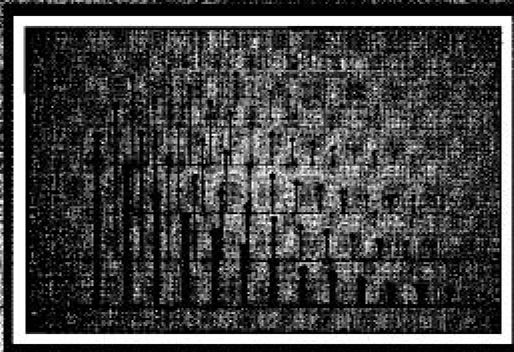
3

The z-Transform



3.0 INTRODUCTION

In this chapter, we develop the z-transform representation of a sequence and study how the properties of a sequence are related to the properties of its z-transform. The z-transform for discrete-time signals is the counterpart of the Laplace transform for continuous-time signals, and they each have a similar relationship to the corresponding Fourier transform. One motivation for introducing this generalization is that the Fourier transform does not converge for all sequences, and it is useful to have a generalization of the Fourier transform that encompasses a broader class of signals. A second advantage is that in analytical problems, the z-transform notation is often more convenient than the Fourier transform notation.

3.1 z-TRANSFORM

The Fourier transform of a sequence $x[n]$ was defined in Chapter 2 as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{3.1}$$

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{3.2}$$

This equation is, in general, an infinite sum or infinite power series, with z considered to be a complex variable. Sometimes it is useful to consider Eq. (3.2) as an operator that transforms a sequence into a function. That is, the *z-transform operator* $\mathcal{Z}\{\cdot\}$, defined as

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z), \tag{3.3}$$

transforms the sequence $x[n]$ into the function $X(z)$, where z is a continuous complex variable. The unique correspondence between a sequence and its z-transform will be indicated by the notation

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z). \tag{3.4}$$

The z-transform, as we have defined it in Eq. (3.2), is often referred to as the *two-sided* or *bilateral z-transform*, in contrast to the *one-sided* or *unilateral z-transform*, which is defined as

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{3.5}$$

Clearly, the bilateral and unilateral transforms are identical if $x[n] = 0$ for $n < 0$, but they differ otherwise. We shall give a brief introduction to the properties of the unilateral z-transform in Section 3.6.

It is evident from a comparison of Eqs. (3.1) and (3.2) that there is a close relationship between the Fourier transform and the z-transform. In particular, if we replace the complex variable z in Eq. (3.2) with the complex quantity $e^{j\omega}$, then the z-transform reduces to the Fourier transform. This is the motivation for the notation $X(e^{j\omega})$ for the Fourier transform. When it exists, the Fourier transform is simply $X(z)$ with $z = e^{j\omega}$. This corresponds to restricting z to have unity magnitude; i.e., for $|z| = 1$, the z-transform corresponds to the Fourier transform. More generally, we can express the complex variable z in polar form as

$$z = re^{j\omega}.$$

With z expressed in this form, Eq. (3.2) becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n},$$

or

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}. \tag{3.6}$$

Equation (3.6) can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence r^{-n}. For $r = 1$, Eq. (3.6) reduces to the Fourier transform of $x[n]$.

Since the z-transform is a function of a complex variable, it is convenient to describe and interpret it using the complex z-plane. In the z-plane, the contour corresponding to $|z| = 1$ is a circle of unit radius, as illustrated in Figure 3.1. This contour, referred to as the *unit circle*, is the set of points $z = e^{j\omega}$ for $0 \le \omega < 2\pi$. The z-transform evaluated on the unit circle corresponds to the Fourier transform. Note that ω is the angle between the vector from the origin to a point z on the unit circle and the real axis

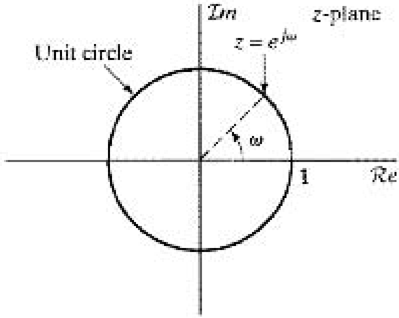


Figure 3.1 The unit circle in the complex z-plane.

of the complex z-plane. If we evaluate $X(z)$ at points on the unit circle in the z-plane beginning at $z = 1$ (i.e., $\omega = 0$) through $z = j$ (i.e., $\omega = \pi/2$) to $z = -1$ (i.e., $\omega = \pi$), we obtain the Fourier transform for $0 \le \omega \le \pi$. Continuing around the unit circle would correspond to examining the Fourier transform from $\omega = \pi$ to $\omega = 2\pi$ or, equivalently, from $\omega = -\pi$ to $\omega = 0$. In Chapter 2, the Fourier transform was displayed on a linear frequency axis. Interpreting the Fourier transform as the z-transform on the unit circle in the z-plane corresponds conceptually to wrapping the linear frequency axis around the unit circle with $\omega = 0$ at $z = 1$ and $\omega = \pi$ at $z = -1$. With this interpretation, the inherent periodicity in frequency of the Fourier transform is captured naturally, since a change of angle of 2π radians in the z-plane corresponds to traversing the unit circle once and returning to exactly the same point.

As we discussed in Chapter 2, the power series representing the Fourier transform does not converge for all sequences; i.e., the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. For any given sequence, the set of values of z for which the z-transform power series converges is called the *region of convergence* (ROC), of the z-transform. As we stated in Section 2.7, if the sequence is absolutely summable, the Fourier transform converges to a continuous function of ω. Applying this criterion to Eq. (3.6) leads to the condition

$$|X(re^{j\omega})| \le \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \tag{3.7}$$

for convergence of the z-transform. From Eq. (3.7) it follows that, because of the multiplication of the sequence by the real exponential r^{-n}, it is possible for the z-transform to converge even if the Fourier transform ($r = 1$) does not. For example, the sequence $x[n] = u[n]$ is not absolutely summable, and therefore, the Fourier transform power series does not converge absolutely. However, $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with an ROC $r = |z| > 1$.

Convergence of the power series of Eq. (3.2) for a given sequence depends only on $|z|$, since $|X(z)| < \infty$ if

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty, \tag{3.8}$$

i.e., the ROC of the power series in Eq. (3.2) consists of all values of z such that the inequality in Eq. (3.8) holds. Thus, if some value of z, say, $z = z_1$, is in the ROC,

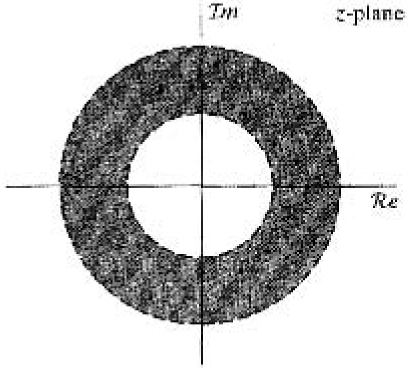


Figure 3.2 The ROC as a ring in the z-plane. For specific cases, the inner boundary can extend inward to the origin, and the ROC becomes a disc. For other cases, the outer boundary can extend outward to infinity.

then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC. As one consequence of this, the ROC will consist of a ring in the z-plane centered about the origin. Its outer boundary will be a circle (or the ROC may extend outward to infinity), and its inner boundary will be a circle (or it may extend inward to include the origin). This is illustrated in Figure 3.2. If the ROC includes the unit circle, then this of course implies convergence of the z-transform for $|z| = 1$, or equivalently, the Fourier transform of the sequence converges. Conversely, if the ROC does not include the unit circle, the Fourier transform does not converge absolutely.

A power series of the form of Eq. (3.2) is a Laurent series. Therefore, a number of elegant and powerful theorems from the theory of functions of a complex variable can be employed in the study of the z-transform. (See Brown and Churchill (2007).) For example, a Laurent series, and therefore the z-transform, represents an analytic function at every point inside the ROC; hence, the z-transform and all its derivatives must be continuous functions of z within the ROC. This implies that if the ROC includes the unit circle, then the Fourier transform and all its derivatives with respect to ω must be continuous functions of ω. Also, from the discussion in Section 2.7, the sequence must be absolutely summable, i.e., a stable sequence.

Uniform convergence of the z-transform requires absolute summability of the exponentially weighted sequence, as stated in Eq. (3.7). Neither of the sequences

$$x_1[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty, \tag{3.9}$$

and

$$x_2[n] = \cos \omega_0 n, \qquad -\infty < n < \infty, \tag{3.10}$$

is absolutely summable. Furthermore, neither of these sequences multiplied by r^{-n} would be absolutely summable for any value of r. Thus, neither of these sequences has a z-transform that converges absolutely for any z. However, we showed in Section 2.7 that even though a sequence such as $x_1[n]$ in Eq. (3.9) is not absolutely summable, it does have finite energy (i.e., it is square-summable), and the Fourier transform converges in the mean-square sense to a discontinuous periodic function. Similarly, the sequence $x_2[n]$ in Eq. (3.10) is neither absolutely nor square summable, but a useful Fourier transform for $x_2[n]$ can be defined using impulse functions (i.e., generalized functions or Dirac delta functions). In both cases, the Fourier transforms are not continuous, infinitely differentiable functions, so they cannot result from evaluating a z-transform on the unit circle. Thus, in such cases it is not strictly correct to think of the Fourier transform as being the z-transform evaluated on the unit circle, although we nevertheless continue to use the notation $X(e^{j\omega})$ always to denote the discrete-time Fourier transform.

The z-transform is most useful when the infinite sum can be expressed in closed form, i.e., when it can be "summed" and expressed as a simple mathematical formula. Among the most important and useful z-transforms are those for which $X(z)$ is equal to a rational function inside the ROC, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}, \tag{3.11}$$

where $P(z)$ and $Q(z)$ are polynomials in z. In general, the values of z for which $X(z) = 0$ are the zeros of $X(z)$, and the values of z for which $X(z)$ is infinite are the poles of $X(z)$. In the case of a rational function as in Eq. (3.11), the zeros are the roots of the numerator polynomial and the poles (for finite values of z) are the roots of the denominator polynomial. For rational z-transforms, a number of important relationships exist between the locations of poles of $X(z)$ and the ROC of the z-transform. We discuss these more specifically in Section 3.2. However, we first illustrate the z-transform with several examples.

Example 3.1 Right-Sided Exponential Sequence

Consider the signal $x[n] = a^n u[n]$, where a denotes a real or complex number. Because it is nonzero only for $n \geq 0$, this is an example of the class of *right-sided* sequences, which are sequences that begin at some time N_1 and have nonzero values only for $N_1 \leq n < \infty$; i.e., they occupy the right side of a plot of the sequence. From Eq. (3.2),

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the ROC, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|. \tag{3.12}$$

To obtain this closed-form expression, we have used the familiar formula for the sum of terms of a geometric series (see Jolley, 1961). The z-transform of the sequence $x[n] = a^n u[n]$ has an ROC for any finite value of $|a|$. For $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \tag{3.13}$$

If $|a| < 1$, the Fourier transform of $x[n] = a^n u[n]$ converges to

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{3.14}$$

However, if $a \geq 1$, the Fourier transform of the right-sided exponential sequence does not converge.

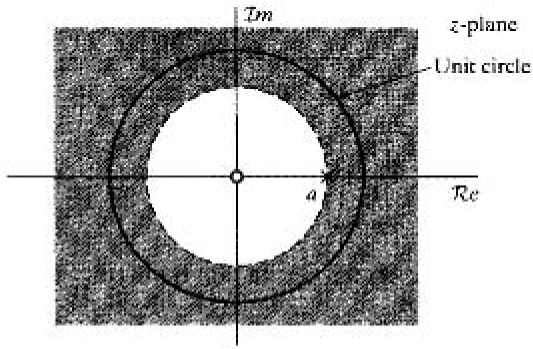


Figure 3.3 Pole–zero plot and ROC for Example 3.1.

In Example 3.1, the infinite sum is equal to a rational function of z inside the ROC. For most purposes, this rational function is a much more convenient representation than the infinite sum. We will see that any sequence that can be represented as a sum of exponentials can equivalently be represented by a rational z-transform. Such a z-transform is determined to within a constant multiplier by its zeros and its poles. For this example, there is one zero, at $z = 0$, and one pole, at $z = a$. The pole–zero plot and the ROC for Example 3.1 are shown in Figure 3.3 where the symbol "∘" denotes the zero and the symbol "×" the pole. For $|a| \geq 1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a, the Fourier transform of the exponentially growing sequence $a^n u[n]$ does not converge.

Example 3.2 Left-Sided Exponential Sequence

Now let

$$x[n] = -a^n u[-n-1] = \begin{cases} -a^n & n \leq -1 \\ 0 & n > -1. \end{cases}$$

Since the sequence is nonzero only for $n \leq -1$, this is a *left-sided* sequence. The z-transform in this case is

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n. \end{aligned} \tag{3.15}$$

If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the last sum in Eq. (3.15) converges, and using again the formula for the sum of terms in a geometric series,

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|. \tag{3.16}$$

The pole–zero plot and ROC for this example are shown in Figure 3.4.

Note that for $|a| < 1$, the sequence $-a^n u[-n-1]$ grows exponentially as $n \to -\infty$, and thus, the Fourier transform does not exist. However, if $|a| > 1$ the Fourier transform is

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \tag{3.17}$$

which is identical in form to Eq. (3.14). At first glance, this would appear to violate the uniqueness of the Fourier transform. However, this ambiguity is resolved if we recall that Eq. (3.14) is the Fourier transform of $a^n u[n]$ if $|a| < 1$, while Eq. (3.17) is the Fourier transform of $-a^n u[-n-1]$ when $|a| > 1$.

