_r = \frac{Q}{Q_o} = \frac{C}{C_o} \qquad [7.2]$$

The increase in the stored charge is due to the polarization of the dielectric by the applied field, as explained below. It is important to remember that when the dielectric medium is inserted, the electric field remains unchanged, provided that the insulator fills the whole space between the plates as shown in Figure 7.1c. The voltage V remains the same and therefore so does the gradient V/d, which means that $\mathcal{E}$ remains constant.

7.1.2 DIPOLE MOMENT AND ELECTRONIC POLARIZATION

An electrical dipole moment is simply a separation between a negative and positive charge of equal magnitude Q as shown in Figure 7.2. If $\mathbf{a}$ is the vector from the negative to the positive charge, the **electric dipole moment** is defined as a vector by

Definition of dipole moment

$$\mathbf{p} = Q\mathbf{a} \qquad [7.3]$$

The region that contains the $+Q$ and $-Q$ charges has zero net charge. Unless the two charge centers coincide, this region will nonetheless, by virtue of the definition in Equation 7.3, contain a dipole moment.

The net charge within a neutral atom is zero. Furthermore, on average, the center of negative charge of the electrons coincides with the positive nuclear charge, which means that the atom has no net dipole moment, as indicated in Figure 7.3a. However, when this atom is placed in an external electric field, it will develop an induced dipole moment. The electrons, being much lighter than the positive nucleus, become easily displaced by the field, which results in the separation of the negative charge center

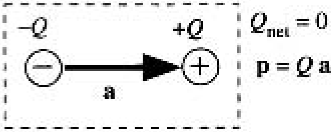

Figure 7.2 The definition of electric dipole moment.

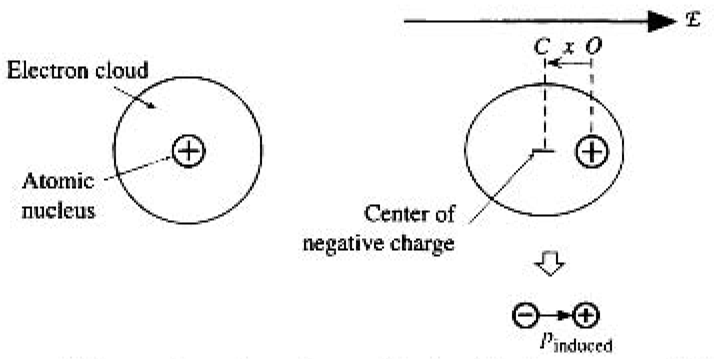

Figure 7.3 The origin of electronic polarization.

(a) A neutral atom in $\mathcal{E} = 0$

(b) Induced dipole moment in a field

from the positive charge center, as shown in Figure 7.3b. This separation of negative and positive charges and the resulting induced dipole moment are termed **polarization.** An atom is said to be **polarized** if it possesses an effective dipole moment, that is, if there is a separation between the centers of negative and positive charge distributions.

The induced dipole moment depends on the electric field causing it. We define a quantity called the **polarizability** α to relate the induced dipole moment p_{induced} to the field $\mathcal{E}$ causing it,

Definition of polarizability

$$p_{\text{induced}} = \alpha\mathcal{E} \tag{7.4}$$

where α is a coefficient called the polarizability of the atom. It depends on the polarization mechanism. Since the polarization of a neutral atom involves the displacement of electrons, α is called **electronic polarization** and denoted as α_e. Inasmuch as the electrons in an atom are not rigidly fixed, all atoms possess a certain amount of electronic polarizability.

In the absence of an electric field, the center of mass of the orbital motions of the electrons coincides with the positively charged nucleus and the electronic dipole moment is zero. Suppose that the atom has Z number of electrons orbiting the nucleus and all the electrons are contained within a certain spherical region. When an electric field $\mathcal{E}$ is applied, the light electrons become displaced in the opposite direction to $\mathcal{E}$, so their center of mass C is shifted by some distance x with respect to the nucleus O, which we take to be the origin, as shown in Figure 7.3b. As the electrons are "pushed" away by the applied field, the Coulombic attraction between the electrons and nuclear charge "pulls in" the electrons. The force on the electrons, due to $\mathcal{E}$, trying to separate them away from the nuclear charge is $Ze\mathcal{E}$. The restoring force F_r, which is the Coulombic attractive force between the electrons and the nucleus, can be taken to be proportional to the displacement x, provided that the latter is small.[1] The restoring force F_r is obviously zero when C coincides with $O\,(x = 0)$. We can write

Restoring force

$$F_r = -\beta x$$

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[1] It may be noticed that even if F_r is a complicated function of x, it can still be expanded in a series in terms of powers of x, that is, x, x^2, x^3, and so on, and for small x only the x term is significant, $F_r = -\beta x$.

where β is a constant and the negative sign indicates that F_r is always directed toward the nucleus O (Figure 7.3b). In equilibrium, the net force on the negative charge is zero or

$$Ze\mathcal{E} = \beta x$$

from which x is known. Therefore the **magnitude** of the induced electronic dipole moment p_e is given by

Electronic polarization

$$p_e = (Ze)x = \left(\frac{Z^2e^2}{\beta}\right)\mathcal{E} \qquad [7.5]$$

As expected, p_e is proportional to the applied field. The electronic dipole moment in Equation 7.5 is valid under static conditions, that is, when the electric field is a dc field. Suppose that we suddenly remove the applied electric field polarizing the atom. There is then only the restoring force $-\beta x$, which always acts to pull the electrons toward the nucleus O. The equation of motion of the negative charge center is then (from force = mass × acceleration)

Equation for simple harmonic motion

$$-\beta x = Zm_e\frac{d^2x}{dt^2}$$

Thus the displacement at any time is

$$x(t) = x_o \cos(\omega_o t)$$

where

Electronic polarization resonance frequency

$$\omega_o = \left(\frac{\beta}{Zm_e}\right)^{1/2} \qquad [7.6]$$

is the oscillation frequency of the center of mass of the electron cloud about the nucleus and x_o is the displacement before the removal of the field. After the removal of the field, the electronic charge cloud executes simple harmonic motion about the nucleus with a natural frequency determined by Equation 7.6; ω_o is called the **electronic polarization resonance frequency.**[2] It is analogous to a mass on a spring being pulled and let go. The system then executes simple harmonic motion. The oscillations of course die out with time. In the atomic case, a sinusoidal displacement implies that the electronic charge cloud has an acceleration

$$\frac{d^2x}{dt^2} = -x_o\omega_o^2\cos(\omega_o t)$$

It is well known from classical electromagnetism that an accelerating charge radiates electromagnetic energy just like a radio antenna. Consequently the oscillating charge

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[2] The term *natural frequency* refers to a system's characteristic frequency of oscillation when it is excited. A mass attached to a spring and then let go will execute simple harmonic motion with a certain natural frequency ω_o. If we then decide to oscillate this mass with an applied force, the maximum energy transfer will occur when the applied force has the same frequency as ω_o; the system will be put in resonance. ω_o is also a *resonant frequency*. Strictly, $\omega = 2\pi f$ is the angular frequency and f is the frequency. It is quite common to simply refer to ω as a frequency because the literature is dominated by ω; the meaning should be obvious within context.

cloud loses energy, and thus its amplitude of oscillation decreases. (Recall that the average energy is proportional to the square of the amplitude of the displacement.)

From the expression derived for p_e in Equation 7.5, we can find the electronic polarizability α_e from Equation 7.4,

Static electronic polarizability

$$\alpha_e = \frac{Ze^2}{m_e\omega_o^2} \tag{7.7}$$

EXAMPLE 7.1

ELECTRONIC POLARIZABILITY Consider the electronic polarizability of inert gas atoms. These atoms have closed shells. Their electronic polarizabilities are listed in Table 7.1. For each type of atom calculate the electronic polarization resonant frequency $f_o = \omega_o/2\pi$, and plot α_e and f_o against the number of electrons Z in the atom. What is your conclusion?

SOLUTION

We can use Equation 7.7 to calculate the resonant frequency $f_o = \omega_o/2\pi$. Taking Ar,

$$\omega_o = \left(\frac{Ze^2}{\alpha_e m_e}\right)^{1/2} = \left[\frac{(18)(1.6\times10^{-19})^2}{(1.7\times10^{-40})(9.1\times10^{-31})}\right]^{1/2} = 5.46\times10^{16}\text{ rad s}^{-1}$$

Table 7.1 Electronic polarizability α_e dependence on Z for the inert element atoms

	Atom					
	He	**Ne**	**Ar**	**Kr**	**Xe**	**Rn***
Z	2	10	18	36	56	
$\alpha_e\times10^{-40}$ (F m^2)	0.18	0.45	1.7	2.7	4.4	5.9
$f_o\times10^{15}$ (Hz)	8.90	12.6	8.69	9.76	9.36	10.2

*Rn (radon) gas is radioactive.

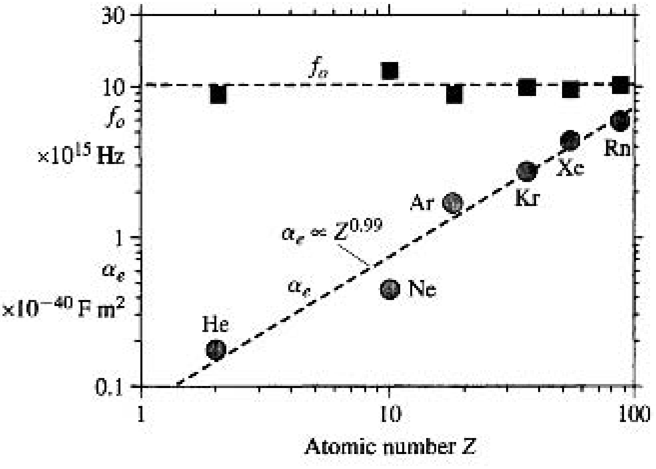

Figure 7.4 Electronic polarizability and its resonance frequency versus the number of electrons in the atom (Z).
The dashed line is the best-fit line.

so that

$$f_o = \frac{\omega_o}{2\pi} = 8.69 \times 10^{15}\ \text{Hz}$$

which is listed in Table 7.1, among other f_o calculations for the other atoms. Such frequencies correspond to the field oscillations in UV light, that is, at optical frequencies. For all practical purposes, electronic polarization occurs very rapidly, that is, on a time scale $1/f_o$ or 10^{-15} s, and we can take the static polarizability α_e to remain the same up to optical frequencies.[3]

Figure 7.4 shows the dependence of α_e and f_o on the number of electrons Z. It is apparent that α_e is nearly linearly proportional to Z, whereas f_o is very roughly constant. It is left as an exercise to show that β increases with Z, which is reasonable since the restoring force was defined as the total force between *all* the electrons and the nucleus when the electrons are displaced.

7.1.3 POLARIZATION VECTOR P

When a material is placed in an electric field, the atoms and the molecules of the material become polarized, so we have a distribution of dipole moments in the material. We can visualize this effect with the insertion of the dielectric slab into the parallel plate capacitor, as depicted in Figure 7.5a. The placement of the dielectric slab into an electric field polarizes the molecules in the material. The induced dipole moments all point in the direction of the field. Consider the polarized medium alone, as shown in Figure 7.5b. In the bulk of the material, the dipoles are aligned head to tail. Every positive charge has a negative charge next to it and vice versa. There is therefore no net charge within the bulk. But the positive charges of the dipoles appearing at the right-hand face are not canceled by negative charges of any dipoles at this face. There is therefore a surface charge $+Q_P$ on the right-hand face that results from the polarization of the medium. Similarly, there is a negative charge $-Q_P$ with the same magnitude appearing on the left-hand face due to the negative charges of the dipoles at this face. We see that charges $+Q_P$ and $-Q_P$ appear on the opposite surfaces of a material when it becomes polarized in an electric field, as shown in Figure 7.5c. These charges are **bound** and are a direct result of the polarization of the molecules. They are termed **surface polarization charges.** Figure 7.5c emphasizes this aspect of dielectric behavior in an electric field by showing the dielectric and its polarization charges only.

We represent the polarization of a medium by a quantity called **polarization P,** which is defined as the total dipole moment per unit volume,

Definition of polarization vector

$$\mathbf{P} = \frac{1}{\text{Volume}}[\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N] \qquad [7.8a]$$

where $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$ are the dipole moments induced at N molecules in the volume. If $\mathbf{p}_{av}$ is the average dipole moment per molecule, then an equivalent definition of $\mathbf{P}$ is

Definition of polarization vector

$$\mathbf{P} = N\mathbf{p}_{av} \qquad [7.8b]$$

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[3] Electronic polarization at optical frequencies controls the optical properties such as the refractive index, as will be covered in Chapter 9.

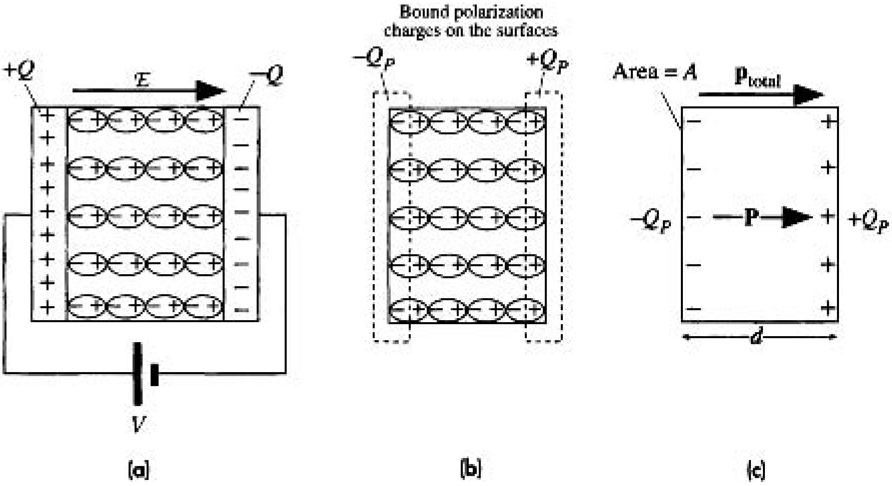

Figure 7.5

(a) When a dielectric is placed in an electric field, bound polarization charges appear on the opposite surfaces.

(b) The origin of these polarization charges is the polarization of the molecules of the medium.

(c) We can represent the whole dielectric in terms of its surface polarization charges $+Q_P$ and $-Q_P$.

where N is the number of molecules per unit volume. There is an important relationship, given below, between **P** and the polarization charges Q_P on the surfaces of the dielectric. It should be emphasized for future discussions that if polarization arises from the effect of the applied field, as shown in Figure 7.5a, which is usually the case, $\mathbf{p}_{av}$ must be the *average dipole moment per atom in the direction of the applied field.* In that case we often also denote $\mathbf{p}_{av}$ as the induced average dipole moment per molecule $\mathbf{p}_{induced}$.

To calculate the polarization **P** for the polarized dielectric in Figure 7.5b, we need to sum all the dipoles in the medium and divide by the volume Ad, as in Equation 7.8a. However, the polarized medium can be simply represented as in Figure 7.5c in terms of surface charge $+Q_P$ and $-Q_P$, which are separated by the thickness distance d. We can view this arrangement as one big dipole moment p_{total} from $-Q_P$ to $+Q_P$. Thus

$$p_{total} = Q_P d$$

Since the polarization is defined as the total dipole moment per unit volume, the magnitude of **P** is

Polarization and bound surface charge density

$$P = \frac{p_{total}}{\text{Volume}} = \frac{Q_P d}{Ad} = \frac{Q_P}{A}$$

But Q_P/A is the **surface polarization charge density** σ_P, so

$$P = \sigma_p \tag{7.9a}$$

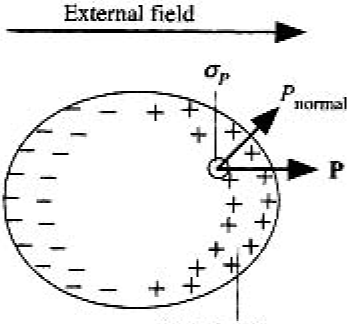

Figure 7.6 Polarization charge density on the surface of a polarized medium is related to the normal component of the polarization vector.

Polarization is a vector and Equation 7.9a only gives its magnitude. For the rectangular slab in Figure 7.5c, the direction of P is normal to the surface. For $+\sigma_p$ (right face), it comes out from the surface and for $-\sigma_P$ (left face), it is directed into the surface. Although Equation 7.9a is derived for one specific geometry, the rectangular slab, it can be generalized as follows. *The charge per unit area appearing on the surface of a polarized medium is equal to the component of the polarization vector normal to this surface.* If P_{normal} is the component of **P** normal to the surface where the polarization charge density is σ_P, as shown in Figure 7.6, then,

Polarization and bound surface charge density

$$P_{\text{normal}} = \sigma_p \tag{7.9b}$$

The polarization P induced in a dielectric medium when it is placed in an electric field depends on the field itself. The induced dipole moment per molecule within the medium depends on the electric field by virtue of Equation 7.4. To express the dependence of P on the field $\mathcal{E}$, we define a quantity called the **electric susceptibility** χ_e by

Definition of electric susceptibility

$$P = \chi_e \varepsilon_o \mathcal{E} \tag{7.10}$$

Equation 7.10 shows an *effect* P due to a *cause* $\mathcal{E}$ and the quantity χ_e relates the effect to its cause. Put differently, χ_e acts as a proportionality constant. It may depend on the field itself, in which case the effect is nonlinearly related to the cause. Further, electronic polarizability is defined by

$$p_{\text{induced}} = \alpha_e \mathcal{E}$$

so

$$P = Np_{\text{induced}} = N\alpha_e \mathcal{E}$$

where N is the number of molecules per unit volume. Then from Equation 7.10, χ_e and α_e are related by

Electric susceptibility and polarization

$$\chi_e = \frac{1}{\varepsilon_o} N\alpha_e \tag{7.11}$$

It is important to recognize the difference between *free* and *polarization* (or *bound*) charges. The charges stored on the metal plates in Figure 7.5a are free because they result from the motion of free electrons in the metal. For example both Q_o and Q, before and after the dielectric insertion in Figure 7.1, are free charges that arrive on the plates from the battery. The polarization charges $+Q_P$ and $-Q_P$, on the other hand, are bound to the molecules. They cannot move within the dielectric or on its surface.

The field $\mathcal{E}$ *before* the dielectric was inserted (Figure 7.1a) is given by

$$\mathcal{E} = \frac{V}{d} = \frac{Q_o}{C_o d} = \frac{Q_o}{\varepsilon_o A} = \frac{\sigma_o}{\varepsilon_o} \qquad [7.12]$$

where $\sigma_o = Q_o/A$ is the **free surface charge density** without any dielectric medium between the plates, as in Figure 7.1a.

After the insertion of the dielectric, this field remains the same V/d, but the free charges on the plates are different. The free surface charge on the plates is now Q. In addition there are bound polarization charges on the dielectric surfaces next to the plates, as shown in Figure 7.5a. It is apparent that the flow of current during the insertion of the dielectric, Figure 7.1b, is due to the additional free charges $Q - Q_o$ needed on the capacitor plates to neutralize the opposite polarity polarization charges Q_P appearing on the dielectric surfaces. The total charge (see Figure 7.5a) due to that on the plate plus that appearing on the dielectric surface, $Q - Q_P$, must be the same as before, Q_o, so that the field, as given by Equation 7.12, does not change inside the dielectric, that is,

$$Q - Q_P = Q_o$$

or

$$Q = Q_o + Q_P$$

Dividing by A, defining $\sigma = Q/A$ as the free surface charge density on the plates with the dielectric inserted, and using Equation 7.12, we obtain

$$\sigma = \varepsilon_o \mathcal{E} + \sigma_P$$

Since $\sigma_P = P$ and $P = \chi_e \varepsilon_o \mathcal{E}$, Equations 7.9 and 7.10, we can eliminate σ_P to obtain

$$\sigma = \varepsilon_o (1 + \chi_e)\mathcal{E}$$

From the definition of the relative permittivity in Equation 7.2 we have

$$\varepsilon_r = \frac{Q}{Q_o} = \frac{\sigma}{\sigma_o}$$

so substituting for σ and using Equation 7.12 we obtain

Relative permittivity and electric susceptibility

$$\varepsilon_r = 1 + \chi_e \qquad [7.13]$$

In terms of electronic polarization, from Equation 7.11, this is

Relative permittivity and polarizability

$$\varepsilon_r = 1 + \frac{N\alpha_e}{\varepsilon_o} \qquad [7.14]$$

The significance of Equation 7.14 is that it relates the microscopic polarization mechanism that determines α_e to the macroscopic property ε_r.

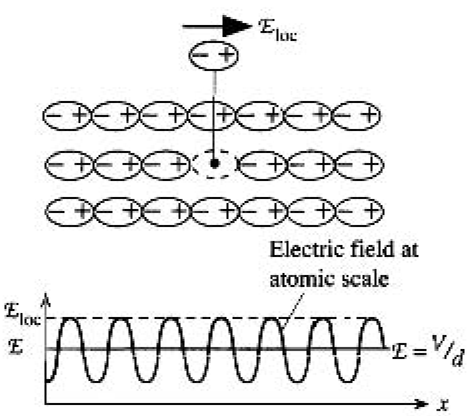

Figure 7.7 The electric field inside a polarized dielectric at the atomic scale is not uniform.
The local field is the actual field that acts on a molecule. It can be calculated by removing that molecule and evaluating the field at that point from the charges on the plates and the dipoles surrounding the point.

7.1.4 LOCAL FIELD $\mathcal{E}_{\text{loc}}$ AND CLAUSIUS–MOSSOTTI EQUATION

Equation 7.14, which relates ε_r to electronic polarizability α_e is only approximate because it assumes that the field acting on an individual atom or molecule is the field $\mathcal{E}$, which is assumed to be uniform within the dielectric. In other words, the induced polarization, $p_{\text{induced}} \propto \mathcal{E}$. However, the induced polarization depends on the actual field experienced by the molecule. It is apparent from Figure 7.5a that there are polarized molecules within the dielectric with their negative and positive charges separated so that the field is not constant *on the atomic scale* as we move through the dielectric. This is depicted in Figure 7.7. The field experienced by an individual molecule is actually different than $\mathcal{E}$, which represents the average field in the dielectric. As soon as the dielectric becomes polarized, the field at some arbitrary point depends not only on the charges on the plates (Q) but also on the orientations of all the other dipoles around this point in the dielectric. When averaged over some distance, say a few thousand molecules, this field becomes $\mathcal{E}$, as shown in Figure 7.7.

The actual field experienced by a molecule in a dielectric is defined as the **local field** and denoted by $\mathcal{E}_{\text{loc}}$. It depends not only on the free charges on the plates but also on the arrangement of all the polarized molecules around this point. In evaluating $\mathcal{E}_{\text{loc}}$ we simply remove the molecule from this point and calculate the field at this point coming from all sources, including neighboring polarized molecules, as visualized in Figure 7.7. $\mathcal{E}_{\text{loc}}$ will depend on the amount of polarization the material has experienced. The greater the polarization, the greater is the local field because there are bigger dipoles around this point. $\mathcal{E}_{\text{loc}}$ depends on the arrangement of polarized molecules around the point of interest and hence depends on the crystal structure. In the simplest case of a material with a cubic crystal structure, or a liquid (no crystal structure), the local field $\mathcal{E}_{\text{loc}}$ acting on a molecule increases with polarization as[4]

$$\mathcal{E}_{\text{loc}} = \mathcal{E} + \frac{1}{3\varepsilon_o} P, \tag{7.15}$$

Lorentz local field in dielectrics

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[4] This field is called the **Lorentz field** and the proof, though not difficult, is not necessary for the present introductory treatment of dielectrics. This local field expression does not apply to dipolar dielectrics discussed in Section 7.3.2.

Equation 7.15 is called the **Lorentz field.** The induced polarization in the molecule now depends on this local field $\mathcal{E}_{\text{loc}}$ rather than the average field $\mathcal{E}$. Thus

$$p_{\text{induced}} = \alpha_e \mathcal{E}_{\text{loc}}$$

The fundamental definition of electric susceptibility by the equation

$$P = \chi_e \varepsilon_o \mathcal{E}$$

is unchanged, which means that $\varepsilon_r = 1 + \chi_e$, Equation 7.13, remains intact. The polarization is defined by $P = N p_{\text{induced}}$, and p_{induced} can be related to $\mathcal{E}_{\text{loc}}$ and hence to $\mathcal{E}$ and P. Then

$$P = (\varepsilon_r - 1)\varepsilon_o \mathcal{E}$$

can be used to eliminate $\mathcal{E}$ and P and obtain a relationship between ε_r and α_e. This is the **Clausius–Mossotti equation,**

Clausius–Mossotti equation

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha_e}{3\varepsilon_o} \qquad [7.16]$$

This equation allows the calculation of the macroscopic property ε_r from microscopic polarization phenomena, namely, α_e.

EXAMPLE 7.2 **ELECTRONIC POLARIZABILITY OF A VAN DER WAALS SOLID** The electronic polarizability of the Ar atom is 1.7×10^{-40} F m^2. What is the static dielectric constant of solid Ar (below 84 K) if its density is 1.8 g cm^{-3}?

SOLUTION

To calculate ε_r we need the number of Ar atoms per unit volume N from the density d. If $M_{\text{at}} = 39.95$ is the relative atomic mass of Ar and N_A is Avogadro's number, then

$$N = \frac{N_A d}{M_{\text{at}}} = \frac{(6.02 \times 10^{23}\ \text{mol}^{-1})(1.8\ \text{g cm}^{-3})}{(39.95\ \text{g mol}^{-1})} = 2.71 \times 10^{22}\ \text{cm}^{-3}$$

with $N = 2.71 \times 10^{28}$ m^{-3} and $\alpha_e = 1.7 \times 10^{-40}$ F m^2, we have

$$\varepsilon_r = 1 + \frac{N\alpha_e}{\varepsilon_o} = 1 + \frac{(2.71 \times 10^{28})(1.7 \times 10^{-40})}{(8.85 \times 10^{-12})} = 1.52$$

If we use the Clausius–Mossotti equation, we get

$$\varepsilon_r = \frac{1 + \dfrac{2N\alpha_e}{3\varepsilon_o}}{1 - \dfrac{N\alpha_e}{3\varepsilon_o}} = 1.63$$

The two values are different by about 7 percent. The simple relationship in Equation 7.14 underestimates the relative permittivity.

7.2 ELECTRONIC POLARIZATION: COVALENT SOLIDS

When a field is applied to a solid substance, the constituent atoms or molecules become polarized, as we visualized in Figure 7.5a. The electron clouds within each atom become shifted by the field, and this gives rise to **electronic polarization.** This type of electronic polarization within an atom, however, is quite small compared with the polarization due to the valence electrons in the covalent bonds within the solid. For example, in crystalline silicon, there are electrons shared with neighboring Si atoms in covalent bonds, as shown in Figure 7.8a. These valence electrons form bonds (*i.e.*, become shared) between the Si atoms because they are already loosely bound to their parent atoms. If this were not the case, the solid would be a van der Waals solid with atoms held together by secondary bonds (*e.g.*, solid Ar below 83.8 K). In the covalent solid, the valence electrons therefore are not rigidly tied to the ionic cores left in the Si atoms. Although intuitively we often view these valence electrons as living in covalent bonds between the ionic Si cores, they nonetheless belong to the whole crystal because they can tunnel from bond to bond and exchange places with each other. We refer to their wavefunctions as delocalized, that is, not localized to any particular Si atom. When an electric field is applied, the negative charge distribution associated with these valence electrons becomes readily shifted with respect to the positive charges of the ionic Si cores, as depicted in Figure 7.8b and the crystal exhibits polarization, or develops a polarization vector. One can appreciate the greater flexibility of electrons in covalent bonds compared with those in individual ionic cores by comparing the energy involved in freeing each. It takes perhaps 1–2 eV to break a covalent bond to free the valence electron, but it takes more than 10 eV to free an electron from an individual ionic Si core. Thus, the valence electrons in the bonds readily respond to an applied field and become displaced. This type of electronic polarization, due to the displacement of electrons in covalent bonds, is responsible for the large dielectric constants of covalent crystals. For example $\varepsilon_r = 11.9$ for the Si crystal and $\varepsilon_r = 16$ for the Ge crystal.

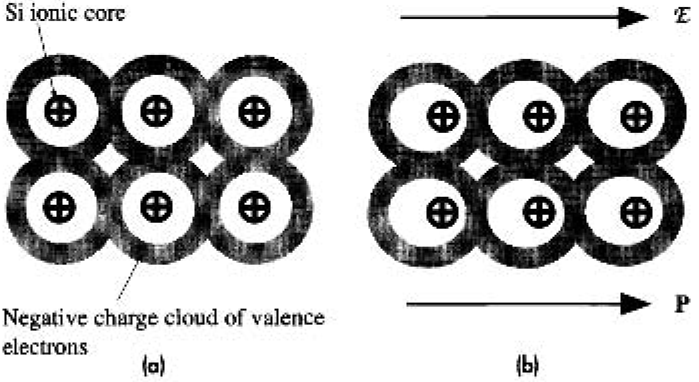

Figure 7.8

(a) Valence electrons in covalent bonds in the absence of an applied field.

(b) When an electric field is applied to a covalent solid, the valence electrons in the covalent bonds are shifted very easily with respect to the positive ionic cores. The whole solid becomes polarized due to the collective shift in the negative charge distribution of the valence electrons.

EXAMPLE 7.3 **ELECTRONIC POLARIZABILITY OF COVALENT SOLIDS** Consider a pure Si crystal that has $\varepsilon_r = 11.9$.

a. What is the electronic polarizability due to valence electrons per Si atom (if one could portion the observed crystal polarization to individual atoms)?

b. Suppose that a Si crystal sample is electroded on opposite faces and has a voltage applied across it. By how much is the local field greater than the applied field?

c. What is the resonant frequency f_o corresponding to ω_o?

From the density of the Si crystal, the number of Si atoms per unit volume, N, is given as $5 \times 10^{28}\ \text{m}^{-3}$.

SOLUTION

a. Given the number of Si atoms, we can apply the Clausius–Mossotti equation to find α_e

$$\alpha_e = \frac{3\varepsilon_o}{N}\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{3(8.85 \times 10^{-12})}{(5 \times 10^{28})}\frac{11.9 - 1}{11.9 + 2} = 4.17 \times 10^{-40}\ \text{F m}^2$$

This is larger, for example, than the electronic polarizability of an isolated Ar atom, which has more electrons. If we were to take the inner electrons in each Si atom as very roughly representing Ne, we would expect their contribution to the overall electronic polarizability to be roughly the same as the Ne atom, which is $0.45 \times 10^{-40}\ \text{F m}^2$.

b. The local field is

$$\mathcal{E}_{\text{loc}} = \mathcal{E} + \frac{1}{3\varepsilon_o}P$$

But, by definition,

$$P = \chi_e\varepsilon_o\mathcal{E} = (\varepsilon_r - 1)\varepsilon_o\mathcal{E}$$

Substituting for P,

$$\mathcal{E}_{\text{loc}} = \mathcal{E} + \frac{1}{3}(\varepsilon_r - 1)\mathcal{E}$$

so the local field with respect to the applied field is

$$\frac{\mathcal{E}_{\text{loc}}}{\mathcal{E}} = \frac{1}{3}(\varepsilon_r + 2) = 4.63$$

The local field is a factor of 4.63 greater than the applied field.

c. Since polarization is due to valence electrons and there are four per Si atom, we can use Equation 7.7,

$$\omega_o = \left(\frac{Ze^2}{m_e\alpha_e}\right)^{1/2} = \left[\frac{4(1.6 \times 10^{-19})^2}{(9.1 \times 10^{-31})(4.17 \times 10^{-40})}\right]^{1/2} = 1.65 \times 10^{16}\ \text{rad s}^{-1}$$

The corresponding resonant frequency is $\omega_o/2\pi$ or 2.6×10^{15} Hz, which is typically associated with electromagnetic waves of wavelength in the ultraviolet region.

7.3 POLARIZATION MECHANISMS

In addition to electronic polarization, we can identify a number of other polarization mechanisms that may also contribute to the relative permittivity.

7.3.1 IONIC POLARIZATION

This type of polarization occurs in ionic crystals such as NaCl, KCl, and LiBr. The ionic crystal has distinctly identifiable ions, for example, Na^+ and Cl^-, located at well-defined lattice sites, so each pair of oppositely charged neighboring ions has a dipole moment. As an example, we consider the one-dimensional NaCl crystal depicted as a chain of alternating Na^+ and Cl^- ions in Figure 7.9a. In the absence of an applied field, the solid has no net polarization because the dipole moments of equal magnitude are lined up head to head and tail to tail so that the net dipole moment is zero. The dipole moment p_+ in the positive x direction has the same magnitude as p_- in the negative x direction, so the net dipole moment

$$p_{\text{net}} = p_+ - p_- = 0$$

In the presence of a field $\mathcal{E}$ along the x direction, however, the Cl^- ions are pushed in the $-x$ direction and the Na^+ ions in the $+x$ direction about their equilibrium positions. Consequently, the dipole moment p_+ in the $+x$ direction *increases* to p'_+ and the dipole moment p_- *decreases* to p'_-, as shown in Figure 7.9b. The net dipole moment is now no longer zero. The net dipole moment, or the average dipole moment, per ion pair is now $(p'_+ - p'_-)$, which depends on the electric field $\mathcal{E}$. Thus the induced average dipole moment per ion pair p_{av} depends on the field $\mathcal{E}$. The ionic polarizability α_i is defined in terms of the local field experienced by the ions,

$$p_{\text{av}} = \alpha_i \mathcal{E}_{\text{loc}} \qquad [7.17]$$

Ionic polarizability

The larger the α_i, the greater the induced dipole moment. Generally, α_i is larger than the electronic polarizability α_e by a factor of 10 or more, which leads to ionic solids having large dielectric constants. The polarization P exhibited by the ionic solid

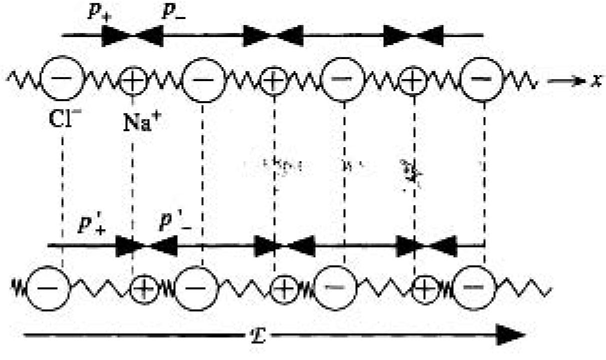

Figure 7.9

(a) A NaCl chain in the NaCl crystal without an applied field. Average or net dipole moment per ion is zero.

(b) In the presence of an applied field, the ions become slightly displaced, which leads to a net average dipole moment per ion.

is therefore given by

$$P = N_i p_{av} = N_i \alpha_i \mathcal{E}_{loc}$$

where N_i is the number of ion pairs per unit volume. By relating the local field to $\mathcal{E}$ and using

$$P = (\varepsilon_r - 1)\varepsilon_o \mathcal{E}$$

Clausius–Mossotti equation for ionic polarization

we can again obtain the Clausius–Mossotti equation, but now due to ionic polarization,

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{1}{3\varepsilon_o} N_i \alpha_i \tag{7.18}$$

Each ion also has a core of electrons that become displaced in the presence of an applied field with respect to their positive nuclei and therefore also contribute to the polarization of the solid. This electronic polarization simply adds to the ionic polarization. Its magnitude is invariably much smaller than the ionic contribution in these solids.

7.3.2 ORIENTATIONAL (DIPOLAR) POLARIZATION

Certain molecules possess permanent dipole moments. For example, the HCl molecule shown in Figure 7.10a has a permanent dipole moment p_o from the Cl^- ion to the H^+ ion. In the liquid or gas phases, these molecules, in the absence of an electric field, are randomly oriented as a result of thermal agitation, as shown in Figure 7.10b. When an electric field $\mathcal{E}$ is applied, $\mathcal{E}$ tries to align the dipoles parallel to itself, as depicted in Figure 7.10c. The Cl^- and H^+ charges experience forces in opposite directions. But the nearly rigid bond between Cl^- and H^+ holds them together, which means that the

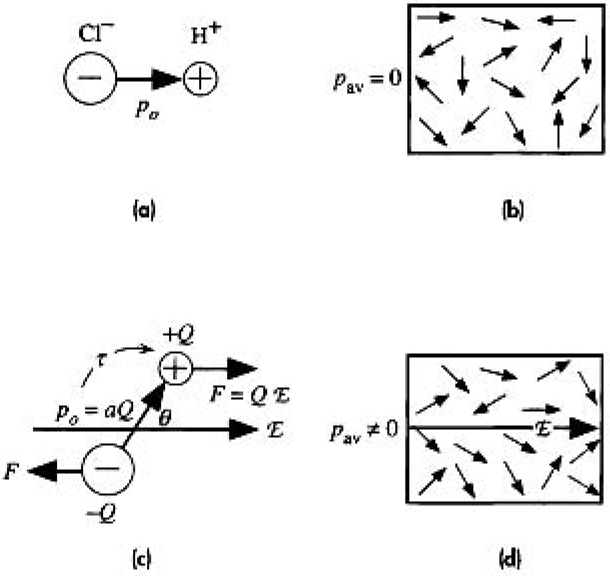

Figure 7.10

(a) A HCl molecule possesses a permanent dipole moment p_o.

(b) In the absence of a field, thermal agitation of the molecules results in zero net average dipole moment per molecule.

(c) A dipole such as HCl placed in a field experiences a torque that tries to rotate it to align p_o with the field $\mathcal{E}$.

