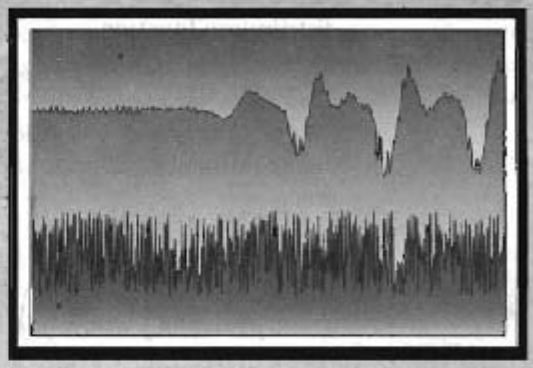
A Random Signals



In this appendix, we collect and summarize a number of results and establish the notation relating to the representation of random signals. We make no attempt here to provide a detailed discussion of the difficult and subtle mathematical issues of the underlying theory. Although our approach is not rigorous, we have summarized the important results and the mathematical assumptions implicit in their derivation. Detailed presentation of the theory of random signals are found in texts such as Davenport (1970), Papoulis (1984), Gray and Davidson (2004), Kay (2006), and Bertsekas and Tsitsiklis (2008).

A.1 DISCRETE-TIME RANDOM PROCESSES

The fundamental concept in the mathematical representation of random signals is that of a *random process*. In our discussion of random processes as models for discrete-time signals, we assume that the reader is familiar with the basic concepts of probability, such as random variables, probability distributions, and averages.

In using the random-process model in practical signal-processing applications, we consider a particular sequence to be one of an ensemble of sample sequences. Given a discrete-time signal, the structure, i.e., the underlying probability law, of the corresponding random process is generally not known and must somehow be inferred. It may be possible to make reasonable assumptions about the structure of the process, or it may be possible to estimate the properties of a random-process representation from a finite segment of a typical sample sequence.

Formally, a random process is an indexed family of random variables $\{\mathbf{x}_n\}$ characterized by a set of probability distribution functions that, in general, may be a function of the index n. In using the concept of a random process as a model for discrete-time

signals, the index n is associated with the time index. In other words, each sample value $x[n]$ of a random signal is assumed to have resulted from a mechanism that is governed by a probability law. An individual random variable $\mathbf{x}_n$ is described by the probability distribution function

$$P_{\mathbf{x}_n}(x_n, n) = \text{Probability}[\mathbf{x}_n \leq x_n], \tag{A.1}$$

where $\mathbf{x}_n$ denotes the random variable and x_n is a particular value of $\mathbf{x}_n$.[1] If $\mathbf{x}_n$ takes on a continuous range of values, it is equivalently specified by the *probability density function*

$$p_{\mathbf{x}_n}(x_n, n) = \frac{\partial P_{\mathbf{x}_n}(x_n, n)}{\partial x_n}, \tag{A.2}$$

or the *probability distribution function*

$$P_{\mathbf{x}_n}(x_n, n) = \int_{-\infty}^{x_n} p_{\mathbf{x}_n}(x, n)dx. \tag{A.3}$$

The interdependence of two random variables $\mathbf{x}_n$ and $\mathbf{x}_m$ of a random process is described by the joint probability distribution function

$$P_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m) = \text{Probability}[\mathbf{x}_n \leq x_n \text{ and } \mathbf{x}_m \leq x_m] \tag{A.4}$$

and by the joint probability density

$$p_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m) = \frac{\partial^2 P_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m)}{\partial x_n \partial x_m}. \tag{A.5}$$

Two random variables are *statistically independent* if knowledge of the value of one does not affect the probability density of the other. If all the random variables of a collection of random variables, $\{\mathbf{x}_n\}$, are statistically independent, then

$$P_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m) = P_{\mathbf{x}_n}(x_n, n) \cdot P_{\mathbf{x}_m}(x_m, m) \qquad m \neq n. \tag{A.6}$$

A complete characterization of a random process requires the specification of all possible joint probability distributions. As we have indicated, these probability distributions may be a function of the time indices m and n. In the case where all the probability distributions are independent of a shift of time origin, the random process is said to be *stationary*. For example, the 2^{nd}-order distribution of a stationary process satisfies

$$P_{\mathbf{x}_n+k,\mathbf{x}_m+k}(x_{n+k}, n+k, x_{m+k}, m+k) = P_{\mathbf{x}_n\mathbf{x}_m}(x_n, n, x_m, m) \qquad \text{for all } k. \tag{A.7}$$

In many of the applications of discrete-time signal processing, random processes serve as models for signals in the sense that a particular signal can be considered a sample sequence of a random process. Although the details of such signals are unpredictable—making a deterministic approach to signal representation inappropriate—certain average properties of the ensemble can be determined, given the probability law of the process. These average properties often serve as a useful, although incomplete, characterization of such signals.

[1] In this appendix, boldface type is used to denote the random variables and regular type denotes dummy variables of probability functions.

A.2 AVERAGES

It is often useful to characterize a random variable by averages such as the mean and variance. Since a random process is an indexed set of random variables, we may likewise characterize the process by statistical averages of the random variables making up the random process. Such averages are called *ensemble averages*. We begin the discussion of averages with some definitions.

A.2.1 Definitions

The average, or mean, of a random process is defined as

$$m_{\mathbf{x}_n} = \mathcal{E}\{\mathbf{x}_n\} = \int_{-\infty}^{\infty} x p_{\mathbf{x}_n}(x, n) dx, \tag{A.8}$$

where $\mathcal{E}$ denotes an operator called *mathematical expectation*. In general, the mean (expected value) may depend on n. In addition, if $g(\cdot)$ is a single-valued function, then $g(\mathbf{x}_n)$ is a random variable, and the set of random variables $\{g(\mathbf{x}_n)\}$ defines a new random process. To compute averages of this new process, we can derive probability distributions of the new random variables. Alternatvely, it can be shown that

$$\mathcal{E}\{g(\mathbf{x}_n)\} = \int_{-\infty}^{\infty} g(x) p_{\mathbf{x}_n}(x, n) dx. \tag{A.9}$$

If the random variables are discrete—i.e., if they have quantized values—the integrals become summations over all possible values of the random variable. In that case $\mathcal{E}\{g(x)\}$ has the form

$$\mathcal{E}\{g(\mathbf{x}_n)\} = \sum_x g(x) \hat{p}_{\mathbf{x}_n}(x, n). \tag{A.10}$$

In cases where we are interested in the relationship between multiple random processes, we must be concerned with multiple sets of random variables. For example, for two sets of random variables, $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_m\}$, the expected value of a function of the two random variables is defined as

$$\mathcal{E}\{g(\mathbf{x}_n, \mathbf{y}_m)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) p_{\mathbf{x}_n, \mathbf{y}_m}(x, n, y, m) dx\, dy, \tag{A.11}$$

where $p_{\mathbf{x}_n, \mathbf{y}_m}(x_m, n, y_m, m)$ is the joint probability density of the random variables $\mathbf{x}_n$ and $\mathbf{y}_m$.

The mathematical expectation operator is a linear operator; that is, it can be shown that

1. $\mathcal{E}\{\mathbf{x}_n + \mathbf{y}_m\} = \mathcal{E}\{\mathbf{x}_n\} + \mathcal{E}\{\mathbf{y}_m\}$; i.e., the average of a sum is the sum of the averages.
2. $\mathcal{E}\{a\mathbf{x}_n\} = a\mathcal{E}\{\mathbf{x}_n\}$; i.e., the average of a constant times $\mathbf{x}_n$ is equal to the constant times the average of $\mathbf{x}_n$.

In general, the average of a product of two random variables is not equal to the product of the averages. When this property holds, however, the two random variables are said to be *linearly independent* or *uncorrelated*. That is, $\mathbf{x}_n$ and $\mathbf{y}_m$ are linearly independent or uncorrelated if

$$\mathcal{E}\{\mathbf{x}_n \mathbf{y}_m\} = \mathcal{E}\{\mathbf{x}_m\} \cdot \mathcal{E}\{\mathbf{y}_m\}. \tag{A.12}$$

It is easy to see from Eqs. (A.11) and (A.12) that a sufficient condition for linear independence is

$$p_{\mathbf{x}_n,\mathbf{y}_m}(x_n, n, y_m, m) = p_{\mathbf{x}_n}(x_n, n) \cdot p_{\mathbf{y}_m}(y_m, m). \tag{A.13}$$

However, Eq. (A.13) is a stronger statement of independence than Eq. (A.12). As previously stated, random variables satisfying Eq. (A.13) are said to be *statistically independent*. If Eq. (A.13) holds for all values of n and m, the random processes $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_m\}$ are said to be statistically independent. Statistically independent random processes are also linearly independent; but the converse is not true: Linear independence does not imply statistical independence.

It can be seen from Eqs. (A.9)–(A.11) that averages generally are functions of the time index. For stationary processes, the mean is the same for all the random variables that constitute the process; i.e., the mean of a stationary process is a constant, which we denote simply m_x.

In addition to the mean of a random process, as defined in Eq. (A.8), a number of other averages are particularly important within the context of signal processing. These are defined next. For notational convenience, we assume that the probability distributions are continuous. Corresponding definitions for discrete random processes can be obtained by applying Eq. (A.10).

The *mean-square* value of $\mathbf{x}_n$ is the average of $|\mathbf{x}_n|^2$; i.e.,

$$\mathcal{E}\{|\mathbf{x}_n|^2\} = \text{mean square} = \int_{-\infty}^{\infty} |x|^2 p_{\mathbf{x}_n}(x, n)dx. \tag{A.14}$$

The mean-square value is sometimes referred to as the *average power*.

The *variance* of $\mathbf{x}_n$ is the mean-square value of $[\mathbf{x}_n - m_{x_n}]$; i.e.,

$$\text{var}[\mathbf{x}_n] = \mathcal{E}\{|(\mathbf{x}_n - m_{x_n})|^2\} = \sigma^2_{\mathbf{x}_n}. \tag{A.15}$$

Since the average of a sum is the sum of the averages, it follows that Eq. (A.15) can be written as

$$\text{var}[\mathbf{x}_n] = \mathcal{E}\{|\mathbf{x}_n|^2\} - |m_{x_n}|^2. \tag{A.16}$$

In general, the mean-square value and the variance are functions of time; however, they are constant for stationary processes.

The mean, mean square, and variance are simple averages that provide only a small amount of information about a process. A more useful average is the *autocorrelation sequence*, which is defined as

$$\begin{aligned}\phi_{xx}[n, m] &= \mathcal{E}\{\mathbf{x}_n \mathbf{x}^*_m\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_n x^*_m p_{\mathbf{x}_n,\mathbf{x}_m}(x_n, n, x_m, m)dx_n\, dx_m,\end{aligned} \tag{A.17}$$

where * denotes complex conjugation. The autocovariance sequence of a random process is defined as

$$\gamma_{xx}[n, m] = \mathcal{E}\{(\mathbf{x}_n - m_{x_n})(\mathbf{x}_m - m_{x_m})^*\}, \tag{A.18}$$

which can be written as

$$\gamma_{xx}[n, m] = \phi_{xx}[n, m] - m_{x_n} m^*_{x_m}. \tag{A.19}$$

Note that, in general, both the autocorrelation and autocovariance are two-dimensional sequences, i.e., functions of two discrete variables.

