Measurement

CHAPTER 1

1–1 Measurement

When plans were being made to lay the first Atlantic telegraph cable, the company in charge of the construction hired a young engineer, William Thomson (1824–1907), as a consultant. To solve some of the problems raised by this undertaking, Thomson made many accurate electrical measurements. Often he used instruments which he himself had invented. His advice, based on his own experiments, was ignored, chiefly because the principles involved were not clearly understood or accepted by those in authority. The subsequent failures of the project later led to a more careful consideration of Thomson's views. Their adoption led to the successful completion of the cable in 1858.* This experience may have helped Thomson form his often quoted view:

> I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of Science, whatever the matter may be.

Although other scientists would deny that they should deal only with ideas that are strictly measurable, none would deny the great importance of measurement to science. Often in the history of science small but significant discrepancies between theory and accurate measurements have led

* In 1892, Thomson, then one of Britain's foremost scientists, was raised to the peerage as Lord Kelvin. Among his other achievements, he was one of the founders of the science of thermodynamics.

to the development of new and more general theories. Such advances in our understanding would not have occurred if scientists had been satisfied with only a qualitative explanation of the phenomena of nature.

1–2 Physical Quantities, Standards, and Units

The building blocks of physics are the physical quantities in terms of which the laws of physics are expressed. Among these are force, time, velocity, density, temperature, charge, magnetic susceptibility, and numerous others. Many of these terms, such as force and temperature, are part of our everyday vocabulary. When these terms are so used, their meanings may be vague or may differ from their scientific meanings.

For the purposes of physics the basic quantities must be defined clearly and precisely. One view is that the definition of a physical quantity has been given when the procedures for measuring that quantity have been given. This is called the *operational point of view* because the definition is, at root, a set of laboratory operations leading to a number with a unit. The operations may include mathematical calculations.

Physical quantities are often divided into *fundamental quantities* and *derived quantities.* Such a division is arbitrary in that a given quantity can be regarded as fundamental in one set of operations and as derived in another. Derived quantities are those whose defining operations are based on other physical quantities. Examples of quantities usually viewed as derived are velocity, acceleration, and volume. Fundamental quantities are not defined in terms of other physical quantities. The number of quantities regarded as fundamental is the minimum number needed to give a consistent and unambiguous description of all the quantities of physics. Examples of quantities usually viewed as fundamental are length and time. Their operational definitions involve two steps: first, the choice of a *standard*, and second, the establishment of procedures for comparing the standard to the quantity to be measured so that a number and a unit are determined as the measure of that quantity.

An ideal standard has two principal characteristics: it is accessible and it is invariable. These two requirements are often incompatible and a compromise has to be made between them. At first greater emphasis was placed on accessibility, but the growing requirements of science and technology introduced the need for greater invariability. The familiar yard, foot, and inch, for example, are descended directly from the human arm, foot, and upper thumb. Today, such rough measures of length are not satisfactory and a much less variable standard must be used even at the expense of accessibility.

Suppose that we have chosen our standard of length to be a bar whose length we *define* as one meter. If by direct comparison of this bar with a second bar we conclude that the second bar is three times as long as the standard, we say that the second bar has a length of three meters. In practice, most quantities cannot be measured by direct comparison to a primary standard. An indirect approach, using more involved procedures, is usually necessary. Certain assumptions are made to relate the results of an indirect measurement to the direct operation.

Suppose, for example, that the distance from a rocket launching station to the surface of the moon must be known at a certain time. One indirect way to determine this distance would be to send out a radar signal from the station which will be reflected from the surface of the moon back to a receiver at the sending station. If the time between sending and receiving the signal is measured and the speed of the radar signal is known, the distance can be obtained as the product of the speed and one-half the time interval. We assume here that the speed of the signal is constant and that the signal has traveled in a straight line. The speed must be measured in a subsidiary experiment, and it is here that the standard of length appears in the operational procedure.

Astronomical distances, such as the distances of stars from the earth, cannot be measured in a direct way. A few stars are close enough so that triangulation measurements can be made. The position of the star with respect to the background of much more distant stars is observed at six-month intervals, when the earth has moved from one point of its orbit to a diametrically opposite point. From these data the desired distance can be obtained using the diameter of the earth's orbit as a baseline. Distances of nebulae many millions of light years from the earth are measured by indirect procedures more involved than triangulation (one light year is approximately 10^{16} meters; see Problem 6).

Just as we use indirect methods for measuring large distances, so we must also use an indirect approach to measure very small distances, such as those within atoms and molecules. The effective radius of the proton, for example, has been measured by particle scattering experiments to be 1.2×10^{-15} meter. Table 1-1 shows the vast range over which length measurements can be made.

