

## Optical Properties of Materials

The way electromagnetic (EM) radiation interacts with matter depends very much on the wavelength of the EM wave. Many familiar types of EM radiation have wavelengths that range over many orders of magnitude. Although radio waves and X-rays are both EM waves, the two interact in a distinctly different way with matter. We tend to think of “light” as the electromagnetic radiation that we can see, that is, wavelengths in the visible range, typically 400 to 700 nm. However, in many applications, light is also used to describe EM waves that can have somewhat shorter or longer wavelengths such as ultraviolet (UV) and infrared (IR) light. For many practical purposes, it is useful to (arbitrarily) define light as EM waves that have wavelengths shorter than very roughly 100  $\mu\text{m}$  but longer than long-wavelength X-rays, roughly 10 nm. Today’s *light wave communications* use EM waves with wavelengths of 1300 and 1550 nm; in the infrared. *Optical properties* of materials are those characteristic properties that determine the interaction of light with matter; the best example being the refractive index  $n$  that determines the speed of light in a medium through  $v = c/n$ , where  $v$  is the speed of light in the medium and  $c$  is the speed of light in free space. The present chapter examines the key optical properties of matter and how these depend on the material and on the characteristics of the EM wave. The refractive index  $n$ , for example, depends on the dielectric polarization mechanisms as well as the wavelength  $\lambda$ . The material’s  $n$ - $\lambda$  behavior is called the **dispersion relation** and is one of the most important characteristics in many optical device applications.

We know from Chapter 3 that, depending on the experiment, we can treat light either as an EM wave, exhibiting typical wave-like properties, or as photons, exhibiting particle-like behavior. In this chapter we will primarily use the wave nature of light, though for absorption of light, the photon interpretation is more appropriate as the photons interact with electrons in the material.

## 9.1 LIGHT WAVES IN A HOMOGENEOUS MEDIUM

We know from well-established experiments that light exhibits typical wave-like properties such as interference and diffraction. We can treat light as an EM wave with time-varying electric and magnetic fields  $E_x$  and  $B_y$ , respectively, which propagate through space in such a way that they are always perpendicular to each other and the direction of propagation  $z$  is as depicted in Figure 9.1. The simplest traveling wave is a sinusoidal wave, which, for propagation along  $z$ , has the general mathematical form,<sup>1</sup>

Traveling  
wave along  $z$

$$E_x = E_o \cos(\omega t - kz + \phi_o) \quad [9.1]$$

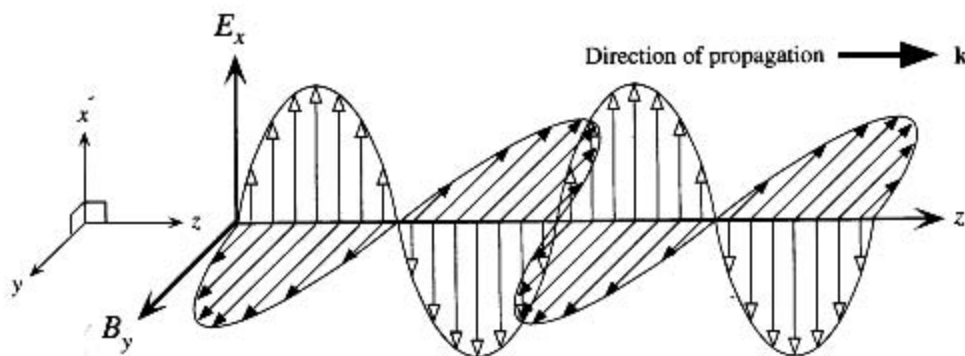
where  $E_x$  is the electric field at position  $z$  at time  $t$ ;  $k$  is the **propagation constant**, or **wavenumber**, given by  $2\pi/\lambda$ , where  $\lambda$  is the wavelength;  $\omega$  is the angular frequency;  $E_o$  is the amplitude of the wave; and  $\phi_o$  is a phase constant which accounts for the fact that at  $t = 0$  and  $z = 0$ ,  $E_x$  may or may not necessarily be zero depending on the choice of origin. The argument  $(\omega t - kz + \phi_o)$  is called the **phase** of the wave and denoted by  $\phi$ . Equation 9.1 describes a **monochromatic plane wave** of infinite extent traveling in the positive  $z$  direction as depicted in Figure 9.2. In any plane perpendicular to the direction of propagation (along  $z$ ), the phase of the wave, according to Equation 9.1, is constant which means that the field in this plane is also constant. A surface over which the phase of a wave is constant is referred to as a **wavefront**. A wavefront of a plane wave is obviously a plane perpendicular to the direction of propagation as shown in Figure 9.2.

We know from electromagnetism that time-varying magnetic fields result in time-varying electric fields (Faraday's law) and vice versa. A time-varying electric field would set up a time-varying magnetic field with the same frequency. According to electromagnetic principles,<sup>2</sup> a traveling electric field  $E_x$  as represented by Equation 9.1 would always be accompanied by a traveling magnetic field  $B_y$  with the same wave frequency and propagation constant ( $\omega$  and  $k$ ) but the directions of the two fields would be orthogonal as in Figure 9.1. Thus, there is a similar traveling wave equation for the magnetic field component  $B_y$ . We generally describe the interaction of a light wave with a nonconducting matter (conductivity,  $\sigma = 0$ ) through the electric field component  $E_x$  rather than  $B_y$  because it is the electric field that displaces the electrons in molecules or ions in the crystal and thereby gives rise to the polarization of matter. However, the two fields are linked, as in Figure 9.1, and there is an intimate relationship between the two fields. The **optical field** refers to the electric field  $E_x$ .

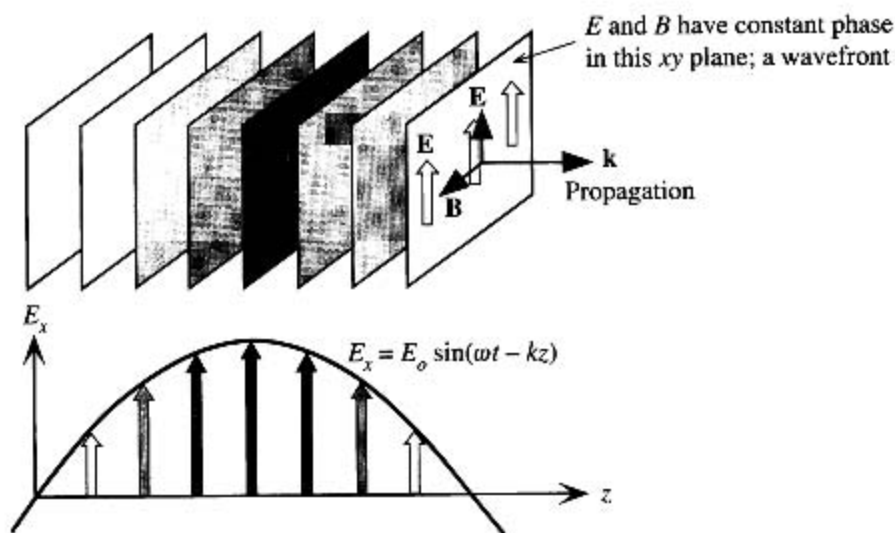
We can also represent a traveling wave using the exponential notation since  $\cos \phi = \text{Re}[\exp(j\phi)]$  where  $\text{Re}$  refers to the real part. We then need to take the real

<sup>1</sup> This chapter uses  $E$  for the electric field which was reserved for energy in previous chapters. There should be no confusion with  $E_g$  that represents the energy bandgap. In addition,  $n$  is used to represent the refractive index rather than the electron concentration.

<sup>2</sup> Maxwell's equations formulate electromagnetic phenomena and provide relationships between the electric and magnetic fields and their space and time derivatives. We only need to use a few selected results from Maxwell's equations without delving into their derivations. The magnetic field  $B$  is also called the magnetic induction or magnetic flux density.



**Figure 9.1** An electromagnetic wave is a traveling wave that has time-varying electric and magnetic fields that are perpendicular to each other and the direction of propagation  $z$ .



**Figure 9.2** A plane EM wave traveling along  $z$ , has the same  $E_x$  (or  $B_y$ ) at any point in a given  $xy$  plane.  
All electric field vectors in a given  $xy$  plane are therefore in phase. The  $xy$  planes are of infinite extent in the  $x$  and  $y$  directions.

part of any complex result at the end of calculations. Thus, we can write Equation 9.1 as

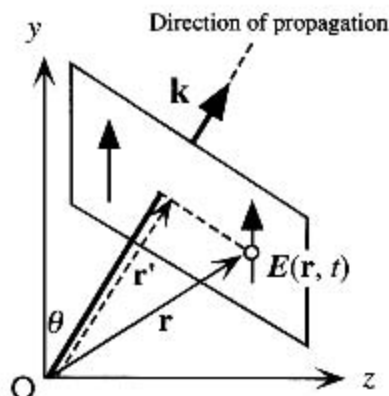
$$E_x(z, t) = \text{Re}[E_o \exp(j\phi_o) \exp j(\omega t - kz)]$$

or

$$E_x(z, t) = \text{Re}[E_c \exp j(\omega t - kz)] \quad [9.2]$$

*Traveling  
wave along  $z$*

where  $E_c = E_o \exp(j\phi_o)$  is a complex number that represents the amplitude of the wave and includes the constant phase information  $\phi_o$ .



**Figure 9.3** A traveling plane EM wave along a direction  $\mathbf{k}$ .

We indicate the direction of propagation with a vector  $\mathbf{k}$ , called the **wavevector**, whose magnitude is the propagation constant  $k = 2\pi/\lambda$ . It is clear that  $\mathbf{k}$  is perpendicular to constant phase planes as indicated in Figure 9.2. When the EM wave is propagating along some arbitrary direction  $\mathbf{k}$ , as indicated in Figure 9.3, then the electric field  $E(\mathbf{r}, t)$  at a point  $\mathbf{r}$  on a plane perpendicular to  $\mathbf{k}$  is

$$E(\mathbf{r}, t) = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_0) \quad [9.3]$$

because the dot product  $\mathbf{k} \cdot \mathbf{r}$  is along the direction of propagation similar to  $kz$ . The dot product is the product of  $\mathbf{k}$  and the projection of  $\mathbf{r}$  onto  $\mathbf{k}$  which is  $\mathbf{r}'$  in Figure 9.3, so  $\mathbf{k} \cdot \mathbf{r} = kr'$ . Indeed, if propagation is along  $z$ ,  $\mathbf{k} \cdot \mathbf{r}$  becomes  $kz$ . In general, if  $\mathbf{k}$  has components  $k_x$ ,  $k_y$ , and  $k_z$  along the  $x$ ,  $y$ , and  $z$  directions, then from the definition of the dot product,  $\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$ .

The time and space evolution of a given phase  $\phi$ , for example, the phase corresponding to a maximum field, according to Equation 9.1 is described by

$$\phi = \omega t - kz + \phi_0 = \text{constant}$$

During a time interval  $\delta t$ , this constant phase (and hence the maximum field) moves a distance  $\delta z$ . The phase velocity of this wave is therefore  $\delta z/\delta t$ . Thus the **phase velocity**  $v$  is

$$v = \frac{dz}{dt} = \frac{\omega}{k} = v\lambda \quad [9.4]$$

where  $v$  is the frequency ( $\omega = 2\pi\nu$ ).

We are frequently interested in the phase difference  $\Delta\phi$  at a given time between two points on a wave (Figure 9.1) that are separated by a certain distance. If the wave is traveling along  $z$  with a wavevector  $k$ , as in Equation 9.1, then the phase difference between two points separated by  $\Delta z$  is simply  $k \Delta z$  since  $\omega t$  is the same for each point. If this phase difference is 0 or multiples of  $2\pi$ , then the two points are in phase. Thus, the phase difference  $\Delta\phi$  can be expressed as  $k \Delta z$  or  $2\pi \Delta z/\lambda$ .

## 9.2 REFRACTIVE INDEX

When an EM wave is traveling in a dielectric medium, the oscillating electric field polarizes the molecules of the medium at the frequency of the wave. Intuitively, the EM wave propagation can be considered to be the propagation of this polarization in the medium. The field and the induced molecular dipoles become coupled. The net effect is that the polarization mechanism delays the propagation of the EM wave. The stronger the interaction between the field and the dipoles, the slower is the propagation of the wave. The relative permittivity  $\epsilon_r$  measures the ease with which the medium becomes polarized, and hence it indicates the extent of interaction between the field and the induced dipoles. For an EM wave traveling in a nonmagnetic dielectric medium of relative permittivity  $\epsilon_r$ , the phase velocity  $v$  is given by

$$v = \frac{1}{\sqrt{\epsilon_r \epsilon_0 \mu_0}} \quad [9.5]$$

Phase velocity in a medium with  $\epsilon_r$

If the frequency  $\nu$  is in the optical frequency range, then  $\epsilon_r$  will be due to electronic polarization as ionic polarization will be too sluggish to respond to the field. However, at the infrared frequencies or below, the relative permittivity also includes a significant contribution from ionic polarization and the phase velocity is slower. For an EM wave traveling in free space,  $\epsilon_r = 1$  and  $v_{\text{vacuum}} = 1/\sqrt{\epsilon_0 \mu_0} = c = 3 \times 10^8 \text{ m s}^{-1}$ , the velocity of light in a vacuum. The ratio of the speed of light in free space to its speed in a medium is called the **refractive index**  $n$  of the medium,

$$n = \frac{c}{v} = \sqrt{\epsilon_r} \quad [9.6]$$

Definition of refractive index

Suppose that in free space  $k_0$  is the wavevector ( $k_0 = 2\pi/\lambda_0$ ) and  $\lambda_0$  is the wavelength, then the wavevector  $k$  in the medium will be  $nk_0$  and the wavelength  $\lambda$  will be  $\lambda_0/n$ . Indeed, we can also define the refractive index in terms of the wavevector  $k$  in the medium with respect to that in a vacuum  $k_0$ ,

$$n = \frac{k}{k_0} \quad [9.7]$$

Definition of refractive index

Equation 9.6 is in agreement with our intuition that light propagates more slowly in a denser medium which has a higher refractive index. We should note that the frequency  $\nu$  remains the same. The refractive index of a medium is not necessarily the same in all directions. In noncrystalline materials such as glasses and liquids, the material structure is the same in all directions and  $n$  does not depend on the direction. The refractive index is then **isotropic**. In crystals, however, the atomic arrangements and interatomic bonding are different along different directions. Crystals, in general, have nonisotropic, or *anisotropic*, properties. Depending on the crystal structure, the relative permittivity  $\epsilon_r$  is different along different crystal directions. This means that, in general, the refractive index  $n$  seen by a propagating EM wave in a crystal will depend on the value of  $\epsilon_r$  along the direction of the oscillating electric field (that is, along the direction of polarization). For example, suppose that the wave in Figure 9.1 is traveling along the  $z$  direction in a particular crystal with its electric field oscillating

along the  $x$  direction. If the relative permittivity along this  $x$  direction is  $\epsilon_{rx}$ , then  $n_x = \sqrt{\epsilon_{rx}}$ . The wave therefore propagates with a phase velocity that is  $c/n_x$ . The variation of  $n$  with direction of propagation and the direction of the electric field depends on the particular crystal structure. With the exception of cubic crystals (such as diamond) all crystals exhibit a degree of optical anisotropy which leads to a number of important applications. Typically noncrystalline solids, such as glasses and liquids, and cubic crystals are **optically isotropic**; they possess only one refractive index for all directions.

**EXAMPLE 9.1**

**RELATIVE PERMITTIVITY AND REFRACTIVE INDEX** Relative permittivity  $\epsilon_r$ , or the dielectric constant, of materials is frequency dependent and further it depends on crystallographic direction since it is easier to polarize the medium along certain directions in the crystal. Glass has no crystal structure; it is amorphous. The relative permittivity is therefore isotropic but nonetheless frequency dependent.

The relationship  $n = \sqrt{\epsilon_r}$  between the refractive index  $n$  and  $\epsilon_r$  must be applied at the same frequency for both  $n$  and  $\epsilon_r$ . The relative permittivity for many materials can be vastly different at high and low frequencies because different polarization mechanisms operate at these frequencies. At low frequencies all polarization mechanisms present can contribute to  $\epsilon_r$ , whereas at optical frequencies only the electronic polarization can respond to the oscillating field. Table 9.1 lists the relative permittivity  $\epsilon_r$  (LF) at low frequencies (e.g., 60 Hz or 1 kHz as would be measured for example using a capacitance bridge in the laboratory) for various materials. It then compares  $\sqrt{\epsilon_r(\text{LF})}$  with  $n$ .

For diamond and silicon there is an excellent agreement between  $\sqrt{\epsilon_r(\text{LF})}$  and  $n$ . Both are covalent solids in which electronic polarization (electronic bond polarization) is the only polarization mechanism at low and high frequencies. Electronic polarization involves the displacement of light electrons with respect to positive ions of the crystal. This process can readily respond to the field oscillations up to optical or even ultraviolet frequencies.

For AgCl and SiO<sub>2</sub>,  $\sqrt{\epsilon_r(\text{LF})}$  is larger than  $n$  because at low frequencies both of these solids possess a degree of ionic polarization. The bonding has a substantial degree of ionic character which contributes to polarization at frequencies below far-infrared wavelengths. (The AgCl crystal has almost all ionic bonding.) In the case of water, the  $\epsilon_r$  (LF) is dominated by orientational or

**Table 9.1** Low-frequency (LF) relative permittivity  $\epsilon_r(\text{LF})$  and refractive index  $n$

Material	$\epsilon_r$ (LF)	$\sqrt{\epsilon_r(\text{LF})}$	$n$ (optical)	Comments
Diamond	5.7	2.39	2.41 (at 590 nm)	Electronic bond polarization up to UV light
Si	11.9	3.44	3.45 (at 2.15 $\mu\text{m}$ )	Electronic bond polarization up to optical frequencies
AgCl	11.14	3.33	2.00 (at 1–2 $\mu\text{m}$ )	Ionic polarization contributes to $\epsilon_r(\text{LF})$
SiO <sub>2</sub>	3.84	2.00	1.46 (at 600 nm)	Ionic polarization contributes to $\epsilon_r(\text{LF})$
Water	80	8.9	1.33 (at 600 nm)	Dipolar polarization contributes to $\epsilon_r(\text{LF})$ , which is large

dipolar polarization which is far too sluggish to respond to high-frequency oscillations of the field at optical frequencies.

It is instructive to consider what factors affect  $n$ . The simplest (and approximate) expression for the relative permittivity is

$$\epsilon_r \approx 1 + \frac{N\alpha}{\epsilon_0} \quad [9.8]$$

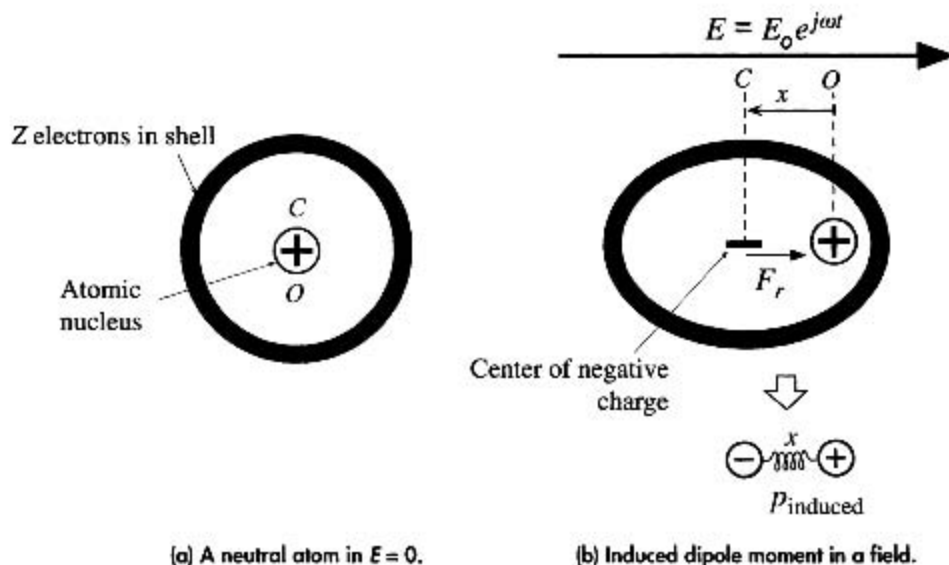
*Relative permittivity and polarizability*

where  $N$  is the number of molecules per unit volume and  $\alpha$  is the polarizability per molecule. Both atomic concentration, or density, and polarizability therefore increase  $n$ . For example, glasses of given type but with greater density tend to have higher  $n$ .

### 9.3 DISPERSION: REFRACTIVE INDEX–WAVELENGTH BEHAVIOR

The refractive index of materials in general depends on the frequency, or the wavelength. This wavelength dependence follows directly from the frequency dependence of the relative permittivity  $\epsilon_r$ . Figure 9.4 shows what happens to an atom in the presence of an oscillating electric field  $E$  which is due to a light wave passing through this location; it may also be due to an applied external field.

In the absence of an electric field and in equilibrium, the center of mass  $C$  of the orbital motions of the electrons coincides with the positively charged nucleus at  $O$  and the net electric dipole moment is zero as indicated in Figure 9.4a. Suppose that the atom has  $Z$  number of electrons orbiting the nucleus and all the electrons are contained



**Figure 9.4** Electronic polarization of an atom. In the presence of a field in the  $+x$  direction, the electrons are displaced in the  $-x$  direction (from  $O$ ), and the restoring force is in the  $+x$  direction.

within a given shell. In the presence of the electric field  $E$ , however, the light electrons become displaced in the opposite direction to the field, so their center of mass  $C$  is shifted by some distance  $x$  with respect to the nucleus  $O$  which we take to be the origin as shown in Figure 9.4b. As the electrons are “pushed” away by the applied field, the Coulombic attraction between the electrons and nuclear charge “pulls in” the electrons. The force on the electrons, due to  $E$ , trying to separate them away from the nuclear charge is  $ZeE$ . The restoring force  $F_r$ , which is the Coulombic attractive force between the electrons and the nucleus, can be taken to be proportional to the displacement  $x$  provided that the latter is small. The reason is that  $F_r = F_r(x)$  can be expanded in powers of  $x$ , and for small  $x$  only the linear term matters. The restoring force  $F_r$  is obviously zero when  $C$  coincides with  $O$  ( $x = 0$ ). We can write  $F_r = -\beta x$  where  $\beta$  is a constant and the negative sign indicates that  $F_r$  is always directed toward the nucleus  $O$ .

First consider applying a dc field. In equilibrium, the *net* force on the negative charge is zero or  $ZeE = \beta x$  from which  $x$  is known. Therefore the *magnitude* of the induced electronic dipole moment is given by

Induced  
electronic  
dc dipole  
moment

$$p_{\text{induced}} = (Ze)x = \frac{Z^2 e^2}{\beta} E \quad [9.9]$$

As expected  $p_{\text{induced}}$  is proportional to the applied field. The electronic dipole moment in Equation 9.9 is valid under static conditions, *i.e.*, when the electric field is a dc field. Suppose that we suddenly remove the applied electric field polarizing the atom. There is then only the restoring force  $-\beta x$ , which always acts to pull the electrons toward the nucleus  $O$ . The equation of motion of the negative charge center is then (force = mass  $\times$  acceleration)

Simple  
harmonic  
motion

$$-\beta x = Zm_e \frac{d^2 x}{dt^2}$$

By solving this differential equation we can show that the displacement at any time is a simple harmonic motion, that is,

$$x(t) = x_0 \cos(\omega_0 t)$$

where the angular frequency of oscillation  $\omega_0$  is

Natural  
frequency  
of the atom

$$\omega_0 = \left( \frac{\beta}{Zm_e} \right)^{1/2} \quad [9.10]$$

In essence, this is the oscillation frequency of the center of mass of the electron cloud about the nucleus and  $x_0$  is the displacement before the removal of the field. After the removal of the field, the electronic charge cloud executes simple harmonic motion about the nucleus with a **natural frequency**  $\omega_0$  determined by Equation 9.10;  $\omega_0$  is also called the **resonance frequency**. The oscillations, of course, die out with time because there is an inevitable loss of energy from an oscillating charge cloud. An oscillating electron is like an oscillating current and loses energy by radiating EM waves; all accelerating charges emit radiation.

Consider now the presence of an oscillating electric field due to an EM wave passing through the location of this atom as in Figure 9.4b. The applied field oscillates



harmonically in the  $+x$  and  $-x$  directions, that is,  $E = E_o \exp(j\omega t)$ . This field will drive and oscillate the electrons about the nucleus. There is again a restoring force  $F_r$  acting on the displaced electrons trying to bring back the electron shell to its equilibrium placement around the nucleus. For simplicity we will again neglect energy losses. Newton's second law for  $Z e$  electrons with mass  $Z m_e$  driven by  $E$  is given by

$$Z m_e \frac{d^2 x}{dt^2} = -Z e E_o \exp(j\omega t) - \beta x \quad [9.11]$$

*Lorentz  
oscillator  
model*

The solution of this equation gives the instantaneous displacement  $x(t)$  of the center of mass of electrons from the nucleus ( $C$  from  $O$ ),

$$x = x(t) = -\frac{e E_o \exp(j\omega t)}{m_e(\omega_o^2 - \omega^2)}$$

The induced electronic dipole moment is then simply given by  $p_{\text{induced}} = -(Z e)x$ . The negative sign is needed because normally  $x$  is measured from negative to positive charge whereas in Figure 9.4b it is measured from the nucleus. By definition, the electronic polarizability  $\alpha_e$  is the induced dipole moment per unit electric field,

$$\alpha_e = \frac{p_{\text{induced}}}{E} = \frac{Z e^2}{m_e(\omega_o^2 - \omega^2)} \quad [9.12]$$

*Electronic  
polarizability*

Thus, the displacement  $x$  and hence electronic polarizability  $\alpha_e$  increase as  $\omega$  increases. Both become very large when  $\omega$  approaches the natural frequency  $\omega_o$ . In practice, charge separation  $x$  and hence polarizability  $\alpha_e$  do not become infinite at  $\omega = \omega_o$  because two factors impose a limit. First, at large  $x$ , the system is no longer linear and this analysis is not valid. Secondly, there is always some energy loss.

Given that the polarizability is frequency dependent as in Equation 9.12, the effect on the refractive index  $n$  is easy to predict. The simplest (and a very rough) relationship between the relative permittivity  $\epsilon_r$  and polarizability  $\alpha_e$  is

$$\epsilon_r = 1 + \frac{N}{\epsilon_o} \alpha_e$$

*Relative  
permittivity  
and  
polarizability*

where  $N$  is the number of atoms per unit volume. Given that the refractive index  $n$  is related to  $\epsilon_r$  by  $n^2 = \epsilon_r$ , it is clear that  $n$  must be frequency dependent, *i.e.*,

$$n^2 = 1 + \left( \frac{N Z e^2}{\epsilon_o m_e} \right) \frac{1}{\omega_o^2 - \omega^2} \quad [9.13]$$

*Dispersion  
relation*

We can also express this in terms of the wavelength  $\lambda$ . If  $\lambda_o = 2\pi c/\omega_o$  is the resonance wavelength, then Equation 9.13 is equivalent to

$$n^2 = 1 + \left( \frac{N Z e^2}{\epsilon_o m_e} \right) \left( \frac{\lambda_o}{2\pi c} \right)^2 \frac{\lambda^2}{\lambda^2 - \lambda_o^2} \quad [9.14]$$

*Dispersion  
relation*

This type of relationship between  $n$  and the frequency  $\omega$ , or wavelength  $\lambda$ , is called the **dispersion relation**. Although the above treatment is grossly simplified, it does nonetheless emphasize that  $n$  will always be wavelength dependent and will exhibit a

Table 9.2 Sellmeier and Cauchy coefficients

	Sellmeier					
	$A_1$	$A_2$	$A_3$	$\lambda_1$ ( $\mu\text{m}$ )	$\lambda_2$ ( $\mu\text{m}$ )	$\lambda_3$ ( $\mu\text{m}$ )
SiO <sub>2</sub> (fused silica)	0.696749	0.408218	0.890815	0.0690660	0.115662	9.900559
86.5% SiO <sub>2</sub> -13.5% GeO <sub>2</sub>	0.711040	0.451885	0.704048	0.0642700	0.129408	9.425478
GeO <sub>2</sub>	0.80686642	0.71815848	0.85416831	0.068972606	0.15396605	11.841931
Sapphire	1.023798	1.058264	5.280792	0.0614482	0.110700	17.92656
Diamond	0.3306	4.3356	—	0.1750	0.1060	—

	Cauchy				
	Range of $h\nu$ (eV)	$n_{-2}$ (eV <sup>2</sup> )	$n_0$	$n_2$ (eV <sup>-2</sup> )	$n_{-4}$ (eV <sup>-4</sup> )
Diamond	0.05–5.47	$-1.07 \times 10^{-3}$	2.378	$8.01 \times 10^{-3}$	$1.04 \times 10^{-4}$
Silicon	0.002–1.08	$-2.04 \times 10^{-8}$	3.4189	$8.15 \times 10^{-2}$	$1.25 \times 10^{-2}$
Germanium	0.002–0.75	$-1.0 \times 10^{-8}$	4.003	$2.2 \times 10^{-1}$	$1.4 \times 10^{-1}$

SOURCE: Sellmeier coefficients combined from various sources. Cauchy coefficients from D. Y. Smith *et al.*, *J. Phys. CM* **13**, 3883, 2001.

substantial increase as the frequency increases toward a natural frequency of the polarization mechanism. In the above example, we considered the electronic polarization of an isolated atom with a well-defined natural frequency  $\omega_0$ . In the crystal, however, the atoms interact, and further we also have to consider the valence electrons in the bonds. The overall result is that  $n$  is a complicated function of the frequency or the wavelength. One possibility is to assume a number of resonant frequencies, that is, not just  $\lambda_0$  but a series of resonant frequencies,  $\lambda_1, \lambda_2, \dots$ , and then sum the contributions arising from each with some weighing factor  $A_1, A_2$ , etc.,

Sellmeier  
equation

$$n^2 = 1 + \frac{A_1 \lambda^2}{\lambda^2 - \lambda_1^2} + \frac{A_2 \lambda^2}{\lambda^2 - \lambda_2^2} + \frac{A_3 \lambda^2}{\lambda^2 - \lambda_3^2} + \dots \quad [9.15]$$

where  $A_1, A_2, A_3$  and  $\lambda_1, \lambda_2$ , and  $\lambda_3$  are constants, called **Sellmeier coefficients**.<sup>3</sup> Equation 9.15 turns out to be quite a useful semiempirical expression for calculating  $n$  at various wavelengths if the Sellmeier coefficients are known. Higher terms involving  $A_4$  and higher  $A$  coefficients can generally be neglected in representing  $n$  versus  $\lambda$  behavior over typical wavelengths of interest. For example, for diamond, we only need the  $A_1$  and  $A_2$  terms. The Sellmeier coefficients are listed in various optical data handbooks.

There is another well-known useful  $n$ - $\lambda$  dispersion relation due originally to Cauchy (1836), which has the short form given by

Cauchy  
short-form  
dispersion  
equation

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} \quad [9.16]$$

<sup>3</sup> This is also known as the Sellmeier-Herzberger formula.

where  $A$ ,  $B$ , and  $C$  are material specific constants. Typically, the Cauchy equation is used in the visible spectrum for various optical glasses. A more general Cauchy dispersion relation is of the form<sup>4</sup>

$$n = n_{-2}(h\nu)^{-2} + n_0 + n_2(h\nu)^2 + n_4(h\nu)^4 \quad [9.17]$$

where  $h\nu$  is the photon energy, and  $n_0$ ,  $n_{-2}$ ,  $n_2$ , and  $n_4$  are constants; values for diamond, Si, and Ge are listed in Table 9.2. The general Cauchy equation is usually applicable over a wide photon energy range.