The autocorrelation sequence is a measure of the dependence between values of the random processes at different times. In this sense, it partially describes the time variation of a random signal. A measure of the dependence between two different random signals is obtained from the cross-correlation sequence. If $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_m\}$ are two random processes, their cross-correlation is

$$\begin{aligned}\phi_{xy}[n, m] &= \mathcal{E}\{\mathbf{x}_n \mathbf{y}_m^*\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy^* p_{\mathbf{x}_n, \mathbf{y}_m}(x, n, y, m) dx\, dy,\end{aligned} \tag{A.20}$$

where $p_{\mathbf{x}_n, \mathbf{y}_m}(x, n, y, m)$ is the joint probability density of $\mathbf{x}_n$ and $\mathbf{y}_m$. The cross-covariance function is defined as

$$\begin{aligned}\gamma_{xy}[n, m] &= \mathcal{E}\{(\mathbf{x}_n - m_{x_n})(\mathbf{y}_m - m_{y_m})^*\} \\ &= \phi_{xy}[n, m] - m_{x_n} m_{y_m}^*.\end{aligned} \tag{A.21}$$

As we have pointed out, the statistical properties of a random process generally vary with time. However, a stationary random process is characterized by an equilibrium condition in which the statistical properties are invariant to a shift of time origin. This means that the 1st-order probability distribution is independent of time. Similarly, all the joint probability functions are also invariant to a shift of time origin; i.e., the 2nd-order joint probability distributions depend only on the time difference $(m - n)$. First-order averages such as the mean and variance are independent of time; 2nd-order averages, such as the autocorrelation $\phi_{xx}[n, m]$, are dependent on the time difference $(m - n)$. Thus, for a stationary process, we can write

$$m_x = \mathcal{E}\{\mathbf{x}_n\}, \tag{A.22}$$

$$\sigma_x^2 = \mathcal{E}\{|(\mathbf{x}_n - m_x)|^2\}, \tag{A.23}$$

both independent of n. If we now denote the time difference by m, we have

$$\phi_{xx}[n + m, n] = \phi_{xx}[m] = \mathcal{E}\{\mathbf{x}_{n+m} \mathbf{x}_n^*\}. \tag{A.24}$$

That is, the autocorrelation sequence of a stationary random process is a one-dimensional sequence, a function of the time difference m.

In many instances, we encounter random processes that are not stationary in the *strict sense*—i.e., their probability distributions are not time invariant—but Eqs. (A.22)–(A.24) still hold. Such random processes are said to be *wide-sense stationary*.

A.2.2 Time Averages

In a signal-processing context, the notion of an ensemble of signals is a convenient mathematical concept that allows us to use the theory of probability to represent the signals. However, in a practical situation, we always have available at most a finite number of finite-length sequences rather than an infinite ensemble of sequences. For example, we might wish to infer the probability law or certain averages of the random-process representation from measurements on a single member of the ensemble. When the probability distributions are independent of time, intuition suggests that the amplitude

distribution (histogram) of a long segment of an individual sequence of samples should be approximately equal to the single probability density that describes each of the random variables of the random-process model. Similarly, the arithmetic average of a large number of samples of a single sequence should be very close to the mean of the process. To formalize these intuitive notions, we define the time average of a random process as

$$\langle \mathbf{x}_n \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_n. \tag{A.25}$$

Similarly, the time autocorrelation sequence is defined as

$$\langle \mathbf{x}_{n+m}\mathbf{x}_n^* \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \mathbf{x}_{n+m}\mathbf{x}_n^*. \tag{A.26}$$

It can be shown that the preceding limits exist if $\{\mathbf{x}_n\}$ is a stationary process with finite mean. As defined in Eqs. (A.25) and (A.26), these time averages are functions of an infinite set of random variables and thus are properly viewed as random variables themselves. However, under the condition known as *ergodicity*, the time averages in Eqs. (A.25) and (A.26) are equal to constants in the sense that the time averages of almost all possible sample sequences are equal to the same constant. Furthermore, they are equal to the corresponding ensemble average.[2] That is, for any single sample sequence $\{x[n]\}$ for $-\infty < n < \infty$,

$$\langle x[n] \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n] = \mathcal{E}\{\mathbf{x}_n\} = m_x \tag{A.27}$$

and

$$\langle x[n+m]x^*[n] \rangle = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} x[n+m]x^*[n] = \mathcal{E}\{\mathbf{x}_{n+m}\mathbf{x}_n^*\} = \phi_{xx}[m]. \tag{A.28}$$

The time-average operator $\langle\cdot\rangle$ has the same properties as the ensemble-average operator $\mathcal{E}\{\cdot\}$. Thus, we generally do not distinguish between the random variable $\mathbf{x}_n$ and its value in a sample sequence, $x[n]$. For example, the expression $\mathcal{E}\{x[n]\}$ should be interpreted as $\mathcal{E}\{\mathbf{x}_n\} = \langle x[n] \rangle$. In general, for *ergodic processes*, time averages equal ensemble averages.

In practice, it is common to assume that a given sequence is a sample sequence of an ergodic random process so that averages can be computed from a single sequence. Of course, we generally cannot compute with the limits in Eqs. (A.27) and (A.28), but instead the quantities

$$\hat{m}_x = \frac{1}{L} \sum_{n=0}^{L-1} x[n], \tag{A.29}$$

$$\hat{\sigma}_x^2 = \frac{1}{L} \sum_{n=0}^{L-1} |x[n] - \hat{m}_x|^2, \tag{A.30}$$

[2] A more precise statement is that the random variables $\langle \mathbf{x}_n \rangle$ and $\langle \mathbf{x}_{n+m}\mathbf{x}_n^* \rangle$ have means equal to m_x and $\phi_{xx}[m]$, respectively, and their variances are zero.

and

$$\langle x[n+m]x^*[n]\rangle_L = \frac{1}{L}\sum_{n=0}^{L-1} x[n+m]x^*[n] \tag{A.31}$$

or similar quantities are often computed as *estimates* of the mean, variance, and autocorrelation. $\hat{m}_x$ and $\hat{\sigma}_x^2$ are referred to as the sample mean and sample variance, respectively. The estimation of averages of a random process from a finite segment of data is a problem of statistics, which we touch on briefly in Chapter 10.

A.3 PROPERTIES OF CORRELATION AND COVARIANCE SEQUENCES OF STATIONARY PROCESSES

Several useful properties of correlation and covariance functions follow in a straightforward way from the definitions. These properties are given in this section.

Consider two real stationary random processes $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ with autocorrelation, autocovariance, cross-correlation, and cross-covariance being given, respectively, by

$$\phi_{xx}[m] = \mathcal{E}\{\mathbf{x}_{n+m}\mathbf{x}_n^*\}, \tag{A.32}$$

$$\gamma_{xx}[m] = \mathcal{E}\{(\mathbf{x}_{n+m} - m_x)(\mathbf{x}_n - m_x)^*\}, \tag{A.33}$$

$$\phi_{xy}[m] = \mathcal{E}\{\mathbf{x}_{n+m}\mathbf{y}_n^*\}, \tag{A.34}$$

$$\gamma_{xy}[m] = \mathcal{E}\{(\mathbf{x}_{n+m} - m_x)(\mathbf{y}_n - m_y)^*\}, \tag{A.35}$$

where m_x and m_y are the means of the two processes. The following properties are easily derived by simple manipulations of the definitions:

Property 1

$$\gamma_{xx}[m] = \phi_{xx}[m] - |m_x|^2, \tag{A.36a}$$

$$\gamma_{xy}[m] = \phi_{xy}[m] - m_x m_y^*. \tag{A.36b}$$

These results follow directly from Eqs. (A.19) and (A.21), and they indicate that the correlation and covariance sequences are identical for zero-mean processes.

Property 2

$$\phi_{xx}[0] = \mathcal{E}[|\mathbf{x}_n|^2] = \text{Mean-square value}, \tag{A.37a}$$

$$\gamma_{xx}[0] = \sigma_x^2 = \text{Variance}. \tag{A.37b}$$

Property 3

$$\phi_{xx}[-m] = \phi_{xx}^*[m], \tag{A.38a}$$

$$\gamma_{xx}[-m] = \gamma_{xx}^*[m], \tag{A.38b}$$

$$\phi_{xy}[-m] = \phi_{yx}^*[m], \tag{A.38c}$$

$$\gamma_{xy}[-m] = \gamma_{yx}^*[m]. \tag{A.38d}$$

Property 4

$$|\phi_{xy}[m]|^2 \leq \phi_{xx}[0]\phi_{yy}[0], \tag{A.39a}$$

$$|\gamma_{xy}[m]|^2 \leq \gamma_{xx}[0]\gamma_{yy}[0]. \tag{A.39b}$$

In particular,

$$|\phi_{xx}[m]| \leq \phi_{xx}[0], \tag{A.40a}$$

$$|\gamma_{xx}[m]| \leq \gamma_{xx}[0]. \tag{A.40b}$$

Property 5. If $\mathbf{y}_n = \mathbf{x}_{n-n_0}$, then

$$\phi_{yy}[m] = \phi_{xx}[m], \tag{A.41a}$$

$$\gamma_{yy}[m] = \gamma_{xx}[m]. \tag{A.41b}$$

Property 6. For many random processes, the random variables become uncorrelated as they become more separated in time. If this is true,

$$\lim_{m\to\infty} \gamma_{xx}[m] = 0, \tag{A.42a}$$

$$\lim_{m\to\infty} \phi_{xx}[m] = |m_x|^2, \tag{A.42b}$$

$$\lim_{m\to\infty} \gamma_{xy}[m] = 0, \tag{A.42c}$$

$$\lim_{m\to\infty} \phi_{xy}[m] = m_x m_y^*. \tag{A.42d}$$

The essence of these results is that the correlation and covariance are finite-energy sequences that tend to die out for large values of m. Thus, it is often possible to represent these sequences in terms of their Fourier transforms or z-transforms.

A.4 FOURIER TRANSFORM REPRESENTATION OF RANDOM SIGNALS

Although the Fourier transform of a random signal does not exist except in a generalized sense, the autocovariance and autocorrelation sequences of such a signal are aperiodic sequences for which the transform does exist. The spectral representation of the correlation functions plays an important role in describing the input–output relations for a linear time-invariant system when the input is a random signal. Therefore, it is of interest to consider the properties of correlation and covariance sequences and their corresponding Fourier and z-transforms.

We define $\Phi_{xx}(e^{j\omega})$, $\Gamma_{xx}(e^{j\omega})$, $\Phi_{xy}(e^{j\omega})$, and $\Gamma_{xy}(e^{j\omega})$ as the DTFTs of $\phi_{xx}[m]$, $\gamma_{xx}[m]$, $\phi_{xy}[m]$, and $\gamma_{xy}[m]$, respectively. Since these functions are all DTFTs of sequences, they must be periodic with period 2π. From Eqs. (A.36a) and (A.36b), it follows that, over one period $|\omega| \leq \pi$,

$$\Phi_{xx}(e^{j\omega}) = \Gamma_{xx}(e^{j\omega}) + 2\pi|m_x|^2\delta(\omega), \qquad |\omega| \leq \pi, \tag{A.43a}$$

and

$$\Phi_{xy}(e^{j\omega}) = \Gamma_{xy}(e^{j\omega}) + 2\pi m_x m_y^* \delta(\omega), \qquad |\omega| \leq \pi. \tag{A.43b}$$

In the case of zero-mean processes ($m_x = 0$ and $m_y = 0$), the correlation and covariance functions are identical so that $\Phi_{xx}(e^{j\omega}) = \Gamma_{xx}(e^{j\omega})$ and $\Phi_{xy}(e^{j\omega}) = \Gamma_{xy}(e^{j\omega})$.