1-3 Reference Frames

The same physical quantity may have different values if it is measured by observers who are moving with respect to each other. The velocity of a train has one value if measured by an observer on the ground, a different value if measured from a speeding car, and the value zero if measured by an observer sitting in the train itself. None of these values has any fundamental advantage over any other; each is equally "correct" from the point of view of the observer making the measurement.

In general, the measured value of a physical quantity depends on the reference frame of the observer who is making the measurement. This is clear enough if the physical quantity is a velocity, as above. It is also true, however, if the physical quantity is, say, a displacement of a particle, a time interval between two events, an electric field, or a magnetic field, although a full appreciation of these four examples must await the study of the theory of relativity.

In the early days of physics it was believed that one particular reference

Table 1–1

SOME MEASURED LENGTHS

	Meters
Distance to the most-distant quasar yet detected[1] (1964)	6×10^{25}
Distance to the nearest nebula (Great Nebula in Andromeda)	2×10^{22}
Radius of our galaxy	6×10^{19}
Distance to the nearest star (Alpha Centauri)	4.3×10^{16}
Mean orbit radius for the most distant planet (Pluto)	5.9×10^{12}
Radius of the sun	6.9×10^{8}
Radius of the earth	6.4×10^{6}
Highest free balloon ascension (1959)	4.6×10^{4}
Height of a man	1.8×10^{0}
Thickness of this book (Part I)	4×10^{-2}
Thickness of a page in this book	1×10^{-4}
Size of a poliomyelitis virus	1.2×10^{-8}
Radius of a hydrogen atom	5.0×10^{-11}
Effective radius of a proton	1.2×10^{-15}

[1] *quasar* = quasi-stellar radio source.

frame, a so-called absolute frame, existed that had some fundamental advantage over all other frames. For an observer at rest in such a frame physical quantities would have their "true" or "absolute" values. This viewpoint has now been abandoned because, over many decades, experimental efforts to find this absolute reference frame have failed completely.

Consider reference frames moving with uniform velocity with respect to each other and with respect to the fixed stars. Such (unaccelerated, nonrotating) reference frames are called *inertial reference frames.* Experiment shows that all inertial reference frames are equivalent for the measurement of physical phenomena. Observers in different frames may obtain different numerical values for measured physical quantities, but *the relationships between the measured quantities, that is, the laws of physics, will be the same for all observers.*

Suppose, for example, that observers in different inertial frames measure the momenta of the particles involved in an atomic collision. They will obtain different numerical values both for the momenta of the individual particles and for the total momentum of the system of particles. Each observer, however, will note that the total momentum of the system of particles, whatever value he measured it to be, is the same after the collision as before. In other words, each observer will note that the collision obeys the *law of conservation of momentum;* we shall discuss this law in detail in Chapter 9.

Although physical laws are the same in all reference frames, the measured values of the physical quantities, as we have seen, may not be. It is important, therefore, that the student always realize what his reference frame is in a particular problem.

1-4 Standard of Length*

The first truly international standard of length was a bar of a platinum-iridium alloy called the *standard meter*, kept at the International Bureau of Weights and Measures near Paris, France. The distance between two fine lines engraved on gold plugs near the ends of the bar (when the bar was at 0.00° C and supported mechanically in a prescribed way) was defined to be *one meter*. Historically, the meter was intended to be a convenient fraction (one ten-millionth) of a distance from pole to equator along the meridian line through Paris. However, accurate measurements taken after the standard meter bar was constructed show that it differs slightly (about 0.023%) from its intended value.

Because the standard meter was not very accessible, accurate master copies of it were made and sent to standardizing laboratories throughout the civilized world. These secondary standards were used to calibrate other, still more accessible, measuring rods. Thus until recently every ruler, micrometer, or vernier caliper derived its legal authority from the standard meter through a complicated chain of comparisons using microscopes and dividing engines. This statement was also true for the *yard* used in English-speaking countries. Since 1959 one yard has been defined, by international agreement, to be

$$1 \text{ yard} = 0.9144 \text{ meter, exactly,}$$

which is equivalent to

$$1 \text{ in.} = 2.54 \text{ cm, exactly.}$$

There are several objections to the meter bar as the primary standard of length: It is potentially destructible, by fire or war, for example; it is not accurately reproducible; it is not very accessible. Most important, the accuracy with which the necessary intercomparisons of length can be made by the technique of comparing fine scratches, using a microscope, is no longer great enough to meet modern requirements of science and technology. The maximum accuracy obtainable with the standard meter as a reference is about 1 part in 10^7; an error of this amount in the borehole of a guidance gyroscope could cause a space shot aimed at the moon to miss by a thousand miles.