(d) In the presence of an applied field, the dipoles try to rotate to align with the field against thermal agitation. There is now a net average dipole moment per molecule along the field.

molecule experiences a torque τ about its center of mass.[5] This torque acts to rotate the molecule to align p_o with $\mathcal{E}$. If all the molecules were to simply rotate and align with the field, the polarization of the solid would be

$$P = Np_o$$

where N is the number of molecules per unit volume. However, due to their thermal energy, the molecules move around randomly and collide with each other and with the walls of the container. These collisions destroy the dipole alignments. Thus the thermal energy tries to randomize the orientations of the dipole moments. A snapshot of the dipoles in the material in the presence of a field can be pictured as in Figure 7.10d in which the dipoles have different orientations. There is, nonetheless, a net average dipole moment per molecule p_{av} that is finite and directed along the field. Thus the material exhibits net polarization, which leads to a dielectric constant that is determined by this **orientational polarization.**

To find the induced average dipole moment p_{av} along $\mathcal{E}$, we need to know the average potential energy E_{dip} of a dipole placed in a field $\mathcal{E}$ and how this compares with the average thermal energy $\frac{5}{2}kT$ per molecule as in the present case of five degrees of freedom. E_{dip} represents the average external work done by the field in aligning the dipoles with the field. If $\frac{5}{2}kT$ is much greater than E_{dip}, then the average thermal energy of collisions will prevent any dipole alignment with the field. If, however, E_{dip} is much greater than $\frac{5}{2}kT$, then the thermal energy is insufficient to destroy the dipole alignments.

A dipole at an angle θ to the field experiences a torque τ that tries to rotate it, as shown in Figure 7.10c. Work done dW by the field in rotating the dipole by $d\theta$ is $\tau\, d\theta$ (as in $F\, dx$). This work dW represents a small change dE in the potential energy of the dipole. No work is done if the dipole is already aligned with $\mathcal{E}$, when $\theta = 0$, which corresponds to the minimum in PE. On the other hand, maximum work is done when the torque has to rotate the dipole from $\theta = 180°$ to $\theta = 0°$ (either clockwise or counterclockwise, it doesn't matter). The torque experienced by the dipole, according to Figure 7.10c, is given by

Torque on a dipole

$$\tau = (F \sin\theta)a \qquad \text{or} \qquad \mathcal{E}p_o \sin\theta$$

where

$$p_o = aQ$$

If we take $PE = 0$ when $\theta = 0$, then the maximum PE is when $\theta = 180°$, or

$$E_{max} = \int_0^{\pi} p_o \mathcal{E} \sin\theta\, d\theta = 2p_o\mathcal{E}$$

The average dipole potential energy is then $\frac{1}{2}E_{max}$ or $p_o\mathcal{E}$. For orientational polarization to be effective, this energy must be greater than the average thermal energy. The average dipole moment p_{av} along $\mathcal{E}$ is directly proportional to the magnitude of p_o itself and also proportional to the average dipole energy to average thermal energy

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[5] The oppositely directed forces also slightly stretch the Cl^-–H^+ bond, but we neglect this effect.

ratio, that is,

$$p_{av} \propto p_o \frac{p_o \mathcal{E}}{\frac{5}{2}kT}$$

If we were to do the calculation properly using Boltzmann statistics for the distribution of dipole energies among the molecules, that is, the probability that the dipole has an energy E is proportional to $\exp(-E/kT)$, then we would find that when $p_o E < kT$ (generally the case),

Average dipole moment in orientational polarization

$$p_{av} = \frac{1}{3}\frac{p_o^2 \mathcal{E}}{kT} \qquad [7.19]$$

It turns out that the intuitively derived expression for p_{av} is roughly the same as Equation 7.19. Strictly, of course, we should use the local field acting on each molecule, in which case $\mathcal{E}$ is simply replaced by $\mathcal{E}_{loc}$. From Equation 7.19 we can define a **dipolar orientational polarizability** α_d per molecule by

Dipolar orientational polarizability

$$\alpha_d = \frac{1}{3}\frac{p_o^2}{kT} \qquad [7.20]$$

It is apparent that, in contrast to the electronic and ionic polarization, dipolar orientational polarization is strongly temperature dependent. α_d decreases with temperature, which means that the relative permittivity ε_r also decreases with temperature. Dipolar orientational polarization is normally exhibited by polar liquids (*e.g.*, water, alcohol, acetone, and various electrolytes) and polar gases (*e.g.*, gaseous HCl and steam). It can also occur in solids if there are permanent dipoles within the solid structure, even if dipolar rotation involves a discrete jump of an ion from one site to another, such as in various glasses.

7.3.3 INTERFACIAL POLARIZATION

Interfacial polarization occurs whenever there is an accumulation of charge at an interface between two materials or between two regions within a material. The simplest example is interfacial polarization due to the accumulation of charges in the dielectric near one of the electrodes, as depicted in Figure 7.11a and b. Invariably materials, however perfect, contain crystal defects, impurities, and various mobile charge carriers such as electrons (*e.g.*, from donor-type impurities), holes, or ionized host or impurity ions. In the particular example in Figure 7.11a, the material has an equal number of positive ions and negative ions, but the positive ions are assumed to be far more mobile. For example, if present, the H^+ ion (which is a proton) and the Li^+ ion in ceramics and glasses are more mobile than negative ions in the structure because they are relatively small. Under the presence of an applied field, these positive ions migrate to the negative electrode. The positive ions, however, cannot leave the dielectric and enter the crystal structure of the metal electrode. They therefore simply pile up at the interface and give rise to a positive space charge near the electrode. These positive charges at the interface attract more electrons to the negative electrode. This additional charge on the electrode,

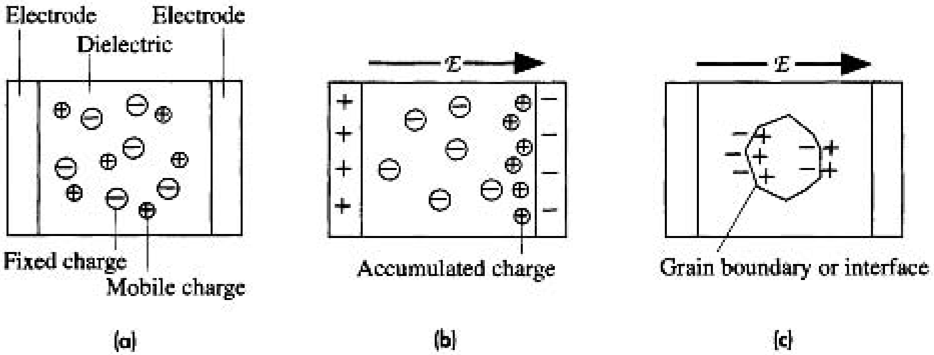

Figure 7.11

(a) A crystal with equal number of mobile positive ions and fixed negative ions. In the absence of a field, there is no net separation between all the positive charges and all the negative charges.

(b) In the presence of an applied field, the mobile positive ions migrate toward the negative electrode and accumulate there. There is now an overall separation between the negative charges and positive charges in the dielectric. The dielectric therefore exhibits interfacial polarization.

(c) Grain boundaries and interfaces between different materials frequently give rise to interfacial polarization.

of course, appears as an increase in the dielectric constant. The term **interfacial polarization** arises because the positive charges accumulating at the interface and the remainder of negative charges in the bulk together constitute dipole moments that appear in the polarization vector **P** (**P** sums all the dipoles within the material per unit volume).

Another typical interfacial polarization mechanism is the trapping of electrons or holes at defects at the crystal surface, at the interface between the crystal and the electrode. In this case we can view the positive charges in Figure 7.11a as holes and negative charges as immobile ionized acceptors. We assume that the contacts are blocking and do not allow electrons or holes to be injected, that is, exchanged between the electrodes and the dielectric. In the presence of a field, the holes drift to the negative electrode and become trapped in defects at the interface, as in Figure 7.11b.

Grain boundaries frequently lead to interfacial polarization as they can trap charges migrating under the influence of an applied field, as indicated in Figure 7.11c. Dipoles between the trapped charges increase the polarization vector. Interfaces also arise in heterogeneous dielectric materials, for example, when there is a dispersed phase within a continuous phase. The principle is then the same as schematically illustrated in Figure 7.11c.

7.3.4 TOTAL POLARIZATION

In the presence of electronic, ionic, and dipolar polarization mechanisms, the average induced dipole moment per molecule will be the sum of all the contributions in terms of the local field,

Total induced dipole moment

$$p_{av} = \alpha_e \mathcal{E}_{loc} + \alpha_i \mathcal{E}_{loc} + \alpha_d \mathcal{E}_{loc}$$

Table 7.2 Typical examples of polarization mechanisms

Example	Polarization	Static ε_r	Comment
Ar gas	Electronic	1.0005	Small N in gases: $\varepsilon_r \approx 1$
Ar liquid ($T < 87.3$ K)	Electronic	1.53	van der Waals bonding
Si crystal	Electronic polarization due to valence electrons	11.9	Covalent solid; bond polarization
NaCl crystal	Ionic	5.90	Ionic crystalline solid
CsCl crystal	Ionic	7.20	Ionic crystalline solid
Water	Orientational	80	Dipolar liquid
Nitromethane (27 °C)	Orientational	34	Dipolar liquid
PVC (polyvinyl chloride)	Orientational	7	Dipole orientations partly hindered in the solid

Each effect adds linearly to the net dipole moment per molecule, a fact verified by experiments. Interfacial polarization cannot be simply added to the above equation as $\alpha_{if}\mathcal{E}_{loc}$ because it occurs at interfaces and cannot be put into an average polarization per molecule in the bulk. Further, the fields are not well defined at the interfaces. In addition, we *cannot* use the simple Lorentz local field approximation for dipolar materials. That is, the Clausius–Mossotti equation does not work with dipolar dielectrics and the calculation of the local field is quite complicated. The dielectric constant ε_r under **electronic** and **ionic polarizations,** however, can be obtained from

Clausius–Mossotti equation

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{1}{3\varepsilon_o}(N_e\alpha_e + N_i\alpha_i) \qquad [7.21]$$

Table 7.2 summarizes the various polarization mechanisms and the corresponding static (or very low frequency) dielectric constant. Typical examples where one mechanism dominates over others are also listed.

EXAMPLE 7.4 **IONIC AND ELECTRONIC POLARIZABILITY** Consider the CsCl crystal which has one Cs^+–Cl^- pair per unit cell and a lattice parameter a of 0.412 nm. The electronic polarizability of Cs^+ and Cl^- ions is 3.35×10^{-40} F m^2 and 3.40×10^{-40} F m^2, respectively, and the mean ionic polarizability per ion pair is 6×10^{-40} F m^2. What is the dielectric constant at low frequencies and that at optical frequencies?

SOLUTION

The CsCl structure has one cation (Cs^+) and one anion (Cl^-) in the unit cell. Given the lattice parameter $a = 0.412 \times 10^{-9}$ m, the number of ion pairs N_i per unit volume is $1/a^3 = 1/(0.412 \times 10^{-9}\text{ m})^3 = 1.43 \times 10^{28}$ m^{-3}. N_i is also the concentration of cations and anions individually. From the Clausius–Mossotti equation,

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{1}{3\varepsilon_o}[N_i\alpha_e(\text{Cs}^+) + N_i\alpha_e(\text{Cl}^-) + N_i\alpha_i]$$

That is,

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{(1.43 \times 10^{28}\ \text{m}^{-3})(3.35 \times 10^{-40} + 3.40 \times 10^{-40} + 6 \times 10^{-40}\ \text{F m}^2)}{3(8.85 \times 10^{-12}\ \text{F m}^{-1})}$$

Solving for ε_r, we find $\varepsilon_r = 7.56$.

At high frequencies—that is, near-optical frequencies—the ionic polarization is too sluggish to allow ionic polarization to contribute to ε_r. Thus, ε_{rop}, relative permittivity at optical frequencies, is given by

$$\frac{\varepsilon_{rop} - 1}{\varepsilon_{rop} + 2} = \frac{1}{3\varepsilon_o}[N_i\alpha_e(\text{Cs}^+) + N_i\alpha_e(\text{Cl}^-)]$$

That is,

$$\frac{\varepsilon_{rop} - 1}{\varepsilon_{rop} + 2} = \frac{(1.43 \times 10^{28}\ \text{m}^{-3})(3.35 \times 10^{-40} + 3.40 \times 10^{-40}\ \text{F m}^2)}{3(8.85 \times 10^{-12}\ \text{F m}^{-1})}$$

Solving for ε_{rop}, we find $\varepsilon_{rop} = 2.71$. Note that experimental values are $\varepsilon_r = 7.20$ at low frequencies and $\varepsilon_{rop} = 2.62$ at high frequencies, very close to calculated values.

7.4 FREQUENCY DEPENDENCE: DIELECTRIC CONSTANT AND DIELECTRIC LOSS

7.4.1 Dielectric Loss

The static dielectric constant is an effect of polarization under dc conditions. When the applied field, or the voltage across a parallel plate capacitor, is a sinusoidal signal, then the polarization of the medium under these ac conditions leads to an ac dielectric constant that is generally different than the static case. As an example we will consider orientational polarization involving dipolar molecules. The sinusoidally varying field changes magnitude and direction continuously, and it tries to line up the dipoles one way and then the other way and so on. If the instantaneous induced dipole moment p per molecule can instantaneously follow the field variations, then at any instant

$$p = \alpha_d \mathcal{E} \qquad [7.22]$$

and the polarizability α_d has its expected maximum value from dc conditions, that is,

$$\alpha_d = \frac{p_o^2}{3kT} \qquad [7.23]$$

There are two factors opposing the immediate alignment of the dipoles with the field. First is that thermal agitation tries to randomize the dipole orientations. Collisions in the gas phase, random jolting from lattice vibrations in the liquid and solid phases, for example, aid the randomization of the dipole orientations. Second, the molecules rotate in a viscous medium by virtue of their interactions with neighbors, which is particularly strong in the liquid and solid states and means that the dipoles cannot respond instantaneously to the changes in the applied field. If the field changes too

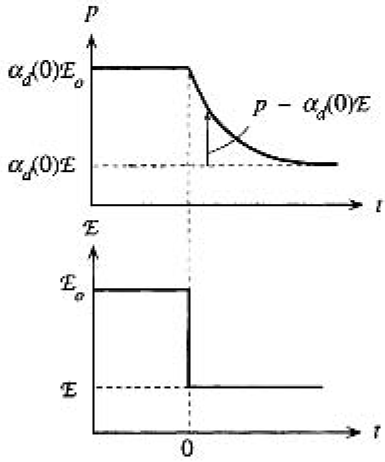

Figure 7.12 The applied dc field is suddenly changed from $\mathcal{E}_o$ to $\mathcal{E}$ at time $t = 0$.
The induced dipole moment p has to decrease from $\alpha_d(0)\mathcal{E}_o$ to a final value of $\alpha_d(0)\mathcal{E}$. The decrease is achieved by random collisions of molecules in the gas.

rapidly, then the dipoles cannot follow the field and, as a consequence, remain randomly oriented. At high frequencies, therefore, α_d will be zero as the field cannot induce a dipole moment. At low frequencies, of course, the dipoles can respond rapidly to follow the field and α_d has its maximum value. It is clear that α_d changes from its maximum value in Equation 7.23 to zero as the frequency of the field is increased. We need to find the behavior of α_d as a function of frequency ω so that we can determine the dielectric constant ε_r by the Clausius–Mossotti equation.

Suppose that after a prolonged application, corresponding to dc conditions, the applied field across the dipolar gaseous medium is suddenly decreased from $\mathcal{E}_o$ to $\mathcal{E}$ at a time we define as zero, as shown in Figure 7.12. The field $\mathcal{E}$ is smaller than $\mathcal{E}_o$, so the induced dc dipole moment per molecule should be smaller and given by $\alpha_d(0)\mathcal{E}$ where $\alpha_d(0)$ is α_d at $\omega = 0$, dc conditions. Therefore, the induced dipole moment per molecule has to decrease, or *relax*, from $\alpha_d(0)\mathcal{E}_o$ to $\alpha_d(0)\mathcal{E}$. In a gas medium the molecules would be moving around randomly and their collisions with each other and the walls of the container randomize the induced dipole per molecule. Thus the decrease, or the **relaxation process,** in the induced dipole moment is achieved by random collisions. Assuming that τ is the average time, called the **relaxation time,** between molecular collisions, then this is the mean time it takes per molecule to randomize the induced dipole moment. If p is the instantaneous induced dipole moment, then $p - \alpha_d(0)\mathcal{E}$ is the *excess* dipole moment, which must eventually disappear to zero through random collisions as $t \to \infty$. It would take an average τ seconds to eliminate the excess dipole moment $p - \alpha_d(0)\mathcal{E}$. The rate at which the induced dipole moment is changing is then $-[p - \alpha_d(0)\mathcal{E}]/\tau$, where the negative sign represents a decrease. Thus,

Dipolar relaxation equation

$$\frac{dp}{dt} = -\frac{p - \alpha_d(0)\mathcal{E}}{\tau} \tag{7.24}$$

Although we did not derive Equation 7.24 rigorously, it is nonetheless a good first-order description of the behavior of the induced dipole moment per molecule in a dipolar medium. Equation 7.24 can be used to obtain the dipolar polarizability under ac conditions. For an ac field, we would write

$$\mathcal{E} = \mathcal{E}_o \sin(\omega t)$$

and solve Equation 7.24, but in engineering we prefer to use an exponential representation for the field

Applied field

$$\mathcal{E} = \mathcal{E}_o \exp(j\omega t)$$

as in ac voltages. In this case the impedance of a capacitor C and an inductor L become $1/j\omega C$ and $j\omega L$, where j represents a phase shift of 90°. With $\mathcal{E} = \mathcal{E}_o \exp(j\omega t)$ in Equation 7.24, we have

Dipole relaxation equation

$$\frac{dp}{dt} = -\frac{p}{\tau} + \frac{\alpha_d(0)}{\tau}\mathcal{E}_o \exp(j\omega t) \qquad [7.25]$$

Solving this we find the induced dipole moment as

$$p = \alpha_d(\omega)\mathcal{E}_o \exp(j\omega t)$$

where $\alpha_d(\omega)$ is given by

Orientational polarizability and frequency

$$\alpha_d(\omega) = \frac{\alpha_d(0)}{1 + j\omega\tau} \qquad [7.26]$$

and represents the orientational polarizability under ac field conditions. Polarizability $\alpha_d(\omega)$ is a complex number that indicates that p and $\mathcal{E}$ are out of phase.[6] Put differently, if N is the number of molecules per unit volume, $P = Np$ and $\mathcal{E}$ are out of phase, as indicated in Figure 7.13a. At low frequencies, $\omega\tau \ll 1$, $\alpha_d(\omega)$ is nearly $\alpha_d(0)$, and p is in phase with $\mathcal{E}$. The rate of relaxation $1/\tau$ is much faster than the frequency of the field or the rate at which the polarization is being changed; p then closely follows $\mathcal{E}$. At very high frequencies, $\omega\tau \gg 1$, the rate of relaxation $1/\tau$ is much slower than the frequency of the field and p can no longer follow the variations in the field.

We can easily obtain the dielectric constant ε_r from $\alpha_d(\omega)$ by using Equation 7.14, which then leads to a complex number for ε_r since α_d itself is a complex number. By convention, we generally write the **complex dielectric constant** as

Complex relative permittivity

$$\varepsilon_r = \varepsilon'_r - j\varepsilon''_r \qquad [7.27]$$

where ε'_r is the real part and ε''_r is the imaginary part, both being frequency dependent, as shown in Figure 7.13b. The real part ε'_r decreases from its maximum value $\varepsilon'_r(0)$, corresponding to $\alpha_d(0)$, to 1 at high frequencies when $\alpha_d = 0$ as $\omega \to \infty$ in Equation 7.26. The imaginary part $\varepsilon''_r(\omega)$ is zero at low and high frequencies but peaks when $\omega\tau = 1$ or when $\omega = 1/\tau$. The real part ε'_r represents the relative permittivity that we would use in calculating the capacitance, as for example in $C = \varepsilon_r\varepsilon_o A/d$. The imaginary part $\varepsilon''_r(\omega)$ represents the energy lost in the dielectric medium as the dipoles are oriented against random collisions one way and then the other way and so on by the field. Consider

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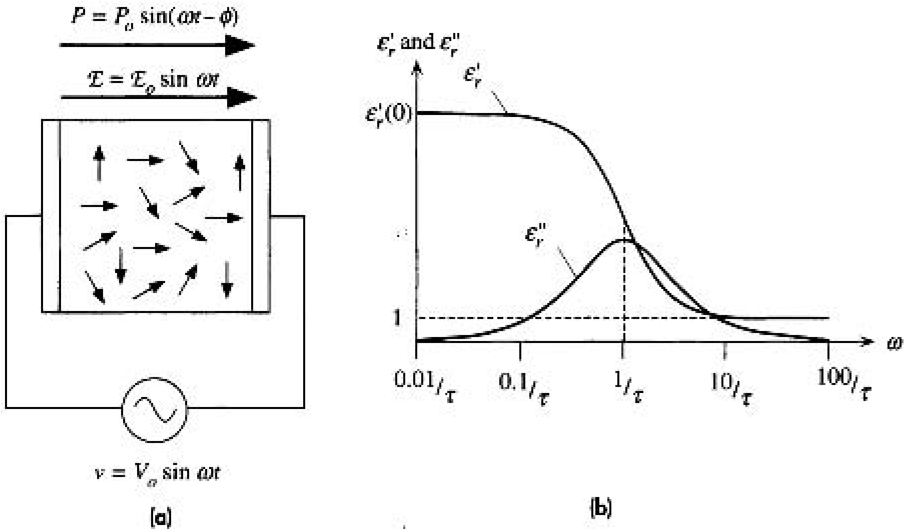
[6] The polarization P lags behind $\mathcal{E}$ by some angle ϕ, that is determined by Equation 7.26 as shown in Figure 7.13.



(a) (b)

Figure 7.13

(a) An ac field is applied to a dipolar medium. The polarization P ($P = Np$) is out of phase with the ac field.

(b) The relative permittivity is a complex number with real (ε_r') and imaginary (ε_r'') parts that exhibit relaxation at $\omega \approx 1/\tau$.

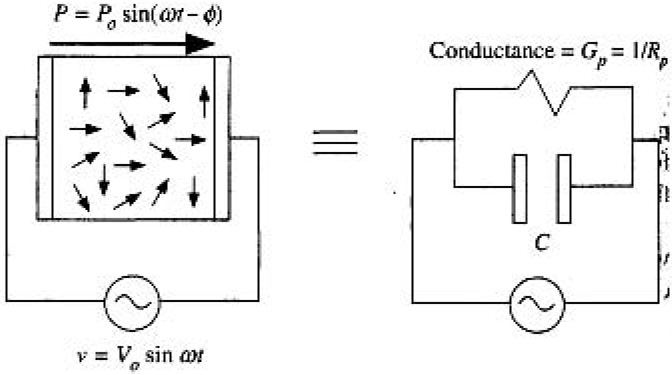


Figure 7.14 The dielectric medium behaves like an ideal (lossless) capacitor of capacitance C, which is in parallel with a conductance G_P.

the capacitor in Figure 7.14, which has this dielectric medium between the plates. Then the admittance Y, *i.e.*, the reciprocal of impedance of this capacitor, with ε_r given in Equation 7.27 is

Admittance of a parallel plate capacitor

$$Y = \frac{j\omega A\varepsilon_o\varepsilon_r(\omega)}{d} = \frac{j\omega A\varepsilon_o\varepsilon_r'(\omega)}{d} + \frac{\omega A\varepsilon_o\varepsilon_r''(\omega)}{d}$$

which can be written as

$$Y = j\omega C + G_P \qquad [7.28]$$

where

$$C = \frac{A\varepsilon_o\varepsilon_r'}{d} \tag{7.29}$$

Equivalent ideal capacitance

and

$$G_P = \frac{\omega A\varepsilon_o\varepsilon_r''}{d} \tag{7.30}$$

Equivalent parallel conductance

is a real number just as if we had a conductive medium with some conductance G_P or resistance $1/G_P$. The admittance of the dielectric medium according to Equation 7.28 is a parallel combination of an ideal, or lossless, capacitor C, with a relative permittivity ε_r', and a resistance of $R_P = 1/G_P$ as indicated in Figure 7.14. Thus the dielectric medium behaves as if C_o and R_P were in parallel. There is no real electric power dissipated in C, but there is indeed real power dissipated in R_P because

$$\text{Input power} = IV = YV^2 = j\omega CV^2 + \frac{V^2}{R_P}$$

and the second term is real. Thus the power dissipated in the dielectric medium is related to ε_r'' and peaks when $\omega = 1/\tau$. The rate of energy storage by the field is determined by ω whereas the rate of energy transfer to molecular collisions is determined by $1/\tau$. When $\omega = 1/\tau$, the two processes, energy storage by the field and energy transfer to random collisions, are then occurring at the same rate, and hence energy is being transferred to heat most efficiently. The peak in ε_r'' versus ω is called a **relaxation peak,** which is at a frequency when the dipole relaxations are at the right rate for maximum power dissipation. This process is known as **dielectric resonance.**

According to Equation 7.28, the magnitude of G_P and hence the energy loss is determined by ε_r''. In engineering applications of dielectrics in capacitors, we would like to minimize ε_r'' for a given ε_r'. We define the relative magnitude of ε_r'' with respect to ε_r' through a quantity, $\tan\delta$, called the **loss tangent** (or **loss factor**), as

$$\tan\delta = \frac{\varepsilon_r''}{\varepsilon_r'} \tag{7.31}$$

Loss tangent

which is frequency dependent and peaks just beyond $\omega = 1/\tau$. The actual value of $1/\tau$ depends on the material, but typically for liquid and solid media it is in the gigahertz range, that is, microwave frequencies. We can easily find the energy per unit time—power—dissipated as dielectric loss in the medium. The resistance R_P represents the dielectric loss, so

$$W_{\text{vol}} = \frac{\text{Power loss}}{\text{Volume}} = \frac{V^2}{R_P} \times \frac{1}{dA} = \frac{V^2}{\dfrac{d}{\omega A\varepsilon_o\varepsilon_r''}} \times \frac{1}{dA} = \frac{V^2}{d^2}\,\omega\varepsilon_o\varepsilon_r''$$

Using Equation 7.31 and $\mathcal{E} = V/d$, we obtain

$$W_{\text{vol}} = \omega\mathcal{E}^2\varepsilon_o\varepsilon_r'\tan\delta \tag{7.32}$$

Dielectric loss per unit volume

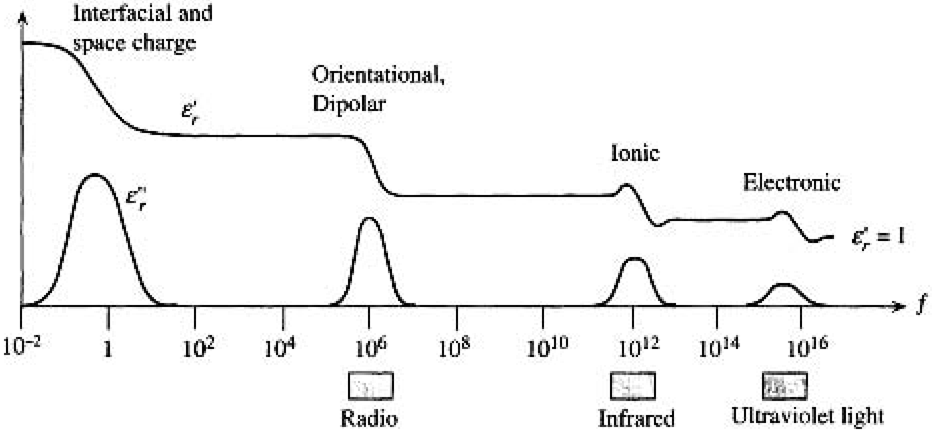
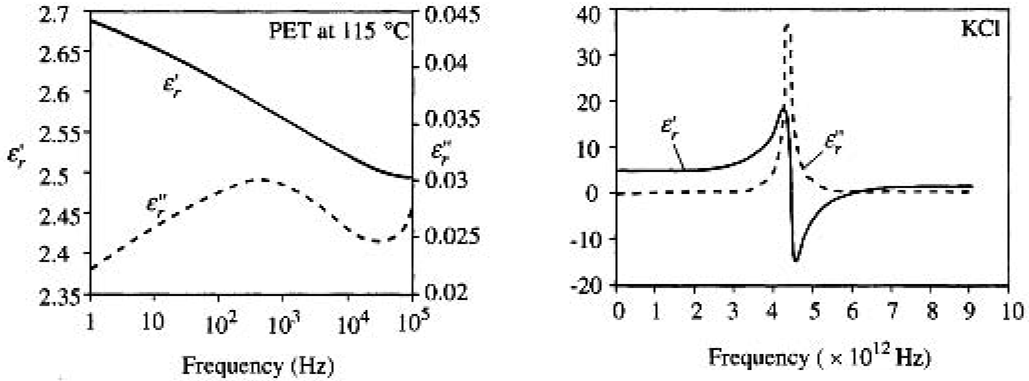


Figure 7.15 The frequency dependence of the real and imaginary parts of the dielectric constant in the presence of interfacial, orientational, ionic, and electronic polarization mechanisms.

Equation 7.32 represents the power dissipated per unit volume in the polarization mechanism: energy lost per unit time to random molecular collisions as heat. It is clear that dielectric loss is influenced by three factors: ω, $\mathcal{E}$, and $\tan\delta$.

Although we considered only orientational polarization, in general a dielectric medium will also exhibit other polarization mechanisms and certainly electronic polarization since there will always be electron clouds around individual atoms, or electrons in covalent bonds. If we were to consider the ionic polarizability in ionic solids, we would also find α_I to be frequency dependent and a complex number. In this case, lattice vibrations in the crystal, typically at frequencies ω_I in the infrared region of the electromagnetic spectrum, will dissipate the energy stored in the induced dipole moments just as energy was dissipated by molecular collisions in the gaseous dipolar medium. Thus, the energy loss will be greatest when the frequency of the polarizing field is the same as the lattice vibration frequency, $\omega = \omega_I$, which tries to randomize the polarization.

We can represent the general features of the frequency dependence of the real and imaginary parts of the dielectric constant as in Figure 7.15. Although the figure shows distinctive peaks in ε_r'' and transition features in ε_r', in reality these peaks and various features are broader. First, there is no single well-defined lattice vibration frequency but instead an allowed range of frequencies just as in solids where there is an allowed range of energies for the electron. Moreover, the polarization effects depend on the crystal orientation. In the case of polycrystalline materials, various peaks in different directions overlap to exhibit a broadened overall peak. At low frequencies the interfacial or space charge polarization features are even broader because there can be a number of conduction mechanisms (different species of charge carriers and different carrier mobilities) for the charges to accumulate at interfaces, each having its own speed. Orientational polarization, especially in many liquid dielectrics at room temperature, typically takes place at radio to microwave frequencies. In some polymeric materials, this type of polarization involves a limited rotation of dipolar side groups



(a)

(b)

Figure 7.16 Real and imaginary parts of the dielectric constant, ε_r' and ε_r'', versus frequency for (a) a polymer, PET, at 115 °C and (b) an ionic crystal, KCl, at room temperature.

Both exhibit relaxation peaks but for different reasons.

SOURCE: Data for (a) from author's own experiments using a dielectric analyzer (DEA), (b) from C. Smart, G. R. Wilkinson, A. M. Karo, and J. R. Hardy, International Conference on Lattice Dynamics, Copenhagen, 1963, as quoted by D. H. Martin, "The Study of the Vibration of Crystal Lattices by Far Infra-Red Spectroscopy," *Advances in Physics*, 14, no. 53–56, 1965, pp. 39–100.

attached to the polymeric chain and can occur at much lower frequencies depending on the temperature. Figure 7.16 shows two typical examples of dielectric behavior, ε_r' and ε_r'' as a function of frequency, for a polymer (PET) and an ionic crystal (KCl). Both exhibit loss peaks, peaks in ε_r'' versus frequency, but for different reasons. The particular polymer, PET (a polyester), exhibits orientational polarization due to dipolar side groups, whereas KCl exhibits ionic polarization due to the displacement of K^+ and Cl^- ions. The frequency of the loss peak in the case of orientational polarization is highly temperature dependent. For the PET example in Figure 7.16 at 115 °C, the peak occurs at around 400 Hz, even below typical radio frequencies.

EXAMPLE 7.5

DIELECTRIC LOSS PER UNIT CAPACITANCE AND THE LOSS ANGLE δ Obtain the dielectric loss per unit capacitance in a capacitor in terms of the loss tangent. Obtain the phase difference between the current through the capacitor and that through R_P. What is the significance of δ?

SOLUTION

We consider the equivalent circuit in Figure 7.14. The power loss in the capacitor is due to R_P. If V is the rms value of the voltage across the capacitor, then the power dissipated per unit capacitance W_{cap} is

$$W_{\text{cap}} = \frac{V^2}{R_P} \times \frac{1}{C} = V^2 \frac{\omega\varepsilon_o\varepsilon_r'' A}{d} \times \frac{d}{\varepsilon_o\varepsilon_r' A} = V^2 \frac{\omega\varepsilon_r''}{\varepsilon_r'}$$

or

$$W_{\text{cap}} = V^2 \omega \tan\delta$$

Table 7.3 Dielectric properties of three insulators

	$f = 60$ Hz			$f = 1$ MHz		
Material	ε_r'	**tan** δ	ω **tan** δ	ε_r'	**tan** δ	ω **tan** δ
Polycarbonate	3.17	9×10^{-4}	0.34	2.96	1×10^{-2}	6.2×10^4
Silicone rubber	3.7	2.25×10^{-2}	8.48	3.4	4×10^{-3}	2.5×10^4
Epoxy with mineral filler	5	4.7×10^{-2}	17.7	3.4	3×10^{-2}	18×10^4

As $\tan\delta$ is frequency dependent and peaks at some frequency, so does the power dissipated per unit capacitance. A clear design objective would be to keep W_{cap} as small as possible. Further, for a given voltage, W_{cap} does not depend on the dielectric geometry. For a given voltage and capacitance, we therefore cannot reduce the power dissipation by simply changing the dimensions of the dielectric.

Consider the rms currents through R_P and C, I_{loss} and I_{cap} respectively, and their ratio,[7]

$$\frac{I_{\text{loss}}}{I_{\text{cap}}} = \frac{V}{R_P} \times \frac{\dfrac{1}{j\omega C}}{V} = \frac{\omega\varepsilon_o\varepsilon_r'' A}{d} \times \frac{d}{j\omega\varepsilon_o\varepsilon_r' A} = -j\tan\delta$$

As expected, the two are 90° out of phase ($-j$) and the loss current (through R_P) is a factor, $\tan\delta$, of the capacitive current (through C). The ratio of I_{cap} and the total current, $I_{\text{total}} = I_{\text{cap}} + I_{\text{loss}}$, is

$$\frac{I_{\text{cap}}}{I_{\text{total}}} = \frac{I_{\text{cap}}}{I_{\text{cap}} + I_{\text{loss}}} = \frac{1}{1 + \dfrac{I_{\text{loss}}}{I_{\text{cap}}}} = \frac{1}{1 - j\tan\delta}$$

The phase angle between I_{cap} and I_{total} is determined by the negative of the phase of the denominator term $(1 - j\tan\delta)$. Thus the phase angle between I_{cap} and I_{total} is δ, where I_{cap} leads I_{total} by δ. δ is also called the **loss angle.** When the loss angle is zero, I_{cap} and I_{total} are equal and there is no loss in the dielectric.

EXAMPLE 7.6 **DIELECTRIC LOSS PER UNIT CAPACITANCE** Consider the three dielectric materials listed in Table 7.3 with their dielectric constant ε_r' (usually simply stated as ε_r) and loss factors $\tan\delta$. At a given voltage, which dielectric will have the lowest power dissipation per unit capacitance at 60 Hz? Is this also true at 1 MHz?

SOLUTION

The power dissipated at a given voltage per unit capacitance depends only on $\omega\tan\delta$, so we do not need to use ε_r'. Calculating $\omega\tan\delta$ or $(2\pi f)\tan\delta$, we find the values listed in the table at 60 Hz and 1 MHz. At 60 Hz, polycarbonate has the lowest power dissipation per unit capacitance, but at 1 MHz it is silicone rubber.

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[7] These currents are phasors, each with a rms magnitude and phase angle.

Table 7.4 Dielectric loss per unit volume for two insulators (κ is the thermal conductivity)

	$f = 60$ Hz			$f = 1$ MHz			
Material	ε'_r	tan δ	Loss (mW cm^{-3})	ε'_r	tan δ	Loss (W cm^{-3})	κ (W cm^{-1} K^{-1})
XLPE	2.3	3×10^{-4}	0.230	2.3	4×10^{-4}	5.12	0.005
Alumina	8.5	1×10^{-3}	2.84	8.5	1×10^{-3}	47.3	0.33

EXAMPLE 7.7

DIELECTRIC LOSS AND FREQUENCY Calculate the heat generated per second due to dielectric loss per cm^3 of cross-linked polyethylene, XLPE (typical power cable insulator), and alumina, Al_2O_3 (typical substrate in thin- and thick-film electronics), at 60 Hz and 1 MHz at a field of 100 kV cm^{-1}. Their properties are given in Table 7.4. What is your conclusion?

SOLUTION

The power dissipated per unit volume is

$$W_{vol} = (2\pi f)E^2\varepsilon_o\varepsilon'_r \tan\delta$$

We can calculate W_{vol} by substituting the properties of individual dielectrics at the given frequency f. For example, for XLPE at 60 Hz,

$$\begin{aligned} W_{vol} &= (2\pi 60 \text{ Hz})(100 \times 10^3 \times 10^2 \text{ V m}^{-1})^2(8.85 \times 10^{-12} \text{ F m}^{-1})(2.3)(3 \times 10^{-4}) \\ &= 230 \text{ W m}^{-3} \end{aligned}$$

We can convert this into per cm^3 by

$$W'_{vol} = \frac{W_{vol}}{10^6} = 0.230 \text{ mW cm}^{-3}$$

which is shown in Table 7.4.