From the inverse Fourier transform equation, it follows that

$$\gamma_{xx}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_{xx}(e^{j\omega})e^{j\omega m}d\omega, \tag{A.44a}$$

$$\phi_{xx}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})e^{j\omega m}d\omega, \tag{A.44b}$$

and, consequently,

$$\mathcal{E}\{|x[n]|^2\} = \phi_{xx}[0] = \sigma_x^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})d\omega, \tag{A.45a}$$

$$\sigma_x^2 = \gamma_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_{xx}(e^{j\omega})d\omega. \tag{A.45b}$$

Sometimes it is notationally convenient to define the quantity

$$P_{xx}(\omega) = \Phi_{xx}(e^{j\omega}), \tag{A.46}$$

in which case Eqs. (A.45a) and (A.45b) are expressed as

$$\mathcal{E}\{|x[n]|^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)d\omega, \tag{A.47a}$$

$$\sigma_x^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)d\omega. \tag{A.47b}$$

Thus, the area under $P_{xx}(\omega)$ for $-\pi \leq \omega \leq \pi$ is proportional to the average power in the signal. In fact, as we discussed in Section 2.10, the integral of $P_{xx}(\omega)$ over a band of frequencies is proportional to the power in the signal in that band. For this reason, the function $P_{xx}(\omega)$ is called the *power density spectrum*, or simply, the *power spectrum*. When $P_{xx}(\omega)$ is a constant independent of ω, the random process is referred to as a white-noise process, or simply, white noise. When $P_{xx}(\omega)$ is constant over a band and zero otherwise, we refer to it as bandlimited white noise.

From Eq. (A.38a), it can be shown that $P_{xx}(\omega) = P_{xx}^*(\omega)$; i.e., $P_{xx}(\omega)$ is always real valued. Furthermore, for real random processes, $\phi_{xx}[m] = \phi_{xx}[-m]$, so in the real case, $P_{xx}(\omega)$ is both real and even; i.e.,

$$P_{xx}(\omega) = P_{xx}(-\omega). \tag{A.48}$$

An additional important property is that the power density spectrum is nonnegative; i.e., $P_{xx}(\omega) \geq 0$ for all ω. This point is discussed in Section 2.10.

The *cross power density spectrum* is defined as

$$P_{xy}(\omega) = \Phi_{xy}(e^{j\omega}). \tag{A.49}$$

This function is generally complex, and from Eq. (A.38c), it follows that

$$P_{xy}(\omega) = P_{yx}^*(\omega). \tag{A.50}$$

Finally, as shown in Section 2.10, if $x[n]$ is a random signal input to a linear time-invariant discrete-time system with frequency response $H(e^{j\omega})$, and if $y[n]$ is the corresponding output, then

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega}) \tag{A.51}$$

and

$$\Phi_{xy}(e^{j\omega}) = H(e^{j\omega}) \Phi_{xx}(e^{j\omega}). \tag{A.52}$$

Example A.1 Noise Power Output of Ideal Lowpass Filter

Suppose that $x[n]$ is a zero-mean white-noise sequence with $\phi_{xx}[m] = \sigma_x^2 \delta[m]$ and power spectrum $\Phi_{xx}(e^{j\omega}) = \sigma_x^2$ for $|\omega| \leq \pi$, and furthermore, assume that $x[n]$ is the input to an ideal lowpass filter with cutoff frequency ω_c. Then from Eq. (A.51), it follows that the output $y[n]$ would be a bandlimited white noise process whose power spectrum would be

$$\Phi_{yy}(e^{j\omega}) = \begin{cases} \sigma_x^2, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases} \tag{A.53}$$

Using the inverse Fourier transform, we obtain the autocorrelation sequence

$$\phi_{yy}[m] = \frac{\sin(\omega_c m)}{\pi m} \sigma_x^2. \tag{A.54}$$

Now, using Eq. (A.45a), we get for the average power of the output,

$$\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \sigma_x^2 d\omega = \sigma_x^2 \frac{\omega_c}{\pi}. \tag{A.55}$$

A.5 USE OF THE z-TRANSFORM IN AVERAGE POWER COMPUTATIONS

To carry out average power calculations using Eq. (A.45a), we must evaluate an integral of the power spectrum as was done in Example A.1. While the integral in that example was easy to evaluate, such integrals in general are difficult to evaluate as real integrals. However, a result based on the z-transform makes the calculation of average output power straightforward in the important case of systems that have rational system functions.

In general, the z-transform can be used to represent the covariance function but not a correlation function. This is because when a signal has nonzero average value, its correlation function will contain an additive constant component that does not have a z-transform representation. When the average value is zero, however, the covariance and correlation functions are, of course, equal. If the z-transform of $\gamma_{xx}[m]$ exists, then since $\gamma_{xx}[-m] = \gamma_{xx}^*[m]$ it follows that in general

$$\Gamma_{xx}(z) = \Gamma_{xx}^*(1/z^*). \tag{A.56}$$

Furthermore, since $\gamma_{xx}[m]$ is two sided and conjugate-symmetric, it follows that the region of convergence of $\Gamma_{xx}(z)$ must be of the form

$$r_a < |z| < \frac{1}{r_a}$$

where necessarily $0 < r_a < 1$. In the important case when $\Gamma_{xx}(z)$ is a rational function of z, Eq. (A.56) implies that the poles and zeros of $\Gamma_{xx}(z)$ must occur in complex-conjugate reciprocal pairs.

The major advantage of the z-transform representation is that when $\Gamma_{xx}(z)$ is a rational function, the average power of the random signal can be computed easily using the relation

$$\mathcal{E}\{|x[n] - m_x|^2\} = \sigma_x^2 = \gamma_{xx}[0] = \left\{ \begin{array}{c} \text{Inverse } z\text{-transform} \\ \text{of } \Gamma_{xx}(z), \\ \text{evaluated for } m = 0 \end{array} \right\}. \tag{A.57}$$

It is straightforward to evaluate the right-hand side of this equation using a method based on the observation that when $\Gamma_{xx}(z)$ is a rational function of z, $\gamma_{xx}[m]$ can be computed for all m by employing a partial fraction expansion. Then to obtain the average power, we can simply evaluate $\gamma_{xx}[m]$ for $m = 0$.

The z-transform is also useful in determining the autocovariance and average power of the output of an LTI system when the input is a random signal. Generalizing Eq. (A.51) leads to

$$\Gamma_{yy}(z) = H(z)H^*(1/z^*)\Gamma_{xx}(z), \tag{A.58}$$

and from the properties of the z-transform and Eq. (A.58), it follows that the autocovariance of the output is the convolution

$$\gamma_{yy}[m] = h[m] * h^*[-m] * \gamma_{xx}[m]. \tag{A.59}$$

This result is particularly useful in quantization noise analysis where we need to compute the average output power when the input to a linear difference equation is a zero-mean white noise signal with average power σ_x^2. Since the autocovariance of such an input is $\gamma_{xx}[m] = \sigma_x^2\delta[m]$, it follows that the autocovariance of the output is $\gamma_{yy}[m] = \sigma_x^2(h[m] * h^*[-m])$, i.e., the covariance of the output is proportional to the deterministic autocorrelation of the impulse response of the LTI system. From this result it follows that

$$\mathcal{E}\{y^2[n]\} = \gamma_{yy}[0] = \sigma_x^2 \sum_{n=-\infty}^{\infty} |h[n]|^2. \tag{A.60}$$

As an alternative to computing the sum of squares of the impulse response sequence, which can be rather difficult for IIR systems, we can apply the method suggested in Eq. (A.57) to obtain $\mathcal{E}\{y^2[n]\}$ from a partial fraction expansion of $\Gamma_{yy}(z)$. Recall that for a white noise input with $\gamma_{xx}[m] = \sigma_x^2\delta[m]$, the z-transform is $\Gamma_{xx}(z) = \sigma_x^2$ so $\Gamma_{yy}(z) = \sigma_x^2 H(z)H^*(1/z^*)$. Therefore, Eq. (A.57) applied to the output of the system gives

$$\mathcal{E}\{y^2[n]\} = \gamma_{yy}[0] = \left\{ \begin{array}{c} \text{Inverse } z\text{-transform of} \\ \Gamma_{yy}(z) = H(z)H^*(1/z^*)\sigma_x^2, \\ \text{evaluated for } m = 0 \end{array} \right\}. \tag{A.61}$$

Now consider the special case of a stable and causal system having a rational system function of the form

$$H(z) = A\frac{\prod_{m=1}^{M}(1 - c_m z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} \qquad |z| > \max_k\{|d_k|\}, \tag{A.62}$$

where $\max_k\{|d_k|\} < 1$ and $M < N$. Such a system function might describe the relationship between an internal round-off noise source and the output of a system implemented with fixed-point arithmetic. Substituting Eq. (A.62) for $H(z)$ in Eq. (A.58) gives

$$\Gamma_{yy}(z) = \sigma_x^2 H(z)H^*(1/z^*) = \sigma_x^2|A|^2\frac{\prod_{m=1}^{M}(1 - c_m z^{-1})(1 - c_m^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}. \tag{A.63}$$

Since we have assumed that $|d_k| < 1$ for all k, all of the original poles are inside the unit circle and therefore the other poles at $(d_k^*)^{-1}$ are at conjugate reciprocal locations outside the unit circle. The region of convergence for $\Gamma_{yy}(z)$ is therefore $\max_k |d_k| < |z| < \min_k |(d_k^*)^{-1}|$. For such rational functions, it can be shown that since $M < N$, the partial fraction expansion has the form

$$\Gamma_{yy}(z) = \sigma_x^2\left(\sum_{k=1}^{N}\left(\frac{A_k}{1 - d_k z^{-1}} - \frac{A_k^*}{1 - (d_k^*)^{-1}z^{-1}}\right)\right). \tag{A.64}$$

where the coefficients are found from

$$A_k = H(z)H^*(1/z^*)(1 - d_k z^{-1})\Big|_{z=d_k}. \tag{A.65}$$

Since the poles at $z = d_k$ are inside the inner boundary of the region of convergence, each of them corresponds to a right-sided sequence, while the poles at $z = (d_k^*)^{-1}$ each correspond to a left-sided sequence. Thus, the autocovariance function corresponding to Eq. (A.64) is

$$\gamma_{yy}[n] = \sigma_x^2\sum_{k=1}^{N}(A_k(d_k)^n u[n] + A_k^*(d_k^*)^{-n}u[-n-1]),$$

from which it follows that we can obtain the average power from

$$\sigma_y^2 = \gamma_{yy}[0] = \sigma_x^2\left(\sum_{k=1}^{N} A_k\right), \tag{A.66}$$

where the quantities A_k are given by Eq. (A.65).

Thus, the computation of the total average power of the output of a system with rational system function and white noise input reduces to the straightforward problem of finding partial fraction expansion coefficients for the z-transform of the output autocorrelation function. The utility of this approach is illustrated by the following example.

Example A.2 Noise Power Output of a 2nd-Order IIR Filter

Consider a system with impulse response

$$h[n] = \frac{r^n \sin\theta(n+1)}{\sin\theta} u[n] \tag{A.67}$$

and system function

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}. \tag{A.68}$$

When the input is white noise with total average power σ_x^2, the z-transform of the autocovariance function of the output is

$$\Gamma_{yy}(z) = \sigma_x^2 \left(\frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}\right)\left(\frac{1}{(1 - re^{-j\theta}z)(1 - re^{j\theta}z)}\right) \tag{A.69}$$

from which we obtain, using Eq. (A.65),

$$\begin{aligned}\mathcal{E}\{y^2[n]\} = \sigma_x^2 \Bigg[&\left(\frac{1}{(1 - re^{-j\theta}z^{-1})}\right)\left(\frac{1}{(1 - re^{-j\theta}z)(1 - re^{j\theta}z)}\right)\Big|_{z=re^{j\theta}} \\ &+ \left(\frac{1}{(1 - re^{j\theta}z^{-1})}\right)\left(\frac{1}{(1 - re^{-j\theta}z)(1 - re^{j\theta}z)}\right)\Big|_{z=re^{-j\theta}} \Bigg].\end{aligned} \tag{A.70}$$

Making the indicated substitutions, placing both terms over a common denominator, and doing some algebra leads to

$$\mathcal{E}\{y^2[n]\} = \sigma_x^2 \left(\frac{1+r^2}{1-r^2}\right)\left(\frac{1}{1 - 2r^2\cos(2\theta) + r^4}\right). \tag{A.71}$$

Thus, using the partial fraction expansion of $\Gamma_{yy}(z)$, we have effectively evaluated the expression

$$\mathcal{E}\{y^2[n]\} = \sigma_x^2 \sum_{n=-\infty}^{\infty} |h[n]|^2 = \sigma_x^2 \sum_{n=0}^{\infty} \left|\frac{r^n \sin\theta(n+1)}{\sin\theta}\right|^2,$$

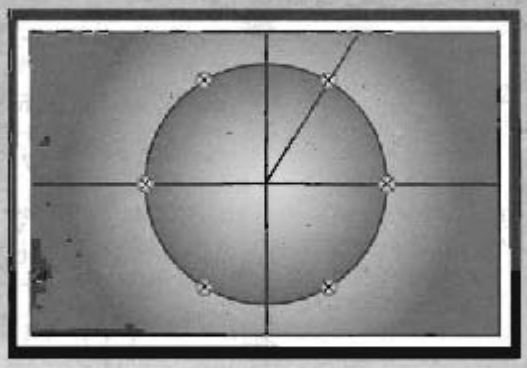
which would be difficult to sum in closed form, and the expression

$$\mathcal{E}\{y^2[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_x^2 |H(e^{j\omega})|^2 d\omega = \frac{\sigma_x^2}{2\pi}\int_{-\pi}^{\pi} \frac{d\omega}{|(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})|^2},$$

which would be difficult to evaluate as an integral over the real variable ω.