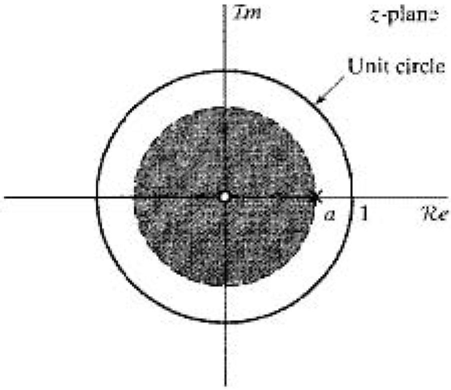


Figure 3.4 Pole–zero plot and ROC for Example 3.2.

Comparing Eqs. (3.12) and (3.16) and Figures 3.3 and 3.4, we see that the sequences and, therefore, the infinite sums are different; however, the algebraic expressions for $X(z)$ and the corresponding pole–zero plots are identical in Examples 3.1 and 3.2. The z-transforms differ only in the ROC. This emphasizes the need for specifying both the algebraic expression and the ROC for the bilateral z-transform of a given sequence. Also, in both examples, the sequences were exponentials and the resulting z-transforms were rational. In fact, as is further suggested by the next example, $X(z)$ will be rational whenever $x[n]$ is a linear combination of real or complex exponentials.

Example 3.3 Sum of Two Exponential Sequences

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]. \tag{3.18}$$

The z-transform is

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left\{\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]\right\} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \end{aligned} \tag{3.19}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}. \tag{3.20}$$

For convergence of $X(z)$, both sums in Eq. (3.19) must converge, which requires that both $\left|\frac{1}{2}z^{-1}\right| < 1$ and $\left|\left(-\frac{1}{3}\right)z^{-1}\right| < 1$ or, equivalently, $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$. Thus, the ROC is the region of overlap, $|z| > \frac{1}{2}$. The pole–zero plot and ROC for the z-transform of each of the individual terms and for the combined signal are shown in Figure 3.5.

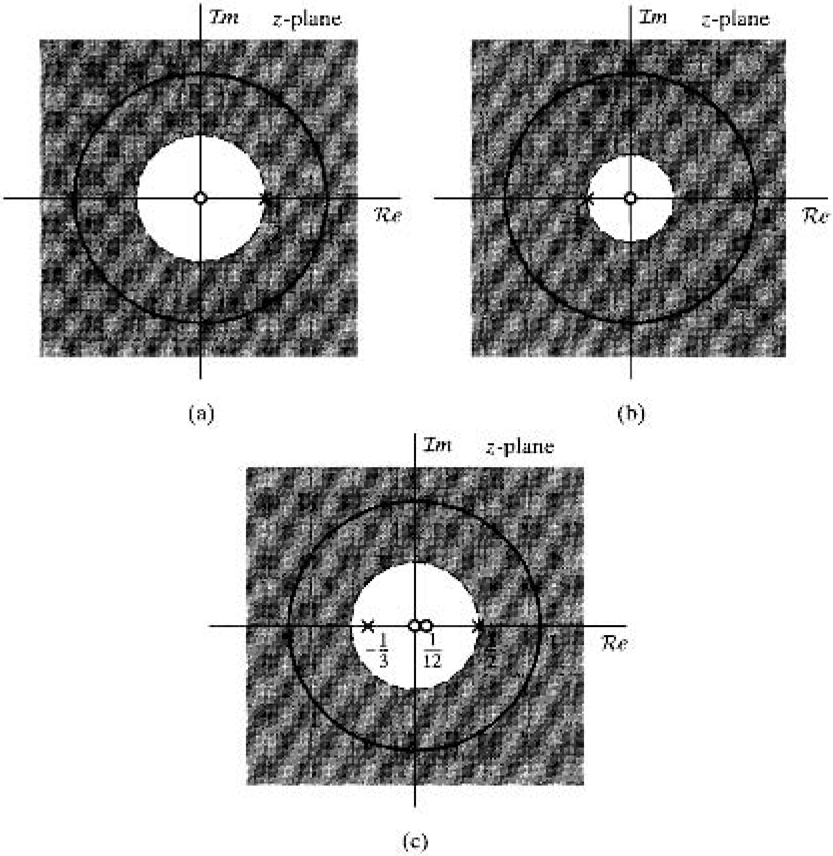


Figure 3.5 Pole–zero plot and ROC for the individual terms and the sum of terms in Examples 3.3 and 3.4. (a) $1/(1-\frac{1}{2}z^{-1})$, $|z| > \frac{1}{2}$. (b) $1/(1+\frac{1}{3}z^{-1})$, $|z| > \frac{1}{3}$. (c) $1/(1-\frac{1}{2}z^{-1}) + 1/(1+\frac{1}{3}z^{-1})$, $|z| > \frac{1}{2}$.

In each of the preceding examples, we started with the definition of the sequence and manipulated each of the infinite sums into a form whose sum could be recognized. When the sequence is recognized as a sum of exponential sequences of the form of Examples 3.1 and 3.2, the z-transform can be obtained much more simply using the fact that the z-transform operator is linear. Specifically, from the definition of the z-transform in Eq. (3.2), if $x[n]$ is the sum of two terms, then $X(z)$ will be the sum of the corresponding z-transforms of the individual terms. The ROC will be the intersection of the individual ROCs, i.e., the values of z for which both individual sums converge. We have already demonstrated the linearity property in obtaining Eq. (3.19) in Example 3.3. Example 3.4 shows how the z-transform in Example 3.3 can be obtained in a much more straightforward manner by expressing $x[n]$ as the sum of two sequences.

Example 3.4 Sum of Two Exponentials (Again)

Again, let $x[n]$ be given by Eq. (3.18). Then using the general result of Example 3.1 with $a = \frac{1}{2}$ and $a = -\frac{1}{3}$, the z-transforms of the two individual terms are easily seen to be

$$\left(\frac{1}{2}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}, \tag{3.21}$$

$$\left(-\frac{1}{3}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}, \tag{3.22}$$

and, consequently,

$$\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{2}, \tag{3.23}$$

as determined in Example 3.3. The pole-zero plot and ROC for the z-transform of each of the individual terms and for the combined signal are shown in Figure 3.5.

All the major points of Examples 3.1–3.4 are summarized in Example 3.5.

Example 3.5 Two-Sided Exponential Sequence

Consider the sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]. \tag{3.24}$$

Note that this sequence grows exponentially as $n \to -\infty$. Using the general result of Example 3.1 with $a = -\frac{1}{3}$, we obtain

$$\left(-\frac{1}{3}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3},$$

and using the result of Example 3.2 with $a = \frac{1}{2}$ yields

$$-\left(\frac{1}{2}\right)^n u[-n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}.$$

Thus, by the linearity of the z-transform,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \frac{1}{3} < |z| \text{ and } |z| < \frac{1}{2},$$
$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}. \quad (3.25)$$

In this case, the ROC is the annular region $\frac{1}{3} < |z| < \frac{1}{2}$. Note that the rational function in this example is identical to the rational function in Example 3.4, but the ROC is different in this case. The pole-zero plot and the ROC for this example are shown in Figure 3.6.

Since the ROC does not contain the unit circle, the sequence in Eq. (3.24) does not have a Fourier transform.

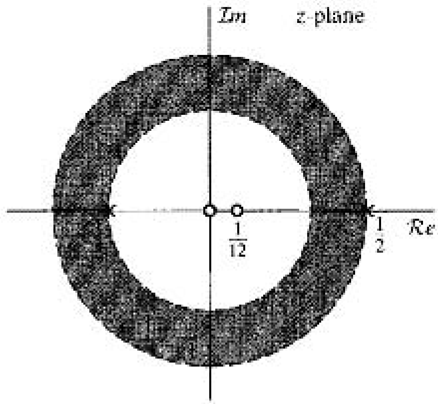


Figure 3.6 Pole-zero plot and ROC for Example 3.5.

In each of the preceding examples, we expressed the z-transform both as a ratio of polynomials in z and as a ratio of polynomials in z^{-1}. From the form of the definition of the z-transform as given in Eq. (3.2), we see that, for sequences that are zero for $n < 0$, $X(z)$ involves only negative powers of z. Thus, for this class of signals, it is particularly convenient for $X(z)$ to be expressed in terms of polynomials in z^{-1} rather than z; however, even when $x[n]$ is nonzero for $n < 0$, $X(z)$ can still be expressed in terms of factors of the form $(1 - az^{-1})$. It should be remembered that such a factor introduces both a pole and a zero, as illustrated by the algebraic expressions in the preceding examples.

These examples show that infinitely long exponential sequences have z-transforms that can be expressed as rational functions of either z or z^{-1}. The case where the sequence has finite length also has a rather simple form. If the sequence is nonzero only in the interval $N_1 \leq n \leq N_2$, the z-transform

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n} \quad (3.26)$$

has no problems of convergence, as long as each of the terms $|x[n]z^{-n}|$ is finite. In general, it may not be possible to express the sum of a finite set of terms in a closed form, but in such cases it may be unnecessary. For example, if $x[n] = \delta[n] + \delta[n-5]$, then $X(z) = 1 + z^{-5}$, which is finite for $|z| > 0$. An example of a case where a finite number of terms can be summed to produce a more compact representation of the z-transform is given in Example 3.6.

Example 3.6 Finite-Length Truncated Exponential Sequence

Consider the signal

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}, \tag{3.27}$$

where we have used the general formula in Eq. (2.55) to obtain a closed-form expression for the sum of the finite series. The ROC is determined by the set of values of z for which

$$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty.$$

Since there are only a finite number of nonzero terms, the sum will be finite as long as az^{-1} is finite, which in turn requires only that $|a| < \infty$ and $z \neq 0$. Thus, assuming that $|a|$ is finite, the ROC includes the entire z-plane, with the exception of the origin ($z = 0$). The pole–zero plot for this example, with $N = 16$ and a real and between zero and unity, is shown in Figure 3.7. Specifically, the N roots of the numerator polynomial are at z-plane locations

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1. \tag{3.28}$$

(Note that these values satisfy the equation $z^N = a^N$, and when $a = 1$, these complex values are the N^{th} roots of unity.) The zero corresponding to $k = 0$ cancels the pole at $z = a$. Consequently, there are no poles other than the $N - 1$ poles at the origin. The remaining zeros are at z-plane locations

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \ldots, N-1. \tag{3.29}$$

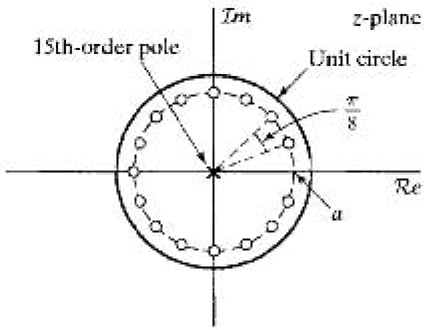


Figure 3.7 Pole–zero plot for Example 3.6 with $N = 16$ and a real such that $0 < a < 1$. The ROC in this example consists of all values of z except $z = 0$.

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

The transform pairs corresponding to some of the preceding examples, as well as a number of other commonly encountered z-transform pairs, are summarized in Table 3.1. We will see that these basic transform pairs are very useful in finding z-transforms given a sequence or, conversely, in finding the sequence corresponding to a given z-transform.

3.2 PROPERTIES OF THE ROC FOR THE z-TRANSFORM

The examples of the previous section suggest that the properties of the ROC depend on the nature of the signal. These properties are summarized in this section with some discussion and intuitive justification. We assume specifically that the algebraic expression for the z-transform is a rational function and that $x[n]$ has finite amplitude, except possibly at $n = \infty$ or $n = -\infty$.

PROPERTY 1: The ROC will either be of the form $0 \leq r_R < |z|$, or $|z| < r_L \leq \infty$, or, in general the annulus, i.e., $0 \leq r_R < |z| < r_L \leq \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Property 1 summarizes the general shape of the ROC. As discussed in Section 3.1, it results from the fact that the condition for convergence of Eq. (3.2) is given by Eq. (3.7) repeated here as

$$\sum_{n=-\infty}^{\infty} |x[n]|r^{-n} < \infty \tag{3.30}$$

where $r = |z|$. Equation (3.30) shows that for a given $x[n]$, convergence is dependent only on $r = |z|$ (i.e., not on the angle of z). Note that if the z-transform converges for $|z| = r_0$, then we may decrease r until the z-transform does not converge. This is the value $|z| = r_R$ such that $|x[n]|r^{-n}$ grows too fast (or decays too slowly) as $n \to \infty$, so that the series is not absolutely summable. This defines r_R. The z-transform cannot converge for $r \leq r_R$ since r^{-n} will grow even faster. Similarly, the outer boundary r_L can be found by increasing r from r_0 and considering what happens when $n \to -\infty$.

Property 2 is a consequence of the fact that Eq. (3.2) reduces to the Fourier transform when $|z| = 1$. Property 3 follows from the recognition that $X(z)$ is infinite at a pole and therefore, by definition, does not converge.

Property 4 follows from the fact that the z-transform of a finite-length sequence is a finite sum of finite powers of z, i.e.,

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}.$$

Therefore, $|X(z)| < \infty$ for all z except $z = 0$ when $N_2 > 0$ and/or $z = \infty$ when $N_1 < 0$.