*Cauchy dispersion equation in photon energy*

**GaAs DISPERSION RELATION** For GaAs, from  $\lambda = 0.89$  to  $4.1 \mu\text{m}$ , the refractive index is given by the following dispersion relation,

$$n^2 = 7.10 + \frac{3.78\lambda^2}{\lambda^2 - 0.2767} \quad [9.18]$$

where  $\lambda$  is in microns ( $\mu\text{m}$ ). What is the refractive index of GaAs for light with a photon energy of 1 eV?

**EXAMPLE 9.2**

*GaAs dispersion relation*

**SOLUTION**

At  $h\nu = 1 \text{ eV}$ ,

$$\lambda = \frac{hc}{h\nu} = \frac{(6.62 \times 10^{-34} \text{ J s})(3 \times 10^8 \text{ m s}^{-1})}{(1 \text{ eV} \times 1.6 \times 10^{-19} \text{ J eV}^{-1})} = 1.24 \mu\text{m}$$

Thus,

$$n^2 = 7.10 + \frac{3.78\lambda^2}{\lambda^2 - 0.2767} = 7.10 + \frac{3.78(1.24)^2}{(1.24)^2 - 0.2767} = 11.71$$

so that  $n = 3.42$

Note that the  $n$  versus  $\lambda$  expression for GaAs is actually a Sellmeier-type formula because when  $\lambda^2 \gg \lambda_1^2$ , then  $A_1$  can be simply lumped with 1 to give  $1 + A_1 = 7.10$ .

**SELMEIER EQUATION AND DIAMOND** The relevant Sellmeier coefficients for diamond are given in Table 9.2. Calculate its refractive index at 550 nm (green light) to three decimal places.

**EXAMPLE 9.3**

**SOLUTION**

The Sellmeier dispersion relation for diamond is

$$\begin{aligned} n^2 &= 1 + \frac{0.3306\lambda^2}{\lambda^2 - (175 \text{ nm})^2} + \frac{4.3356\lambda^2}{\lambda^2 - (106 \text{ nm})^2} \\ &= 1 + \frac{0.3306(550 \text{ nm})^2}{(550 \text{ nm})^2 - (175 \text{ nm})^2} + \frac{4.3356(550 \text{ nm})^2}{(550 \text{ nm})^2 - (106 \text{ nm})^2} = 5.8707 \end{aligned}$$

So that  $n = 2.423$

which is about 0.1 percent different than the experimental value of 2.426.

<sup>4</sup> D. Y. Smith et al., *J. Phys. CM* **13**, 3883, 2001.

## EXAMPLE 9.4

**CAUCHY EQUATION AND DIAMOND** Using the Cauchy coefficients for diamond in Table 9.2, calculate the refractive index at 550 nm.

## SOLUTION

At  $\lambda = 550$  nm, the photon energy is

$$h\nu = \frac{hc}{\lambda} = \frac{(6.62 \times 10^{-34} \text{ J s})(3 \times 10^8 \text{ m s}^{-1})}{550 \times 10^{-9} \text{ m}} \times \frac{1}{1.6 \times 10^{-19} \text{ J eV}^{-1}} = 2.254 \text{ eV}$$

Using the Cauchy dispersion relation for diamond with coefficients from Table 9.2,

$$\begin{aligned} n &= n_{-2}(h\nu)^{-2} + n_0 + n_2(h\nu)^2 + n_4(h\nu)^4 \\ &= (-1.07 \times 10^{-5})(2.254)^{-2} + 2.378 + (8.01 \times 10^{-3})(2.254)^2 + (1.04 \times 10^{-4})(2.254)^4 \\ &= 2.421 \end{aligned}$$

The difference in  $n$  from the value in Example 9.3 is 0.08 percent, and is due to the Cauchy coefficients quoted in Table 9.2 being applicable over a wider wavelength range at the expense of some accuracy.

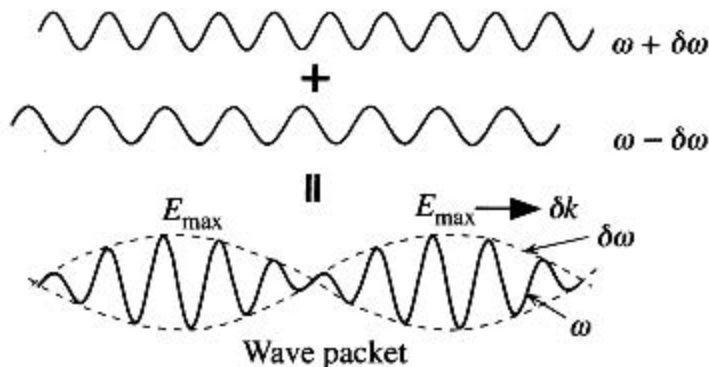
## 9.4 GROUP VELOCITY AND GROUP INDEX

Since there are no perfect monochromatic waves in practice, we have to consider the way in which a group of waves differing slightly in wavelength will travel along the  $z$  direction as depicted in Figure 9.5. When two perfectly harmonic waves of frequencies  $\omega - \delta\omega$  and  $\omega + \delta\omega$  and wavevectors  $k - \delta k$  and  $k + \delta k$  interfere, as shown in Figure 9.5, they generate a **wavepacket** which contains an oscillating field at the mean frequency  $\omega$  that is amplitude modulated by a slowly varying field of frequency  $\delta\omega$ . The maximum amplitude moves with a wavevector  $\delta k$  and thus with a **group velocity** that is given by  $\delta\omega/\delta k$ , that is,

Group  
velocity

$$v_g = \frac{d\omega}{dk} \quad [9.19]$$

**Figure 9.5** Two slightly different wavelength waves traveling in the same direction result in a wave packet that has an amplitude variation that travels at the group velocity.



The group velocity therefore defines the speed with which energy or information is propagated since it defines the speed of the envelope of the amplitude variation. The maximum electric field in Figure 9.5 advances with a velocity  $v_g$ , whereas the phase variations in the electric field are propagating at the phase velocity  $v$ .

Inasmuch as  $\omega = vk$  and the phase velocity  $v = c/n$ , the group velocity in a medium can be readily evaluated from Equation 9.19. In a vacuum, obviously  $v$  is simply  $c$  and independent of the wavelength or  $k$ . Thus for waves traveling in a vacuum,  $\omega = ck$  and the group velocity is

$$v_g(\text{vacuum}) = \frac{d\omega}{dk} = c = \text{Phase velocity} \quad [9.20]$$

*Group velocity in a vacuum*

On the other hand, suppose that  $v$  depends on the wavelength or  $k$  by virtue of  $n$  being a function of the wavelength as in the case for glasses. Then,

$$\omega = vk = \left[ \frac{c}{n(\lambda)} \right] \left( \frac{2\pi}{\lambda} \right) \quad [9.21]$$

where  $n = n(\lambda)$  is a function of the wavelength. The group velocity  $v_g$  in a medium, from differentiating Equation 9.21 in Equation 9.19, is approximately given by

$$v_g(\text{medium}) = \frac{d\omega}{dk} = \frac{c}{n - \lambda \frac{dn}{d\lambda}}$$

This can be written as

$$v_g(\text{medium}) = \frac{c}{N_g} \quad [9.22]$$

*Group velocity in a medium*

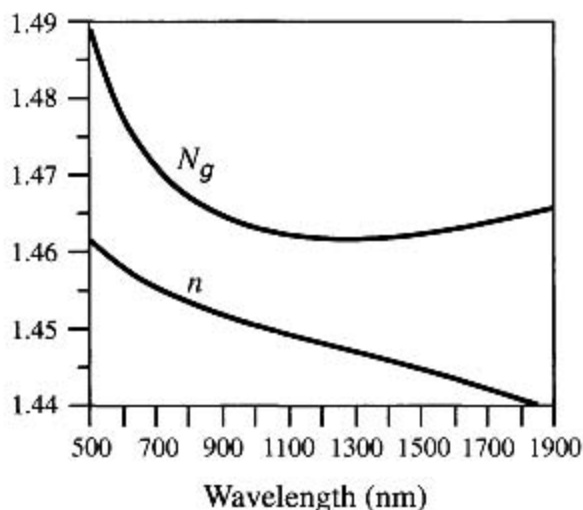
where

$$N_g = n - \lambda \frac{dn}{d\lambda} \quad [9.23]$$

*Group index*

is defined as the **group index** of the medium. Equation 9.23 defines the group refractive index  $N_g$  of a medium and determines the effect of the medium on the group velocity via Equation 9.22.

In general, for many materials the refractive index  $n$  and hence the group index  $N_g$  depend on the wavelength of light by virtue of the relative permittivity  $\epsilon_r$  being frequency dependent. Then both the phase velocity  $v$  and the group velocity  $v_g$  depend on the wavelength and the medium is called a **dispersive medium**. The refractive index  $n$  and the group index  $N_g$  of pure SiO<sub>2</sub> (silica) glass are important parameters in optical fiber design in optical communications. Both of these parameters depend on the wavelength of light as shown in Figure 9.6. Around 1300 nm,  $N_g$  is at a minimum which means that for wavelengths close to 1300 nm,  $N_g$  is wavelength independent. Thus, light waves with wavelengths around 1300 nm travel with the same group velocity and do not experience dispersion. This phenomenon is significant in the propagation of light in glass fibers used in optical communications.



**Figure 9.6** Refractive index  $n$  and the group index  $N_g$  of pure  $\text{SiO}_2$  (silica) glass as a function of wavelength.

### EXAMPLE 9.5

**GROUP VELOCITY** Consider two sinusoidal waves which are close in frequency, that is, waves of frequencies  $\omega - \delta\omega$  and  $\omega + \delta\omega$  as in Figure 9.5. Their wavevectors will be  $k - \delta k$  and  $k + \delta k$ . The resultant wave will be

$$E_x(z, t) = E_o \cos[(\omega - \delta\omega)t - (k - \delta k)z] + E_o \cos[(\omega + \delta\omega)t - (k + \delta k)z]$$

By using the trigonometric identity  $\cos A + \cos B = 2 \cos[\frac{1}{2}(A - B)] \cos[\frac{1}{2}(A + B)]$  we arrive at

$$E_x(z, t) = 2E_o \cos[(\delta\omega)t - (\delta k)z] \cos[\omega t - kz]$$

As depicted in Figure 9.5, this represents a sinusoidal wave of frequency  $\omega$  which is amplitude modulated by a very slowly varying sinusoidal of frequency  $\delta\omega$ . The system of waves, that is, the modulation, travels along  $z$  at a speed determined by the modulating term  $\cos[(\delta\omega)t - (\delta k)z]$ . The maximum in the field occurs when  $[(\delta\omega)t - (\delta k)z] = 2m\pi = \text{constant}$  ( $m$  is an integer), which travels with a velocity

Group  
velocity

$$\frac{dz}{dt} = \frac{\delta\omega}{\delta k} \quad \text{or} \quad v_g = \frac{d\omega}{dk}$$

This is the group velocity of the waves, as stated in Equation 9.19, since it determines the speed of propagation of the maximum electric field along  $z$ .

### EXAMPLE 9.6

**GROUP AND PHASE VELOCITIES** Consider a light wave traveling in a pure  $\text{SiO}_2$  (silica) glass medium. If the wavelength of light is 1300 nm and the refractive index at this wavelength is 1.447, what is the phase velocity, group index ( $N_g$ ), and group velocity ( $v_g$ )?

#### SOLUTION

The phase velocity is given by

$$v = \frac{c}{n} = \frac{3 \times 10^8 \text{ m s}^{-1}}{1.447} = 2.073 \times 10^8 \text{ m s}^{-1}$$

From Figure 9.6, at  $\lambda = 1300$  nm,  $N_g = 1.462$ , so

$$v_g = \frac{c}{N_g} = \frac{3 \times 10^8 \text{ m s}^{-1}}{1.462} = 2.052 \times 10^8 \text{ m s}^{-1}$$

The group velocity is  $\sim 0.7$  percent smaller than the phase velocity.

## 9.5 MAGNETIC FIELD: IRRADIANCE AND POYNTING VECTOR

Although we have considered the electric field component  $E_x$  of the EM wave, we should recall that the magnetic field (magnetic induction) component  $B_y$  always accompanies  $E_x$  in an EM wave propagation. In fact, if  $v$  is the phase velocity of an EM wave in an isotropic dielectric medium and  $n$  is the refractive index, then according to electromagnetism, at all times and anywhere in an EM wave,<sup>5</sup>

$$E_x = vB_y = \frac{c}{n}B_y \quad [9.24]$$

*Fields in an EM wave*

where  $v = (\epsilon_0 \epsilon_r \mu_0)^{-1/2}$  and  $n = \sqrt{\epsilon_r}$ . Thus, the two fields are simply and intimately related for an EM wave propagating in an isotropic medium. Any process that alters  $E_x$  also intimately changes  $B_y$  in accordance with Equation 9.24.

As the EM wave propagates in the direction of the wavevector  $\mathbf{k}$  as shown in Figure 9.7, there is an energy flow in this direction. The wave brings with it electromagnetic energy. A small region of space where the electric field is  $E_x$  has an energy density, that is, energy per unit volume, given by  $\frac{1}{2}\epsilon_0 \epsilon_r E_x^2$ . Similarly, a region of space where the magnetic field is  $B_y$  has an energy density  $\frac{1}{2}B_y^2/\mu_0$ . Since the two fields are related by Equation 9.24, the energy densities in the  $E_x$  and  $B_y$  fields are the same,

$$\frac{1}{2}\epsilon_0 \epsilon_r E_x^2 = \frac{1}{2\mu_0} B_y^2 \quad [9.25]$$

*Energy densities in an EM wave*

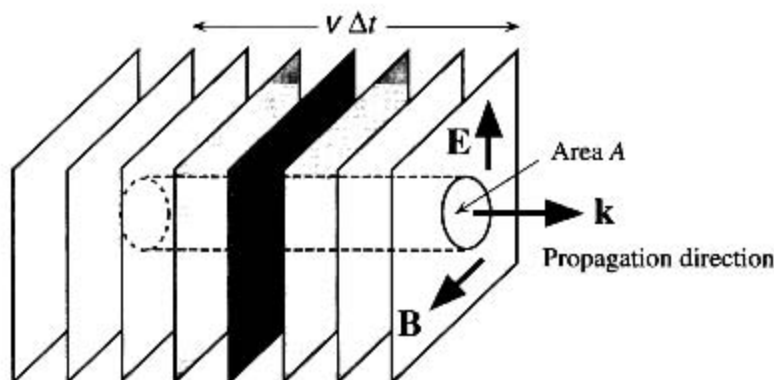
The total energy density in the wave is therefore  $\epsilon_0 \epsilon_r E_x^2$ . Suppose that an ideal “energy meter” is placed in the path of the EM wave so that the receiving area  $A$  of this meter is perpendicular to the direction of propagation. In a time interval  $\Delta t$ , a portion of the wave of spatial length  $v \Delta t$  crosses  $A$  as shown in Figure 9.7. Thus, a volume  $Av \Delta t$  of the EM wave crosses  $A$  in time  $\Delta t$ . The energy in this volume consequently becomes received. If  $S$  is the EM power flow per unit area,

$$S = \text{Energy flow per unit time per unit area}$$

giving,

$$S = \frac{(Av \Delta t) (\epsilon_0 \epsilon_r E_x^2)}{A \Delta t} = v \epsilon_0 \epsilon_r E_x^2 = v^2 \epsilon_0 \epsilon_r E_x B_y \quad [9.26]$$

<sup>5</sup> This is actually a statement of Faraday's law for EM waves. In vector notation it is often expressed as  $\omega \mathbf{B} = \mathbf{k} \times \mathbf{E}$ .



**Figure 9.7** A plane EM wave traveling along  $\mathbf{k}$  crosses an area  $A$  at right angles to the direction of propagation. In time  $\Delta t$ , the energy in the cylindrical volume  $A v \Delta t$  (shown dashed) flows through  $A$ .

In an isotropic medium, the energy flow is in the direction of wave propagation. If we use the vectors  $\mathbf{E}$  and  $\mathbf{B}$  to represent the electric and magnetic fields in the EM wave, then the wave propagates in a direction  $\mathbf{E} \times \mathbf{B}$ , because this direction is perpendicular to both  $\mathbf{E}$  and  $\mathbf{B}$ . The EM power flow per unit area in Equation 9.26 can be written as

*Poynting vector*

$$\mathbf{S} = v^2 \epsilon_0 \epsilon_r \mathbf{E} \times \mathbf{B} \quad [9.27]$$

where  $\mathbf{S}$ , called the **Poynting vector**, represents the energy flow per unit time per unit area in a direction determined by  $\mathbf{E} \times \mathbf{B}$  (direction of propagation). Its magnitude, power flow per unit area, is called the **irradiance**.<sup>6</sup>

The field  $E_x$  at the receiver location (say,  $z = z_1$ ) varies sinusoidally which means that the energy flow also varies sinusoidally. The irradiance in Equation 9.26 is the **instantaneous irradiance**. If we write the field as  $E_x = E_o \sin(\omega t)$  and then calculate the average irradiance by averaging  $S$  over one period, we would find the **average irradiance**,

*Average irradiance (intensity)*

$$I = S_{\text{average}} = \frac{1}{2} v \epsilon_0 \epsilon_r E_o^2 \quad [9.28]$$

Since  $v = c/n$  and  $\epsilon_r = n^2$  we can write Equation 9.28 as

*Average irradiance (intensity)*

$$I = S_{\text{average}} = \frac{1}{2} c \epsilon_0 n E_o^2 = (1.33 \times 10^{-3}) n E_o^2 \quad [9.29]$$

The instantaneous irradiance can only be measured if the power meter can respond more quickly than the oscillations of the electric field, and since this is in the

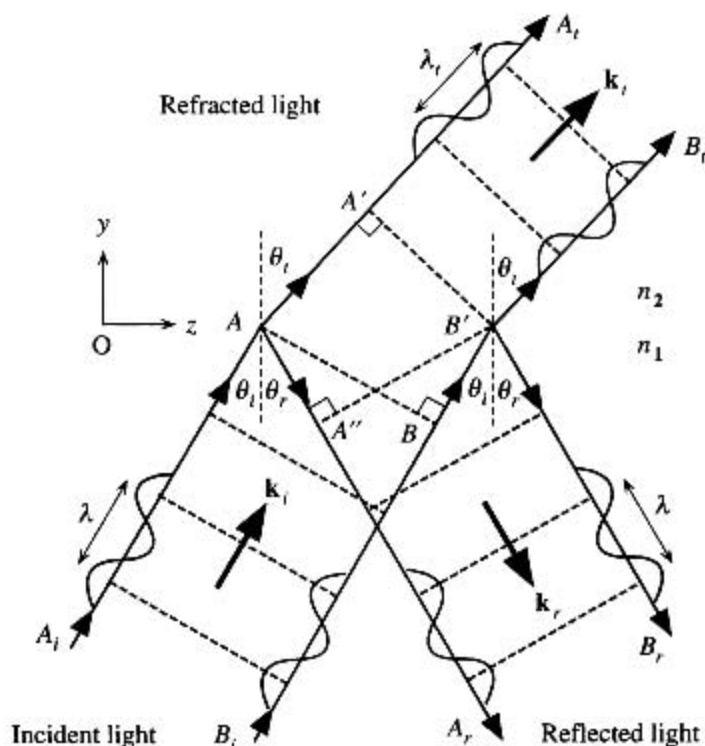
<sup>6</sup> The term *intensity* is widely used and interpreted by many engineers as power flow per unit area even though the strictly correct term is *irradiance*. Many optoelectronic data books simply use intensity to mean irradiance.



optical frequencies range, all practical measurements invariably yield the average irradiance because all detectors have a response rate much slower than the frequency of the wave.

## 9.6 SNELL'S LAW AND TOTAL INTERNAL REFLECTION (TIR)

We consider a traveling plane EM wave in a medium (1) of refractive index  $n_1$  propagating toward a medium (2) with a refractive index  $n_2$ . Constant phase fronts are joined with broken lines, and the wavevector  $\mathbf{k}_i$  is perpendicular to the wave fronts as shown in Figure 9.8. When the wave reaches the plane boundary between the two media, a transmitted wave in medium 2 and a reflected wave in medium 1 appear. The transmitted wave is called the **refracted light**. The angles,  $\theta_i$ ,  $\theta_t$ ,  $\theta_r$  define the directions of the incident, transmitted, and reflected waves, respectively, with respect to the normal to the boundary plane as shown in Figure 9.8. The wavevectors of the reflected and transmitted waves are denoted as  $\mathbf{k}_r$  and  $\mathbf{k}_t$ , respectively. Since both the incident and reflected waves are in the same medium, the magnitudes of  $\mathbf{k}_r$  and  $\mathbf{k}_i$  are the same,  $k_r = k_i$ .



**Figure 9.8** A light wave traveling in a medium with a greater refractive index ( $n_1 > n_2$ ) suffers reflection and refraction at the boundary.

Simple arguments based on constructive interference can be used to show that there can only be one reflected wave that occurs at an angle equal to the incidence angle. The two waves along  $A_i$  and  $B_i$  are in phase. When these waves are reflected to become waves  $A_r$  and  $B_r$ , then they must still be in phase, otherwise they will interfere destructively and destroy each other. The only way the two waves can stay in phase is if  $\theta_r = \theta_i$ . All other angles lead to the waves  $A_r$  and  $B_r$  being out of phase and interfering destructively.

The refracted waves  $A_t$  and  $B_t$  are propagating in a medium of refracted index  $n_2 (< n_1)$  that is different than  $n_1$ . Hence the waves  $A_t$  and  $B_t$  have different velocities than  $A_i$  and  $B_i$ . We consider what happens to a wavefront such as  $AB$ , corresponding perhaps to the maximum field, as it propagates from medium 1 to 2. We recall that the points  $A$  and  $B$  on this front are always in phase. During the time it takes for the phase  $B$  on wave  $B_t$  to reach  $B'$ , phase  $A$  on wave  $A_t$  has progressed to  $A'$ . The wavefront  $AB$  thus becomes the front  $A'B'$  in medium 2. Unless the two waves at  $A'$  and  $B'$  still have the same phase, there will be no transmitted wave.  $A'$  and  $B'$  points on the front are only in phase for one particular transmitted angle  $\theta_t$ .

If it takes time  $t$  for the phase at  $B$  on wave  $B_t$  to reach  $B'$ , then  $BB' = v_1 t = ct/n_1$ . During this time  $t$ , the phase  $A$  has progressed to  $A'$  where  $AA' = v_2 t = ct/n_2$ .  $A'$  and  $B'$  belong to the same front just like  $A$  and  $B$ , so  $AB$  is perpendicular to  $\mathbf{k}_i$  in medium 1 and  $A'B'$  is perpendicular to  $\mathbf{k}_t$  in medium 2. From geometrical considerations,  $AB' = BB'/\sin \theta_i$  and  $AB' = AA'/\sin \theta_t$ , so

$$AB' = \frac{v_1 t}{\sin \theta_i} = \frac{v_2 t}{\sin \theta_t}$$

or

Snell's law

$$\frac{\sin \theta_i}{\sin \theta_t} = \frac{v_1}{v_2} = \frac{n_2}{n_1} \quad [9.30]$$

This is **Snell's law**<sup>7</sup> which relates the angles of incidence and refraction to the refractive indices of the media.

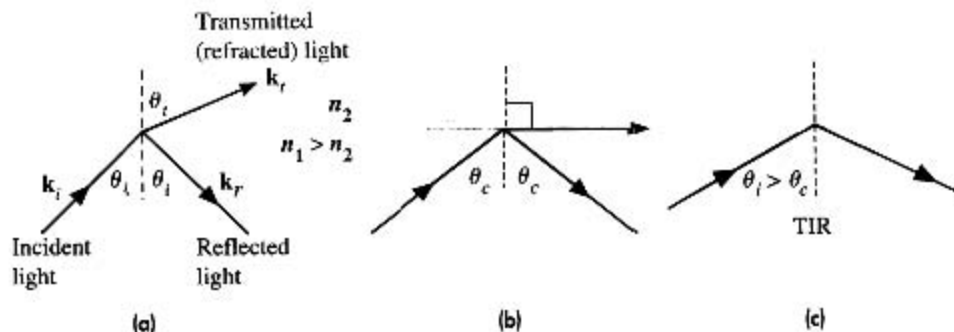
If we consider the reflected wave, the wave front  $AB$  becomes  $A''B'$  in the reflected wave. In time  $t$ , phase  $B$  moves to  $B'$  and  $A$  moves to  $A''$ . Since they must still be in phase to constitute the reflected wave,  $BB'$  must be equal to  $AA''$ . Suppose it takes time  $t$  for the wavefront  $B$  to move to  $B'$  (or  $A$  to  $A''$ ). Then, since  $BB' = AA'' = v_1 t$ , from geometrical considerations,

$$AB' = \frac{v_1 t}{\sin \theta_i} = \frac{v_1 t}{\sin \theta_r}$$

so that  $\theta_i = \theta_r$ . The angles of incidence and reflection are the same.

When  $n_1 > n_2$ , then obviously the transmitted angle is greater than the incidence angle as apparent in Figure 9.8. When the refraction angle  $\theta_t$  reaches  $90^\circ$ , the incidence

<sup>7</sup> Willebrord van Roijen Snell (1581–1626), a Dutch physicist and mathematician, was born in Leiden and eventually became a professor at Leiden University. He obtained his refraction law in 1621 which was published by René Descartes in France in 1637; it is not known whether Descartes knew of Snell's law or formulated it independently.



**Figure 9.9** Light wave traveling in a more dense medium strikes a less dense medium. Depending on the incidence angle with respect to  $\theta_c$ , determined by the ratio of the refractive indices, the wave may be transmitted (refracted) or reflected.

(a)  $\theta_i < \theta_c$ .

(b)  $\theta_i = \theta_c$ .

(c)  $\theta_i > \theta_c$  and total internal reflection (TIR).

angle is called the **critical angle**  $\theta_c$  which is given by

$$\sin \theta_c = \frac{n_2}{n_1} \quad [9.31]$$

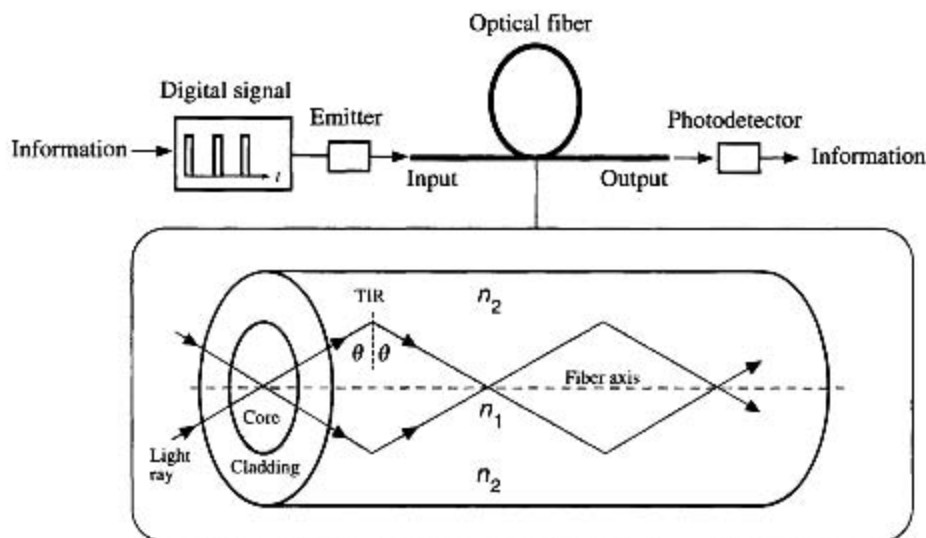
When the incidence angle  $\theta_i$  exceeds  $\theta_c$ , then there is no transmitted wave but only a reflected wave. The latter phenomenon is called **total internal reflection (TIR)**. The effect of increasing the incidence angle is shown in Figure 9.9. It is the TIR phenomenon that leads to the propagation of waves in a dielectric medium surrounded by a medium of smaller refractive index as in optical waveguides (e.g., optical fibers).

*Critical angle  
for total  
internal  
reflection  
(TIR)*

**OPTICAL FIBERS IN COMMUNICATIONS** Figure 9.10 shows a simplified view of a modern optical communications system. Information is converted into a digital signal (e.g., current pulses) which drives a light emitter such as a semiconductor laser. The light pulses from the emitter are coupled into an **optical fiber**, which acts as a light guide. The optical fiber is a very thin glass fiber [made of silica ( $\text{SiO}_2$ )], almost as thin as your hair, that is able to optically guide the light pulses to their destination. The photodetector at the destination converts the light pulses into an electric signal, which is then decoded into the original information.

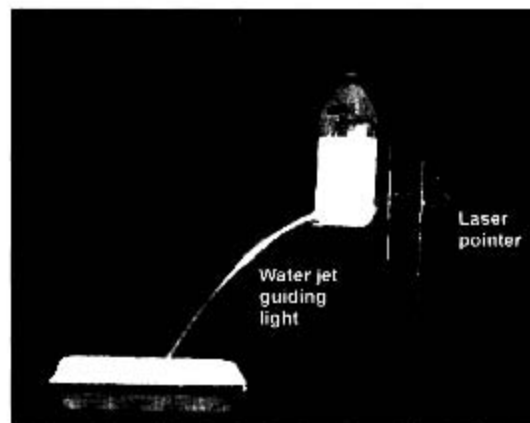
**EXAMPLE 9.7**

The **core** of the optical fiber has a higher refractive index than the surrounding region, which is called the **cladding** as shown in Figure 9.10. Optical fibers for short-distance applications (e.g., communications in local area networks within a large building) usually have a core region that has a diameter of about  $100 \mu\text{m}$ , and the whole fiber would be about  $150\text{--}200 \mu\text{m}$  in diameter. The core and cladding refractive indices,  $n_1$  and  $n_2$ , respectively, are normally only 1–3 percent different. The light propagates along the fiber core because light rays experience total internal reflections at the core-cladding interface as shown in Figure 9.10. Only those light rays that can exercise TIR travel along the fiber length and can reach the destination. Consider a fiber with  $n_1(\text{core}) = 1.455$ , and  $n_2(\text{cladding}) = 1.440$ . The critical angle for a ray traveling



**Figure 9.10** An optical fiber link for transmitting digital information in communications. The fiber core has a higher refractive index, so the light travels along the fiber inside the fiber core by total internal reflection at the core-cladding interface.

A small hole is made in a plastic bottle full of water to generate a water jet. When the hole is illuminated with a laser beam (from a green laser pointer), the light is guided by total internal reflections along the jet to the tray. Light guiding by a water jet was demonstrated by John Tyndall in 1854 to the Royal Institution. [Water with air bubbles was used to increase the visibility of light, since air bubbles scatter light.]



in the core is

$$\theta_c = \arcsin\left(\frac{n_2}{n_1}\right) = \arcsin\left(\frac{1.440}{1.455}\right) = 81.8^\circ$$

Those light rays that have angles  $\theta > \theta_c$  satisfy TIR and can propagate along the fiber.<sup>8</sup> Notice that the ray angles with respect to the fiber axis are less than  $8.2^\circ$ .

<sup>8</sup> The light propagation in an optical fiber is much more complicated than the simple zigzagging of light rays with TIRs at the core-cladding interface. The waves in the core have to satisfy not only TIR but also have to avoid destructive interference so that they are not destroyed as they travel along the guide; see for example, S. O. Kasap, *Optoelectronics and Photonics: Principles and Practices*, Upper Saddle River: Prentice Hall, 2001, chap. 2.