The result of Example A.2 is an illustration of the power of the partial fraction method in evaluating average power formulas. In Chapter 6, we make use of this technique in the analysis of quantization effects in the implementation of digital filters.

B Continuous-Time Filters



The techniques discussed in Chapter 7 for designing IIR digital filters rely on the availability of appropriate continuous-time filter designs. In this appendix,we briefly summarize the characteristics of several classes of lowpass filter approximations that we referred to in Chapter 7. More detailed discussions of these classes of filters appear in Guillemin (1957), Weinberg (1975) and Parks and Burrus (1987), and extensive design tables and formulas are found in Zverev (1967). Design programs for all the common continuous-time approximations and transformations to digital filters are available in MATLAB, Simulink, and LabVIEW.

B.1 BUTTERWORTH LOWPASS FILTERS

Butterworth lowpass filters are defined by the property that the magnitude response is maximally flat in the passband. For an N^{th}-order lowpass filter, this means that the first $(2N-1)$ derivatives of the magnitude-squared function are zero at $\Omega = 0$. Another property is that the magnitude response is monotonic in the passband and the stopband. The magnitude-squared function for a continuous-time Butterworth lowpass filter has the form

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}. \tag{B.1}$$

This function is plotted in Figure B.1.

As the parameter N in Eq. (B.1) increases, the filter characteristics become sharper: that is, they remain close to unity over more of the passband and become close to zero more rapidly in the stopband, although the magnitude-squared function at the cutoff frequency Ω_c will always be equal to one-half because of the nature of Eq. (B.1). The

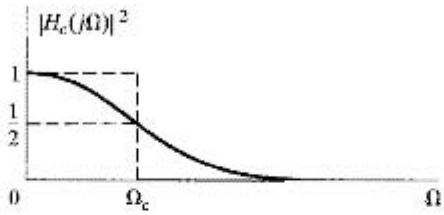


Figure B.1 Magnitude-squared function for continuous-time Butterworth filter.

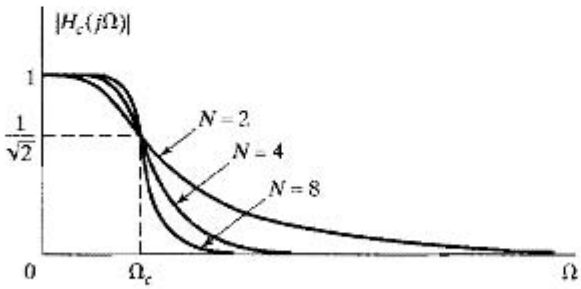


Figure B.2 Dependence of Butterworth magnitude characteristics on the order N.

dependence of the Butterworth filter characteristic on the parameter N is indicated in Figure B.2, which shows $|H_c(j\Omega)|$ for several values of N.

From the magnitude-squared function in Eq. (B.1), we observe by substituting $j\Omega = s$ that $H_c(s)H_c(-s)$ must be of the form

$$H_c(s)H_c(-s) = \frac{1}{1 + (s/j\Omega_c)^{2N}}. \tag{B.2}$$

The roots of the denominator polynomial (the poles of the magnitude-squared function) are therefore located at values of s satisfying $1 + (s/j\Omega_c)^{2N} = 0$; i.e.,

$$s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{(j\pi/2N)(2k+N-1)}, \qquad k = 0, 1, \ldots, 2N-1. \tag{B.3}$$

Thus, there are $2N$ poles equally spaced in angle on a circle of radius Ω_c in the s-plane. The poles are symmetrically located with respect to the imaginary axis. A pole never falls on the imaginary axis, and one occurs on the real axis for N odd, but not for N even. The angular spacing between the poles on the circle is π/N radians. For example, for $N = 3$, the poles are spaced by $\pi/3$ radians, or 60 degrees, as indicated in Figure B.3. To determine the system function of the analog filter to associate with the Butterworth magnitude-squared function, we perform the factorization $H_c(s)H_c(-s)$. The poles of the magnitude-squared function always occur in pairs; i.e., if there is a pole at $s = s_k$, then a pole also occurs at $s = -s_k$. Consequently, to construct $H_c(s)$ from the magnitude-squared function, we would choose the one pole from each such pair. To obtain a stable and causal filter, we should choose all the poles on the left-half-plane part of the s-plane.

With this approach, $H_c(s)$ would be

$$H_c(s) = \frac{\Omega_c^3}{(s + \Omega_c)(s - \Omega_c e^{j2\pi/3})(s - \Omega_c e^{-j2\pi/3})},$$

which can be written as

$$H_c(s) = \frac{\Omega_c^3}{s^3 + 2\Omega_c s^2 + 2\Omega_c s + \Omega_c^3}.$$

In general the numerator of $H_c(s)$ would be Ω_c^N to ensure that $|H_c(0) = 1$.

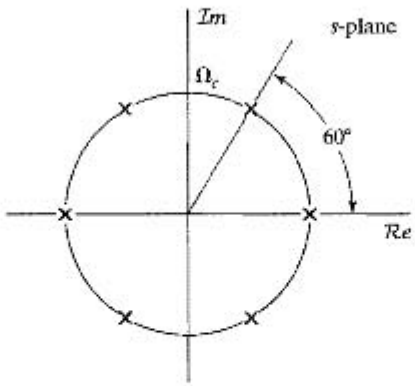


Figure B.3 s-plane pole locations for the magnitude-squared function of 3rd-order Butterworth filter.

B.2 CHEBYSHEV FILTERS

In a Butterworth filter, the magnitude response is monotonic in both the passband and the stopband. Consequently, if the filter specifications are in terms of maximum passband and stopband approximation error, the specifications are exceeded toward the low-frequency end of the passband and above the stopband cutoff frequency. A more efficient approach, which usually leads to a lower order filter, is to distribute the accuracy of the approximation uniformly over the passband or the stopband (or both). This is accomplished by choosing an approximation that has an equiripple behavior rather than a monotonic behavior. The class of Chebyshev filters has the property that the magnitude of the frequency response is either equiripple in the passband and monotonic in the stopband (referred to as a type I Chebyshev filter) or monotonic in the passband and equiripple in the stopband (a type II Chebyshev filter). The frequency response of a type I Chebyshev filter is shown in Figure B.4. The magnitude-squared function for this filter is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 V_N^2(\Omega/\Omega_c)}, \tag{B.4}$$

where $V_N(x)$ is the N^{th}-order Chebyshev polynomial defined as

$$V_N(x) = \cos(N \cos^{-1} x). \tag{B.5}$$

For example, for $N = 0$, $V_0(x) = 1$; for $N = 1$, $V_1(x) = \cos(\cos^{-1} x) = x$; for $N = 2$, $V_2(x) = \cos(2\cos^{-1} x) = 2x^2 - 1$; and so on.

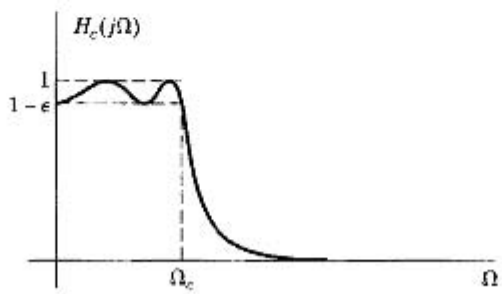


Figure B.4 Type I Chebyshev lowpass filter approximation.

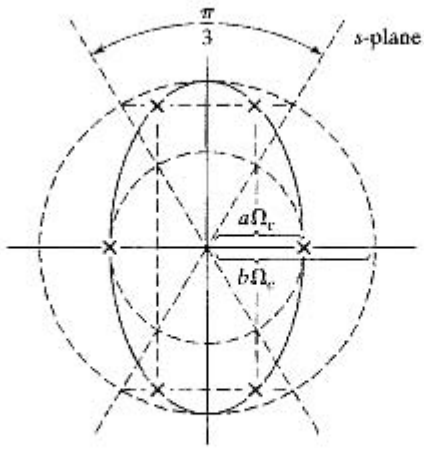


Figure B.5 Location of poles for the magnitude-squared function of 3rd-order type I lowpass Chebyshev filter.

From Eq. (B.5), which defines the Chebyshev polynomials, it is straightforward to obtain a recurrence formula from which $V_{N+1}(x)$ can be obtained from $V_N(x)$ and $V_{N-1}(x)$. By applying trigonometric identities to Eq. (B.5), it follows that

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x). \tag{B.6}$$

From Eq. (B.5), we note that $V_N^2(x)$ varies between zero and unity for $0 < x < 1$. For $x > 1$, $\cos^{-1} x$ is imaginary, so $V_N(x)$ behaves as a hyperbolic cosine and consequently increases monotonically. Referring to Eq. (B.4), we see that $|H_c(j\Omega)|^2$ ripples between 1 and $1/(1+\varepsilon^2)$ for $0 \le \Omega/\Omega_c \le 1$ and decreases monotonically for $\Omega/\Omega_c > 1$. Three parameters are required to specify the filter: ε, Ω_c, and N. In a typical design, ε is specified by the allowable passband ripple and Ω_c is specified by the desired passband cutoff frequency. The order N is then chosen so that the stopband specifications are met.

The poles of the Chebyshev filter lie on an ellipse in the s-plane. As shown in Figure B.5, the ellipse is defined by two circles whose diameters are equal to the minor and major axes of the ellipse. The length of the minor axis is $2a\Omega_c$, where

$$a = \tfrac{1}{2}(\alpha^{1/N} - \alpha^{-1/N}) \tag{B.7}$$

with

$$\alpha = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}, \tag{B.8}$$

The length of the major axis is $2b\Omega_c$, where

$$b = \tfrac{1}{2}(\alpha^{1/N} + \alpha^{-1/N}). \tag{B.9}$$

To locate the poles of the Chebyshev filter on the ellipse, we first identify the points on the major and minor circles equally spaced in angle with a spacing of π/N in such a way that the points are symmetrically located with respect to the imaginary axis and such that a point never falls on the imaginary axis and a point occurs on the real axis for N odd but not for N even. This division of the major and minor circles corresponds exactly to the manner in which the circle is divided in locating the poles of a Butterworth filter as in Eq. (B.3). The poles of a Chebyshev filter fall on the ellipse, with the ordinate

specified by the points identified on the major circle and the abscissa specified by the points identified on the minor circle. In Figure B.5, the poles are shown for $N = 3$.