From similar calculations we can obtain the heat generated per second per cm^3 as shown in Table 7.4. The heats at 60 Hz are small. The thermal conductivity of the insulation and its connecting electrodes can remove the heat without substantially increasing the temperature of the insulation. At 1 MHz, the heats generated are not trivial. One has to remove 5.12 W of heat from 1 cm^3 of XLPE and 47.3 W from 1 cm^3 of alumina. The thermal conductivity κ of XLPE is about 0.005 W cm^{-1} K^{-1}, whereas that of alumina is almost 100 times larger, 0.33 W cm^{-1} K^{-1}. The poor thermal conductivity of polyethylene means that 5.12 W of heat cannot be conducted away easily and it will raise the temperature of the insulation until dielectric breakdown ensues. In the case of alumina, 47.3 W of heat will substantially increase the temperature. *Dielectric loss is the mechanism by which microwave ovens heat food.* Dielectric heating at high frequencies is used in industrial applications such as heating plastics and drying wood.

7.4.2 DEBYE EQUATIONS, COLE–COLE PLOTS, AND EQUIVALENT SERIES CIRCUIT

Consider a dipolar dielectric in which there are both orientational and electronic polarizations, α_d and α_e, respectively, contributing to the overall polarizability. Electronic polarization α_e will be independent of frequency over the typical frequency range of operation of a dipolar dielectric, well below optical frequencies. At high frequencies, orientational polarization will be too sluggish too respond, $\alpha_d = 0$, and the ε_r will be $\varepsilon_{r\infty}$. (The subscript "infinity" simply means high frequencies where orientational polarization is negligible.) The dielectric constant and polarizabilities are generally related through[8]

Dielectric constant of a dipolar material

$$\varepsilon_r = 1 + \frac{N}{\varepsilon_o}\alpha_e + \frac{N}{\varepsilon_o}\alpha_d(\omega) = \varepsilon_{r\infty} + \frac{N}{\varepsilon_o}\alpha_d(\omega)$$

where we have combined 1 and α_e terms to represent the high frequency ε_r as $\varepsilon_{r\infty}$. Further $N\alpha_d(0)/\varepsilon_o$ determines the contribution of orientational polarization to the static dielectric constant $\varepsilon_{r\mathrm{dc}}$, so that $N\alpha_d(0)/\varepsilon_o$ is simply $(\varepsilon_{r\mathrm{dc}} - \varepsilon_{r\infty})$. Substituting for the frequency dependence of $\alpha_d(\omega)$ from Equation 7.26, and writing ε_r in terms of real and imaginary parts,

Dipolar dielectric constant

$$\varepsilon_r' - j\varepsilon_r'' = \varepsilon_{r\infty} + \frac{N}{\varepsilon_o}\frac{\alpha_d(0)}{1 + j\omega\tau} = \varepsilon_{r\infty} + \frac{(\varepsilon_{r\mathrm{dc}} - \varepsilon_{r\infty})}{1 + j\omega\tau} \qquad [7.33]$$

We can eliminate the complex denominator by multiplying both the denominator and numerator of the right-hand side by $1 - j\omega\tau$ and equate real and imaginary parts to obtain what are known as **Debye equations:**

Debye equations for real and imaginary parts

$$\varepsilon_r' = \varepsilon_{r\infty} + \frac{\varepsilon_{r\mathrm{dc}} - \varepsilon_{r\infty}}{1 + (\omega\tau)^2} \qquad [7.34a]$$

Debye equations for real and imaginary parts

and

$$\varepsilon_r'' = \frac{(\varepsilon_{r\mathrm{dc}} - \varepsilon_{r\infty})(\omega\tau)}{1 + (\omega\tau)^2} \qquad [7.34b]$$

Equations 7.34a and b reflect the behavior of ε_r' and ε_r'' as a function of frequency shown in Figure 7.13b. The imaginary part ε_r'' that represents the dielectric loss exhibits a peak at $\omega = 1/\tau$ which is called a **Debye loss peak.** Many dipolar gases and some liquids with dipolar molecules exhibit this type of behavior. In the case of solids the peak is typically much broader because we cannot represent the losses in terms of just one single well-defined relaxation time τ; the relaxation in the solid is usually represented by a distribution of relaxation times. Further, the simple relaxation process that is described in Equation 7.25 assumes that the dipoles do not influence each other either through their electric fields or through their interactions with the lattice; that is, they are not coupled. In solids, the dipoles can also couple, which complicates the relaxation process. Nonetheless, there are also many solids whose dielectric relaxation can be approximated by a nearly Debye relaxation or by slightly modifying Equation 7.33.

In dielectric studies of materials it is quite common to find a plot of the imaginary part (ε_r'') versus the real part (ε_r') as a function of frequency ω. Such plots are called **Cole–Cole plots** after their originators. The Debye equations 7.34a and b obviously

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[8] This simple relationship is used because the Lorentz local field equation does not apply in dipolar dielectrics and the local field problem is particularly complicated in these dielectrics.

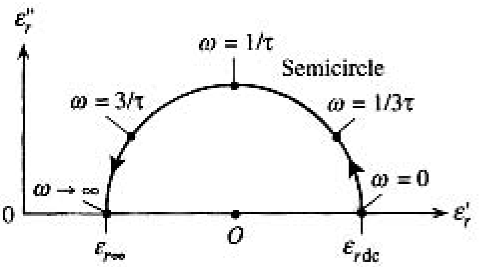

Figure 7.17 Cole–Cole plot is a plot of ε_r'' versus ε_r' as a function of frequency ω.

As the frequency is changed from low to high, the plot traces out a semicircle.

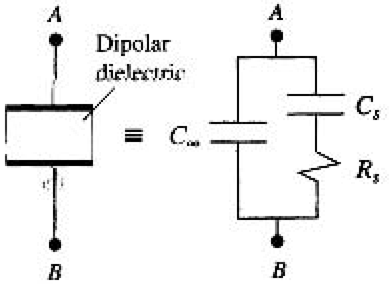

Figure 7.18 A capacitor with a dipolar dielectric and its equivalent circuit in terms of an ideal Debye relaxation.

provide the necessary values for ε_r' and ε_r'' to be plotted for the present simple dipolar relaxation mechanism that has only a single relaxation time τ. In fact, by simply putting in $\tau = 1$ second, we can calculate and plot ε_r'' versus ε_r' for $\omega = 0$ (dc) to $\omega \to \infty$ as shown in Figure 7.17. The result is a *semicircle.* While for certain substances, such as gases and some liquids, the Cole–Cole plots do indeed generate a semicircle, for many dielectrics, the curve is typically flattened and asymmetric, and not a semicircle.[9]

The Debye equations lead to a particular RC circuit representation of a dielectric material that is quite useful. Suppose that we have a resistance R_s in series with a capacitor C_s, both of which are in parallel with the capacitor C_∞ as in Figure 7.18. If we were to write down the equivalent admittance of this circuit, we would find that it corresponds to Equation 7.33, that is, the Debye equation. (The circuit mathematics is straightforward and is not reproduced here.) The reader may wonder why this circuit is different than the general model shown in Figure 7.14. Any series R_s and C_s circuit can be transformed to be equivalent to a parallel R_p and C_p (or G_p and C in Figure 7.14) circuit as is well known in circuit theory; the relationships between the elements depend on the frequency. Many electrolytic capacitors are frequently represented by an equivalent series R_s and C_s circuit as in Figure 7.18. If A is the area and d is the thickness of a parallel plate capacitor with a dipolar dielectric, then

Equivalent circuit of a Debye dielectric

$$C_\infty = \frac{\varepsilon_o \varepsilon_{r\infty} A}{d} \qquad C_s = \frac{\varepsilon_o(\varepsilon_{r\text{dc}} - \varepsilon_{r\infty})A}{d} \qquad \text{and} \qquad R_s = \frac{\tau}{C_s} \tag{7.35}$$

Notice that in this circuit model, R_s, C_s, and C_∞ do not depend on the frequency, which is only true for an ideal Debye dielectric, that with a single relaxation time τ.

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[9] The departure is simply due to the fact that a simple relaxation process with a single relaxation time cannot describe the dielectric behavior accurately. (A good overview of non-Debye relaxations is given by Andrew Jonscher in *J. Phys D*, **32**, R57, 1999.)

EXAMPLE 7.8

NEARLY DEBYE RELAXATION There are some dielectric solids that exhibit nearly Debye relaxation. One example is the $La_{0.7}Sr_{0.3}MnO_3$ ceramic whose relaxation peak and Cole–Cole plots are similar to those shown in Figures 7.13b and 7.17,[10] especially in the high-frequency range past the resonance peak. $La_{0.7}Sr_{0.3}MnO_3$'s low frequency (ε_{rdc}) and high frequency ($\varepsilon_{r\infty}$) dielectric constants are 3.6 and 2.58, respectively, where *low* and *high* refer, respectively, to frequencies far below and above the Debye relaxation peak, *i.e.*, ε_{rdc} and $\varepsilon_{r\infty}$. The Debye loss peak occurs at 6 kHz. Calculate ε_r' and the dielectric loss factor $\tan\delta$ at 29 kHz.

SOLUTION

The loss peak occurs when $\omega_o = 1/\tau$, so that $\tau = 1/\omega_o = 1/(2\pi 6000) = 26.5\ \mu\text{s}$. We can now calculate the real and imaginary parts of ε_r at 29 kHz,

$$\varepsilon_r' = \varepsilon_{r\infty} + \frac{\varepsilon_{rdc} - \varepsilon_{r\infty}}{1 + (\omega\tau)^2} = 2.58 + \frac{3.6 - 2.58}{1 + [(2\pi)(29 \times 10^3)(26.5 \times 10^{-6})]^2} = 2.62$$

$$\varepsilon_r'' = \frac{(\varepsilon_{rdc} - \varepsilon_{r\infty})(\omega\tau)}{1 + (\omega\tau)^2} = \frac{(3.6 - 2.58)[(2\pi)(29 \times 10^3)(26.5 \times 10^{-6})]}{1 + [(2\pi)(29 \times 10^3)(26.5 \times 10^{-6})]^2} = 0.202$$

and hence

$$\tan\delta = \frac{\varepsilon_r''}{\varepsilon_r'} = \frac{0.202}{2.62} = 0.077$$

which is close to the experimental value of 0.084.

This example was a special case of nearly Debye relaxation. Debye equations have been modified over the years to account for the broad relaxation peaks that have been observed, particularly in polymeric dielectric, by writing the complex ε_r as

Non-Debye relaxation

$$\varepsilon_r = \varepsilon_{r\infty} + \frac{\varepsilon_{rdc} - \varepsilon_{r\infty}}{[1 + (j\omega\tau)^{\alpha}]^{\beta}} \qquad [7.36]$$

where α and β are constants, typically less than unity (setting $\alpha = \beta = 1$ generates the Debye equations). Such equations are useful in engineering for predicting ε_r at any frequency from a few known values at various frequencies, as highlighted in this simple nearly Debye example. Further, if τ dependence on the temperature T is known (often τ is thermally activated), then we can predict ε_r at any ω and T.

7.5 GAUSS'S LAW AND BOUNDARY CONDITIONS

An important fundamental theorem in electrostatics is Gauss's law, which relates the integration of the electric field over a surface to the total charge enclosed. It can be derived from Coulomb's law, or the latter can be derived from Gauss's law. Suppose $\mathcal{E}_n$ is the electric field normal to a small surface area dA on a closed surface, as shown in Figure 7.19; then summing $\mathcal{E}_n\, dA$ products over the whole surface gives total net charge Q_{total} inside it,

Gauss's law

$$\oint_{\text{Surface}} \mathcal{E}_n\, dA = \frac{Q_{\text{total}}}{\varepsilon_o} \qquad [7.37]$$

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[10] Z. C. Xia *et al.*, *J. Phys. Cond. Matter*, **13**, 4359, 2001. The origin of the dipolar activity in this ceramic is quite complex and involves an electron hopping (jumping) from a Mn^{3+} to Mn^{4+} ion; we do not need the physical details in the example.

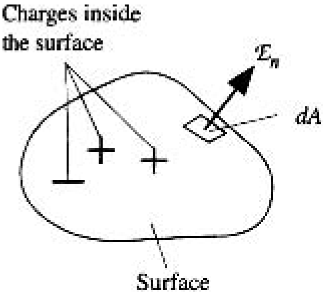


Figure 7.19 Gauss's law.
The surface integral of the electric field normal to the surface is the total charge enclosed. The field is positive if it is coming out, negative if it is going into the surface.

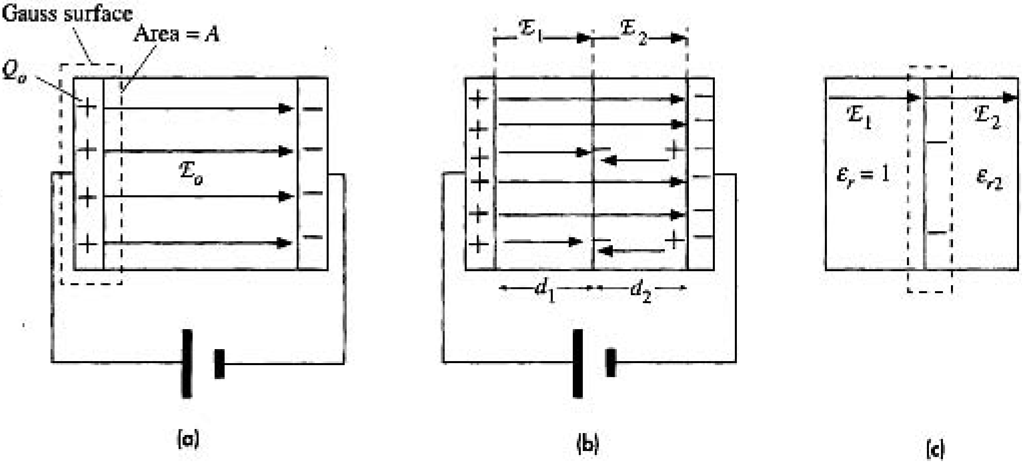


Figure 7.20

(a) The Gauss surface is a very thin rectangular surface just surrounding the positive electrode and enclosing the positive charges Q_o. The field cuts only the face just inside the capacitor.

(b) A solid dielectric occupies part of the distance between the plates. The vacuum (air)–dielectric boundary is parallel to the plates and normal to the fields $\mathcal{E}_1$ and $\mathcal{E}_2$.

(c) A thin rectangular Gauss surface at the boundary encloses the negative polarization charges.

where the circle on the integral sign represents integrating over the whole surface (any shape) enclosing the charges constituting Q_{total} as shown in Figure 7.19. The total charge Q_{total} includes *all charges*, both free charges and bound polarization charges. Gauss's law is one of the most useful laws for calculating electric fields in electrostatics, more so than the Coulomb law with which the reader is probably more familiar. The surface can be of any shape as long as it contains the charges. We generally choose convenient surfaces to simplify the integral in Equation 7.37, and these convenient surfaces are called Gauss surfaces. It should be noted from Figure 7.19 that the field $\mathcal{E}_n$ is coming *out* from the surface.

As an example, we can consider the field in the parallel plate capacitor in Figure 7.20a with no dielectric medium. We draw a thin rectangular Gauss surface (a hypothetical surface) just enclosing the positive electrode that contains the free charges $+Q_o$ on the plate. The field $\mathcal{E}_o$ is normal to the inner face (area A) of the Gauss surface.

Further, we can assume that $\mathcal{E}_o$ is uniform across the plate surface, which means that the integral of $\mathcal{E}_n\,dA$ in Equation 7.37 over the surface is simply $\mathcal{E}_o A$. There is no field on the other faces of this rectangular Gauss surface. Then from Equation 7.37,

$$\mathcal{E}_o A = \frac{Q_o}{\varepsilon_o}$$

which gives

$$\mathcal{E}_o = \frac{\sigma_o}{\varepsilon_o} \tag{7.38}$$

where

$$\sigma_o = \frac{Q_o}{A}$$

is the free surface charge density. This is the same as the field we calculated using $\mathcal{E}_o = V/d$ and $Q_o = CV$.

An important application of Gauss's law is determining what happens at boundaries between dielectric materials. The simplest example is the insertion of a dielectric slab to only partially fill the distance between the plates, as shown in Figure 7.20b. The applied voltage remains the same, but the field is no longer uniform between the plates. There is an air–dielectric boundary. The field is different in the air and dielectric regions. Suppose that the field is $\mathcal{E}_1$ in the air region and $\mathcal{E}_2$ in the dielectric region. Both these fields are normal to the boundary by the choice of the dielectric shape (faces parallel to the plates). As a result of polarization, bound surface charges $+A\sigma_P$ and $-A\sigma_P$ appear on the surfaces of the dielectric slab, as shown in Figure 7.20b, where $\sigma_P = P$, the polarization in the dielectric. We draw a very narrow rectangular Gauss surface that encompasses the air–dielectric interface and hence the surface polarization charges $-A\sigma_P$ as shown in Figure 7.20c. The field coming *in* at the left face in air is $\mathcal{E}_1$ (taken as negative) and the field coming *out* at the right face in the dielectric is $\mathcal{E}_2$. The surface integral $\mathcal{E}_n\,dA$ and Gauss's law become

$$\mathcal{E}_2 A - \mathcal{E}_1 A = \frac{-(A\sigma_P)}{\varepsilon_o}$$

or

$$\mathcal{E}_1 = \mathcal{E}_2 + \frac{P}{\varepsilon_o}$$

The polarization P and the field $\mathcal{E}_2$ in the dielectric are related by

$$P = \varepsilon_o \chi_{e2} \mathcal{E}_2$$

or

$$P = \varepsilon_o(\varepsilon_{r2} - 1)\mathcal{E}_2$$

where χ_{e2} is the electrical susceptibility and ε_{r2} is the relative permittivity of the inserted dielectric. Then, substituting for P, we can relate $\mathcal{E}_1$ and $\mathcal{E}_2$,

$$\mathcal{E}_1 = \mathcal{E}_2 + (\varepsilon_{r2} - 1)\mathcal{E}_2$$

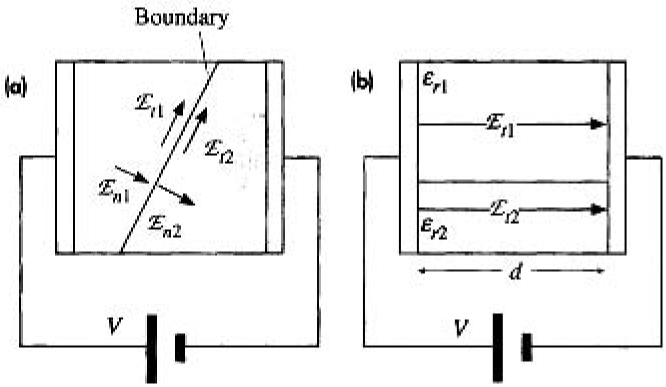


Figure 7.21
(a) Boundary conditions between dielectrics.
(b) The case for $\mathcal{E}_{t1} = \mathcal{E}_{t2}$.

or

$$\mathcal{E}_1 = \varepsilon_{r2}\mathcal{E}_2$$

The field in the air part is $\mathcal{E}_1$ and the relative permittivity is 1. The example in Figure 7.20b involved a boundary between air (vacuum) and a dielectric solid, and the boundary was parallel to the plates and hence normal to the fields $\mathcal{E}_1$ and $\mathcal{E}_2$. A more general expression can be shown to relate the normal components of the electric field, shown as $\mathcal{E}_{n1}$ and $\mathcal{E}_{n2}$ in Figure 7.21a, on either side of a boundary by

General boundary condition

$$\varepsilon_{r1}\mathcal{E}_{n1} = \varepsilon_{r2}\mathcal{E}_{n2} \qquad [7.39]$$

There is a second boundary condition that relates the tangential components of the electric field, shown as $\mathcal{E}_{t1}$ and $\mathcal{E}_{t2}$ in Figure 7.21a, on either side of a boundary. These tangential fields must be equal.

General boundary condition

$$\mathcal{E}_{t1} = \mathcal{E}_{t2} \qquad [7.40]$$

We can readily appreciate this boundary condition by examining the fields in a parallel plate capacitor, which has two dielectrics longitudinally filling the space between the plates but with a boundary parallel to the field, as shown in Figure 7.21b. The field in each, $\mathcal{E}_{t1}$ and $\mathcal{E}_{t2}$, is parallel to the boundary. The voltage across each longitudinal dielectric slab is the same, and since $\mathcal{E} = dV/dx$, the field in each is the same, $\mathcal{E}_{t1} = \mathcal{E}_{t2} = V/d$.

The above boundary conditions are widely used in explaining dielectric behavior when boundaries are involved. For example, consider a small disk-shaped cavity within a solid dielectric between two electrodes, as depicted in Figure 7.22. The disk-shaped cavity has its face perpendicular to the electric field. Suppose that the dielectric length d is 1 cm and the cavity size is on the scale of micrometers. The average field within the dielectric will still be close to V/d because in integrating the field $\mathcal{E}(x)$ to find the voltage across the dielectric, the contribution from a tiny distance of a few microns will be negligible compared with contributions coming over the rest of the 1 cm. But the field within the cavity will not be the same as the average field $\mathcal{E}_1$ in the dielectric. If $\varepsilon_{r1} = 5$ for the dielectric medium and the cavity has air, then at the cavity face we have

$$\varepsilon_{r2}\mathcal{E}_2 = \varepsilon_{r1}\mathcal{E}_1$$

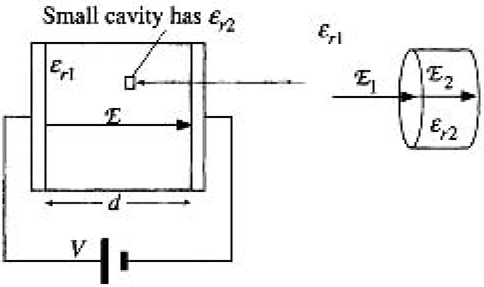


Figure 7.22 Field in the cavity is higher than the field in the solid.

which gives

$$\mathcal{E}_2 = 5\left(\frac{V}{d}\right)$$

Air insulation in a 100 micron (0.1 mm) thick cavity breaks down when $\mathcal{E}_2$ is typically 100 kV cm^{-1}. From $\mathcal{E}_2 = 5(V/d)$, a voltage of 20 kV will result in the breakdown of air in the cavity and hence a discharge current. This is called a **partial discharge** as only a partial breakdown of the insulation, that in the cavity, has occurred between the electrodes. Under an ac voltage, the discharge in the cavity can often be sustained by the capacitive current through the surrounding dielectric. Without this cavity, the dielectric would accept a greater voltage across it, which in this case is typically greater than 100 kV.

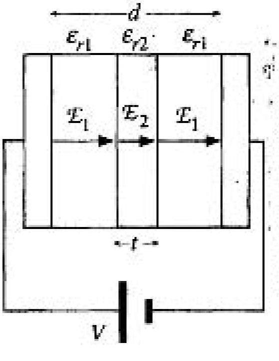
EXAMPLE 7.9 **FIELD INSIDE A THIN DIELECTRIC WITHIN A SECOND DIELECTRIC** When the dielectric fills the whole space between the plates of a capacitor, the net field within the dielectric is the same as before, $\mathcal{E} = V/d$. Explain what happens when a dielectric slab of thickness $t \ll d$ is inserted in the middle of the space between the plates, as shown in Figure 7.23. What is the field inside the dielectric?

SOLUTION

The problem is illustrated in Figure 7.23 and has symmetry in that the field in air on either side of the dielectric is the same and $\mathcal{E}_1$. The boundary conditions give

$$\varepsilon_{r1}\mathcal{E}_1 = \varepsilon_{r2}\mathcal{E}_2$$

Figure 7.23 A thin slab of dielectric is placed in the middle of a parallel plate capacitor.
The field inside the thin slab is $\mathcal{E}_2$.



Further, the integral of the field from one plate to the other must be V because $dV/dx = \mathcal{E}$. Examining Figure 7.23, we see that the integration is

$$\mathcal{E}_1(d - t) + \mathcal{E}_2 t = V$$

We now have to eliminate $\mathcal{E}_1$ between the previous two equations and obtain $\mathcal{E}_2$, which can be done by algebraic manipulation,

$$\mathcal{E}_2 = \frac{\varepsilon_{r1}}{\varepsilon_{r2} - \dfrac{t}{d}(\varepsilon_{r2} - \varepsilon_{r1})}\left(\frac{V}{d}\right) \tag{7.41}$$

If $t \ll d$, then this approximates to

$$\mathcal{E}_2 = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\left(\frac{V}{d}\right) \quad \text{and} \quad \mathcal{E}_1 = \left(\frac{V}{d}\right) \quad (t \ll d) \tag{7.42}$$

Clearly $\mathcal{E}_1$ in the air space remains the same as the applied field V/d. Since $\varepsilon_{r1} = 1$ (air) and $\varepsilon_{r2} > 1$, $\mathcal{E}_2$ in the thin dielectric slab is smaller than the applied field V/d. On the other hand, if we have air space between two dielectric slabs, then the field in this air space will be greater than the field inside the two dielectric slabs. Indeed, if the applied voltage is sufficiently large, the field in the air gap can cause dielectric breakdown of this region.

EXAMPLE 7.10

GAUSS'S LAW WITHIN A DIELECTRIC AND FREE CHARGES Gauss's law in Equation 7.37 contains the total charge Q_{total}, enclosed within the surface. Generally, these enclosed charges are free charges Q_{free}, due to the free carriers on the electrode, and bound charges Q_P, due to polarization charges on the dielectric surface. Apply Gauss's law using a Gaussian rectangular surface enclosing the left electrode and the dielectric surface in Figure 7.24. Show that the electric field $\mathcal{E}$ in the dielectric can be expressed in terms of free charges only, Q_{free}, through

Free charges and field in a dielectric

$$\oint_{\text{Surface}} \mathcal{E}_n \, dA = \frac{Q_{\text{free}}}{\varepsilon_o \varepsilon_r} \tag{7.43}$$

where ε_r is the relative permittivity of the dielectric medium.

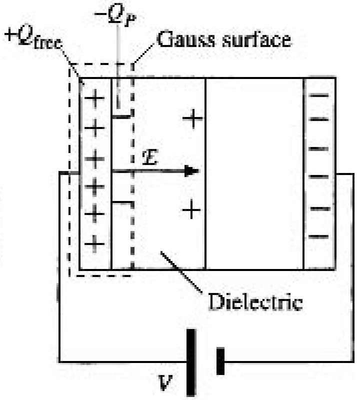


Figure 7.24 A convenient Gauss surface for calculating the field inside the dielectric is a very thin rectangular surface enclosing the surface of the dielectric.

The total charges enclosed are the free charges on the electrodes and the polarization charges on the surface of the dielectric.

SOLUTION

We apply Gauss's law to a hypothetical rectangular surface enclosing the left electrode and the dielectric surface. The field $\mathcal{E}$ in the dielectric is normal and outwards at the Gauss surface in Figure 7.24. Thus $\mathcal{E}_n = \mathcal{E}$ in the left-hand side of Equation 7.37.

$$\varepsilon_o A\mathcal{E} = Q_{\text{total}} = Q_{\text{free}} - Q_P = Q_{\text{free}} - AP = Q_{\text{free}} - A\varepsilon_o(\varepsilon_r - 1)\mathcal{E}$$

where we have used $P = \varepsilon_o(\varepsilon_r - 1)\mathcal{E}$. Rearranging,

$$\varepsilon_o\varepsilon_r A\mathcal{E} = Q_{\text{free}}$$

Since $A\mathcal{E}$ is effectively the surface integral of $\mathcal{E}_n$, the above corresponds to writing Gauss's law in a dielectric in terms of free charges as

$$\oint_{\text{Surface}} \mathcal{E}_n\, dA = \frac{Q_{\text{free}}}{\varepsilon_o\varepsilon_r}$$

The above equation assumes that polarization P and $\mathcal{E}$ are linearly related,

$$P = \varepsilon_o(\varepsilon_r - 1)\mathcal{E}$$

We note that if we only use free charges in Gauss's law, then we simply multiply ε_o by the dielectric constant of the medium. The above proof is by no means a rigorous derivation.

7.6 DIELECTRIC STRENGTH AND INSULATION BREAKDOWN

7.6.1 DIELECTRIC STRENGTH: DEFINITION

A defining property of a dielectric medium is not only its ability to increase capacitance but also, and equally important, its insulating behavior or low conductivity so that the charges are not simply conducted from one plate of the capacitor to the other through the dielectric. Dielectric materials are widely used as insulating media between conductors at different voltages to prevent the ionization of air and hence current flashovers between conductors. The voltage across a dielectric material and hence the field within it cannot, however, be increased without limit. Eventually a voltage is reached that causes a substantial current to flow between the electrodes, which appears as a short between the electrodes and leads to what is called **dielectric breakdown.** In gaseous and many liquid dielectrics, the breakdown does not generally permanently damage the material. This means that if the voltage causing breakdown is removed, then the dielectric can again sustain voltages until the voltage is sufficiently high to cause breakdown again. In solid dielectrics the breakdown process invariably leads to the formation of a permanent conducting channel and hence to permanent damage. The **dielectric strength** $\mathcal{E}_{\text{br}}$ is the maximum field that can be applied to an insulating medium without causing dielectric breakdown. Beyond $\mathcal{E}_{\text{br}}$, dielectric breakdown takes place. The dielectric strength of solids depends on a number of factors besides simply the molecular structure, such as the impurities in the material, microstructural defects (*e.g.*, microvoids), sample geometry, nature of the electrodes, temperature, and ambient conditions (*e.g.*, humidity), as well as the duration and frequency of the applied field. Dielectric strength

Table 7.5 Dielectric strength; typical values at room temperature and 1 atm

Dielectric Medium	Dielectric Strength	Comments
Atmosphere at 1 atm pressure	31.7 kV cm^{-1} at 60 Hz	1 cm gap. Breakdown by electron avalanche by impact ionization.
SF_6 gas	79.3 kV cm^{-1} at 60 Hz	Used in high-voltage circuit breakers to avoid discharges.
Polybutene	>138 kV cm^{-1} at 60 Hz	Liquid dielectric used as oil filler and HV pipe cables.
Transformer oil	128 kV cm^{-1} at 60 Hz	
Amorphous silicon dioxide (SiO_2) in MOS technology	10 MV cm^{-1} dc	Very thin oxide films without defects. Intrinsic breakdown limit.
Borosilicate glass	10 MV cm^{-1} duration of 10 μs 6 MV cm^{-1} duration of 30 s	Intrinsic breakdown. Thermal breakdown.
Polypropylene	295–314 kV cm^{-1}	Likely to be thermal breakdown or electrical treeing.

is different under dc and ac conditions. There are also **aging effects** that slowly degrade the properties of the insulator and reduce the dielectric strength. For engineers involved in insulation, the dielectric strength of solids is therefore one of the most difficult parameters to interpret and use. For example, the breakdown field also depends on the thickness of the insulation because thicker insulators have more volume and hence a greater probability of containing a microstructural defect (*e.g.*, a microcavity) that can initiate a dielectric breakdown. Table 7.5 shows some typical dielectric strengths for various dielectrics used in electrical insulation. Unpressurized gases have lower breakdown strengths than liquids and solids.

7.6.2 DIELECTRIC BREAKDOWN AND PARTIAL DISCHARGES: GASES

Due to cosmic radiation, there are always a few free electrons in a gas. If the field is sufficiently large, then one of these electrons can be accelerated to sufficiently large kinetic energies to impact ionize a neutral gas molecule and produce an additional free electron and a positively charged gas ion. Both the first and liberated electrons are now available to accelerate in the field again and further impact ionize more neutral gas molecules, and so on. Thus, an avalanche of impact ionization processes creates many free electrons and positive gas ions in the gas, which give rise to a discharge current between the electrodes. The process is similar to avalanche breakdown in a reverse-biased *pn* junction. The breakdown in gases depends on the pressure. The concentration of gas molecules is greater at higher pressures. This means that the mean separation between molecules, and, hence, the mean free path of a free electron, is shorter. Shorter mean free paths inhibit the free electrons from accelerating to reach impact ionization energies unless the field is increased. Thus, generally, $\mathcal{E}_{br}$ increases with the gas pressure. The 60 Hz breakdown field for an air gap of 1 cm at room temperature and at atmospheric pressure is about 31.7 kV cm^{-1}. On the other hand, the gas sulfurhexafluoride, SF_6, has

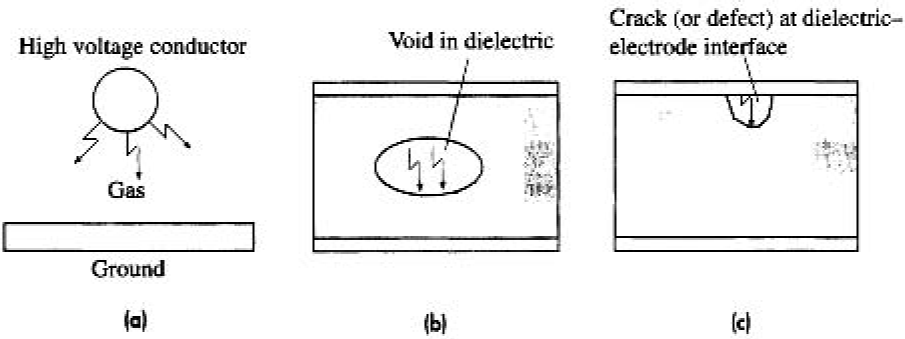


Figure 7.25

(a) The field is greatest on the surface of the cylindrical conductor facing the ground. If the voltage is sufficiently large, this field gives rise to a corona discharge.

(b) The field in a void within a solid can easily cause partial discharge.

(c) The field in the crack at the solid–metal interface can also lead to a partial discharge.

a dielectric strength of 79.3 kV cm^{-1} and an even higher strength when pressurized. SF_6 is therefore used instead of air in high-voltage circuit breakers.

A **partial discharge** occurs when only a local region of the dielectric is exhibiting discharge, so the discharge does not directly connect the two electrodes. For example, for the cylindrical conductor carrying a high voltage above a grounded plate, as in Figure 7.25a, the electric field is greatest on the surface of the conductor facing the ground. This field initiates discharge locally in this region because the field is sufficiently high to give rise to an electron avalanche effect. Away from the conductor, however, the field is not sufficiently strong to continue the electron avalanche discharge. This type of local discharge in high field regions is termed **corona discharge.** Voids and cracks occurring within solid dielectrics and discontinuities at the dielectric–electrode interface can also lead to partial discharges as the field in these voids is higher than the average field in the dielectric, and, further, the dielectric strength in the gas (*e.g.*, atmosphere) in the void is less than that of the continuous solid insulation. Figure 7.25b and c depict two examples of partial discharges occurring in voids, one inside the solid (perhaps an air or gas bubble introduced during the processing of the dielectric) and the other (perhaps in the form of a crack) at the solid–electrode interface. In practice, a variety of factors can lead to microvoids and microcavities inside solids as well as at interfaces. Partial discharges in these voids physically and chemically erode the surrounding dielectric region and lead to an overall deterioration of the dielectric strength. If uncontrolled, they can eventually give rise to a major breakdown.

7.6.3 DIELECTRIC BREAKDOWN: LIQUIDS

The processes that lead to the breakdown of insulation in liquids are not as clear as the electron avalanche effect in gases. In impure liquids with small conductive particles in suspension, it is believed that these impurities coalesce end to end to form a conducting bridge between the electrodes and thereby give rise to discharge. In some liquids, the discharge initiates as partial discharges in gas bubbles entrapped in the liquid. These partial discharges can locally raise the temperature and vaporize more of the liquid and hence increase the size of the bubble. The eventual discharge can be a series of partial discharges in entrapped gas bubbles. Moisture absorption and absorption of gases from the ambient generally deteriorate the dielectric strength. Oxidation of certain liquids, such as oils, with time produces more acidic and hence higher conductivity inclusions or regions that eventually give discharge. In some liquids, the discharge involves the emission of a large number of electrons from the electrode into the liquid due to field emission at high fields. This is a discharge process by electrode injection.

7.6.4 DIELECTRIC BREAKDOWN: SOLIDS

There are various major mechanisms that can lead to dielectric breakdown in solids. The most likely mechanism depends on the dielectric material's condition and sometimes on extrinsic factors such as the ambient conditions, moisture absorption being a typical example.

Intrinsic Breakdown or Electronic Breakdown The most common type of electronic breakdown is an **electron avalanche breakdown.** A free electron in the conduction band (CB) of a dielectric in the presence of a large field can be accelerated to sufficiently large energies to collide with and ionize a host atom of the solid. The electron gains an energy $e\mathcal{E}_{br}\ell$ when it moves a distance ℓ under an applied field $\mathcal{E}_{br}$. If this energy is greater than the bandgap energy E_g, then the electron, as a result of a collision with the lattice vibrations, can excite an electron from the valence band to the conduction band, that is, break a bond. Both the primary and the released electron can further impact ionize other host atoms and thereby generate an electron avalanche effect that leads to a substantial current. The initial conduction electrons for the avalanche are either present in the CB or are injected from the metal into the CB as a result of field-assisted thermal emission from the Fermi energy in the metal to the CB in the dielectric. Taking typical values, $E_g \approx 5$ eV and ℓ to be of the order of the mean free path for lattice scattering, say ~50 nm, one finds $\mathcal{E}_{br} \approx 1$ MV cm^{-1}. Obviously, $\mathcal{E}_{br}$ depends on the choice of ℓ, but its order of magnitude indicates voltages that are quite large. This type of breakdown represents an upper theoretical limit that is probably approached by only certain dielectrics—those that have practically no defects. Usually, microstructural defects lead to a lower dielectric strength than the limit indicated by intrinsic breakdown. Silicon dioxide (SiO_2) films with practically no structural defects in present MOS (metal-oxide-semiconductor) capacitors (as in the gates of MOSFETs) probably exhibit an intrinsic breakdown.

If dielectric breakdown does not occur by an electron avalanche effect (perhaps due to short mean free paths in the insulator), then another insulation breakdown mechanism is the enormous increase in the injection of electrons from the metal electrode into the insulator at very high fields as a result of field-assisted emission.[11] It has

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[11] The emission of electrons by tunneling from an electrode in the presence of a large field was treated in Chapter 4 as Fowler–Nordheim field emission.

been proposed that insulation breakdown under short durations in some thin polymer films is due to tunneling injection.