## 9.7 FRESNEL'S EQUATIONS

### 9.7.1 AMPLITUDE REFLECTION AND TRANSMISSION COEFFICIENTS

Although the ray picture with constant phase wave fronts is useful in understanding refraction and reflection, to obtain the magnitude of the reflected and refracted waves and their relative phases, we need to consider the electric field in the light wave. The electric field in the wave must be perpendicular to the direction of propagation as shown in Figure 9.11. We can resolve the field  $E_i$  of the incident wave into two components, one in the plane of incidence  $E_{i\parallel}$  and the other perpendicular to the plane of incidence  $E_{i\perp}$ . The **plane of incidence** is defined as the plane containing the incident and the reflected rays which in Figure 9.11 corresponds to the plane of the paper.<sup>9</sup> Similarly for both the reflected and transmitted waves, we will have field components parallel and perpendicular to the plane of incidence, *i.e.*,  $E_{r\parallel}$ ,  $E_{r\perp}$  and  $E_{t\parallel}$ ,  $E_{t\perp}$ .

As apparent from Figure 9.11, the incident, transmitted, and reflected waves all have a wavevector component along the  $z$  direction; that is, they have an effective velocity along  $z$ . The fields  $E_{i\perp}$ ,  $E_{r\perp}$ , and  $E_{t\perp}$  are all perpendicular to the  $z$  direction. These waves are called **transverse electric field (TE) waves**. On the other hand, waves with  $E_{i\parallel}$ ,  $E_{r\parallel}$ , and  $E_{t\parallel}$  only have their magnetic field components perpendicular to the  $z$  direction and these are called **transverse magnetic field (TM) waves**.

We will describe the incident, reflected, and refracted waves by the exponential representation of a traveling wave, *i.e.*,

$$E_i = E_{i0} \exp j(\omega t - \mathbf{k}_i \cdot \mathbf{r}) \quad [9.32]$$

$$E_r = E_{r0} \exp j(\omega t - \mathbf{k}_r \cdot \mathbf{r}) \quad [9.33]$$

$$E_t = E_{t0} \exp j(\omega t - \mathbf{k}_t \cdot \mathbf{r}) \quad [9.34]$$

*Incident wave*

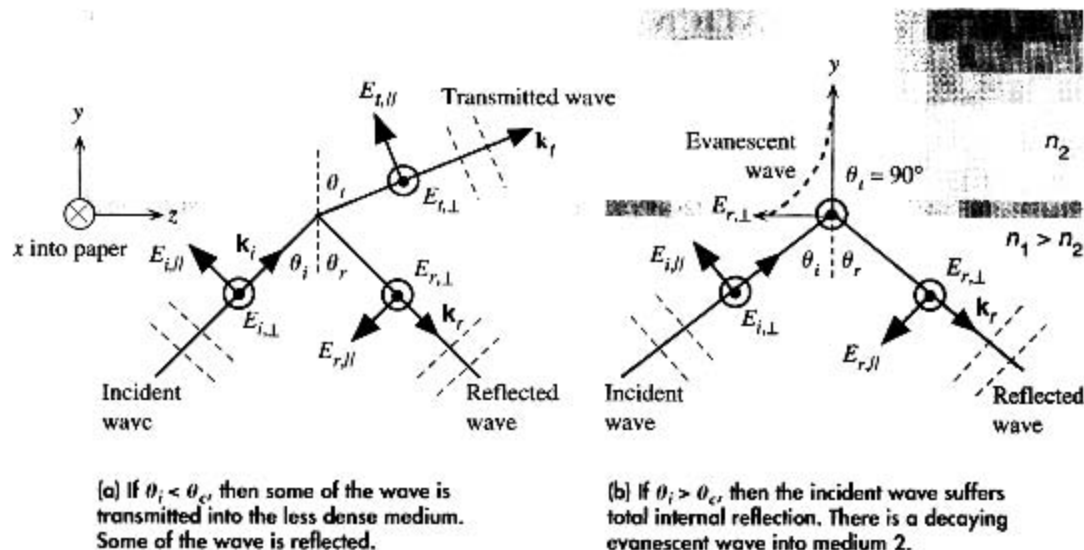
*Reflected  
wave*

*Transmitted  
wave*

where  $\mathbf{r}$  is the position vector; the wavevectors  $\mathbf{k}_i$ ,  $\mathbf{k}_r$ , and  $\mathbf{k}_t$  describe, respectively, the directions of the incident, reflected, and transmitted waves; and  $E_{i0}$ ,  $E_{r0}$ , and  $E_{t0}$  are the respective amplitudes. Any phase changes such as  $\phi_r$  and  $\phi_t$  in the reflected and transmitted waves with respect to the phase of the incident wave are incorporated into the complex amplitudes  $E_{r0}$  and  $E_{t0}$ . Our objective is to find  $E_{r0}$  and  $E_{t0}$  with respect to  $E_{i0}$ .

We should note that similar equations can be stated for the magnetic field components in the incident, reflected, and transmitted waves, but these will be perpendicular to the corresponding electric fields. The electric and magnetic fields anywhere on the wave must be perpendicular to each other as a requirement of electromagnetic wave theory. This means that with  $E_{\parallel}$  in the EM wave we have a magnetic field  $B_{\perp}$  associated

<sup>9</sup>The definitions of the field components follow those of S. G. Lipson et al., *Optical Physics*, 3rd ed., Cambridge, MA, Cambridge University Press, 1995, and Grant Fowles, *Introduction to Modern Optics*, 2nd ed., New York, Dover Publications, Inc., 1975, whose clear treatments of this subject are highly recommended. The majority of the authors use a different convention which leads to different signs later in the equations; Fresnel's equations are related to the specific electric field directions from which they are derived.



**Figure 9.11** Light wave traveling in a more dense medium strikes a less dense medium.

The plane of incidence is the plane of the paper and is perpendicular to the flat interface between the two media. The electric field is normal to the direction of propagation. It can be resolved into perpendicular ( $\perp$ ) and parallel ( $\parallel$ ) components.

with it such that  $B_{\perp} = (n/c)E_{\perp}$ . Similarly  $E_{\perp}$  will have a magnetic field  $B_{\parallel}$  associated with it such that  $B_{\parallel} = (n/c)E_{\parallel}$ .

There are two useful fundamental rules in electromagnetism that govern the behavior of the electric and magnetic fields at a boundary between two dielectric media which we can arbitrarily label as 1 and 2. These rules are called boundary conditions. The first states that the electric field that is tangential to the boundary surface  $E_{\text{tangential}}$  must be continuous across the boundary from medium 1 to 2, *i.e.*, at the boundary  $y = 0$  in Figure 9.11,

Boundary  
condition

$$E_{\text{tangential}}(1) = E_{\text{tangential}}(2) \quad [9.35]$$

The second rule is that the tangential component of the magnetic field  $B_{\text{tangential}}$  to the boundary must be likewise continuous from medium 1 to 2 provided that the two media are nonmagnetic (relative permeability  $\mu_r = 1$ ),

Boundary  
condition

$$B_{\text{tangential}}(1) = B_{\text{tangential}}(2) \quad [9.36]$$

Using these boundary conditions for the fields at  $y = 0$ , and the relationship between the electric and magnetic fields, we can find the reflected and transmitted waves in terms of the incident wave. The boundary conditions can only be satisfied if the reflection and incidence angles are equal,  $\theta_r = \theta_i$ , and the angles for the transmitted and incident waves obey Snell's law,  $n_1 \sin \theta_i = n_2 \sin \theta_r$ .

Applying the boundary conditions to the EM wave going from medium 1 to 2, the amplitudes of the reflected and transmitted waves can be readily obtained in terms of

$n_1$ ,  $n_2$ , and the incidence angle  $\theta_i$  alone.<sup>10</sup> These relationships are called **Fresnel's equations**. If we define  $n = n_2/n_1$ , as the relative refractive index of medium 2 to that of 1, then the **reflection** and **transmission coefficients** for  $E_{\perp}$  are

$$r_{\perp} = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos \theta_i - (n^2 - \sin^2 \theta_i)^{1/2}}{\cos \theta_i + (n^2 - \sin^2 \theta_i)^{1/2}} \quad [9.37] \quad \text{Reflection coefficient}$$

and

$$t_{\perp} = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2 \cos \theta_i}{\cos \theta_i + (n^2 - \sin^2 \theta_i)^{1/2}} \quad [9.38] \quad \text{Transmission coefficient}$$

There are corresponding coefficients for the  $E_{\parallel}$  fields with corresponding **reflection** and **transmission coefficients**  $r_{\parallel}$  and  $t_{\parallel}$ :

$$r_{\parallel} = \frac{E_{r0,\parallel}}{E_{i0,\parallel}} = \frac{(n^2 - \sin^2 \theta_i)^{1/2} - n^2 \cos \theta_i}{(n^2 - \sin^2 \theta_i)^{1/2} + n^2 \cos \theta_i} \quad [9.39] \quad \text{Reflection coefficient}$$

$$t_{\parallel} = \frac{E_{t0,\parallel}}{E_{i0,\parallel}} = \frac{2n \cos \theta_i}{n^2 \cos \theta_i + (n^2 - \sin^2 \theta_i)^{1/2}} \quad [9.40] \quad \text{Transmission coefficient}$$

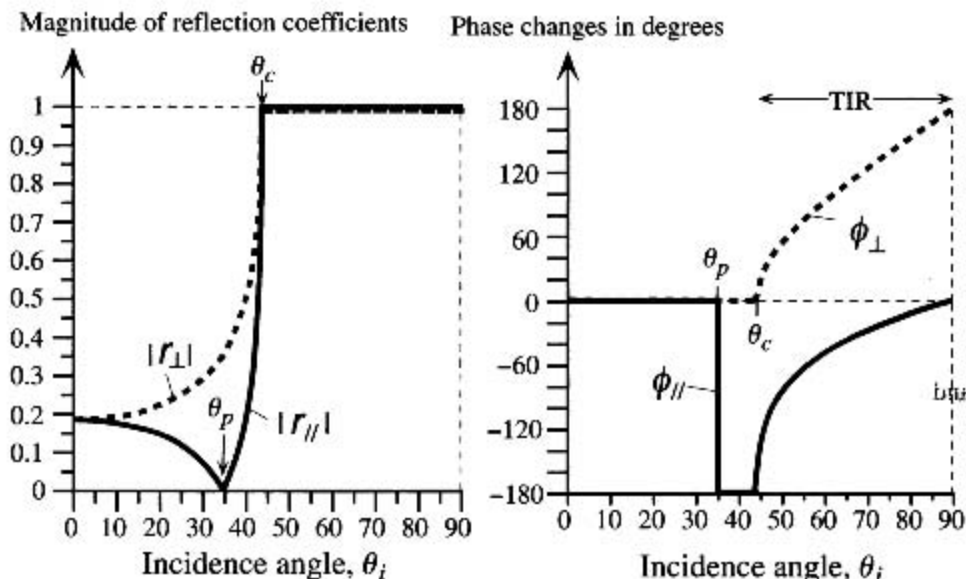
Further, the reflection and transmission coefficients are related by

$$r_{\parallel} + n t_{\parallel} = 1 \quad \text{and} \quad r_{\perp} + t_{\perp} = 1 \quad [9.41] \quad \text{Transmission and reflection}$$

The significance of these equations is that they allow the amplitudes and phases of the reflected and transmitted waves to be determined from the coefficients  $r_{\perp}$ ,  $r_{\parallel}$ ,  $t_{\perp}$ , and  $t_{\parallel}$ . For convenience we take  $E_{i0}$  to be a real number so that the phase angles of  $r_{\perp}$  and  $t_{\perp}$  correspond to the **phase changes** measured with respect to the incident wave. For example, if  $r_{\perp}$  is a complex quantity, then we can write this as  $r_{\perp} = |r_{\perp}| \exp(-j\phi_{\perp})$  where  $|r_{\perp}|$  and  $\phi_{\perp}$  represent the relative amplitude and phase of the reflected wave with respect to the incident wave for the field perpendicular to the plane of incidence. Of course, when  $r_{\perp}$  is a real quantity, then a positive number represents no phase shift and a negative number is a phase shift of  $180^\circ$  (or  $\pi$ ). As with all waves, a negative sign corresponds to a  $180^\circ$  phase shift. Complex coefficients can only be obtained from Fresnel's equations if the terms under the square roots become negative, and this can only happen when  $n < 1$  (or  $n_1 > n_2$ ), and also when  $\theta_i > \theta_c$ , the critical angle. Thus, phase changes other than  $0$  or  $180^\circ$  occur only when there is total internal reflection.

Figure 9.12a shows how the magnitudes of the reflection coefficients  $|r_{\perp}|$  and  $|r_{\parallel}|$  vary with the incidence angle  $\theta_i$  for a light wave traveling from a more dense medium,  $n_1 = 1.44$ , to a less dense medium,  $n_2 = 1.00$ , as predicted by Fresnel's equations. Figure 9.12b shows the changes in the phase of the reflected wave,  $\phi_{\perp}$  and  $\phi_{\parallel}$ , with  $\theta_i$ . The critical angle  $\theta_c$  as determined from  $\sin \theta_c = n_2/n_1$  in this case is  $44^\circ$ . It is clear that for incidence close to normal (small  $\theta_i$ ), there is no phase change in the reflected wave. For

<sup>10</sup> These equations are readily available in any electromagnetism textbook. Their derivation from the two boundary conditions involves extensive algebraic manipulation which we will not carry out here. The electric and magnetic field components on both sides of the boundary are resolved tangentially to the boundary surface and the boundary conditions are then applied. We then use such relations as  $\cos \theta_t = [1 - \sin^2 \theta_i]^{1/2}$  and  $\sin \theta_t$  as determined by Snell's law, etc.



**Figure 9.12** Internal reflection.

(a) Magnitude of the reflection coefficients  $r_{\parallel}$  and  $r_{\perp}$  versus the angle of incidence  $\theta_i$  for  $n_1 = 1.44$  and  $n_2 = 1.00$ . The critical angle is  $44^\circ$ .

(b) The corresponding phase changes  $\phi_{\parallel}$  and  $\phi_{\perp}$  versus incidence angle.

example, putting normal incidence ( $\theta_i = 0$ ) into Fresnel's equations, we find

Normal  
incidence

$$r_{\parallel} = r_{\perp} = \frac{n_1 - n_2}{n_1 + n_2} \quad [9.42]$$

This is a positive quantity for  $n_1 > n_2$  which means that the reflected wave suffers no phase change. This is confirmed by  $\phi_{\perp}$  and  $\phi_{\parallel}$  in Figure 9.12b. As the incidence angle increases, eventually  $r_{\parallel}$  becomes zero at an angle of about  $35^\circ$ . We can find this special incidence angle, labeled as  $\theta_p$ , by solving the Fresnel equation, Equation 9.39, for  $r_{\parallel} = 0$ . The field in the reflected wave is then always perpendicular to the plane of incidence and hence well-defined. This special angle is called the **polarization angle** or **Brewster's angle** and from Equation 9.39 is given by

Brewster's  
polarization  
angle

$$\tan \theta_p = \frac{n_2}{n_1} \quad [9.43]$$

The reflected wave is then said to be **linearly polarized** because it contains *electric field oscillations that are contained within a well-defined plane* which is perpendicular to the plane of incidence and also to the direction of propagation. Electric field oscillations in **unpolarized light**, on the other hand, can be in any one of an infinite number of directions that are perpendicular to the direction of propagation. In linearly polarized light, however, the field oscillations are contained within a well-defined plane. Light emitted from many light sources such as a tungsten light bulb or an LED diode is



unpolarized and the field is randomly oriented in a direction that is perpendicular to the direction of propagation.

For incidence angles greater than  $\theta_p$  but smaller than  $\theta_c$ , Fresnel's equation, Equation 9.39, gives a negative number for  $r_{\parallel}$  which indicates a phase shift of  $180^\circ$  as shown in  $\phi_{\parallel}$  in Figure 9.12b. The magnitudes of both  $r_{\perp}$  and  $r_{\parallel}$  increase with  $\theta_i$  as apparent in Figure 9.12a. At the critical angle and beyond (past  $44^\circ$  in Figure 9.12), *i.e.*, when  $\theta_i \geq \theta_c$ , the magnitudes of both  $r_{\perp}$  and  $r_{\parallel}$  go to unity, so the reflected wave has the same amplitude as the incident wave. The incident wave has suffered **total internal reflection (TIR)**. When  $\theta_i > \theta_c$ , in the presence of TIR, the Equations 9.37 to 9.40 are complex quantities because then  $\sin \theta_i > n$  and the terms under the square roots become negative. The reflection coefficients become complex quantities of the type  $r_{\perp} = 1 \cdot \exp(-j\phi_{\perp})$  and  $r_{\parallel} = 1 \cdot \exp(-j\phi_{\parallel})$  with the phase angles  $\phi_{\perp}$  and  $\phi_{\parallel}$  being other than 0 or  $180^\circ$ . The reflected wave therefore suffers phase changes  $\phi_{\perp}$  and  $\phi_{\parallel}$  in the components  $E_{\perp}$  and  $E_{\parallel}$ . These phase changes depend on the incidence angle, as apparent in Figure 9.12b, and on  $n_1$  and  $n_2$ .

Examination of Equation 9.37 for  $r_{\perp}$  shows that for  $\theta_i > \theta_c$ , we have  $|r_{\perp}| = 1$ , but the phase change  $\phi_{\perp}$  is given by

$$\tan\left(\frac{1}{2}\phi_{\perp}\right) = \frac{(\sin^2 \theta_i - n^2)^{1/2}}{\cos \theta_i} \quad 9.44 \quad \text{Phase change in TIR}$$

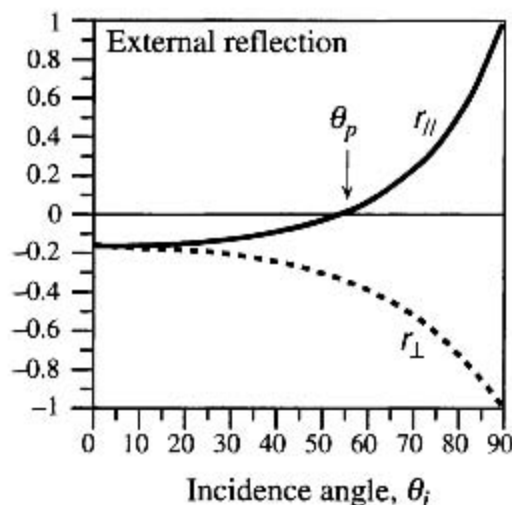
For the  $E_{\parallel}$  component, the phase change  $\phi_{\parallel}$  is given by

$$\tan\left(\frac{1}{2}\phi_{\parallel} + \frac{1}{2}\right) = \frac{(\sin^2 \theta_i - n^2)^{1/2}}{n^2 \cos \theta_i} \quad 9.45 \quad \text{Phase change in TIR}$$

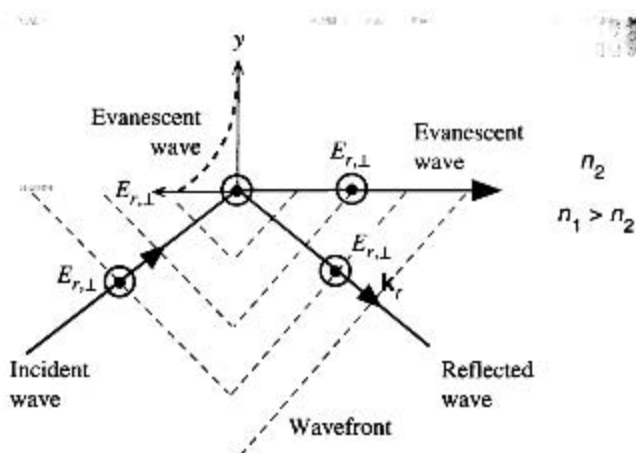
We can summarize that, in internal reflection ( $n_1 > n_2$ ), the amplitude of the reflected wave from TIR is equal to the amplitude of the incident wave but its phase has shifted by an amount determined by Equations 9.44 and 9.45.<sup>11</sup> The fact that  $\phi_{\parallel}$  has an additional  $\pi$  shift which makes  $\phi_{\parallel}$  negative for  $\theta_i > \theta_c$  is due to the choice for the direction of the reflected optical field  $E_{r,\parallel}$  in Figure 9.11. (This  $\pi$  shift can be ignored if we simply invert  $E_{r,\parallel}$ .)

The reflection coefficients in Figure 9.12 considered the case in which  $n_1 > n_2$ . When light approaches the boundary from the higher index side, that is,  $n_1 > n_2$ , the reflection is said to be **internal reflection** and *at normal incidence there is no phase change*. On the other hand, if light approaches the boundary from the lower index side, that is,  $n_1 < n_2$ , then it is called **external reflection**. Thus in external reflection light becomes reflected by the surface of an optically denser (higher refractive index) medium. There is an important difference between the two. Figure 9.13 shows how the reflection coefficients  $r_{\perp}$  and  $r_{\parallel}$  depend on the incidence angle  $\theta_i$  for external reflection ( $n_1 = 1$  and  $n_2 = 1.44$ ). At normal incidence, both coefficients are negative, which means that *in external reflection at normal incidence there is a phase shift of  $180^\circ$* . Further,  $r_{\parallel}$  goes through zero at the **Brewster angle**  $\theta_p$  given by Equation 9.43. At this angle of incidence, the reflected wave is polarized in the  $E_{\perp}$  component only. Transmitted light in both internal reflection (when  $\theta_i < \theta_c$ ) and external reflection does not experience a phase shift.

<sup>11</sup> It should be apparent that the concepts and the resulting equations apply to a well-defined linearly polarized light wave.



**Figure 9.13** The reflection coefficients  $r_{\parallel}$  and  $r_{\perp}$  versus angle of incidence  $\theta_i$  for  $n_1 = 1.00$  and  $n_2 = 1.44$ .



**Figure 9.14** When  $\theta_i > \theta_c$ , for a plane wave that is reflected, there is an evanescent wave at the boundary propagating along  $z$ .

What happens to the transmitted wave when  $\theta_i > \theta_c$ ? According to the boundary conditions, there must still be an electric field in medium 2; otherwise, the boundary conditions cannot be satisfied. When  $\theta_i > \theta_c$ , the field in medium 2 is a wave that travels near the surface of the boundary along the  $z$  direction as depicted in Figure 9.14. The wave is called an **evanescent wave** and advances along  $z$  with its field decreasing as we move into medium 2, *i.e.*,

Evanescent  
wave

$$E_{t,\perp}(y, z, t) \propto e^{-\alpha_2 y} \exp j(\omega t - k_{i2} z) \quad [9.46]$$

where  $k_{i2} = k_i \sin \theta_i$  is the wavevector of the incident wave along the  $z$  axis, and  $\alpha_2$  is an **attenuation coefficient** for the electric field penetrating into medium 2,

Attenuation  
of evanescent  
wave

$$\alpha_2 = \frac{2\pi n_2}{\lambda} \left[ \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_i - 1 \right]^{1/2} \quad [9.47]$$

where  $\lambda$  is the free-space wavelength. According to Equation 9.46, the evanescent wave travels along  $z$  and has an amplitude that decays exponentially as we move from the boundary into medium 2 (along  $y$ ) as shown in Figure 9.11b. The field of the evanescent wave is  $e^{-1}$  in medium 2 when  $y = 1/\alpha_2 = \delta$  which is called the **penetration depth**. It is not difficult to show that the evanescent wave is correctly predicted by Snell's law when  $\theta_i > \theta_c$ . The evanescent wave propagates along the boundary (along  $z$ ) with the same speed as the  $z$  component velocity of the incident and reflected waves. In Equations 9.32 to 9.34 we had assumed that the incident and reflected waves were *plane waves*, that is, of infinite extent. If we were to extend the plane wavefronts on the reflected wave, these would cut the boundary as shown in Figure 9.14. The evanescent wave traveling along  $z$  can be thought of as arising from these plane wavefronts at the boundary as in Figure 9.14. (The evanescent wave is important in light propagation in optical waveguides such as in optical fibers.) If the incident wave is a narrow beam of light (e.g., from a laser pointer), then the reflected beam would have the same cross section. There would still be an evanescent wave at the boundary, but it would exist only within the cross-sectional area of the reflected beam at the boundary.

## 9.7.2 INTENSITY, REFLECTANCE, AND TRANSMITTANCE

It is frequently necessary to calculate the intensity or irradiance of the reflected and transmitted waves when light traveling in a medium of index  $n_1$  is incident at a boundary where the refractive index changes to  $n_2$ . In some cases we are simply interested in normal incidence where  $\theta_i = 0^\circ$ . For example, in laser diodes light is reflected from the ends of an optical cavity where there is a change in the refractive index.

**Reflectance**  $R$  measures the intensity of the reflected light with respect to that of the incident light and can be defined separately for electric field components parallel and perpendicular to the plane of incidence. The reflectances  $R_\perp$  and  $R_\parallel$  are defined by

$$R_\perp = \frac{|E_{r0,\perp}|^2}{|E_{i0,\perp}|^2} = |r_\perp|^2 \quad \text{and} \quad R_\parallel = \frac{|E_{r0,\parallel}|^2}{|E_{i0,\parallel}|^2} = |r_\parallel|^2 \quad (9.48)$$

From Equations 9.37 to 9.40 with normal incidence, these are simply given by

$$R = R_\perp = R_\parallel = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 \quad (9.49)$$

*Reflectance  
at normal  
incidence*

Since a glass medium has a refractive index of around 1.5, this means that typically 4 percent of the incident radiation on an air-glass surface will be reflected back.

**Transmittance**  $T$  relates the intensity of the transmitted wave to that of the incident wave in a similar fashion to the reflectance. We must, however, consider that the transmitted wave is in a different medium and further its direction with respect to the boundary is also different by virtue of refraction. For normal incidence, the incident and transmitted beams are normal and the transmittances are defined and given by

$$T_\perp = \frac{n_2 |E_{t0,\perp}|^2}{n_1 |E_{i0,\perp}|^2} = \left( \frac{n_2}{n_1} \right) |t_\perp|^2 \quad \text{and} \quad T_\parallel = \frac{n_2 |E_{t0,\parallel}|^2}{n_1 |E_{i0,\parallel}|^2} = \left( \frac{n_2}{n_1} \right) |t_\parallel|^2 \quad (9.50)$$

Transmittance at normal incidence

or

$$T = T_{\perp} = T_{\parallel} = \frac{4n_1n_2}{(n_1 + n_2)^2} \quad [9.51]$$

Further, the fraction of light reflected and fraction transmitted must add to unity. Thus  $R + T = 1$ .

### EXAMPLE 9.8

**REFLECTION OF LIGHT FROM A LESS DENSE MEDIUM (INTERNAL REFLECTION)** A ray of light which is traveling in a glass medium of refractive index  $n_1 = 1.460$  becomes incident on a less dense glass medium of refractive index  $n_2 = 1.440$ . Suppose that the free-space wavelength ( $\lambda$ ) of the light ray is 1300 nm.

- What should be the minimum incidence angle for TIR?
- What is the phase change in the reflected wave when  $\theta_i = 87^\circ$  and when  $\theta_i = 90^\circ$ ?
- What is the penetration depth of the evanescent wave into medium 2 when  $\theta_i = 80^\circ$  and when  $\theta_i = 90^\circ$ ?

#### SOLUTION

- The critical angle  $\theta_c$  for TIR is given by  $\sin \theta_c = n_2/n_1 = 1.440/1.460$ , so  $\theta_c = 80.51^\circ$ .
- Since the incidence angle  $\theta_i > \theta_c$ , there is a phase shift in the reflected wave. The phase change in  $E_{r,\perp}$  is given by  $\phi_{\perp}$ . With  $n_1 = 1.460$ ,  $n_2 = 1.440$ , and  $\theta_i = 87^\circ$ ,

$$\begin{aligned} \tan\left(\frac{1}{2}\phi_{\perp}\right) &= \frac{(\sin^2 \theta_i - n_2^2)^{1/2}}{\cos \theta_i} = \frac{\left[\sin^2(87^\circ) - \left(\frac{1.440}{1.460}\right)^2\right]^{1/2}}{\cos(87^\circ)} \\ &= 2.989 = \tan\left[\frac{1}{2}(143.0^\circ)\right] \end{aligned}$$

so the phase change is  $143^\circ$ . For the  $E_{r,\parallel}$  component, the phase change is

$$\tan\left(\frac{1}{2}\phi_{\parallel} + \frac{1}{2}\pi\right) = \frac{(\sin^2 \theta_i - n_2^2)^{1/2}}{n_2 \cos \theta_i} = \frac{1}{n_2} \tan\left(\frac{1}{2}\phi_{\perp}\right)$$

so

$$\tan\left(\frac{1}{2}\phi_{\parallel} + \frac{1}{2}\pi\right) = \left(\frac{n_1}{n_2}\right)^2 \tan\left(\frac{\phi_{\perp}}{2}\right) = \left(\frac{1.460}{1.440}\right)^2 \tan\left[\frac{1}{2}(143^\circ)\right]$$

which gives

$$\phi_{\parallel} = 143.95^\circ - 180^\circ = -36.05^\circ$$

We can repeat the calculation with  $\theta_i = 90^\circ$  to find  $\phi_{\perp} = 180^\circ$  and  $\phi_{\parallel} = 0^\circ$ .

Note that as long as  $\theta_i > \theta_c$ , the magnitude of the reflection coefficients are unity. Only the phase changes.

- The amplitude of the evanescent wave as it penetrates into medium 2 is

$$E_{i,\perp}(y, t) \approx E_{r,\perp} \exp(-\alpha_2 y)$$

We ignore the  $z$  dependence,  $\exp j(\omega t - k_z z)$ , as this only gives a propagating property along  $z$ . The field strength drops to  $e^{-1}$  when  $y = 1/\alpha_2 = \delta$ , which is called the **penetration**

**depth.** The attenuation constant  $\alpha_2$  is

$$\alpha_2 = \frac{2\pi n_2}{\lambda} \left[ \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_i - 1 \right]^{1/2}$$

i.e.,

$$\alpha_2 = \frac{2\pi(1.440)}{(1300 \times 10^{-9} \text{ m})} \left[ \left( \frac{1.460}{1.440} \right)^2 \sin^2(87^\circ) - 1 \right]^{1/2} = 1.104 \times 10^6 \text{ m}^{-1}$$

so the penetration depth is  $\delta = 1/\alpha_2 = 1/(1.104 \times 10^6 \text{ m}) = 9.06 \times 10^{-7} \text{ m}$ , or  $0.906 \mu\text{m}$ . For  $90^\circ$ , repeating the calculation we find  $\alpha_2 = 1.164 \times 10^6 \text{ m}^{-1}$ , so  $\delta = 1/\alpha_2 = 0.859 \mu\text{m}$ . We see that the penetration is greater for smaller incidence angles. The values for the refractive indices and wavelength are typical of those values found in optical fiber communications.

**REFLECTION AT NORMAL INCIDENCE. INTERNAL AND EXTERNAL REFLECTION** Consider the reflection of light at normal incidence on a boundary between a glass medium of refractive index 1.5 and air of refractive index 1.

**EXAMPLE 9.9**

- If light is traveling from air to glass, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?
- If light is traveling from glass to air, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?
- What is the polarization angle in the external reflection in part (a)? How would you make a polaroid device that polarizes light based on the polarization angle?