A type II Chebyshev lowpass filter can be related to a type I filter through a transformation. Specifically, if in Eq. (B.4) we replace the term $\varepsilon^2 V_N^2(\Omega/\Omega_c)$ by its reciprocal and also replace the argument of V_N^2 by its reciprocal, we obtain

$$|H_c(j\Omega)|^2 = \frac{1}{1 + [\varepsilon^2 V_N^2(\Omega_c/\Omega)]^{-1}}. \tag{B.10}$$

This is the analytic form for the type II Chebyshev lowpass filter. One approach to designing a type II Chebyshev filter is to first design a type I filter and then apply the transformation of Eq. (B.10).

B.3 ELLIPTIC FILTERS

If we distribute the error uniformly across the entire passband or across the entire stopband, as in the Chebyshev cases, we are able to meet the design specifications with a lower order filter than if we permit a monotonically varying error in the passband and stopband, as in the Butterworth case. We note that in the type I Chebyshev approximation, the stopband error decreases monotonically with frequency, raising the possibility of further improvements if we distribute the stopband error uniformly across the stopband. This suggests the lowpass filter approximation in Figure B.6. Indeed, it can be shown (Papoulis, 1957) that this type of approximation (i.e., equiripple error in the passband and the stopband) is the best that can be achieved for a given filter order N, in the sense that for given values of Ω_p, δ_1, and δ_2, the transition band $(\Omega_s - \Omega_p)$ is as small as possible.

This class of approximations, referred to as elliptic filters, has the form

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 U_N^2(\Omega)}, \tag{B.11}$$

where $U_N(\Omega)$ is a Jacobian elliptic function. To obtain equiripple error in both the passband and the stopband, elliptic filters must have both poles and zeros. As can be seen from Figure B.6, such a filter will have zeros on the $j\Omega$-axis of the s-plane. A discussion of elliptic filter design, even on a superficial level, is beyond the scope of this appendix. The reader is referred to the texts by Guillemin (1957), Storer (1957), Gold and Rader (1969) and Parks and Burrus (1987) for more detailed discussions.

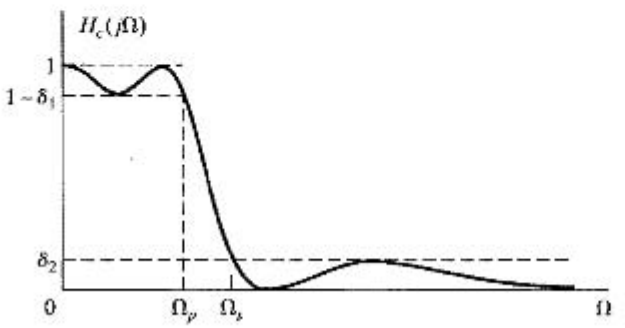


Figure B.6 Equiripple approximation in both passband and stopband.

C

Answers to Selected Basic Problems

This appendix contains the answers to the first 20 basic problems in Chapter 2 through 10.

Answers to Basic Problems in Chapter 2

2.1. **(a)** Always (2), (3), (5). If $g[n]$ is bounded, (1).

(b) (3).

(c) Always (1), (3), (4). If $n_0 = 0$, (2) and (5).

(d) Always (1), (3), (4). If $n_0 = 0$, (5). If $n_0 \geq 0$, (2).

(e) (1), (2), (4), (5).

(f) Always (1), (2), (4), (5). If $b = 0$, (3).

(g) (1), (3).

(h) (1), (5).

2.2. **(a)** $N_4 = N_0 + N_2$, $N_5 = N_1 + N_3$.

(b) At most $N + M - 1$ nonzero points.

2.3.

$$y[n] = \begin{cases} \dfrac{a^n}{1-a}, & n < 0, \\ \dfrac{1}{1-a}, & n \geq 0. \end{cases}$$

2.4. $y[n] = 8[(1/2)^n - (1/4)^n]u[n]$.

2.5. **(a)** $y_h[n] = A_1(2)^n + A_2(3)^n$.

(b) $h[n] = 2(3^n - 2^n)u[n]$.

(c) $s[n] = [-8(2)^{(n-1)} + 9(3)^{(n-1)} + 1]u[n]$.

2.6. (a)

$$H(e^{j\omega}) = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}.$$

(b) $y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3]$.

2.7. (a) Periodic, $N = 12$.

(b) Periodic, $N = 8$.

(c) Not periodic.

(d) Not periodic.

2.8. $y[n] = 3(-1/2)^n u[n] + 2(1/3)^n u[n]$.

2.9. (a)

$$h[n] = 2\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right]u[n],$$

$$H(e^{j\omega}) = \frac{\frac{1}{3}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}},$$

$$s[n] = \left[-2\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n + 1\right]u[n].$$

(b) $y_h[n] = A_1(1/2)^n + A_2(1/3)^n$.

(c) $y[n] = 4(1/2)^n - 3(1/3)^n - 2(1/2)^n u[-n-1] + 2(1/3)^n u[-n-1]$. Other answers are possible.

2.10. (a)

$$y[n] = \begin{cases} a^{-1}/(1 - a^{-1}), & n \geq -1, \\ a^n/(1 - a^{-1}), & n \leq -2. \end{cases}$$

(b)

$$y[n] = \begin{cases} 1, & n \geq 3, \\ 2^{(n-3)}, & n \leq 2. \end{cases}$$

(c)

$$y[n] = \begin{cases} 1, & n \geq 0, \\ 2^n, & n \leq -1. \end{cases}$$

(d)

$$y[n] = \begin{cases} 0, & n \geq 9, \\ 1 - 2^{(n-9)}, & 8 \geq n \geq -1, \\ 2^{(n+1)} - 2^{(n-9)}, & -2 \geq n. \end{cases}$$

2.11. $y[n] = 2\sqrt{2}\sin(\pi(n+1)/4)$.

2.12. (a) $y[n] = n!u[n]$.

(b) The system is linear.

(c) The system is not time invariant.

2.13. **(a)**, **(b)**, and **(c)** are eigenfunctions of stable LTI systems.

2.14. **(a)** (iv).

(b) (i).

(c) (iii), $h[n] = (1/2)^n u[n]$.

2.15. **(a)** Not LTI. Inputs $\delta[n]$ and $\delta[n-1]$ violate TI.

(b) Not causal. Consider $x[n] = \delta[n-1]$.

(c) Stable.

2.16. **(a)** $y_h[n] = A_1(1/2)^n + A_2(-1/4)^n$.

(b) Causal: $h_c[n] = 2(1/2)^n u[n] + (-1/4)^n u[n]$.
Anticausal: $h_{ac}[n] = -2(1/2)^n u[-n-1] - (-1/4)^n u[-n-1]$.

(c) $h_c[n]$ is absolutely summable, $h_{ac}[n]$ is not.

(d) $y_p[n] = (1/3)(-1/4)^n u[n] + (2/3)(1/2)^n u[n] + 4(n+1)(1/2)^{(n+1)} u[n+1]$.

2.17. **(a)**

$$R(e^{j\omega}) = e^{-j\omega M/2} \frac{\sin\left(\omega\left(\frac{M+1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}.$$

(b) $W(e^{j\omega}) = (1/2)R(e^{j\omega}) - (1/4)R(e^{j(\omega-2\pi/M)}) - (1/4)R(e^{j(\omega+2\pi/M)})$.

2.18. Systems (a) and (b) are causal.

2.19. Systems (b), (c), (e), and (f) are stable.

2.20. **(a)** $h[n] = (-1/a)^{n-1} u[n-1]$.

(b) The system will be stable for $|a| > 1$.

Answers to Basic Problems in Chapter 3

3.1. **(a)** $\dfrac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$.

(b) $\dfrac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$.

(c) $\dfrac{-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$.

(d) $1, \quad$ all z.

(e) $z^{-1}, \quad z \neq 0$.

(f) $z, \quad |z| < \infty$.

(g) $\dfrac{1-\left(\frac{1}{2}\right)^{10} z^{-10}}{1-\frac{1}{2}z^{-1}}, \quad |z| \neq 0$.

3.2. $X(z) = \dfrac{(1-z^{-N})^2}{(1-z^{-1})^2}$.

3.3. **(a)** $X_a(z) = \dfrac{z^{-1}(\alpha - \alpha^{-1})}{(1-\alpha z^{-1})(1-\alpha^{-1}z^{-1})}$, ROC: $|\alpha| < |z| < |\alpha^{-1}|$.

(b) $X_b(z) = \dfrac{1-z^{-N}}{1-z^{-1}}$, ROC: $z \neq 0$.

(c) $X_c(z) = \dfrac{(1-z^{-N})^2}{(1-z^{-1})^2}$, ROC: $z \neq 0$.

3.4. **(a)** $(1/3) < |z| < 2$, two sided.

(b) Two sequences. $(1/3) < |z| < 2$ and $2 < |z| < 3$.

(c) No. Causal sequence has $|z| > 3$, which does not include the unit circle.

3.5. $x[n] = 2\delta[n+1] + 5\delta[n] - 4\delta[n-1] - 3\delta[n-2]$.

3.6. **(a)** $x[n] = \left(-\frac{1}{2}\right)^n u[n]$, Fourier transform exists.

(b) $x[n] = -(-\frac{1}{2})^n u[-n-1]$. Fourier transform does not exist.

(c) $x[n] = 4\left(-\frac{1}{2}\right)^n u[n] - 3\left(-\frac{1}{4}\right)^n u[n]$. Fourier transform exists.

(d) $x[n] = \left(-\frac{1}{2}\right)^n u[n]$, Fourier transform exists.

(e) $x[n] = -(a^{-(n+1)})u[n] + a^{-(n-1)}u[n-1]$. Fourier transform exists if $|a| > 1$.

3.7. **(a)** $H(z) = \dfrac{1-z^{-1}}{1+z^{-1}}$, $|z| > 1$.

(b) ROC$\{Y(z)\} = |z| > 1$.

(c) $y[n] = \left[-\frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}(-1)^n\right]u[n]$.

3.8. **(a)** $h[n] = \left(-\frac{3}{4}\right)^n u[n] - \left(-\frac{3}{4}\right)^{n-1} u[n-1]$.

(b) $y[n] = \frac{8}{13}\left(-\frac{3}{4}\right)^n u[n] - \frac{8}{13}\left(\frac{1}{3}\right)^n u[n]$.

(c) The system is stable.

3.9. **(a)** $|z| > (1/2)$.

(b) Yes. The ROC includes the unit circle.

(c) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1-2z^{-1}}$, ROC: $|z| < 2$.

(d) $h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$.

3.10. **(a)** $|z| > \frac{3}{4}$.

(b) $0 < |z| < \infty$.

(c) $|z| < 2$.

(d) $|z| > 1$.

(e) $|z| < \infty$.

(f) $\frac{1}{2} < |z| < \sqrt{13}$.

3.11. **(a)** Causal.
(b) Not causal.
(c) Causal.
(d) Not causal.

3.12. **(a)**

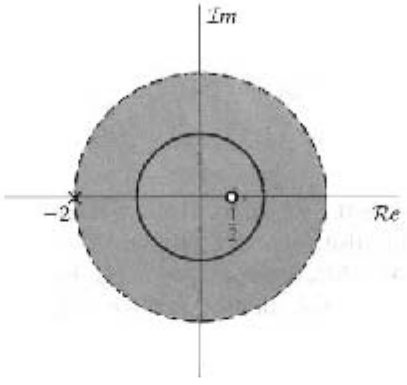


Figure P3.12

(b)

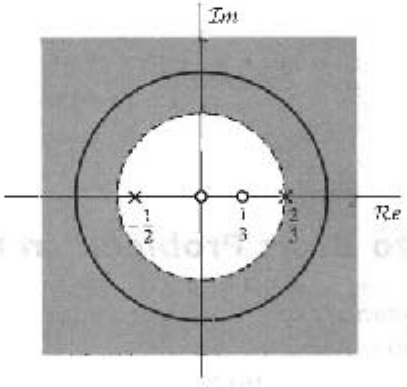


Figure P3.12

(c)

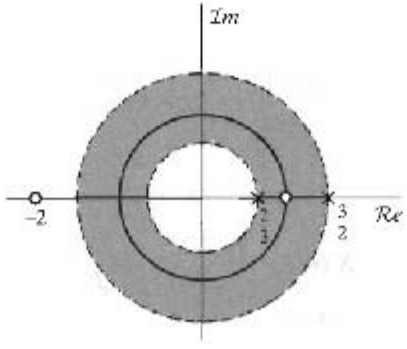


Figure P3.12

3.13. $g[11] = -\frac{1}{11!} + \frac{3}{9!} - \frac{2}{7!}$.