Thermal Breakdown Finite conductivity of the insulation means that there is Joule heat $\sigma \mathcal{E}^2$ being released within the solid. Further, at high frequencies, the dielectric loss, $V^2\omega \tan \delta$, becomes especially significant. For example, the work done by the external field in rotating the dipoles is transferred more frequently to random molecular collisions as heat as the frequency of the field increases. Both conduction and dielectric losses therefore generate heat within the dielectric. If this heat cannot be removed from the solid sufficiently quickly by thermal conduction (or by other means), then the temperature of the dielectric will increase. The increase in the temperature invariably increases the conductivity of an insulator. The increase in the conductivity then leads to more Joule heating and hence further rises in the temperature and so on. If the heat cannot be conducted away to limit the temperature, then the result is a thermal runaway condition in which the temperature and the current increase until a discharge occurs through various sections of the solid. As a consequence of sample inhomogeneities, frequently thermal runaway is severe in certain parts of the solid that become hot spots and suffer local melting and physical and chemical erosion. Hot spots are those local regions or inhomogeneities where σ or ε_r'' is larger or where the thermal conductivity is too poor to remove the heat generated. Local breakdown at various hot spots eventually leads to a conducting channel connecting the opposite electrodes and hence to a dielectric breakdown. Since it takes time to raise the temperature of the dielectric, due to the heat capacity, this breakdown process has a marked thermal lag. The time to achieve thermal breakdown depends on the heat generated, and hence on $\mathcal{E}^2$. Conversely, this means that the dielectric strength $\mathcal{E}_{br}$ depends on the duration of application of the field. For example, at 70 °C, pyrex has an $\mathcal{E}_{br}$ of typically 9 MV cm^{-1} if the applied field duration is kept short, not more than 1 ms or so. If the field is kept for 30 s, then the breakdown field is only 2.5 MV cm^{-1}. Dielectric breakdown in various ceramics and glasses at high frequencies has been attributed directly to thermal breakdown. A characteristic feature of thermal breakdown is not only the thermal lag, the time dependence, but also the temperature dependence. Thermal breakdown is facilitated by increasing the temperature of the dielectric, which means that $\mathcal{E}_{br}$ decreases with temperature.

Electromechanical Breakdown and Electrofracture A dielectric medium between oppositely charged electrodes experiences compressional forces because the opposite charges $+Q$ and $-Q$ on the plates attract each other, as depicted in Figure 7.26. As the voltage increases, so does the compressive load, and the dielectric becomes squeezed, or the thickness d gets smaller. At each stage, the increase in the compressive load is normally balanced by the elastic deformation of the insulation to a new smaller thickness. However, if the elastic modulus is sufficiently small, then compressive loads cannot be simply balanced by the elasticity of the solid, and there is a mechanical runaway for the following reasons. The decrease in d, due to the compressive load, leads to a higher field ($\mathcal{E} = V/d$) and also to more charges on the electrodes ($Q = CV$, $C = \varepsilon_o\varepsilon_r A/d$). This in turn leads to a greater compressive load, which further decreases d, and so on, until the shear stresses within the insulation cause the insulation to flow plastically (for example, by viscous deformation). Eventually, the insulation breaks down. In addition, the increase in $\mathcal{E}$ as d gets

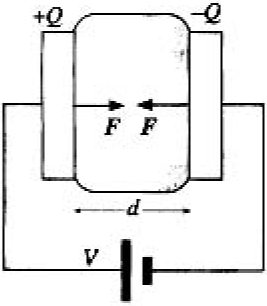
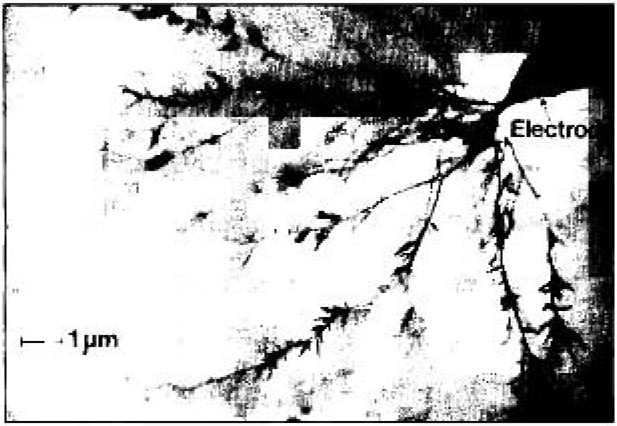


Figure 7.26 An exaggerated schematic illustration of a soft dielectric medium experiencing strong compressive forces due to the applied voltage.

smaller results in more Joule ($\sigma\mathcal{E}^2$) and dielectric-loss heating ($\omega\mathcal{E}^2 \tan\delta$) in the dielectric, which increases the temperature and hence lowers the elastic modulus and viscosity, thereby further deteriorating the mechanical stability. It is also possible for the field during the mechanical deformation of the dielectric to reach the thermal breakdown field, in which case the dielectric failure is not truly a mechanical breakdown mechanism though initiated by mechanical deformations. Another possibility is the initiation and growth of internal cracks (perhaps filamentary cracks) by internal stresses around inhomogeneous regions inside the dielectric. For example, an imperfection or a tiny cavity experiences shear stresses and also large local electric fields. Combined effects of both large shear stresses and large electric fields eventually lead to crack propagation and mechanical and, hence, dielectric failure. This type of process is sometimes called **electrofracture.** It is generally believed that certain thermoplastic polymers suffer from electromechanical dielectric breakdown, especially close to their softening temperatures. Polyethylene and polyisobutylene have been cited as examples.

Internal Discharges These are partial discharges that take place in microstructural voids, cracks, or pores within the dielectric where the gas atmosphere (usually air) has lower dielectric strength. A porous ceramic, for example, would experience partial discharges if the applied field is sufficiently large. The discharge current in a void,



Electrical breakdown by *treeing* (formation of discharge channels) in a low-density polyethylene insulation when a 50 Hz, 20 kV (rms) voltage is applied for 200 minutes to an electrode embedded in the insulation.

SOURCE: J. W. Billing and D. J. Groves, *Proceedings of the Institution of Electrical Engineers*, **212**, 1451, 1974.

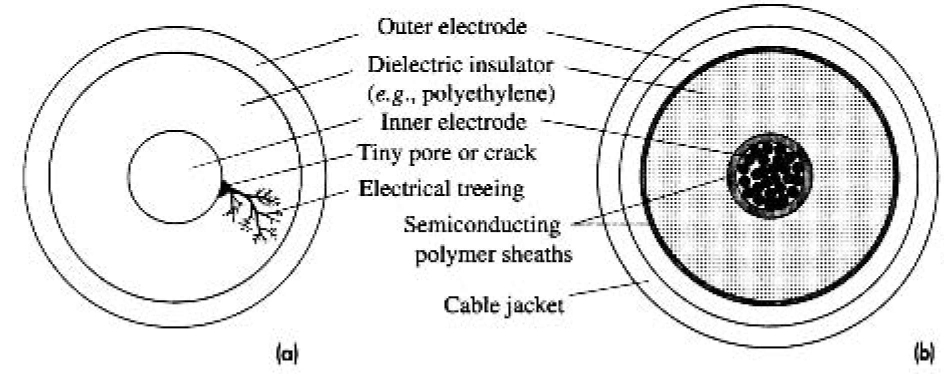


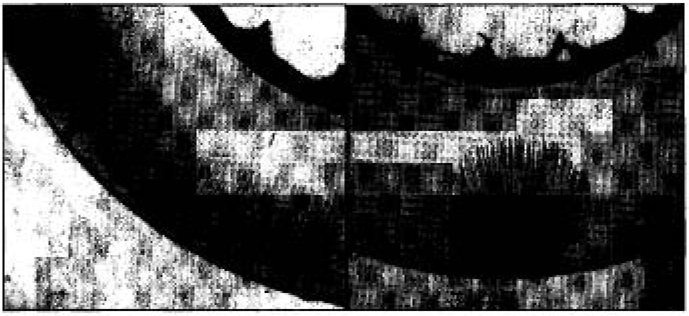
Figure 7.27

(a) A schematic illustration of electrical treeing breakdown in a high-voltage coaxial cable that was initiated by a partial discharge in the void at the inner conductor–dielectric interface.

(b) A schematic diagram of a typical high-voltage coaxial cable with semiconducting polymer layers around the inner conductor and around the outer surface of the dielectric.

such as those in Figure 7.25b and c, can be easily sustained under ac conditions, which accounts for the severity of this type of breakdown mechanism under ac conditions. Initially, the pore size (or the number of pores) may be small and the partial discharge insignificant, but with time the partial discharge erodes the internal surfaces of the void. Partial discharges can locally melt the insulator and can easily cause chemical transformations. Eventually, and usually, an **electrical tree** type of discharge develops from a partial discharge that has been eroding the dielectric, as depicted in Figure 7.27a for a high-voltage cable in which there is a tiny void at the interface between the dielectric and the inner conductor (generated perhaps by the differential thermal expansion of the electrode and polymeric insulation). The erosion of the dielectric by the partial discharge propagates like a branching tree. The "tree branches" are erosion channels—hollow filaments of various sizes—in which gaseous discharge takes place and forms a conducting channel during operation.

In the case of a coaxial high-voltage cable in Figure 7.27a, the dielectric is usually a polymer, polyethylene (PE) being one of the most popular. The electric field is maximum at the surface of the inner conductor, which is the reason for the initiation of most electrical trees near this surface. Electrical treeing is substantially controlled by having semiconductive polymer layers or sheaths surrounding the inner conductor and the outer surface of the insulator, as shown in Figure 7.27b. For flexibility, the inner conductor is frequently multicored, or stranded, rather than solid. Due to the extrusion process used to draw the insulation, the semiconductive polymer sheaths are bonded to the insulation. There are therefore practically no microvoids at the interfaces between the insulator and the semiconducting sheath. Further, these semiconducting polymer sheaths are sufficiently conductive to become "part of the electrodes." Both the conductor and the adjacent semiconductor are roughly at the same voltage, which means that there is no breakdown in the semiconductor–conductor interfaces. There is normally an outer jacket (e.g., PVC) to protect the cable.



Some typical water trees found in field aged cables. Left: Trees in a cable with tape and graphite insulation. Right: Trees in a cable with strippable insulation.

SOURCE: P. Werellius, P. Tharning, R. Eriksson, B. Holmgren. J. Gafvert, "Dielectric Spectroscopy for Diagnosis of Water Tree Deterioration in XLPE Cables," *IEEE Transactions on Dielectrics and Electrical Insulation*, vol. **8**, February 2001, p. 34, figure 10 (© IEEE, 2001).

Insulation Aging It is well recognized that during service, the properties of an insulating material become degraded and eventually dielectric breakdown occurs at a field below that predicted by experiments on fresh forms of the insulation. **Aging** is a term used to describe, in a general sense, the deterioration in the properties of the insulation. Aging therefore determines the useful life of the insulation. There are many factors that either directly or indirectly affect the properties and performance of an insulator in service. Even in the absence of an electric field, the insulation will experience physical and chemical aging whereby its physical and chemical properties change considerably. An insulation that is subjected to temperature and mechanical stress variations can develop structural defects, such as microcracks, which are quite damaging to the dielectric strength, as mentioned above. Irradiation by ionizing radiation such as X-rays, exposure to severe ambient conditions such as excessive humidity, ozone, and many other external conditions, through various chemical processes, deteriorate the chemical structure and properties of an insulator. This is generally much more severe for polymers than ceramics, but it is not practical to use a solid ceramic insulation in a coaxial power cable. Oxidation of a polymeric insulation with time is another form of chemical aging and is well-known to degrade the insulation performance. This is the reason for adding various antioxidants into semicrystalline polymers for use in insulation. The chemical aging processes are generally accelerated with temperature. In service, the insulation also experiences electrical aging as a result of the effects of the field on the properties of the insulation. For example, dc fields can disassociate and transport various ions in the structure and thereby slowly change the structure and properties of the insulation. Electrical trees develop as a result of electrical aging because, in service, the ac field gives rise to continual partial discharges in an internal or surface microcavity, which then erodes the region around it and slowly grows like a branching tree. In well-manufactured insulation systems, electrical treeing has been substantially reduced or eliminated from microvoids. A form of electrical aging that is currently in vogue is **water treeing,** which eventually leads to electrical treeing. The definition of a water tree, as viewed under an optical microscope, is a diffused bushy (or broccoli) type growth that consists of millions of microscopic voids (per mm^3) containing water or aqueous electrolyte. They invariably occur in moist environments and are relatively nonconducting, which means that they do not themselves lead to a direct discharge.

External Discharges There are many examples where the surface of the insulation becomes contaminated by ambient conditions such as excessive moisture, deposition

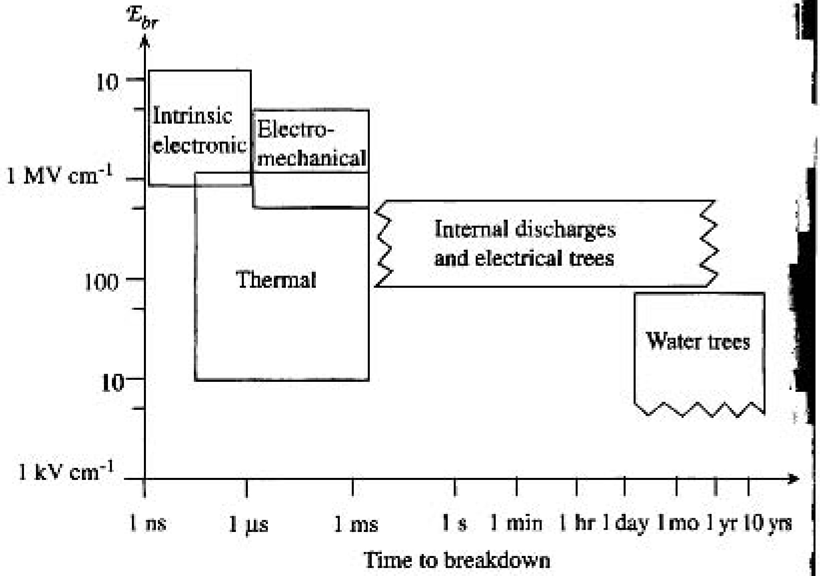


Figure 7.28 Time to breakdown and the field at breakdown $\mathcal{E}_{br}$ are interrelated and depend on the mechanism that causes the insulation breakdown.

External discharges have been excluded.

SOURCE: Based on L. A. Dissado and J. C. Fothergill, *Electrical Degradation and Breakdown in Polymers*, United Kingdom: Peter Peregrinus Ltd. for IEE, 1992, p. 63.

of pollutants, dirt, dust, and salt spraying. Eventually the contaminated surface develops sufficient conductance to allow discharge between the electrodes at a field below the normal breakdown strength of the insulator. This type of dielectric breakdown over the surface of the insulation is termed **surface tracking.**

It is apparent that there are a number of dielectric breakdown mechanisms and the one that causes eventual breakdown depends not only on the properties and quality of the material but also on the operating conditions, environmental factors being no less important. Figure 7.28 provides an illustrative diagram showing the relationship between the breakdown field and the time to breakdown. An insulation that can withstand large fields for a very short duration will break down at a lower field if the duration of the field increases. The breakdown mechanism is also likely to change from being intrinsic to being, perhaps, thermal. When insulation breakdown occurs in times beyond a few days, it is generally attributed to the degradation of the insulation, which eventually leads to a breakdown through, most probably, electrical treeing. It is also apparent that it is not possible to clearly identify a specific dielectric breakdown mechanism for a given material.

EXAMPLE 7.11 **DIELECTRIC BREAKDOWN IN A COAXIAL CABLE** Consider the coaxial cable in Figure 7.29 with a and b defining the radii of the inner and outer conductors.

a. Using Gauss's law, find the capacitance of the coaxial cable.

b. What is the electric field at r from the center of the cable ($r > a$)? Where is the field maximum?

c. Consider two candidate materials for the dielectric insulation: cross-linked polyethylene (XLPE) and silicone rubber. Suppose that the inner conductor diameter is 5 mm and the insulation thickness is also 5 mm. What is the voltage that will cause dielectric breakdown in each insulator?

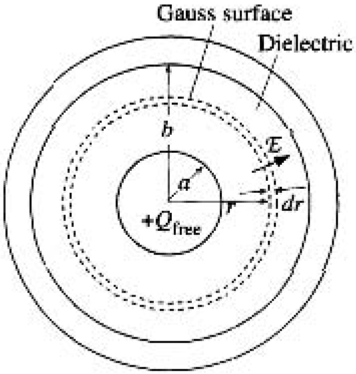


Figure 7.29 A schematic diagram for the calculation of the capacitance of a coaxial cable and the field at point r from the axis.

Consider an infinitesimally thin cylindrical shell of radius r and thickness dr in the dielectric and concentrically around the inner conductor. This surface is chosen as the Gauss surface. The voltage across the dielectric thickness dr is dV. The field $\mathcal{E} = -dV/dr$.

d. What typical voltage will initiate a partial discharge in a small air pore (perhaps formed during mechanical and thermal stressing) at the inner conductor–insulator interface? Assume that the breakdown field for air at 1 atm and gap spacing around 0.1 mm is about 100 kV cm^{-1}.

SOLUTION

Consider a cylindrical shell of thickness dr of the dielectric as shown in Figure 7.29. Suppose that the voltage across the shell thickness is dV. Then the field $\mathcal{E}$ at r is $-dV/dr$ (this is the definition of $\mathcal{E}$). Suppose that Q_{free} is the free charge on the inner conductor. We take a Gauss surface that is a cylinder of radius r and concentric with the inner conductor as depicted in Figure 7.29. The surface area A of this cylinder is $2\pi rL$ where L is the length of the cable. The field at the surface, at distance r, is $\mathcal{E}$, which is normal to A and coming out of A. Then from Equation 7.43

$$\mathcal{E}(2\pi rL) = \frac{Q_{\text{free}}}{\varepsilon_o\varepsilon_r} \qquad [7.44]$$

Thus

$$-\frac{dV}{dr} = \frac{Q_{\text{free}}}{\varepsilon_o\varepsilon_r 2\pi rL}$$

This can be integrated from $r = a$, where the voltage is V, to b, where $V = 0$. Then

$$V = \frac{Q_{\text{free}}}{\varepsilon_o\varepsilon_r 2\pi L}\ln\left(\frac{b}{a}\right) \qquad [7.45]$$

We can obtain the capacitance of the coaxial cable from $C_{\text{coax}} = Q_{\text{free}}/V$, which is

Capacitance of a coaxial cable

$$C_{\text{coax}} = \frac{\varepsilon_o\varepsilon_r 2\pi L}{\ln\left(\dfrac{b}{a}\right)} \qquad [7.46]$$

The capacitance per unit length can be calculated using $a = 2.5$ mm and

$$b = a + \text{Thickness} = 7.5 \text{ mm}$$

and the appropriate dielectric constants, $\varepsilon_r = 2.3$ for XLPE and 3.7 for silicone rubber. The values are around 100–200 pF per meter, as listed in the fourth column in Table 7.6.

Table 7.6 Dielectric insulation candidates for a coaxial cable

Dielectric	ε_r (60 Hz)	Strength (60 Hz) (kV cm^{-1})	C (60 Hz) (pF m^{-1})	Breakdown Voltage (kV)	Voltage for Partial Discharge in a Microvoid (kV)
XLPE	2.3	217	116	59.6	11.9
Silicone rubber	3.7	158	187	43.4	7.4

The electric field $\mathcal{E}$ follows directly when we substitute for Q_{free} from Equation 7.45 into Equation 7.44.

Field in a coaxial cable

$$\mathcal{E} = \frac{V}{r \ln\left(\dfrac{b}{a}\right)} \tag{7.47}$$

Equation 7.47 is valid for r from a to b (there is no field within the conductors). The field is maximum where $r = a$,

Maximum field in a coaxial cable

$$\mathcal{E}_{\max} = \frac{V}{a \ln\left(\dfrac{b}{a}\right)} \tag{7.48}$$

The breakdown voltage V_{br} is reached when this maximum field $\mathcal{E}_{\max}$ reaches the dielectric strength or the breakdown field $\mathcal{E}_{\text{br}}$

Breakdown voltage

$$V_{\text{br}} = \mathcal{E}_{\text{br}} a \ln\left(\frac{b}{a}\right) \tag{7.49}$$

The breakdown voltages calculated from Equation 7.49 are listed in the fifth column in Table 7.6. Although the values are high, it must be remembered that, due to a number of other factors such as insulation aging, one cannot expect the cable to withstand these voltages forever.

If there is an air cavity or bubble at the inner conductor to dielectric surface, then the field in this gaseous space will be $\mathcal{E}_{\text{air}} \approx \varepsilon_r \mathcal{E}_{\max}$, where $\mathcal{E}_{\max}$ is the field at $r = a$. Air breakdown occurs when

$$\mathcal{E}_{\text{air}} = \mathcal{E}_{\text{air-br}} = 100 \text{ kV cm}^{-1}$$

at 1 atm and 25 °C for a 0.1 mm gap. Then $\mathcal{E}_{\max} \approx \mathcal{E}_{\text{air-br}}/\varepsilon_r$. The corresponding voltage from Equation 7.48 is

$$V_{\text{air-br}} \approx \frac{\mathcal{E}_{\text{air-br}}}{\varepsilon_r} a \ln\left(\frac{b}{a}\right)$$

The voltages for partial discharges for the two coaxial cables are shown in the sixth column of Table 7.6. It should be noted that these voltages will only give partial discharges contained within microvoids and will not normally lead to the immediate breakdown of the insulation. The partial discharges erode the cavities and also release vapor from the polymer that accumulates in the cavities. Thus, gaseous content and pressure in a cavity will change as the partial discharge continues. For example, the pressure buildup will increase the breakdown field and elevate the voltage for partial breakdown. Eventual degradation is likely to lead to electrical treeing.

We should also note that the actual field in the air cavity depends on the shape of the cavity, and the above treatment is only valid for a thin disk-like cavity lying perpendicular to the field (see Section 7.9, Additional Topics).

7.7 CAPACITOR DIELECTRIC MATERIALS

7.7.1 TYPICAL CAPACITOR CONSTRUCTIONS

The selection criteria of dielectric materials for capacitors depend on the capacitance value, frequency of application, maximum tolerable loss, and maximum working voltage, with size and cost being additional external constraints. Requirements for high-voltage power capacitors are distinctly different than those used in small integrated circuits. Large capacitance values are more easily obtained at low frequencies because low-frequency polarization mechanisms such as interfacial and dipolar polarization make a substantial contribution to the dielectric constant. At high frequencies, it becomes more difficult to achieve large capacitances and at the same time maintain acceptable low dielectric loss, inasmuch as the dielectric loss per unit volume is $\varepsilon_o\varepsilon_r'\omega\mathcal{E}^2\tan\delta$.

The bar-chart diagrams in Figures 7.30 and 7.31 provide some typical examples of dielectrics for a range of capacitance values and for a range of usable frequencies. For example, electrolytic dielectrics characteristically provide capacitances between one to thousands of microfarads, but their frequency response is typically limited to below

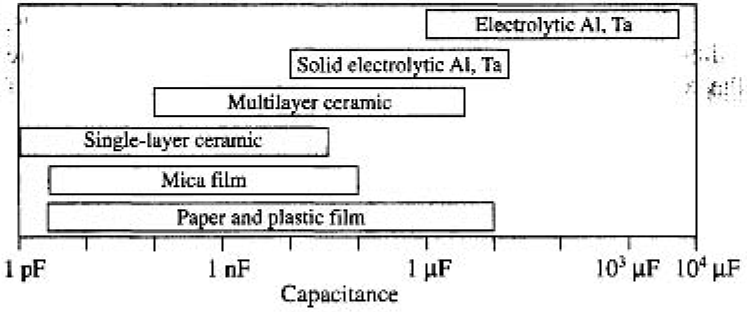


Figure 7.30 Examples of dielectrics that can be used for various capacitance values.

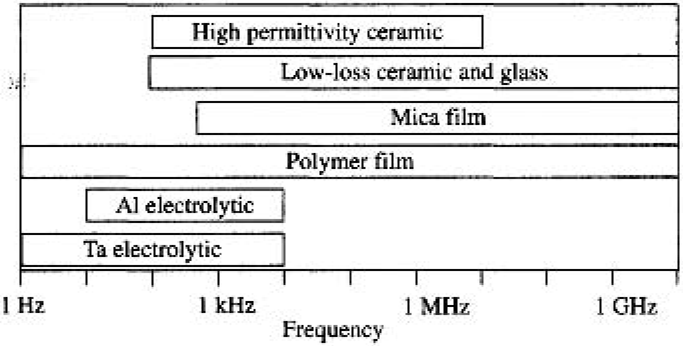


Figure 7.31 Examples of dielectrics that can be used in various frequency ranges.

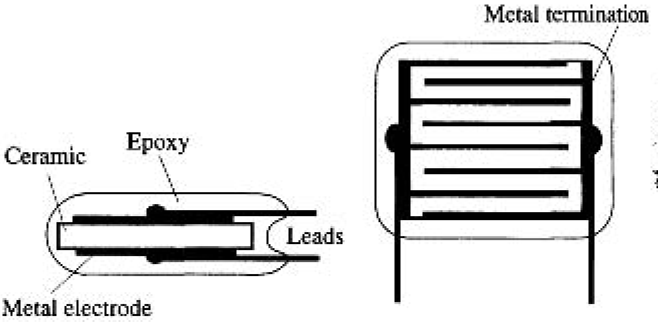


Figure 7.32 Single- and multilayer dielectric capacitors.

(a) Single-layer ceramic capacitor (e.g., disk capacitors)

(b) Multilayer ceramic capacitor (stacked ceramic layers)

10 kHz. On the other hand, polymeric film capacitors typically have values less than 10 μF but a frequency response that is flat well into the gigahertz range.

We can understand the principles utilized in capacitor design from the capacitance of a parallel plate capacitor,

$$C = \frac{\varepsilon_o \varepsilon_r A}{d} \qquad [7.50]$$

where ε_r infers ε_r'. Large capacitances can be achieved by using high ε_r dielectrics, thin dielectrics, and large areas. There are various commercial ceramics, usually a mixture of various oxides or ferroelectric ceramics, that have high dielectric constants, ranging up to several thousands. These are typically called high-K (or high-κ), where K (or κ) stands for the relative permittivity. A ceramic dielectric with $\varepsilon_r = 10$, d of perhaps 10 μm, and an area of 1 cm^2 has a capacitance of 885 pF. Figure 7.32a shows a typical single-layer ceramic capacitor. The thin ceramic disk or plate has suitable metal electrodes, and the whole structure has been encapsulated in an epoxy by dipping it in a thermosetting resin. The epoxy coating prevents moisture from degrading the dielectric properties of the ceramic (increasing ε_r'' and the loss, $\tan\delta$). One way to increase the capacitance is to connect N number of these in parallel, and this is done in a space-efficient way by using the multilayer ceramic structure shown in Figure 7.32b. In this case there are N electroded dielectric layers. Each ceramic has offset metal electrodes that align with the opposite sides of the plate and make contact with the metal terminations on these sides. The result is N number of parallel plate capacitors. There is therefore an effective use of volume as the surface area of the component stays the same but the height increases to at least Nd. By using multilayer ceramic structures, capacitances up to a few hundred microfarads have been recently obtained.

Many wide-frequency-range capacitors utilize **polymeric thin films** for two reasons. Although ε_r is typically 2 to 3 (less than those for many ceramics), it is constant over a wide frequency range. The dielectric loss $\varepsilon_o \varepsilon_r \omega \mathcal{E}^2 \tan\delta$ becomes significant at high frequencies and polymers have low $\tan\delta$ values. Low ε_r values mean that one has to find a space-efficient way of constructing polymer film capacitors. One method is shown in Figure 7.33a and b for constructing a metallized film polymer capacitor. Two polymeric

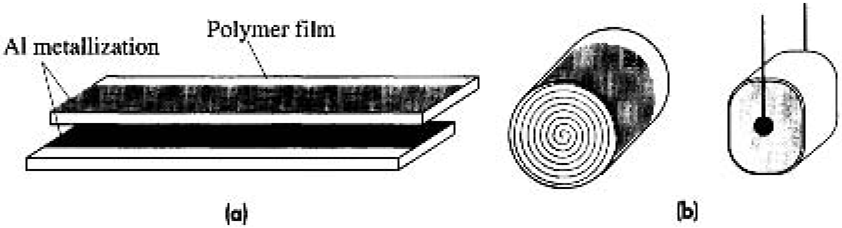


Figure 7.33 Two polymer tapes in (a), each with a metallized film electrode on the surface (offset from each other), can be rolled together (like a Swiss roll) to obtain a polymer film capacitor as in (b).

As the two separate metal films are lined at opposite edges, electroding is done over the whole side surface.

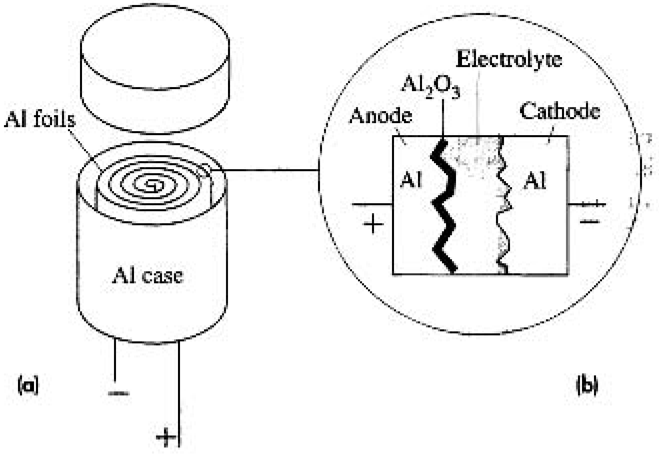


Figure 7.34 Aluminum electrolytic capacitor.

tapes have metallized electrodes (typically vacuum deposited Al) on one surface, leaving a margin on one side. These metal film electrodes have been offset in opposite directions so that they line up with the opposite sides of the tapes. The two tapes together are rolled up (like a Swiss-roll cake) and the opposite sides are electroded using suitable conducting glues or other means. The concept is therefore similar to the multilayer ceramic capacitor except that the layers are rolled up to form a circular cross section. It is also possible to cut and stack the layers as in the multilayer ceramic construction.

Electrolytic capacitors provide large values of capacitance while maintaining a tolerable size. There are various types of electrolytic capacitors. In aluminum electrolytic capacitors, the metal electrodes are two Al foils, typically 50–100 μm thick, that are separated by a porous paper medium soaked with a liquid electrolyte. The two foils together are wound into a cylindrical form and held within a cylindrical case, as shown in Figure 7.34a. Contrary to intuition, the paper-soaked electrolyte is not the dielectric. The dielectric medium is the thin alumina Al_2O_3 layer grown on the roughened surface of one of the foils, as shown in Figure 7.34b. This foil is then called the anode (+ terminal). Both Al foils are etched to obtain rough surfaces, which increases the surface area

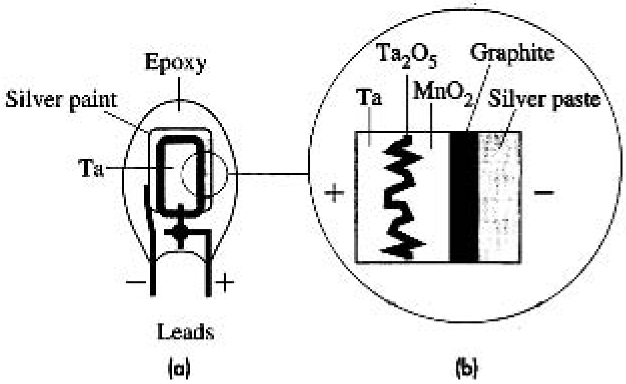


Figure 7.35 Solid electrolyte tantalum capacitor.
(a) A cross section without fine detail.
(b) An enlarged section through the Ta capacitor.

compared with smooth surfaces. The capacitor is called electrolytic because the Al_2O_3 layer is grown electrolytically on one of the foils and is typically 0.1 μm in thickness. This small thickness and the large surface area are responsible for the large capacitance. The electrolyte is conducting and serves to heal local minor breakdowns in the Al_2O_2 by an electrolytic reaction, provided that the anode has been positively biased. The capacitive behavior is due to the Al/(Al_2O_3)/electrolyte structure. Furthermore, Al/Al_2O_3 contact is like a metal to *p*-type semiconductor contact and has rectifying properties. It must be reverse-biased to prevent charge injection into the Al_2O_3 and hence conduction through the capacitor. Thus the Al must be connected to the positive terminal, which makes it the anode. When the electrolytic Al capacitor in Figure 7.34b is oppositely biased, it becomes conducting.

Electrolytic capacitors using liquid electrolytes tend to dry up over a long period, which is a disadvantage. **Solid electrolyte tantalum capacitors** overcome the drying-up problem by using a solid electrolyte. The structure of a typical solid Ta capacitor is shown in Figure 7.35a and b. The anode (+ electrode) is a porous (sintered) Ta pellet that has the surface anodized to obtain a thin surface layer of tantalum pentoxide, Ta_2O_5, which is the dielectric medium (with $\varepsilon_r' = 28$). The Ta pellet with Ta_2O_5 is then coated with a thick solid electrolyte, in this case MnO_2. Subsequently, graphite and silver paste layers are applied. Leads are then attached and the whole construction is molded into a resin chip. Solid tantalum capacitors are widely used in numerous electronics applications due to their small size, temperature and time stability, and high reliability.

7.7.2 DIELECTRICS: COMPARISON

The **capacitance per unit volume** C_{vol}, which characterizes the **volume efficiency** of a dielectric, can be obtained by dividing C by Ad,

Capacitance per unit volume

$$C_{\text{vol}} = \frac{\varepsilon_o \varepsilon_r}{d^2} \qquad [7.51]$$

It is clear that large capacitances require high dielectric constants and thin dielectrics. We should note that d appears as d^2, so the importance of d cannot be understated.

Table 7.7 Comparison of dielectrics for capacitor applications

	Capacitor Name					
	Polypropylene	**Polyester**	**Mica**	**Aluminum, Electrolytic**	**Tantalum, Electrolytic, Solid**	**High-*K* Ceramic**
Dielectric	Polymer film	Polymer film	Mica	Anodized Al_2O_3 film	Anodized Ta_2O_5 film	X7R $BaTiO_3$ base
ε_r'	2.2–2.3	3.2–3.3	6.9	8.5	27	2000
$\tan\delta$	4×10^{-4}	4×10^{-3}	2×10^{-4}	0.05–0.1	0.01	0.01
$\mathcal{E}_{br}$ (kV mm^{-1}) dc	100–350	100–300	50–300	400–1000	300–600	10
d (typical minimum) (μm)	3–4	1	2–3	0.1	0.1	10
C_{vol} (μF cm^{-3})	2	30	15	7500*	24,000*	180
$R_p = 1/G_p$(kΩ) for $C = 1$ μF, $f = 1$ kHz	400	40	800	1.5–3	16	16
E_{vol} (mJ cm^{-3})†	10	15	8	1000	1200	100
Polarization	Electronic	Electronic and dipolar	Ionic	Ionic	Ionic	Large ionic displacement

* Proper volumetric calculations must also consider the volumes of electrodes and the electrolyte necessary for these dielectrics to work; hence the number would have to be decreased.

† E_{vol} depends very sensitively on $\mathcal{E}_{br}$ and the choice of η; hence it can vary substantially.

NOTES: Values are typical. Assume $\eta = 3$. The table is for comparison purposes only. Breakdown fields are typical dc values and can vary substantially, by at least an order of magnitude; $\mathcal{E}_{br}$ depends on the thickness, material quality, and the duration of the applied voltage. Polyester is PET, or polyethylene terephthalate. Mica is potassium aluminosilicate, a muscovite crystal. X7R is the name of a particular $BaTiO_3$-based ceramic solid solution.

Although mica has a higher ε_r than polymer films, the latter can be made quite thin, a few microns, which leads to a greater capacitance per unit volume. The reason that electrolytic aluminum capacitors can achieve large capacitance per unit volume is that d can be made very thin over a large surface area by using the liquid electrolyte to heal minor local dielectric breakdowns. Table 7.7 shows a selection of dielectric materials for capacitor applications and compares the "volume efficiency" C_{vol} based on a typical minimum thickness that a convenient process can handle. It is apparent that, compared with polymeric films, ceramics have substantial volume efficiency as a result of large dielectric constants (high-*K* ceramics) in some cases and as a consequence of a thin dielectric thickness in other cases (Al_2O_3).

Another engineering consideration in selecting a dielectric is the working voltage. Although d can be decreased to obtain large capacitances per unit volume, this also decreases the working voltage. The maximum voltage that can be applied to a capacitor depends on the breakdown field of the dielectric medium $\mathcal{E}_{br}$, which itself is a highly variable quantity. A safe working voltage must be some safety factor η less than the breakdown voltage $\mathcal{E}_{br}d$. Thus, if V_m is the maximum safe working voltage, then the maximum energy that can be stored per unit volume is given by

Maximum energy per unit volume

$$E_{vol} = \frac{1}{2}CV_m^2 \times \frac{1}{Ad} = \frac{\varepsilon_o\varepsilon_r'}{2\eta^2}\mathcal{E}_{br}^2 \qquad [7.52]$$

It is clear that both ε_r' and $\mathcal{E}_{br}$ of the dielectric are significant in determining the energy storage ability of the capacitor. Moreover, at the maximum working voltage, the rate of dielectric loss per unit volume in the capacitor becomes

Dielectric loss per unit volume

$$W_{vol} = \frac{\mathcal{E}_{br}^2}{\eta^2}\omega\varepsilon_o\varepsilon_r' \tan\delta \qquad [7.53]$$

Those materials that have relatively higher tan δ exhibit greater dielectric losses. Although dielectric losses may be small at low frequencies, at high frequencies they become quite significant. Table 7.7 compares the energy storage efficiency E_{vol} and tan δ for various dielectrics. It seems that ceramics have a better energy storage efficiency than polymers. High-K ceramics tend to have large tan δ values and suffer from greater dielectric loss. Polystyrene and polypropylene have particularly low tan δ as the polarization mechanism is due to electronic polarization and the dielectric losses are the least. Indeed, polystyrene and polypropylene capacitors have found applications in high-quality audio electronics. Equations 7.52 and 7.53 should be used with care, because the breakdown field $\mathcal{E}_{br}$ can depend on the thickness d, among many other factors, including the quality of the dielectric material. For example, for polypropylene insulation, $\mathcal{E}_{br}$ is typically quoted as roughly 50 kV mm^{-1} (500 kV cm^{-1}), whereas for thin films (*e.g.*, 25 μm), over short durations, $\mathcal{E}_{br}$ can be as high as 200 kV mm^{-1}. Further, in some cases, $\mathcal{E}_{br}$ is more suitably defined in terms of the maximum allowable leakage current, that is, a field at which the dielectric is sufficiently conducting.