Properties 5 and 6 are special cases of Property 1. To interpret Property 5 for rational z-transforms, note that a sequence of the form

$$x[n] = \sum_{k=1}^{N} A_k (d_k)^n u[n] \tag{3.31}$$

is an example of a right-sided sequence composed of exponential sequences with amplitudes A_k and exponential factors d_k. While this is not the most general right-sided sequence, it will suffice to illustrate Property 5. More general right-sided sequences can be formed by adding finite-length sequences or shifting the exponential sequences by finite amounts; however, such modifications to Eq. (3.31) would not change our conclusions. Invoking the linearity property, the z-transform of $x[n]$ in Eq. (3.31) is

$$X(z) = \sum_{k=1}^{N} \underbrace{\frac{A_k}{1 - d_k z^{-1}}}_{|z| > |d_k|}. \tag{3.32}$$

Note that for values of z that lie in all of the individual ROCs, $|z| > |d_k|$, the terms can be combined into one rational function with common denominator

$$\prod_{k=1}^{N} (1 - d_k z^{-1});$$

i.e., the poles of $X(z)$ are located at $z = d_1, \ldots, d_N$. Assume for convenience that the poles are ordered so that d_1 has the smallest magnitude, corresponding to the innermost pole, and d_N has the largest magnitude, corresponding to the outermost pole. The least rapidly increasing of these exponentials, as n increases, is the one corresponding to the innermost pole, i.e., d_1, and the most slowly decaying (or most rapidly growing) is the one corresponding to the outermost pole, i.e., d_N. Not surprisingly, d_N determines the inner boundary of the ROC which is the intersection of the regions $|z| > |d_k|$. That is, the ROC of the z-transform of a right-sided sum of exponential sequences is

$$|z| > |d_N| = \max_k |d_k| = r_R, \tag{3.33}$$

i.e., the ROC is outside the outermost pole, extending to infinity. If a right-sided sequence begins at $n = N_1 < 0$, then the ROC will not include $|z| = \infty$.

Another way of arriving at Property 5 is to apply Eq. (3.30) to Eq. (3.31) obtaining

$$\sum_{n=0}^{\infty} \left| \sum_{k=1}^{N} A_k (d_k)^n \right| r^{-n} \le \sum_{k=1}^{N} |A_k| \left(\sum_{n=0}^{\infty} |d_k / r|^n \right) < \infty, \tag{3.34}$$

which shows that convergence is guaranteed if all the sequences $|d_k/r|^n$ are absolutely summable. Again, since $|d_N|$ is the largest pole magnitude, we choose $|d_N/r| < 1$, or $r > |d_N|$.

For Property 6, which is concerned with left-sided sequences, an exactly parallel argument can be carried out for a sum of left-sided exponential sequences to show that the ROC will be defined by the pole with the smallest magnitude. With the same assumption on the ordering of the poles, the ROC will be

$$|z| < |d_1| = \min_k |d_k| = r_L, \tag{3.35}$$

i.e., the ROC is inside the innermost pole. If the left-sided sequence has nonzero values for positive values of n, then the ROC will not include the origin, $z = 0$. Since $x[n]$ now extends to $-\infty$ along the negative n-axis, r must be restricted so that for each d_k, the exponential sequence $(d_k r^{-1})^n$ decays to zero as *n decreases* toward $-\infty$.

For right-sided sequences, the ROC is dictated by the exponential weighting r^{-n} required to have all exponential terms decay to zero for increasing n; for left-sided sequences, the exponential weighting must be such that all exponential terms decay to zero for decreasing n. Property 7 follows from the fact that for two-sided sequences, the exponential weighting needs to be balanced, since if it decays too fast for increasing n, it may grow too quickly for decreasing n and vice versa. More specifically, for two-sided sequences, some of the poles contribute only for $n > 0$ and the rest only for $n < 0$. The ROC is bounded on the inside by the pole with the largest magnitude that contributes for $n > 0$ and on the outside by the pole with the smallest magnitude that contributes for $n < 0$.

Property 8 is intuitively suggested by our discussion of Properties 4 through 7. Any infinite two-sided sequence can be represented as a sum of a right-sided part (say, for $n \geq 0$) and a left-sided part that includes everything not included in the right-sided part. The right-sided part will have an ROC given by Eq. (3.33), while the ROC of the left-sided part will be given by Eq. (3.35). The ROC of the entire two-sided sequence must be the intersection of these two regions. Thus, if such an intersection exists, it will always be a simply connected annular region of the form

$$r_R < |z| < r_L.$$

There is a possibility of no overlap between the ROCs of the right- and left-sided parts; i.e., $r_L < r_R$. In such cases, the z-transform of the sequence simply does not exist.

Example 3.7 Non-Overlapping Regions of Convergence

An example is the sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1].$$

Applying the corresponding entries from Table 3.1 separately to each part leads to

$$X(z) = \underbrace{\frac{1}{1 - \frac{1}{2}z^{-1}}}_{|z| > \frac{1}{2}} + \underbrace{\frac{1}{1 + \frac{1}{3}z^{-1}}}_{|z| < \frac{1}{3}}.$$

Since there is no overlap between $|z| > \frac{1}{2}$ and $|z| < \frac{1}{3}$, we conclude that $x[n]$ has no z-transform (nor Fourier transform) representation.

As we indicated in comparing Examples 3.1 and 3.2, the algebraic expression or pole–zero pattern does not completely specify the z-transform of a sequence; i.e., the ROC must also be specified. The properties considered in this section limit the possible ROCs that can be associated with a given pole–zero pattern. To illustrate, consider the pole–zero pattern shown in Figure 3.8(a). From Properties 1, 3, and 8, there are only four possible choices for the ROC. These are indicated in Figures 3.8(b), (c), (d), and (e), each being associated with a different sequence. Specifically, Figure 3.8(b) corresponds

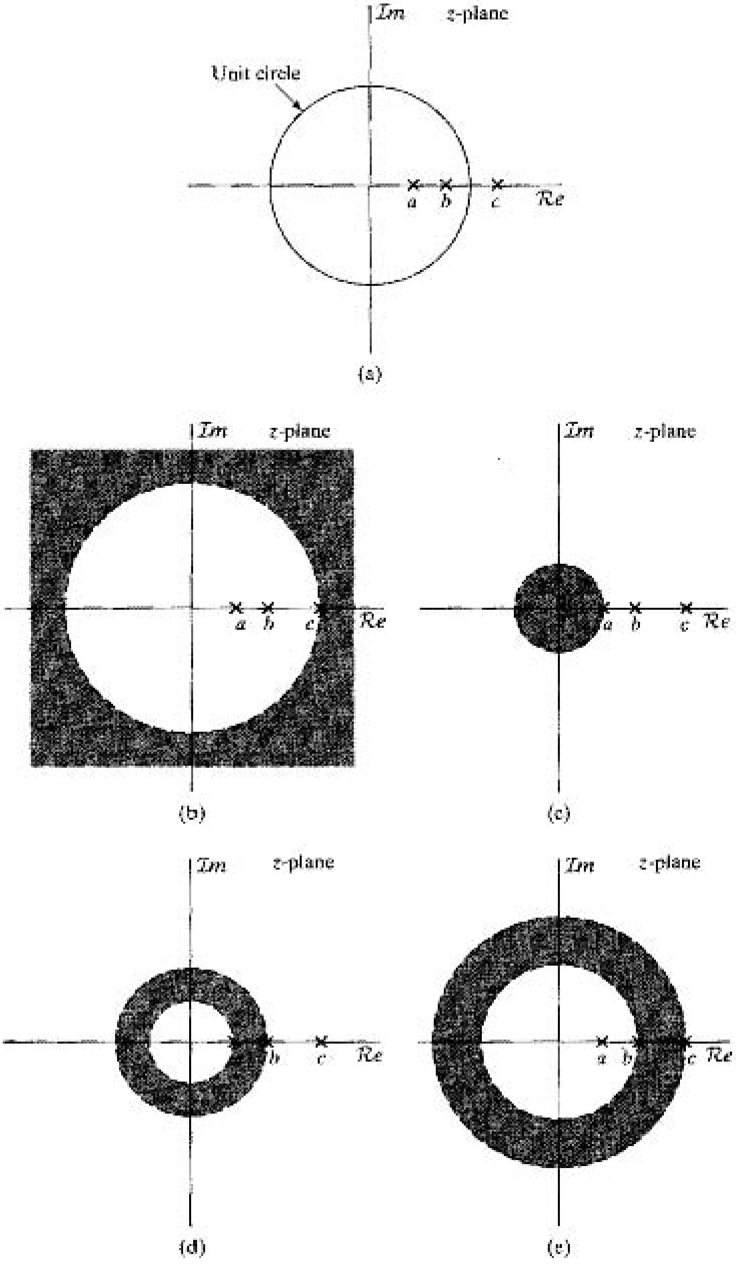


Figure 3.8 Examples of four z-transforms with the same pole–zero locations, illustrating the different possibilities for the ROC, each of which corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

to a right-sided sequence, Figure 3.8(c) to a left-sided sequence, and Figures 3.8(d) and 3.8(e) to two different two-sided sequences. If we assume, as indicated in Figure 3.8(a), that the unit circle falls between the pole at $z = b$ and the pole at $z = c$, then the only one of the four cases for which the Fourier transform would converge is that in Figure 3.8(e).

In representing a sequence through its z-transform, it is sometimes convenient to specify the ROC implicitly through an appropriate time-domain property of the sequence. This is illustrated in Example 3.8.

Example 3.8 Stability, Causality, and the ROC

Consider an LTI system with impulse response $h[n]$. As we will discuss in more detail in Section 3.5, the z-transform of $h[n]$ is called the *system function* of the LTI system. Suppose that $H(z)$ has the pole–zero plot shown in Figure 3.9. There are three possible ROCs consistent with Properties 1–8 that can be associated with this pole–zero plot; i.e., $|z| < \frac{1}{2}$, $\frac{1}{2} < |z| < 2$, and $|z| > 2$. However, if we state in addition that the system is stable (or equivalently, that $h[n]$ is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus, stability of the system and Properties 1–8 imply that the ROC is the region $\frac{1}{2} < |z| < 2$. Note that as a consequence, $h[n]$ is two sided; therefore, the system is not causal.

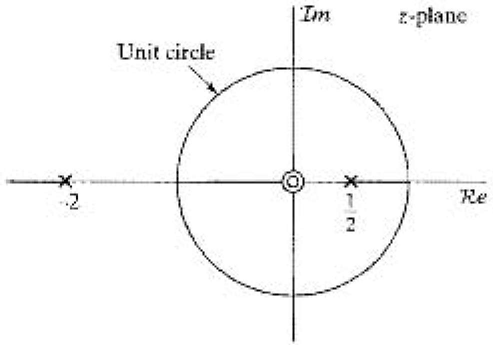


Figure 3.9 Pole–zero plot for the system function in Example 3.8.

If we state instead that the system is causal, and therefore that $h[n]$ is right sided, Property 5 would require that the ROC be the region $|z| > 2$. Under this condition, the system would not be stable; i.e., for this specific pole–zero plot, there is no ROC that would imply that the system is both stable and causal.

3.3 THE INVERSE z-TRANSFORM

In using the z-transform for analysis of discrete-time signals and systems, we must be able to move back and forth between time-domain and z-domain representations. Often, this analysis involves finding the z-transform of sequences and, after some manipulation of the algebraic expressions, finding the inverse z-transform. The inverse z-transform is the following complex contour integral:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz, \tag{3.36}$$

where C represents a closed contour within the ROC of the z-transform. This integral expression can be derived using the Cauchy integral theorem from the theory of complex variables. (See Brown and Churchill, 2007 for a discussion of the topics of Laurent series and complex integration theorems, all of which are relevant to an in-depth study of fundamental mathematical foundations of the z-transform.) However, for the typical kinds of sequences and z-transforms that we will encounter in the analysis of discrete LTI systems, less formal procedures are sufficient and preferable to techniques based on evaluation of Eq. (3.36). In Sections 3.3.1–3.3.3, we consider some of these procedures, specifically the inspection method, partial fraction expansion, and power series expansion.

3.3.1 Inspection Method

The inspection method consists simply of becoming familiar with, or recognizing "by inspection," certain transform pairs. For example, in Section 3.1, we evaluated the z-transform for sequences of the form $x[n] = a^n u[n]$, where a can be either real or complex. Sequences of this form arise quite frequently, and consequently, it is particularly useful to make direct use of the transform pair

$$a^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| > |a|. \tag{3.37}$$

If we need to find the inverse z-transform of

$$X(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2}, \tag{3.38}$$

and we recall the z-transform pair of Eq. (3.37), we would recognize "by inspection" the associated sequence as $x[n] = \left(\frac{1}{2}\right)^n u[n]$. If the ROC associated with $X(z)$ in Eq. (3.38) had been $|z| < \frac{1}{2}$, we can recall transform pair 6 in Table 3.1 to find by inspection that $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$.

Tables of z-transforms, such as Table 3.1, are invaluable in applying the inspection method. If the table is extensive, it may be possible to express a given z-transform as a sum of terms, each of whose inverse is given in the table. If so, the inverse transform (i.e., the corresponding sequence) can be written from the table.

3.3.2 Partial Fraction Expansion

As already described, inverse z-transforms can be found by inspection if the z-transform expression is recognized or tabulated. Sometimes, $X(z)$ may not be given explicitly in an available table, but it may be possible to obtain an alternative expression for $X(z)$ as a sum of simpler terms, each of which is tabulated. This is the case for any rational function, since we can obtain a partial fraction expansion and easily identify the sequences corresponding to the individual terms.