3.14. $A_1 = A_2 = 1/2, \quad \alpha_1 = -1/2, \quad \alpha_2 = 1/2$.

3.15. $h[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-10])$. The system is causal.

3.16. **(a)** $H(z) = \dfrac{1 - 2z^{-1}}{1 - \frac{2}{3}z^{-1}}, \quad |z| > \frac{2}{3}$.

(b) $h[n] = \left(\frac{2}{3}\right)^n u[n] - 2\left(\frac{2}{3}\right)^{(n-1)} u[n-1]$.

(c) $y[n] - \frac{2}{3}y[n-1] = x[n] - 2x[n-1]$.

(d) The system is stable and causal.

3.17. $h[0]$ can be 0, 1/3, or 1. To be painstakingly literal, $h[0]$ can also be 2/3, due to the impulse response $h[n] = (2/3)(2)^n u[n] - (1/3)(1/2)^n u[-n-1]$, which satisfies the difference equation but has no ROC. This noncausal system with no ROC can be implemented as the parallel combination of its causal and anticausal components.

3.18. **(a)** $h[n] = -2\delta[n] + \frac{1}{3}(-\frac{1}{2})^n u[n] + \frac{8}{3}u[n]$.

(b) $y[n] = \dfrac{18}{5}2^n$.

3.19. **(a)** $|z| > 1/2$.

(b) $1/3 < |z| < 2$.

(c) $|z| > 1/3$.

3.20. **(a)** $|z| > 2/3$.

(b) $|z| > 1/6$.

Answers to Basic Problems in Chapter 4

4.1. $x[n] = \sin(\pi n/2)$.

4.2. $\Omega_0 = 250\pi, 1750\pi$.

4.3. **(a)** $T = 1/12{,}000$. **(b)** Not unique. $T = 5/12{,}000$.

4.4. **(a)** $T = 1/100$. **(b)** Not unique. $T = 11/100$.

4.5. **(a)** $T \leq 1/10{,}000$. **(b)** 625 Hz. **(c)** 1250 Hz.

4.6. **(a)** $H_c(j\Omega) = 1/(a + j\Omega)$.

(b) $H_d(e^{j\omega}) = T/(1 - e^{-aT}e^{-j\omega})$.

(c) $|H_d(e^{j\omega})| = T/(1 + e^{-aT})$.

4.7. **(a)**

$$X_c(j\Omega) = S_c(j\Omega)(1 + \alpha e^{-j\Omega\tau_d}),$$

$$X(e^{j\omega}) = \left(\frac{1}{T}\right) S_c\left(\frac{j\omega}{T}\right)\left(1 + \alpha e^{-j\omega\tau_d/T}\right) \qquad \text{for } |\omega| \leq \pi.$$

(b) $H(e^{j\omega}) = 1 + \alpha e^{-j\omega\tau_d/T}$.

(c) (i) $h[n] = \delta[n] + \alpha\delta[n-1]$.

(ii) $h[n] = \delta[n] + \alpha\frac{\sin(\pi(n-1/2))}{\pi(n-1/2)}$.

4.8. **(a)** $T \leq 1/20{,}000$.

(b) $h[n] = Tu[n]$.

(c) $TX\left(e^{j\omega}\right)|_{\omega=0}$.

(d) $T \leq 1/10{,}000$.

4.9. **(a)** $X\left(e^{j(\omega+\pi)}\right) = X\left(e^{j(\omega+\pi-\pi)}\right) = X\left(e^{j\omega}\right)$.

(b) $x[3] = 0$.

(c) $x[n] = \begin{cases} y[n/2], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$

4.10. **(a)** $x[n] = \cos(2\pi n/3)$.

(b) $x[n] = -\sin(2\pi n/3)$.

(c) $x[n] = \sin(2\pi n/5)/(\pi n/5000)$.

4.11. **(a)** $T = 1/40$, $T = 9/40$.

(b) $T = 1/20$, unique.

4.12. **(a)** (i) $y_c(t) = -6\pi \sin(6\pi t)$.

(ii) $y_c(t) = -6\pi \sin(6\pi t)$.

(b) (i) Yes.

(ii) No.

4.13. **(a)** $y[n] = \sin\left(\frac{\pi n}{2} - \frac{\pi}{4}\right)$.

(b) Same $y[n]$.

(c) $h_c(t)$ has no effect on T.

4.14. **(a)** No.

(b) Yes.

(c) No.

(d) Yes.

(e) Yes. (No information is lost; however, the signal cannot be recovered by the system in Figure P3.21.)

4.15. **(a)** Yes.

(b) No.

(c) Yes.

4.16. **(a)** $M/L = 5/2$, unique.

(b) $M/L = 2/3$; unique.

4.17. **(a)** $\tilde{x}_d[n] = (4/3)\sin(\pi n/2)/(\pi n)$.

(b) $\tilde{x}_d[n] = 0$.

4.18. **(a)** $\omega_0 = 2\pi/3$.

(b) $\omega_0 = 3\pi/5$.

(c) $\omega_0 = \pi$.

4.19. $T \leq \pi/\Omega_0$.

4.20. **(a)** $F_s \geq 2000$ Hz.

(b) $F_s \geq 4000$ Hz.

Answers to Basic Problems in Chapter 5

5.1. $x[n] = y[n]$, $\omega_c = \pi$.

5.2. **(a)** Poles: $z = 3, 1/3$, Zeros: $z = 0, \infty$.

(b) $h[n] = -(3/8)(1/3)^n u[n] - (3/8)3^n u[-n-1]$.

5.3. **(a)**, **(d)** are the impulse responses.

5.4. **(a)** $H(z) = \dfrac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$, $|z| > 3/4$.

(b) $h[n] = (3/4)^n u[n] - 2(3/4)^{n-1}u[n-1]$.

(c) $y[n] - (3/4)y[n-1] = x[n] - 2x[n-1]$.

(d) Stable and causal.

5.5. **(a)** $y[n]-(7/12)y[n-1]+(1/12)y[n-2] = 3x[n]-(19/6)x[n-1]+(2/3)x[n-2]$.

(b) $h[n] = 3\delta[n] - (2/3)(1/3)^{n-1}u[n-1] - (3/4)(1/4)^{n-1}u[n-1]$.

(c) Stable.

5.6. **(a)** $X(z) = \dfrac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$, $\dfrac{1}{2} < |z| < 2$.

(b) $\frac{1}{2} < |z| < 2$.

(c) $h[n] = \delta[n] - \delta[n-2]$.

5.7. **(a)** $H(z) = \dfrac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}$, $|z| > \dfrac{3}{4}$.

(b) $h[n] = -(2/5)(1/2)^n u[n] + (7/5)(-3/4)^n u[n]$.

(c) $y[n] + (1/4)y[n-1] - (3/8)y[n-2] = x[n] - x[n-1]$.

5.8. **(a)** $H(z) = \dfrac{z^{-1}}{1-\frac{3}{2}z^{-1}-z^{-2}}$, $|z| > 2$.

(b) $h[n] = -(2/5)(-1/2)^n u[n] + (2/5)(2)^n u[n]$.

(c) $h[n] = -(2/5)(-1/2)^n u[n] - (2/5)(2)^n u[-n-1]$.

5.9.

$$h[n] = \left[-\frac{4}{3}(2)^{n-1} + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1}\right]u[-n], \qquad |z| < \frac{1}{2},$$

$$h[n] = -\frac{4}{3}(2)^{n-1}u[-n] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[n-1], \qquad \frac{1}{2} < |z| < 2,$$

$$h[n] = \frac{4}{3}(2)^{n-1}u[n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1}u[n-1], \qquad |z| > 2.$$

5.10. $H_i(z)$ cannot be causal and stable. The zero of a $H(z)$ at $z = \infty$ is a pole of $H_i(z)$. The existence of a pole at $z = \infty$ implies that the system is not causal.

5.11. **(a)** Cannot be determined.

(b) Cannot be determined.

(c) False.

(d) True.

5.12. **(a)** Stable.

(b)

$$H_1(z) = -9\frac{(1+0.2z^{-1})\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}{(1-j0.9z^{-1})(1+j0.9z^{-1})},$$

$$H_{ap}(z) = \frac{\left(z^{-1}-\frac{1}{3}\right)\left(z^{-1}+\frac{1}{3}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}.$$

5.13. $H_1(z)$, $H_3(z)$, and $H_4(z)$ are allpass systems.

5.14. **(a)** 5.

(b) $\frac{1}{2}$.

5.15. **(a)** $\alpha = 1, \beta = 0, A(e^{j\omega}) = 1 + 4\cos(\omega)$. The system is a generalized linear-phase system but not a linear-phase system, because $A(e^{j\omega})$ is not nonnegative for all ω.

(b) Not a generalized linear-phase or a linear-phase system.

(c) $\alpha = 1, \beta = 0, A(e^{j\omega}) = 3 + 2\cos(\omega)$. Linear phase, since $|H(e^{j\omega})| = A(e^{j\omega}) \geq 0$ for all ω.

(d) $\alpha = 1/2, \beta = 0, A(e^{j\omega}) = 2\cos(\omega/2)$. Generalized linear phase, because $A(e^{j\omega})$ is not nonnegative at all ω.

(e) $\alpha = 1, \beta = \pi/2, A(e^{j\omega}) = 2\sin(\omega)$. Generalized linear phase, because $\beta \neq 0$.

5.16. $h[n]$ is not necessarily causal. Both $h[n] = \delta[n-\alpha]$ and $h[n] = \delta[n+1] + \delta[n-(2\alpha+1)]$ will have this phase.

5.17. $H_2(z)$ and $H_3(z)$ are minimum-phase systems.

5.18. **(a)** $H_{min}(z) = \dfrac{2\left(1-\frac{1}{2}z^{-1}\right)}{1+\frac{1}{3}z^{-1}}$.

(b) $H_{min}(z) = 3\left(1-\dfrac{1}{2}z^{-1}\right)$.

(c) $H_{min}(z) = \dfrac{9}{4}\dfrac{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)^2}$.

5.19. $h_1[n]: 2, h_2[n]: 3/2, h_3[n]: 2, h_4[n]: 3, h_5[n]: 3, h_6[n]: 7/2$.

5.20. Systems $H_1(z)$ and $H_3(z)$ have a linear phase and can be implemented by a real-valued difference equation.

Answers to Basic Problems in Chapter 6

6.1. **Network 1**:

$$H(z) = \frac{1}{1-2r\cos\theta z^{-1}+r^2z^{-2}}.$$

Network 2:

$$H(z) = \frac{r \sin \theta z^{-1}}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}.$$

Both systems have the same denominators and thus the same poles.

6.2. $y[n] - 3y[n-1] - y[n-2] - y[n-3] = x[n] - 2x[n-1] + x[n-2]$.

6.3. The system in Part (d) is the same as that in Part (a).

6.4. (a)

$$H(z) = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

(b)

$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = 2x[n] + \frac{1}{4}x[n-1].$$

6.5. (a)

$$y[n] - 4y[n-1] + 7y[n-3] + 2y[n-4] = x[n].$$

(b)

$$H(z) = \frac{1}{1 - 4z^{-1} + 7z^{-3} + 2z^{-4}}.$$

(c) Two multiplications and four additions.