**SOLUTION**

- The light travels in air and becomes partially reflected at the surface of the glass which corresponds to external reflection. Thus  $n_1 = 1$  and  $n_2 = 1.5$ . Then,

$$r_{\perp} = r_{\parallel} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2$$

This is negative which means that there is a  $180^\circ$  phase shift. The reflectance ( $R$ ), which gives the fractional reflected power, is

$$R = r_{\perp}^2 = 0.04 \quad \text{or} \quad 4\%$$

- The light travels in glass and becomes partially reflected at the glass-air interface which corresponds to internal reflection. Thus  $n_1 = 1.5$  and  $n_2 = 1$ . Then,

$$r_{\perp} = r_{\parallel} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1.5 - 1}{1.5 + 1} = 0.2$$

There is no phase shift. The reflectance is again 0.04 or 4 percent. In both cases (a) and (b), the amount of reflected light is the same.

- Light is traveling in air and is incident on the glass surface at the polarization angle. Here  $n_1 = 1$ ,  $n_2 = 1.5$ , and  $\tan \theta_p = (n_2/n_1) = 1.5$ , so  $\theta_p = 56.3^\circ$ .

If we were to reflect light from a glass plate keeping the angle of incidence at  $56.3^\circ$ , then the reflected light will be polarized with an electric field component perpendicular to the plane of incidence. The transmitted light will have the field greater in the plane of incidence; that is,

it will be partially polarized. By using a stack of glass plates one can increase the polarization of the transmitted light. (This type of *pile-of-plates polarizer* was invented by Dominique F. J. Arago in 1812.)

**EXAMPLE 9.10**

**ANTIREFLECTION COATINGS ON SOLAR CELLS** When light is incident on the surface of a semiconductor, it becomes partially reflected. Partial reflection is an important consideration in solar cells where transmitted light energy into the semiconductor device is converted to electric energy. The refractive index of Si is about 3.5 at wavelengths around 700–800 nm. Thus the reflectance with  $n_1(\text{air}) = 1$  and  $n_2(\text{Si}) \approx 3.5$  is

$$R = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left( \frac{1 - 3.5}{1 + 3.5} \right)^2 = 0.309$$

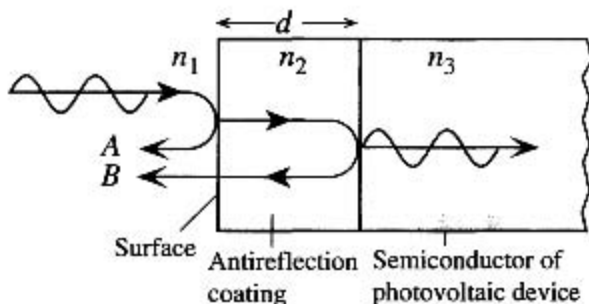
This means that 30 percent of the light is reflected and is not available for conversion to electric energy, a considerable reduction in the efficiency of the solar cell.

However, we can coat the surface of the semiconductor device with a thin layer of a dielectric material such as  $\text{Si}_3\text{N}_4$  (silicon nitride) that has an intermediate refractive index. Figure 9.15 illustrates how the thin dielectric coating reduces the reflected light intensity. In this case  $n_1(\text{air}) = 1$ ,  $n_2(\text{coating}) \approx 1.9$ , and  $n_3(\text{Si}) = 3.5$ . Light is first incident on the air-coating surface, and some of it becomes reflected; this reflected wave is shown as *A* in Figure 9.15. Wave *A* has experienced a  $180^\circ$  phase change on reflection as this is an external reflection. The wave that enters and travels in the coating then becomes reflected at the coating-semiconductor surface. This wave, which is shown as *B*, also suffers a  $180^\circ$  phase change since  $n_3 > n_2$ . When wave *B* reaches *A*, it has suffered a total delay of traversing the thickness  $d$  of the coating twice. The phase difference is equivalent to  $k_c(2d)$  where  $k_c = 2\pi/\lambda_c$  is the wavevector in the coating and is given by  $2\pi/\lambda_c$  where  $\lambda_c$  is the wavelength in the coating. Since  $\lambda_c = \lambda/n_2$ , where  $\lambda$  is the free-space wavelength, the phase difference  $\Delta\phi$  between *A* and *B* is  $(2\pi n_2/\lambda)(2d)$ . To reduce the reflected light, *A* and *B* must interfere destructively, and this requires the phase difference to be  $\pi$  or odd multiples of  $\pi$ ,  $m\pi$  where  $m = 1, 3, 5, \dots$  is an odd integer. Thus

$$\left( \frac{2\pi n_2}{\lambda} \right) 2d = m\pi \quad \text{or} \quad d = m \left( \frac{\lambda}{4n_2} \right)$$

Thus, the thickness of the coating must be multiples of the quarter wavelength in the coating and depends on the wavelength.

**Figure 9.15** Illustration of how an antireflection coating reduces the reflected light intensity.



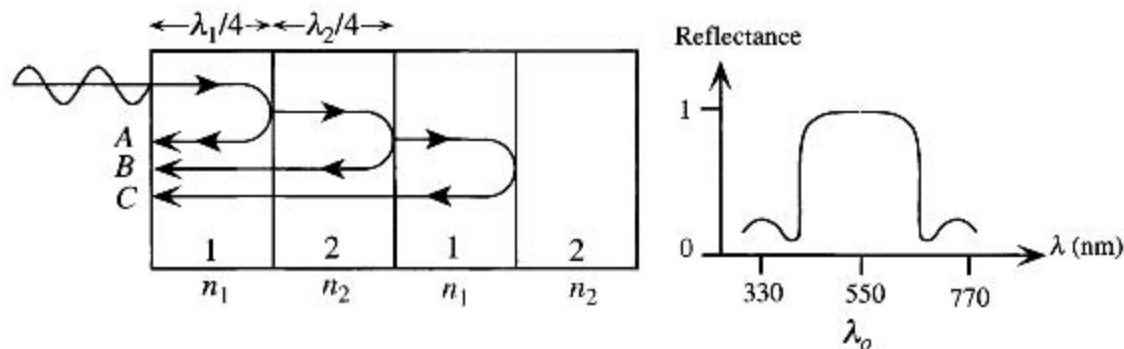
To obtain a good degree of destructive interference between waves  $A$  and  $B$ , the two amplitudes must be comparable. It turns out that we need  $n_2 = \sqrt{n_1 n_3}$ . When  $n_2 = \sqrt{n_1 n_3}$ , then the reflection coefficient between the air and coating is equal to that between the coating and the semiconductor. In this case we would need  $\sqrt{3.5}$  or 1.87. Thus,  $\text{Si}_3\text{N}_4$  is a good choice as an antireflection coating material on Si solar cells.

Taking the wavelength to be 700 nm,  $d = (700 \text{ nm})/[4(1.9)] = 92.1 \text{ nm}$  or odd multiples of  $d$ .

**DIELECTRIC MIRRORS** A dielectric mirror consists of a stack of dielectric layers of alternating refractive indices as schematically illustrated in Figure 9.16 where  $n_1$  is smaller than  $n_2$ . The thickness of each layer is a quarter wavelength or  $\lambda_{\text{layer}}/4$ , where  $\lambda_{\text{layer}}$  is the wavelength of light in that layer, or  $\lambda_0/n$ , where  $\lambda_0$  is the free-space wavelength at which the mirror is required to reflect the incident light and  $n$  is the refractive index of the layer. Reflected waves from the interfaces interfere constructively and give rise to a substantial reflected light. If there are a sufficient number of layers, the reflectance can approach unity at the wavelength  $\lambda_0$ . Figure 9.16 also shows schematically a typical reflectance versus wavelength behavior of a dielectric mirror with many layers.

## EXAMPLE 9.11

The reflection coefficient  $r_{12}$  for light in layer 1 being reflected at the 1–2 boundary is  $r_{12} = (n_1 - n_2)/(n_1 + n_2)$  and is a negative number indicating a  $\pi$  phase change. The reflection coefficient for light in layer 2 being reflected at the 2–1 boundary is  $r_{21} = (n_2 - n_1)/(n_1 + n_2)$  which is  $-r_{12}$  (positive) indicating no phase change. Thus the reflection coefficient alternates in sign through the mirror. Consider two arbitrary waves  $A$  and  $B$  which are reflected at two consecutive interfaces. The two waves are therefore already out of phase by  $\pi$  due to reflections at the different boundaries. Further, wave  $B$  travels an additional distance which is twice  $(\lambda_2/4)$  before reaching wave  $A$  and therefore experiences a phase change equivalent to  $2(\lambda_2/4)$  or  $\lambda_2/2$ , that is,  $\pi$ . The phase difference between  $A$  and  $B$  is then  $\pi + \pi$  or  $2\pi$ . Thus waves  $A$  and  $B$  are in phase and *interfere constructively*. We can similarly show that waves  $B$  and  $C$  also interfere constructively and so on, so all reflected waves from the consecutive boundaries interfere constructively. After several layers (depending on  $n_1$  and  $n_2$ ), the transmitted intensity will be very small and the reflected light intensity will be close to unity. Dielectric mirrors are widely used in modern vertical cavity surface emitting semiconductor lasers.



**Figure 9.16** Schematic illustration of the principle of the dielectric mirror with many low and high refractive index layers and its reflectance.

## 9.8 COMPLEX REFRACTIVE INDEX AND LIGHT ABSORPTION

Generally when light propagates through a material, it becomes *attenuated* in the direction of propagation as illustrated in Figure 9.17. We distinguish between *absorption* and *scattering* both of which give rise to a loss of intensity in the regular direction of propagation. In **absorption**, the loss in the power in the propagating EM wave is due to the conversion of light energy to other forms of energy, *e.g.*, lattice vibrations (heat) during the polarization of the molecules of the medium, local vibrations of impurity ions, and excitation of electrons from the valence band to the conduction band. On the other hand, **scattering** is a process by which the energy from a propagating EM wave is redirected as secondary EM waves in various directions away from the original direction of propagation; this is discussed in Section 9.11.

It is instructive to consider what happens when a monochromatic light wave such as

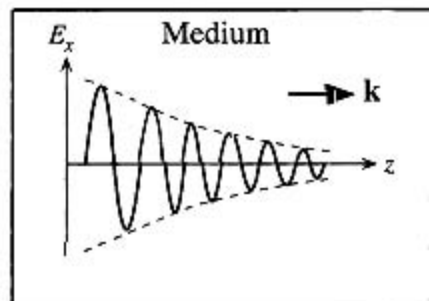
$$E = E_0 \exp j(\omega t - kz) \quad [9.52]$$

is propagating in a dielectric medium. The electric field  $E$  in Equation 9.52 is either parallel to  $x$  or  $y$  since propagation is along  $z$ . As the wave travels through the medium, the molecules become polarized. This polarization effect is represented by the *relative permittivity*  $\epsilon_r$  of the medium. If there were no losses in the polarization process, then the relative permittivity  $\epsilon_r$  would be a real number and the corresponding refractive index  $n = \sqrt{\epsilon_r}$  would also be a real number. However, we know that there are always some losses in all polarization processes. For example, when the ions of an ionic crystal are displaced from their equilibrium positions by an alternating electric field and made to oscillate, some of the energy from the electric field is coupled and converted to lattice vibrations (intuitively, “sound” and heat). These losses are generally accounted for by describing the whole medium in terms of a **complex relative permittivity** (or **dielectric constant**)  $\epsilon_r$ , that is,

$$\epsilon_r = \epsilon_r' - j\epsilon_r'' \quad [9.53]$$

where the real part  $\epsilon_r'$  determines the polarization of the medium with losses ignored and the imaginary part  $\epsilon_r''$  describes the losses in the medium. For a lossless medium, obviously  $\epsilon_r = \epsilon_r'$ . The loss  $\epsilon_r''$  depends on the frequency of the wave and usually peaks at certain natural (resonant) frequencies. If the medium has a finite conductivity

**Figure 9.17** Attenuation of light in the direction of propagation.



Lossless  
propagation

Complex  
dielectric  
constant



(e.g., due to a small number of conduction electrons), then there will be a Joule loss due to the electric field in the wave driving these conduction electrons. This type of light attenuation is called **free carrier absorption**. In such cases,  $\epsilon_r''$  and  $\sigma$  are related by

$$\epsilon_r'' = \frac{\sigma}{\epsilon_0 \omega} \quad [9.54] \quad \text{Conduction loss}$$

where  $\epsilon_0$  is the absolute permittivity and  $\sigma$  is the conductivity at the frequency of the EM wave. Since  $\epsilon_r$  is a complex quantity, we should also expect to have a *complex refractive index*.

An EM wave that is traveling in a medium and experiencing attenuation due to absorption can be generally described by a **complex propagation constant**  $k$ , that is,

$$k = k' - jk'' \quad [9.55] \quad \text{Complex propagation constant}$$

where  $k'$  and  $k''$  are the real and imaginary parts. If we put Equation 9.55 into Equation 9.52, we will find the following,

$$E = E_0 \exp(-k''z) \exp j(\omega t - k'z) \quad [9.56] \quad \text{Attenuated propagation}$$

The amplitude decays exponentially while the wave propagates along  $z$ . The **real**  $k'$  part of the complex propagation constant (wavevector) describes the propagation characteristics, e.g., phase velocity  $v = \omega/k'$ . The **imaginary**  $k''$  part describes the rate of attenuation along  $z$ . The intensity  $I$  at any point along  $z$  is

$$I \propto |E|^2 \propto \exp(-2k''z)$$

so the rate of change in the intensity with distance is

$$\frac{dI}{dz} = -2k''I \quad [9.57] \quad \text{Imaginary part } k''$$

where the negative sign represents attenuation.

Suppose that  $k_0$  is the propagation constant in a vacuum. This is a real quantity as a plane wave suffers no loss in free space. The **complex refractive index**  $N$  with real part  $n$  and imaginary part  $K$  is defined as the ratio of the complex propagation constant in a medium to propagation constant in free space,

$$N = n - jK = \frac{k}{k_0} = \left(\frac{1}{k_0}\right)(k' - jk'') \quad [9.58a] \quad \text{Complex refractive index}$$

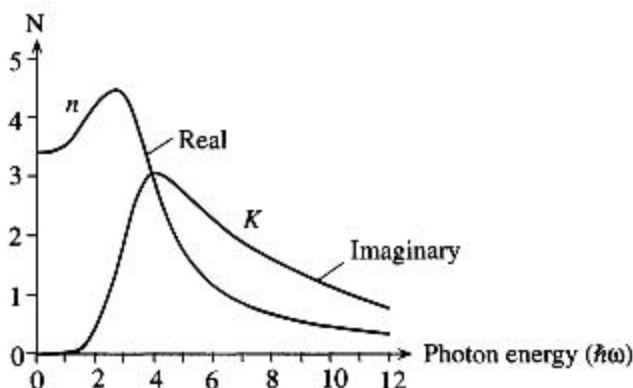
i.e.,

$$n = \frac{k'}{k_0} \quad \text{and} \quad K = \frac{k''}{k_0} \quad [9.58b] \quad \text{Refractive index and extinction coefficient}$$

The real part  $n$  is simply and generally called the **refractive index** and  $K$  is called the **extinction coefficient**. In the absence of attenuation,

$$k'' = 0 \quad k = k' \quad \text{and} \quad N = n = \frac{k}{k_0} = \frac{k'}{k_0}$$

**Figure 9.18** Optical properties of an amorphous silicon film in terms of real ( $n$ ) and imaginary ( $K$ ) parts of the complex refractive index.



We know that in the absence of loss, the relationship between the refractive index  $n$  and the relative permittivity  $\epsilon_r$  is  $n = \sqrt{\epsilon_r}$ . This relationship is also valid in the presence of loss except that we must use complex refractive index and complex relative permittivity, that is,

$$N = n - jK = \sqrt{\epsilon_r} = \sqrt{\epsilon_r' - j\epsilon_r''} \quad [9.59]$$

By squaring both sides we can relate  $n$  and  $K$  directly to  $\epsilon_r'$  and  $\epsilon_r''$ . The final result is

$$n^2 - K^2 = \epsilon_r' \quad \text{and} \quad 2nK = \epsilon_r'' \quad [9.60]$$

Optical properties of materials are typically reported either by showing the frequency dependences of  $n$  and  $K$  or  $\epsilon_r'$  and  $\epsilon_r''$ . Clearly we can use Equation 9.60 to obtain one set of properties from the other. Figure 9.18 shows the real ( $n$ ) and imaginary ( $K$ ) parts of the complex refractive index of amorphous silicon (noncrystalline form of Si) as a function of photon energy ( $h\nu$ ). For photon energies below the bandgap energy,  $K$  is negligible and  $n$  is close to 3.5. Both  $n$  and  $K$  change strongly as the photon energy increases far beyond the bandgap energy.

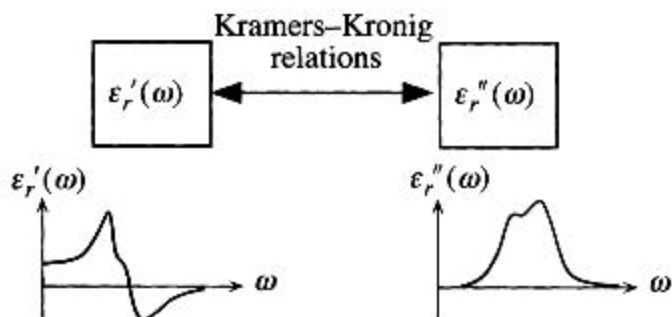
If we know the frequency dependence of the real part  $\epsilon_r'$  of the relative permittivity of a material, we can also determine the frequency dependence of the imaginary part  $\epsilon_r''$ , and vice versa. This may seem remarkable, but it is true provided that we know the frequency dependence of either the real or imaginary part over as wide a range of frequencies as possible (ideally from dc to infinity) and the material is *linear*, i.e., it has a relative permittivity that is independent of the applied field; the polarization response must be linearly proportional to the applied field.<sup>12</sup> The relationships that relate the real and imaginary parts of the relative permittivity are called **Kramers–Kronig relations**. If  $\epsilon_r'(\omega)$  and  $\epsilon_r''(\omega)$  represent the frequency dependences of the real and imaginary parts, respectively, then one can be determined from the other as depicted schematically in Figure 9.19.

The optical properties  $n$  and  $K$  can be determined by measuring the reflectance from the surface of a material as a function of polarization and the angle of incidence (based on Fresnel's equations).

<sup>12</sup> In addition the material system should be passive—contain no sources of energy.

Complex refractive index

Complex refractive index



**Figure 9.19** Kramers-Kronig relations allow frequency dependences of the real and imaginary parts of the relative permittivity to be related to each other. The material must be a linear system.

It is instructive to mention that the reflection and transmission coefficients that we derived in Section 9.7 were based on using a real refractive index, that is, neglecting losses. We can still use the reflection and transmission coefficients if we simply use the complex refractive index  $N$  instead of  $n$ . For example, consider a light wave traveling in free space incident on a material at normal incidence ( $\theta_i = 90^\circ$ ). The reflection coefficient is now

$$r = -\frac{N-1}{N+1} = -\frac{n-jK-1}{n-jK+1} \quad [9.61] \quad \text{Reflection coefficient}$$

The reflectance is then

$$R = \left| \frac{n-jK-1}{n-jK+1} \right|^2 = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2} \quad [9.62] \quad \text{Reflectance}$$

which reduce to the usual forms when the extinction coefficient  $K = 0$ .

**COMPLEX REFRACTIVE INDEX** Spectroscopic ellipsometry measurements on a silicon crystal at a wavelength of 826.6 nm show that the real and imaginary parts of the complex relative permittivity are 13.488 and 0.038, respectively. Find the complex refractive index, the reflectance and the absorption coefficient  $\alpha$  at this wavelength, and the phase velocity.

**EXAMPLE 9.12**

**SOLUTION**

We know that  $\epsilon_r' = 13.488$  and  $\epsilon_r'' = 0.038$ . Thus, from Equation 9.60, we have

$$n^2 - K^2 = 13.488 \quad \text{and} \quad 2nK = 0.038$$

We can take  $K$  from the second equation and substitute for it in the first equation,

$$n^2 - \left( \frac{0.038}{2n} \right)^2 = 13.488$$

This is a quadratic equation in  $n^2$  that can be easily solved on a calculator to find  $n = 3.67$ . Once we know  $n$ , we can find  $K = 0.038/2n = 0.00517$ . If we simply take the square root of the real part of  $\epsilon_r$ , we would still find  $n = 3.67$ , because the extinction coefficient  $K$  is small. The reflectance of the Si crystal is

$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2} = \frac{(3.67-1)^2 + 0.00517^2}{(3.67+1)^2 + 0.00517^2} = 0.327$$

which is the same as simply using  $(n - 1)^2 / (n + 1)^2 = 0.327$ , because  $K$  is small.

The absorption coefficient  $\alpha$  describes the loss in the light intensity  $I$  via  $I = I_0 \exp(-\alpha z)$ . By virtue of Equation 9.57,

$$\alpha = 2k'' = 2k_0 K = 2 \left( \frac{2\pi}{826.6 \times 10^{-9}} \right) (0.00517) = 7.9 \times 10^4 \text{ m}^{-1}$$

Almost all of this absorption is due to band-to-band absorption (photogeneration of electron-hole pairs).

The phase velocity is given by

$$v = \frac{c}{n} = \frac{3 \times 10^8 \text{ m s}^{-1}}{3.67} = 8.17 \times 10^7 \text{ m s}^{-1}$$

**EXAMPLE 9.13**

**COMPLEX REFRACTIVE INDEX OF InP** An InP crystal has a refractive index (real part)  $n$  of 3.549 at a wavelength of 620 nm (photon energy of 2 eV). The reflectance of the air-InP crystal surface at this wavelength is 0.317. Calculate the extinction coefficient  $K$  and the absorption coefficient  $\alpha$  of InP at this wavelength.

**SOLUTION**

The reflectance  $R$  is given by

$$R = \frac{(n - 1)^2 + K^2}{(n + 1)^2 + K^2} \quad \text{or} \quad 0.317 = \frac{(3.549 - 1)^2 + K^2}{(3.549 + 1)^2 + K^2}$$

which on solving gives  $K = 0.302$ .

The absorption coefficient is

$$\alpha = 2k_0 K = 2 \left( \frac{2\pi}{620 \times 10^{-9}} \right) (0.302) = 6.1 \times 10^6 \text{ m}^{-1}$$

**EXAMPLE 9.14**

**FREE CARRIER ABSORPTION COEFFICIENT AND CONDUCTIVITY** Consider a semiconductor sample with a conductivity  $\sigma$ , and a refractive index  $n$ . Show that the absorption coefficient due to free carrier absorption (due to conductivity) is given by

$$\alpha = \left( \frac{1}{c\epsilon_0} \right) \frac{\sigma}{n}$$

An  $n$ -type Ge has a resistivity of about  $5 \times 10^{-3} \Omega \text{ m}$ . Calculate the imaginary part  $\epsilon_r''$  of the relative permittivity at a wavelength of 10  $\mu\text{m}$  where the refractive index is 4. Find the attenuation coefficient  $\alpha$  due to free carrier absorption.

**SOLUTION**

The relationship between the conductivity and the absorption coefficient is given by

$$\epsilon_r'' = \frac{\sigma}{\epsilon_0 \omega} \quad [9.63]$$

The relationship between the imaginary part  $\epsilon_r''$  of the relative permittivity and the extinction coefficient  $K$  is

$$2nK = \epsilon_r''$$

where  $n$  is the refractive index (the real part of  $N$ ). Since the absorption coefficient from Example 9.13 is

$$\alpha = 2k'' = 2k_o K = 2 \left( \frac{2\pi}{\lambda} \right) \left( \frac{\epsilon_r''}{2n} \right)$$

then 
$$\alpha = \left( \frac{\omega}{c} \right) \frac{\epsilon_r''}{n} \quad [9.64]$$

Absorption  
and  
imaginary  
relative  
permittivity

where  $\omega$  is the angular frequency of the EM radiation,  $\omega = 2\pi c/\lambda$ . Substituting for  $\alpha$  in terms of  $\epsilon_r''$  gives

$$\alpha = \left( \frac{1}{c\epsilon_o} \right) \frac{\sigma}{n} \quad [9.65]$$

Absorption  
and  
conductivity

The frequency  $\omega$  is

$$\omega = \frac{2\pi c}{\lambda} = \left[ \frac{2\pi (3 \times 10^8 \text{ m s}^{-1})}{10 \times 10^{-6} \text{ m}} \right] = 1.88 \times 10^{14} \text{ rad s}^{-1}$$

The relationship between the conductivity and  $\epsilon_r''$  is given by

$$\epsilon_r'' = \frac{\sigma}{\epsilon_o \omega} = \left[ \frac{(5 \times 10^{-3} \Omega \text{ m})^{-1}}{(8.85 \times 10^{-12} \text{ F m}^{-1})(1.88 \times 10^{14} \text{ rad s}^{-1})} \right]$$

i.e., 
$$\epsilon_r'' = 0.120$$

The absorption coefficient due to free carriers is given by

$$\alpha = \left( \frac{1}{c\epsilon_o} \right) \frac{\sigma}{n} = \left[ \frac{1}{(3 \times 10^8 \text{ m s}^{-1})(8.85 \times 10^{-12} \text{ F m}^{-1})} \right] \frac{(5 \times 10^{-3} \Omega \text{ m})^{-1}}{4} = 1.9 \times 10^4 \text{ m}^{-1}$$

**COMPLEX REFRACTIVE INDEX AND RESONANCE ABSORPTION** Equation 9.12 is a simple expression for the electronic polarizability  $\alpha_e$  due to an oscillating field. It is based on the *Lorentz model* in which there is a restoring force acting against polarization of the atom or the molecule.  $\omega_o$  is a *resonant frequency*, or a natural frequency, associated with this type of electronic polarization. The same type of expression will also apply to *ionic polarization*, except that the resonant frequency  $\omega_o$  will be lower, and the mass  $m_e$  has to be changed to an effective mass of the ions.<sup>13</sup> In practice there will be some loss mechanism that absorbs energy from the oscillating field and dissipates it. For example, in ionic polarization, this would involve energy transfer from light to lattice vibrations. In mechanics it is well known that the loss forces (frictional forces) are always proportional to the velocity  $dx/dt$ . If we include the energy loss in ac polarization, Equation 9.11 would have an additional term  $-\gamma dx/dt$  on the right-hand side. If we then follow the same steps to obtain  $\alpha_e$ , we would find

$$\alpha_e = \frac{Ze^2}{m_e(\omega_o^2 - \omega^2 + j\gamma\omega)} \quad [9.66]$$

Electronic  
polarizability  
with loss

which is a complex number with real and imaginary parts ( $\alpha_e = \alpha_e' - j\alpha_e''$ ).

<sup>13</sup> Both electronic and ionic polarizabilities have similar expressions. The ionic polarizability in an oscillating field was derived in Chapter 7, and looks almost exactly like Equation 9.66.

Since  $\alpha_e$  is a complex quantity, so is  $\epsilon_r$ , and hence the refractive index. Consider the simplest relationship between the relative permittivity  $\epsilon_r$  and polarizability  $\alpha_e$ ,

Relative  
permittivity

$$\epsilon_r = 1 + \frac{N}{\epsilon_0} \alpha_e \quad [9.67]$$

where  $N$  is the number of atoms per unit volume (or ion pairs per unit volume for ionic polarization). Thus, the relative permittivity is a *complex quantity*, that is  $\epsilon_r = \epsilon_r' - j\epsilon_r''$ . We can substitute from Equation 9.66 into 9.67, and also use the fact that when  $\omega = 0$ ,  $\epsilon_r = \epsilon_{rdc}$ , to obtain a simple expression for  $\epsilon_r$ ,

Complex  
relative  
permittivity

$$\epsilon_r = 1 + \frac{\epsilon_{rdc} - 1}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\gamma\omega}{\omega_0^2}} \quad [9.68]$$

The relationship between the complex refractive index  $N$  and the complex relative permittivity  $\epsilon_r$  is

Complex  
refractive  
index

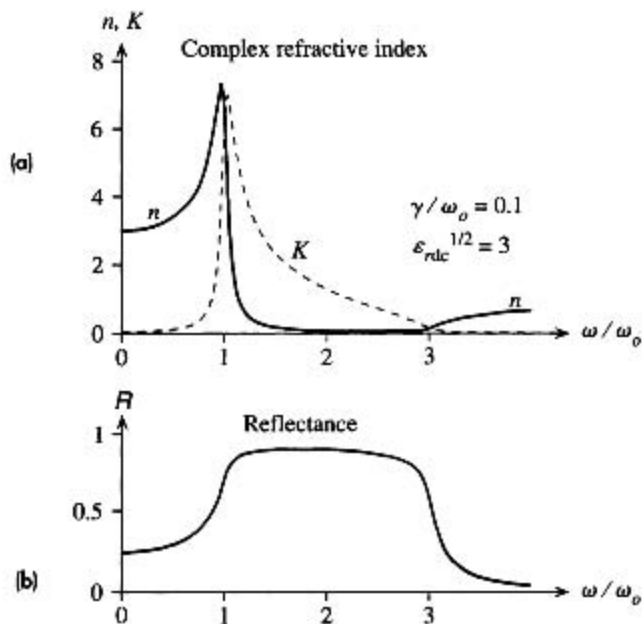
$$N = n - jK = \epsilon_r^{1/2} = (\epsilon_r' - j\epsilon_r'')^{1/2} \quad [9.69]$$

Suppose for simplicity we consider ionic polarization, and we set  $\epsilon_{rdc} = 9$  and  $\gamma = 0.1\omega_0$  (reasonable values for ionic polarization). We can calculate  $\epsilon_r$  from Equation 9.68 for any choice of  $\omega/\omega_0$  (or for  $\omega$  by taking  $\omega_0 = 1$ ), and then calculate  $N$ , that is  $n$  and  $K$ . (Our calculator or the math program must be able to handle complex numbers.) Figure 9.20a shows the dependence of  $n$  and  $K$  on the frequency  $\omega/\omega_0$  for the simple Lorentz oscillator model in Equation 9.68. Notice how  $n$  and the extinction coefficient  $K$  peak close to  $\omega = \omega_0$ .

The reflectance from Equation 9.62 is plotted in Figure 9.20b as  $R$  versus  $\omega/\omega_0$ . It is apparent that  $R$  reaches its maximum value at a frequency slightly above  $\omega = \omega_0$ , and then remains high until  $\omega$  reaches nearly  $3\omega_0$ ; the *reflectance is substantial while absorption is strong*. It may

**Figure 9.20**

- (a) Refractive index and extinction coefficient versus normalized frequency,  $\omega/\omega_0$ .  
(b) Reflectance versus normalized frequency.

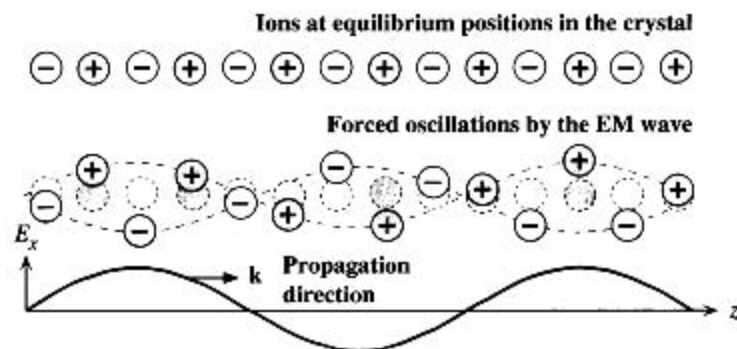


seem strange that the crystal is both highly reflecting and highly absorbing. The light that is incident is strongly reflected, and the light that is inside the crystal becomes strongly absorbed. This phenomenon is known as **infrared reflectance**, and occurs over a band of frequencies, called the **Reststrahlen band**; in the present case from  $\omega_0$  to roughly  $3\omega_0$ .