The temperature stability of a capacitor is determined by the temperature dependences of ε_r' and tan δ, which are controlled by the dominant polarization mechanism. For example, polar polymers have permanent dipole groups attached to the polymer chains as in polyethyleneterephthalate (PET). In the absence of an applied field, these dipoles are randomly oriented and also restricted in their rotations by neighboring chains, as depicted in Figure 7.36a. In the presence of an applied dc field, as in Figure 7.36b, some very limited rotation enables partial dipolar (orientational) polarization to take place. Typically, at room temperature, dipolar contribution to ε_r under ac conditions, however, is small because restricted and hindered rotation prevents the dipoles to closely follow the ac field. Close to the softening temperature of the polymer, the molecular motions become easier and, further, there is more volume between chains for the dipoles to rotate. The dipolar side groups and polarized chains become capable of responding to the field. They can align with the field and also follow the field variations, as shown in Figure 7.36c. Dipolar contribution to ε_r is substantial even at high frequencies. Both ε_r' and tan δ therefore increase with temperature. Thus, polar polymers exhibit temperature dependent ε_r and tan δ, which reflect in the properties of the capacitor.

On the other hand, in nonpolar polymers such as polystyrene and polypropylene, the polarization is due to electronic polarization and ε_r and tan δ remain relatively constant. Thus polystyrene and polypropylene capacitors are more stable compared with PET (polyester) capacitors. The change in the capacitance with temperature is measured by the **temperature coefficient of capacitance** (TCC), which is defined as the fractional (or percentage) change in the capacitance per unit temperature change. The temperature controls not only ε_r but also the linear expansion of the dielectric,

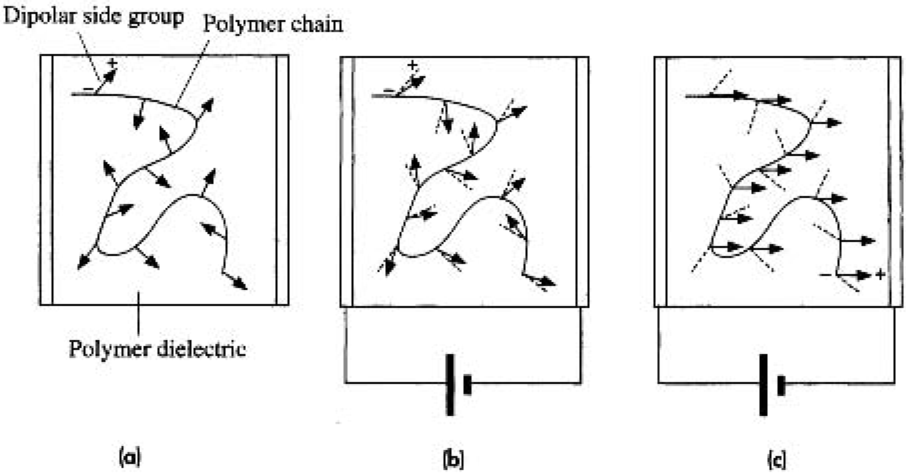


Figure 7.36

(a) A polymer dielectric that has dipolar side groups attached to the polymer chains. With no applied field, the dipoles are randomly oriented.

(b) In the presence of an applied field, some very limited rotation enables dipolar polarization to take place.

(c) Near the softening temperature of the polymer, the molecular motions are rapid and there is also sufficient volume between chains for the dipoles to align with the field. The dipolar contribution to ε_r is substantial, even at high frequencies.

which changes the dimensions A and d. For example, polystyrene, polycarbonate, and mica capacitors are particularly stable with small TCC values. Plastic capacitors are typically limited to operations well below their melting temperatures, which is one of their main drawbacks. The specified operating temperature, for example, from −55 °C to 125 °C, for many of the ceramic capacitors is often a limitation of the epoxy coating of the capacitor rather than the actual limitation of the ceramic material. In many capacitors, the working voltage has to be derated for operation at high temperatures and high frequencies because $\mathcal{E}_{br}$ decreases with ambient temperature and the frequency of the applied field. For example, a 1000 V dc polypropylene capacitor will have a substantially lower ac working voltage, *e.g.*, 100 V at 10 kHz.

EXAMPLE 7.12

DIELECTRIC LOSS AND EQUIVALENT CIRCUIT OF A POLYESTER CAPACITOR AT 1 kHz Figure 7.37 shows the temperature dependence of ε_r' and $\tan\delta$ for a polyester film. Calculate the equivalent circuit at 25 °C at 1 kHz for a 560 pF PET capacitor that uses a 0.5 micron thick polyester film. What happens to these values at 100 °C?

SOLUTION

From Figure 7.37 at 25 °C, $\varepsilon_r' = 2.60$ and $\tan\delta \approx 0.002$. The capacitance C at 25 °C is given as 560 pF. The equivalent parallel conductance G_P, representing the dielectric loss, is given by

$$G_P = \frac{\omega A \varepsilon_o \varepsilon_r' \tan\delta}{d} = \omega C \tan\delta$$

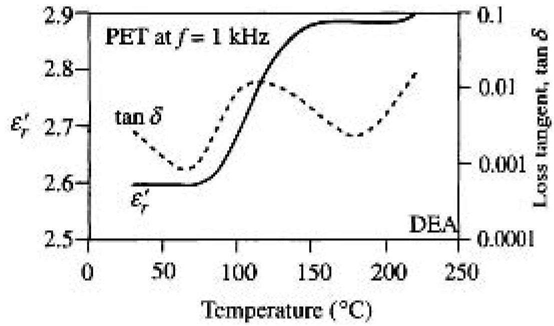


Figure 7.37 Real part of the dielectric constant ε_r' and loss tangent, tan δ, at 1 kHz versus temperature for PET.

SOURCE: Data obtained by Kasap and Maeda [1995] using a dielectric analyzer (DEA).

Substituting

$$\omega = 2\pi f = 2000\pi$$

and $\tan\delta = 0.002$, we get

$$G_P = (2000\pi)(560 \times 10^{-12})(0.002) = 7.04 \times 10^{-9}\frac{1}{\Omega}$$

This is equivalent to a resistance of 142 MΩ. The equivalent circuit is an ideal (lossless) capacitor of 560 pF in parallel with a 142 MΩ resistance (this resistance value decreases with the frequency).

At 100 °C, $\varepsilon_r' = 2.69$ and $\tan\delta \approx 0.01$, so the new capacitance is

$$C_{100\,°\mathrm{C}} = C_{25\,°\mathrm{C}}\frac{\varepsilon_r(100\,°\mathrm{C})}{\varepsilon_r(25\,°\mathrm{C})} = (560\text{ pF})\frac{2.69}{2.60} = 579\text{ pF}$$

The equivalent parallel conductance at 100 °C is

$$G_P = (2000\pi)(579 \times 10^{-12})(0.01) = 3.64 \times 10^{-8}\frac{1}{\Omega}$$

This is equivalent to a resistance of 27.5 MΩ. The equivalent circuit is an ideal (lossless) capacitor of 579 pF in parallel with a 27.5 MΩ resistance.

7.8 PIEZOELECTRICITY, FERROELECTRICITY, AND PYROELECTRICITY

7.8.1 PIEZOELECTRICITY

Certain crystals, for example, quartz (crystalline SiO_2) and $BaTiO_3$, become polarized when they are mechanically stressed. Charges appear on the surfaces of the crystal, as depicted in Figure 7.38a and b. Appearance of surface charges leads to a voltage difference between the two surfaces of the crystal. The same crystals also exhibit mechanical strain or distortion when they experience an electric field, as shown in Figure 7.38c and d. The direction of mechanical deformation (*e.g.*, extension or compression) depends on

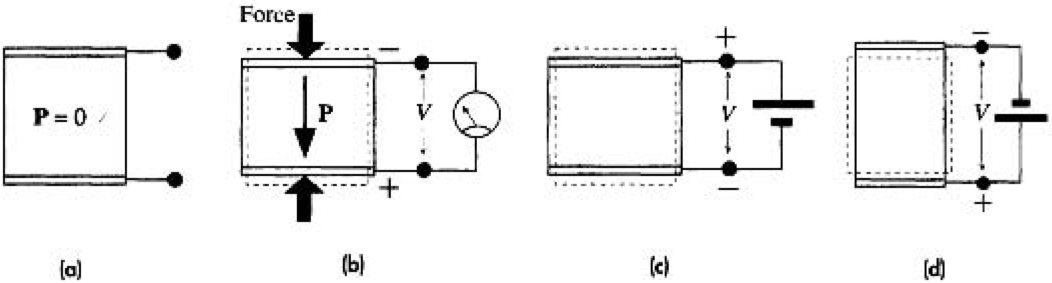


Figure 7.38 The piezoelectric effect.

(a) A piezoelectric crystal with no applied stress or field.

(b) The crystal is strained by an applied force that induces polarization in the crystal and generates surface charges.

(c) An applied field causes the crystal to become strained. In this case the field compresses the crystal.

(d) The strain changes direction with the applied field and now the crystal is extended.

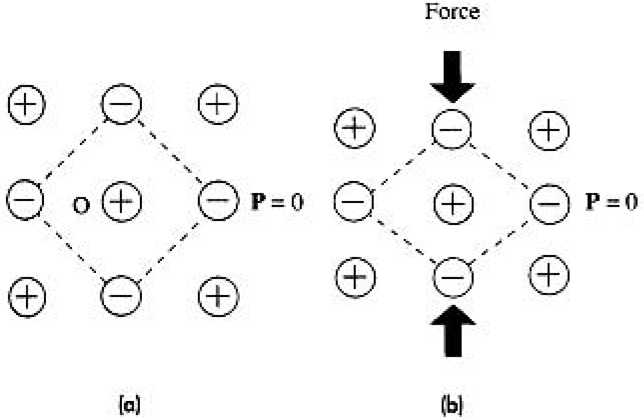


Figure 7.39 A NaCl-type cubic unit cell has a center of symmetry.

(a) In the absence of an applied force, the centers of mass for positive and negative ions coincide.

(b) This situation does not change when the crystal is strained by an applied force.

the direction of the applied field, or the polarity of the applied voltage. The two effects are complementary and define **piezoelectricity.**

Only certain crystals can exhibit piezoelectricity because the phenomenon requires a special crystal structure—that which has no center of symmetry. Consider a NaCl-type cubic unit cell in Figure 7.39a. We can describe the whole crystal behavior by examining the properties of the unit cell. This unit cell has a **center of symmetry** at O because if we draw a vector from O to any charge and then draw the reverse vector, we will find the same type of charge. Indeed, any point on any charge is a center of symmetry. Many similar cubic crystals (not all) possess a center of symmetry. When unstressed, the center of mass of the negative charges at the corners of the unit cell coincides with the positive charge at the center, as shown in Figure 7.39a. There is therefore no net polarization in the unit cell and **P** = 0. Under stress, the unit cell becomes strained, as shown in Figure 7.39b, but the center of mass of the negative charges still coincides with the positive charge and the net polarization is still zero. Thus, the strained crystal

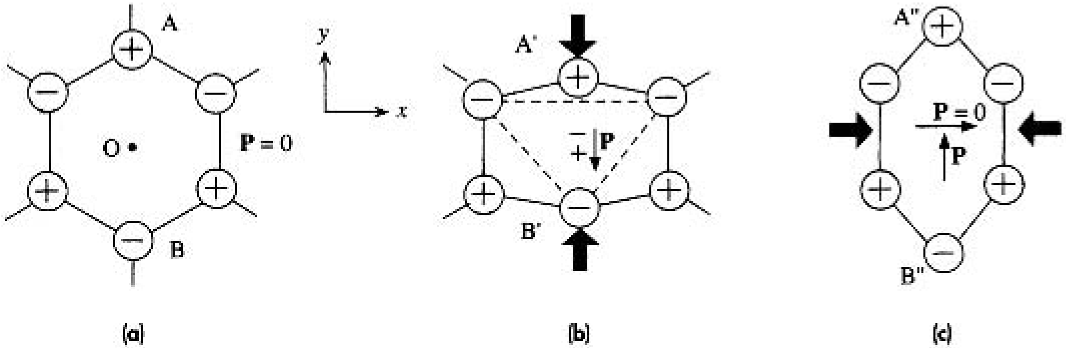


Figure 7.40 A hexagonal unit cell has no center of symmetry.

(a) In the absence of an applied force, the centers of mass for positive and negative ions coincide.

(b) Under an applied force in the y direction, the centers of mass for positive and negative ions are shifted, which results in a net dipole moment, **P**, along y.

(c) When the force is along a different direction, along x, there may not be a resulting net dipole moment in that direction though there may be a net **P** along a different direction (y).

still has $\mathbf{P} = 0$. This result is generally true for all crystals that have a center of symmetry. The centers of mass of negative and positive charges in the unit cell remain coincident when the crystal is strained.

Piezoelectric crystals have no center of symmetry. For example, the hexagonal unit cell shown in Figure 7.40a exhibits no center of symmetry. If we draw a vector from point O to any charge and then reverse the vector, we will find an opposite charge. The unit cell is said to be **noncentrosymmetric.** When unstressed, as shown in Figure 7.40a, the center of mass of the negative charges coincides with the center of mass of the positive charges, both at O. However, when the unit cell is stressed, as shown in Figure 7.40b, the positive charge at A and the negative charge at B both become displaced inwards to A′ and B′, respectively. The two centers of mass therefore become shifted and there is now a net polarization **P**. Thus, an applied stress produces a net polarization **P** in the unit cell, and in this case **P** appears to be in the same direction as the applied stress, along y.

The direction of the induced polarization depends on the direction of the applied stress. When the same unit cell in Figure 7.40a is stressed along x, as illustrated in Figure 7.40c, there is no induced dipole moment along this direction because there is no net displacement of the centers of mass in the x direction. However, the stress causes the atoms A and B to be displaced outwards to A″ and B″, respectively, and this results in the shift of the centers of mass away from each other along y. In this case, an applied stress along x results in an induced polarization along y. Generally, an applied stress in one direction can give rise to induced polarization in other crystal directions. Suppose that T_j is the applied mechanical stress along some j direction and P_i is the induced polarization along some i direction; then the two are linearly related by

Piezoelectric effect

$$P_i = d_{ij} T_j \tag{7.54}$$

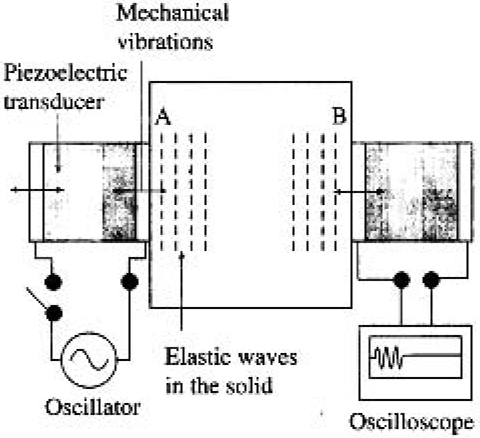


Figure 7.41 Piezoelectric transducers are widely used to generate ultrasonic waves in solids and also to detect such mechanical waves. The transducer on the left is excited from an ac source and vibrates mechanically. These vibrations are coupled to the solid and generate elastic waves. When the waves reach the other end, they mechanically vibrate the transducer on the right, which converts the vibrations to an electrical signal.

where d_{ij} are called **the piezoelectric coefficients.** Reversing the stress reverses the polarization. Although we did not specifically consider shear stresses in Figure 7.40, they, as well as tensile stresses, can also induce a net polarization, which means that T in Equation 7.54 can also represent shear stresses. The converse piezoelectric effect is that between an induced strain S_j along j and an applied electric field $\mathcal{E}_i$ along i,

Converse piezoelectric effect

$$S_j = d_{ij}\mathcal{E}_i \qquad [7.55]$$

The coefficients d_{ij} in Equations 7.54 and 7.55 are the same.[12]

As apparent from the foregoing discussions and Figure 7.38, piezoelectric crystals are essentially electromechanical transducers because they convert an electrical signal, an electric field, to a mechanical signal, strain, and vice versa. They are used in many engineering applications that involve electromechanical conversions, as in ultrasonic transducers, microphones, accelerometers, and so forth. Piezoelectric transducers are widely used to generate ultrasonic waves in solids and also to detect such mechanical waves, as illustrated in Figure 7.41. The transducer is simply a piezoelectric crystal, for example, quartz, that is appropriately cut and electroded to generate the desired types of mechanical vibrations (*e.g.*, longitudinal or transverse vibrations). The transducer on the left is attached to the surface A of the solid under examination, as shown in Figure 7.41. It is excited from an ac source, which means that it mechanically vibrates. These vibrations are coupled to the solid by a proper coupling medium (typically grease) and generate mechanical waves or elastic waves that propagate away from A. They are called **ultrasonic waves** as their frequencies are typically above the audible range. When the waves reach the other end, B, they mechanically vibrate the transducer attached to B, which converts the vibrations to an electrical signal that can readily be displayed on an oscilloscope. In this trivial example, one can easily measure the time it takes for elastic waves to travel in the solid from A to B and hence determine the ultrasonic velocity of the waves since the distance AB is

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[12] The equivalence of the coefficients in Equations 7.54 and 7.55 can be shown by using thermodynamics and is not considered in this textbook. For rigorous piezoelectric definitions see IEEE Standard 176-1987 (*IEEE Trans. on Ultrasonics, Ferroelectrics and Frequency Control*, September 1996).

known. From the ultrasonic velocity one can determine the elastic constants (Young's modulus) of the solid. Furthermore, if there are internal imperfections such as cracks in the solid, then they reflect or scatter the ultrasonic waves. These reflections can lead to echoes that can be detected by suitably located transducers. Such ultrasonic testing methods are widely used for nondestructive evaluations of solids in mechanical engineering.

It is clear that an important engineering factor in the use of piezoelectric transducers is the electromechanical coupling between electrical and mechanical energies. The **electromechanical coupling factor** k is defined in terms of k^2 by

Electromechanical coupling factor

$$k^2 = \frac{\text{Electrical energy converted to mechanical energy}}{\text{Input of electrical energy}} \qquad [7.56a]$$

or equivalently by

Electromechanical coupling factor

$$k^2 = \frac{\text{Mechanical energy converted to electrical energy}}{\text{Input of mechanical energy}} \qquad [7.56b]$$

Table 7.8 summarizes some typical piezoelectric materials with some applications. The so-called PZT ceramics are widely used in many piezoelectric applications. PZT stands for lead zirconate titanate and the ceramic is a solid solution of lead zirconate, $PbZrO_3$, and lead titanate, $PbTiO_3$, so its composition is $PbTi_{1-x}Zr_xO_3$ where x is determined by the extent of the solid solution but typically is around 0.5. PZT piezoelectric components are manufactured by sintering, which is a characteristic ceramic manufacturing process in which PZT powders are placed in a mold and subjected to a pressure at high temperatures. During sintering the ceramic powders are fused through interdiffusion. The final properties depend not only on the composition of the solid solution but also on the manufacturing process, which controls the average grain size or polycrystallinity. Electrodes are deposited onto the final ceramic component, which is then poled by the application of a temporary electric field to induce it to become

Table 7.8 Piezoelectric materials and some typical values for d and k

Crystal	d (m V^{-1})	k	Comment
Quartz (crystal SiO_2)	2.3×10^{-12}	0.1	Crystal oscillators, ultrasonic transducers, delay lines, filters
Rochelle salt ($NaKC_4H_4O_6 \cdot 4H_2O$)	350×10^{-12}	0.78	
Barium titanate ($BaTiO_3$)	190×10^{-12}	0.49	Accelerometers
PZT, lead zirconate titanate ($PbTi_{1-x}Zr_xO_3$)	480×10^{-12}	0.72	Wide range of applications including earphones, microphones, spark generators (gas lighters, car ignition), displacement transducers, accelerometers
Polyvinylidene fluoride (PVDF)	18×10^{-12}	—	Must be poled; heated, put in an electric field and then cooled. Large area and inexpensive

piezoelectric. **Poling** refers to the application of a temporary electric field, generally at an elevated temperature, to align the polarizations of various grains and thereby develop piezoelectric behavior.

EXAMPLE 7.13

PIEZOELECTRIC SPARK GENERATOR The piezoelectric spark generator, as used in various applications such as lighters and car ignitions, operates by stressing a piezoelectric crystal to generate a high voltage which is discharged through a spark gap in air as schematically shown in Figure 7.42a. Consider a piezoelectric sample in the form of a cylinder as in Figure 7.42a. Suppose that the piezoelectric coefficient $d = 250 \times 10^{-12}$ m V^{-1} and $\varepsilon_r = 1000$. The piezoelectric cylinder has a length of 10 mm and a diameter of 3 mm. The spark gap is in air and has a breakdown voltage of about 3.5 kV. What is the force required to spark the gap? Is this a realistic force?

SOLUTION

We need to express the induced voltage in terms of the applied force. If the applied stress is T, then the induced polarization P is

$$P = dT = d\frac{F}{A}$$

Induced polarization P leads to induced surface polarization charges given by $Q = AP$. If C is the capacitance, then the induced voltage is

$$V = \frac{Q}{C} = \frac{AP}{\left(\dfrac{\varepsilon_o\varepsilon_r A}{L}\right)} = \frac{LP}{\varepsilon_o\varepsilon_r} = \frac{L\left(d\dfrac{F}{A}\right)}{\varepsilon_o\varepsilon_r} = \frac{dLF}{\varepsilon_o\varepsilon_r A}$$

Therefore, the required force is

$$F = \frac{\varepsilon_o\varepsilon_r AV}{dL} = \frac{(8.85 \times 10^{-12} \times 1000)\pi(1.5 \times 10^{-3})^2(3500)}{(250 \times 10^{-12})(10 \times 10^{-3})} = 87.6 \text{ N}$$

This force can be applied by squeezing by hand an appropriate lever arrangement; it is the weight of 9 kg. The force must be applied quickly because the piezoelectric charge generated

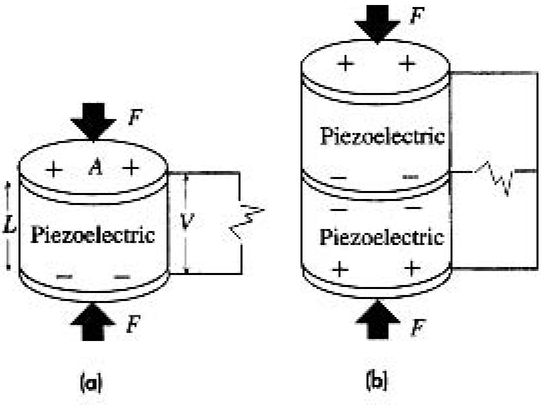

Figure 7.42 The piezoelectric spark generator.

will leak away (or become neutralized) if the charge is generated too slowly; many spark igniters use mechanical impact. The *energy* in the spark depends on the amount of charge generated. This can increase by using two piezoelectric crystals back to back as in Figure 7.42b, which is a more practical arrangement for a spark generator. The induced voltage per unit force V/F is proportional to $d/(\varepsilon_o\varepsilon_r)$ which is called the **piezoelectric voltage coefficient.** In general, if an applied stress $T = F/A$ induces a field $\mathcal{E} = V/L$ in a piezoelectric crystal, then the effect is related to the cause by the piezoelectric voltage coefficient g,

Piezoelectric voltage coefficient

$$\mathcal{E} = gT \qquad [7.57]$$

It is left as an exercise to show that $g = d/(\varepsilon_o\varepsilon_r)$.

7.8.2 PIEZOELECTRICITY: QUARTZ OSCILLATORS AND FILTERS

One of the most important applications of the piezoelectric quartz crystal in electronics is in the frequency control of oscillators and filters. Consider a suitably cut thin plate of a quartz crystal that has thin gold electrodes on the opposite faces. Suppose that we set up mechanical vibrations in the crystal by connecting the electrodes to an ac source, as in Figure 7.43a. It is possible to set up a mechanical resonance, or mechanical standing waves, in the crystal if the wavelength λ of the waves and the length ℓ along which the waves are traveling satisfy the condition for standing waves:

Mechanical standing waves

$$n\left(\frac{1}{2}\lambda\right) = \ell \qquad [7.58]$$

where n is an integer.

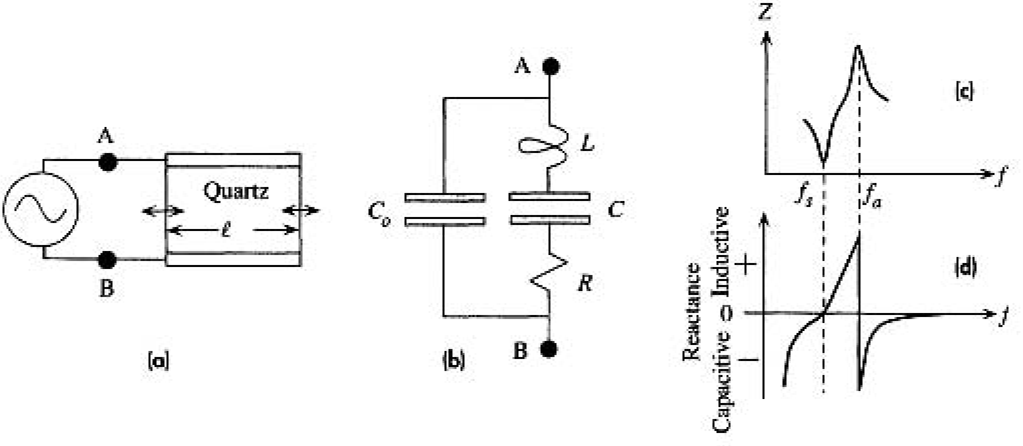

Figure 7.43 When a suitably cut quartz crystal with electrodes is excited by an ac voltage as in (a), it behaves as if it has the equivalent circuit in (b).

(c) and (d) The magnitude of the impedance Z and reactance (both between A and B) versus frequency, neglecting losses.

The frequency of these mechanical vibrations f_s is given by $f_s = v/\lambda$, where v is the velocity of the waves in the medium and λ is the wavelength. These mechanical vibrations in quartz experience very small losses and therefore have a high-quality factor Q, which means that resonance can only be set up if the frequency of the excitation, the electrical frequency, is close to f_s. Because of the coupling of energy between the electrical excitation and mechanical vibrations through the piezoelectric effect, mechanical vibrations appear like a series LCR circuit to the ac source, as shown in Figure 7.43b. This LCR series circuit has an impedance that is minimum at the **mechanical resonant frequency** f_s, given by

Mechanical resonant frequency

$$f_s = \frac{1}{2\pi\sqrt{LC}} \qquad [7.59]$$

In this series LCR circuit, L represents the mass of the transducer, C the stiffness, and R the losses or mechanical damping. Since the quartz crystal has electrodes at opposite faces, there is, in addition, the parallel plate capacitance C_o between the electrodes. Thus, the whole equivalent circuit is C_o in *parallel* with LCR, as in Figure 7.43b. As far as L is concerned, C_o and C are in series. There is a second higher resonant frequency f_a, called the **antiresonant frequency,** that is due to L resonating with C and C_o in series,

Antiresonant frequency

$$f_a = \frac{1}{2\pi\sqrt{LC'}} \qquad [7.60]$$

where

$$\frac{1}{C'} = \frac{1}{C_o} + \frac{1}{C}$$

The impedance between the terminals of the quartz crystal has the frequency dependence shown in Figure 7.43c. The two frequencies f_s and f_a are called the series and

Various quartz crystal "oscillators." Left to right: Raltron 40 MHz; a natural quartz crystal (South Dakota); Phillips 27 MHz; a cutaway view of a typical crystal oscillator.

parallel resonant frequencies, respectively. It is apparent that around f_a, the crystal behaves like a filter with a high Q value. If we were to examine the reactance of the crystal, whether it is behaving capacitively or inductively, we would find the behavior in Figure 7.43d, where positive reactance refers to an inductive and negative reactance to a capacitive behavior. Between f_s and f_a the crystal behaves inductively, and capacitively outside this range. Indeed, between f_s and f_a the response of the transducer is controlled by the mass of the crystal. This property has been utilized by electrical engineers in designing quartz oscillators.

In quartz oscillators, the crystal is invariably used in one of two modes. First, it can be used at f_s where it behaves as a resistance of R without any reactance. The circuit is designed so that oscillations can take place only when the crystal in the circuit exhibits no reactance or phase change—in other words, at f_s. Outside this frequency, the crystal introduces reactance or phase changes that do not lead to sustained oscillations. In a different mode of operation, the oscillator circuit is designed to make use of the **inductance** of the crystal just above f_s. Oscillations are maintained close to f_s because even very large changes in the inductance result in small changes in the frequency between f_s and f_a.

EXAMPLE 7.14 **THE QUARTZ CRYSTAL AND ITS EQUIVALENT CIRCUIT** From the following equivalent definition of the coupling coefficient,

$$k^2 = \frac{\text{Mechanical energy stored}}{\text{Total energy stored}}$$

show that

$$k^2 = 1 - \frac{f_s^2}{f_a^2}$$

Given that typically for an X-cut quartz crystal, $k = 0.1$, what is f_a for $f_s = 1$ MHz? What is your conclusion?

SOLUTION

C represents the mechanical mass where the mechanical energy is stored, whereas C_o is where the electrical energy is stored. If V is the applied voltage, then

$$k^2 = \frac{\text{Mechanical energy stored}}{\text{Total energy stored}} = \frac{\frac{1}{2}CV^2}{\frac{1}{2}CV^2 + \frac{1}{2}C_oV^2} = \frac{C}{C + C_o} = 1 - \frac{f_s^2}{f_a^2}$$

Rearranging this equation, we find

$$f_a = \frac{f_s}{\sqrt{1 - k^2}} = \frac{1\text{ MHz}}{\sqrt{1 - (0.1)^2}} = 1.005\text{ MHz}$$

Thus, $f_a - f_s$ is only 5 kHz. The two frequencies f_s and f_a in Figure 7.43d are very close. An oscillator designed to oscillate at f_s, that is, at 1 MHz, therefore, cannot drift far (for example, a few kHz) because that would change the reactance enormously, which would upset the oscillation conditions.

EXAMPLE 7.15

QUARTZ CRYSTAL AND ITS INDUCTANCE A typical 1 MHz quartz crystal has the following properties:

$$f_s = 1\text{ MHz} \qquad f_a = 1.0025\text{ MHz} \qquad C_o = 5\text{ pF} \qquad R = 20\ \Omega$$

What are C and L in the equivalent circuit of the crystal? What is the quality factor Q of the crystal, given that

$$Q = \frac{1}{2\pi f_s RC}$$

SOLUTION

The expression for f_s is

$$f_s = \frac{1}{2\pi\sqrt{LC}}$$

From the expression for f_a, we have

$$f_a = \frac{1}{2\pi\sqrt{LC'}} = \frac{1}{2\pi\sqrt{L\dfrac{CC_o}{C + C_o}}}$$

Dividing f_a by f_s eliminates L, and we get

$$\frac{f_a}{f_s} = \sqrt{\frac{C + C_o}{C_o}}$$

so that C is

$$C = C_o\left[\left(\frac{f_a}{f_s}\right)^2 - 1\right] = (5\text{ pF})(1.0025^2 - 1) = 0.025\text{ pF}$$

Thus

$$L = \frac{1}{C(2\pi f_s)^2} = \frac{1}{0.025 \times 10^{-12}(2\pi\,10^6)^2} = 1.01\text{ H}$$

This is a substantial inductance, and the enormous increase in the inductive reactance above f_s is intuitively apparent. The quality factor

$$Q = \frac{1}{2\pi f_s RC} = 3.18 \times 10^5$$

is very large.

7.8.3 FERROELECTRIC AND PYROELECTRIC CRYSTALS

Certain crystals are permanently polarized even in the absence of an applied field. The crystal already possesses a finite polarization vector due to the separation of positive and negative charges in the crystal. These crystals are called **ferroelectric.**[13] Barium

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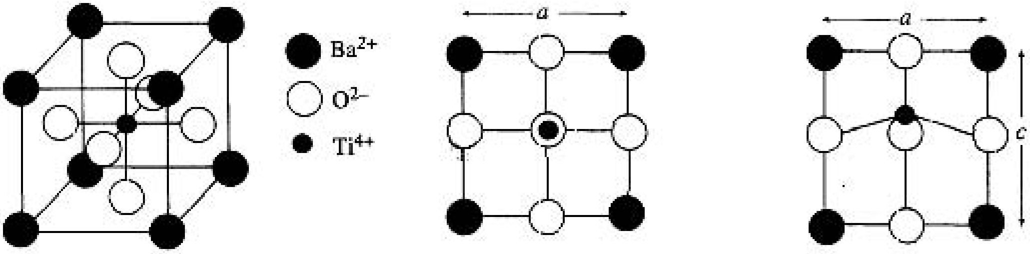
[13] In analogy with the ferromagnetic crystals that already possess magnetization.



(a) $BaTiO_3$ cubic crystal structure above 130 °C

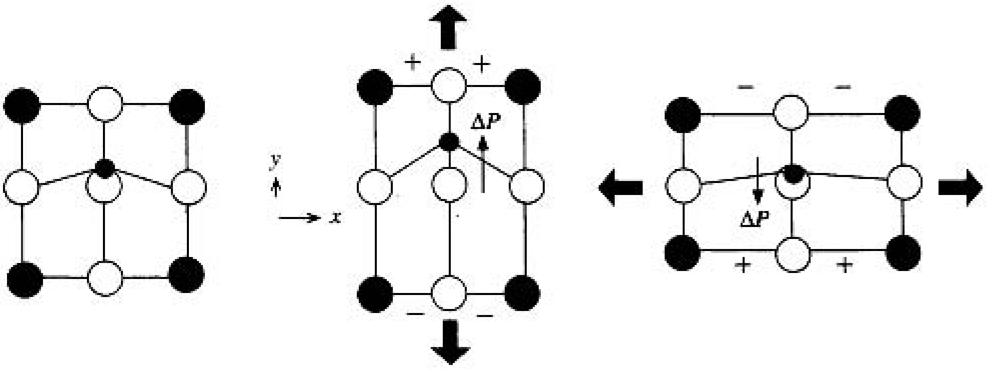
(b) $BaTiO_3$ cubic structure above 130 °C

(c) $BaTiO_3$ tetragonal structure below 130 °C

Figure 7.44 $BaTiO_3$ has different crystal structures above and below 130 °C that lead to different dielectric properties.

titanate ($BaTiO_3$) is probably the best cited example. Above approximately 130 °C, the crystal structure of $BaTiO_3$ has a cubic unit cell, as shown in Figure 7.44a. The centers of mass of the negative charges (O^{2-}) and the positive charges, Ba^{2+} and Ti^{4+}, coincide at the Ti^{4+} ion, as shown in Figure 7.44b. There is therefore no net polarization and $\mathbf{P} = 0$. Above 130 °C, therefore, the barium titanate crystal exhibits no permanent polarization and is not ferroelectric. However, below 130 °C, the structure of barium titanate is tetragonal, as shown in Figure 7.44c, in which the Ti^{4+} atom is not located at the center of mass of the negative charges. The crystal is therefore polarized by the separation of the centers of mass of the negative and positive charges. The crystal possesses a finite polarization vector **P** and is ferroelectric. The critical temperature above which ferroelectric property is lost, in this case 130 °C, is called **the Curie temperature** (T_C). Below the Curie temperature, the whole crystal becomes spontaneously polarized. The onset of spontaneous polarization is accompanied by the distortion of the crystal structure, as shown by the change from Figure 7.44b to Figure 7.44c. The spontaneous displacement of the Ti^{4+} ion below the Curie temperature elongates the cubic structure, which becomes tetragonal. It is important to emphasize that we have only described an observation and not the reasons for the spontaneous polarization of the whole crystal. The development of the permanent dipole moment below the Curie temperature involves long-range interactions between the ions outside the simple unit cell pictured in Figure 7.44. The energy of the crystal is lower when the Ti^{4+} ion in each unit cell is slightly displaced along the c direction, as in Figure 7.44c, which generates a dipole moment in each unit cell. The interaction energy of these dipoles when all are aligned in the same direction lowers the energy of the whole crystal. It should be mentioned that the distortion of the crystal that takes place when spontaneous polarization occurs just below T_C is very small relative to the dimensions of the unit cell. For $BaTiO_3$, for example, c/a is 1.01 and the displacement of the Ti^{4+} ion from the center is only 0.012 nm, compared with $a = 0.4$ nm.