(d) No. It requires at least four delays to implement a 4^{th}-order system.

6.6.

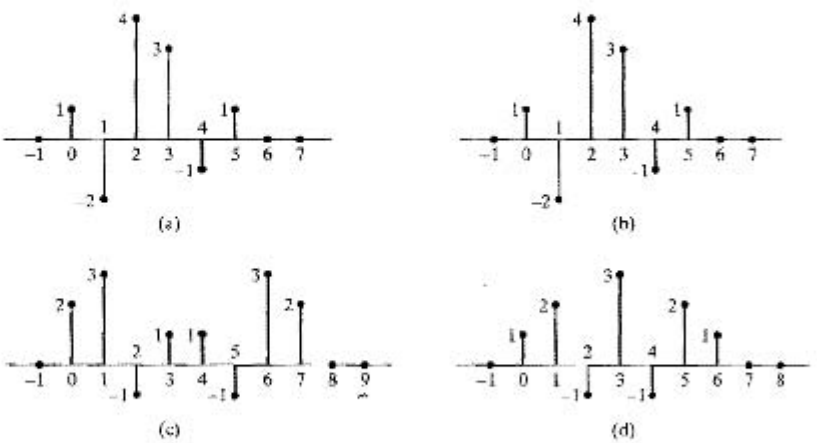


Figure P6.6

6.7.

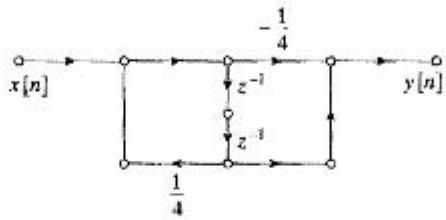


Figure P6.7

6.8. $y[n] - 2y[n-2] = 3x[n-1] + x[n-2]$.

6.9. **(a)** $h[1] = 2$.

(b) $y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3]$.

6.10. **(a)**

$$y[n] = x[n] + v[n-1].$$

$$v[n] = 2x[n] + \frac{1}{2}y[n] + w[n-1].$$

$$w[n] = x[n] + \frac{1}{2}y[n].$$

(b)

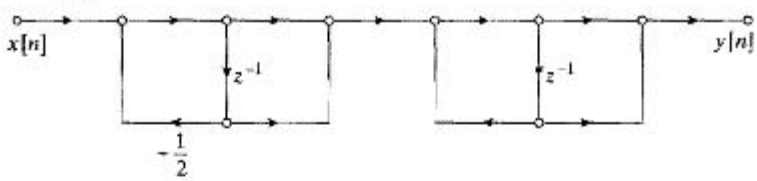


Figure P6.10

(c) The poles are at $z = -1/2$ and $z = 1$. Since the second pole is on the unit circle, the system is not stable.

6.11. **(a)**

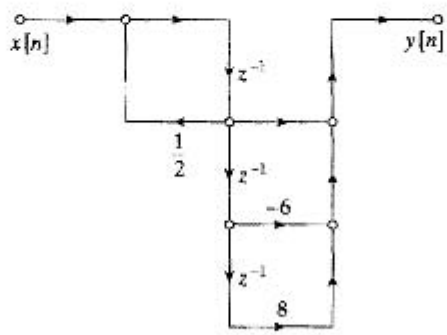


Figure P6.11

(b)

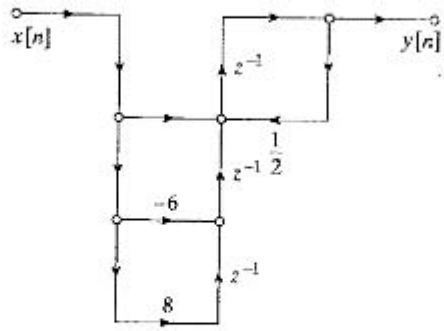


Figure P6.11

6.12. $y[n] - 8y[n-1] = -2x[n] + 6x[n-1] + 2x[n-2]$.

6.13.

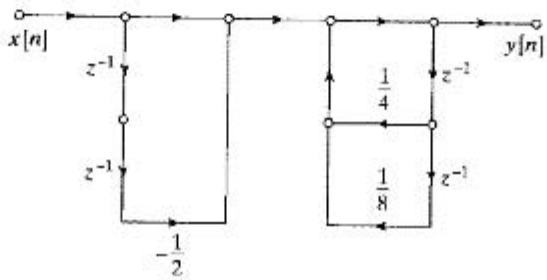


Figure P6.13

6.14.

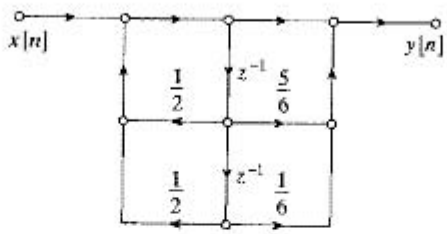


Figure P6.14

6.15.

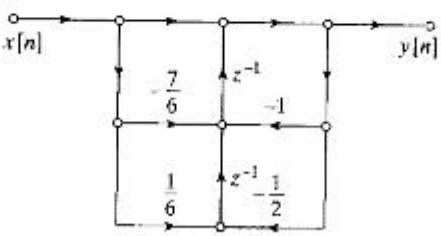


Figure P6.15

6.16. **(a)**

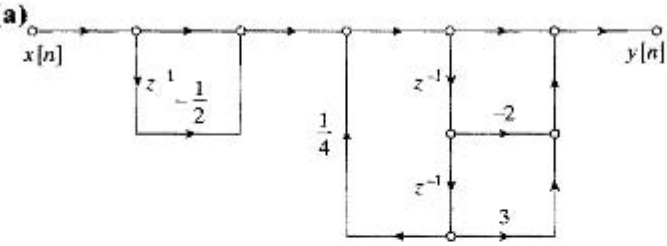


Figure P6.16

(b) Both systems have the system function

$$H(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1} + 3z^{-2})}{1 - \frac{1}{4}z^{-2}}.$$

6.17. **(a)**

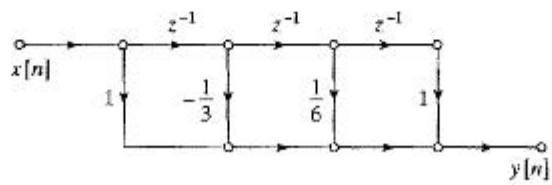


Figure P6.17-1

(b)

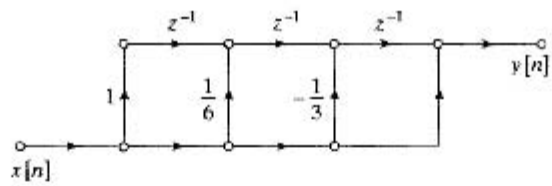


Figure P6.17-2

6.18. If $a = 2/3$, the overall system function is

$$H(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

If $a = -2$, the overall system function is

$$H(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

6.19.

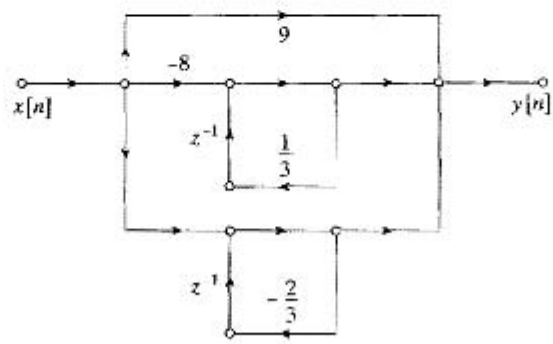


Figure P6.19

6.20.

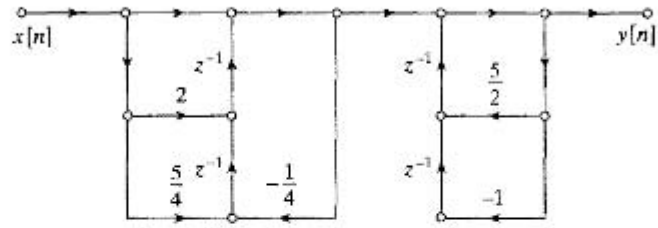


Figure P6.20

Answers to Basic Problems in Chapter 7

7.1. (a)

$$H_1(z) = \frac{1 - e^{-aT}\cos(bT)z^{-1}}{1 - 2e^{-aT}\cos(bT)z^{-1} + e^{-2aT}z^{-2}}, \quad \text{ROC: } |z| > e^{-aT}.$$

(b)

$$H_2(z) = (1 - z^{-1})S_2(z), \quad \text{ROC: } |z| > e^{-aT}, \text{ where}$$

$$S_2(z) = \frac{a}{a^2 + b^2}\frac{1}{1 - z^{-1}} - \frac{1}{2(a + jb)}\frac{1}{1 - e^{-(a+jb)T}z^{-1}} - \frac{1}{2(a - jb)}\frac{1}{1 - e^{-(a-jb)T}z^{-1}}.$$

(c) They are not equal.

7.2. (a)

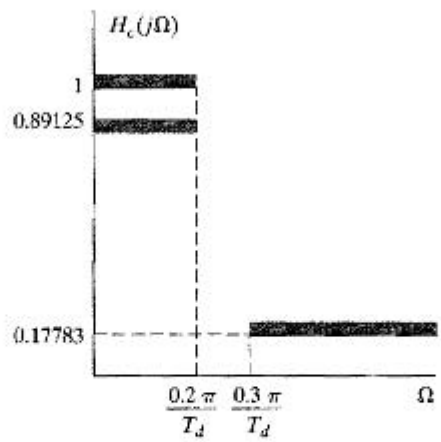


Figure P7.2

(b) $N = 6$, $\Omega_c T_d = 0.7032$.

(c) The poles in the s-plane are on a circle of radius $R = 0.7032/T_d$. They map to poles in the z-plane at $z = e^{sT_d}$. The factors of T_d cancel out, leaving the pole locations in the z-plane for $H(z)$ independent of T_d.

7.3. **(a)** $\hat{\delta}_2 = \delta_2/(1+\delta_1)$, $\hat{\delta}_1 = 2\delta_1/(1+\delta_1)$.

(b)

$$\delta_2 = 0.18806,\ \delta_1 = 0.05750$$

$$H(z) = \frac{0.3036 - 0.4723z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.2660 + 1.2114z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.9624 - 0.6665z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}$$

(c) Use the same δ_1 and δ_2.

$$H(z) = \frac{0.0007802(1+z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})(1 - 0.9044z^{-1} + 0.2155z^{-2})}.$$

7.4. **(a)**

$$H_c(s) = \frac{1}{s+0.1} - \frac{0.5}{s+0.2}.$$

The answer is not unique. Another possibility is

$$H_c(s) = \frac{1}{s+0.1+j2\pi} - \frac{0.5}{s+0.2+j2\pi}.$$

(b)

$$H_c(s) = \frac{2(1+s)}{0.1813 + 1.8187s} - \frac{1+s}{0.3297 + 1.6703s}.$$

This answer is unique.

7.5. **(a)** $M + 1 = 91$, $\beta = 3.3953$.

(b) $M/2 = 45$.

(c) $h_d[n] = \dfrac{\sin[0.625\pi(n-45)]}{\pi(n-45)} - \dfrac{\sin[0.3\pi(n-45)]}{\pi(n-45)}$.

7.6. **(a)** $\delta = 0.03$, $\beta = 2.181$.

(b) $\Delta\omega = 0.05\pi$, $M = 63$.

7.7.

$$0.99 \le |H(e^{j\omega})| \le 1.01, \qquad |\omega| \le 0.2\pi,$$

$$|H(e^{j\omega})| \le 0.01, \qquad 0.22\pi \le |\omega| \le \pi$$

7.8. **(a)** Six alternations. $L = 5$, so this does not satisfy the alternation theorem and is not optimal.