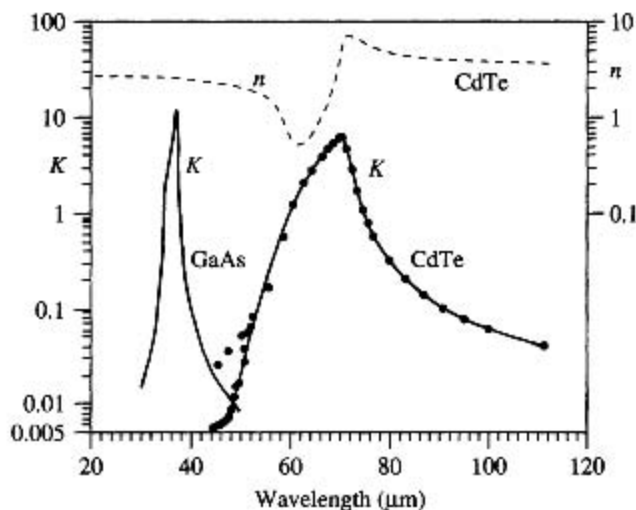
## 9.9 LATTICE ABSORPTION

In optical absorption, some of the energy from the propagating EM wave is converted to other forms of energy, for example, to heat by the generation of lattice vibrations. There are a number of absorption processes that dissipate the energy from the wave. One important mechanism is called **lattice absorption (Reststrahlen absorption)** and involves the vibrations of the lattice atoms as illustrated in Figure 9.21. The crystal in this example consists of ions, and as an EM wave propagates it displaces the oppositely charged ions in opposite directions and forces them to vibrate at the frequency of the wave. In other words, the medium experiences *ionic polarization*. It is the displacements of these ions that give rise to ionic polarization and its contribution to the relative permittivity  $\epsilon_r$ . As the ions and hence the lattice is made to vibrate by the passing EM wave, as shown in Figure 9.21, some energy is coupled into the natural lattice vibrations of the solid. This energy peaks when the frequency of the wave is close to the natural lattice vibration frequencies. Typically these frequencies are in the *infrared region*. Most of the energy is then absorbed from the EM wave and converted to lattice vibrational energy (heat). We associate this absorption with the resonance peak or relaxation peak of ionic polarization loss (imaginary part of the relative permittivity  $\epsilon_r''$ ).

Figure 9.22 shows the infrared resonance absorption peaks in the extinction coefficient  $K$  versus wavelength characteristics of GaAs and CdTe; both crystals have substantial ionic bonding. These absorption peaks in Figure 9.22 are usually called **Reststrahlen bands** because absorption occurs over a band of frequencies (even though the band may be narrow), and in some cases may even have identifiable features. Indeed, if we were to plot the reflectance ( $R$ ) versus wavelength, it would be similar to that shown in Figure 9.20b, and the band would be identified with the high reflectance region.



**Figure 9.21** Lattice absorption through a crystal. The field in the EM wave oscillates the ions which consequently generate "mechanical" waves in the crystal; energy is thereby transferred from the wave to lattice vibrations.



**Figure 9.22** Lattice or Reststrahlen absorption in CdTe and GaAs in terms of the extinction coefficient versus wavelength. For reference,  $n$  versus  $\lambda$  for CdTe is also shown.

Although Figure 9.21 depicts an ionic solid to visualize absorption due to lattice waves, energy from a passing EM wave can also be absorbed by various ionic impurities in a medium as these charges can couple to the electric field and oscillate. Bonding between an oscillating ion and the neighboring atoms causes the mechanical oscillations of the ion to be coupled to neighboring atoms. This leads to a generation of lattice waves which takes away energy from the EM wave.

### EXAMPLE 9.16

**RESTSTRAHLEN ABSORPTION** Figure 9.22 shows the infrared extinction coefficient  $K$  of GaAs and CdTe. Consider CdTe. Calculate the absorption coefficient  $\alpha$  and the reflectance  $R$  of CdTe at the Reststrahlen peak, and also at 50  $\mu\text{m}$  and at 100  $\mu\text{m}$ . What is your conclusion?

#### SOLUTION

At the resonant peak,  $\lambda \approx 72 \mu\text{m}$ ,  $K \approx 6$ , and  $n \approx 5$ , so the corresponding free-space wavevector is

$$k_o = \frac{2\pi}{\lambda} = \frac{2\pi}{72 \times 10^{-6} \text{ m}} = 8.7 \times 10^4 \text{ m}^{-1}$$

The absorption coefficient  $\alpha$ , by definition, is  $2k''$  in Equation 9.57, so

$$\alpha = 2k'' = 2k_o K = 2(8.7 \times 10^4 \text{ m}^{-1})(6) = 1.0 \times 10^6 \text{ m}^{-1}$$

which corresponds to an *absorption depth*  $1/\alpha$  of about 1  $\mu\text{m}$ . The reflectance is

$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2} = \frac{(5-1)^2 + 6^2}{(5+1)^2 + 6^2} = 0.72 \quad \text{or} \quad 72\%$$

Repeating the above calculations at  $\lambda = 50 \mu\text{m}$ , we get  $\alpha = 8.3 \times 10^2 \text{ m}^{-1}$ , and  $R = 0.11$  or 11 percent. There is a sharp increase in the reflectance from 11 to 72 percent as we approach the resonant peak. At  $\lambda = 100 \mu\text{m}$ ,  $\alpha = 6.3 \times 10^3 \text{ m}^{-1}$  and  $R = 0.31$  or 31 percent, which is again smaller than the peak reflectance.  $R$  is maximum around the Reststrahlen peak.



## 9.10 BAND-TO-BAND ABSORPTION

The photon absorption process for photogeneration, that is, the creation of electron–hole pairs (EHPs), requires the photon energy to be at least equal to the bandgap energy  $E_g$  of the semiconductor material to excite an electron from the valence band (VB) to the conduction band (CB). The **upper cut-off wavelength** (or the threshold wavelength)  $\lambda_g$  for photogenerative absorption is therefore determined by the bandgap energy  $E_g$  of the semiconductor, so  $h(c/\lambda_g) = E_g$  or

$$\lambda_g (\mu\text{m}) = \frac{1.24}{E_g (\text{eV})} \quad (9.70)$$

*Cut-off  
wavelength  
and bandgap*

For example, for Si,  $E_g = 1.12$  eV and  $\lambda_g$  is  $1.11 \mu\text{m}$  whereas for Ge,  $E_g = 0.66$  eV and the corresponding  $\lambda_g = 1.87 \mu\text{m}$ . It is clear that Si photodiodes cannot be used for optical communications at  $1.3$  and  $1.55 \mu\text{m}$ , whereas Ge photodiodes are commercially available for use at these wavelengths. Table 9.3 lists some typical bandgap energies and the corresponding cut-off wavelengths of various photodiode semiconductor materials.

Incident photons with wavelengths shorter than  $\lambda_g$  become absorbed as they travel in the semiconductor, and the light intensity, which is proportional to the number of photons, decays exponentially with distance into the semiconductor. The light intensity  $I$  at a distance  $x$  from the semiconductor surface is given by

$$I(x) = I_0 \exp(-\alpha x) \quad (9.71)$$

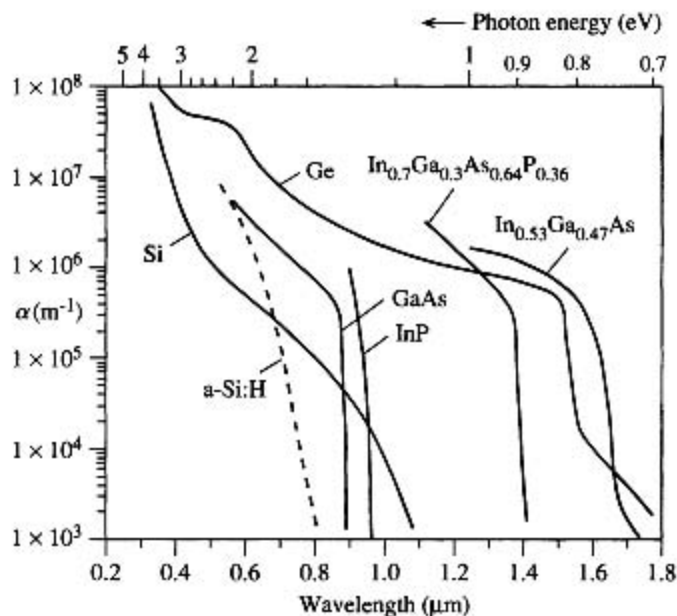
*Absorption  
coefficient*

where  $I_0$  is the intensity of the incident radiation and  $\alpha$  is the **absorption coefficient** that depends on the photon energy or wavelength  $\lambda$ . The absorption coefficient  $\alpha$  is a material property. Most of the photon absorption (63%) occurs over a distance  $1/\alpha$ , and  $1/\alpha$  is called the **penetration depth**  $\delta$ . Figure 9.23 shows the  $\alpha$  versus  $\lambda$  characteristics of various semiconductors where it is apparent that the behavior of  $\alpha$  with the wavelength  $\lambda$  depends on the semiconductor material.

Absorption in semiconductors can be understood in terms of the behavior of the electron energy ( $E$ ) with the electron momentum ( $\hbar k$ ) in the crystal, called the **crystal**

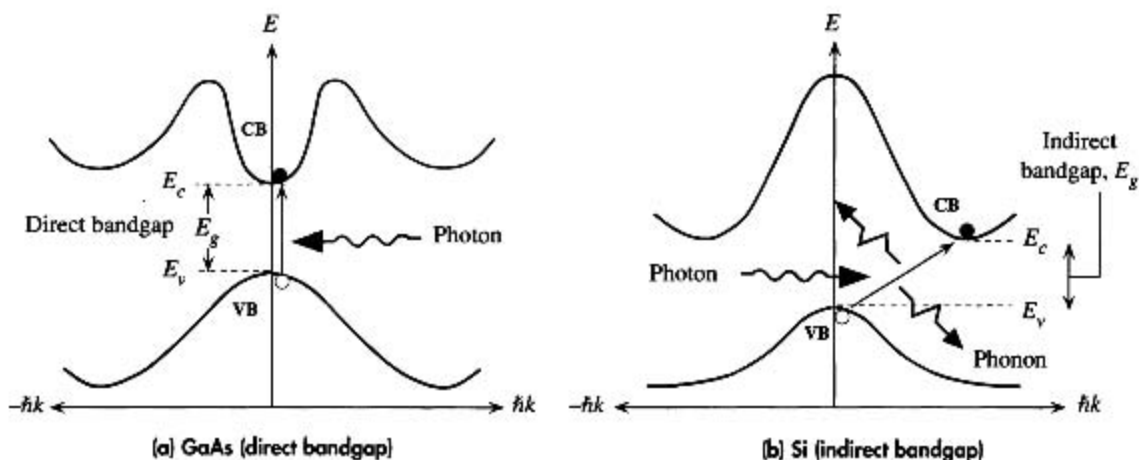
**Table 9.3** Bandgap energy  $E_g$  at 300 K, cut-off wavelength  $\lambda_g$ , and type of bandgap (D = direct and I = indirect) for some photodetector materials

Semiconductor	$E_g$ (eV)	$\lambda_g$ ( $\mu\text{m}$ )	Type
InP	1.35	0.91	D
GaAs <sub>0.88</sub> Sb <sub>0.12</sub>	1.15	1.08	D
Si	1.12	1.11	I
In <sub>0.7</sub> Ga <sub>0.3</sub> As <sub>0.64</sub> P <sub>0.36</sub>	0.89	1.4	D
In <sub>0.53</sub> Ga <sub>0.47</sub> As	0.75	1.65	D
Ge	0.66	1.87	I
InAs	0.35	3.5	D
InSb	0.18	7	D



**Figure 9.23** Absorption coefficient  $\alpha$  versus wavelength  $\lambda$  for various semiconductors.

SOURCE: Data selectively collected and combined from various sources.



**Figure 9.24** Electron energy  $E$  versus crystal momentum  $\hbar k$  and photon absorption.

(a) Photon absorption in a direct bandgap semiconductor.

(b) Photon absorption in an indirect bandgap semiconductor (VB = valence band; CB = conduction band).

**momentum.** If  $k$  is the wavevector of the electron's wavefunction in the crystal, then the momentum of the electron within the crystal is  $\hbar k$ .  $E$  versus  $\hbar k$  behaviors for electrons in the conduction and valence bands of direct and indirect bandgap semiconductors are shown in Figure 9.24a and b, respectively. In **direct bandgap** semiconductors such as III–V semiconductors (e.g., GaAs, InAs, InP, GaP) and in many of their alloys (e.g., InGaAs, GaAsSb) the photon absorption process is a direct process

which requires no assistance from lattice vibrations. The photon is absorbed and the electron is excited directly from the valence band to the conduction band without a change in its  $k$ -vector, or its crystal momentum  $\hbar k$ , inasmuch as the photon momentum is very small. The change in the electron momentum from the valence to the conduction band is

$$\hbar k_{\text{CB}} - \hbar k_{\text{VB}} = \text{Photon momentum} \approx 0$$

This process corresponds to a vertical transition on the electron energy ( $E$ ) versus electron momentum ( $\hbar k$ ) diagram as shown in Figure 9.24a. The absorption coefficient of these semiconductors rises sharply with decreasing wavelength from  $\lambda_g$  as apparent for GaAs and InP in Figure 9.23.

In **indirect bandgap** semiconductors such as Si and Ge, the photon absorption for photon energies near  $E_g$  requires the absorption and emission of lattice vibrations, that is, **phonons**,<sup>14</sup> during the absorption process as shown in Figure 9.24. If  $K$  is the wavevector of a lattice wave (lattice vibrations travel in the crystal), then  $\hbar K$  represents the momentum associated with such a lattice vibration; that is,  $\hbar K$  is a **phonon momentum**. When an electron in the valence band is excited to the conduction band, there is a change in its momentum in the crystal, and this change in the momentum cannot be supplied by the momentum of the incident photon which is very small. Thus, the momentum difference must be balanced by a phonon momentum,

$$\hbar k_{\text{CB}} - \hbar k_{\text{VB}} = \text{Phonon momentum} = \hbar K$$

The absorption process is said to be **indirect** as it depends on lattice vibrations which in turn depend on the temperature. Since the interaction of a photon with a valence electron needs a third body, a lattice vibration, the probability of photon absorption is not as high as in a direct transition. Furthermore, the cut-off wavelength is not as sharp as for direct bandgap semiconductors. During the absorption process, a phonon may be absorbed or emitted. If  $\vartheta$  is the frequency of the lattice vibrations, then the phonon energy is  $h\vartheta$ . The photon energy is  $h\nu$  where  $\nu$  is the photon frequency. Conservation of energy requires that

$$h\nu = E_g \pm h\vartheta$$

Thus, the onset of absorption does not exactly coincide with  $E_g$ , but typically it is very close to  $E_g$  inasmuch as  $h\vartheta$  is small ( $< 0.1$  eV). The absorption coefficient initially rises slowly with decreasing wavelength from about  $\lambda_g$  as apparent in Figure 9.23 for Si and Ge.

**FUNDAMENTAL ABSORPTION** A GaAs infrared LED emits at about 860 nm. A Si photodetector is to be used to detect this radiation. What should be the thickness of the Si crystal that absorbs most of this radiation?

**EXAMPLE 9.17**

<sup>14</sup> As much as an electromagnetic radiation is quantized in terms of photons, lattice vibrations in the crystal are quantized in terms of phonons. A phonon is a quantum of lattice vibration. If  $K$  is the wavevector of a vibrational wave in a crystal lattice and  $\omega$  is its angular frequency, then the momentum of the wave is  $\hbar K$  and its energy is  $\hbar\omega$ .

## SOLUTION

According to Figure 9.23, at  $\lambda \approx 0.8 \mu\text{m}$ , Si has  $\alpha \approx 6 \times 10^4 \text{ m}^{-1}$ , so the absorption depth

$$\delta = \frac{1}{\alpha} = \frac{1}{6 \times 10^4 \text{ m}^{-1}} = 1.7 \times 10^{-5} \text{ m} \quad \text{or} \quad 17 \mu\text{m}$$

If the crystal thickness is  $\delta$ , then 63 percent of the radiation will be absorbed. If the thickness is  $2\delta$ , then the fraction of absorbed radiation, from Equation 9.71, will be

$$\text{Fraction of absorbed radiation} = 1 - \exp[-\alpha(2\delta)] = 0.86 \quad \text{or} \quad 86\%$$

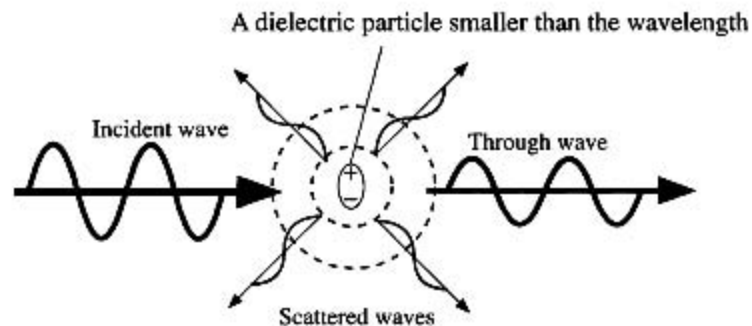
## 9.11 LIGHT SCATTERING IN MATERIALS

Scattering of an EM wave implies that a portion of the energy in a light beam is directed away from the original direction of propagation as illustrated for a small dielectric particle scattering a light beam in Figure 9.25. There are various types of scattering processes.

Consider what happens when a propagating wave encounters a molecule, or a small dielectric particle (or region), which is smaller than the wavelength. The electric field in the wave polarizes the particle by displacing the lighter electrons with respect to the heavier positive nuclei. The electrons in the molecule couple and oscillate with the electric field in the wave (ac electronic polarization). The oscillation of charge “up” and “down,” or the oscillation of the induced dipole, radiates EM waves all around the molecule as depicted in Figure 9.25. We should remember that an oscillating charge is like an alternating current which always radiates EM waves (like an antenna). The net effect is that the incident wave becomes partially reradiated in different directions and hence loses intensity in its original direction of propagation. We may think of the process as the particle absorbing some of the energy via electronic polarization and reradiating it in different directions. It may be thought that the scattered waves constitute a spherical wave emanating from the scattering molecule, but this is not generally the case as the reemitted radiation depends on the shape and polarizability of the molecule in different directions. We assumed a small particle so that at any time the field has no spatial variation through the particle, whose polarization then oscillates with the electric field oscillation. Whenever the size of the scattering region, whether an inhomogeneity or a small

**Figure 9.25** Rayleigh scattering involves the polarization of a small dielectric particle or a region that is much smaller than the light wavelength.

The field forces dipole oscillations in the particle (by polarizing it), which leads to the emission of EM waves in “many” directions so that a portion of the light energy is directed away from the incident beam.



particle or a molecule, is much smaller than the wavelength  $\lambda$  of the incident wave, the scattering process is generally termed **Rayleigh scattering**. In this type of scattering, typically the particle size is smaller than one-tenth of the wavelength.

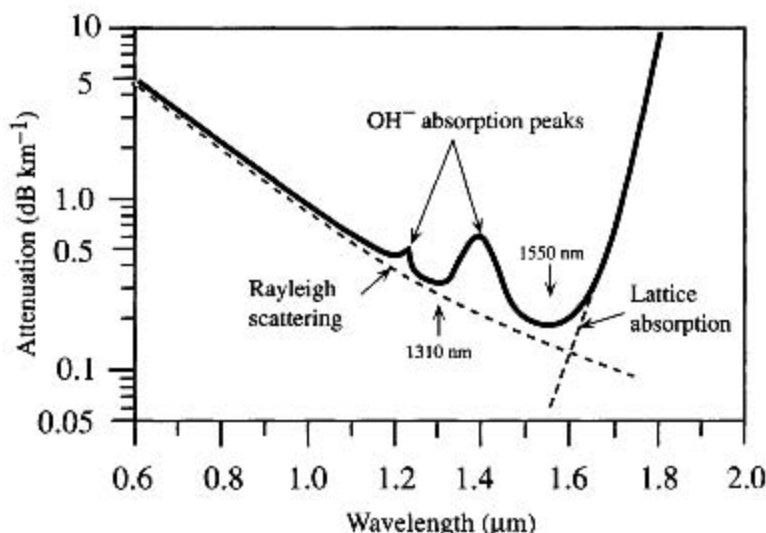
Rayleigh scattering of waves in a medium arises whenever there are small inhomogeneous regions in which the refractive index is different than the medium (which has some average refractive index). This means a local change in the relative permittivity and polarizability. The result is that the small inhomogeneous region acts like a small dielectric particle and scatters the propagating wave in different directions. In the case of optical fibers, dielectric inhomogeneities arise from fluctuations in the relative permittivity that is part of the intrinsic glass structure. As the fiber is drawn by freezing a liquid-like flow, random thermodynamic fluctuations in the composition and structure that occur in the liquid state become frozen into the solid structure. Consequently, the glass fiber has small fluctuations in the relative permittivity which leads to Rayleigh scattering. Nothing can be done to eliminate Rayleigh scattering in glasses as it is part of their intrinsic structure.

It is apparent that the scattering process involves electronic polarization of the molecule or the dielectric particle. We know that this process couples most of the energy at ultraviolet frequencies where the dielectric loss due to electronic polarization is maximum and the loss is due to EM wave radiation. Therefore, as the frequency of light increases, the scattering becomes more severe. In other words, *scattering decreases with increasing wavelength*. For example, blue light which has a shorter wavelength than red light is scattered more strongly by air molecules. When we look at the sun directly, it appears yellow because the blue light has been scattered in the direct light more than the red light. When we look at the sky in any direction but the sun, our eyes receive scattered light which appears blue; hence the sky is blue. At sunrise and sunset, the rays from the sun have to traverse the longest distance through the atmosphere and have the most blue light scattered which gives the sun its red color at these times.

## 9.12 ATTENUATION IN OPTICAL FIBERS

As light propagates through an optical fiber, it becomes attenuated by a number of processes that depend on the wavelength of light. Figure 9.26 shows the attenuation coefficient, as dB per km, of a typical silica-glass-based optical fiber as a function of wavelength. The sharp increase in the attenuation at wavelengths beyond  $1.6 \mu\text{m}$  in the *infrared* region is due to energy absorption by "lattice vibrations" of the constituent ions of the glass material. Fundamentally, energy absorption in this region corresponds to the stretching of the Si-O bonds in ionic polarization induced by the EM wave. Absorption increases with wavelength as we approach the resonance wavelength of the Si-O bond which is around  $9 \mu\text{m}$ . In the case of Ge-O glasses, this is further away, around  $11 \mu\text{m}$ . There is another intrinsic material absorption in the region below  $500 \text{ nm}$ , not shown in Figure 9.26, which is due to photons exciting electrons from the valence band to the conduction band of the glass.

There is a marked attenuation peak centered at  $1.4 \mu\text{m}$ , and a barely discernible minor peak at about  $1.24 \mu\text{m}$ . These attenuation regions arise from the presence of hydroxyl ions as impurities in the glass structure inasmuch as it is difficult to remove all



**Figure 9.26** Illustration of typical attenuation versus wavelength characteristics of a silica-based optical fiber.

There are two communications channels at 1310 and 1550 nm.

traces of hydroxyl (water) products during fiber production. Further, hydrogen atoms can easily diffuse into the glass structure at high temperatures during production which leads to the formation of hydrogen bonds in the silica structure and OH ions. Energy is absorbed mainly by the stretching vibrations of the OH bonds within the silica structure which has a fundamental resonance in the infrared region (beyond  $2.7 \mu\text{m}$ ) but overtones or harmonics at lower wavelengths (or higher frequencies). The first overtone at around  $1.4 \mu\text{m}$  is the most significant as can be seen in Figure 9.26. The second overtone is around  $1 \mu\text{m}$ , and in high-quality fibers this is negligible. A combination of the first overtone of the OH vibration and the fundamental vibrational frequency of  $\text{SiO}_2$  gives rise to a minor loss peak at around  $1.24 \mu\text{m}$ . There are two important windows in the attenuation versus wavelength behavior where the attenuation exhibits minima. The window at around  $1.3 \mu\text{m}$  is the region between two neighboring  $\text{OH}^-$  absorption peaks. This window is widely used in optical communications at 1310 nm. The window at around  $1.55 \mu\text{m}$  is between the first harmonic absorption of  $\text{OH}^-$  and the infrared lattice absorption tail and represents the lowest attenuation. Current technological drive is to use this window for long-haul communications. It can be seen that it is important to keep the hydroxyl content in the fiber within tolerable levels.

There is a background attenuation process that decreases with wavelength and is due to the Rayleigh scattering of light by the local variations in the refractive index. Glass has a noncrystalline or an amorphous structure which means that there is no long-range order to the arrangement of the atoms but only a short-range order, typically a few bond lengths. The glass structure is as if the structure of the melt has been suddenly frozen. We can only define the number of bonds a given atom in the structure will have. Random variations in the bond angle from atom to atom lead to a disordered structure. There is therefore a random local variation in the density over a few bond lengths which leads to fluctuations in the refractive index over few atomic lengths. These random fluctuations in the refractive index give rise to light scattering and hence

light attenuation along the fiber. It should be apparent that since a degree of structural randomness is an intrinsic property of the glass structure, this scattering process is unavoidable and represents the lowest attenuation possible through a glass medium. As one may surmise, attenuation by scattering in a medium is minimum for light propagating through a "perfect" crystal. In this case the only scattering mechanisms will be due to thermodynamic defects (vacancies) and the random thermal vibrations of the lattice atoms.

As mentioned above, the Rayleigh scattering process decreases with wavelength and, according to Lord Rayleigh, it is inversely proportional to  $\lambda^4$ . The expression for the attenuation  $\alpha_R$  in a single component glass due to Rayleigh scattering is approximately given by

$$\alpha_R \approx \frac{8\pi^3}{3\lambda^4} (n^2 - 1)^2 \beta_T k T_f \quad [9.72]$$

*Rayleigh  
scattering  
in silica*

where  $\lambda$  is the free-space wavelength,  $n$  is the refractive index at the wavelength of interest,  $\beta_T$  is the isothermal compressibility (at  $T_f$ ) of the glass,  $k$  is the Boltzmann constant, and  $T_f$  is a quantity called the *fictive temperature* (roughly the *softening temperature of glass*) where the liquid structure during the cooling of the fiber is frozen to become the glass structure. Fiber is drawn at high temperatures, and as the fiber cools eventually the temperature drops sufficiently for the atomic motions to be so sluggish that the structure becomes essentially "frozen-in" and remains like this even at room temperature. Thus  $T_f$  marks the temperature below which the liquid structure is frozen, and hence the density fluctuations are also frozen into the glass structure. It is apparent that Rayleigh scattering represents the lowest attenuation one can achieve using a glass structure. By proper design, the attenuation window at 1.5  $\mu\text{m}$  may be lowered to approach the Rayleigh scattering limit.

**RAYLEIGH SCATTERING LIMIT** What is the attenuation due to Rayleigh scattering at around the  $\lambda = 1.55 \mu\text{m}$  window given that pure silica ( $\text{SiO}_2$ ) has the following properties:  $T_f = 1730^\circ\text{C}$  (softening temperature),  $\beta_T = 7 \times 10^{-11} \text{m}^2 \text{N}^{-1}$  (at high temperatures),  $n = 1.4446$  at 1.5  $\mu\text{m}$ ?

**EXAMPLE 9.18**

**SOLUTION**

We simply calculate the Rayleigh scattering attenuation using

$$\alpha_R \approx \frac{8\pi^3}{3\lambda^4} (n^2 - 1)^2 \beta_T k T_f$$

so

$$\begin{aligned} \alpha_R &\approx \frac{8\pi^3}{3(1.55 \times 10^{-6})^4} (1.4446^2 - 1)^2 (7 \times 10^{-11}) (1.38 \times 10^{-23}) (1730 + 273) \\ &= 3.27 \times 10^{-5} \text{m}^{-1} \quad \text{or} \quad 3.27 \times 10^{-2} \text{km}^{-1} \end{aligned}$$

Attenuation in dB per km is then

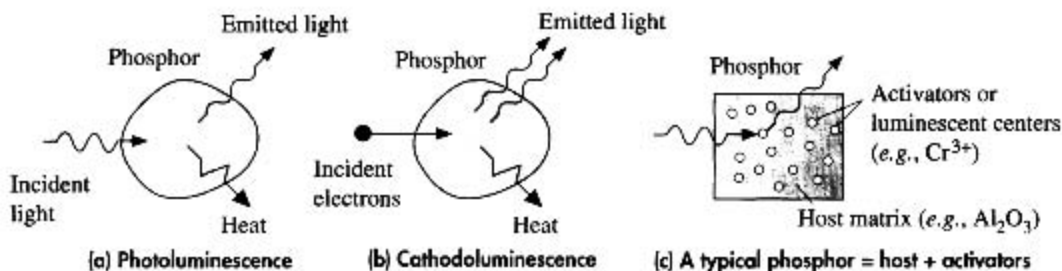
$$\alpha_{\text{dB}} = 4.34\alpha_R = (4.34)(3.27 \times 10^{-2} \text{km}^{-1}) = 0.142 \text{dB km}^{-1}$$

This represents the lowest possible attenuation for a silica glass fiber at 1.55  $\mu\text{m}$ .

## 9.13 LUMINESCENCE, PHOSPHORS, AND WHITE LEDS

We know from our general experience that certain substances, known as *phosphors*, can absorb light and then reemit light even after the excitation light source has been turned off; this is an example of luminescence. In general, **luminescence** is the emission of light by a material, called a **phosphor**, due to the absorption and conversion of energy into electromagnetic radiation as illustrated in Figure 9.27a and b. The luminescent radiation emitted by the phosphor material is considered to be quite separate from the thermal radiation emitted by virtue of its temperature. Luminescence is light emitted by a nonthermal source when it is excited, in contrast to the emission of radiation from a heated object such as the tungsten filament of a light bulb; the latter is called **incandescence**. Typically the emission of light occurs from certain dopants, impurities, or even defects, called **luminescent** or **luminescence centers**, purposefully introduced into a **host matrix**, which may be a crystal or glass as shown in Figure 9.27c. The luminescent center is also called an **activator**. There are many examples of phosphors. For example, in ruby, the  $\text{Cr}^{3+}$  ions are the luminescent centers in the sapphire ( $\text{Al}_2\text{O}_3$ ) crystal host.  $\text{Cr}^{3+}$  ions can absorb UV or violet light and then emit red light. This phosphor system is written as  $\text{Al}_2\text{O}_3:\text{Cr}^{3+}$ . The excitation and emission involves only the  $\text{Cr}^{3+}$  ion. In other cases, the activator excitation may also involve the host as discussed later.