An important and technologically useful characteristic of a ferroelectric crystal is its ability to be poled. Above 130 °C there is no permanent polarization in the crystal. If we apply a temporary field $\mathcal{E}$ and let the crystal cool to below 130 °C, we can induce the spontaneous polarization **P** to develop along the field direction. In other words, we would define the c axis by imposing a temporary external field. This process is called poling. The c axis is the polar axis along which **P** develops. It is also called the



(a) $BaTiO_3$ tetragonal structure below 130 °C (b) $BaTiO_3$ crystal under stress along y (c) $BaTiO_3$ crystal under stress along x

Figure 7.45 Piezoelectric properties of $BaTiO_3$ below its Curie temperature.

ferroelectric axis. Since below the Curie temperature the ferroelectric crystal already has a permanent polarization, it is not possible to use the expression

$$P = \varepsilon_o(\varepsilon_r - 1)\mathcal{E}$$

to define a relative permittivity. Suppose that we use a ferroelectric crystal as a dielectric medium between two parallel plates. Since any change ΔP normal to the plates changes the stored charge, what is of significance to the observer is the change in the polarization. We can appreciate this by noting that $C = Q/V$ is not a good definition of capacitance if there are already charges on the plates, even in the absence of voltage.[14] We then prefer a definition of C based on $\Delta Q/\Delta V$ where ΔQ is the change in stored charge due to a change ΔV in the voltage. Similarly, we define the relative permittivity ε_r in this case in terms of the change ΔP in P induced by $\Delta\mathcal{E}$ in the field $\mathcal{E}$,

$$\Delta P = \varepsilon_o(\varepsilon_r - 1)\,\Delta\mathcal{E}$$

An applied field along the a axis can displace the Ti^{4+} ion more easily than that along the c axis, and experiments show that $\varepsilon_r \approx 4100$ along a is much greater than $\varepsilon_r \approx 160$ along c. Because of their large dielectric constants, ferroelectric ceramics are used as high-K dielectrics in capacitors.

All ferroelectric crystals are also piezoelectric, but the reverse is not true: not all piezoelectric crystals are ferroelectric. When a stress along y is applied to the $BaTiO_3$ crystal in Figure 7.45a, the crystal is stretched along y, as a result of which the Ti^{4+} atom becomes displaced, as shown in Figure 7.45b. There is, however, no shift in the center of mass of the negative charges, which means that there is a change ΔP in the polarization vector along y. Thus, the applied stress induces a change in the polarization, which is a piezoelectric effect. If the stress is along x, as illustrated in Figure 7.45c, then the change in the polarization is along y. In both cases, ΔP is proportional to the stress, which is a characteristic of the piezoelectric effect.

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[14] A finite Q on the plates of a capacitor when $V = 0$ implies an infinite capacitance, $C = \infty$. However, $C = dQ/dV$ definition avoids this infinity.

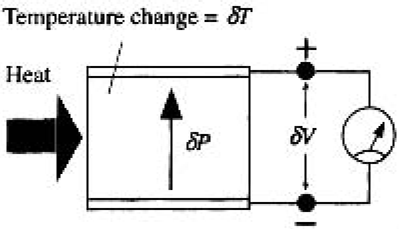

Figure 7.46 The heat absorbed by the crystal increases the temperature by δT, which induces a change δP in the polarization.

This is the pyroelectric effect. The change δP gives rise to a change δV in the voltage that can be measured.

The barium titanate crystal in Figure 7.44 is also said to be pyroelectric because when the temperature increases, the crystal expands and the relative distances of ions change. The Ti^{4+} ion becomes shifted, which results in a change in the polarization. Thus, a temperature change δT induces a change δP in the polarization of the crystal. This is called **pyroelectricity,** which is illustrated in Figure 7.46. The magnitude of this effect is quantized by the **pyroelectric coefficient** p, which is defined by

Pyroelectric coefficient

$$p = \frac{dP}{dT} \tag{7.61}$$

A few typical pyroelectric crystals and their pyroelectric coefficients are listed in Table 7.9. Very small temperature changes, even in thousandths of degrees, in the material can develop voltages that can be readily measured. For example, for a PZT-type pyroelectric ceramic in Table 7.9, taking $\delta T = 10^{-3}$ K and $p \approx 380 \times 10^{-6}$, we find $\delta P = 3.8 \times 10^{-7}$ C m^{-2}. From

$$\delta P = \varepsilon_o(\varepsilon_r - 1)\,\delta\mathcal{E}$$

with $\varepsilon_r = 290$, we find

$$\delta\mathcal{E} = 148 \text{ V m}^{-1}$$

If the distance between the faces of the ceramic where the charges are developed is 0.1 mm, then

$$\delta V = 0.0148 \text{ V} \qquad \text{or} \qquad 15 \text{ mV}$$

Table 7.9 Some pyroelectric (and also ferroelectric) crystals and typical properties

Material	ε_r'	tan δ	Pyroelectric Coefficient ($\times 10^{-6}$ C m^{-2} K^{-1})	Curie Temperature (°C)
$BaTiO_3$	4100 ⊥ polar axis; 160 //polar axis	7×10^{-3}	20	130
$LiTaO_3$	47	5×10^{-3}	230	610
PZT modified for pyroelectric	290	2.7×10^{-3}	380	230
PVDF, polymer	12	0.01	27	80

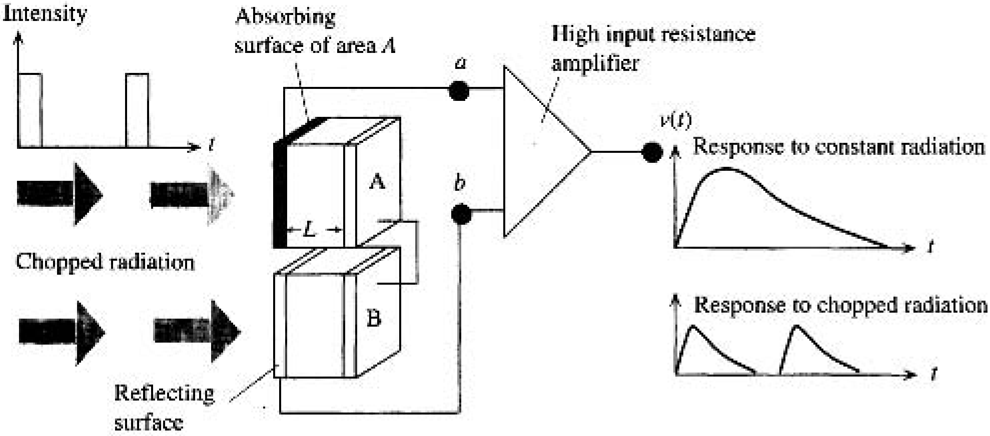

Figure 7.47 The pyroelectric detector.

Radiation is absorbed in the detecting element, A, which generates a pyroelectric voltage that is measured by the amplifier. The second element, B, has a reflecting electrode and does not absorb the radiation. It is a dummy element that compensates for the piezoelectric effects. Piezoelectric effects generate equal voltages in both A and B, which cancel each other across *a* and *b*, the input of the amplifier.

which can be readily measured. Pyroelectric crystals are widely used as infrared detectors. Any infrared radiation that can raise the temperature of the crystal even by a thousandth of a degree can be detected. For example, many intruder alarms use pyroelectric detectors because as the human or animal intruder passes by the view of detector, the infrared radiation from the warm body raises the temperature of the pyroelectric detector, which generates a voltage that actuates an alarm.

Figure 7.47 shows a simplified schematic circuit for a pyroelectric radiation detector. The detecting element, labeled A, is actually a thin crystal or ceramic (or even a polymer) of a pyroelectric material that has electrodes on opposite faces. Pyroelectric materials are also piezoelectric and therefore also sensitive to stresses. Thus, pressure fluctuations, for example, vibrations from the detector mount or sound waves, interfere with the response of the detector to radiation alone. These can be compensated for by having a second dummy detector B that has a reflecting coating and is subjected to the same vibrations (air and mount), as depicted in Figure 7.47. Thus, there are two elements in the detector, one with an absorbing surface, detecting element A, and the other with a reflecting surface, compensating element B. Stress fluctuations give rise to the same piezoelectric voltage in both, which then cancel each other between *a* and *b* at the input of the amplifier. When radiation is incident, then only the detecting element absorbs the radiation, becomes warmer, and hence generates a pyroelectric voltage. This voltage appears directly across *a* and *b*. As the incident radiation warms the detecting element and increases its temperature, the pyroelectric voltage increases with time. Eventually the temperature reaches a steady-state value determined by heat losses from the element. We therefore expect the pyroelectric voltage to reach a constant value as well. However, the problem is that a constant pyroelectric voltage cannot be sustained because the surface charges slowly become neutralized or leak away.

The constant radiation is therefore normally chopped to subject the detector to periodic bursts of radiation, as shown in Figure 7.47. The pyroelectric voltage is then a changing function of time, which is readily measured and related to the power in the incident radiation.

Many pyroelectric applications refer to a pyroelectric current that is generated by the temperature rise. There is another way to look at the pyroelectric phenomenon instead of considering the induced pyroelectric voltage that is created across the crystal (Figure 7.46). The induced polarization δP in a small time interval δt, due to the change δT in the temperature, generates an induced polarization charge density δP on the crystal's surfaces. This charge density δP flows in a time interval δt, and hence generates an *induced polarization current density* J_p to flow, *i.e.*,

Pyroelectric current density

$$J_p = \frac{dP}{dt} = p\frac{dT}{dt} \qquad [7.62]$$

J_p in Equation 7.62 is called the **pyroelectric current density** and depends on the rate of change of the temperature dT/dt brought about by the absorption of radiation.

Most pyroelectric detectors are characterized by their **current responsivity** $\mathcal{R}_I$ defined as the pyroelectric current generated per unit input radiation power,

Pyroelectric current responsivity

$$\mathcal{R}_I = \frac{\text{Pyroelectric current generated}}{\text{Input radiation power}} = \frac{J_p}{\mathcal{I}} \qquad [7.63]$$

where $\mathcal{I}$ is the radiation intensity (W m^{-2}); $\mathcal{R}_I$ is quoted in A W^{-1}. If the pyroelectric current generated by the crystal flows into the self-capacitance of the crystal itself (no external resistors or capacitors connected, and the voltmeter is an ideal meter), it charges the self-capacitance to generate the observed voltage δV in Figure 7.46. The **pyroelectric voltage responsivity** $\mathcal{R}_V$ is defined similarly to Equation 7.63 but considers the voltage that is developed upon receiving the input radiation:

Pyroelectric voltage responsivity

$$\mathcal{R}_V = \frac{\text{Pyroelectric output voltage generated}}{\text{Input radiation power}} \qquad [7.64]$$

The output voltage that is generated depends not only on the pyroelectric crystal's dielectric properties, but also on the input impedance of the amplifier, and can be quite complicated. A typical commercial LiTaO$_3$ pyroelectric detector has a current responsivity of 0.1–1 μA/W.

EXAMPLE 7.16 **A PYROELECTRIC RADIATION DETECTOR** Consider the radiation detector in Figure 7.47 but with a single element A. Suppose that the radiation is chopped so that the radiation is passed to the detector for a time Δt seconds every τ seconds, where $\Delta t \ll \tau$. If Δt is sufficiently small, then the temperature rise ΔT is small and hence the heat losses are negligible during Δt. Using the heat capacity to find the temperature change during Δt, relate the magnitude of the voltage ΔV to the incident radiation intensity $\mathcal{I}$. What is your conclusion?

Consider a PZT-type pyroelectric material with a density of about 7 g cm^{-3} and a specific heat capacity of about 380 J K^{-1} kg^{-1}. If $\Delta t = 0.2$ s and the minimum voltage that can be detected above the background noise is 1 mV, what is the minimum radiation intensity that can be measured?

SOLUTION

Suppose that the radiation of intensity $\mathcal{I}$ is received during a time interval Δt and delivers an amount of energy ΔH to the pyroelectric detector. This energy ΔH, in the absence of any heat losses, increases the temperature by ΔT. If c is the specific heat capacity (heat capacity per unit mass) and ρ is the density,

$$\Delta H = (AL\rho)c\,\Delta T$$

where A is the surface area and L the thickness of the detector. The change in the polarization ΔP is

$$\Delta P = p\,\Delta T = \frac{p\,\Delta H}{AL\rho c}$$

The change in the surface charge ΔQ is

$$\Delta Q = A\,\Delta P = \frac{p\,\Delta H}{L\rho c}$$

This change in the surface charge gives a voltage change ΔV across the electrodes of the detector. If $C = \varepsilon_o \varepsilon_r A/L$ is the capacitance of the pyroelectric crystal,

$$\Delta V = \frac{\Delta Q}{C} = \frac{p\,\Delta H}{L\rho c} \times \frac{L}{\varepsilon_o \varepsilon_r A} = \frac{p\,\Delta H}{A\rho c \varepsilon_o \varepsilon_r}$$

The absorbed energy (heat) ΔH during Δt depends on the intensity of incident radiation. Incident intensity $\mathcal{I}$ is the energy arriving per unit area per unit time. In time Δt, $\mathcal{I}$ delivers an energy $\Delta H = \mathcal{I} A\,\Delta t$. Substituting for ΔH in the expression for ΔV, we find

Pyroelectric detector output voltage

$$\Delta V = \frac{p\,\mathcal{I}\,\Delta t}{\rho c \varepsilon_r \varepsilon_o} = \left(\frac{p}{\rho c \varepsilon_r \varepsilon_o}\right)\mathcal{I}\,\Delta t \qquad [7.65]$$

The parameters in the parentheses are material properties and reflect the "goodness" of the pyroelectric material for the application. We should emphasize that in deriving Equation 7.65 we did not consider any heat losses that will prevent the rise of the temperature indefinitely. If Δt is short, then the temperature change will be small and heat losses negligible.

For a PZT-type pyroelectric, we can take $p = 380 \times 10^{-6}$ C m^{-2} K^{-1}, $\varepsilon_r = 290$, $c = 380$ J K^{-1} kg^{-1}, and $\rho = 7 \times 10^3$ kg m^{-3}, and then from Equation 7.65 with $\Delta V = 0.001$ V and $\Delta t = 0.2$ s, we have

$$\mathcal{I} = \left(\frac{p}{\rho c \varepsilon_o \varepsilon_r}\right)^{-1} \frac{\Delta V}{\Delta t} = \left(\frac{380 \times 10^{-6}}{(7000)(380)(290)(8.85 \times 10^{-12})}\right)^{-1} \frac{0.001}{0.2}$$

$$= 0.090 \text{ W m}^{-2} \qquad \text{or} \qquad 9\,\mu\text{W cm}^{-2}$$

We have assumed that all the incident radiation $\mathcal{I}$ is absorbed by the pyroelectric crystal. In practice, only a fraction η (called the *emissivity* of the surface), that is, $\eta\,\mathcal{I}$, will be absorbed instead of $\mathcal{I}$. We also assumed that the output voltage ΔV is developed totally across the pyroelectric element capacitance; that is, the amplifier's input impedance (parallel combination of its input capacitance and resistance) is negligible compared with that of the pyroelectric crystal. As stated, we also neglected all heat losses from the pyroelectric crystal so that the absorbed radiation simply increases the crystal's temperature. These simplifying assumptions lead to the maximum signal ΔV that can be generated from a given input radiation signal $\mathcal{I}$ as stated in Equation 7.65. It is left as an exercise to show that Equation 7.65 can also be easily derived by starting from Equation 7.62 for the pyroelectric current density J_p, and have J_p charge up the capacitance $C = \varepsilon_o \varepsilon_r A/L$ of the crystal.

ADDITIONAL TOPICS

7.9 ELECTRIC DISPLACEMENT AND DEPOLARIZATION FIELD

Electric Displacement (D) and Free Charges Consider a parallel plate capacitor with free space between the plates, as shown in Figure 7.48a, which has been charged to a voltage V_o by connecting it to a battery of voltage V_o. The battery has been suddenly removed, which has left the free positive and negative charges Q_{free} on the plates. These charges are free in the sense that they can be conducted away. An ideal electrometer (with no leakage current) measures the total charge on the positive plate (or voltage of the positive plate with respect to the negative plate). The voltage across the plates is V_o and the capacitance is C_o. The field in the free space between the plates is

Electric field without dielectric

$$\mathcal{E}_o = \frac{Q_{\text{free}}}{\varepsilon_o A} = \frac{V_o}{d} \qquad [7.66]$$

where d is the separation of the plates.

When we insert a dielectric to fit between the plates, the field polarizes the dielectric and polarization charges $-Q_P$ and $+Q_P$ appear on the left and right surfaces of the dielectric, as shown in Figure 7.48b. As there is no battery to supply more free charges, the net charge on the left plate (positive plate) becomes $Q_{\text{free}} - Q_P$. Similarly the net negative charge on the right plate becomes $-Q_{\text{free}} + Q_P$. The field inside the dielectric is no longer $\mathcal{E}_o$ but less because induced polarization charges have the opposite polarity to the original free charges and the net charge on each plate has been reduced. The new field can be found by applying Gauss's law. Consider a Gauss surface just enclosing the left plate and the surface region of the dielectric with its negative polarization charges, as shown in Figure 7.49. Then Gauss's law gives

Gauss's law with dielectric

$$\oint_{\text{Surface}} \varepsilon_o \mathcal{E}\, dA = Q_{\text{total}} = Q_{\text{free}} - Q_P \qquad [7.67]$$

where A is the plate area (same as dielectric surface area) and we take the field $\mathcal{E}$ to be normal to the surface area dA, as indicated in Figure 7.49. If the polarization charge is

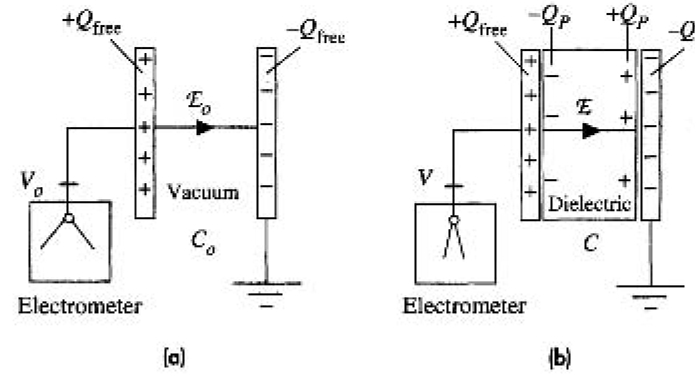

Figure 7.48

(a) Parallel plate capacitor with free space between plates that has been charged to a voltage V_o. There is no battery to maintain the voltage constant across the capacitor. The electrometer measures the voltage difference across the plates and, in principle, does not affect the measurement.

(b) After the insertion of the dielectric, the voltage difference is V, less than V_o, and the field in the dielectric is $\mathcal{E}$ less than $\mathcal{E}_o$.

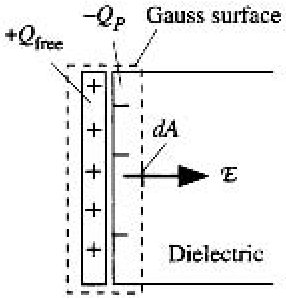

Figure 7.49 A Gauss surface just around the left plate and within the dielectric, encompassing both $+Q_{\text{free}}$ and $-Q_P$.

dQ_P over a small surface area dA of the dielectric, then the polarization charge density σ_P at this point is defined as

$$\sigma_P = \frac{dQ_P}{dA}$$

For uniform polarization, the charge distribution is Q_P/A, as we have used previously. Since $\sigma_P = P$, where P is the polarization vector, we can write

$$P = \frac{dQ_P}{dA}$$

and therefore express Q_P as

$$Q_P = \oint_{\text{Surface}} P\,dA \tag{7.68}$$

We can now substitute for Q_P in Equation 7.67 and take this term to the left-hand side to add the two surface integrals. The right-hand side is left with only Q_{free}. Thus,

$$\oint_{\text{Surface}} (\varepsilon_o\mathcal{E} + P)\,dA = Q_{\text{free}} \tag{7.69}$$

What is important here is that the surface integration of the quantity $\varepsilon_o\mathcal{E} + P$ is always equal to the total free charges on the surface. Whatever the dielectric material, this integral is always Q_{free}. It becomes convenient to define $\varepsilon_o\mathcal{E} + P$ as a usable quantity, called the **electric displacement** and denoted as D, that is,

Definition of electric displacement

$$D = \varepsilon_o\mathcal{E} + P \tag{7.70}$$

Then, Gauss's law in terms of free charges alone in Equation 7.69 becomes

Gauss's law for free charges

$$\oint_{\text{Surface}} D\,dA = Q_{\text{free}} \tag{7.71}$$

In Equation 7.71 we take D to be normal to the surface area dA as in the case of $\mathcal{E}$ in Gauss's law. Equation 7.71 provides a convenient way to calculate the electric displacement D, from which one should be able to determine the field. We should note that, in general, $\mathcal{E}$ is a vector and so is P, so the definition in Equation 7.70 is strictly in terms of vectors. Inasmuch as the electric displacement depends only on free charges, as a vector it starts at negative free charges and finishes on positive free charges.

Equation 7.71 for D defines it in terms of $\mathcal{E}$ and P, but we can express D in terms of the field $\mathcal{E}$ in the dielectric alone. The polarization P and $\mathcal{E}$ are related by the definition of the relative permittivity ε_r,

$$P = \varepsilon_o(\varepsilon_r - 1)\mathcal{E}$$

Substituting for P in Equation 7.70 and rearranging, we find that D is simply given by

Electric displacement and the field

$$D = \varepsilon_o\varepsilon_r\mathcal{E} \qquad [7.72]$$

We should note that this simple equation applies in an isotropic medium where the field along one direction, for example, x, does not generate polarization along a different direction, for example, y. In those cases, Equation 7.72 takes a tensor form whose mathematics is beyond the scope of this book.

We can now apply Equation 7.71 for a Gauss surface surrounding the left plate,

$$D = \frac{Q_{\text{free}}}{A} = \varepsilon_o\mathcal{E}_o \qquad [7.73]$$

where we used Equation 7.66 to replace Q_{free}. Thus D does not change when we insert the dielectric because the same free charges are still on the plates (they cannot be conducted away anywhere). The new field $\mathcal{E}$ between the plates after the insertion of the dielectric is

$$\mathcal{E} = \frac{1}{\varepsilon_o\varepsilon_r}D = \frac{1}{\varepsilon_r}\mathcal{E}_o \qquad [7.74]$$

The original field is reduced by the polarization of the dielectric. We should recall that the field does *not* change in the case where the parallel plate capacitor is connected to a battery that keeps the voltage constant across the plates and supplies additional free charges (ΔQ_{free}) to make up for the induced opposite-polarity polarization charges.

Gauss's law in Equation 7.71 in terms of D and the enclosed free charges Q_{free} can also be written in terms of the field $\mathcal{E}$, but including the relative permittivity, because D and $\mathcal{E}$ are related by Equation 7.72. Using Equation 7.72, Equation 7.71 becomes

Gauss's law for free charges

$$\oint_{\text{Surface}} \varepsilon_o\varepsilon_r\mathcal{E}\,dA = Q_{\text{free}}$$

For an isotropic medium where ε_r is the same everywhere,

Gauss's law in an isotropic dielectric

$$\oint_{\text{Surface}} \mathcal{E}\,dA = \frac{Q_{\text{free}}}{\varepsilon_o\varepsilon_r} \qquad [7.75]$$

As before, $\mathcal{E}$ in the surface integral is taken as normal to dA everywhere. Equation 7.75 is a convenient way of evaluating the field from the free charges alone, given the dielectric constant of the medium.

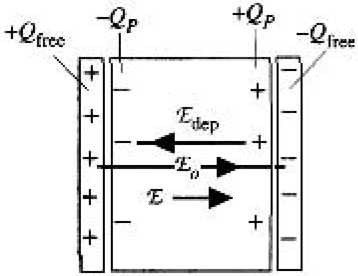

Figure 7.50 The field inside the dielectric can be considered to be the sum of the field due to the free charges (Q_{free}) and a field due to the polarization of the dielectric, called the depolarization field.

The Depolarizing Field We can view the field $\mathcal{E}$ as arising from two electric fields: that due to the free charges $\mathcal{E}_o$ and that due to the polarization charges, denoted as $\mathcal{E}_{\text{dep}}$. These two fields are indicated in Figure 7.50. $\mathcal{E}_o$ is called the **applied field** as it is due to the free charges that have been put on the plates. It starts and ends at free charges on the plates. The field due to polarization charges starts and ends at these bound charges and is in the *opposite* direction to the $\mathcal{E}_o$. Although $\mathcal{E}_o$ polarizes the molecules of the medium, $\mathcal{E}_{\text{dep}}$, being in the opposite direction, tries to depolarize the medium. It is called the **depolarizing field** (and hence the subscript). Thus the field inside the medium is

$$\mathcal{E} = \mathcal{E}_o - \mathcal{E}_{\text{dep}} \tag{7.76}$$

The depolarizing field depends on the amount of polarization since it is determined by $+Q_P$ and $-Q_P$. For the dielectric plate in Figure 7.50, we know the field $\mathcal{E}$ is $\mathcal{E}_o/\varepsilon_r$, so we can eliminate $\mathcal{E}_o$ in Equation 7.76 and relate $\mathcal{E}_{\text{dep}}$ directly to $\mathcal{E}$,

$$\mathcal{E}_{\text{dep}} = \mathcal{E}(\varepsilon_r - 1)$$

However, the polarization P is related to the field $\mathcal{E}$ by

$$P = \varepsilon_o(\varepsilon_r - 1)\mathcal{E}$$

which means that the depolarization field is

Depolarizing field in a dielectric plate

$$\mathcal{E}_{\text{dep}} = \frac{1}{\varepsilon_o}P \tag{7.77}$$

As we expected, the depolarizing field is proportional to the polarization P. We should emphasize that $\mathcal{E}_{\text{dep}}$ is in the *opposite direction* to $\mathcal{E}$ and P and Equation 7.77 is for magnitudes only. If we write it as a vector equation, then we must introduce a negative sign to give $\mathcal{E}_{\text{dep}}$ a direction opposite to that of P. Moreover, the relationship in Equation 7.77 is special to the dielectric plate geometry in Figure 7.50. In general, the depolarizing field is still proportional to the polarization, as in Equation 7.77, but it is given by

Depolarizing field in a dielectric

$$\mathcal{E}_{\text{dep}} = \frac{N_{\text{dep}}}{\varepsilon_o}P \tag{7.78}$$

where N_{dep} is a numerical factor called the **depolarization factor.** It takes into account the shape of the dielectric and the variation in the polarization within the medium. For

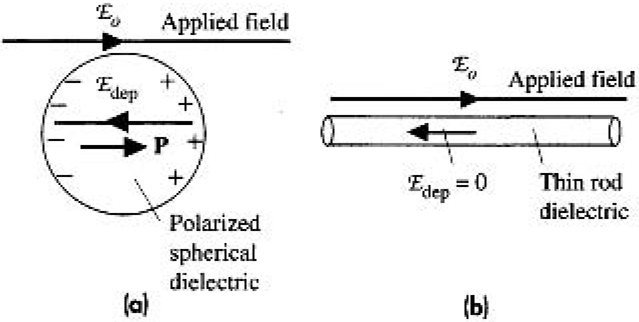

Figure 7.51
(a) Polarization and the depolarizing field in a spherical-shaped dielectric placed in an applied field.
(b) Depolarization field in a thin rod placed in an applied field is nearly zero.

a dielectric plate placed perpendicularly to an external field, $N_{dep} = 1$, as we found in Equation 7.77. For the spherical dielectric medium as in Figure 7.51a, $N_{dep} = \frac{1}{3}$. For a long thin dielectric rod placed with its axis along the applied field, as in Figure 7.51b, $N_{dep} \approx 0$ and becomes exactly zero as the diameter shrinks to zero. N_{dep} is always between 0 and 1. If we know N_{dep}, we can determine the field inside the dielectric, for example, in a small spherical cavity within an insulation given the external field.

7.10 LOCAL FIELD AND THE LORENTZ EQUATION

When a dielectric medium is placed in an electric field, it becomes polarized and there is a macroscopic, or an average, field $\mathcal{E}$ in the medium. The actual field at an atom, called the **local field** $\mathcal{E}_{loc}$, however, is not the same as the average field as illustrated in Figure 7.7.

Consider a dielectric plate polarized by placing it between the plates of a capacitor as shown in Figure 7.52a. The macroscopic field $\mathcal{E}$ in the dielectric is given by the applied field $\mathcal{E}_o$ due to the free charges Q_{free} on the plates, and the depolarization field due to P, or polarization charges on the dielectric plate surfaces A. Since we have a plate

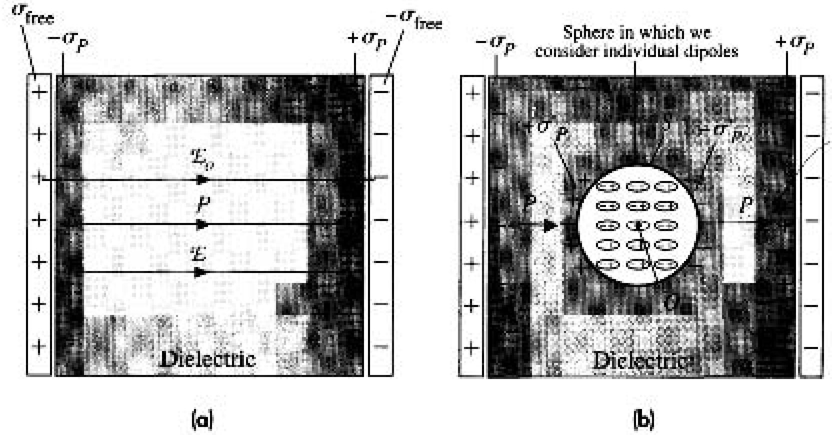

Figure 7.52
(a) The macroscopic field $\mathcal{E}$ is determined by the applied field $\mathcal{E}_o$ and the depolarization field due to P.
(b) Calculation of the local field involves making a hypothetical spherical cavity S inside the dielectric. This produces polarization surface charges on the inside surface S of the cavity. The effects of the dipoles inside the cavity are treated individually.

dielectric, the depolarization field is P/ε_o, so

$$\mathcal{E} = \mathcal{E}_o - \mathcal{E}_{\text{dep}} = \mathcal{E}_o - \frac{1}{\varepsilon_o}P$$

Consider the field at some atomic site, point O, but with the atom itself removed. We evaluate the field at O coming from all the charges except the atom at O itself since we are looking at the field experienced by this atom (the atom cannot become polarized by its own field). We then cut a (hypothetical) spherical cavity S centered at O and consider the atomic polarizations individually within the spherical cavity. In other words, the effects of the dipoles in the cavity are treated separately from the remaining dielectric medium which is now left with a spherical cavity. This remaining dielectric is considered as a continuous medium but with a spherical cavity. Its dielectric property is represented by its polarization vector $\mathbf{P}$. Because of the cavity, we must now put polarization charges on the inner surface S of this cavity as illustrated in Figure 7.52b. This may seem surprising, but we should remember that we are treating the effects of the atomic dipoles within the cavity individually and separately by cutting out a spherical cavity from the medium and thereby introducing a surface S.

The field at O comes from four sources:

1. Free charges Q_{free} on the electrodes, represented by $\mathcal{E}_o$.
2. Polarization charges on the plate surfaces A, represented by $\mathcal{E}_{\text{dep}}$.
3. Polarization charges on the inner surface of the spherical cavity S, represented by $\mathcal{E}_S$.
4. Individual dipoles within the cavity, represented by $\mathcal{E}_{\text{dipoles}}$.

Thus,

$$\mathcal{E}_{\text{loc}} = \mathcal{E}_o + \mathcal{E}_{\text{dep}} + \mathcal{E}_S + \mathcal{E}_{\text{dipoles}}$$

Since the first two terms make up the macroscopic field, we can write this as

Local field in a crystal

$$\mathcal{E}_{\text{loc}} = \mathcal{E} + \mathcal{E}_S + \mathcal{E}_{\text{dipoles}}$$

The field from the individual dipoles surrounding O depends on the positions of these atomic dipoles which depend on the crystal structure. For cubic crystals, amorphous solids (*e.g.*, glasses), or liquids, effects of these dipoles around O cancel each other and $\mathcal{E}_{\text{dipoles}} = 0$. Thus,

Local field in a cubic crystal or a non-crystalline material

$$\mathcal{E}_{\text{loc}} = \mathcal{E} + \mathcal{E}_S \qquad [7.79]$$

We are then left with evaluating the field due to polarization charges on the inner surface S of the cavity. This field comes from polarization charges on the surface S. Consider a thin spherical shell on surface S as shown in Figure 7.53 which makes an angle θ with O. The radius of this shell is $a \sin\theta$, whereas its width (or thickness) is $a\,d\theta$. The surface area dS is then $(2\pi a \sin\theta)(a\,d\theta)$. The polarization charge dQ_P on this spherical shell surface is $P_n\,dS$ where P_n is the polarization vector normal to the surface dS. Thus,

$$dQ_P = P_n\,dS = (P\cos\theta)(2\pi a\sin\theta)(a\,d\theta)$$

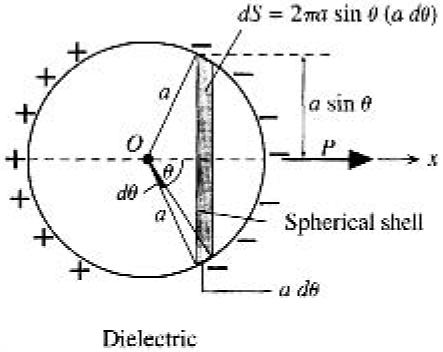

Figure 7.53 Calculation of the field due to polarization charges on the inner surface S of the spherical cavity.
Consider a spherical shell of radius a. The surface area is $dS = 2\pi a \sin\theta\ (a\,d\theta)$.

But the field at O from dQ_P is given from electrostatics as

$$d\mathcal{E}_S = \frac{dQ_P}{4\pi\varepsilon_o a^2} = \frac{(P\cos\theta)(2\pi a\ \sin\theta)(a\,d\theta)}{4\pi\varepsilon_o a^2}$$

To find the total field coming from the whole surface S we have to integrate $d\mathcal{E}_S$ from $\theta = 0$ to $\theta = \pi$,

$$\mathcal{E}_S = \int_0^{\pi} \frac{(P\cos\theta)(\sin\theta)}{2\varepsilon_o}\,d\theta$$

which integrates to

Field in a spherical cavity

$$\mathcal{E}_S = \frac{1}{3\varepsilon_o}P \qquad [7.80]$$

Local field in a cubic crystal or noncrystalline material

The local field by Equation 7.79 is

$$\mathcal{E}_{\text{loc}} = \mathcal{E} + \frac{1}{3\varepsilon_o}P \qquad [7.81]$$

Equation 7.81 is the **Lorentz relation** for the local field in terms of the polarization P of the medium and is valid for cubic crystals and noncrystalline materials, such as glasses. It does *not* apply to dipolar dielectrics in which the local field can be quite complicated.

7.11 DIPOLAR POLARIZATION

Consider a medium with molecules that have permanent dipole moments. Each permanent dipole moment is p_o. In the presence of an electric field the dipoles try to align perfectly with the field, but random thermal collisions, *i.e.*, thermal agitation, act against this perfect alignment. A molecule that manages to rotate and align with the field finds itself later colliding with another molecule and losing its alignment. We are interested in the mean dipole moment in the presence of an applied field taking into

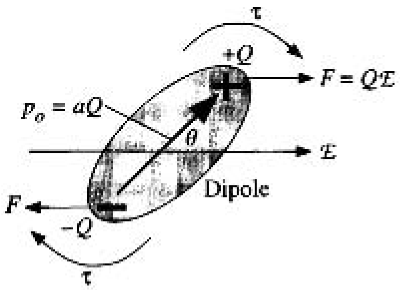

Figure 7.54 In the presence of an applied field a dipole tries to rotate to align with the field against thermal agitation.

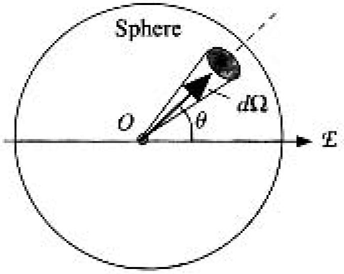

Figure 7.55 The dipole is pointing within a solid angle dΩ.

account the thermal energies of the molecules and their random collisions. We will assume that the probability that a molecule has an energy E is given by the Boltzmann factor, $\exp(-E/kT)$.

Consider an arbitrary dipolar molecule in an electric field as in Figure 7.54 with its dipole moment p_o at an angle θ with the field $\mathcal{E}$. The torque experienced by the dipole is given by $\tau = (F \sin\theta)a$ or $\mathcal{E}p_o \sin\theta$ where $p_o = aQ$. The potential energy E at an angle θ is given by integrating $\tau\, d\theta$,

Potential energy of a dipole at an angle θ

$$E = \int_0^{\theta} p_o\mathcal{E} \sin\theta \, d\theta = -p_o\mathcal{E}\cos\theta + p_o\mathcal{E}$$

Inasmuch as the PE depends on the orientation, there is a certain probability of finding a dipole oriented at this angle as determined by the Boltzmann distribution. The fraction f of molecules oriented at θ is proportional to $\exp(-E/kT)$,

Boltzmann distribution

$$f \propto \exp\left(\frac{p_o\mathcal{E}\cos\theta}{kT}\right) \qquad [7.82]$$

The initial orientation of the dipole should be considered in three dimensions and not as in the two-dimensional illustration in Figure 7.54. In three dimensions we use solid angles, and the fraction f then represents the fraction of molecules pointing in a direction defined by a small solid angle $d\Omega$ as shown in Figure 7.55. The whole sphere around the dipole corresponds to a solid angle of 4π. Furthermore, we need to find the average dipole moment along $\mathcal{E}$ as this will be the induced net dipole moment by the field. The dipole moment along $\mathcal{E}$ is $p_o\cos\theta$. Then from the definition of the average

$$p_{\text{av}} = \frac{\int_0^{4\pi} (p_o\cos\theta) f \, d\Omega}{\int_0^{4\pi} f \, d\Omega} \qquad [7.83]$$

where f is the Boltzmann factor given in Equation 7.82 and depends on $\mathcal{E}$ and θ. The final result of this integration is a special function called the **Langevin function** which

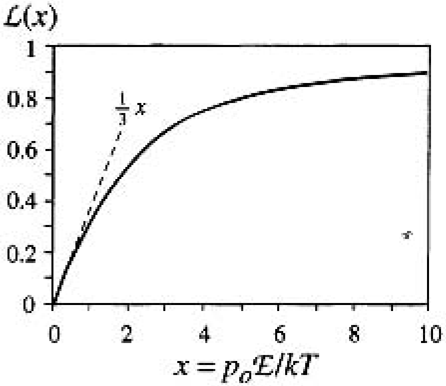

Figure 7.56 The Langevin function.

is denoted as $\mathcal{L}(x)$ where x is the argument of the function (not the x coordinate). The integration of Equation 7.83 then gives

Average dipole moment and the Langevin function

$$p_{av} = p_o\mathcal{L}(x) \qquad \text{and} \qquad x = \frac{\mathcal{E}}{kT} \tag{7.84}$$

The behavior of the Langevin function is shown in Figure 7.56. At the highest fields $\mathcal{L}(x)$ tends toward saturation at unity. Then, $p_{av} = p_o$, which corresponds to nearly all the dipoles aligning with the field, so increasing the field cannot increase p_{av} anymore. In the low field region, p_{av} increases linearly with the field. In practice, the applied fields are such that all dipolar polarizations fall into this linear behavior region where the Langevin function $\mathcal{L}(x) \approx \frac{1}{3}x$. Then Equation 7.84 becomes

Average induced dipole in orientational polarization

$$p_{av} = \frac{1}{3}\frac{p_o^2\mathcal{E}}{kT} \tag{7.85}$$

The **dipolar** or **orientational polarizability** is then simply

Dipolar or orientational polarizability

$$\alpha_d = \frac{1}{3}\frac{p_o^2}{kT} \tag{7.86}$$

7.12 IONIC POLARIZATION AND DIELECTRIC RESONANCE

In ionic polarization, as shown in Figure 7.9, the applied field displaces the positive and negative ions in opposite directions, which results in a net dipole moment per ion, called the *induced dipole moment* p_i per ion. We can calculate the ionic polarizability α_i and the ionic contribution to the relative permittivity as a function of frequency by applying an ac field of the form $\mathcal{E} = \mathcal{E}_o \exp(j\omega t)$.