To see how to obtain a partial fraction expansion, let us assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}; i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{3.39}$$

Such z-transforms arise frequently in the study of LTI systems. An equivalent expression is

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}. \tag{3.40}$$

Equation (3.40) explicitly shows that for such functions, there will be M zeros and N poles at nonzero locations in the finite z-plane assuming a_0, b_0, a_N, and b_M are nonzero. In addition, there will be either $M - N$ poles at $z = 0$ if $M > N$ or $N - M$ zeros at $z = 0$ if $N > M$. In other words, z-transforms of the form of Eq. (3.39) always have the same number of poles and zeros in the finite z-plane, and there are no poles or zeros at $z = \infty$. To obtain the partial fraction expansion of $X(z)$ in Eq. (3.39), it is most convenient to note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}, \tag{3.41}$$

where the c_ks are the nonzero zeros of $X(z)$ and the d_ks are the nonzero poles of $X(z)$. If $M < N$ and the poles are all 1^{st}-order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{3.42}$$

Obviously, the common denominator of the fractions in Eq. (3.42) is the same as the denominator in Eq. (3.41). Multiplying both sides of Eq. (3.42) by $(1 - d_k z^{-1})$ and evaluating for $z = d_k$ shows that the coefficients, A_k, can be found from

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}. \tag{3.43}$$

Example 3.9 2nd-Order z-Transform

Consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}. \tag{3.44}$$

The pole-zero plot for $X(z)$ is shown in Figure 3.10. From the ROC and Property 5, Section 3.2, we see that $x[n]$ is a right-sided sequence. Since the poles are both 1^{st}-order, $X(z)$ can be expressed in the form of Eq. (3.42); i.e.,

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

From Eq. (3.43),

$$A_1 = \left(1 - \frac{1}{4}z^{-1}\right)X(z)\Big|_{z=1/4} = \frac{(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}\Bigg|_{z=1/4} = -1,$$

$$A_2 = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\Big|_{z=1/2} = \frac{(1 - \frac{1}{2}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}\Bigg|_{z=1/2} = 2.$$

(Observe that the common factors between the numerator and denominator must be canceled before evaluating the above expressions for A_1 and A_2.) Therefore,

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

Since $x[n]$ is right sided, the ROC for each term extends outward from the outermost pole. From Table 3.1 and the linearity of the z-transform, it then follows that

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

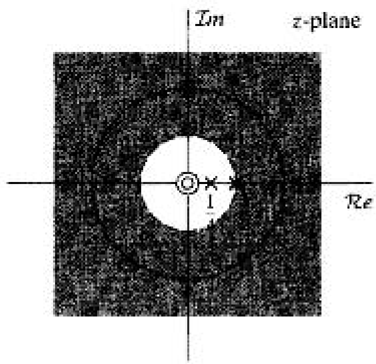


Figure 3.10 Pole-zero plot and ROC for Example 3.9.

Clearly, the numerator that would result from adding the terms in Eq. (3.42) would be at most of degree $(N-1)$ in the variable z^{-1}. If $M \geq N$, then a polynomial must be added to the right-hand side of Eq. (3.42), the order of which is $(M-N)$. Thus, for $M \geq N$, the complete partial fraction expansion would have the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{3.45}$$

If we are given a rational function of the form of Eq. (3.39), with $M \geq N$, the B_rs can be obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator. The A_ks can still be obtained with Eq. (3.43).

If $X(z)$ has multiple-order poles and $M \geq N$, Eq. (3.45) must be further modified. In particular, if $X(z)$ has a pole of order s at $z = d_i$ and all the other poles are 1st-order, then Eq. (3.45) becomes

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}. \tag{3.46}$$

The coefficients A_k and B_r are obtained as before. The coefficients C_m are obtained from the equation

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1 - d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}. \tag{3.47}$$

Equation (3.46) gives the most general form for the partial fraction expansion of a rational z-transform expressed as a function of z^{-1} for the case $M \geq N$ and for d_i a pole of order s. If there are several multiple-order poles, then there will be a term like the third sum in Eq. (3.46) for each multiple-order pole. If there are no multiple-order poles, Eq. (3.46) reduces to Eq. (3.45). If the order of the numerator is less than the order of the denominator ($M < N$), then the polynomial term disappears from Eqs. (3.45) and (3.46) leading to Eq. (3.42).

It should be noted that we could have achieved the same results by assuming that the rational z-transform was expressed as a function of z instead of z^{-1}. That is, instead of factors of the form $(1 - az^{-1})$, we could have considered factors of the form $(z - a)$. This would lead to a set of equations similar in form to Eqs. (3.41)–(3.47) that would be convenient for use with a table of z-transforms expressed in terms of z. Since we find it most convenient to express Table 3.1 in terms of z^{-1}, the development we pursued is more useful.

To see how to find the sequence corresponding to a given rational z-transform, let us suppose that $X(z)$ has only 1st-order poles, so that Eq. (3.45) is the most general form of the partial fraction expansion. To find $x[n]$, we first note that the z-transform operation is linear, so that the inverse transform of individual terms can be found and then added together to form $x[n]$.

The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, i.e., terms of the form $B_r \delta[n-r]$. The fractional terms correspond to exponential sequences. To decide whether a term

$$\frac{A_k}{1 - d_k z^{-1}}$$

corresponds to $(d_k)^n u[n]$ or $-(d_k)^n u[-n-1]$, we must use the properties of the ROC that were discussed in Section 3.2. From that discussion, it follows that if $X(z)$ has only simple poles and the ROC is of the form $r_R < |z| < r_L$, then a given pole d_k will correspond to a right-sided exponential $(d_k)^n u[n]$ if $|d_k| \leq r_R$, and it will correspond to a left-sided exponential if $|d_k| \geq r_L$. Thus, the ROC can be used to sort the poles, with all poles inside the inner boundary r_R corresponding to right-sided sequences and all the poles outside the outer boundary corresponding to left-sided sequences. Multiple-order poles also are divided into left-sided and right-sided contributions in the same way. The use of the ROC in finding inverse z-transforms from the partial fraction expansion is illustrated by the following examples.

Example 3.10 Inverse by Partial Fractions

To illustrate the case in which the partial fraction expansion has the form of Eq. (3.45), consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}, \qquad |z| > 1. \tag{3.48}$$

The pole–zero plot for $X(z)$ is shown in Figure 3.11. From the ROC and Property 5, Section 3.2, it is clear that $x[n]$ is a right-sided sequence. Since $M = N = 2$ and the poles are all 1^{st}-order, $X(z)$ can be represented as

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}.$$

The constant B_0 can be found by long division:

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & \underline{z^{-2}-3z^{-1}+2} \\ & \qquad 5z^{-1}-1 \end{array}$$

Since the remainder after one step of long division is of degree 1 in the variable z^{-1}, it is not necessary to continue to divide. Thus, $X(z)$ can be expressed as

$$X(z) = 2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}. \tag{3.49}$$

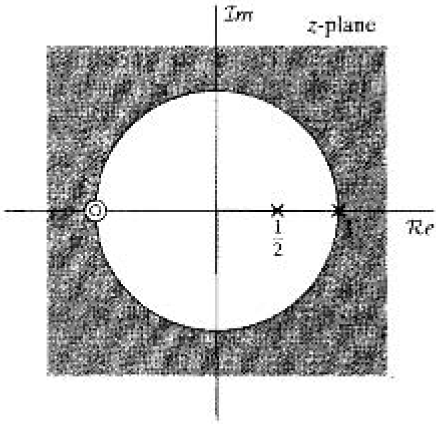


Figure 3.11 Pole–zero plot for the z-transform in Example 3.10.

Now the coefficients A_1 and A_2 can be found by applying Eq. (3.43) to Eq. (3.48) or, equivalently, Eq. (3.49). Using Eq. (3.49), we obtain

$$A_1 = \left[\left(2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}\right)\left(1-\frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}\right)(1-z^{-1})\right]_{z=1} = 8.$$

Therefore,

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}. \tag{3.50}$$

From Table 3.1, we see that since the ROC is $|z| > 1$,

$$2 \overset{\mathcal{Z}}{\longleftrightarrow} 2\delta[n],$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u[n],$$

$$\frac{1}{1-z^{-1}} \overset{\mathcal{Z}}{\longleftrightarrow} u[n].$$

Thus, from the linearity of the z-transform,

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

In Section 3.4, we will discuss and illustrate a number of properties of the z-transform that, in combination with the partial fraction expansion, provide a means for determining the inverse z-transform from a given rational algebraic expression and associated ROC, even when $X(z)$ is not exactly in the form of Eq. (3.41). The examples of this section were simple enough so that the computation of the partial fraction expansion was not difficult. However, when $X(z)$ is a rational function with high-degree polynomials in numerator and denominator, the computations to factor the denominator and compute the coefficients become much more difficult. In such cases, software tools such as MATLAB can implement the computations with ease.

3.3.3 Power Series Expansion

The defining expression for the z-transform is a Laurent series where the sequence values $x[n]$ are the coefficients of z^{-n}. Thus, if the z-transform is given as a power series in the form

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots, \end{aligned} \tag{3.51}$$

we can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}. We have already used this approach in finding the inverse transform of the polynomial part of the partial fraction expansion when $M \geq N$. This approach is also very useful for finite-length sequences where $X(z)$ may have no simpler form than a polynomial in z^{-1}.

Example 3.11 Finite-Length Sequence

Suppose $X(z)$ is given in the form

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}). \tag{3.52}$$

Although $X(z)$ is obviously a rational function of z, it is really not a rational function in the form of Eq. (3.39). Its only poles are at $z = 0$, so a partial fraction expansion according to the technique of Section 3.3.2 is not appropriate. However, by multiplying the factors of Eq. (3.52), we can express $X(z)$ as

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}.$$

Therefore, by inspection, $x[n]$ is seen to be

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Equivalently,

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1].$$

In finding z-transforms of a sequence, we generally seek to sum the power series of Eq. (3.51) to obtain a simpler mathematical expression, e.g., a rational function. If we wish to use the power series to find the sequence corresponding to a given $X(z)$ expressed in closed form, we must expand $X(z)$ back into a power series. Many power series have been tabulated for transcendental functions such as log, sin, sinh, etc. In some cases, such power series can have a useful interpretation as z-transforms, as we illustrate in Example 3.12. For rational z-transforms, a power series expansion can be obtained by long division, as illustrated in Example 3.13.

Example 3.12 Inverse Transform by Power Series Expansion

Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|. \tag{3.53}$$

Using the Taylor series expansion for $\log(1 + x)$ with $|x| < 1$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

Therefore,

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases} \tag{3.54}$$

When $X(z)$ is the ratio of polynomials, it is sometimes useful to obtain a power series by long division of the polynomials.

Example 3.13 Power Series Expansion by Long Division

Consider the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|. \tag{3.55}$$

Since the ROC is the exterior of a circle, the sequence is a right-sided one. Furthermore, since $X(z)$ approaches a finite constant as z approaches infinity, the sequence is causal. Thus, we divide, so as to obtain a series in powers of z^{-1}. Carrying out the long division, we obtain

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad a^2z^{-2} \quad \cdots \end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots.$$

Hence, $x[n] = a^n u[n]$.

By dividing the highest power of z^{-1} in the denominator into the highest power of the numerator in Example 3.13, we obtained a series in z^{-1}. An alternative is to express the rational function as a ratio of polynomials in z and then divide. This leads to a power series in z from which the corresponding left-sided sequence can be determined.

3.4 *z*-TRANSFORM PROPERTIES

Many of the mathematical properties of the z-transform are particularly useful in studying discrete-time signals and systems. For example, these properties are often used in conjunction with the inverse z-transform techniques discussed in Section 3.3 to obtain the inverse z-transform of more complicated expressions. In Section 3.5 and Chapter 5 we will see that the properties also form the basis for transforming linear constant-coefficient difference equations to algebraic equations in terms of the transform variable z, the solution to which can then be obtained using the inverse z-transform. In this section, we consider some of the most frequently used properties. In the following discussion, $X(z)$ denotes the z-transform of $x[n]$, and the ROC of $X(z)$ is indicated by R_x; i.e.,

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \text{ROC} = R_x.$$

As we have seen, R_x represents a set of values of z such that $r_R < |z| < r_L$. For properties that involve two sequences and associated z-transforms, the transform pairs will be denoted as

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \qquad \text{ROC} = R_{x_1},$$

$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \qquad \text{ROC} = R_{x_2}.$$

3.4.1 Linearity

The linearity property states that

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2},$$

and follows directly from the z-transform definition, Eq. (3.2); i.e.,

$$\sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])z^{-n} = a \underbrace{\sum_{n=-\infty}^{\infty} x_1[n]z^{-n}}_{|z| \in R_{x_1}} + b \underbrace{\sum_{n=-\infty}^{\infty} x_2[n]z^{-n}}_{|z| \in R_{x_2}}.$$

As indicated, to split the z-transform of a sum into the sum of corresponding z-transforms, z must be in both ROCs. Therefore, the ROC is at least the intersection of the individual ROCs. For sequences with rational z-transforms, if the poles of $aX_1(z) + bX_2(z)$ consist of all the poles of $X_1(z)$ and $X_2(z)$ (i.e., if there is no pole–zero cancellation), then the ROC will be exactly equal to the overlap of the individual ROCs. If the linear combination is such that some zeros are introduced that cancel poles, then the ROC may be larger. A simple example of this occurs when $x_1[n]$ and $x_2[n]$ are of infinite duration, but the linear combination is of finite duration. In this case the ROC of the linear combination is the entire z-plane, with the possible exception of $z = 0$ or $z = \infty$. An example was given in Example 3.6, where $x[n]$ can be expressed as

$$x[n] = a^n\,(u[n] - u[n-N]) = a^n u[n] - a^n u[n-N].$$

Both $a^n u[n]$ and $a^n u[n-N]$ are infinite-extent right-sided sequences, and their z-transforms have a pole at $z = a$. Therefore, their individual ROCs would both be $|z| > |a|$. However, as shown in Example 3.6, the pole at $z = a$ is canceled by a zero at $z = a$, and therefore, the ROC extends to the entire z-plane, with the exception of $z = 0$.