Luminescence is normally categorized according to the source of excitation energy. **Photoluminescence** involves excitation by photons (light) as in Figure 9.27a. **X-ray luminescence** involves incident X-rays exciting a phosphor to emit light. **Cathodoluminescence**, as shown in Figure 9.27b, is light emission when the excitation is the bombardment of the phosphor with energetic electrons as in TV cathode ray tubes. **Electroluminescence** is light emission due to the passage of an electric current. Electroluminescence in semiconductive materials appears as a result of an excited electron transiting down to the ground energy level, which would correspond to the recombination of an electron and a hole; the excited electron is the conduction band (CB), and its ground state corresponds to a hole in the valence band (VB). The direct electron-hole recombination mechanism generally occurs very quickly. For example, typical minority carrier lifetimes are in the range of nanoseconds, so light emission from a semiconductor stops within nanoseconds after the removal of the excitation. Such quick luminescence processes occurring over a nanosecond time scale or shorter are normally identified as **fluorescence**. The emission of light from a fluorescent tube



**Figure 9.27** Photoluminescence, cathodoluminescence, and a typical phosphor.





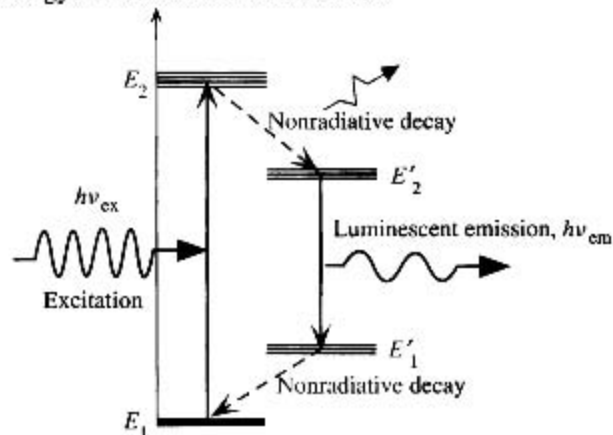
This flashlight uses a white LED instead of an incandescent light bulb. The flashlight can operate continuously for 200 hours and can project an intense spot over 30 ft. White LEDs use a phosphor to generate yellow light from the blue light emitted from the LED's semiconductor chip. The mixture of blue and yellow light appears as white.

is actually a fluorescence process. The tube contains a gas mixture of argon and mercury. The Ar and Hg gas atoms become excited by the electrical discharge process and emit light mainly in the ultraviolet region. This UV light is absorbed by the fluorescent coating on the inside of the tube. The excited activators in the phosphor coating then emit radiation in the visible region. A number of phosphors are used to obtain "white" light from the tube.

There are also phosphors from which light emission may continue for milliseconds to hours after the cessation of excitation. These slow luminescence processes are normally referred to as **phosphorescence** (also known as *afterglow*).

Many phosphors are based on activators doped into a host matrix; for example,  $\text{Eu}^{3+}$  (europium ion) in a  $\text{Y}_2\text{O}_3$  (yttrium oxide) matrix is a widely used modern phosphor. When excited by UV radiation, it provides an efficient luminescence emission in the red (around 613 nm). It is used as the red-emitting phosphor in color TV tubes and in modern tricolor fluorescent lamps. In very general terms, we can represent the energy of an activator in a host matrix by the highly simplified energy diagram in Figure 9.28.

Energy of luminescent center in host



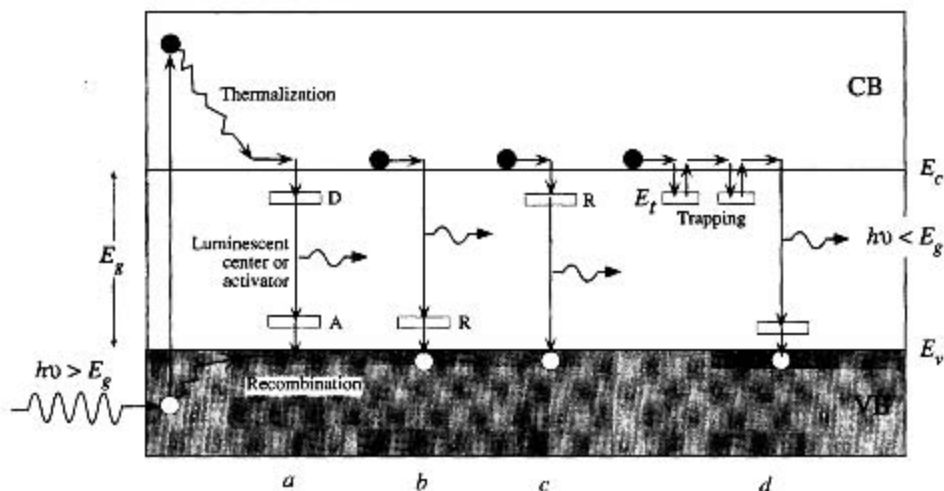
**Figure 9.28** Photoluminescence: light absorption, excitation, nonradiative decay and light emission, and return to the ground state  $E_1$ .

The energy levels have been displaced horizontally for clarity.

The ground state of the activator is  $E_1$ . Upon excitation by an incident radiation of suitable energy  $h\nu_{\text{ex}}$  the activator becomes excited to  $E_2$ . From this energy level, it decays, or *relaxes*, down relatively quickly (on a time scale of the order of picoseconds) to an energy level  $E'_2$  by emitting phonons or lattice vibrations. This type of decay is called *radiationless* or *nonradiative decay*. From  $E'_2$ , the activator decays down to  $E'_1$  by emitting a photon (spontaneous emission), which is the emitted luminescent radiation. The emitted photon energy is  $h\nu_{\text{em}}$ , which is less than the excitation photon energy  $h\nu_{\text{ex}}$ . The return from  $E'_1$  to the ground state  $E_1$  involves phonon emissions. Further, for some activators,  $E'_1$  is either very close to  $E_1$ , or it is  $E_1$ . The energy levels such as  $E_2$ ,  $E'_2$ ,  $E'_1$ , etc., are not well-defined single levels but involve finely spaced multilevels. The higher levels may form multilevel narrow energy "bands." In this example, the activator absorbed the incident radiation and was directly excited, which is known as **activator excitation**. The  $\text{Cr}^{3+}$  ions in  $\text{Al}_2\text{O}_3:\text{Cr}^{3+}$  can be excited directly by blue light and would then emit in the red. There are many phosphors in which the excitation involves the host. In **host excitation**, the host matrix absorbs the incident radiation and transfers the energy to the activator, which then becomes excited to  $E_2$  in Figure 9.28, and so on. In X-ray phosphors, for example, the X-rays are absorbed by the host, which subsequently transfers the energy to the activators. It is apparent from Figure 9.28 that the emitted radiation ( $h\nu_{\text{em}}$ ) has a *longer* wavelength than the exciting radiation ( $h\nu_{\text{ex}}$ ), that is,  $h\nu_{\text{em}} < h\nu_{\text{ex}}$ . The downshift in the light frequency from absorbed to emitted radiation is called the **Stoke's shift**. It should be emphasized that the energy levels of the activator (as shown in Figure 9.28) also depend on the host, because the internal electric fields within the host crystal act on the activator and shift these levels up and down. The emission characteristics depend firstly on the activator, and secondly on the host.

There are a number of host excitation mechanisms. In one possible process, which involves a semiconductor host, as depicted in Figure 9.29, an incident photon initially excites a valence band (VB) electron to the conduction band (CB). The electron then thermalizes, *i.e.*, loses the excess energy as it collides with lattice vibrations, and falls close to  $E_c$ , and wanders around in the crystal. In one process, *a* in Figure 9.29, the electron can be captured into an excited state D of a luminescent center or an activator. The electron then falls down in energy to the ground state A of the activator releasing a photon, which is the luminescent emission. The electron at the ground state then recombines with a hole in the VB. Thus the activator acts as a **radiative recombination center**. In some cases D and A may be separate centers representing *donor* and *acceptor*-like centers, hence the labels D and A. In other cases, the radiative recombination center may simply be a single energy level in the bandgap, which is shown as R in Figure 9.29. The electron can emit a photon as it is captured into R, shown as process *b* in Figure 9.29, or emit the photon after it is captured by R, as it recombines with a hole, shown as process *c* in Figure 9.29. Processes *a* and *b* occur in various ZnS-based phosphors. For example, in ZnS: $\text{Cu}^+$  phosphors, the activator is  $\text{Cu}^+$ , which has an energy level at A in Figure 9.29. The luminescent emission is enhanced by using a coactivator, such as Al in ZnS: $\text{Cu}^+$ . Al acts as a shallow donor D, and the luminescence is due to process *a* in Figure 9.29.

There may also be traps in the semiconductor because of various crystal defects, or there may be added impurities. The electron can become captured by a trap at a localized energy level  $E_t$  in the bandgap, but close to  $E_c$ . These electron traps temporarily



**Figure 9.29** Optical absorption generates an EHP.

Both carriers thermalize. There are a number of recombination processes via a dopant that can result in a luminescent emission.

capture an electron from the conduction band and thereby immobilize it. The time the electron spends trapped at  $E_t$  depends on the energy depth of the trap from the conduction band,  $E_c - E_t$ . After a while a strong lattice vibration returns the electron back into the conduction band (by thermal excitation). The time interval between photogeneration and recombination can be relatively long if the electron remains captured at  $E_t$  for a considerable length of time. In fact, the electron may become trapped and de-trapped many times before it finally recombines, so the emission of light can persist for a relatively long time after the cessation of excitation (e.g., milliseconds or longer) as indicated by process  $d$  in Figure 9.29.

It is also possible to excite electrons into the CB by bombarding the material with a high-energy electron beam, which leads to cathodoluminescence. Color CRT displays are typically coated uniformly with three sets of phosphor dots which exhibit cathodoluminescence in the blue, red, and green wavelengths. In electroluminescence, an electric current, either ac or dc, is used to inject electrons into the CB which then recombine with holes and emit light. For example, passing a current through certain semiconducting phosphors such as ZnS doped with Mn causes light emission by electroluminescence. The emission of light from a light emitting diode (LED) is an example of **injection electroluminescence** in which the applied voltage causes charge carrier injection and recombination in a device (diode) that has a junction between a  $p$ -type and an  $n$ -type semiconductor.

Zinc sulfide with various activators has been one of the traditional phosphors. The ZnS:Ag<sup>+</sup> in which Ag<sup>+</sup> is the activator, is still used as a blue emitting phosphor, though in some cases Cd is substituted for some of the Zn. ZnS:Cu<sup>+</sup> emits in the green, which is also a useful phosphor. Most modern phosphors, on the other hand, have been based on using rare earth activators in various hosts. For example, Y<sub>2</sub>O<sub>3</sub>:Eu<sup>3+</sup> absorbs UV

Table 9.4 Selected phosphor examples

Phosphor	Activator	Useful Emission	Example Excitation	Comment or Application
$\text{Y}_2\text{O}_3:\text{Eu}^{3+}$	$\text{Eu}^{3+}$	Red	UV	Fluorescent lamp, color TV
$\text{BaMgAl}_{10}\text{O}_{17}:\text{Eu}^{2+}$	$\text{Eu}^{2+}$	Blue	UV	Fluorescent lamp
$\text{CeMgAl}_{11}\text{O}_{19}:\text{Tb}^{3+}$	$\text{Tb}^{3+}$	Green	UV	Fluorescent lamp
$\text{Y}_3\text{Al}_5\text{O}_{12}:\text{Ce}^{3+}$	$\text{Ce}^{3+}$	Yellow	Blue, violet	White LED
$\text{Sr}_2\text{SiO}_4:\text{Eu}^{3+}$	$\text{Eu}^{3+}$	Yellow	Violet	White LED (experimental)
$\text{ZnS}:\text{Ag}^+$	$\text{Ag}^+$	Blue	Electron beam	Color TV blue phosphor
$\text{Zn}_{0.68}\text{Cd}_{0.32}\text{S}:\text{Ag}^+$	$\text{Ag}^+$	Green	Electron beam	Color TV green phosphor
$\text{ZnS}:\text{Cu}^+$	$\text{Cu}^+$	Green	Electron beam	Color TV green phosphor

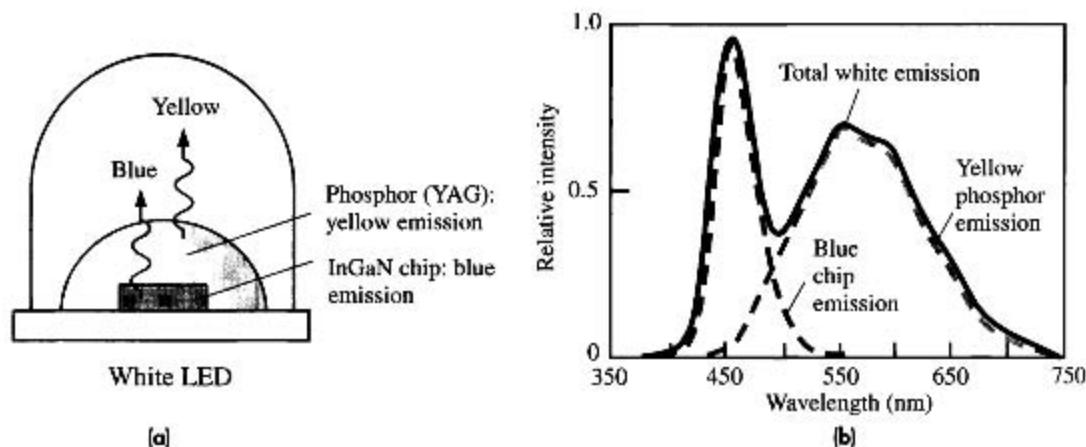


Figure 9.30

(a) A typical "white" LED structure.

(b) The spectral distribution of light emitted by a white LED. Blue luminescence is emitted by the GaInN chip and "yellow" phosphorescence or luminescence is produced by a phosphor. The combined spectrum looks "white."

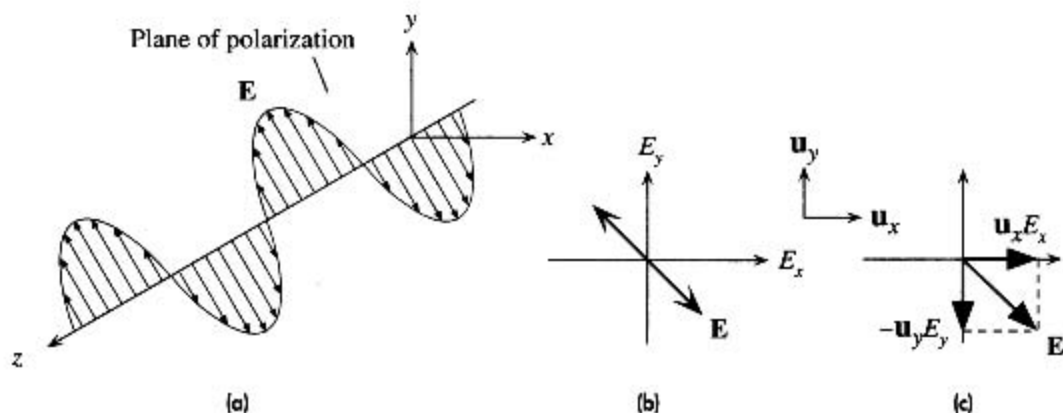
radiation and emits in the red.  $\text{Y}_3\text{Al}_5\text{O}_{12}:\text{Ce}^{3+}$  absorbs blue light and emits yellow light. Some of the most popular activators are  $\text{Eu}^{3+}$  for red,  $\text{Eu}^{2+}$  for blue, and  $\text{Tb}^{3+}$  for green. Table 9.4 summarizes a number of phosphors commonly used in various applications.

Recent inexpensive white LEDs that have appeared on the market seem to emit white light by emitting a mixture of blue and yellow light which are registered visually by the eye as appearing white. (Yellow consists of red and green mixed together, so mixing blue and yellow generates "white.") The production of white LEDs became possible due to development of bright blue-emitting LEDs based on gallium-indium-nitride (GaInN). The white LED uses a semiconductor chip emitting at a short wavelength (blue, violet, or ultraviolet) and a phosphor to convert some of the blue light to yellow light as depicted in Figure 9.30a. The phosphor absorbs light from the diode

and undergoes luminescent emission at a longer wavelength. Obviously, the quality and spectral characteristics of the combined emission vary with different designs; Figure 9.30b shows example spectra involved in the blue and yellow emissions and the overall “white” emission from a white LED. Typical phosphors have been based on yttrium-aluminum- ( $\text{Y}_3\text{Al}_5\text{O}_{12}$ ) garnets (YAGs) as the host material. This host is doped with one of the rare earth elements for the activator. Cerium is a common dopant element in YAG phosphors; that is, the phosphor is  $\text{Y}_3\text{Al}_5\text{O}_{12}:\text{Ce}^{3+}$ , which is able to efficiently absorb the blue and emit the yellow. White LEDs are soon expected to challenge the existing incandescent sources for general lighting.

## 9.14 POLARIZATION

A propagating EM wave has its electric and magnetic fields at right angles to the direction of propagation. If we place a  $z$  axis along the direction of propagation, then the electric field can be in any direction in the plane perpendicular to the  $z$  axis. The term **polarization** of an EM wave describes the behavior of the electric field vector in the EM wave as it propagates through a medium. If the oscillations of the electric field at all times are contained within a well-defined line, then the EM wave is said to be **linearly polarized** as shown in Figure 9.31a. The field vibrations and the direction of propagation ( $z$ ) define a plane of polarization (plane of vibration), so linear polarization implies a wave that is **plane-polarized**. By contrast, if a beam of light has waves with the  $E$  field in each in a random direction but perpendicular to  $z$ , then this light beam is *unpolarized*. A light beam can be linearly polarized by passing the beam



**Figure 9.31**

- (a) A linearly polarized wave has its electric field oscillations defined along a line perpendicular to the direction of propagation  $z$ . The field vector  $\mathbf{E}$  and  $z$  define a *plane of polarization*.  
 (b) The  $E$ -field oscillations are contained in the plane of polarization.  
 (c) A linearly polarized light at any instant can be represented by the superposition of two fields  $E_x$  and  $E_y$  with the right magnitude and phase.

through a *polarizer*, such as a polaroid sheet, a device that only passes electric field oscillations lying on a well-defined plane parallel to its transmission axis.

Suppose that we arbitrarily place the  $x$  and  $y$  axes and describe the electric field in terms of its components  $E_x$  and  $E_y$  along  $x$  and  $y$  (we are justified to do this because  $E_x$  and  $E_y$  are perpendicular to  $z$ ). To find the electric field in the wave at any space and time location, we add  $E_x$  and  $E_y$  *vectorially*. Both  $E_x$  and  $E_y$  can individually be described by a wave equation which must have the same angular frequency  $\omega$  and wavenumber  $k$ . However, we must include a phase difference  $\phi$  between the two:

$$E_x = E_{x0} \cos(\omega t - kz) \quad [9.73]$$

and

$$E_y = E_{y0} \cos(\omega t - kz + \phi) \quad [9.74]$$

where  $\phi$  is the phase difference between  $E_y$  and  $E_x$ ;  $\phi$  can arise if one of the components is delayed (retarded).

The linearly polarized wave in Figure 9.31a has the  $\mathbf{E}$  oscillations at  $-45^\circ$  to the  $x$  axis as shown in Figure 9.31b. We can generate this field by choosing  $E_{x0} = E_{y0}$  and  $\phi = \pm 180^\circ (\pm\pi)$  in Equations 9.73 and 9.74. Put differently,  $E_x$  and  $E_y$  have the same magnitude, but they are out of phase by  $180^\circ$ . If  $\mathbf{u}_x$  and  $\mathbf{u}_y$  are the unit vectors along  $x$  and  $y$ , using  $\phi = \pi$  in Equation 9.74, the field in the wave is

$$\mathbf{E} = \mathbf{u}_x E_x + \mathbf{u}_y E_y = \mathbf{u}_x E_{x0} \cos(\omega t - kz) - \mathbf{u}_y E_{y0} \cos(\omega t - kz)$$

or

$$\mathbf{E} = \mathbf{E}_0 \cos(\omega t - kz) \quad [9.75]$$

where

$$\mathbf{E}_0 = \mathbf{u}_x E_{x0} - \mathbf{u}_y E_{y0} \quad [9.76]$$

Equations 9.75 and 9.76 state that the vector  $\mathbf{E}_0$  is at  $-45^\circ$  to the  $x$  axis and propagates along the  $z$  direction.

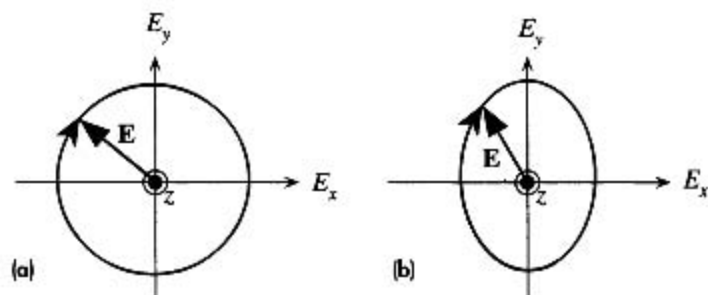
There are many choices for the behavior of the electric field besides the simple linear polarization in Figure 9.31. For example, if the magnitude of the field vector  $\mathbf{E}$  remains constant but its tip at a given location on  $z$  traces out a circle by rotating in a clockwise sense with time, as observed by the receiver of the wave, then the wave is said to be **right circularly polarized**<sup>15</sup> as in Figure 9.32. If the rotation of the tip of  $\mathbf{E}$  is counterclockwise, the wave is said to be **left circularly polarized**. From Equations 9.73 and 9.74, it should be apparent that a right circularly polarized wave has  $E_{x0} = E_{y0} = A$  (an amplitude) and  $\phi = \pi/2$ . This means that,

$$E_x = A \cos(\omega t - kz) \quad [9.77]$$

and

$$E_y = -A \sin(\omega t - kz) \quad [9.78]$$

<sup>15</sup> There is a difference in this definition in optics and engineering. The definition here follows that in optics which is more prevalent in optoelectronics.

**Figure 9.32**

(a) A right circularly polarized light that is traveling along  $z$  [out of paper]. The field vector  $\mathbf{E}$  is always at right angles to  $z$ , rotates clockwise around  $z$  with time, and traces out a full circle over one wavelength of distance propagated.

(b) An elliptically polarized light.

It is relatively straightforward to show that Equations 9.77 and 9.78 represent a circle that is

$$E_x^2 + E_y^2 = A^2 \quad (9.79)$$

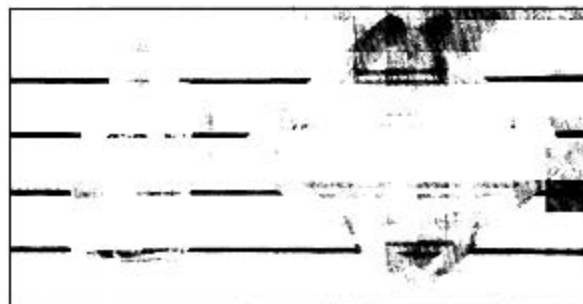
as shown in Figure 9.32.

When the phase difference  $\phi$  is other than  $0$ ,  $\pm\pi$ , or  $\pm\pi/2$ , the resultant wave is **elliptically polarized** and the tip of the vector in Figure 9.32 traces out an *ellipse*.

## 9.15 OPTICAL ANISOTROPY

An important characteristic of crystals is that many of their properties depend on the crystal direction; that is, crystals are generally anisotropic. The dielectric constant  $\epsilon_r$  depends on electronic polarization which involves the displacement of electrons with respect to positive atomic nuclei. Electronic polarization depends on the crystal direction inasmuch as it is easier to displace electrons along certain crystal directions. *This means that the refractive index  $n$  of a crystal depends on the direction of the electric field in the propagating light beam.* Consequently, the velocity of light in a crystal depends on the direction of propagation and on the state of its polarization, *i.e.*, the direction of the electric field. Most noncrystalline materials, such as glasses and liquids, and all cubic crystals are **optically isotropic**, that is, the refractive index is the same in all directions. For all classes of crystals excluding cubic structures, the refractive index depends on the propagation direction and the state of polarization. The result of optical anisotropy is that, except along certain special directions, any unpolarized light ray entering such a crystal breaks into two different rays with different polarizations and phase velocities. When we view an image through a calcite crystal, an optically anisotropic crystal, we see two images, each constituted by light of different polarization passing through the crystal, whereas there is only one image through an optically isotropic crystal as depicted in Figure 9.33. Optically anisotropic crystals are called **birefringent** because an incident light beam may be doubly refracted.

Experiments and theories on "most anisotropic crystals," *i.e.*, those with the highest degree of anisotropy, show that we can describe light propagation in terms of *three* refractive indices, called **principal refractive indices**  $n_1$ ,  $n_2$ , and  $n_3$ , along three mutually orthogonal directions in the crystal, say  $x$ ,  $y$ , and  $z$ , called **principal axes**. These



**Figure 9.33** A line viewed through a cubic sodium chloride (halite) crystal (optically isotropic) and a calcite crystal (optically anisotropic).

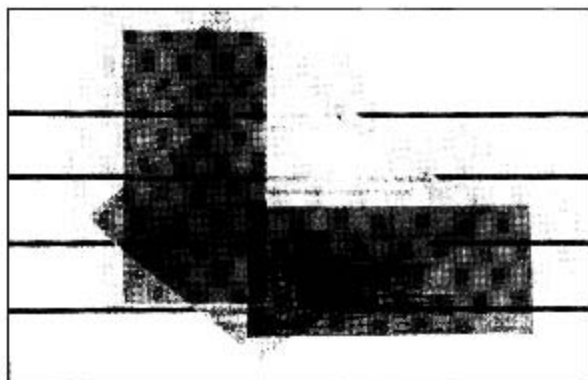
indices correspond to the polarization state of the EM wave along these axes. In addition, anisotropic crystals may possess one or two optic axes. An **optic axis** is a special direction in the crystal along which the velocity of propagation does *not* depend on the state of polarization. The propagation velocity along the optic axis is the same whatever the polarization of the EM wave.

Crystals that have three distinct principal indices also have *two* optic axes and are called **biaxial crystals**. On the other hand, **uniaxial crystals** have two of their principal indices the same ( $n_1 = n_2$ ) and have only *one* optic axis. Table 9.5 summarizes crystal classifications according to optical anisotropy. Uniaxial crystals, such as quartz, that have  $n_3 > n_1$ , are called **positive**, and those such as calcite that have  $n_3 < n_1$  are called **negative** uniaxial crystals.

**Table 9.5** Principal refractive indices of some optically isotropic and anisotropic crystals (near 589 nm, yellow Na-D line)

<i>Optically Isotropic</i>	$n = n_o$		
Glass (crown)	1.510		
Diamond	2.417		
Fluorite (CaF <sub>2</sub> )	1.434		
<i>Uniaxial—Positive</i>	$n_o$	$n_e$	
Ice	1.309	1.3105	
Quartz	1.5442	1.5533	
Rutile (TiO <sub>2</sub> )	2.616	2.903	
<i>Uniaxial—Negative</i>	$n_o$	$n_e$	
Calcite (CaCO <sub>3</sub> )	1.658	1.486	
Tourmaline	1.669	1.638	
Lithium niobate (LiNbO <sub>3</sub> )	2.29	2.20	
<i>Biaxial</i>	$n_1$	$n_2$	$n_3$
Mica (muscovite)	1.5601	1.5936	1.5977





**Figure 9.34** Two polaroid analyzers are placed with their transmission axes, along the long edges, at right angles to each other.

The ordinary ray, undeflected, goes through the left polarizer, whereas the extraordinary wave, deflected, goes through the right polarizer. The two waves therefore have orthogonal polarizations.

### 9.15.1 UNIAXIAL CRYSTALS AND FRESNEL'S OPTICAL INDICATRIX

For our discussions of optical anisotropy, we will consider uniaxial crystals such as calcite and quartz. All experiments and theories lead to the following basic principles.<sup>16</sup>

Any EM wave entering an anisotropic crystal splits into two orthogonal linearly polarized waves that travel with different phase velocities; that is, they experience different refractive indices. These two orthogonally polarized waves in uniaxial crystals are called **ordinary** (*o*) and **extraordinary** (*e*) waves. The *o*-wave has the same phase velocity in all directions and behaves like an ordinary wave in which the field is perpendicular to the phase propagation direction. The *e*-wave has a phase velocity that depends on its direction of propagation and its state of polarization, and further the electric field in the *e*-wave is not necessarily perpendicular to the phase propagation direction. These two waves propagate with the same velocity only along a special direction called the **optic axis**. The *o*-wave is always perpendicularly polarized to the optic axis and obeys the usual Snell's law.

The two images observed through the calcite crystal in Figure 9.33 are due to *o*-waves and *e*-waves being refracted differently, so when they emerge from the crystal they have been separated. Each ray constitutes an image, but the field directions are **orthogonal**. The fact that this is so is easily demonstrated by using two polaroid analyzers with their transmission axes at right angles as in Figure 9.34. If we were to view an object along the optic axis of the crystal, we would not see two images because the two rays would experience the same refractive index.

As mentioned, we can represent the optical properties of a crystal in terms of three refractive indices along three orthogonal axes, the principal axes of the crystal, shown as *x*, *y*, and *z* in Figure 9.35a. These are special axes along which the polarization vector and the electric field are parallel. (Put differently, the electric displacement<sup>17</sup> **D** and the electric field **E** vectors are parallel.) The refractive indices along these *x*, *y*, and *z* axes are the principal indices  $n_1$ ,  $n_2$ , and  $n_3$ , respectively, for electric

<sup>16</sup> These statements can be proved by solving Maxwell's equations in an anisotropic medium.

<sup>17</sup> Electric displacement **D** at any point is defined by  $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$  where **E** is the electric field and **P** is the polarization at that point.

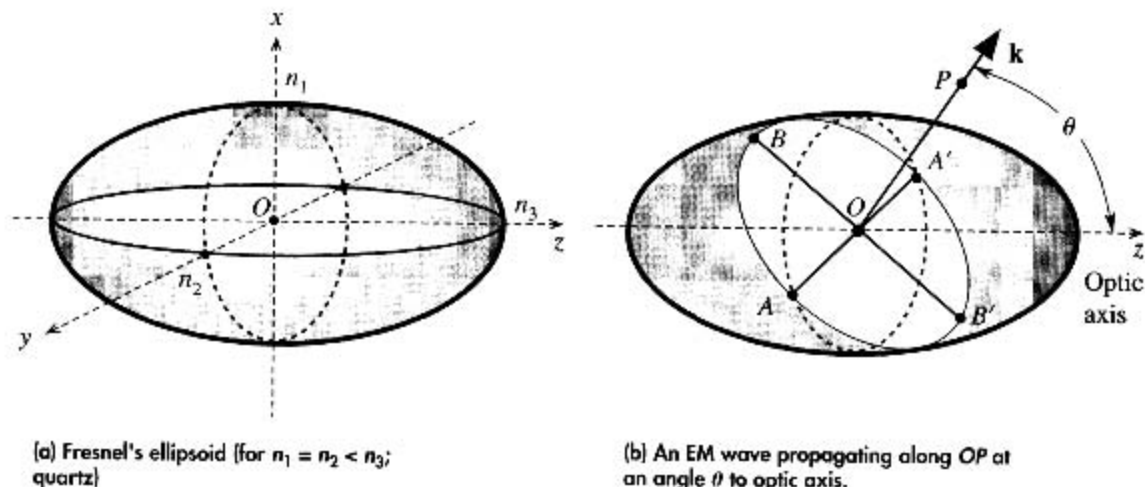


Figure 9.35

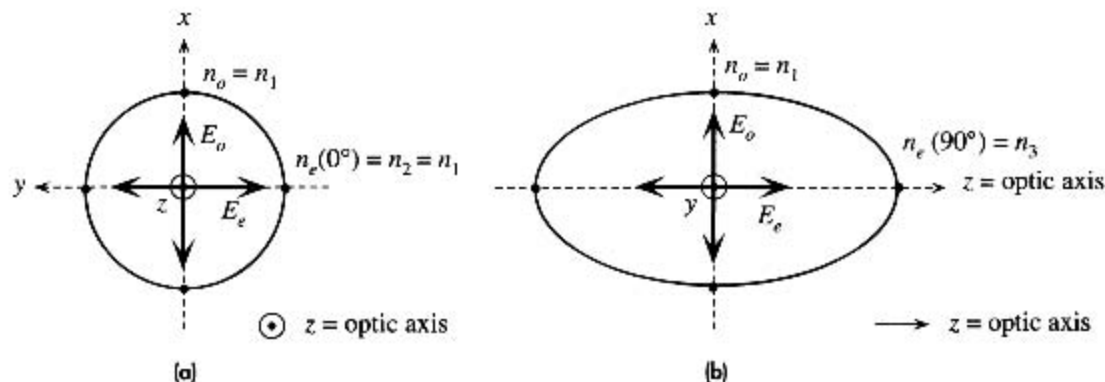
field oscillations along these directions (not to be confused with the wave propagation direction). For example, for a wave with a polarization parallel to the  $x$  axes, the refractive index is  $n_1$ .