(b) Seven alternations, which satisfies the alternation theorem for $L = 5$.

7.9. $\omega_c = 0.4\pi$.

7.10. $\omega_c = 2.3842$ rad.

7.11. $\Omega_c = 2\pi(1250)$ rad/sec.

7.12. $\Omega_c = 2000$ rad/sec.

7.13. $T = 50\,\mu$s. This T is unique.

7.14. $T = 1.46$ ms. This T is unique.

7.15. Hamming and Hanning: $M+1=81$. Blackman: $M+1=121$.

7.16. $\beta = 2.6524$, $M = 181$.

7.17.

$$\begin{array}{rl} |H_c(j\Omega)| < 0.02, & |\Omega| \le 2\pi(20) \text{ rad/sec}, \\ 0.95 < |H_c(j\Omega)| < 1.05, & 2\pi(30) \le |\Omega| \le 2\pi(70) \text{ rad/sec}, \\ |H_c(j\Omega)| < 0.001, & 2\pi(75) \text{ rad/sec} \le |\Omega|. \end{array}$$

7.18.

$$\begin{array}{rl} |H_c(j\Omega)| < 0.04, & |\Omega| \le 324.91 \text{ rad/sec}, \\ 0.995 < |H_c(j\Omega)| < 1.005, & |\Omega| \ge 509.52 \text{ rad/sec}. \end{array}$$

7.19. $T = 0.41667$ ms. This T is unique.

7.20. True.

Answers to Basic Problems in Chapter 8

8.1. **(a)** $x[n]$ is periodic with period $N = 6$.

(b) T will not avoid aliasing.

(c)

$$\tilde{X}[k] = 2\pi \begin{cases} a_0 + a_6 + a_{-6}, & k = 0, \\ a_1 + a_7 + a_{-5}, & k = 1, \\ a_2 + a_8 + a_{-4}, & k = 2, \\ a_3 + a_9 + a_{-3} + a_{-9}, & k = 3, \\ a_4 + a_{-2} + a_{-8}, & k = 4, \\ a_5 + a_{-1} + a_{-7}, & k = 5. \end{cases}$$

8.2. **(a)**

$$\tilde{X}_3[k] = \begin{cases} 3\tilde{X}[k/3], & \text{for } k = 3\ell, \\ 0, & \text{otherwise}. \end{cases}$$

(b)

$$\tilde{X}[k] = \begin{cases} 3, & k = 0, \\ -1, & k = 1. \end{cases}$$

$$\tilde{X}_3[k] = \begin{cases} 9, & k = 0, \\ 0, & k = 1, 2, 4, 5, \\ -3, & k = 3. \end{cases}$$

8.3. **(a)** $\tilde{x}_2[n]$.

(b) None of the sequences.

(c) $\tilde{x}_1[n]$ and $\tilde{x}_3[n]$.

8.4. (a)

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}.$$

(b)

$$\tilde{X}[k] = \frac{1}{1 - \alpha e^{-j(2\pi/N)k}}.$$

(c)

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi k/N)}.$$

8.5. (a) $X[k] = 1$.

(b) $X[k] = W_N^{kn_0}$.

(c)

$$X[k] = \begin{cases} N/2, & k = 0, N/2, \\ 0, & \text{otherwise.} \end{cases}$$

(d)

$$X[k] = \begin{cases} N/2, & k = 0, \\ e^{-j(\pi k/N)(N/2-1)}(-1)^{(k-1)/2}\frac{1}{\sin(k\pi/N)}, & k \text{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$

(e)

$$X[k] = \frac{1 - a^N}{1 - aW_N^k}.$$

8.6. (a)

$$X(e^{j\omega}) = \frac{1 - e^{j(\omega_0 - \omega)N}}{1 - e^{j(\omega_0 - \omega)}}.$$

(b)

$$X[k] = \frac{1 - e^{j\omega_0 N}}{1 - e^{j\omega_0} W_N^k}.$$

(c)

$$X[k] = \begin{cases} N, & k = k_0 \\ 0, & \text{otherwise.} \end{cases}$$

8.7.

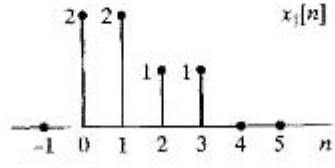


Figure P8.7

8.8.

$$y[n] = \begin{cases} \frac{1024}{1023}\left(\frac{1}{2}\right)^n, & 0 \le n \le 9, \\ 0, & \text{otherwise.} \end{cases}$$

8.9. **(a)** **1.** Let $x_1[n] = \sum_m x[n + 5m]$ for $n = 0, 1, \ldots 4$.

2. Let $X_1[k]$ be the five-point FFT of $x_1[n]$. $M = 5$.

3. $X_1[2]$ is $X(e^{j\omega})$ at $\omega = 4\pi/5$.

(b) Define $x_2[n] = \Sigma_m W_{27}^{-(n+9m)} x[n + 9m]$ for $n = 0, \ldots, 8$. Compute $X_2[k]$, the 9-point DFT of $x_2[n]$. $X_2[2] = X(e^{j\omega})\big|_{\omega=10\pi/27}$.

8.10. $X_2[k] = (-1)^k X_1[k]$.

8.11.

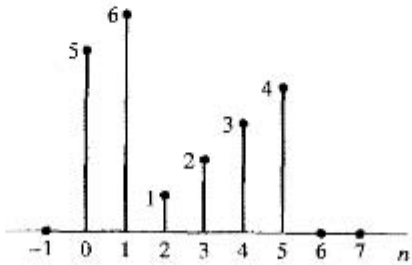


Figure P8.11

8.12. **(a)**

$$X[k] = \begin{cases} 2, & k = 1, 3, \\ 0, & k = 0, 2. \end{cases}$$

(b)

$$H[k] = \begin{cases} 15, & k = 0, \\ -3 + j6, & k = 1, \\ -5, & k = 2, \\ -3 - j6, & k = 3. \end{cases}$$

(c) $y[n] = -3\delta[n] - 6\delta[n - 1] + 3\delta[n - 2] + 6\delta[n - 3]$.

(d) $y[n] = -3\delta[n] - 6\delta[n - 1] + 3\delta[n - 2] + 6\delta[n - 3]$.

8.13.

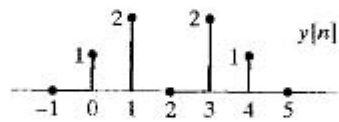


Figure P8.13

8.14. $x_3[2] = 9$.

8.15. $a = -1$. This is unique.

8.16. $b = 3$. This is unique.

8.17. $N = 9$.

8.18. $c = 2$.

8.19. $m = 2$. This is not unique. Any $m = 2 + 6\ell$ for integer ℓ works.

8.20. $N = 5$. This is unique.

Answers to Basic Problems in Chapter 9

9.1. If the input is $(1/N)X[((-n))_N]$, the output of the DFT program will be $x[n]$, the IDFT of $X[k]$.

9.2.

$$X = AD - BD + CA - DA = AC - BD$$
$$Y = AD - BD + BC + BD = BC + AD.$$

9.3.

$$y[32] = X(e^{-j2\pi(7/32)}) = X(e^{j2\pi(25/32)}).$$

9.4. $\omega_k = 7\pi/16$.

9.5.

$$a = -\sqrt{2}$$
$$b = -e^{-j(6\pi/8)}.$$

9.6. **(a)** The gain is $-W_N^2$.

(b) There is one path. In general, there is only one path from any input sample to any output sample.

(c) By tracing paths, we see

$$X[2] = x[0] \cdot 1 + x[1]W_8^2 - x[2] - x[3]W_8^2 + \ldots$$
$$x[4] + x[5]W_8^2 - x[6] - x[7]W_8^2.$$

9.7. **(a)** Store $x[n]$ in $A[\cdot]$ in bit-reversed order, and $D[\cdot]$ will contain $X[k]$ in sequential (normal) order.

(b)

$$D[r] = \begin{cases} 8, & r = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(c)

$$C[r] = \begin{cases} 1, & r = 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

9.8. **(a)** $N/2$ butterflies with $2^{(m-1)}$ different coefficients.

(b) $y[n] = W_N^{2^{v-m}} y[n-1] + x[n]$.

(c) Period: 2^m, Frequency: $2\pi 2^{-m}$.

9.9. Statement 1.

9.10.

$$y[n] = X(e^{j\omega})|_{\omega=(2\pi/7)+(2\pi/21)(n-19)}.$$

9.11. **(a)** 2^{m-1}.

(b) 2^m.

9.12. $r[n] = e^{-j(2\pi/19)n} W^{n^2/2}$ where $W = e^{-j(2\pi/10)}$.

9.13. $x[0], x[8], x[4], x[12], x[2], x[10], x[6], x[14], x[1], x[9], x[5], x[13], x[3], x[11], x[7], x[15]$.

9.14. False.

9.15. $m = 1$.

9.16.

$$r = \begin{cases} 0, & m = 1, \\ 0, 4, & m = 2, \\ 0, 2, 4, 6, & m = 3, \\ 0, 1, 2, 3, 4, 5, 6, 7, & m = 4. \end{cases}$$

9.17. $N = 64$.

9.18. $m = 3$ or 4.

9.19. Decimation-in-time.

9.20. 1021 is prime, so the program must implement the full DFT equations and cannot exploit any FFT algorithm. The computation time goes as N^2. Contrastingly, 1024 is a power of 2 and can exploit the $N \log N$ computation time of the FFT.

Answers to Basic Problems in Chapter 10

10.1. **(a)** $f = 1500$ Hz.

(b) $f = -2000$ Hz.

10.2. $N = 2048$ and $10000 \text{ Hz} < f < 10240$ Hz.

10.3. **(a)** $T = 2\pi k_0/(N\Omega_0)$.

(b) Not unique. $T = (2\pi/\Omega_0)(1 - k_0/N)$.

10.4.

$$X_c(j2\pi(4200)) = 5 \times 10^{-4}$$
$$X_c(-j2\pi(4200)) = 5 \times 10^{-4}$$
$$X_c(j2\pi(1000)) = 10^{-4}$$
$$X_c(-j2\pi(1000)) = 10^{-4}$$

10.5. $L = 1024$.

10.6. $x_2[n]$ will have two distinct peaks.

10.7. $\Delta\Omega = 2\pi(2.44)$ rad/sec.

10.8. $N \geq 1600$.

10.9.

$$X_0[k] = \begin{cases} 18, & k = 3, 33, \\ 0, & \text{otherwise.} \end{cases}$$

$$X_1[k] = \begin{cases} 18, & k = 9, 27, \\ 0, & \text{otherwise.} \end{cases}$$

$$X_r[k] = 0 \text{ for } r \neq 0, 1.$$

10.10. $\omega_0 = 0.25\pi$ rad/sample, $\lambda = \pi/76000$ rad/sample^2.

10.11. $\Delta f = 9.77$ Hz.

10.12. The peaks will not have the same height. The peak from the rectangular window will be bigger.

10.13. **(a)** $A = 21$ dB.

(b) Weak components will be visible if their amplitude exceeds 0.0891.

10.14. **(a)** 320 samples.

(b) 400 DFT/second.

(c) $N = 256$.

(d) 62.5 Hz.

10.15. **(a)** $X[200] = 1 - j$.

(b)

$$X(j2\pi(4000)) = 5 \times 10^{-5}(1 - j)$$

$$X(-j2\pi(4000)) = 5 \times 10^{-5}(1 + j).$$

10.16. Rectangular, Hanning, Hamming, and Bartlett windows work.

10.17. $T > 1/1024$ sec.

10.18. $x_2[n]$, $x_3[n]$, $x_6[n]$.

10.19. Methods 2 and 5 will improve the resolution.

10.20. $L = M + 1 = 262$.