We have already exploited the linearity property in our previous discussion of the use of the partial fraction expansion for evaluating the inverse z-transform. With that procedure, $X(z)$ is expanded into a sum of simpler terms, and through linearity, the inverse z-transform is the sum of the inverse transforms of each of these terms.

3.4.2 Time Shifting

The time-shifting property is,

$$x[n-n_0] \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z), \qquad \text{ROC} = R_x \text{ (except for the possible addition or deletion of } z=0 \text{ or } z=\infty).$$

The quantity n_0 is an integer. If n_0 is positive, the original sequence $x[n]$ is shifted right, and if n_0 is negative, $x[n]$ is shifted left. As in the case of linearity, the ROC can be changed, since the factor z^{-n_0} can alter the number of poles at $z = 0$ or $z = \infty$.

The derivation of this property follows directly from the z-transform expression in Eq. (3.2). Specifically, if $y[n] = x[n-n_0]$, the corresponding z-transform is

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}.$$

With the substitution of variables $m = n - n_0$,

$$\begin{aligned} Y(z) &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} \\ &= z^{-n_0}\sum_{m=-\infty}^{\infty} x[m]z^{-m}, \end{aligned}$$

or

$$Y(z) = z^{-n_0}X(z).$$

The time-shifting property is often useful, in conjunction with other properties and procedures, for obtaining the inverse z-transform. We illustrate with an example.

Example 3.14 Shifted Exponential Sequence

Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \frac{1}{4}.$$

From the ROC, we identify this as corresponding to a right-sided sequence. We can first rewrite $X(z)$ in the form

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}. \tag{3.56}$$

This z-transform is of the form of Eq. (3.41) with $M = N = 1$, and its expansion in the form of Eq. (3.45) is

$$X(z) = -4 + \frac{4}{1 - \frac{1}{4}z^{-1}}. \tag{3.57}$$

From Eq. (3.57), it follows that $x[n]$ can be expressed as

$$x[n] = -4\delta[n] + 4\left(\frac{1}{4}\right)^n u[n]. \tag{3.58}$$

An expression for $x[n]$ can be obtained more directly by applying the time-shifting property. First, $X(z)$ can be written as

$$X(z) = z^{-1}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right), \qquad |z| > \frac{1}{4}. \tag{3.59}$$

From the time-shifting property, we recognize the factor z^{-1} in Eq. (3.59) as being associated with a time shift of one sample to the right of the sequence $\left(\frac{1}{4}\right)^n u[n]$; i.e.,

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]. \tag{3.60}$$

It is easily verified that Eqs. (3.58) and (3.60) are the same for all values of n; i.e., they represent the same sequence.

3.4.3 Multiplication by an Exponential Sequence

The exponential multiplication property is

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \qquad \text{ROC} = |z_0| R_x.$$

The notation $\text{ROC} = |z_0|R_x$ signifies that the ROC is R_x scaled by the number $|z_0|$; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then $|z_0|R_x$ is the set of values of z such that $|z_0|r_R < |z| < |z_0|r_L$.

This property is easily shown simply by substituting $z_0^n x[n]$ into Eq. (3.2). As a consequence of the exponential multiplication property, all the pole–zero locations are scaled by a factor z_0, since, if $X(z)$ has a pole (or zero) at $z = z_1$, then $X(z/z_0)$ will have a pole (or zero) at $z = z_0 z_1$. If z_0 is a positive real number, the scaling can be interpreted as a shrinking or expanding of the z-plane; i.e., the pole and zero locations change along radial lines in the z-plane. If z_0 is complex with unity magnitude, so that $z_0 = e^{j\omega_0}$, the scaling corresponds to a rotation in the z-plane by an angle of ω_0; i.e., the pole and zero locations change in position along circles centered at the origin. This in turn can be interpreted as a frequency shift or translation of the discrete-time Fourier transform, which is associated with the modulation in the time domain by the complex exponential sequence $e^{j\omega_0 n}$. That is, if the Fourier transform exists, this property has the form

$$e^{j\omega_0 n}x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

Example 3.15 Exponential Multiplication

Starting with the transform pair

$$u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \qquad |z| > 1, \tag{3.61}$$

we can use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n)u[n], \qquad r > 0. \tag{3.62}$$

First, $x[n]$ is expressed as

$$x[n] = \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n].$$

Then, using Eq. (3.61) and the exponential multiplication property, we see that

$$\frac{1}{2}(re^{j\omega_0})^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{\frac{1}{2}}{1-re^{j\omega_0}z^{-1}}, \qquad |z| > r,$$

$$\frac{1}{2}(re^{-j\omega_0})^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{\frac{1}{2}}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r.$$

From the linearity property, it follows that

$$\begin{aligned} X(z) &= \frac{\frac{1}{2}}{1-re^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r \\ &= \frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}, \qquad |z| > r. \end{aligned} \tag{3.63}$$

3.4.4 Differentiation of $X(z)$

The differentiation property states that

$$nx[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

This property is verified by differentiating the z-transform expression of Eq. (3.2); i.e., for

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

we obtain

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$

We illustrate the use of the differentiation property with two examples.

Example 3.16 Inverse of Non-Rational z-Transform

In this example, we use the differentiation property together with the time-shifting property to determine the inverse z-transform considered in Example 3.12. With

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|,$$

we first differentiate to obtain a rational expression:

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}.$$

From the differentiation property,

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}}, \qquad |z| > |a|. \tag{3.64}$$

The inverse transform of Eq. (3.64) can be obtained by the combined use of the z-transform pair of Example 3.1, the linearity property, and the time-shifting property. Specifically, we can express $nx[n]$ as

$$nx[n] = a(-a)^{n-1}u[n-1].$$

Therefore,

$$x[n] = (-1)^{n+1}\frac{a^n}{n}u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \log(1 + az^{-1}), \qquad |z| > |a|.$$

The result of Example 3.16 will be useful in our discussion of the cepstrum in Chapter 13.

Example 3.17 2nd-Order Pole

As another example of the use of the differentiation property, let us determine the z-transform of the sequence

$$x[n] = na^n u[n] = n(a^n u[n]).$$

From the z-transform pair of Example 3.1 and the differentiation property, it follows that

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1 - az^{-1}}\right), \qquad |z| > |a|$$

$$= \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| > |a|.$$

Therefore,

$$na^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|.$$

3.4.5 Conjugation of a Complex Sequence

The conjugation property is expressed as

$$x^*[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x.$$

This property follows in a straightforward manner from the definition of the z-transform, the details of which are left as an exercise (Problem 3.54).

3.4.6 Time Reversal

The time-reversal property is given by

$$x^*[-n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}.$$

The notation ROC=$1/R_x$ implies that R_x is inverted; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then the ROC for $X^*(1/z^*)$ is the set of values of z such that $1/r_L < |z| < 1/r_R$. Thus, if z_0 is in the ROC for $x[n]$, then $1/z_0^*$ is in the ROC for the z-transform of $x^*[-n]$. If the sequence $x[n]$ is real or we do not conjugate a complex sequence, the result becomes

$$x[-n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(1/z), \qquad \text{ROC} = \frac{1}{R_x}.$$

As with the conjugation property, the time-reversal property follows easily from the definition of the z-transform, and the details are left as an exercise (Problem 3.54).

Note that if z_0 is a pole (or zero) of $X(z)$, then $1/z_0$ will be a pole (or zero) of $X(1/z)$. The magnitude of $1/z_0$ is simply the reciprocal of the magnitude of z_0. However, the angle of $1/z_0$ is the negative of the angle of z_0. When the poles and zeros of $X(z)$ are all real or in complex conjugate pairs, as they must be when $x[n]$ is real, this complex conjugate pairing is maintained.

Example 3.18 Time-Reversed Exponential Sequence

As an example of the use of the property of time reversal, consider the sequence

$$x[n] = a^{-n}u[-n],$$

which is a time-reversed version of $a^n u[n]$. From the time-reversal property, it follows that

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

Note that the z-transform of $a^n u[n]$ has a pole at $z = a$, while $X(z)$ has a pole at $1/a$.

3.4.7 Convolution of Sequences

According to the convolution property,

$$x_1[n] * x_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}.$$

To derive this property formally, we consider

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k],$$

so that

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n}. \end{aligned}$$

If we interchange the order of summation (which is allowed for z in the ROC),

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}.$$

Changing the index of summation in the second sum from n to $m = n - k$, we obtain

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\} z^{-k} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] \underbrace{X_2(z)}_{|z| \in R_{x_2}} z^{-k} = \left(\sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \right) X_2(z) \end{aligned}$$

Thus, for values of z inside the ROCs of both $X_1(z)$ and $X_2(z)$, we can write

$$Y(z) = X_1(z)X_2(z),$$

where the ROC includes the intersection of the ROCs of $X_1(z)$ and $X_2(z)$. If a pole that borders on the ROC of one of the z-transforms is canceled by a zero of the other, then the ROC of $Y(z)$ may be larger.

The use of the z-transform for evaluating convolutions is illustrated by the following example.

Example 3.19 Convolution of Finite-Length Sequences

Suppose that

$$x_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

is a finite-length sequence to be convolved with the sequence $x_2[n] = \delta[n] - \delta[n-1]$. The corresponding z-transforms are

$$X_1(z) = 1 + 2z^{-1} + z^{-2}$$

and $X_2(z) = 1 - z^{-1}$. The convolution $y[n] = x_1[n] * x_2[n]$ has z-transform

$$\begin{aligned} Y(z) &= X_1(z)X_2(z) = (1 + 2z^{-1} + z^{-2})(1 - z^{-1}) \\ &= 1 + z^{-1} - z^{-2} - z^{-3}. \end{aligned}$$

Since the sequences are both of finite length, the ROCs are both $|z| > 0$ and therefore so is the ROC of $Y(z)$. From $Y(z)$, we conclude by inspection of the coefficients of the polynomial that

$$y[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3].$$

The important point of this example is that convolution of finite-length sequences is equivalent to polynomial multiplication. Conversely, the coefficients of the product of two polynomials are obtained by discrete convolution of the polynomial coefficients.

The convolution property plays a particularly important role in the analysis of LTI systems as we will discuss in more detail in Section 3.5 and Chapter 5. An example of the use of the z-transform for computing the convolution of two infinite-duration sequences is given in Section 3.5.

3.4.8 Summary of Some z-Transform Properties

We have presented and discussed a number of the theorems and properties of z-transforms, many of which are useful in manipulating z-transforms in the analysis of discrete-time systems. These properties and a number of others are summarized for convenient reference in Table 3.2.

3.5 z-TRANSFORMS AND LTI SYSTEMS

The properties discussed in Section 3.4 make the z-transform a very useful tool for discrete-time system analysis. Since we shall rely on the z-transform extensively in Chapter 5 and later chapters, it is worthwhile now to illustrate how the z-transform can be used in the representation and analysis of LTI systems.

Recall from Section 2.3 that an LTI system can be represented as the convolution $y[n] = x[n] * h[n]$ of the input $x[n]$ with $h[n]$, where $h[n]$ is the response of the system to the unit impulse sequence $\delta[n]$. From the convolution property of Section 3.4.7, it follows that the z-transform of $y[n]$ is

$$Y(z) = H(z)X(z) \tag{3.65}$$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1}\cap R_{x_2}$

where $H(z)$ and $X(z)$ are the z-transforms of $h[n]$ and $x[n]$ respectively. In this context, the z-transform $H(z)$ is called the *system function* of the LTI system whose impulse response is $h[n]$.

The computation of the output of an LTI system using the z-transform is illustrated by the following example.

Example 3.20 Convolution of Infinite-Length Sequences

Let $h[n]=a^n u[n]$ and $x[n]=Au[n]$. To use the z-transform to evaluate the convolution $y[n]=x[n]*h[n]$, we begin by finding the corresponding z-transforms as

$$H(z)=\sum_{n=0}^{\infty}a^n z^{-n}=\frac{1}{1-az^{-1}},\qquad |z|>|a|,$$

and

$$X(z)=\sum_{n=0}^{\infty}Az^{-n}=\frac{A}{1-z^{-1}},\qquad |z|>1.$$

The z-transform of the convolution $y[n]=x[n]*h[n]$ is therefore

$$Y(z)=\frac{A}{(1-az^{-1})(1-z^{-1})}=\frac{Az^2}{(z-a)(z-1)},\qquad |z|>1,$$

where we assume that $|a| < 1$ so that the overlap of the ROCs is $|z| > 1$.