The refractive index associated with a particular EM wave in a crystal can be determined by using Fresnel's *refractive index ellipsoid*, called the **optical indicatrix**,<sup>18</sup> which is a refractive index surface placed in the center of the principal axes, as shown in Figure 9.35a, where the  $x$ ,  $y$ , and  $z$  axis intercepts are  $n_1$ ,  $n_2$ , and  $n_3$ . If all three indices were the same,  $n_1 = n_2 = n_3 = n_o$ , we would have a spherical surface and all electric field polarization directions would experience the same refractive index  $n_o$ . Such a spherical surface would represent an optically isotropic crystal. For positive uniaxial crystals such as quartz,  $n_1 = n_2 < n_3$ , which is the ellipsoid example shown in Figure 9.35a.

Suppose that we wish to find the refractive indices experienced by a wave traveling with an arbitrary wavevector  $\mathbf{k}$ , which represents the direction of phase propagation. This phase propagation direction is shown as  $OP$  in Figure 9.35b and is at an angle  $\theta$  to the  $z$  axis. We place a plane perpendicular to  $OP$  and passing through the center  $O$  of the indicatrix. This plane intersects the ellipsoid surface in a curve  $ABA'B'$  which is an *ellipse*. The major ( $BOB'$ ) and minor ( $AOA'$ ) axes of this ellipse determine the field oscillation directions and the refractive indices associated with this wave. Put differently, the original wave is now represented by two orthogonally polarized EM waves.

The line  $AOA'$ , the *minor axis*, corresponds to the polarization of the ordinary wave, and its semiaxis  $AA'$  is the refractive index  $n_o = n_2$  of this  $o$ -wave. The electric displacement and the electric field are in the same direction and parallel to  $AOA'$ . If

<sup>18</sup> There are various names in the literature with various subtle nuances: the Fresnel ellipsoid, optical indicatrix, index ellipsoid, reciprocal ellipsoid, Poincaré ellipsoid, ellipsoid of wave normals.



**Figure 9.36**  $E_o = E_{o\text{-wave}}$  and  $E_e = E_{e\text{-wave}}$ .  
 (a) Wave propagation along the optic axis.  
 (b) Wave propagation normal to the optic axis.

we were to change the direction of  $OP$ , we would always find the same minor axis, *i.e.*,  $n_o$  is either  $n_1$  or  $n_2$  whatever the orientation of  $OP$  (try orientating  $OP$  to be along  $y$  and along  $x$ ). This means that the  $o$ -wave always experiences the same refractive index in all directions. (The  $o$ -wave behaves just like an ordinary wave, hence the name.)

The line  $BOB'$  in Figure 9.35b, the *major axis*, corresponds to the electric displacement field ( $\mathbf{D}$ ) oscillations in the extraordinary wave, and its semiaxis  $OB$  is the refractive index  $n_e(\theta)$  of this  $e$ -wave. This refractive index is smaller than  $n_3$  but greater than  $n_2 (= n_o)$ . The  $e$ -wave therefore travels more slowly than the  $o$ -wave in this particular direction and in this crystal. If we change the direction of  $OP$ , we find that the length of the major axis changes with the  $OP$  direction. Thus,  $n_e(\theta)$  depends on the wave direction  $\theta$ . As apparent,  $n_e = n_o$  when  $OP$  is along the  $z$  axis, that is, when the wave is traveling along  $z$  as in Figure 9.36a. This direction is the *optic axis*, and all waves traveling along the optic axis have the same phase velocity whatever their polarization. When the  $e$ -wave is traveling along the  $y$  axis, or along the  $x$  axis,  $n_e(\theta) = n_3 = n_e$  and the  $e$ -wave has its slowest phase velocity as shown in Figure 9.36b. Along any  $OP$  direction that is at an angle  $\theta$  to the optic axis, the  $e$ -wave has a refractive index  $n_e(\theta)$  given by

$$\frac{1}{n_e(\theta)^2} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2} \quad [9.80]$$

*Refractive index of the e-wave*

Clearly, for  $\theta = 0^\circ$ ,  $n_e(0^\circ) = n_o$  and for  $\theta = 90^\circ$ ,  $n_e(90^\circ) = n_e$ .

The major axis  $BOB'$  in Figure 9.35b determines the  $e$ -wave polarization by defining the direction of the displacement vector  $\mathbf{D}$  and not  $\mathbf{E}$ . Although  $\mathbf{D}$  is perpendicular to  $\mathbf{k}$ , this is not true for  $\mathbf{E}$ . The electric field  $\mathbf{E}_{e\text{-wave}}$  of the  $e$ -wave is orthogonal to that of the  $o$ -wave, and it is in the plane determined by  $\mathbf{k}$  and the optic axis.  $\mathbf{E}_{e\text{-wave}}$  is orthogonal to  $\mathbf{k}$  only when the  $e$ -wave propagates along one of the principal axes. In birefringent crystals it is usual to take the *ray direction* as the direction of energy flow,

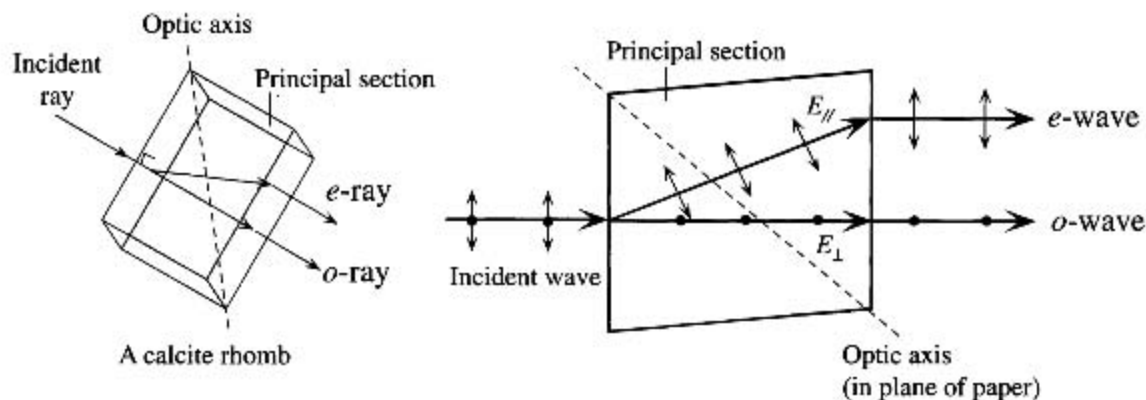
that is the direction of the Poynting vector ( $\mathbf{S}$ ). The  $\mathbf{E}_{e\text{-wave}}$  is then orthogonal to the ray direction. For the  $o$ -wave, the wavefront propagation direction  $\mathbf{k}$  is the same as the energy flow direction  $\mathbf{S}$ . For the  $e$ -wave, however, the wavefront propagation direction  $\mathbf{k}$  is not the same as the energy flow direction  $\mathbf{S}$ .

### 9.15.2 BIREFRINGENCE OF CALCITE

Consider a calcite crystal ( $\text{CaCO}_3$ ) which is a negative uniaxial crystal and also well known for its double refraction. When the surfaces of a calcite crystal have been cleaved, that is, cut along certain crystal planes, the crystal attains a shape that is called a *cleaved form* and the crystal faces are rhombohedrons (parallelogram with  $78.08^\circ$  and  $101.92^\circ$ ). A cleaved form of the crystal is called a *calcite rhomb*. A plane of the calcite rhomb that contains the optical axis and is normal to a pair of opposite crystal surfaces is called a *principal section*.

Consider what happens when an unpolarized or natural light enters a calcite crystal at *normal* incidence and thus also normal to a principal section to this surface, but at an angle to the optic axis as shown in Figure 9.37. The ray breaks into ordinary ( $o$ ) and extraordinary ( $e$ ) waves with mutually orthogonal polarizations. The waves propagate in the plane of the principal section as this plane also contains the incident light. The  $o$ -wave has its field oscillations perpendicular to the optic axis. It obeys Snell's law which means that it enters the crystal undeflected. Thus the direction of  $E$ -field oscillations must come out of the paper so that it is normal to the optic axis and also to the direction of propagation. The field  $E_\perp$  in the  $o$ -ray is shown as dots, oscillating into and out of the paper.

The  $e$ -wave has a polarization orthogonal to the  $o$ -wave and in the principal section. The  $e$ -wave polarization is in the plane of the paper, indicated as  $E_{\parallel}$ , in Figure 9.37. It travels with a different velocity and diverges from the  $o$ -wave. Clearly, the



**Figure 9.37** An EM wave that is off the optic axis of a calcite crystal splits into two waves called ordinary and extraordinary waves.

These waves have orthogonal polarizations and travel with different velocities. The  $o$ -wave has a polarization that is always perpendicular to the optical axis.

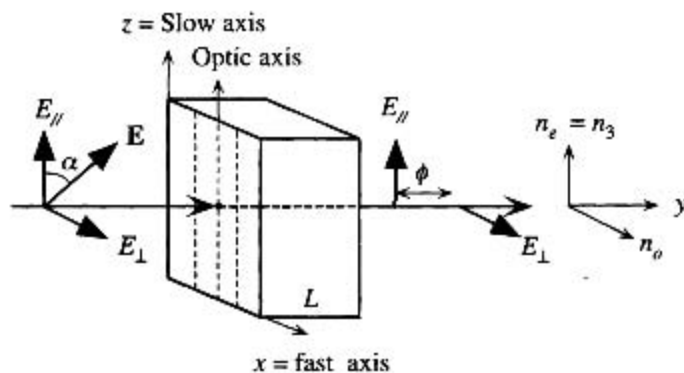
$e$ -wave does not obey the usual Snell's law inasmuch as the angle of refraction is not zero. We can determine the  $e$ -ray direction by noting that the  $e$ -wave propagates sideways as in Figure 9.37b at right angles to  $E_{\parallel}$ .

### 9.15.3 DICHOISM

In addition to the variation in the refractive index, some anisotropic crystals also exhibit **dichroism**, a phenomenon in which the optical absorption in a substance depends on the direction of propagation and the state of polarization of the light beam. A dichroic crystal is an optically anisotropic crystal in which either the  $e$ -wave or the  $o$ -wave is heavily attenuated (absorbed). This means that a light wave of arbitrary polarization entering a dichroic crystal emerges with a well-defined polarization because the other orthogonal polarization would have been attenuated. Generally dichroism depends on the wavelength of light. For example, in a tourmaline (aluminum borosilicate) crystal, the  $o$ -wave is much more heavily absorbed with respect to the  $e$ -wave.

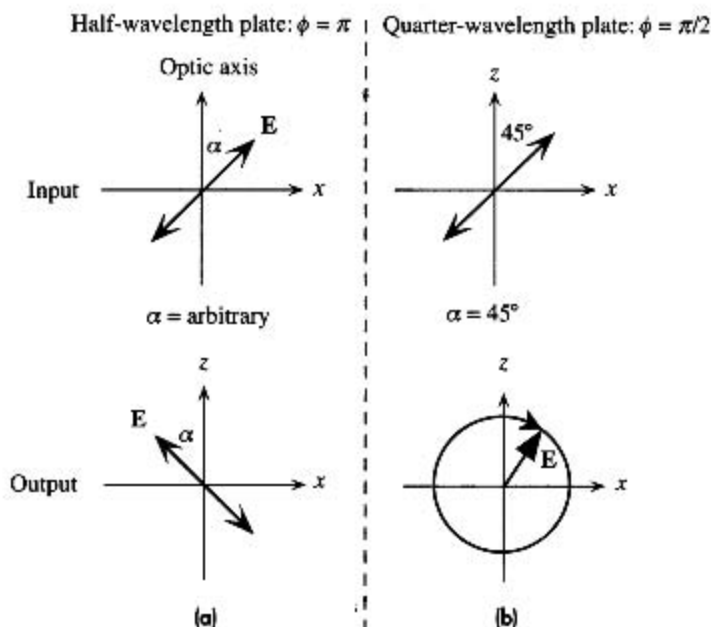
## 9.16 BIREFRINGENT RETARDING PLATES

Consider a positive uniaxial crystal such as a quartz ( $n_e > n_o$ ) plate that has the optic axis (taken along  $z$ ) parallel to the plate faces as in Figure 9.38. Suppose that a *linearly polarized* wave is normally incident on a plate face. If the field  $\mathbf{E}$  is parallel to the optic axis (shown as  $E_{\parallel}$ ), then this wave will travel through the crystal as an  $e$ -wave with a velocity  $c/n_e$  slower than the  $o$ -wave since  $n_e > n_o$ . Thus, the optic axis is the "slow axis" for waves polarized parallel to it. If  $\mathbf{E}$  is at right angles to the optic axis (shown as  $E_{\perp}$ ), then this wave will travel with a velocity  $c/n_o$ , which will be the fastest velocity in the crystal. Thus the axis perpendicular to the optic axis (say  $x$ ) will be the "fast axis" for polarization along this direction. When a light ray enters a crystal at normal incidence to the optic axis and plate surface, then the  $o$ - and  $e$ -waves travel along the same direction as shown in Figure 9.38. We can of course resolve a linear polarization at an angle  $\alpha$  to  $z$  into  $E_{\perp}$  and  $E_{\parallel}$ . The  $o$ -wave corresponds to the propagation of  $E_{\perp}$  and the  $e$ -wave to the propagation of  $E_{\parallel}$  in the crystal. When the light comes out at the



**Figure 9.38** A retarder plate.

The optic axis is parallel to the plate face. The  $o$ - and  $e$ -waves travel in the same direction but at different speeds.



**Figure 9.39** Input and output polarizations of light through (a) a half-wavelength plate and (b) through a quarter-wavelength plate.

opposite face, these two components  $E_{\perp}$  and  $E_{\parallel}$  would have been phase shifted by  $\phi$ . Depending on the initial angle  $\alpha$  of  $\mathbf{E}$  and the length of the crystal, which determines the total phase shift  $\phi$  through the plate, the emerging beam can have its initial linear polarization rotated, or changed into an elliptically or circularly polarized light as summarized in Figure 9.39.

If  $L$  is the thickness of the plate, then the  $o$ -wave experiences a phase change given by  $k_{o\text{-wave}}L$  through the plate where  $k_{o\text{-wave}}$  is the wavevector of the  $o$ -wave;  $k_{o\text{-wave}} = (2\pi/\lambda)n_o$ , where  $\lambda$  is the free-space wavelength. Similarly, the  $e$ -wave experiences a phase change  $(2\pi/\lambda)n_eL$  through the plate. Thus, the phase difference  $\phi$  between the orthogonal components  $E_{\perp}$  and  $E_{\parallel}$  of the emerging beam is

$$\phi = \frac{2\pi}{\lambda}(n_e - n_o)L \quad [9.81]$$

The phase difference  $\phi$  expressed in terms of full wavelengths is called the **retardation** of the plate. For example, a phase difference  $\phi$  of  $180^\circ$  is a **half-wavelength retardation**.

The polarization of the exiting-beam depends on the crystal-type,  $(n_e - n_o)$ , and the plate thickness  $L$ . We know that depending on the phase difference  $\phi$  between the orthogonal components of the field, the EM wave can be linearly, circularly, or elliptically polarized.

A **half-wave plate retarder** has a thickness  $L$  such that the phase difference  $\phi$  is  $\pi$  or  $180^\circ$ , corresponding to a half wavelength ( $\lambda/2$ ) of retardation. The result is that  $E_{\parallel}$  is delayed by  $180^\circ$  with respect to  $E_{\perp}$ . If we add the emerging  $E_{\perp}$  and  $E_{\parallel}$  with this phase shift  $\phi$ ,  $\mathbf{E}$  would be at an angle  $-\alpha$  to the optic axis and still linearly polarized.  $\mathbf{E}$  has been rotated counterclockwise through  $2\alpha$ .

A **quarter-wave plate retarder** has a thickness  $L$  such that the phase difference  $\phi$  is  $\pi/2$  or  $90^\circ$ , corresponding to a quarter wavelength  $\frac{1}{4}\lambda$ . If we add the emerging  $E_\perp$  and  $E_\parallel$  with this phase shift  $\phi$ , the emerging light will be elliptically polarized if  $0 < \alpha < 45^\circ$  and circularly polarized if  $\alpha = 45^\circ$ .

**QUARTZ HALF-WAVE PLATE** What should be the thickness of a half-wave quartz plate for a wavelength  $\lambda \approx 707$  nm given the extraordinary and ordinary refractive indices are  $n_o = 1.541$  and  $n_e = 1.549$ ?

**EXAMPLE 9.19****SOLUTION**

Half-wavelength retardation is a phase difference of  $\pi$ , so from Equation 9.81

$$\phi = \frac{2\pi}{\lambda}(n_e - n_o)L = \pi$$

giving

$$L = \frac{\frac{1}{2}\lambda}{(n_e - n_o)} = \frac{\frac{1}{2}(707 \times 10^{-9} \text{ m})}{(1.549 - 1.541)} = 44.2 \text{ } \mu\text{m}$$

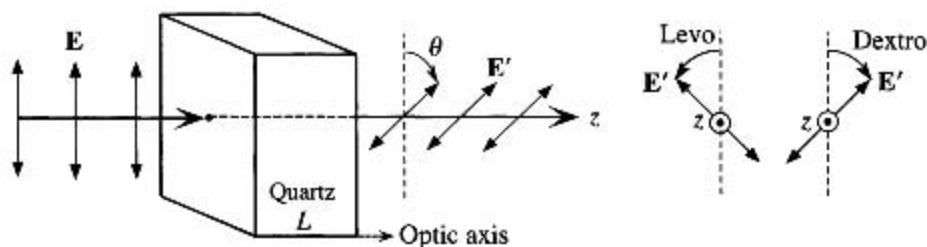
This is roughly the thickness of a sheet of paper.

## 9.17 OPTICAL ACTIVITY AND CIRCULAR BIREFRINGENCE

When a linearly polarized light wave is passed through a quartz crystal along its optic axis, it is observed that the emerging wave has its **E**-vector (plane of polarization) rotated, which is illustrated in Figure 9.40. This rotation increases continuously with the distance traveled through the crystal (about  $21.7^\circ$  per mm of quartz). The rotation of the plane of polarization by a substance is called **optical activity**. In very simple intuitive terms, optical activity occurs in materials in which the electron motions induced by the external electromagnetic field follows spiraling or helical paths (orbits).<sup>19</sup> Electrons flowing in helical paths resemble a current flowing in a coil and thus possess a magnetic moment. The optical field in light therefore induces oscillating magnetic moments which can be either parallel or antiparallel to the induced oscillating electric dipoles. Wavelets emitted from these oscillating induced magnetic and electric dipoles interfere to constitute a forward wave that has its optical field rotated either clockwise or counterclockwise.

If  $\theta$  is the angle of rotation, then  $\theta$  is proportional to the distance  $L$  propagated in the optically active medium as depicted in Figure 9.40. For an observer receiving the wave through quartz, the rotation of the plane of polarization may be *clockwise* (to the right) or *counterclockwise* (to the left) which are called *dextrorotatory* and *levorotatory* forms of optical activity. The structure of quartz is such that atomic arrangements spiral around the optic axis either in clockwise or counterclockwise sense. Quartz thus occurs in two distinct crystalline forms, right-handed and left-handed, which exhibit

<sup>19</sup> The explanation of optical activity involves examining both induced magnetic and electric dipole moments which will not be described here in detail.



**Figure 9.40** An optically active material such as quartz rotates the plane of polarization of the incident wave: The optical field  $\mathbf{E}$  rotated to  $\mathbf{E}'$ .  
If we reflect the wave back into the material,  $\mathbf{E}'$  rotates back to  $\mathbf{E}$ .

dextrorotatory and levorotatory types of optical activity, respectively. Although we used quartz as an example, there are many substances that are optically active, including various biological substances and even some liquid solutions (e.g., corn syrup) that contain various organic molecules with a rotatory power.

The **specific rotatory power** ( $\theta/L$ ) is defined as the extent of rotation per unit distance traveled in the optically active substance. Specific rotatory power depends on the wavelength. For example, for quartz this is  $49^\circ$  per mm at 400 nm but  $17^\circ$  per mm at 650 nm.

Optical activity can be understood in terms of left and right circularly polarized waves traveling at different velocities in the crystal, *i.e.*, experiencing different refractive indices. Due to the helical twisting of the molecular or atomic arrangements in the crystal, the velocity of a circularly polarized wave depends on whether the optical field rotates clockwise or counterclockwise. A vertically polarized light with a field  $\mathbf{E}$  at the input can be thought of as two right- and left-handed circularly polarized waves  $\mathbf{E}_R$  and  $\mathbf{E}_L$  that are symmetrical with respect to the  $y$  axis, *i.e.*, at any instant  $\alpha = \beta$ , as shown in Figure 9.41. If they travel at the same velocity through the crystal, then they remain symmetrical with respect to the vertical ( $\alpha = \beta$  remains the same) and the resultant is still a vertically polarized light. If, however, these travel at different velocities through a medium, then at the output  $\mathbf{E}'_L$  and  $\mathbf{E}'_R$  are no longer symmetrical with respect to the vertical,  $\alpha' \neq \beta'$ , and their resultant is a vector  $\mathbf{E}'$  at an angle  $\theta$  to the  $y$  axis.

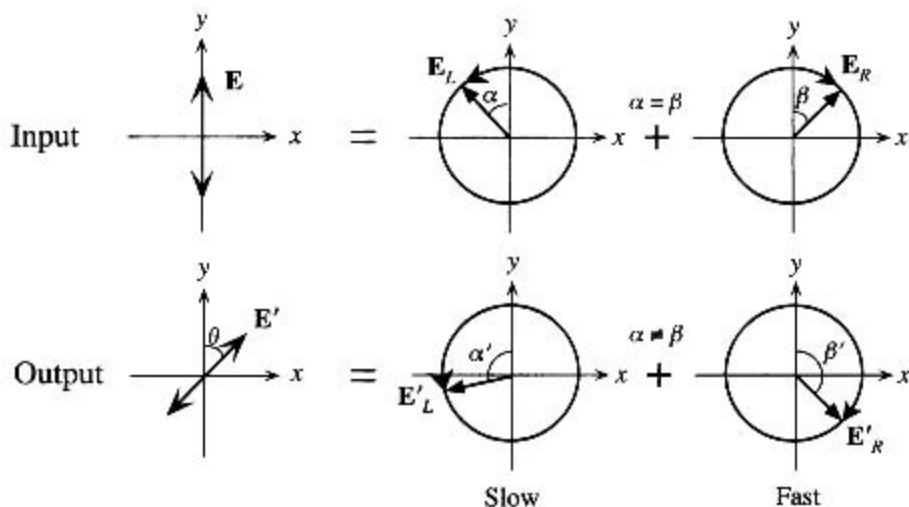
Suppose that  $n_R$  and  $n_L$  are the refractive indices experienced by the right- and left-handed circularly polarized light, respectively. After traversing the crystal length  $L$ , the phase difference between the two optical fields  $\mathbf{E}'_R$  and  $\mathbf{E}'_L$  at the output leads to a new optical field  $\mathbf{E}'$  that is  $\mathbf{E}$  rotated by  $\theta$ , given by

$$\theta = \frac{\pi}{\lambda}(n_L - n_R)L \quad [9.82]$$

where  $\lambda$  is the free-space wavelength. For a left-handed quartz crystal, and for 589 nm light propagation along the optic axis,  $n_R = 1.54427$  and  $n_L = 1.54420$ , which means  $\theta$  is about  $21.4^\circ$  per mm of crystal.

In a **circularly birefringent** medium, the right- and left-handed circularly polarized waves propagate with different velocities and experience different refractive indices  $n_R$  and  $n_L$ . Since optically active materials naturally rotate the optical field, it is





**Figure 9.41** Vertically polarized wave at the input can be thought of as two right- and left-handed circularly polarized waves that are symmetrical; i.e., at any instant  $\alpha = \beta$ . If these travel at different velocities through a medium, then at the output they are no longer symmetric with respect to  $y$ ,  $\alpha \neq \beta$ , and the result is a vector  $E'$  at an angle  $\theta$  to  $y$ .

not unreasonable to expect that a circularly polarized light with its optical field rotating in the same sense as the optical activity will find it easier to travel through the medium. Thus, an optically active medium possesses different refractive indices for right- and left-handed circularly polarized light and exhibits circular birefringence. It should be mentioned that if the direction of the light wave is reversed in Figure 9.40, the ray simply retraces itself and  $E'$  becomes  $E$ .

## ADDITIONAL TOPICS

### 9.18 ELECTRO-OPTIC EFFECTS<sup>20</sup>

Electro-optic effects refer to changes in the refractive index of a material induced by the application of an external electric field, which therefore “modulates” the optical properties. We can apply such an external field by placing electrodes on opposite faces of a crystal and connecting these electrodes to a battery. The presence of such a field distorts the electron motions in the atoms or molecules of the substance or distorts the crystal structure resulting in changes in the optical properties. For example, an applied external field can cause an optically isotropic crystal such as GaAs to become birefringent. In this case, the field induces principal axes and an optic axis. Typically changes in the refractive index are small. The frequency of the applied field has to be such that

<sup>20</sup> An extensive discussion and applications of the electro-optic effects may be found in S. O. Kasap, *Optoelectronics and Photonics: Principles and Practices*, Prentice Hall, 2001, Upper Saddle River, NJ, ch. 7.

the field appears static over the time scale it takes for the medium to change its properties, that is, respond, as well as for any light to cross the substance. The electro-optic effects are classified according to first- and second-order effects.

If we were to take the refractive index  $n$  to be a function of the applied electric field  $E$ , that is,  $n = n(E)$ , we can of course expand this as a Taylor series in  $E$ . The new refractive index  $n'$  is

Field induced  
refractive  
index

$$n' = n + a_1 E + a_2 E^2 + \dots \quad [9.83]$$

where the coefficients  $a_1$  and  $a_2$  are called the *linear* electro-optic effect and *second-order* electro-optic effect coefficients. Although we would expect even higher terms in the expansion in Equation 9.83, these are generally very small and their effects negligible within the highest practical fields. The change in  $n$  due to the first  $E$  term is called the **Pockels effect**. The change in  $n$  due to the second  $E^2$  term is called the **Kerr effect**,<sup>21</sup> and the coefficient  $a_2$  is generally written as  $\lambda K$  where  $K$  is called the Kerr coefficient. Thus, the two effects are

Pockels effect

$$\Delta n = a_1 E \quad [9.84]$$

and

Kerr effect

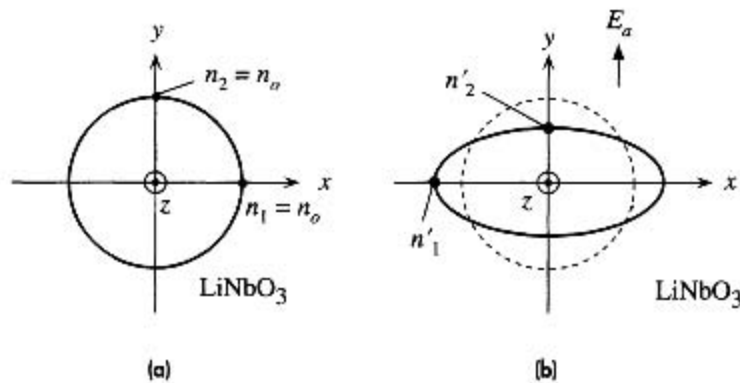
$$\Delta n = a_2 E^2 = (\lambda K) E^2 \quad [9.85]$$

All materials exhibit the Kerr effect. It may be thought that we will always find some (nonzero) value for  $a_1$  for all materials, but this is not true and only certain crystalline materials exhibit the Pockels effect. If we apply a field  $\mathbf{E}$  in one direction and then reverse the field and apply  $-\mathbf{E}$ , then according to Equation 9.84,  $\Delta n$  should change sign. If the refractive index increases for  $\mathbf{E}$ , it must decrease for  $-\mathbf{E}$ . Reversing the field should *not* lead to an identical effect (the same  $\Delta n$ ). The structure has to respond differently to  $\mathbf{E}$  and  $-\mathbf{E}$ . There must therefore be some *asymmetry* in the structure to distinguish between  $\mathbf{E}$  and  $-\mathbf{E}$ . In a noncrystalline material,  $\Delta n$  for  $\mathbf{E}$  would be the same as  $\Delta n$  for  $-\mathbf{E}$  as all directions are equivalent in terms of dielectric properties. Thus  $a_1 = 0$  for all noncrystalline materials (such as glasses and liquids). Similarly, if the crystal structure has a center of symmetry, then reversing the field direction has an identical effect and  $a_1$  is again zero. Only crystals that are **noncentrosymmetric**<sup>22</sup> exhibit the Pockels effect. For example, a NaCl crystal (centrosymmetric) exhibits no Pockels effect, but a GaAs crystal (noncentrosymmetric) does.

The Pockels effect expressed in Equation 9.84 is an oversimplification because in reality we have to consider the effect of an applied field along a particular crystal direction on the refractive index for light with a given propagation direction and polarization. For example, suppose that  $x$ ,  $y$ , and  $z$  are the principal axes of a crystal with refractive indices  $n_1$ ,  $n_2$ , and  $n_3$  along these directions. For an optically isotropic crystal, these would be the same whereas for a uniaxial crystal such as LiNbO<sub>3</sub>  $n_1 = n_2 \neq n_3$  as depicted in the  $xy$  cross section in Figure 9.42a. Suppose that we suitably apply a voltage across a crystal and thereby apply an external dc field  $E_x$ . In the Pockels effect, the field will modify the

<sup>21</sup> John Kerr (1824–1907) was a Scottish physicist who was a faculty member at Free Church Training College for Teachers, Glasgow (1857–1901) where he set up an optics laboratory and demonstrated the Kerr effect (1875).

<sup>22</sup> A crystal is a center of symmetry about a point  $O$ , if any atom (or point) with a position vector  $\mathbf{r}$  from  $O$  also appears when we invert  $\mathbf{r}$ , that is, take  $-\mathbf{r}$ .

**Figure 9.42**

(a) Cross section of the optical indicatrix with no applied field,  $n_1 = n_2 = n_o$ .  
 (b) Applied field along  $y$  in  $\text{LiNbO}_3$  modifies the indicatrix and changes  $n_1$  and  $n_2$  to  $n'_1$  and  $n'_2$ .

optical indicatrix. The exact effect depends on the crystal structure. For example, a crystal like GaAs, optically isotropic with a spherical indicatrix, becomes *birefringent* with two different refractive indices. In the case of  $\text{LiNbO}_3$  (lithium niobate), which is an optoelectronically important uniaxial crystal, a field  $E_a$  along the  $y$  direction changes the principal refractive indices  $n_1$  and  $n_2$  (both equal to  $n_o$ ) to  $n'_1$  and  $n'_2$  as illustrated in Figure 9.42b. Moreover, in some crystals such as KDP ( $\text{KH}_2\text{PO}_4$ , potassium dihydrogen phosphate), the field  $E_a$  along  $z$  rotates the principal axes by  $45^\circ$  about  $z$  and changes the principal indices. Rotation of principal axes in  $\text{LiNbO}_3$  is small and can be neglected.