Consider two oppositely charged neighboring ions, *e.g.*, Na^+ and Cl^-, which experience forces $Q\mathcal{E}$ in opposite directions where Q is the magnitude of the ionic charge of each ion as shown in Figure 7.57. The bond between the ions becomes

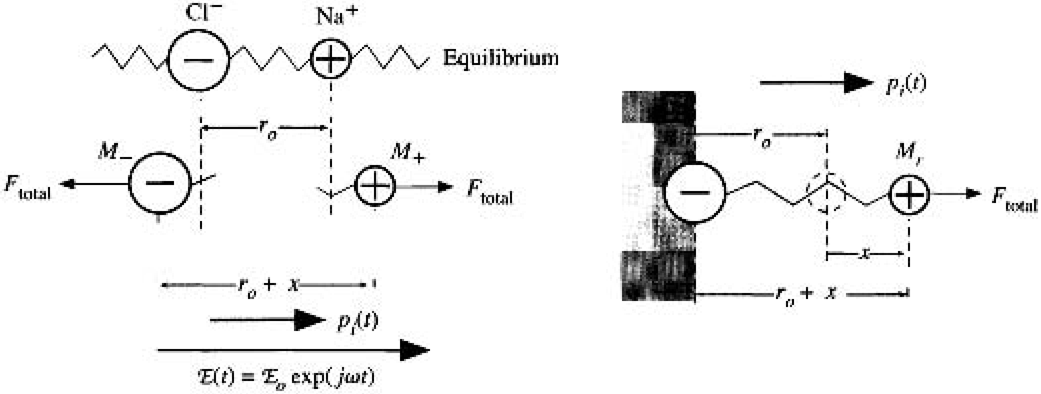

Figure 7.57 Consider a pair of oppositely charged ions. In the presence of an applied field $\mathcal{E}$ along x, the Na^+ and Cl^- ions are displaced from each other by a distance x. The net average (or induced) dipole moment is p_i.

stretched, and the two ions become displaced from the equilibrium separation r_o to a new separation $r_o + x$ as depicted in Figure 7.57. The force $F = Q\mathcal{E}$ of the applied field is the polarizing force, which causes the relative displacement. We take F to be along the x direction. The applied force is resisted by a **restoring force** F_r that is due to the stretching of the bond (Hooke's law) and is proportional to the amount of bond stretching, *i.e.*, $F_r = -\beta x$ where β is the **spring constant** associated with the ionic bond (easily calculated from the potential energy curve of the bond), and the negative sign ensures that F_r is directed in the opposite direction to the applied force. Thus, the net force acting on the ions is $Q\mathcal{E} - \beta x$. As the ions are oscillated by the applied force, they couple some of the energy in the applied field to lattice vibrations and this energy is then lost as heat (lattice vibrations) in the crystal. As in classical mechanics, this type of energy loss through a coupling mechanism can be represented as a **frictional force** (force associated with losses) F_{loss} that acts against the effect of the applied force. This frictional force is proportional to the velocity of the ions or dx/dt, so it is written as $F_{\text{loss}} = -\gamma(dx/dt)$ where γ is a proportionality constant that depends on the exact mechanism for the energy loss from the field, and the negative sign ensures that it is opposing the applied field. The total (net) force on the ions is

$$F_{\text{total}} = F + F_r + F_{\text{loss}} = Q\mathcal{E} - \beta x - \gamma\frac{dx}{dt}$$

Total force

Normally we would examine the equations of motion (Newton's second law) under forced oscillation for each ion separately, and then we would use the results to find the overall extension x. An equivalent procedure (as well known in mechanics) is to keep one ion stationary and allow the other one to oscillate with a reduced mass M_r, which is $M_r = (M_+M_-)/(M_+ + M_-)$ where M_+ and M_- are the masses of Na^+ and Cl^- ions, respectively. For example, we can simply examine the oscillations of the Na^+-ion within the reference frame of the Cl^--ion (kept "stationary") and attach a reduced mass M_r to Na$^+$ as depicted in Figure 7.57. Then Newton's second law gives

Forced oscillations of Na$^+$–Cl$^-$ ion pair

$$M_r \frac{d^2x}{dt^2} = Q\mathcal{E} - \beta x - \gamma \frac{dx}{dt} \tag{7.87}$$

It is convenient to put M_r and β together into a new constant ω_I which represents the **resonant or natural angular frequency** of the ionic bond, or the natural oscillations when the applied force is removed. Defining $\omega_I = (\beta/M_r)^{1/2}$ and γ_I as γ per unit reduced mass, *i.e.*, $\gamma_I = \gamma/M_r$, we have

Forced dipole oscillator, ionic polarization

$$\frac{d^2x}{dt^2} + \gamma_I \frac{dx}{dt} + \omega_I^2 x = \frac{Q}{M_r}\mathcal{E}_o \exp(j\omega t) \tag{7.88}$$

Equation 7.88 is a second-order differential equation for the induced displacement x of a pair of neighboring ions about the equilibrium separation as a result of an applied force $Q\mathcal{E}$. It is called the *forced oscillator* equation and is well known in mechanics. (The same equation would describe the damped motion of a ball attached to a spring in a viscous medium and oscillated by an applied force.) The solution to Equation 7.88 will give the displacement $x = x_o \exp(j\omega t)$, which will have the same time dependence as $\mathcal{E}$ but *phase shifted;* that is, x_o will be a complex number. The *relative* displacement of the ions from the equilibrium gives rise to a *net* or **induced polarization** $p_i = Qx$. Thus Equation 7.88 can be multiplied by Q to represent the forced oscillations of the induced dipole. Equation 7.88 is also called the **Lorentz dipole oscillator model.**

The induced dipole p_i will also be phase shifted with respect to the applied force $Q\mathcal{E}$. When we divide p_i by the applied field $\mathcal{E}$, we get the **ionic polarizability** α_i, given by

Ionic polarizability

$$\alpha_i = \frac{p_i}{\mathcal{E}} = \frac{Qx}{\mathcal{E}} = \frac{Q^2}{M_r\left(\omega_I^2 - \omega^2 + j\gamma_I\omega\right)} \tag{7.89}$$

It can be seen that the polarizability is also a complex number as we expect; there is a phase shift between $\mathcal{E}$ and induced p_i. It therefore has real α_i' and imaginary α_i'' parts and can be written as $\alpha_i = \alpha_i' - j\alpha_i''$. We note that, by convention, the imaginary part is written with a minus sign to keep α_i'' as a positive quantity. Further, when $\omega = 0$, under dc conditions, the ionic polarizability $\alpha_i(0)$ from Equation 7.89 is

DC ionic polarizability

$$\alpha_i(0) = \frac{Q^2}{M_r\omega_I^2} \tag{7.90}$$

The dc polarizability is a real quantity as there can be no phase shift under dc conditions. We can then write the ionic polarizability in Equation 7.89 in terms of the normalized frequency (ω/ω_I) as

AC ionic polarizability

$$\alpha_i(\omega) = \frac{\alpha_i(0)}{\left[1 - \left(\frac{\omega}{\omega_I}\right)^2 + j\left(\frac{\gamma_I}{\omega_I}\right)\left(\frac{\omega}{\omega_I}\right)\right]} \tag{7.91}$$

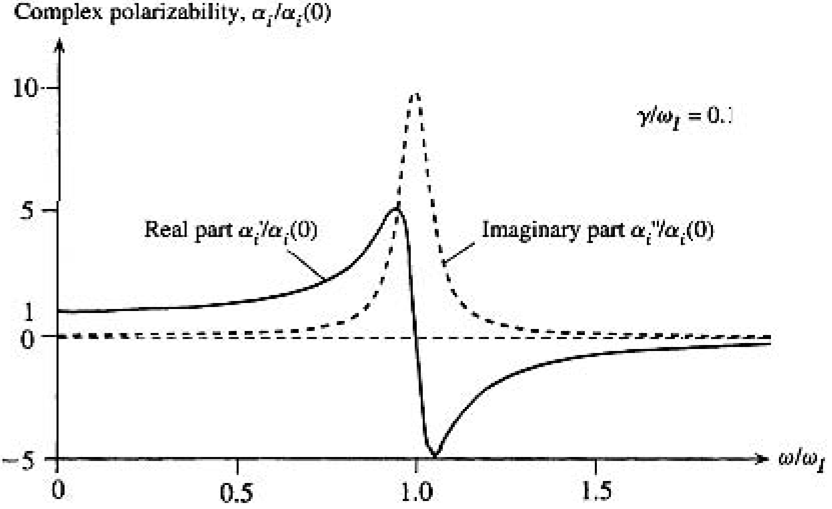

Figure 7.58 A schematic representation of the frequency dependence of the real and imaginary parts of normalized polarizability $\alpha_i/\alpha_i(0)$ versus ω/ω_I.

The dependences of the real and imaginary parts of α_i on the frequency of the field are shown in Figure 7.58 in terms of the normalized frequency (ω/ω_I) for one particular value of the loss factor, $\gamma_I = 0.1\omega_I$. Note that α_i'' peaks at a frequency very close to the ionic bond resonant frequency ω_I (it is exactly ω_I when $\gamma_I = 0$). The sharpness and magnitude of the α_i'' peak depends on the loss factor γ_I. The peak is sharper and higher for smaller γ_I. Notice that α_i' is nearly constant at frequencies lower than ω_I. Indeed, in a dc field, $\alpha_i' = \alpha_i(0)$. But, through ω_I, α_i' shows a rapid change from positive to negative values and then it tends toward zero for frequencies greater than ω_I.

Zero or negative α_i' should not be disconcerting since the actual magnitude of the polarizability is $|\alpha_i| = (\alpha_i'^2 + \alpha_i''^2)^{1/2}$, which is always positive through ω_I and maximum at ω_I. The phase of α_i however changes through ω_I. The phase of α_i, and hence the phase of the polarization with respect to the field, are zero at low frequencies $(\omega \ll \omega_I)$. As the frequency increases, the polarization lags behind the field and the phase of α_i becomes more negative. At $\omega = \omega_I$, the polarization lags behind the field by 90°. However, the rate of change of polarization is in phase with the field oscillations, which leads to a maximum energy transfer. At high frequencies, well above ω_I, the ions cannot respond to the rapidly changing field and the coupling between the field and the ions is negligible. The peak in the α_i'' versus ω behavior around $\omega = \omega_I$ is what is called the **dielectric resonance peak,** and in this particular case it is called the **ionic polarization relaxation peak** and is due to the strong coupling of the applied field with the natural vibrations of the ionic bond at $\omega = \omega_I$.

The resulting relative permittivity ε_r can be found from the Clausius–Mossotti equation. But we also have to consider the electronic polarizability α_e of the two types of ions since this type of polarization operates up to optical frequencies $(\omega \gg \omega_I)$, which means that

Dielectric constant of an ionic solid

$$\frac{\varepsilon_r(\omega) - 1}{\varepsilon_r(\omega) + 2} = \frac{N_i}{3\varepsilon_o}[\alpha_i + \alpha_{e+} + \alpha_{e-}] \qquad [7.92]$$

where N_i is the concentrations of negative and positive ion pairs (assuming an equal number of positive and negative ions), and α_{e+} and α_{e-} are the electronic polarizabilities of the negative and positive ion species, respectively. Inasmuch as α_i is a complex quantity, so is the relative permittivity $\varepsilon_r(\omega)$. We can express Equation 7.92 differently by noting that at very high frequencies, $\omega \gg \omega_I$, $\alpha_i = 0$, and the relative permittivity is then denoted as ε_{rop}. Equation 7.92 then becomes

Dispersion relation for ionic polarization

$$\frac{\varepsilon_r(\omega)-1}{\varepsilon_r(\omega)+2}-\frac{\varepsilon_{rop}-1}{\varepsilon_{rop}+2}=\frac{N_I\alpha_I}{3\varepsilon_o}=\frac{N_IQ^2}{3\varepsilon_o M_r\left(\omega_I^2-\omega^2+j\gamma_I\omega\right)} \qquad [7.93]$$

This is called the **dielectric dispersion relation** between the relative permittivity, due to ionic polarization, and the frequency of the electric field. Figure 7.16b shows the behavior of $\varepsilon_r(\omega)$ with frequency for KCl where ε_r'' peaks at $\omega = \omega_I = 2\pi(4.5 \times 10^{12})$ rad s^{-1} and ε_r' exhibits sharp changes around this frequency. It is clear that as ω gets close to ω_I, there are rapid changes in $\varepsilon_r(\omega)$. The resonant frequencies (ω_I) for ionic polarization relaxations are typically in the infrared frequency range, and the "applied" field in the crystal is then due to a propagating electromagnetic (EM) wave rather than an ac applied field between two external electrodes placed on the crystal.[15]

It should be mentioned that electronic polarization can also be described by the Lorentz oscillator model, and can also be represented by Equation 7.91 if we appropriately replace α_i by α_e and interpret ω_I and γ_I as the resonant frequency and loss factor involved in electronic polarization.

EXAMPLE 7.17

IONIC POLARIZATION RESONANCE IN KCl Consider a KCl crystal which has the FCC crystal structure and the following properties. The optical dielectric constant is 2.19, the dc dielectric constant is 4.84, and the lattice parameter a is 0.629 nm. Calculate the dc ionic polarizability $\alpha_i(0)$. Estimate the ionic resonance absorption frequency and compare the value with the experimentally observed resonance at 4.5×10^{12} Hz in Figure 7.16b. The atomic masses of K and Cl are 39.09 and 35.45 g mol^{-1}, respectively.

SOLUTION

At optical frequencies the dielectric constant ε_{rop} is determined by electronic polarization. At low frequencies and under dc conditions, the dielectric constant ε_{rdc} is determined by both electronic and ionic polarization. If N_i is the concentration of negative and positive ion pairs, then equation 7.93 becomes

$$\frac{\varepsilon_{rdc}-1}{\varepsilon_{rdc}+2}=\frac{\varepsilon_{rop}-1}{\varepsilon_{rop}+2}+\frac{1}{3\varepsilon_o}N_I\alpha_I(0)$$

There are four negative and positive ion pairs per unit cell, and the cell dimension is a. The concentration of negative and positive ion pairs N_i is

$$N_I=\frac{4}{a^3}=\frac{4}{(0.629\times10^{-9}\ \mathrm{m})^3}=1.61\times10^{28}\ \mathrm{m}^{-3}$$

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[15] More rigorous theories of ionic polarization would consider the interactions of a propagating electromagnetic wave with various phonon modes within the crystal, which is beyond the scope of this book.

Substituting $\varepsilon_{rdc} = 4.84$ and $\varepsilon_{rop} = 2.19$ and N_i in Equation 7.93

$$\alpha_i(0) = \frac{3\varepsilon_o}{N_i}\left[\frac{\varepsilon_{rdc}-1}{\varepsilon_{rdc}+2} - \frac{\varepsilon_{rop}-1}{\varepsilon_{rop}+2}\right] = \frac{3(8.85\times10^{-12})}{1.61\times10^{28}}\left[\frac{4.84-1}{4.84+2} - \frac{2.19-1}{2.19+2}\right]$$

we find

$$\alpha_i(0) = 4.58\times10^{-40}\ \mathrm{F\ m^2}$$

The relationship between $\alpha_i(0)$ and the resonance absorption frequency involves the reduced mass M_r of the K^+–Cl^- ion pair,

$$M_r = \frac{M_+M_-}{M_+ + M_-} = \frac{(39.09)(35.45)(10^{-3})}{(39.09+35.45)(6.022\times10^{23})} = 3.09\times10^{-26}\ \mathrm{kg}$$

At $\omega = 0$, the polarizability is given by Equation 7.90, so the resonance absorption frequency ω_I is

$$\omega_I = \left[\frac{Q^2}{M_r\alpha_i(0)}\right]^{1/2} = \left[\frac{(1.6\times10^{-19})^2}{(3.09\times10^{-26})(4.58\times10^{-40})}\right]^{1/2} = 4.26\times10^{13}\ \mathrm{rad\ s^{-1}}$$

or

$$f_I = \frac{\omega_I}{2\pi} = 6.8\times10^{12}\ \mathrm{Hz}$$

This is about a factor of 1.5 greater than the observed resonance absorption frequency of 4.5×10^{12} Hz. Typically one accounts for the difference by noting that the actual ionic charges may not be exactly $+e$ on K^+ and $-e$ on Cl^-, but Q is effectively $0.76e$. Taking $Q = 0.76e$ makes $f_I = 5.15\times10^{12}$ Hz, only 14 percent greater than the observed value. A closer agreement can be obtained by refining the simple theory and considering how many effective dipoles there are in the unit cell along the direction of the applied field.

7.13 DIELECTRIC MIXTURES AND HETEROGENEOUS MEDIA

Many dielectrics are composite materials; that is, they are mixtures of two or more different types of dielectric materials with different relative permittivities and loss factors. The simplest example is a porous dielectric which has small air pores randomly dispersed within the bulk of the material as shown in Figure 7.59a (analogous to a random raisin pudding). Another example would be a dielectric material composed of two distinctly different phases that are randomly mixed, as shown in Figure 7.59b, somewhat like a Swiss cheese that has air bubbles. We often need to find the overall or the **effective dielectric constant** ε_{reff} of the mixture, which is not a trivial problem.[16] This overall ε_{reff} can then be used to treat the mixture as if it were one dielectric substance with this particular dielectric constant; for example, the capacitance can be calculated

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[16] The theories that try to represent a heterogeneous medium in terms of effective quantities are called *effective medium theories* (or approximations). The theory of finding an effective dielectric constant of a mixture has intrigued many famous scientists in the past. Over the years, many quite complicated mixture rules have been developed, and there is no shortage of formulas in this field. Many engineers however still tend to use simple empirical rules to model a composite dielectric. The primary reason is that many theoretical mixture rules depend on the exact knowledge of the geometrical shapes, sizes, and distributions of the mixed phases.

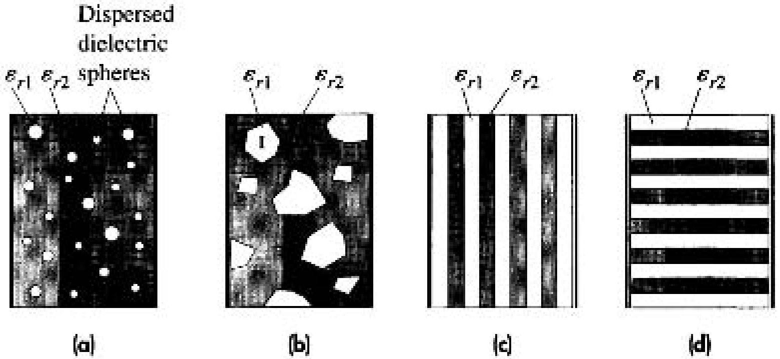


Figure 7.59 Heterogeneous dielectric media examples.
(a) Dispersed dielectric spheres in a dielectric matrix.
(b) A heterogeneous medium with two distinct phases I and II.
(c) Series mixture rule.
(d) Parallel mixture rule.

from $C = \varepsilon_o \varepsilon_{\text{reff}} A/d$ by simply using $\varepsilon_{\text{reff}}$. It should be emphasized that if mixing occurs at the atomic level so that the material is essentially a *solid solution*, then, in principle, the Clausius–Mossotti equation can be used in which we simply add the polarizabilities of each species of atoms or ions weighted by their concentration. (We did this for CsCl in Example 7.4.) The present problem examines **heterogeneous materials,** and hence excludes such solid solutions.

The theoretical treatment of mixtures can be quite complicated since one has to consider not only individual dielectric properties but also the geometrical shapes, sizes, and distributions of the two (or more) phases present in the composite material. In many cases, empirical rules that have been shown to work have been used to predict $\varepsilon_{\text{reff}}$. Consider a heterogeneous dielectric that has two mixed phases I and II with dielectric constants ε_{r1} and ε_{r2}, and volume fractions v_1 and v_2, respectively, $(v_1 + v_2 = 1)$ as in Figure 7.59b. One simple and useful mixture rule is

Generalized mixture rule

$$\varepsilon_{\text{reff}}^n = v_1 \varepsilon_{r1}^n + v_2 \varepsilon_{r2}^n \qquad [7.94]$$

where n is an index (a constant), usually determined empirically, that depends on the type of mixture. If we have a parallel stack of plates of I and II in alternating (or in random) sequence between the two electrodes, this would be like many series-connected dielectrics and n would be -1. If the phases are in parallel as plates of I and II stacked on top of each other, as shown in Figure 7.59d, then n is 1. As n approaches 0, Equation 7.94 can be shown to be equivalent to a **logarithmic mixture** rule:

Lichtenecker formula

$$\ln \varepsilon_{\text{reff}} = v_2 \ln \varepsilon_{r1} + v_2 \ln \varepsilon_{r2} \qquad [7.95]$$

which is known as the **Lichtenecker formula** (1926). Although its scientific basis is not strong, it has shown remarkable applicability to various heterogeneous media; perhaps due to the fact that it is a kind of compromise between the two extreme limits of series and parallel mixtures.

There is one particular mixture rule for dispersed dielectric spheres (with ε_{r1}), such as air pores, in a continuous dielectric matrix (with ε_{r2}), that works quite well for volume fractions up to about 20 percent, called the **Maxwell–Garnett formula**

Maxwell–Garnett formula

$$\frac{\varepsilon_{\text{reff}} - \varepsilon_{r2}}{\varepsilon_{\text{reff}} + 2\varepsilon_{r2}} = v_1 \frac{\varepsilon_{r1} - \varepsilon_{r2}}{\varepsilon_{r1} + 2\varepsilon_{r2}} \qquad [7.96]$$

The Maxwell–Garnett equation can predict the effective dielectric constant of many different types of dielectrics that have dispersed pores. There are other mixture rules, but the above are some of the common types.

EXAMPLE 7.18

LOW-κ POROUS DIELECTRICS FOR MICROELECTRONICS It was mentioned in Chapter 2 that today's high transistor density ICs have multilayers of metal interconnect lines that are separated by an **interlayer dielectric** (ILD). The speed of the chip (as limited by the RC time constant) depends on the overall interconnect capacitance, which depends on the relative permittivity $\varepsilon_{r\text{ILD}}$ of the ILD. The traditional ILD material has been SiO_2 with $\varepsilon_r = 3.9$. There is much research interest in finding suitable low-κ (also called low-k) materials for such ILD applications, especially in ultralarge-scale integration (ULSI). What is the required porosity in SiO_2 if its effective relative permittivity is to be 2.5?

SOLUTION

The Maxwell–Garnett equation is particularly useful for such porous media calculations. Substituting $\varepsilon_{r2} = 3.9$, $\varepsilon_{r1} = 1$ (air pores), and setting $\varepsilon_{\text{reff}} = 2.5$ in Equation 7.96 we have

$$\frac{2.5 - 3.9}{2.5 + 2(3.9)} = v_1 \frac{1 - 3.9}{1 + 2(3.9)}$$

and solving gives

$$v_1 = 0.412, \qquad \text{or} \qquad 41\% \text{ porosity}$$

Such porosity is achievable but it may have side effects such as poorer mechanical properties and lower breakdown voltage. Note that the Lichtenecker formula gives 32.6 percent porosity. As apparent from this example, there is a distinct advantage in starting with a dielectric that has a low initial ε_r, and then using porosity to lower ε_r further. For example, if we start with $\varepsilon_r = 3$, then the same 41 percent porosity will yield $\varepsilon_{\text{reff}} = 2.05$. Many polymeric materials have ε_r values ~ 2.5 and have been candidate materials for low-κ ILD applications in microelectronics.

CD Selected Topics and Solved Problems

Selected Topics

Static Dielectric Constant of Materials
Piezoelectric Materials and Devices: Elementary Concepts
Real and Imaginary Dielectric Constant
Conduction in Solid Insulating Materials

Solved Problems

Static Electronic Polarizability
Relative Permittivity of an Ionic Crystal at Low and Optical Frequencies
Piezoelectric Coefficients
Piezoelectric Spark Generator
Electric Field in Coaxial Cables: Double Layer Insulation for Controlling the Maximum Field

DEFINING TERMS

Boundary conditions relate the normal and tangential components of the electric field next to the boundary. The tangential component must be continuous through the boundary. Suppose that $\mathcal{E}_{n1}$ is the normal component of the field in medium 1 at the boundary and ε_{r1} is the relative permittivity in medium 1. Using a similar notation for medium 2, then the boundary condition is $\varepsilon_{r1}\mathcal{E}_{n1} = \varepsilon_{r2}\mathcal{E}_{n2}$.

Clausius–Mossotti equation relates the dielectric constant (ε_r), a macroscopic property, to the polarizability (α), a microscopic property.

Complex relative permittivity ($\varepsilon_r' + j\varepsilon_r''$) has a real part ($\varepsilon_r'$) that determines the charge storage ability and an imaginary part (ε_r'') that determines the energy losses in the material as a result of the polarization mechanism. The real part determines the capacitance through $C = \varepsilon_o\varepsilon_r' A/d$ and the imaginary part determines the electric power dissipation per unit volume as heat by $\mathcal{E}^2\omega\varepsilon_o\varepsilon_r''$.

Corona discharge is a local discharge in a gaseous atmosphere where the field is sufficiently high to cause dielectric breakdown, for example, by avalanche ionization.

Curie temperature T_C is the temperature above which ferroelectricity disappears, that is, the spontaneous polarization of the crystal is lost.

Debye equations attempt to describe the frequency response of the complex relative permittivity $\varepsilon_r' + j\varepsilon_r''$ of a dipolar medium through the use of a single relaxation time τ to describe the sluggishness of the dipoles driven by the external ac field.

Dielectric is a material in which energy can be stored by the polarization of the molecules. It is a material that increases the capacitance or charge storage ability of a capacitor. Ideally, it is a nonconductor of electrical charge so that an applied field does not cause a flow of charge but instead relative displacement of opposite charges and hence polarization of the medium.

Dielectric loss is the electrical energy lost as heat in the polarization process in the presence of an applied ac field. The energy is absorbed from the ac voltage and converted to heat during the polarization of the molecules. It should not be confused with conduction loss $\sigma\mathcal{E}^2$ or V^2/R.

Dielectric strength is the maximum field ($\mathcal{E}_{\text{br}}$) that can be sustained in a dielectric beyond which dielectric breakdown ensues; that is, there is a large conduction current through the dielectric shorting the plates.

Dipolar (orientational) polarization arises when randomly oriented polar molecules in a dielectric are rotated and aligned by the application of a field so as to give rise to a net average dipole moment per molecule. In the absence of the field, the dipoles (polar molecules) are randomly oriented and there is no average dipole moment per molecule. In the presence of the field, the dipoles are rotated, some partially and some fully, to align with the field and hence give rise to a net dipole moment per molecule.

Dipolar relaxation equation describes the time response of the induced dipole moment per molecule in a dipolar material in the presence of a time-dependent applied field. The response of the dipoles depends on their relaxation time, which is the mean time required to dissipate the stored electrostatic energy in the dipole alignment to heat through lattice vibrations or molecular collisions.

Dipole relaxation (dielectric resonance) occurs when the frequency of the applied ac field is such that there is maximum energy transfer from the ac voltage source to heat in the dielectric through the alternating polarization and depolarization of the molecules by the ac field. The stored electrostatic energy is dissipated through molecular collisions and lattice vibrations (in solids). The peak occurs when the angular frequency of the ac field is the reciprocal of the relaxation time.

Electric dipole moment exists when a positive charge $+Q$ is separated from a negative charge $-Q$. Even though the net charge is zero, there is nonetheless an electric dipole moment $\mathbf{p}$ given by $\mathbf{p} = Q\mathbf{x}$ where $\mathbf{x}$ is the distance vector from $-Q$ to $+Q$. Just as two charges exert a Coulombic force on each other, two dipoles also exert a force on each other that depends on the magnitudes of the dipoles, their separation, and orientation.

Electric susceptibility (χ_e) is a material quantity that measures the extent of polarization in the material per unit field. It relates the amount of polarization P at a point in the dielectric to the field $\mathcal{E}$ at that point via $P = \chi_e \varepsilon_o \mathcal{E}$. If ε_r is the relative permittivity, then $\chi_e = \varepsilon_r - 1$. Vacuum has no electric susceptibility.

Electromechanical breakdown and electrofracture are breakdown processes that directly or indirectly involve electric field–induced mechanical weakening, for example, crack propagation, or mechanical deformation that eventually lead to dielectric breakdown.

Electronic bond polarization is the displacement of valence electrons in the bonds in covalent solids (*e.g.*, Ge, Si). It is a collective displacement of the electrons in the bonds with respect to the positive nuclei.

Electronic polarization is the displacement of the electron cloud of an atom with respect to the positive nucleus. Its contribution to the relative permittivity of a solid is usually small.

External discharges are discharges or shorting currents over the surface of the insulator when the conductance of the surface increases as a result of surface contamination, for example, excessive moisture, deposition of pollutants, dirt, dust, and salt spraying. Eventually the contaminated surface develops sufficient conductance to allow discharge between the electrodes at a field below the normal breakdown strength of the insulator. Dielectric breakdown over the surface of an insulation is termed **surface tracking.**

Ferroelectricity is the occurrence of spontaneous polarization in certain crystals such as barium titanate ($BaTiO_3$). Ferroelectric crystals have a permanent polarization **P** as a result of spontaneous polarization. The direction of **P** can be defined by the application of an external field.

Gauss's law is a fundamental law of physics that relates the surface integral of the electric field over a closed (hypothetical) surface to the sum of all the charges enclosed within the surface. If $\mathcal{E}_n$ is the field normal to a small surface area dA and Q_{total} is the enclosed total charge, then over the whole closed surface $\varepsilon_o \oint \mathcal{E}_n \, dA = Q_{\text{total}}$.

Induced polarization is the polarization of a molecule as a result of its placement in an electric field. The induced polarization is along the direction of the field. If the molecule is already polar, then induced polarization is the additional polarization that arises due to the applied field alone and it is directed along the field.

Insulation aging is a term used to describe the physical and chemical deterioration in the properties of the insulation so that its dielectric breakdown characteristics worsen with time. Aging therefore determines the useful life of the insulation.

Interfacial polarization occurs whenever there is an accumulation of charge at an interface between two materials or between two regions within a material. Grain boundaries and electrodes are regions where charges generally accumulate and give rise to this type of polarization.

Internal discharges are partial discharges that take place in microstructural voids, cracks, or pores within the dielectric where the gas atmosphere (usually air) has lower dielectric strength. A porous ceramic, for example, would experience partial discharges if the field is sufficiently large. Initially, the pore size (or the number of pores) may be small and the partial discharge insignificant, but with time the partial discharge erodes the internal surfaces of the void. Eventually (and usually) an *electrical tree* type of discharge develops from a partial discharge that has been eroding the dielectric. The erosion of the dielectric by the partial discharge propagates like a branching tree. The "tree branches" are erosion channels, filaments of various sizes, in which gaseous discharge takes place and forms a conducting channel during operation.

Intrinsic breakdown or **electronic breakdown** commonly involves the avalanche multiplication of electrons (and holes in solids) by impact ionization in the presence of high electric fields. The large number of free carriers generated by the avalanche of impact ionizations leads to a runaway current between the electrodes and hence to insulation breakdown.

Ionic polarization is the relative displacement of oppositely charged ions in an ionic crystal that results in the polarization of the whole material. Typically, ionic polarization is important in ionic crystals below the infrared wavelengths.

Local field ($\mathcal{E}_{\text{loc}}$) is the true field experienced by a molecule in a dielectric that arises from the free charges on the plates and all the induced dipoles surrounding the molecule. The true field at a molecule is not simply the applied field (V/d) because of the field of the neighboring induced dipoles.

Loss tangent or **tan** δ is the ratio of the dielectric constant's imaginary part to the real part, $\varepsilon_r''/\varepsilon_r'$. The angle δ is the phase angle between the capacitive current and the total current. If there is no dielectric loss, then the two currents are the same and $\delta = 0$.

Partial discharge occurs when only a local region of the dielectric is exhibiting discharge, so the discharge does not directly connect the two electrodes.

Piezoelectric material has a noncentrosymmetric crystal structure that leads to the generation of a polarization vector *P*, or charges on the crystal surfaces, upon the application of a mechanical stress. When strained, a piezoelectric crystal develops an internal field and therefore exhibits a voltage difference between two of its faces.

PLZT, lead lanthanum zirconate titanate, is a PZT-type material with lanthanum occupying the Pb site.

Polarizability (α) is the ability of an atom or molecule to become polarized in the presence of an electric field. It is induced polarization in the molecule per unit field along the field direction.

Polarization is the separation of positive and negative charges in a system so that there is a net electric dipole moment per unit volume.

Polarization vector (P) measures the extent of polarization in a unit volume of dielectric matter. It is the vector sum of dielectric dipoles per unit volume. If $\mathbf{p}$ is the average dipole moment per molecule and n is the number of molecules per unit volume, then $\mathbf{P} = n\mathbf{p}$. In a polarized dielectric matter (*e.g.*, in an electric field), the bound surface charge density σ_p due to polarization is equal to the normal component of $\mathbf{P}$ at that point, $\sigma_p = P_{\text{normal}}$.

Poling is the application of a temporary electric field to a piezoelectric (or ferroelectric) material, generally at an elevated temperature, to align the polarizations of various grains and thereby develop piezoelectric behavior.

Pyroelectric material is a polar dielectric (such as barium titanate) in which a temperature change ΔT induces a proportional change ΔP in the polarization, that is, $\Delta P = p\,\Delta T$, where p is the pyroelectric coefficient of the crystal.

PZT is a general acronym for the lead zirconate titanate ($PbZrO_3$-$PbTiO_3$ or $PbTi_{0.48}Zr_{0.52}O_3$) family of crystals.

***Q*-factor** or **quality factor** for an impedance is the ratio of its reactance to its resistance. The *Q*-factor of a capacitor is X_c/R_p where $X_c = 1/\omega C$ and R_p is the equivalent parallel resistance that represents the dielectric and conduction losses. The *Q*-factor of a resonant circuit measures the circuit's peak response at the resonant frequency and also its bandwidth. The greater the *Q*, the higher the peak response and the narrower the bandwidth. For a series *RLC* resonant circuit,

$$Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o C R}$$

where ω_o is the resonant angular frequency, $\omega_o = 1/\sqrt{LC}$. The width of the resonant response curve between half-power points is $\Delta\omega = \omega_o/Q$.

Relative permittivity (ε_r) or **dielectric constant** of a dielectric is the fractional increase in the stored charge per unit voltage on the capacitor plates due to the presence of the dielectric between the plates (the whole space between the plates is assumed to be filled). Alternatively, we can define it as the fractional increase in the capacitance of a capacitor when the insulation between the plates is changed from a vacuum to a dielectric material, keeping the geometry the same.

Relaxation time (τ) is a characteristic time that determines the sluggishness of the dipole response to an applied field. It is the mean time for the dipole to lose its alignment with the field due to its random interactions with the other molecules through molecular collisions, lattice vibrations, and so forth.

Surface tracking is an *external dielectric breakdown* that occurs over the surface of the insulation.

Temperature coefficient of capacitance (TCC) is the fractional change in the capacitance per unit temperature change.

Thermal breakdown is a breakdown process that involves thermal runaway, which leads to a runaway current or discharge between the electrodes. If the heat generated by dielectric loss, due to ε_r'', or Joule heating, due to finite σ, cannot be removed sufficiently rapidly, then the temperature of the dielectric rises, which increases the conductivity and the dielectric loss. The increases in ε_r'' and σ lead to more heat generation and a further rise in the temperature, so thermal runaway ensues, followed by either a large shorting current or local thermal decomposition of the insulation accompanied by a partial discharge in this region.

Transducer is a device that converts electrical energy into another form of usable energy or vice versa. For example, piezoelectric transducers convert electrical energy to mechanical energy and vice versa.