The poles and zeros of $Y(z)$ are plotted in Figure 3.12, and the ROC is seen to be the overlap region. The sequence $y[n]$ can be obtained by determining the inverse z-transform. The partial fraction expansion of $Y(z)$ is

$$Y(z) = \frac{A}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right) \qquad |z| > 1.$$

Therefore, taking the inverse z-transform of each term yields

$$y[n] = \frac{A}{1-a}(1-a^{n+1})u[n].$$

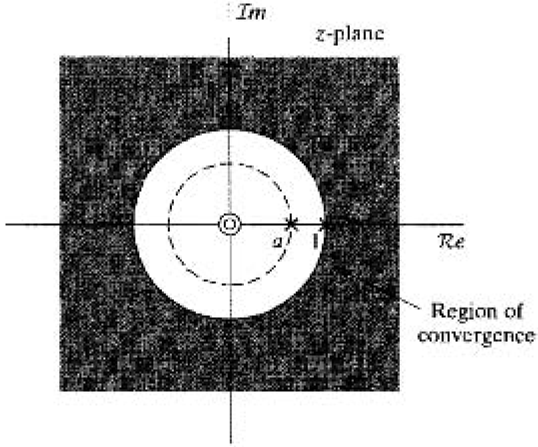


Figure 3.12 Pole–zero plot for the z-transform of the convolution of the sequences $u[n]$ and $a^n u[n]$ (assuming $|a| < 1$).

The z-transform is particularly useful in the analysis of LTI systems described by difference equations. Recall that in Section 2.5, we showed that difference equations of the form

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k], \tag{3.66}$$

behave as causal LTI systems when the input is zero prior to $n = 0$ and initial rest conditions are imposed prior to the time when the input becomes nonzero; i.e.,

$$y[-N], y[-N+1], \ldots, y[-1]$$

are all assumed to be zero. The difference equation with assumed initial rest conditions defines the LTI system, but it is also of interest to know the system function. If we apply the linearity property (Section 3.4.1) and the time-shift property (Section 3.4.2) to Eq. (3.66), we obtain

$$Y(z) = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z) + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}X(z). \tag{3.67}$$

Solving for $Y(z)$ in terms of $X(z)$ and the parameters of the difference equation yields

$$Y(z) = \left(\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \right) X(z), \tag{3.68}$$

and from a comparison of Eqs. (3.65) and (3.68) it follows that for the LTI system described by Eq. (3.66), the system function is

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{3.69}$$

Since the system defined by the difference equation of Eq. (3.66) is a causal system, our discussion in Section 3.2 leads to the conclusion that $H(z)$ in Eq. (3.69) must have an ROC of the form $|z| > r_R$, and since the ROC can contain no poles, r_R must be equal to the magnitude of pole of $H(z)$ that is farthest from the origin. Furthermore, the discussion in Section 3.2 also confirms that if $r_R < 1$, i.e., all poles are inside the unit circle, then the system is stable and the frequency response of the system is obtained by setting $z = e^{j\omega}$ in Eq. (3.69).

Note that if Eq. (3.66) is expressed in the equivalent form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{3.70}$$

then Eq. (3.69), which gives the system function (and frequency response for stable systems) as a ratio of polynomials in the variable z^{-1}, can be written down directly by observing that the numerator is the z-transform representation of the coefficient and delay terms involving the input, whereas the denominator represents the coefficients and delays of the terms involving the output. Similarly, given the system function as a ratio of polynomials in z^{-1} as in Eq. (3.69), it is straightforward to write down the difference equation in the form of Eq. (3.70) and then write it in the form of Eq. (3.66) for recursive implementation.

Example 3.21 1st-Order System

Suppose that a causal LTI system is described by the difference equation

$$y[n] = ay[n-1] + x[n]. \tag{3.71}$$

By inspection, it follows that the system function for this system is

$$H(z) = \frac{1}{1 - az^{-1}}, \tag{3.72}$$

with ROC $|z| > |a|$, from which it follows from entry 5 of Table 3.1 that the impulse response of the system is

$$h[n] = a^n u[n]. \tag{3.73}$$

Finally, if $x[n]$ is a sequence with a rational z-transform such as $x[n] = Au[n]$, we can find the output of the system in three distinct ways. (1) We can iterate the difference equation in Eq. (3.71). In general, this approach could be used with any input and would generally be used to implement the system, but it would not lead directly to a closed-form solution valid for all n even if such expression exists. (2) We could evaluate the convolution of $x[n]$ and $h[n]$ explicitly using the techniques illustrated in Section 2.3. (3) Since the z-transforms of both $x[n]$ and $h[n]$ are rational functions of z, we can use the partial fraction method of Section 3.3.2 to find a closed-form expression for the output valid for all n. In fact, this was done in Example 3.20.

We shall have much more use for the z-transform in Chapter 5 and subsequent chapters. For example, in Section 5.2.3, we shall obtain general expressions for the impulse response of an LTI system with rational system function, and we shall show how the frequency response of the system is related to the locations of the poles and zeros of $H(z)$.

3.6 THE UNILATERAL z-TRANSFORM

The z-transform, as defined by Eq. (3.2), and as considered so far in this chapter, is more explicitly referred to as the bilateral z-transform or the two-sided z-transform. In contrast, the *unilateral* or *one-sided z-transform* is defined as

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{3.74}$$

The unilateral z-transform differs from the bilateral z-transform in that the lower limit of the sum is always fixed at zero, regardless of the values of $x[n]$ for $n < 0$. If $x[n] = 0$ for $n < 0$, the unilateral and bilateral z-transforms are identical, whereas, if $x[n]$ is not zero for all $n < 0$, they will be different. A simple example illustrates this.

Example 3.22 Unilateral Transform of an Impulse

Suppose that $x_1[n] = \delta[n]$. Then it is clear from Eq. (3.74) that $\mathcal{X}_1(z) = 1$, which is identical to the bilateral z-transform of the impulse. However, consider $x_2[n] = \delta[n+1] = x_1[n+1]$. This time using Eq. (3.74) we find that $\mathcal{X}_2(z) = 0$, whereas the bilateral z-transform would be $X_2(z) = zX_1(z) = z$.

Because the unilateral transform in effect ignores any left-sided part, the properties of the ROC of the unilateral z-transform will be the same as those of the bilateral transform of a right-sided sequence obtained by assuming that the sequence values are zero for $n < 0$. That is, the ROC for all unilateral z-transforms will be of the form $|z| > r_R$, and for rational unilateral z-transforms, the boundary of the ROC will be defined by the pole that is farthest from the origin of the z-plane.

In digital signal processing applications, difference equations of the form of Eq. (3.66) are generally employed with initial rest conditions. However, in some situations, noninitial rest conditions may occur. In such cases, the linearity and time-shifting properties of the unilateral z-transform are particularly useful tools. The linearity property is identical to that of the bilateral z-transform (Property 1 in Table 3.2). The time-shifting property is different in the unilateral case because the lower limit in the unilateral transform definition is fixed at zero. To illustrate how to develop this property, consider a sequence $x[n]$ with unilateral z-transform $\mathcal{X}(z)$ and let $y[n] = x[n-1]$. Then, by definition

$$\mathcal{Y}(z) = \sum_{n=0}^{\infty} x[n-1]z^{-n}.$$

With the substitution of summation index $m = n - 1$, we can write $\mathcal{Y}(z)$ as

$$\mathcal{Y}(z) = \sum_{m=-1}^{\infty} x[m]z^{-(m+1)} = x[-1] + z^{-1} \sum_{m=0}^{\infty} x[m]z^{-m},$$

so that

$$\mathcal{Y}(z) = x[-1] + z^{-1}\mathcal{X}(z). \tag{3.75}$$

Thus, to determine the unilateral z-transform of a delayed sequence, we must provide sequence values that are ignored in computing $\mathcal{X}(z)$. By a similar analysis, it can be shown that if $y[n] = x[n-k]$, where $k > 0$, then

$$\begin{aligned}\mathcal{Y}(z) &= x[-k] + x[-k+1]z^{-1} + \ldots + x[-1]z^{-k+1} + z^{-k}\mathcal{X}(z) \\ &= \sum_{m=1}^{k} x[m-k-1]z^{-m+1} + z^{-k}\mathcal{X}(z). \end{aligned} \tag{3.76}$$

The use of the unilateral z-transform to solve for the output of a difference equation with nonzero initial conditions is illustrated by the following example.

Example 3.23 Effect of Nonzero Initial Conditions

Consider a system described by the linear constant-coefficient difference equation

$$y[n] - ay[n-1] = x[n], \tag{3.77}$$

which is the same as the system in Examples 3.20 and 3.21. Assume that $x[n] = 0$ for $n < 0$ and the initial condition at $n = -1$ is denoted $y[-1]$. Applying the unilateral z-transform to Eq. (3.77) and using the linearity property as well as the time-shift property in Eq. (3.75), we have

$$\mathcal{Y}(z) - ay[-1] - az^{-1}\mathcal{Y}(z) = \mathcal{X}(z).$$

Solving for $\mathcal{Y}(z)$ we obtain

$$\mathcal{Y}(z) = \frac{ay[-1]}{1 - az^{-1}} + \frac{1}{1 - az^{-1}}\mathcal{X}(z). \tag{3.78}$$

Note that if $y[-1] = 0$ the first term disappears, and we are left with $\mathcal{Y}(z) = H(z)\mathcal{X}(z)$, where

$$H(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

is the system function of the LTI system corresponding to the difference equation in Eq. (3.77) when iterated with initial rest conditions. This confirms that initial rest conditions are necessary for the iterated difference equation to behave as an LTI system. Furthermore, note that if $x[n] = 0$ for all n, the output will be equal to

$$y[n] = y[-1]a^{n+1} \qquad n \geq -1.$$

This shows that if $y[-1] \neq 0$, the system does not behave linearly because the scaling property for linear systems [Eq. (2.23b)] requires that when the input is zero for all n, the output must likewise be zero for all n.

To be more specific, suppose that $x[n] = Au[n]$ as in Example 3.20. We can determine an equation for $y[n]$ for $n \geq -1$ by noting that the unilateral z-transform of $x[n] = Au[n]$ is

$$\mathcal{X}(z) = \frac{A}{1 - z^{-1}}, \qquad |z| > 1$$

so that Eq. (3.78) becomes

$$\mathcal{Y}(z) = \frac{ay[-1]}{1 - az^{-1}} + \frac{A}{(1 - az^{-1})(1 - z^{-1})}. \tag{3.79}$$

Applying the partial fraction expansion technique to Eq. (3.79) gives

$$\mathcal{Y}(z) = \frac{ay[-1]}{1 - az^{-1}} + \frac{\frac{A}{1-a}}{1 - z^{-1}} + \frac{\frac{aA}{1-a}}{1 - az^{-1}},$$

from which it follows that the complete solution is

$$y[n] = \begin{cases} y[-1] & n = -1 \\ \underbrace{y[-1]a^{n+1}}_{\text{ZIR}} + \underbrace{\frac{A}{1-a}\left(1 - a^{n+1}\right)}_{\text{ZICR}} & n \geq 0 \end{cases} \tag{3.80}$$

Equation (3.80) shows that the system response is composed of two parts. The zero input response (ZIR) is the response when the input is zero (in this case when $A = 0$). The zero initial conditions response (ZICR) is the part that is directly proportional to the input (as required for linearity). This part remains when $y[-1] = 0$. In Problem 3.49, this decomposition into ZIR and ZICR components is shown to hold for any difference equation of the form of Eq. (3.66).

3.7 SUMMARY

In this chapter, we have defined the z-transform of a sequence and shown that it is a generalization of the Fourier transform. The discussion focused on the properties of the z-transform and techniques for obtaining the z-transform of a sequence and vice versa. Specifically, we showed that the defining power series of the z-transform may converge when the Fourier transform does not. We explored in detail the dependence of the shape of the ROC on the properties of the sequence. A full understanding of the properties of the ROC is essential for successful use of the z-transform. This is particularly true in developing techniques for finding the sequence that corresponds to a given z-transform, i.e., finding inverse z-transforms. Much of the discussion focused on z-transforms that are rational functions in their region of convergence. For such functions, we described a technique of inverse transformation based on the partial fraction expansion of $X(z)$. We also discussed other techniques for inverse transformation, such as the use of tabulated power series expansions and long division.

An important part of the chapter was a discussion of some of the many properties of the z-transform that make it useful in analyzing discrete-time signals and systems. A variety of examples demonstrated how these properties can be used to find direct and inverse z-transforms.

Problems

Basic Problems with Answers

3.1. Determine the z-transform, including the ROC, for each of the following sequences:

(a) $\left(\frac{1}{2}\right)^n u[n]$

(b) $-\left(\frac{1}{2}\right)^n u[-n-1]$

(c) $\left(\frac{1}{2}\right)^n u[-n]$

(d) $\delta[n]$

(e) $\delta[n-1]$

(f) $\delta[n+1]$

(g) $\left(\frac{1}{2}\right)^n (u[n]-u[n-10])$.

3.2. Determine the z-transform of the sequence

$$x[n] = \begin{cases} n, & 0 \le n \le N-1, \\ N, & N \le n. \end{cases}$$

3.3. Determine the z-transform of each of the following sequences. Include with your answer the ROC in the z-plane and a sketch of the pole–zero plot. Express all sums in closed form; α can be complex.

(a) $x_a[n] = \alpha^{|n|}, \quad 0 < |\alpha| < 1$.

(b) $x_b[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$

(c) $x_c[n] = \begin{cases} n+1, & 0 \le n \le N-1, \\ 2N-1-n, & N \le n \le 2(N-1), \\ 0, & \text{otherwise.} \end{cases}$

Hint: Note that $x_b[n]$ is a rectangular sequence and $x_c[n]$ is a triangular sequence. First, express $x_c[n]$ in terms of $x_b[n]$.

3.4. Consider the z-transform $X(z)$ whose pole–zero plot is as shown in Figure P3.4.

(a) Determine the ROC of $X(z)$ if it is known that the Fourier transform exists. For this case, determine whether the corresponding sequence $x[n]$ is right sided, left sided, or two sided.