As an example consider a wave propagating along the  $z$  direction (optic axis) in a  $\text{LiNbO}_3$  crystal. This wave will experience the same refractive index ( $n_1 = n_2 = n_o$ ) whatever the polarization as in Figure 9.42a. However, in the presence of an applied field  $E_a$  parallel to the principal  $y$  axis as in Figure 9.42b, the light propagates as two orthogonally polarized waves (parallel to  $x$  and  $y$ ) experiencing different refractive indices  $n'_1$  and  $n'_2$ . The applied field thus *induces a birefringence* for light traveling along the  $z$  axis. (The field induced rotation of the principal axes in this case, though present, is small and can be neglected.) Before the field  $E_a$  is applied, the refractive indices  $n_1$  and  $n_2$  are both equal to  $n_o$ . The Pockels effect then gives the new refractive indices  $n'_1$  and  $n'_2$  in the presence of  $E_a$  as

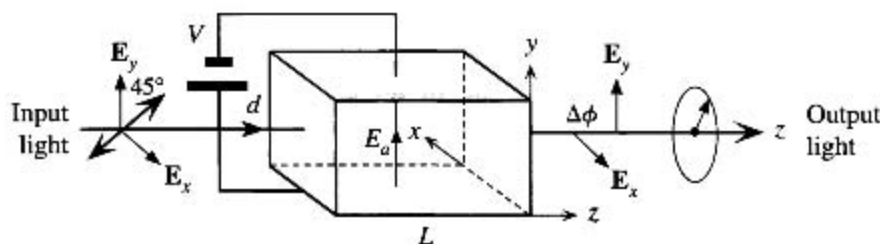
$$n'_1 \approx n_1 + \frac{1}{2}n_1^3 r_{22} E_a \quad \text{and} \quad n'_2 \approx n_2 - \frac{1}{2}n_2^3 r_{22} E_a \quad [9.86] \quad \text{Pockels effect}$$

where  $r_{22}$  is a constant, called a **Pockels coefficient**, that depends on the crystal structure and the material. The reason for the seemingly unusual subscript notation is that there are more than one constant and these are elements of a tensor that represents the optical response of the crystal to an applied field along a particular direction with respect to the principal axes (the exact theory is more mathematical than intuitive). We therefore have to use the correct Pockels coefficients for the refractive index changes for a given crystal and a given field direction.<sup>23</sup> If the field were along  $z$ , the Pockels coefficient in Equation 9.86 would be  $r_{13}$ . Table 9.6 shows some typical values for Pockels coefficients of various crystals.

<sup>23</sup> The reader should not be too concerned with the subscripts but simply interpret them as identifying the right Pockels coefficient value for the particular electro-optic problem at hand.

**Table 9.6** Pockels ( $r$ ) and Kerr ( $K$ ) coefficients in various materials

Material	Crystal	Indices	Pockels Coefficients $\times 10^{-12}$ m/V	Comment
LiNbO <sub>3</sub>	Uniaxial	$n_o = 2.272$ $n_e = 2.187$	$r_{13} = 8.6$ ; $r_{33} = 30.8$ $r_{22} = 3.4$ ; $r_{31} = 28$	$\lambda \approx 500$ nm
KDP	Uniaxial	$n_o = 1.512$ $n_e = 1.470$	$r_{41} = 8.8$ ; $r_{63} = 10.5$	$\lambda \approx 546$ nm
GaAs	Isotropic	$n_o = 3.6$	$r_{41} = 1.5$	$\lambda \approx 546$ nm

**Figure 9.43** Transverse Pockels cell phase modulator. A linearly polarized input light into an electro-optic crystal emerges as a circularly polarized light.

It is clear that the control of the refractive index by an external applied field (and hence a voltage) is a distinct advantage that enables the phase change through a Pockels crystal to be controlled or modulated; such a **phase modulator** is called a *Pockels cell*. In the *longitudinal Pockels cell phase modulator* the applied field is in the direction of light propagation, whereas in the *transverse phase modulator* the applied field is transverse to the direction of light propagation.

Consider the transverse phase modulator in Figure 9.43. In this example, the applied electric field  $E_a = V/d$  is applied parallel to the  $y$  direction, normal to the direction of light propagation along  $z$ . Suppose that the incident beam is linearly polarized (shown as  $\mathbf{E}$ ) say at  $45^\circ$  to the  $y$  axes. We can represent the incident light in terms of polarizations ( $\mathbf{E}_x$  and  $\mathbf{E}_y$ ) along the  $x$  and  $y$  axes. These components  $\mathbf{E}_x$  and  $\mathbf{E}_y$  experience refractive indices  $n'_1$  and  $n'_2$ , respectively. Thus, when  $\mathbf{E}_x$  traverses the distance  $L$ , its phase changes by  $\phi_1$ ,

$$\phi_1 = \frac{2\pi n'_1}{\lambda} L = \frac{2\pi L}{\lambda} \left( n_o + \frac{1}{2} n_o^3 r_{22} \frac{V}{d} \right)$$

When the component  $\mathbf{E}_y$  traverses the distance  $L$ , its phase changes by  $\phi_2$ , given by a similar expression except that  $r_{22}$  changes sign. Thus the phase change  $\Delta\phi$  between the two field components is

$$\Delta\phi = \phi_1 - \phi_2 = \frac{2\pi}{\lambda} n_o^3 r_{22} \frac{L}{d} V \quad [9.87]$$

The applied voltage thus inserts an adjustable phase difference  $\Delta\phi$  between the two field components. The polarization state of the output wave can therefore be controlled by the applied voltage and the Pockels cell is a **polarization modulator**. We can change the medium from a quarter-wave to a half-wave plate by simply adjusting  $V$ . The voltage  $V = V_{\lambda/2}$ , the **half-wave voltage**, corresponds to  $\Delta\phi = \pi$  and generates a half-wave plate.

## CD Selected Topics and Solved Problems

### Selected Topics

Real and Imaginary Dielectric Constant  
Optical Dispersion and Absorption

### Solved Problems

Fresnel's Equations  
Complex Refractive Index and Light Absorption  
Dispersion: Refractive Index versus Wavelength Behavior

## DEFINING TERMS

**Absorption** is the loss in the power of electromagnetic radiation that is traveling in a medium. The loss is due to the conversion of light energy to other forms of energy, such as lattice vibrations (heat) during the polarization of the molecules of the medium, local vibrations of impurity ions, excitation of electrons from the valence band to the conduction band, and so on.

**Activator** is a luminescent center in a host crystal or glass in which it is excited, by some external excitation such as UV light; following excitation, the activator emits radiation to return to its ground state, or become de-excited.

**Anisotropy (optical)** refers to the fact that the refractive index  $n$  of a crystal depends on the direction of propagation of light and on the state of its polarization, that is, the direction of the electric field.

**Antireflection coating** is a thin dielectric layer coated on an optical device or component to reduce the reflection of light and increase the transmitted light intensity.

**Attenuation** is the decrease in the optical power (or irradiance) of a traveling wave in the direction of propagation due to absorption and scattering.

**Attenuation coefficient**  $\alpha$  represents the spatial rate of attenuation of an EM wave along the direction of

propagation. If  $P_o$  is the optical power at some location  $O$ , and if it is  $P$  at a distance  $L$  from  $O$  along the direction of propagation, then  $P = P_o \exp(-\alpha L)$ .

**Birefringent crystals** such as calcite are optically anisotropic which leads to an incident light beam becoming separated into ordinary and extraordinary waves with orthogonal polarizations; incident light becomes doubly refracted because these two waves experience different refractive indices  $n_o$  and  $n_e$ .

**Brewster's angle** or **polarization angle** ( $\theta_p$ ) is the angle of incidence that results in the reflected wave having no electric field in the plane of incidence (plane defined by the incident ray and the normal to the surface). The electric field oscillations in the reflected wave are in the plane perpendicular to the plane of incidence.

**Circularly birefringent medium** is a medium in which right and left circularly polarized waves propagate with different velocities and experience different refractive indices  $n_R$  and  $n_L$ .

**Circularly polarized light** is light where the magnitude of the field vector  $\mathbf{E}$  remains constant but its tip at a given location on the direction of propagation traces out a circle by rotating either in a clockwise sense, *right circularly polarized*, with time, as observed by the

receiver of the wave, or in a counterclockwise sense, *left circularly polarized*.

**Complex propagation constant** ( $k' - jk''$ ) describes the propagation characteristics of an electromagnetic wave that is experiencing attenuation as it travels in a lossy medium. If  $k = k' - jk''$  is the complex propagation constant, then the electric field component of a plane wave traveling in a lossy medium can be described by

$$E = E_0 \exp(-k''z) \exp j(\omega t - k'z)$$

The amplitude decays exponentially while the wave propagates along  $z$ . The *real*  $k'$  part of the complex propagation constant (wavevector) describes the propagation characteristics, that is, the phase velocity  $v = \omega/k'$ . The *imaginary*  $k''$  part describes the rate of attenuation along  $z$ .

**Complex refractive index**  $N$  with real part  $n$  and imaginary part  $K$  is defined as the ratio of the complex propagation constant  $k$  in a medium to propagation constant  $k_0$  in free space,

$$N = n - jK = \frac{k}{k_0} = \left(\frac{1}{k_0}\right)(k' - jk'')$$

The real part  $n$  is simply called the refractive index, and  $K$  is called the *extinction coefficient*.

**Critical angle** ( $\theta_c$ ) is the angle of incidence that results in a refracted wave at  $90^\circ$  when the incident wave is traveling in a medium of lower refractive index and is incident at a boundary with a material with a higher refractive index.

**Dielectric mirror** is made from alternating high and low refractive index quarter-wave-thick multilayers such that constructive interference of partially reflected waves gives rise to a high degree of wavelength-selective reflectance.

**Dispersion relation** is a *relationship* between the refractive index  $n$  and the wavelength  $\lambda$  of the EM wave,  $n = n(\lambda)$ ; the wavelength usually refers to the free-space wavelength. The relationship between the angular frequency  $\omega$  and the propagation constant  $k$ , the  $\omega$ - $k$  curve, is also called the dispersion relation.

**Dispersive medium** has a refractive index  $n$  that depends on the wavelength; that is,  $n$  is not a constant.

**Electro-optic effects** refer to changes in the refractive index of a material induced by the application of an

external electric field, which therefore "modulates" the optical properties; the applied field is not the electric field of any light wave, but a separate external field.

**Extinction coefficient** is the imaginary part of the complex refractive index  $N$ .

**Fluorescence** is luminescence that occurs over very short time scales, usually less than  $10^{-8}$  seconds (or 10 ns). In fluorescence, the onset and decay of luminescent emission, due to the onset and cessation of excitation of the phosphor, is very short, appearing to be almost instantaneous.

**Fresnel's equations** describe the amplitude and phase relationships between the incident, reflected, and transmitted waves at a dielectric-dielectric interface in terms of the refractive indices of the two media and the angle of incidence.

**Group index** ( $N_g$ ) represents the factor by which the group velocity of a group of waves in a dielectric medium is reduced with respect to propagation in free space,  $N_g = c/v_g$  where  $v_g$  is the group velocity.

**Group velocity** ( $v_g$ ) is the velocity at which energy, or information, is transported by a group of waves;  $v_g$  is determined by  $d\omega/dk$  whereas phase velocity is determined by  $\omega/k$ .

**Instantaneous irradiance** is the instantaneous flow of energy per unit time per unit area and is given by the instantaneous value of the Poynting vector  $S$ .

**Irradiance (average)** is the average flow of energy per unit time per unit area where averaging is typically carried out by the light detector (over many oscillation periods). Average irradiance can also be defined mathematically by the average value of the Poynting vector  $S$ . The *instantaneous irradiance* can only be measured if the power meter can respond more quickly than the oscillations of the electric field, and since this is in the optical frequencies range, all practical measurements invariably yield the average irradiance.

**Kerr effect** is a second-order effect in which the change in the refractive index  $n$  depends on the square of the electric field, that is,  $\Delta n = a_2 E^2$ , where  $a_2$  is a material dependent constant.

**Kramers-Kronig relations** relate the real and imaginary parts of the relative permittivity. If we know the complete frequency dependence of the real part  $\epsilon'_r(\omega)$ ,

using the Kramer–Kronig relation, we can find the frequency dependence of the imaginary part  $\epsilon_r''(\omega)$ .

**Luminescence** is the emission of light by a material, called a **phosphor**, due to the absorption and conversion of energy into electromagnetic radiation. Typically the emission of light occurs from certain dopant impurities or even defects, called **luminescent** or **luminescence centers** or **activators** purposefully introduced into a **host matrix**, which may be a crystal or glass, which can accept the activators. **Photoluminescence** involves excitation by photons (light). **Cathodoluminescence** is light emission when the excitation is the bombardment of the phosphor with energetic electrons as in TV cathode ray tubes. **Electroluminescence** is light emission due to the passage of an electric current as in the LED.

**Optic axis** is an axis in the crystal structure along which there is no double refraction for light propagation along this axis.

**Optical activity** is the rotation of the plane of polarization of plane polarized light by a substance such as quartz.

**Optical indicatrix** (Fresnel's ellipsoid) is a refractive index surface placed in the center of the principal axes  $x$ ,  $y$ , and  $z$  of a crystal; the axis intercepts are  $n_1$ ,  $n_2$ , and  $n_3$ . We can represent the optical properties of a crystal in terms of three refractive indices along three orthogonal axes, the *principal axes* of the crystal,  $x$ ,  $y$ , and  $z$ .

**Phase** of a traveling wave is the quantity  $(kx - \omega t)$  which determines the amplitude of the wave at position  $x$  and at time  $t$  given the propagation constant  $k (= 2\pi/\lambda)$  and angular frequency  $\omega$ . In three dimensions it is the quantity  $(\mathbf{k} \cdot \mathbf{r} - \omega t)$  where  $\mathbf{k}$  is the wavevector and  $\mathbf{r}$  is the position vector.

**Phase velocity** is the rate at which a given phase on a traveling wave advances. It represents the velocity of a given phase rather than the velocity at which information is carried by the wave. Two consecutive peaks of a wave are separated by a wavelength  $\lambda$ , and it takes a time period  $1/\nu$  for one peak to reach the next (or the time separation of two consecutive peaks at one location); then the phase velocity is defined as  $v = \lambda\nu$ .

**Phosphor** is a substance made of an activator and a host matrix (crystal or glass) that exhibits luminescence upon suitable excitation.

**Phosphorescence** is a slow luminescence process in which luminescent emission occurs well after the cessation of excitation, even after minutes or hours.

**Pockels effect** is a linear change in the refractive index  $n$  of a crystal due to an application of an external electric field  $E$ , other than the field of the light wave, that is,  $\Delta n = a_1 E$ , where  $a_1$  is a constant that depends on the crystal structure.

**Polarization** of an EM wave describes the behavior of the electric field vector in the EM wave as it propagates through a medium. If the oscillations of the electric field at all times are contained within a well-defined line, then the EM wave is said to be *linearly polarized*. The field vibrations and the direction of propagation, e.g.,  $z$  direction, define a *plane of polarization* (plane of vibration), so linear polarization implies a wave that is plane-polarized.

**Poynting vector** ( $\mathbf{S}$ ) represents the energy flow per unit time per unit area in a direction determined by  $\mathbf{E} \times \mathbf{B}$  (direction of propagation),  $\mathbf{S} = \mathbf{v}^2 \epsilon_0 \epsilon_r \mathbf{E} \times \mathbf{B}$ . Its magnitude, power flow per unit area, is called the irradiance.

**Principal axes** of the crystal, normally labeled,  $x$ ,  $y$ , and  $z$ , are special axes along which the polarization vector and the electric field are parallel. Put differently, the electric displacement  $D$  and the electric field  $E$  vectors are parallel. The refractive indices along these  $x$ ,  $y$ , and  $z$  axes are the principal indices  $n_1$ ,  $n_2$ , and  $n_3$ , respectively, for electric field oscillations along these directions (not to be confused with the wave propagation direction).

**Reflectance** is the fraction of power in the reflected electromagnetic wave with respect to the incident power.

**Reflection coefficient** is the ratio of the amplitude of the reflected EM wave to that of the incident wave. It can be positive, negative, or a complex number which then represents a phase change.

**Refraction** is a change in the direction of a wave when it enters a medium with a different refractive index. A wave that is incident at a boundary between two media with different refractive indices experiences refraction and changes direction in passing from one to the other medium.

**Refractive index**  $n$  of an optical medium is the ratio of the velocity of light in a vacuum to its velocity in the medium  $n = c/v$ .

**Retarding plates** are optical devices that change the state of polarization of an incident light beam. For example, when a linearly polarized light enters a *quarter-wave plate*, it emerges from the device either as circularly or elliptically polarized light, depending on the angle of the incident electric field with respect to the optic axis of the retarder plate.

**Scattering** is a process by which the energy from a propagating EM wave is redirected as secondary EM waves in various directions away from the original direction of propagation. There are a number of scattering processes. In Rayleigh scattering, fluctuations in the refractive index, inhomogeneities, etc., lead to the scattering of light that decreases with the wavelength as  $\lambda^4$ .

**Snell's law** is a law that relates the angles of incidence and refraction when an EM wave traveling in one medium becomes refracted as it enters an adjacent medium. If light is traveling in a medium with index  $n_1$  is incident on a medium of index  $n_2$ , and if the angles of incidence and refraction (transmission) are  $\theta_i$  and  $\theta_r$ , then according to Snell's law,

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{n_2}{n_1}$$

**Specific rotatory power** is defined as the amount of rotation of the optical field in a linearly polarized light per unit distance traveled in the optically active substance.

**Stoke's shift** in luminescence is the shift down in the frequency of the emitted radiation with respect to that of the exciting radiation.

**Total internal reflection (TIR)** is the total reflection of a wave traveling in a medium when it is incident at a boundary with another medium of lower refractive index. The angle of incidence must be greater than the critical angle  $\theta_c$  which depends on the refractive indices  $\sin \theta_c > n_2/n_1$ .

**Transmission coefficient** is the ratio of the amplitude of the transmitted wave to that of the incident wave when the incident wave traveling in a medium meets a boundary with a different medium (different refractive index).

**Transmittance** is the fraction of transmitted intensity when a wave traveling in a medium is incident at a boundary with a different medium (different refractive index).

**Wavefront** is a surface where all the points have the same phase. A wavefront on a plane wave is an infinite plane perpendicular to the direction of propagation.

**Wavenumber** or **propagation constant** is defined as  $2\pi/\lambda$  where  $\lambda$  is the wavelength. It is the phase shift in the wave over a distance of unit length.

**Wavepacket** is a group of waves with slightly different frequencies traveling together and forming a "group." This wavepacket travels with a group velocity  $v_g$  that depends on the slope of  $\omega$  versus  $k$  characteristics of the wavepacket, i.e.,  $v_g = d\omega/dk$ .

**Wavevector** is a vector denoted as  $\mathbf{k}$  that describes the direction of propagation of a wave and has the magnitude of the wavenumber,  $k = 2\pi/\lambda$ .

## QUESTIONS AND PROBLEMS

**9.1 Refractive index and relative permittivity** Using  $n = \sqrt{\epsilon_r}$ , calculate the refractive index  $n$  of the materials in the table given their low-frequency relative permittivities  $\epsilon_r$  (LF). What is your conclusion?

	Material			
	a-Se	Ge	NaCl	MgO
$\epsilon_r$ (LF)	6.4	16.2	5.90	9.83
$n$ ( $\sim 1\text{--}5 \mu\text{m}$ )	2.45	4.0	1.54	1.71



- 9.2 Refractive index and bandgap** Diamond, silicon, and germanium all have the same diamond unit cell. All three are covalently bonded solids. Their refractive indices ( $n$ ) and energy bandgaps ( $E_g$ ) are shown in the table. (a) Plot  $n$  versus  $E_g$  and (b) plot  $n^4$  versus  $1/E_g$ . What is your conclusion? According to **Moss's rule**, very roughly,

$$n^4 E_g \approx K = \text{Constant}$$

*Moss's rule*

What is the value of  $K$ ?

	Material		
	Diamond	Silicon	Germanium
Bandgap, $E_g$ (eV)	5	1.1	0.66
$n$	2.4	3.46	4.0

- \*9.3 Temperature coefficient of refractive index** Suppose that we could write the relationship between the refractive index  $n$  (at frequencies much less than ultraviolet light) and the bandgap  $E_g$  of a semiconductor as suggested by Hervé and Vandamme,

$$n^2 = 1 + \left( \frac{A}{E_g + B} \right)^2$$

where  $E_g$  is in eV,  $A = 13.6$  eV, and  $B = 3.4$  eV. ( $B$  depends on the incident photon energy.) Temperature dependence in  $n$  results from  $dE_g/dT$  and  $dB/dT$ . Show that the temperature coefficient of refractive index (TCRI) is given by,<sup>24</sup>

$$\text{TCRI} = \frac{1}{n} \cdot \frac{dn}{dT} = -\frac{(n^2 - 1)^{3/2}}{13.6 n^2} \left[ \frac{dE_g}{dT} + B' \right]$$

*Hervé–Vandamme relationship*

where  $B' = dB/dT$ . Given that  $B' = 2.5 \times 10^{-5}$  eV K<sup>-1</sup>, calculate TCRI for two semiconductors: Si with  $n \approx 3.5$  and  $dE_g/dT \approx -3 \times 10^{-4}$  eV K<sup>-1</sup>, and AlAs with  $n \approx 3.2$  and  $dE_g/dT \approx -4 \times 10^{-4}$  eV K<sup>-1</sup>.

- 9.4 Sellmeier dispersion equation** Using the Sellmeier equation and the coefficients in Table 9.2, calculate the refractive index of fused silica (SiO<sub>2</sub>) and germania (GeO<sub>2</sub>) at 1550 nm. Which is larger, and why?
- 9.5 Dispersion ( $n$  versus  $\lambda$ ) in GaAs** By using the dispersion relation for GaAs, calculate the refractive index  $n$  and the group index  $N_g$  of GaAs at a wavelength of 1300 nm.
- 9.6 Cauchy dispersion equation** Using the Cauchy coefficients and the general Cauchy equation, calculate the refractive index of a silicon crystal at 200  $\mu\text{m}$  and at 2  $\mu\text{m}$ , over two orders of magnitude wavelength change. What is your conclusion? Would you expect a significant change in  $n$  for  $\hbar\omega > E_g$ ?
- 9.7 Cauchy dispersion relation for zinc selenide** ZnSe is a II–VI semiconductor and a very useful optical material used in various applications such as optical windows (especially high-power laser windows), lenses, prisms, etc. It transmits over 0.50 to 19  $\mu\text{m}$ .  $n$  in the 1–11  $\mu\text{m}$  range described by a Cauchy expression of the form

$$n = 2.4365 + \frac{0.0485}{\lambda^2} + \frac{0.0061}{\lambda^4} - 0.0003\lambda^2$$

*ZnSe dispersion relation*

in which  $\lambda$  is in  $\mu\text{m}$ . What is ZnSe's refractive index  $n$  and group index  $N_g$  at 5  $\mu\text{m}$ ?

- \*9.8 Dispersion ( $n$  versus  $\lambda$ )** Consider an atom in the presence of an oscillating electric field as in Figure 9.4. The applied field oscillates harmonically in the  $+x$  and  $-x$  directions and is given by  $E = E_0 \exp(j\omega t)$ . The energy losses can be represented by a frictional force whose magnitude is proportional to the velocity

<sup>24</sup> P. J. L. Hervé and L. K. J. Vandamme, *J. Appl. Phys.*, 77, 5476, 1995 and references therein.

$dx/dt$ . If  $\gamma$  is the proportionality constant per electron and per unit electron mass, then Newton's second law for  $Z$  electrons in the polarized atom is

$$Zm_e \frac{d^2x}{dt^2} = -ZeE_0 \exp(j\omega t) - Zm_e\omega_0^2 x - Zm_e\gamma \frac{dx}{dt}$$

where  $\omega_0 = (\beta/Zm_e)^{1/2}$  is the natural frequency of the system composed of  $Z$  electrons and a  $+Ze$  nucleus and  $\beta$  is a force constant for the restoring Coulombic force between the electrons and the nucleus. Show that the electronic polarizability  $\alpha_e$  is

Electronic polarizability

$$\alpha_e = \frac{p_{\text{induced}}}{E} = \frac{Ze^2}{m_e(\omega_0^2 - \omega^2 + j\gamma\omega)}$$

What does a complex polarizability represent? Since  $\alpha_e$  is a complex quantity, so is  $\epsilon_r$  and hence the refractive index. By writing the complex refractive index  $N = \sqrt{\epsilon_r}$  where  $\epsilon_r$  is related to  $\alpha_e$  by the Clausius-Mossotti equation, show that

Complex refractive index

$$\frac{N^2 - 1}{N^2 + 2} = \frac{NZe^2}{3\epsilon_0 m_e(\omega_0^2 - \omega^2 + j\gamma\omega)}$$

where  $N$  is the number of atoms per unit volume. What are your conclusions?

- 9.9 Dispersion and diamond** Consider applying the simple electronic polarizability and Clausius-Mossotti equations to diamond. Neglecting losses,

$$\alpha_e = \frac{Ze^2}{m_e(\omega_0^2 - \omega^2)}$$

and

Dispersion in diamond

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{NZe^2}{3\epsilon_0 m_e(\omega_0^2 - \omega^2)}$$

For diamond we can take  $Z = 4$  (valence electrons only as these are the most responsive),  $N = 1.8 \times 10^{29}$  atoms  $\text{m}^{-3}$ ,  $\epsilon_r(\text{DC}) = 5.7$ . Find  $\omega_0$ , and then find the refractive index at  $\lambda = 0.5 \mu\text{m}$  and  $5 \mu\text{m}$ .

- 9.10 Electric and magnetic fields in light** The intensity (irradiance) of the red laser beam from a He-Ne laser in air has been measured to be about  $1 \text{ mW cm}^{-2}$ . What are the magnitudes of the electric and magnetic fields? What are the magnitudes if this  $1 \text{ mW cm}^{-2}$  beam were in a glass medium with a refractive index  $n = 1.45$  and still had the same intensity?

- 9.11 Reflection of light from a less dense medium (internal reflection)** A ray of light which is traveling in a glass medium of refractive index  $n_1 = 1.450$  becomes incident on a less dense glass medium of refractive index  $n_2 = 1.430$ . Suppose that the free-space wavelength ( $\lambda$ ) of the light ray is  $1 \mu\text{m}$ .
- What should be the minimum incidence angle for TIR?
  - What is the phase change in the reflected wave when  $\theta_i = 85^\circ$  and when  $\theta_i = 90^\circ$ ?
  - What is the penetration depth of the evanescent wave into medium 2 when  $\theta_i = 85^\circ$  and when  $\theta_i = 90^\circ$ ?

- 9.12 Internal and external reflection at normal incidence** Consider the reflection of light at normal incidence on a boundary between a GaAs crystal medium of refractive index 3.6 and air of refractive index 1.

- If light is traveling from air to GaAs, what is the reflection coefficient and the intensity of the reflected light in terms of the incident light?
- If light is traveling from GaAs to air, what is the reflection coefficient and the intensity of the reflected light in terms of the incident light?

- 9.13 Antireflection coating**

- Consider three dielectric media with flat and parallel boundaries with refractive indices  $n_1$ ,  $n_2$ , and  $n_3$ . Show that for normal incidence the reflection coefficient between layers 1 and 2 is the same as that between layers 2 and 3 if  $n_2 = \sqrt{n_1 n_3}$ . What is the significance of this?

- b. Consider a Si photodiode that is designed for operation at 900 nm. Given a choice of two possible antireflection coatings, SiO<sub>2</sub> with a refractive index of 1.5 and TiO<sub>2</sub> with a refractive index of 2.3, which would you use and what would be the thickness of the antireflection coating you chose? The refractive index of Si is 3.5.
- 9.14 Optical fibers in communications** Optical fibers for long-haul applications usually have a core region that has a diameter of about 10 μm, and the whole fiber would be about 125 μm in diameter. The core and cladding refractive indices,  $n_1$  and  $n_2$ , respectively, are normally only 0.3–0.5 percent different. Consider a fiber with  $n_1(\text{core}) = 1.4510$ , and  $n_2(\text{cladding}) = 1.4477$ , both at 1550 nm. What is the maximum angle that a light ray can make with the fiber axis if it is still to propagate along the fiber?
- 9.15 Optical fibers in communications** Consider a short-haul optical fiber that has  $n_1(\text{core}) = 1.455$  and  $n_2(\text{cladding}) = 1.440$  at 870 nm. Assume the core-cladding interface behaves like the flat interface between two infinite media as in Figure 9.11. Consider a ray that is propagating that has an angle of incidence 85° at the core-cladding interface. Can this ray exercise total internal reflection? What would be its penetration depth into the cladding?
- 9.16 Complex refractive index** Spectroscopic ellipsometry measurements on a silicon crystal at a wavelength of 620 nm show that the real and imaginary parts of the complex relative permittivity are 15.2254 and 0.172, respectively. Find the complex refractive index. What is the reflectance and absorption coefficient at this wavelength? What is the phase velocity?
- 9.17 Complex refractive index** Spectroscopic ellipsometry measurements on a germanium crystal at a photon energy of 1.5 eV show that the real and imaginary parts of the complex relative permittivity are 21.56 and 2.772, respectively. Find the complex refractive index. What is the reflectance and absorption coefficient at this wavelength? How do your calculations match with the experimental values of  $n = 4.653$  and  $K = 0.298$ ,  $R = 0.419$  and  $\alpha = 4.53 \times 10^6 \text{ m}^{-1}$ ?
- 9.18** An *n*-type germanium sample has a conductivity of about  $300 \Omega^{-1} \text{ m}^{-1}$ . Calculate the imaginary part  $\epsilon''$  of the relative permittivity at a wavelength of 20 μm. Find the attenuation coefficient  $\alpha$  due to free carrier absorption. The refractive index of germanium at the specified wavelength is  $n = 4$ .
- 9.19 Reststrahlen absorption in CdTe** Figure 9.22 shows the infrared extinction coefficient  $K$  of CdTe. Calculate the absorption coefficient  $\alpha$  and the reflectance  $R$  of CdTe at 60 μm and 80 μm.
- 9.20 Reststrahlen absorption in GaAs** Figure 9.22 shows the infrared extinction coefficient  $K$  of GaAs as a function of wavelength. Optical measurements show that  $K$  peaks at  $\lambda = 37.1 \mu\text{m}$  where  $K \approx 11.6$  and  $n \approx 6.6$ . Calculate the absorption coefficient  $\alpha$  and the reflectance  $R$  at this wavelength.
- 9.21 Fundamental absorption** Consider the semiconductors in Figure 9.23, and those semiconductors listed in Table 9.3.
- Which semiconductors can be candidates for a photodetector that can detect light in optical communications at 1550 nm?
  - For amorphous Si (*a*-Si), one definition of an *optical gap* is the photon energy that results in an optical absorption coefficient  $\alpha$  of  $10^4 \text{ cm}^{-1}$ . What is the optical gap of *a*-Si in Figure 9.23?
  - Consider a solar cell from crystalline Si. What is the absorption depth of light at 1000 nm, and at 500 nm?
- 9.22 Quartz half-wave plate** What are the possible thicknesses of a half-wave quartz plate for a wavelength  $\lambda \approx 1.01 \mu\text{m}$  given the extraordinary and ordinary refractive indices are  $n_o = 1.534$  and  $n_e = 1.543$ , respectively?
- 9.23 Pockels cell modulator** What should be the aspect ratio  $d/L$  for the transverse LiNiO<sub>3</sub> phase modulator in Figure 9.43 that will operate at a free-space wavelength of 1.3 μm and will provide a phase shift  $\Delta\phi$  of  $\pi$  (half wavelength) between the two field components propagating through the crystal for an applied voltage of 20 V? The Pockels coefficient  $r_{22}$  is  $3.2 \times 10^{-12} \text{ m/V}$  and  $n_o = 2.2$ .

*Free carrier  
absorption*