The suggestion that the length of a light wave be used as a length standard was first made in 1864 by Hippolyte Louis Fizeau (1819-1896). The later development of the *interferometer* (see Chapter 43) provided scientists with a precision optical device in which light waves can be used as a length

---

* See "The Metre" by H. Barrell, in *Contemporary Physics*, Vol. 3, p. 415, 1962, for an excellent discussion of the standard of length.

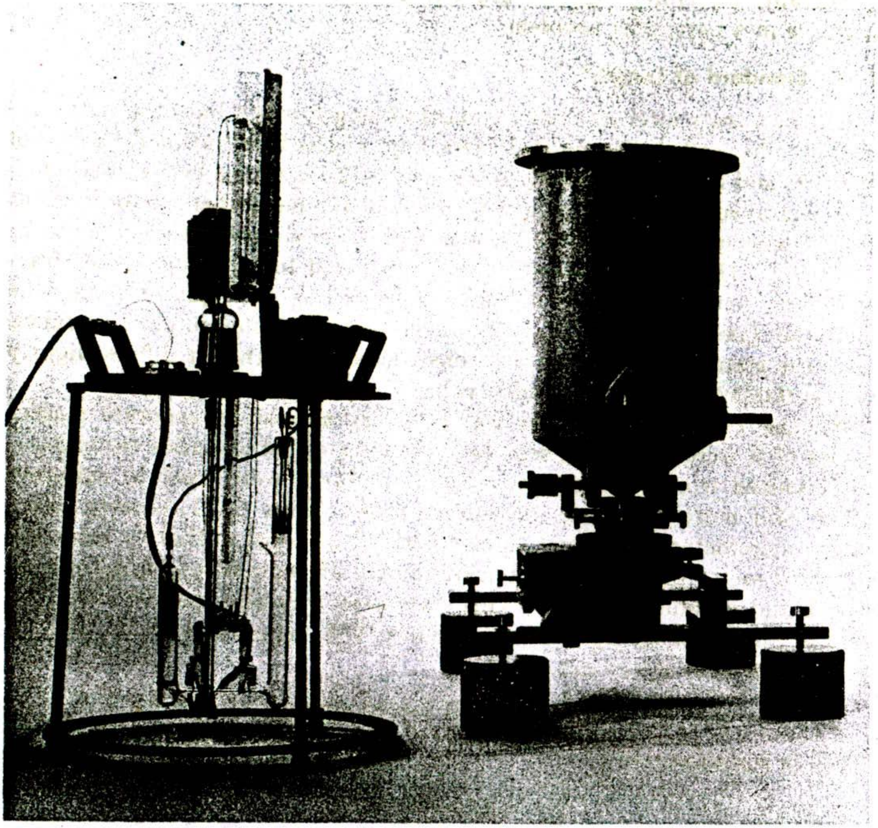


Fig. 1–1 A Kr^{86} light source shown removed from the container in which it is housed. In operation the lamp is cooled with liquid nitrogen. (Courtesy the National Physical Laboratories, Teddington, England. Crown copyright reserved.)

comparison probe. Light waves are about 5×10^{-5} cm long and length measurements of bars some centimeters long can be made to a very small fraction of a wavelength. An accuracy of 1 part in 10^9 in the intercomparison of lengths using light waves is inherently possible. As the need for this increased accuracy in length comparisons arose, efforts were made to determine the best light source.

In 1961 an atomic standard of length was adopted by international agreement. The wavelength in vacuum of a particular orange radiation (identified by the spectroscopic notation $2p_{10} - 5d_5$) emitted by atoms of a particular isotope of krypton (Kr^{86}) in an electrical discharge was chosen. Specifically, one meter is now defined to be 1,650,763.73 wavelengths of this light. This number of wavelengths was arrived at by carefully measuring the length of the standard meter bar in terms of these light waves. This comparison was done so that the new standard, based on the wave length of light, would be as consistent as possible with the old standard, based on the meter bar. Figure 1–1 shows a krypton-86 light source, used as the basis of the length standard.

The choice of an atomic standard offers advantages other than increased precision in length measurements. The atoms that generate light are available everywhere and all atoms of a given species are identical and emit light of the same wavelength. Hence such an atomic standard is both accessible and invariable. The particular wavelength chosen is uniquely characteristic of krypton-86 and is very sharply defined. This isotope can be obtained with great purity relatively easily and cheaply.

1–5 Standard of Time

The measurement of time has two different aspects. For civil and for some scientific purposes, we want to know the time of day so that we can order events in sequence. In most scientific work, we want to know how long an event lasts. Alternatively, if we are dealing with an oscillating system such as a microwave oscillator or an acoustic resonator, we want to know its frequency of oscillation. Thus any time standard must be able to answer both the question "What time is it?" and the two related questions "How long does it last?" or "What is its frequency?"* Table 1–2 shows the wide range of time intervals that can be measured.