(b) How many possible two-sided sequences have the pole–zero plot shown in Figure P3.4?

(c) Is it possible for the pole–zero plot in Figure P3.4 to be associated with a sequence that is both stable and causal? If so, give the appropriate ROC.

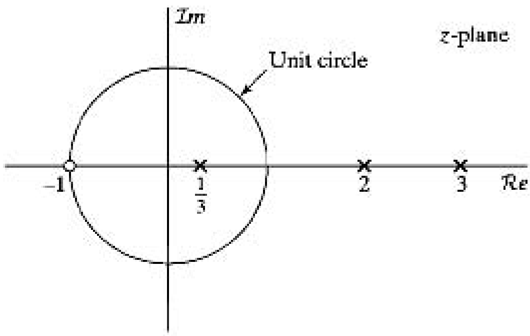


Figure P3.4

3.5. Determine the sequence $x[n]$ with z-transform

$$X(z) = (1 + 2z)(1 + 3z^{-1})(1 - z^{-1}).$$

3.6. Following are several z-transforms. For each, determine the inverse z-transform using both methods—partial fraction expansion and power series expansion—discussed in Section 3.3. In addition, indicate in each case whether the Fourier transform exists.

(a) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| > \dfrac{1}{2}$

(b) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| < \dfrac{1}{2}$

(c) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \qquad |z| > \dfrac{1}{2}$

(d) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \qquad |z| > \dfrac{1}{2}$

(e) $X(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}, \qquad |z| > |1/a|$

3.7. The input to a causal LTI system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)}.$$

(a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the ROC.

(b) What is the ROC for $Y(z)$?

(c) Determine $y[n]$.

3.8. The system function of a causal LTI system is

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}.$$

The input to this system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1].$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output $y[n]$.
(c) Is the system stable? That is, is $h[n]$ absolutely summable?

3.9. A causal LTI system has impulse response $h[n]$, for which the z-transform is

$$H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}.$$

(a) What is the ROC of $H(z)$?
(b) Is the system stable? Explain.
(c) Find the z-transform $X(z)$ of an input $x[n]$ that will produce the output

$$y[n] = -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1].$$

(d) Find the impulse response $h[n]$ of the system.

3.10. Without explicitly solving for $X(z)$, find the ROC of the z-transform of each of the following sequences, and determine whether the Fourier transform converges:

(a) $x[n] = \left[\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right]u[n-10]$
(b) $x[n] = \begin{cases} 1, & -10 \le n \le 10, \\ 0, & \text{otherwise}, \end{cases}$
(c) $x[n] = 2^n u[-n]$
(d) $x[n] = \left[\left(\frac{1}{4}\right)^{n+4} - (e^{j\pi/3})^n\right]u[n-1]$
(e) $x[n] = u[n+10] - u[n+5]$
(f) $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n] + (2+3j)^{n-2}u[-n-1]$.

3.11. Following are four z-transforms. Determine which ones *could* be the z-transform of a *causal* sequence. Do not evaluate the inverse transform. You should be able to give the answer by inspection. Clearly state your reasons in each case.

(a) $\dfrac{(1-z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)}$

(b) $\dfrac{(z-1)^2}{\left(z-\frac{1}{2}\right)}$

(c) $\dfrac{\left(z-\frac{1}{4}\right)^5}{\left(z-\frac{1}{2}\right)^6}$

(d) $\dfrac{\left(z-\frac{1}{4}\right)^6}{\left(z-\frac{1}{2}\right)^5}$

3.12. Sketch the pole–zero plot for each of the following z-transforms and shade the ROC:

(a) $X_1(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + 2z^{-1}}$, ROC: $|z| < 2$

(b) $X_2(z) = \dfrac{1 - \frac{1}{3}z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{2}{3}z^{-1}\right)}$, $x_2[n]$ causal

(c) $X_3(z) = \dfrac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6}z^{-1} + z^{-2}}$, $x_3[n]$ absolutely summable.

3.13. A causal sequence $g[n]$ has the z-transform

$$G(z) = \sin(z^{-1})(1 + 3z^{-2} + 2z^{-4}).$$

Find $g[11]$.

3.14. If $H(z) = \dfrac{1}{1 - \frac{1}{4}z^{-2}}$ and $h[n] = A_1\alpha_1^n u[n] + A_2\alpha_2^n u[n]$, determine the values of A_1, A_2, α_1, and α_2.

3.15. If $H(z) = \dfrac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}}$ for $|z| > 0$, is the corresponding LTI system causal? Justify your answer.

3.16. When the input to an LTI system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1],$$

the corresponding output is

$$y[n] = 5\left(\frac{1}{3}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the pole(s) and zero(s) of $H(z)$ and indicate the ROC.
(b) Find the impulse response $h[n]$ of the system.
(c) Write a difference equation that is satisfied by the given input and output.
(d) Is the system stable? Is it causal?

3.17. Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1].$$

Determine all possible values for the system's impulse response $h[n]$ at $n = 0$.

3.18. A causal LTI system has the system function

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)(1 - z^{-1})}.$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output of this system, $y[n]$, for the input

$$x[n] = 2^n.$$

3.19. For each of the following pairs of input z-transform $X(z)$ and system function $H(z)$, determine the ROC for the output z-transform $Y(z)$:

(a)

$$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

(b)

$$X(z) = \frac{1}{1 - 2z^{-1}}, \qquad |z| < 2$$

$$H(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

(c)

$$X(z) = \frac{1}{\left(1 - \frac{1}{5}z^{-1}\right)\left(1 + 3z^{-1}\right)}, \qquad \frac{1}{5} < |z| < 3$$

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

3.20. For each of the following pairs of input and output z-transforms $X(z)$ and $Y(z)$, determine the ROC for the system function $H(z)$:

(a)

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$Y(z) = \frac{1}{1 + \frac{2}{3}z^{-1}}, \qquad |z| > \frac{2}{3}$$

(b)

$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}}, \qquad |z| < \frac{1}{3}$$

$$Y(z) = \frac{1}{\left(1 - \frac{1}{6}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}, \qquad \frac{1}{6} < |z| < \frac{1}{3}$$

Basic Problems

3.21. A causal LTI system has the following system function:

$$H(z) = \frac{4 + 0.25z^{-1} - 0.5z^{-2}}{(1 - 0.25z^{-1})(1 + 0.5z^{-1})}$$

(a) What is the ROC for $H(z)$?

(b) Determine if the system is stable or not.
(c) Determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$.
(d) Use a partial fraction expansion to determine the impulse response $h[n]$.
(e) Find $Y(z)$, the z-transform of the output, when the input is $x[n] = u[-n-1]$. Be sure to specify the ROC for $Y(z)$.
(f) Find the output sequence $y[n]$ when the input is $x[n] = u[-n-1]$.

3.22. A causal LTI system has system function

$$H(z) = \frac{1-4z^{-2}}{1+0.5z^{-1}}.$$

The input to this system is

$$x[n] = u[n] + 2\cos\left(\frac{\pi}{2}n\right) \qquad -\infty < n < \infty.$$

Determine the output $y[n]$ for large positive n; i.e., find an expression for $y[n]$ that is asymptotically correct as n gets large. (*Of course, one approach is to find an expression for $y[n]$ that is valid for all n, but you should see an easier way.*)

3.23. Consider an LTI system with impulse response

$$h[n] = \begin{cases} a^n, & n \geq 0, \\ 0, & n < 0, \end{cases}$$

and input

$$x[n] = \begin{cases} 1, & 0 \leq n \leq (N-1), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the output $y[n]$ by explicitly evaluating the discrete convolution of $x[n]$ and $h[n]$.
(b) Determine the output $y[n]$ by computing the inverse z-transform of the product of the z-transforms of $x[n]$ and $h[n]$.

3.24. Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3}{1+\frac{1}{3}z^{-1}}.$$

Suppose $x[n]$, the input to the system, is a unit step sequence.

(a) Determine the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.
(b) Determine the output $y[n]$ by computing the inverse z-transform of $Y(z)$.

3.25. Sketch each of the following sequences and determine their z-transforms, including the ROC:

(a) $\displaystyle\sum_{k=-\infty}^{\infty} \delta[n-4k]$

(b) $\frac{1}{2}\left[e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right]u[n]$

3.26. Consider a right-sided sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})} = \frac{z^2}{(z-a)(z-b)}.$$

In Section 3.3, we considered the determination of $x[n]$ by carrying out a partial fraction expansion, with $X(z)$ considered as a ratio of polynomials in z^{-1}. Carry out a partial fraction expansion of $X(z)$, considered as a ratio of polynomials in z, and determine $x[n]$ from this expansion.

3.27. Determine the unilateral z-transform, including the ROC, for each of the following sequences:

(a) $\delta[n]$
(b) $\delta[n-1]$
(c) $\delta[n+1]$
(d) $\left(\frac{1}{2}\right)^n u[n]$
(e) $-\left(\frac{1}{2}\right)^n u[-n-1]$
(f) $\left(\frac{1}{2}\right)^n u[-n]$
(g) $\{\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\}u[n]$
(h) $\left(\frac{1}{2}\right)^{n-1} u[n-1]$

3.28. If $\mathcal{X}(z)$ denotes the unilateral z-transform of $x[n]$, determine, in terms of $\mathcal{X}(z)$, the unilateral z-transform of the following:

(a) $x[n-2]$
(b) $x[n+1]$
(c) $\sum_{m=-\infty}^{n} x[m]$

3.29. For each of the following difference equations and associated input and initial conditions, determine the response $y[n]$ for $n \geq 0$ by using the unilateral z-transform.

(a) $y[n] + 3y[n-1] = x[n]$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$
$$y[-1] = 1$$

(b) $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1]$

$$x[n] = u[n]$$
$$y[-1] = 0$$

(c) $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1]$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$
$$y[-1] = 1$$

Advanced Problems

3.30. A causal LTI system has system function

$$H(z) = \frac{1 - z^{-1}}{1 - 0.25z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}.$$

(a) Determine the output of the system when the input is $x[n] = u[n]$.
(b) Determine the input $x[n]$ so that the corresponding output of the above system is $y[n] = \delta[n] - \delta[n-1]$.
(c) Determine the output $y[n]$ when the input is $x[n] = \cos(0.5\pi n)$ for $-\infty < n < \infty$. You may leave your answer in any convenient form.

3.31. Determine the inverse z-transform of each of the following. In parts (a)–(c), use the methods specified. (In part (d), use any method you prefer.)

(a) Long division:

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}, \qquad x[n] \text{ a right-sided sequence}$$

(b) Partial fraction:

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}, \qquad x[n] \text{ stable}$$

(c) Power series:

$$X(z) = \ln(1 - 4z), \qquad |z| < \frac{1}{4}$$

(d) $X(z) = \dfrac{1}{1 - \frac{1}{3}z^{-3}}, \qquad |z| > (3)^{-1/3}$

3.32. Using any method, determine the inverse z-transform for each of the following:

(a) $X(z) = \dfrac{1}{\left(1 + \frac{1}{2}z^{-1}\right)^2 (1 - 2z^{-1})(1 - 3z^{-1})}$,
($x[n]$ is a stable sequence)
(b) $X(z) = e^{z^{-1}}$
(c) $X(z) = \dfrac{z^3 - 2z}{z - 2}$, ($x[n]$ is a left-sided sequence)

3.33. Determine the inverse z-transform of each of the following. You should find the z-transform properties in Section 3.4 helpful.

(a) $X(z) = \dfrac{3z^{-3}}{\left(1 - \frac{1}{4}z^{-1}\right)^2}$, $x[n]$ left sided
(b) $X(z) = \sin(z)$, ROC includes $|z| = 1$
(c) $X(z) = \dfrac{z^7 - 2}{1 - z^{-7}}$, $|z| > 1$

3.34. Determine a sequence $x[n]$ whose z-transform is $X(z) = e^z + e^{1/z}$, $z \neq 0$.

3.35. Determine the inverse z-transform of

$$X(z) = \log(1 - 2z), \qquad |z| < \frac{1}{2},$$

by

(a) using the power series

$$\log(1 - x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}, \qquad |x| < 1;$$

(b) first differentiating $X(z)$ and then using the derivative to recover $x[n]$.

3.36. For each of the following sequences, determine the z-transform and ROC, and sketch the pole–zero diagram:

(a) $x[n] = a^n u[n] + b^n u[n] + c^n u[-n-1], \qquad |a| < |b| < |c|$

(b) $x[n] = n^2 a^n u[n]$

(c) $x[n] = e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n] - e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n-1]$.

3.37. The pole–zero diagram in Figure P3.37 corresponds to the z-transform $X(z)$ of a causal sequence $x[n]$. Sketch the pole–zero diagram of $Y(z)$, where $y[n] = x[-n+3]$. Also, specify the ROC for $Y(z)$.

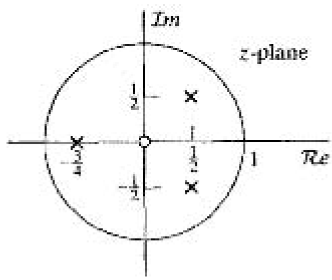


Figure P3.37

3.38. Let $x[n]$ be the sequence with the pole–zero plot shown in Figure P3.38. Sketch the pole–zero plot for:

(a) $y[n] = \left(\frac{1}{2}\right)^n x[n]$

(b) $w[n] = \cos\left(\frac{\pi n}{2}\right)x[n]$

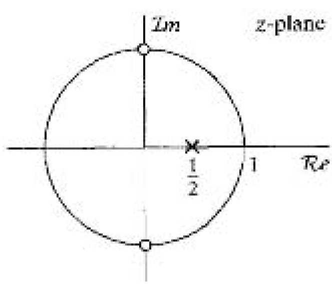


Figure P3.38

3.39. Determine the unit step response of the causal system for which the z-transform of the impulse response is

$$H(z) = \frac{1 - z^3}{1 - z^4}.$$

3.40. If the input $x[n]$ to an LTI system is $x[n] = u[n]$, the output is

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1].$$

(a) Find $H(z)$, the z-transform of the system impulse response, and plot its pole–zero diagram.
(b) Find the impulse response $h[n]$.
(c) Is the system stable?
(d) Is the system causal?