QUESTIONS AND PROBLEMS

7.1 Relative permittivity and polarizability

a. Show that the local field is given by

$$\mathcal{E}_{\text{loc}} = \mathcal{E}\left(\frac{\varepsilon_r + 2}{3}\right)$$

Local field

b. Amorphous selenium (a-Se) is a high-resistivity semiconductor that has a density of approximately 4.3 g cm^{-3} and an atomic number and mass of 34 and 78.96, respectively. Its relative permittivity at 1 kHz has been measured to be 6.7. Calculate the relative magnitude of the local field in a-Se. Calculate the polarizability per Se atom in the structure. What type of polarization is this? How will ε_r depend on the frequency?

c. If the electronic polarizability of an isolated atom is given by

$$\alpha_e \approx 4\pi\varepsilon_o r_o^3$$

where r_o is the radius of the atom, then calculate the electronic polarizability of an isolated Se atom, which has $r_o = 0.12$ nm, and compare your result with that for an atom in a-Se. Why is there a difference?

7.2 Electronic polarization and SF$_6$ Because of its high dielectric strength, SF$_6$ (sulfur hexafluoride) gas is widely used as an insulator and a dielectric in HV applications such as HV transformers, switches, circuit breakers, transmission lines, and even HV capacitors. The SF$_6$ gas at 1 atm and at room temperature has a dielectric constant of 1.0015. The number of SF$_6$ molecules per unit volume N can be found by the gas law, $P = (N/N_A)RT$. Calculate the electronic polarizability α_e of the SF$_6$ molecule. How does α_e compare with the α_e versus Z line in Figure 7.4? (Note: The SF$_6$ molecule has no net dipole. Assume that the overall polarizability of SF$_6$ is due to electronic polarization.)

7.3 Electronic polarization in liquid xenon Liquid xenon has been used in radiation detectors. The density of the liquid is 3.0 g cm^{-3}. What is the relative permittivity of liquid xenon given its electronic polarizability in Table 7.1? (The experimental ε_r is 1.96.)

7.4 Relative permittivity, bond strength, bandgap, and refractive index Diamond, silicon, and germanium are covalent solids with the same crystal structure. Their relative permittivities are shown in Table 7.10.

a. Explain why ε_r increases from diamond to germanium.

b. Calculate the polarizability per atom in each crystal and then plot polarizability against the elastic modulus Y (Young's modulus). Should there be a correlation?

c. Plot the polarizability from part (*b*) against the bandgap energy E_g. Is there a relationship?

d. Show that the refractive index n is $\sqrt{\varepsilon_r}$. When does this relationship hold and when does it fail?

e. Would your conclusions apply to ionic crystals such as NaCl?

Table 7.10 Properties of diamond, Si, and Ge

	ε_r	M_{at}	Density (g cm^{-3})	α_e	Y (GPa)	E_g (eV)	n
Diamond	5.8	12	3.52		827	5.5	2.42
Si	11.9	28.09	2.33		190	1.12	3.45
Ge	16	72.61	5.32		75.8	0.67	4.09

7.5 **Dipolar liquids** Given the static dielectric constant of water as 80, its high-frequency dielectric constant (due to electronic polarization) as 4, and its density as 1 g cm^{-3}, calculate the permanent dipole moment p_o per water molecule assuming that it is the orientational and electronic polarization of individual molecules that gives rise to the dielectric constant. Use both the simple relationship in Equation 7.14 where the local field is the same as the macroscopic field and also the Clausius–Mossotti equation and compare your results with the permanent dipole moment of the water molecule which is 6.1×10^{-30} C m. What is your conclusion? What is ε_r calculated from the Clausius–Mossotti equation taking the true p_o (6.1×10^{-30} C m) of a water molecule? (Note: Static dielectric constant is due to both orientational and electronic polarization. The Clausius–Mossotti equation does not apply to dipolar materials because the local field is not described by the Lorentz field.)

7.6 **Dielectric constant of water vapor or steam** The isolated water molecule has a permanent dipole p_o of 6.1×10^{-30} C m. The electronic polarizability α_e of the water molecule under dc conditions is about 4×10^{-40} C m. What is the dielectric constant of steam at a pressure of 10 atm (10×10^5 Pa) and at a temperature of 400 °C? [Note: The number of water molecules per unit volume N can be found from the simple gas law, $P = (N/N_A)RT$. The Clausius–Mossotti equation does not apply to orientational polarization. Since N is small, use Equation 7.14.]

7.7 **Dipole moment in a nonuniform electric field** Figure 7.60 shows an electric dipole moment p in a nonuniform electric field. Suppose the gradient of the field is $d\mathcal{E}/dx$ at the dipole p, and the dipole is oriented to be along the direction of increasing $\mathcal{E}$ as in Figure 7.60. Show that the *net force* acting on this dipole is given by

Net force on a dipole

$$F = p\frac{d\mathcal{E}}{dx}$$

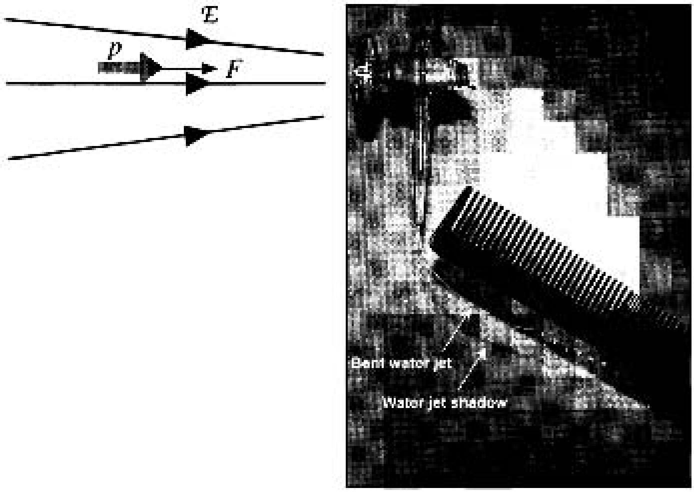


Figure 7.60
Left: A dipole moment in a nonuniform field experiences a net force F that depends on the dipole moment p and the field gradient $d\mathcal{E}/dx$.
Right: When a charged comb (by combing hair) is brought close to a water jet, the field from the comb polarizes the liquid by orientational polarization. The induced polarization vector P and hence the liquid is attracted to the comb where the field is higher.

Which direction is the force? What happens to this net force when the dipole moment is facing the direction of decreasing field? Given that a dipole normally also experiences a torque as described in Section 7.3.2, explain qualitatively what happens to a randomly placed dipole in a nonuniform electric field. Explain the experimental observation of bending a flow of water by a nonuniform field from a charged comb as shown in the photograph in Figure 7.60? (Remember that a dielectric medium placed in a field develops polarization P directed along the field.)

7.8 **Ionic and electronic polarization** Consider a CsBr crystal that has the CsCl unit cell crystal structure (one Cs^+–Br^- pair per unit cell) with a lattice parameter (a) of 0.430 nm. The electronic polarizability of Cs^+ and Br^- ions are 3.35×10^{-40} F m^2 and 4.5×10^{-40} F m^2, respectively, and the mean ionic polarizability per ion pair is 5.8×10^{-40} F m^2. What is the low-frequency dielectric constant and that at optical frequencies?

7.9 **Electronic and ionic polarization in KCl** KCl has the same crystal structure as NaCl. KCl's lattice parameter is 0.629 nm. The ionic polarizability per ion pair (per K^+–Cl^- ion) is 4.58×10^{-40} F m^2. The electronic polarizability of K^+ is 1.26×10^{-40} F m^2 and that of Cl^- is 3.41×10^{-40} F m^2. Calculate the dielectric constant under dc operation and at optical frequencies. Experimental values are 4.84 and 2.19.

7.10 **Debye relaxation** We will test the Debye equations for approximately calculating the real and imaginary parts of the dielectric constant of water just above the freezing point at 0.2 °C. Assume the following values in the Debye equations for water: $\varepsilon_{rdc} = 87.46$ (dc), $\varepsilon_{r\infty} = 4.87$ (at $f = 300$ GHz well beyond the relaxation peak), and $\tau = 1/\omega_o = (2\pi 9.18\text{ GHz})^{-1} = 0.017$ ns. Calculate the real and imaginary, ε_r' and ε_r'', parts of ε_r for water at frequencies in Table 7.11, and plot both the experimental values and your calculations on a linear–log plot (frequency on the log axis). What is your conclusion? (Note: It is possible to obtain a better agreement by using two relaxation times or using more sophisticated models.)

Table 7.11 Dielectric properties of water at 0.2 °C

	f (GHz)												
	0.3	**0.5**	**1**	**1.5**	**3**	**5**	**9.18**	**10**	**20**	**40**	**70**	**100**	**300**
ε_r'	87.46	87.25	86.61	85.34	76.20	68.19	46.13	42.35	19.69	10.16	7.20	6.14	4.87
ε_r''	2.60	4.50	8.85	13.18	24.28	34.53	40.55	40.24	30.23	17.68	11.15	8.31	3.68

I SOURCE: Data extracted from R. Buchner et al., *Chem. Phys. Letts*, **306**, 57, 1999.

***7.11** **Debye and non-Debye relaxation and Cole–Cole plots** Consider the Debye equation

$$\varepsilon_r = \varepsilon_{r\infty} + \frac{\varepsilon_{rdc} - \varepsilon_{r\infty}}{1 + j\omega\tau}$$

Debye relaxation

and also the **generalized dielectric relaxation** equation, which "stretches" (broadens) the Debye function,

$$\varepsilon_r = \varepsilon_{r\infty} + \frac{\varepsilon_{rdc} - \varepsilon_{r\infty}}{[1 + (j\omega\tau)^{\alpha}]^{\beta}}$$

Generalized dielectric relaxation

Take $\tau = 1$, $\varepsilon_{rdc} = 5$, $\varepsilon_{r\infty} = 2$, and $\alpha = 0.8$, and $\beta = 1$. Plot the real and imaginary parts of ε_r versus frequency (on a log scale) for both functions from $\omega = 0, 0.1/\tau, 1/3\tau, 1/\tau, 3/\tau$, and 10τ. For the same ω values, plot ε_r'' versus ε_r' (Cole–Cole plot) for both functions using a graph in which the x and y axes have the same divisions. What is your conclusion?

7.12 **Equivalent circuit of a polyester capacitor** Consider a 1 nF polyester capacitor that has a polymer (PET) film thickness of 1 µm. Calculate the equivalent circuit of this capacitor at 50 °C and at 120 °C for operation at 1 kHz. What is your conclusion?

7.13 **Student microwaves mashed potatoes** A microwave oven uses electromagnetic waves at 2.48 GHz to heat food by dielectric loss, that is, making use of ε_r'' of the food material, which normally has substantial water content. An undergraduate student microwaves 10 cm^3 of mashed potatoes in 60 seconds. The microwave generates an rms field of $\mathcal{E}_{\text{rms}}$ of 200 V cm^{-1} in mashed potatoes. At 2.48 GHz, mashed potatoes have $\varepsilon_r'' = 21$. Calculate the average power dissipated per cm^3, and also the total energy dissipated heating the food. (Note: You can use $\mathcal{E}_{\text{rms}}$ instead of E in Equation 7.32.)

7.14 **Dielectric loss per unit capacitance** Consider the three dielectric materials listed in Table 7.12 with the real and imaginary dielectric constants ε_r' and ε_r''. At a given voltage, which dielectric will have the lowest power dissipation per unit capacitance at 1 kHz and at an operating temperature of 50 °C? Is this also true at 120 °C?

Table 7.12 Dielectric properties of three insulators at 1 kHz

	$T = 50\ ^\circ\text{C}$		$T = 120\ ^\circ\text{C}$	
Material	ε_r'	ε_r''	ε_r'	ε_r''
Polycarbonate	2.47	0.003	2.535	0.003
PET	2.58	0.003	2.75	0.027
PEEK	2.24	0.003	2.25	0.003

| SOURCE: Data taken using a DEA by Kasap and Nomura [1995].

7.15 **Parallel and series equivalent circuits** Figure 7.61 shows simplified parallel and series equivalent circuits for a capacitor. The elements R_p and C_p in the parallel circuit and the elements R_s and C_s in the series circuit are related. We can write down the impedance Z_{AB} between the terminals A and B for both the circuits, and then equate $Z_{AB}(\text{parallel}) = Z_{AB}(\text{series})$. Show that

Equivalent series resistance and capacitance

$$R_s = \frac{R_p}{1 + (\omega R_p C_p)^2} \quad \text{and} \quad C_s = C_p\left[1 + \frac{1}{(\omega R_p C_p)^2}\right]$$

and similarly by considering the admittance (1/impedance),

Equivalent series resistance and capacitance

$$R_p = R_s\left[1 + \frac{1}{(\omega R_s C_s)^2}\right] \quad \text{and} \quad C_p = \frac{C_s}{1 + (\omega R_s C_s)^2}$$

A 10 nF capacitor operating at 1 MHz has a parallel equivalent resistance of 100 kΩ. What are C_s and R_s?

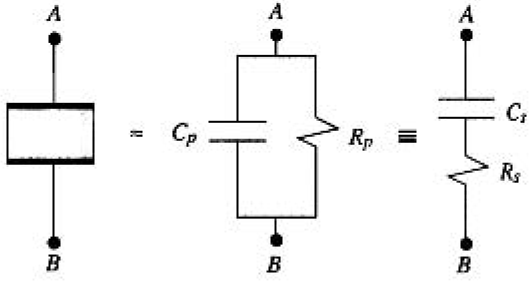


Figure 7.61 An equivalent parallel R_p and C_p circuit is equivalent to a series R_s and C_s circuit. The elements R_p and C_p in the parallel circuit are related to the elements R_s and C_s in the series circuit.

7.16 **Tantalum capacitors** Electrolytic capacitors tend to be modeled by a series $R_s + j\omega C_s$ equivalent circuit. A nominal 22 μF Ta capacitor (22 μF at low frequencies) has the following properties at 10 kHz:

$\varepsilon_r' \approx 20$ (at this frequency), $\tan\delta \approx 0.05$, dielectric thickness $d = 0.16$ µm, effective area $A = 150$ cm^2. Calculate C_p, R_p, C_s, and R_s.

7.17 **Tantalum versus niobium oxide capacitors** Niobium oxide (Nb_2O_5) is a competing dielectric to Ta_2O_5 (the dielectric in the tantalum capacitor). The dielectric constants are 41 for Nb_2O_5 and 27 for Ta_2O_5. For operation at the same voltage, the Ta_2O_5 thickness is 0.17 µm, and that of Nb_2O_5 is 0.25 µm. Explain why the niobium oxide capacitor is superior (or inferior) to the Ta capacitor. (Use a quantitative argument, such as the capacitance per unit volume.) What other factors would you consider if you were choosing between the two?

***7.18** **TCC of a polyester capacitor** Consider the parallel plate capacitor equation

$$C = \frac{\varepsilon_o \varepsilon_r x y}{z}$$

where ε_r is the relative permittivity (or ε_r'), x and y are the side lengths of the dielectric so that xy is the area A, and z is the thickness of the dielectric. The quantities ε_r, x, y, and z change with temperature. By differentiating this equation with respect to temperature, show that the **temperature coefficient of capacitance** (TCC) is

Temperature coefficient of capacitance

$$\text{TCC} = \frac{1}{C}\frac{dC}{dT} = \frac{1}{\varepsilon_r}\frac{d\varepsilon_r}{dT} + \lambda$$

where λ is the linear expansion coefficient defined by

$$\lambda = \frac{1}{L}\frac{dL}{dT}$$

where L stands for any length of the material (x, y, or z). Assume that the dielectric is isotropic and λ is the same in all directions. Using ε_r' versus T behavior in Figure 7.62 and taking $\lambda = 50 \times 10^{-6}$ K^{-1} as a typical value for polymers, predict the TCC at room temperature and at 10 kHz.

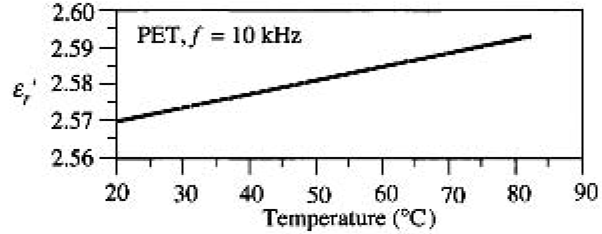


Figure 7.62 Temperature dependence of ε_r' at 10 kHz.

SOURCE: Data taken by Kasap and Maeda (1995).

7.19 **Dielectric breakdown of gases and Paschen curves** Dielectric breakdown in gases typically involves the avalanche ionization of the gas molecules by energetic electrons accelerated by the applied field. The mean free path between collisions must be sufficiently long to allow the electrons to gain sufficient energy from the field to impact ionize the gas molecules. The breakdown voltage V_{br} between two electrodes depends on the distance d between the electrodes as well as the gas pressure P, as shown in Figure 7.63. V_{br} versus Pd plots are called **Paschen curves.** We consider gaseous insulation, air and SF_6, in an HV switch.

a. What is the breakdown voltage between two electrodes of a switch separated by a 5 mm gap with air at 1 atm when the gaseous insulation is air and when it is SF_6?

b. What are the breakdown voltages in the two cases when the pressure is 10 times greater? What is your conclusion?

c. At what pressure is the breakdown voltage a minimum?

d. What air gap spacing d at 1 atm gives the minimum breakdown voltage?

e. What would be the reasons for preferring gaseous insulation over liquid or solid insulation?

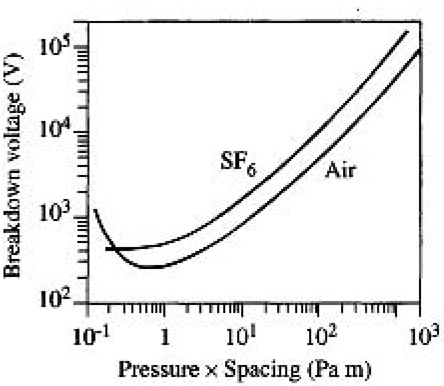


Figure 7.63 Breakdown voltage versus [pressure × electrode spacing] (Paschen curves).

***7.20** **Capacitor design** Consider a nonpolarized 100 nF capacitor design at 60 Hz operation. Note that there are three candidate dielectrics, as listed in Table 7.13.

a. Calculate the volume of the 100 nF capacitor for each dielectric, given that they are to be used under low voltages and each dielectric has its minimum fabrication thickness. Which one has the smallest volume?

b. How is the volume affected if the capacitor is to be used at a 500 V application and the maximum field in the dielectric must be a factor of 2 less than the dielectric strength? Which one has the smallest volume?

c. At a 500 V application, what is the power dissipated in each capacitor at 60 Hz operation? Which one has the lowest dissipation?

Table 7.13 Comparison of dielectric properties at 60 Hz (typical values)

	Polymer Film PET	**Ceramic TiO_2**	**High-K Ceramic ($BaTiO_3$ based)**
Name	Polyester	Polycrystalline titania	X7R
ε_r'	3.2	90	1800
$\tan\delta$	5×10^{-3}	4×10^{-4}	5×10^{-2}
$\mathcal{E}_{br}$ (kV cm^{-1})	150	50	100
Typical minimum thickness	1–2 μm	10 μm	10 μm

***7.21** **Dielectric breakdown in a coaxial cable** Consider a coaxial underwater high-voltage cable as in Figure 7.64a. The current flowing through the inner conductor generates heat, which has to flow through the dielectric insulation to the outer conductor where it will be carried away by conduction and convection. We will assume that steady state has been reached and the inner conductor is carrying a dc current I. Heat generated per unit second $Q' = dQ/dt$ by Joule heating of the inner conductor is

Rate of heat generation

$$Q' = RI^2 = \frac{\rho L I^2}{\pi a^2} \qquad [7.97]$$

where ρ is the resistivity, a the radius of the conductor, and L the cable length.

This heat flows radially out from the inner conductor through the dielectric insulator to the outer conductor, then to the ambient. This heat flow is by thermal conduction through the dielectric. The rate of heat flow Q' depends on the temperature difference $T_i - T_o$ between the inner and outer conductors;

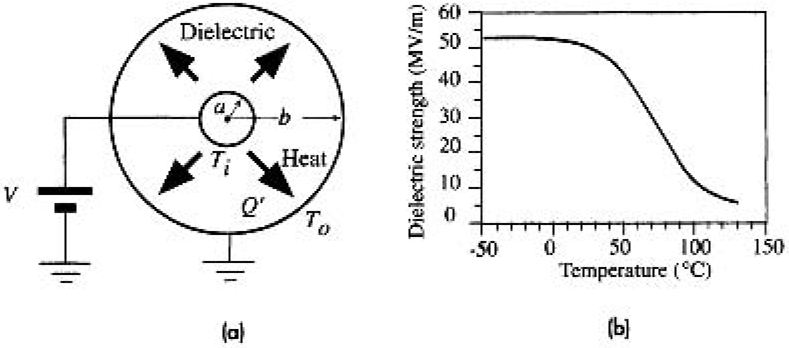


Figure 7.64

(a) The Joule heat generated in the core conductor flows outward radially through the dielectric material.

(b) Typical temperature dependence of the dielectric strength of a polyethylene-based polymeric insulation.

on the sample geometry (a, b, and L); and on the thermal conductivity κ of the dielectric. From elementary thermal conduction theory, this is given by

Rate of heat conduction

$$Q' = (T_i - T_o)\frac{2\pi\kappa L}{\ln\left(\frac{b}{a}\right)} \qquad [7.98]$$

The inner core temperature T_i rises until, in the steady state, the rate of Joule heat generation by the electric current in Equation 7.97 is just removed by the rate of thermal conduction through the dielectric insulation, given by Equation 7.98.

a. Show that the inner conductor temperature is

Steady-state inner conductor temperature

$$T_i = T_o + \frac{\rho I^2}{2\pi^2 a^2 \kappa}\ln\left(\frac{b}{a}\right) \qquad [7.99]$$

b. The breakdown occurs at the maximum field point, which is at $r = a$, just outside the inner conductor and is given by (see Example 7.11).

Maximum field in a coaxial cable

$$\mathcal{E}_{\max} = \frac{V}{a\ln\left(\frac{b}{a}\right)} \qquad [7.100]$$

The dielectric breakdown occurs when $\mathcal{E}_{\max}$ reaches the dielectric strength $\mathcal{E}_{br}$. However the dielectric strength $\mathcal{E}_{br}$ for many polymeric insulation materials depends on the temperature, and generally it decreases with temperature, as shown for a typical example in Figure 7.64b. If the load current I increases, then more heat Q' is generated per second and this leads to a higher inner core temperature T_i by virtue of Equation 7.99. The increase in T_i with I eventually lowers $\mathcal{E}_{br}$ so much that it becomes equal to $\mathcal{E}_{\max}$ and the insulation breaks down (thermal breakdown). Suppose that a certain coaxial cable has an aluminum inner conductor of diameter 10 mm and resistivity 27 nΩ m. The insulation is 3 mm thick and is a polyethylene-based polymer whose long-term dc dielectric strength is shown in Figure 7.64b. Suppose that the cable is carrying a voltage of 40 kV and the outer shield temperature is the ambient temperature, 25 °C. Given that the thermal conductivity of the polymer is about 0.3 W K^{-1} m^{-1}, at what dc current will the cable fail?

c. Rederive T_i in Equation 7.99 by considering that ρ depends on the temperature as $\rho = \rho_o[1 + \alpha_o(T - T_o)]$ (Chapter 2). Recalculate the maximum current in b given that $\alpha_o = 3.9 \times 10^{-3}$ °C^{-1} at 25 °C.

7.22 **Piezoelectricity** Consider a quartz crystal and a PZT ceramic filter both designed for operation at $f_s = 1$ MHz. What is the bandwidth of each? Given Young's modulus (Y), density (ρ) for each, and that the filter is a disk with electrodes and is oscillating radially, what is the diameter of the disk for each material? For quartz, $Y = 80$ GPa and $\rho = 2.65$ g cm^{-3}. For PZT, $Y = 70$ GPa and $\rho = 7.7$ g m^{-3}.

Assume that the velocity of mechanical oscillations in the crystal is $v = \sqrt{Y/\rho}$ and the wavelength $\lambda = v/f_s$. Consider only the fundamental mode ($n = 1$).

Piezoelectric voltage coefficient

7.23 **Piezoelectric voltage coefficient** The application of a stress T to a piezoelectric crystal leads to a polarization P and hence to an electric field $\mathcal{E}$ in the crystal such that

$$\mathcal{E} = gT$$

where g is the *piezoelectric voltage coefficient*. If $\varepsilon_o\varepsilon_r$ is the permittivity of the crystal, show that

$$g = \frac{d}{\varepsilon_o\varepsilon_r}$$

A $BaTiO_3$ sample, along a certain direction (called 3), has $d = 190$ pC N^{-1}, and its $\varepsilon_r \approx 1900$ along this direction. What do you expect for its g coefficient for this direction and how does this compare with the measured value of approximately 0.013 m^2 C^{-1}?

7.24 **Piezoelectricity and the piezoelectric bender**

a. Consider using a piezoelectric material in an application as a mechanical positioner where the displacements are expected to be small (as in a scanning tunneling microscope). For the piezoelectric plate shown in Figure 7.65a, we will take $L = 20$ mm, $W = 10$ mm, and D (thickness) = 0.25 mm. Under an applied voltage of V, the plate changes length, width, and thickness according to the piezoelectric coefficients d_{ij}, relating the applied field along i to the resulting strain along j.

Suppose we define direction 3 along the thickness D and direction 1 along the length L, as shown in Figure 7.65a. Show that the changes in the thickness and length are

Piezoelectric effects

$$\delta D = d_{33}\, V$$

$$\delta L = \left(\frac{L}{D}\right) d_{31}\, V$$

Given $d_{33} \approx 500 \times 10^{-12}$ m V^{-1} and $d_{31} \approx -250 \times 10^{-12}$ m V^{-1}, calculate the changes in the length and thickness for an applied voltage of 100 V. What is your conclusion?

b. Consider two oppositely poled and joined ceramic plates, A and B, forming a bimorph, as shown in Figure 7.65b. This piezoelectric bimorph is mounted as a cantilever; one end is fixed and the other end is free to move. Oppositely poled means that the electric field elongates A and contracts B, and the two relative motions *bend* the plate. The displacement h of the tip of the cantilever is given by

Piezoelectric bending

$$h = \frac{3}{2} d_{31} \left(\frac{L}{D}\right)^2 V$$

What is the deflection of the cantilever for an applied voltage of 100 V? What is your conclusion?

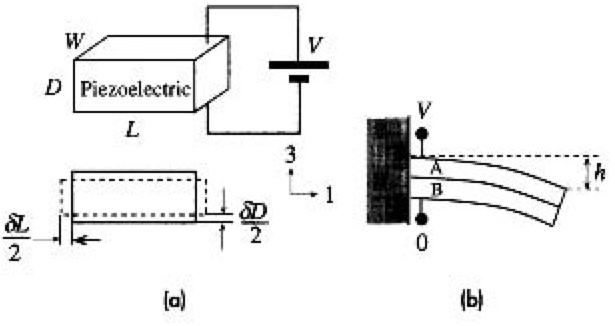


Figure 7.65
(a) A mechanical positioner using a piezoelectric plate under an applied voltage of V.
(b) A cantilever-type piezoelectric bender. An applied voltage bends the cantilever.

7.25 **Piezoelectricity** The wavelength λ of mechanical oscillations in a piezoelectric slab satisfies

$$n\left(\frac{1}{2}\lambda\right) = L$$

where n is an integer, L is the length of the slab along which mechanical oscillations are set up, and the wavelength λ is determined by the frequency f and velocity v of the waves. The ultrasonic wave velocity v depends on Young's modulus Y as

$$v = \left(\frac{Y}{\rho}\right)^{1/2}$$

where ρ is the density. For quartz, $Y = 80$ GPa and $\rho = 2.65$ g cm^{-3}. Considering the fundamental mode ($n = 1$), what are practical dimensions for crystal oscillators operating at 1 kHz and 1 MHz?

7.26 **Pyroelectric detectors** Consider two different radiation detectors using PZT and PVDF as pyroelectric materials whose properties are summarized in Table 7.14. The receiving area is 4 mm^2. The thicknesses of the PZT ceramic and the PVDF polymer film are 0.1 mm and 0.005 mm, respectively. In both cases the incident radiation is chopped periodically to allow the radiation to pass for a duration of 0.05 s.

a. Calculate the magnitude of the output voltage for each detector if both receive a radiation of intensity 10 μW cm^{-2}. What is the corresponding current in the circuit? In practice, what would limit the magnitude of the output voltage?

b. What is the minimum detectable radiation intensity if the minimum detectable signal voltage is 10 nV?

Table 7.14 Properties of PZT and PVDF

	ε'_r	Pyroelectric Coefficient ($\times 10^{-6}$ C m^{-2} K^{-1})	Density (g cm^{-3})	Heat Capacity (J K^{-1} g^{-1})
PZT	290	380	7.7	0.3
PVDF	12	27	1.76	1.3

7.27 **$LiTaO_3$ pyroelectric detector** $LiTaO_3$ (lithium tantalate) detectors are available commercially. $LiTaO_3$ has the following properties: pyroelectric coefficient $p \approx 200 \times 10^{-6}$ nC m^{-2} K^{-1}, density $\rho = 7.5$ g cm^{-3}, specific heat capacity $c_s = 0.43$ J K^{-1} g^{-1}. A particular detector has a cylindrical crystal with a diameter of 10 mm and thickness of 0.2 mm. Suppose we chop the input radiation and allow the radiation to fall on the detector for short periods of time. Each input radiation pulse has a duration of $\Delta t = 10$ ms. (The time between the radiation pulses is long, so consider only the response of the detector to a single pulse of radiation.) Suppose that all the incident radiation is absorbed. If the input radiation has an intensity of 10 μW cm^{-2}, calculate the pyroelectric current, and the maximum possible output voltage that can be generated assuming that the input impedance of the amplifier is sufficiently large to be negligible. What is the current responsivity of this detector? What are the major assumptions in your calculation of the voltage signal?

*7.28 **Pyroelectric detectors** Consider a typical pyroelectric radiation detector circuit as shown in Figure 7.66. The FET circuit acts as a voltage follower (source follower). The resistance R_1 represents the input resistance of the FET in parallel with a bias resistance that is usually inserted between the gate and source. C_1 is the overall input capacitance of the FET including any stray capacitance but excluding the capacitance of the pyroelectric detector. Suppose that the incident radiation intensity is constant and equal to $\mathcal{I}$. Emissivity η of a surface characterizes what fraction of the incident radiation that is absorbed? $\eta\mathcal{I}$ is the energy absorbed per unit area per unit time. Some of the absorbed energy will increase

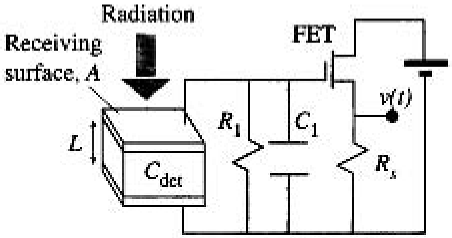


Figure 7.66 A pyroelectric detector with an FET voltage follower circuit.

the temperature of the detector and some of it will be lost to surroundings by thermal conduction and convection. Let the detector receiving area be A, thickness be L, density be ρ, and specific heat capacity (heat capacity per unit mass) be c. The heat losses will be proportional to the temperature difference between the detector temperature T and the ambient temperature T_o, as well as the surface area A (much greater than L). Energy balance requires that

Rate of increase in the internal energy (heat content) of the detector
= Rate of energy absorption − Rate of heat losses

that is,

$$(AL\rho)c\frac{dT}{dt} = A\eta I - KA(T - T_o)$$

where K is a constant of proportionality that represents the heat losses and hence depends on the thermal conductivity κ. If the heat loss involves pure thermal conduction from the detector surface to the detector base (detector mount), then $K = \kappa/L$. In practice, this is generally not the case and $K = \kappa/L$ is an oversimplification.

a. Show that the temperature of the detector rises exponentially as

Detector temperature

$$T = T_o + \frac{\eta I}{K}\left[1 - \exp\left(-\frac{t}{\tau_{\text{th}}}\right)\right]$$

where τ_{th} is a **thermal time constant** defined by $\tau_{\text{th}} = L\rho c/K$. Further show that for very small K, this equation simplifies to

$$T = T_o + \frac{\eta I}{L\rho c}t$$

b. Show that temperature change dT in time dt leads to a pyroelectric current i_p given by

Pyroelectric current

$$i_p = Ap\frac{dT}{dt} = \frac{Ap\eta I}{L\rho c}\exp\left(-\frac{t}{\tau_{\text{th}}}\right)$$

where p is the pyroelectric coefficient. What is the initial current?

c. The voltage across the FET and hence the output voltage $v(t)$ is given by

Pyroelectric detector output voltage

$$v(t) = V_o\left[\exp\left(-\frac{t}{\tau_{\text{th}}}\right) - \exp\left(-\frac{t}{\tau_{\text{el}}}\right)\right]$$

where V_o is a constant and τ_{el} is the **electrical time constant** given by R_1C_t, where C_t, total capacitance, is $(C_1 + C_{\text{det}})$, where C_{det} is the capacitance of the detector. Consider a particular PZT pyroelectric detector with an area of 1 mm^2 and a thickness of 0.05 mm. Suppose that this PZT has $\varepsilon_r = 250$, $\rho = 7.7$ g cm^{-3}, $c = 0.3$ J K^{-1} g^{-1}, and $\kappa = 1.5$ W K^{-1} m^{-1}. The detector is connected to an FET circuit that has $R_1 = 10$ MΩ and $C_1 = 3$ pF. Taking the thermal conduction loss constant K as κ/L, and $\eta = 1$, calculate τ_{th} and τ_{el}. Sketch schematically the output voltage. What is your conclusion?

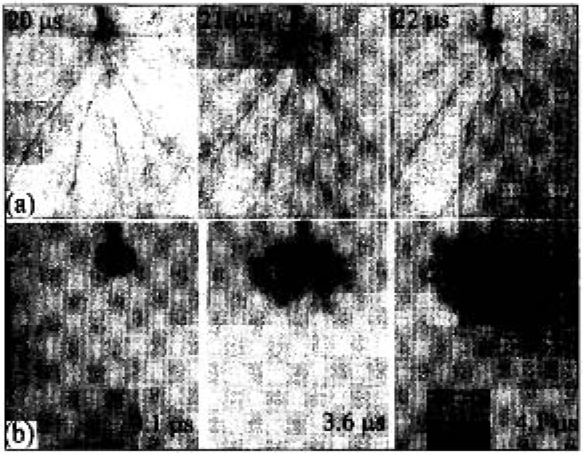
7.29 **Spark generator design** Design a PLZT piezoelectric spark generator using two back-to-back PLZT crystals that provide a 60 μJ spark in an air gap of 0.5 mm from a force of 50 N. At 1 atm in an air gap of 0.5 mm, the breakdown voltage is about 3000 V. The design will need to specify the dimensions of the crystal and the dielectric constant. Assume that the piezoelectric voltage coefficient is 0.023 V m N^{-1}.

7.30 **Ionic polarization resonance in CsCl** Consider a CsCl crystal which has the following properties. The optical dielectric constant is 2.62, the dc dielectric constant is 7.20, and the lattice parameter *a* is 0.412 nm. There is only one ion pair (Cs^+–Cl^-) in the cubic-type unit cell. Calculate (estimate) the ionic resonance absorption frequency and compare the value with the experimentally observed resonance at 3.1×10^{12} Hz. What effective value of *Q* would bring the calculated value to within 10 percent of the experimental value?

7.31 **Low-κ porous dielectrics for microelectronics** Interconnect technologies need lower ε_r interlayer dielectrics (ILDs) to minimize the interconnect capacitances. These materials are called **low-κ dielectrics.**

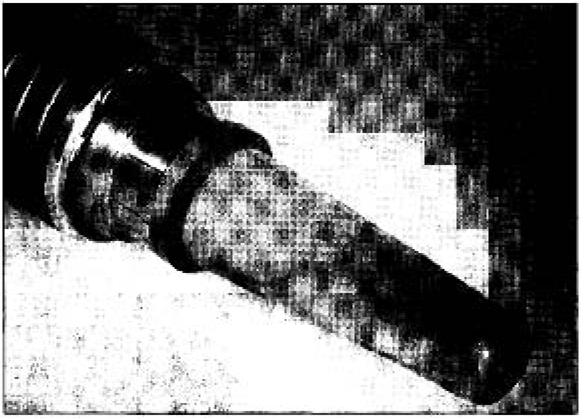
a. Consider fluorinated silicon dioxide, also known as fluorosilicate glass (FSG). Its ε_r is 3.2. What would be the effective dielectric constant if the ILD is 40 percent porous?

b. What should be the starting ε_r if we need an effective ε_r less than 2 and the porosity cannot exceed 40 percent?



Tree and bush type electrical discharge structures. (a) Voltage $V =$ 160 kV, gap spacing $d = 0.06$ m at various times. (b) Dense bush discharge structure, $V = 300$ kV, $d = 0.06$ m at various times.

SOURCE: V. Lopatin, M. D. Noskov, R. Badent, K. Kist, A. J. Swab, "Positive Discharge Development in Insulating Oil: Optical Observation and Simulation," *IEEE Trans. on Dielec. and Elec. Insulation*, vol. 5, no. 2, 1998, p. 251, figure 2. (© IEEE, 1998)



Coaxial cable connector with traces of corona discharge; electrical treeing.

SOURCE: M. Mayer and G. H. Schröder, "Coaxial 30 kV Connectors for the RG220/U Cable: 20 Years of Operational Experience," *IEEE Electrical Insulation Magazine*, vol. **16**, March/April 2000, p. 11, figure 6. (© IEEE, 2000)

This small neodymium-iron-boron permanent magnet (diameter about the same as one-cent coin) is capable of lifting up to 10 pounds. Nd-Fe-B magnets typically have large $(BH)_{max}$ values (200–275 kJ m^{-3}).

In 1986 J. George Bednorz (right) and K. Alex Müller, at IBM Research Laboratories in Zurich, discovered that a copper oxide based ceramic-type compound (La-Ba-Cu-O) which normally has high resistivity becomes superconducting when cooled below 35 K. This Nobel prize winning discovery opened a new era of high-temperature-superconductivity research; now there are various ceramic compounds that are superconducting above the liquid nitrogen (an inexpensive cryogen) temperature (77 K).

SOURCE: IBM Zürich Research Laboratories.