3.41. Consider a sequence $x[n]$ for which the z-transform is

$$X(z) = \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{4}}{1 - 2z^{-1}}$$

and for which the ROC includes the unit circle. Determine $x[0]$ using the initial-value theorem (see Problem 3.57).

3.42. In Figure P3.42, $H(z)$ is the system function of a causal LTI system.

(a) Using z-transforms of the signals shown in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.
(b) For the special case $H(z) = z^{-1}/(1 - z^{-1})$, determine $H_1(z)$ and $H_2(z)$.
(c) Is the system $H(z)$ stable? Are the systems $H_1(z)$ and $H_2(z)$ stable?

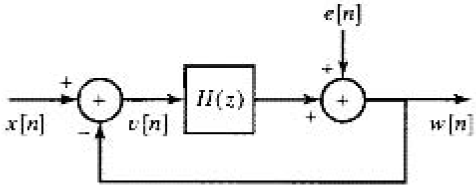


Figure P3.42

3.43. In Figure P3.43, $h[n]$ is the impulse response of the LTI system within the inner box. The input to system $h[n]$ is $v[n]$, and the output is $w[n]$. The z-transform of $h[n]$, $H(z)$, exists in the following ROC:

$$0 < r_{min} < |z| < r_{max} < \infty.$$

(a) Can the LTI system with impulse response $h[n]$ be bounded input, bounded output stable? If so, determine inequality constraints on r_{min} and r_{max} such that it is stable. If not, briefly explain why.
(b) Is the overall system (in the large box, with input $x[n]$ and output $y[n]$) LTI? If so, find its impulse response $g[n]$. If not, briefly explain why.

(c) Can the overall system be BIBO stable? If so, determine inequality constraints relating α, $r_{\min}$, and $r_{\max}$ such that it is stable. If not, briefly explain why.

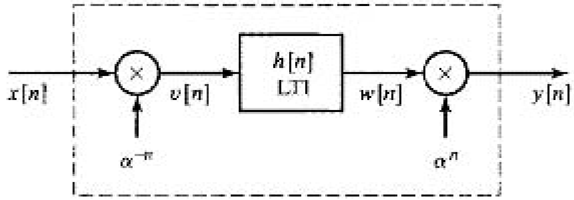


Figure P3.43

3.44. A causal and stable LTI system S has its input $x[n]$ and output $y[n]$ related by the linear constant-coefficient difference equation

$$y[n] + \sum_{k=1}^{10} \alpha_k y[n-k] = x[n] + \beta x[n-1].$$

Let the impulse response of S be the sequence $h[n]$.

(a) Show that $h[0]$ must be nonzero.

(b) Show that α_1 can be determined from knowledge of β, $h[0]$, and $h[1]$.

(c) If $h[n] = (0.9)^n \cos(\pi n/4)$ for $0 \le n \le 10$, sketch the pole–zero plot for the system function of S, and indicate the ROC.

3.45. When the input to an LTI system is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1],$$

the output is

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the ROC.

(b) Find the impulse response $h[n]$ of the system.

(c) Write the difference equation that characterizes the system.

(d) Is the system stable? Is it causal?

3.46. The following information is known about an LTI system:

(i) The system is causal.

(ii) When the input is

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1],$$

then the z-transform of the output is

$$Y(z) = \frac{1 - z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

(a) Find the z-transform of $x[n]$.
(b) What are the possible choices for the ROC of $Y(z)$?
(c) What are the possible choices for a linear constant-coefficient difference equation used to describe the system?
(d) What are the possible choices for the impulse response of the system?

3.47. Let $x[n]$ be a discrete-time signal with $x[n] = 0$ for $n \leq 0$ and z-transform $X(z)$. Furthermore, given $x[n]$, let the discrete-time signal $y[n]$ be defined by

$$y[n] = \begin{cases} \frac{1}{n}x[n], & n > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $Y(z)$ in terms of $X(z)$.
(b) Using the result of part (a), find the z-transform of

$$w[n] = \frac{1}{n + \delta[n]}u[n-1].$$

3.48. The signal $y[n]$ is the output of an LTI system with impulse response $h[n]$ for a given input $x[n]$. Throughout the problem, assume that $y[n]$ is stable and has a z-transform $Y(z)$ with the pole–zero diagram shown in Figure P3.48-1. The signal $x[n]$ is stable and has the pole–zero diagram shown in Figure P3.48-2.

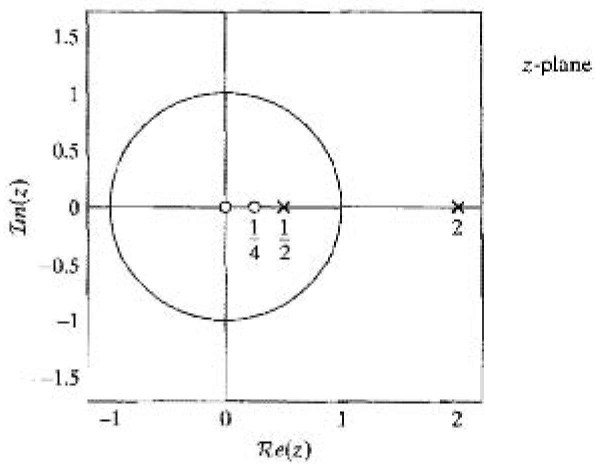


Figure P3.48-1

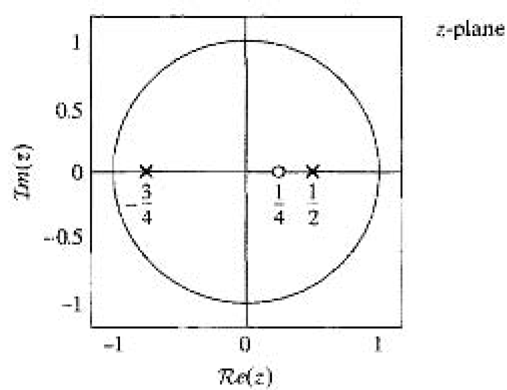


Figure P3.48-2

(a) What is the ROC, $Y(z)$?
(b) Is $y[n]$ left sided, right sided, or two sided?
(c) What is the ROC of $X(z)$?
(d) Is $x[n]$ a causal sequence? That is, does $x[n] = 0$ for $n < 0$?
(e) What is $x[0]$?
(f) Draw the pole-zero plot of $H(z)$, and specify its ROC.
(g) Is $h[n]$ anticausal? That is, does $h[n] = 0$ for $n > 0$?

3.49. Consider the difference equation of Eq. (3.66).

(a) Show that with nonzero initial conditions the unilateral z-transform of the output of the difference equation is

$$\mathcal{Y}(z) = -\frac{\sum_{k=1}^{N} a_k \left(\sum_{m=1}^{k} y[m-k-1] z^{-m+1} \right)}{\sum_{k=0}^{N} a_k z^{-k}} + \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \mathcal{X}(z).$$

(b) Use the result of (a) to show that the output has the form

$$y[n] = y_{ZIR}[n] + y_{ZICR}[n]$$

where $y_{ZIR}[n]$ is the output when the input is zero for all n and $y_{ZICR}[n]$ is the output when the initial conditions are all zero.

(c) Show that when the initial conditions are all zero, the result reduces to the result that is obtained with the bilateral z-transform.

Extension Problems

3.50. Let $x[n]$ denote a causal sequence; i.e., $x[n] = 0$, $n < 0$. Furthermore, assume that $x[0] \neq 0$ and that the z-transform is a rational function.

(a) Show that there are no poles or zeros of $X(z)$ at $z = \infty$, i.e., that $\lim_{z\to\infty} X(z)$ is nonzero and finite.

(b) Show that the number of poles in the finite z-plane equals the number of zeros in the finite z-plane. (The finite z-plane excludes $z = \infty$.)

3.51. Consider a sequence with z-transform $X(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials in z. If the sequence is absolutely summable and if all the roots of $Q(z)$ are inside the unit circle, is the sequence necessarily causal? If your answer is yes, clearly explain. If your answer is no, give a counterexample.

3.52. Let $x[n]$ be a causal stable sequence with z-transform $X(z)$. The *complex cepstrum* $\hat{x}[n]$ is defined as the inverse transform of the logarithm of $X(z)$; i.e.,

$$\hat{X}(z) = \log X(z) \overset{\mathcal{Z}}{\longleftrightarrow} \hat{x}[n],$$

where the ROC of $\hat{X}(z)$ includes the unit circle. (Strictly speaking, taking the logarithm of a complex number requires some careful considerations. Furthermore, the logarithm of a valid z-transform may not be a valid z-transform. For now, we assume that this operation is valid.)

Determine the complex cepstrum for the sequence

$$x[n] = \delta[n] + a\delta[n-N], \qquad \text{where } |a| < 1.$$

3.53. Assume that $x[n]$ is real and even; i.e., $x[n] = x[-n]$. Further, assume that z_0 is a zero of $X(z)$; i.e., $X(z_0) = 0$.

(a) Show that $1/z_0$ is also a zero of $X(z)$.

(b) Are there other zeros of $X(z)$ implied by the information given?

3.54. Using the definition of the z-transform in Eq. (3.2), show that if $X(z)$ is the z-transform of $x[n] = x_R[n] + jx_I[n]$, then

(a) $x^*[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$

(b) $x[-n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(1/z)$

(c) $x_R[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2}[X(z) + X^*(z^*)]$

(d) $x_I[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2j}[X(z) - X^*(z^*)]$.

3.55. Consider a *real* sequence $x[n]$ that has all the poles and zeros of its z-transform inside the unit circle. Determine, in terms of $x[n]$, a *real* sequence $x_1[n]$ not equal to $x[n]$, but for which $x_1[0] = x[0]$, $|x_1[n]| = |x[n]|$, and the z-transform of $x_1[n]$ has all its poles and zeros inside the unit circle.

3.56. A real finite-duration sequence whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle is uniquely specified to within a positive scale factor by its Fourier transform phase (Hayes et al., 1980).

An example of zeros at conjugate reciprocal pair locations is $z = a$ and $(a^*)^{-1}$. Even though we can generate sequences that do not satisfy the preceding set of conditions, almost any sequence of practical interest satisfies the conditions and therefore is uniquely specified to within a positive scale factor by the phase of its Fourier transform.

Consider a sequence $x[n]$ that is real, that is zero outside $0 \le n \le N-1$, and whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle. We wish to develop an algorithm that reconstructs $cx[n]$ from $\angle X(e^{j\omega})$, the Fourier transform phase of $x[n]$, where c is a positive scale factor.

(a) Specify a set of $(N-1)$ linear equations, the solution to which will provide the recovery of $x[n]$ to within a positive or negative scale factor from $\tan\{\angle X(e^{j\omega})\}$. You do not have to prove that the set of $(N-1)$ linear equations has a unique solution. Further, show that if we know $\angle X(e^{j\omega})$ rather than just $\tan\{\angle X(e^{j\omega})\}$, the sign of the scale factor can also be determined.

(b) Suppose

$$x[n] = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ 2, & n = 1, \\ 3, & n = 2, \\ 0, & n \ge 3. \end{cases}$$

Using the approach developed in part (a), demonstrate that $cx[n]$ can be determined from $\angle X(e^{j\omega})$, where c is a positive scale factor.

3.57. For a sequence $x[n]$ that is zero for $n < 0$, use Eq. (3.2) to show that

$$\lim_{z\to\infty} X(z) = x[0].$$

This result is called the *initial value theorem*. What is the corresponding theorem if the sequence is zero for $n > 0$?

3.58. The aperiodic autocorrelation function for a real-valued stable sequence $x[n]$ is defined as

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k].$$

(a) Show that the z-transform of $c_{xx}[n]$ is

$$C_{xx}(z) = X(z)X(z^{-1}).$$

Determine the ROC for $C_{xx}(z)$.

(b) Suppose that $x[n] = a^n u[n]$. Sketch the pole-zero plot for $C_{xx}(z)$, including the ROC. Also, find $c_{xx}[n]$ by evaluating the inverse z-transform of $C_{xx}(z)$.

(c) Specify another sequence, $x_1[n]$, that is not equal to $x[n]$ in part (b), but that has the same autocorrelation function, $c_{xx}[n]$, as $x[n]$ in part (b).

(d) Specify a third sequence, $x_2[n]$, that is not equal to $x[n]$ or $x_1[n]$, but that has the same autocorrelation function as $x[n]$ in part (b).

3.59. Determine whether or not the function $X(z) = z^*$ can correspond to the z-transform of a sequence. Clearly explain your reasoning.

3.60. Let $X(z)$ denote a ratio of polynomials in z; i.e.,

$$X(z) = \frac{B(z)}{A(z)}.$$

Show that if $X(z)$ has a 1^{st}-order pole at $z = z_0$, then the residue of $X(z)$ at $z = z_0$ is equal to

$$\frac{B(z_0)}{A'(z_0)},$$

where $A'(z_0)$ denotes the derivative of $A(z)$ evaluated at $z = z_0$.