Any phenomenon that repeats itself can be used as a measure of time;

* See "Accurate Measurement of Time" by Louis Essen, in *Physics Today*, July 1960, for an excellent discussion of the standard of time.

Table 1–2

SOME MEASURED TIME INTERVALS

	Seconds
Age of the earth	1.3×10^{17}
Age of the pyramid of Cheops	1.5×10^{11}
Human life expectancy (USA)	2×10^{9}
Time of earth's orbit around the sun (1 year)	3.1×10^{7}
Time of earth's rotation about its axis (1 day)	8.6×10^{4}
Period of the Echo II satellite	5.1×10^{3}
Half-life of the free neutron	7.0×10^{2}
Time between normal heartbeats	8.0×10^{-1}
Period of concert-A tuning fork	2.3×10^{-3}
Half-life of the muon	2.2×10^{-6}
Period of oscillation of 3-cm microwaves	1.0×10^{-10}
Typical period of rotation of a molecule	1×10^{-12}
Half-life of the neutral pion	2.2×10^{-16}
Period of oscillation of a 1-Mev gamma ray (calculated)	4×10^{-21}
Time for a fast elementary particle to pass through a medium-sized nucleus (calculated)	2×10^{-23}

the measurement consists of counting the repetitions. An oscillating pendulum, coiled spring, or quartz crystal can be used, for example. Of the many repetitive phenomena occurring in nature, the rotation of the earth on its axis, which determines the length of the day, has been used as a time standard from earliest times. It is still the basis of our civil and legal time standard, one (mean solar) second being defined to be 1/86,400 of a (mean solar) day. Time defined in terms of the rotation of the earth is called *universal time* (*UT*).

In 1956, for reasons that will follow, the International Congress of Weights and Measures redefined the second, for scientific purposes requiring high precision, in terms of the earth's orbital motion about the sun. More particularly, they defined the second to be the fraction 1/31,556,925.9747 of the tropical year 1900; the selection of a particular earth orbit in the definition automatically makes the time standard *invariable*. Time defined in terms of the earth's orbital motion is called *ephemeris time* (*ET*).

Both *UT* and *ET* must be determined by astronomical observations. Since these observations must be extended over several weeks (for *UT*) or several years (for *ET*), a good secondary terrestrial clock, calibrated by the astronomical observations, is needed. *Quartz crystal clocks*, based on the electrically sustained natural periodic vibrations of a quartz wafer, serve

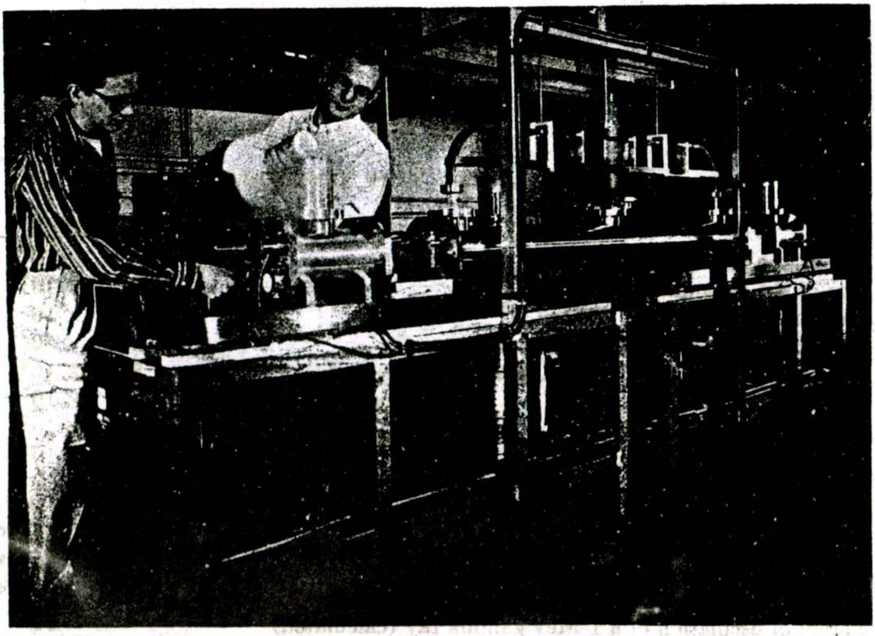


Fig. 1–2 This cesium atomic clock at the Boulder Laboratories of the National Bureau of Standards measures frequency and time intervals to an accuracy equivalent to the loss of less than 1 sec in 3000 years.

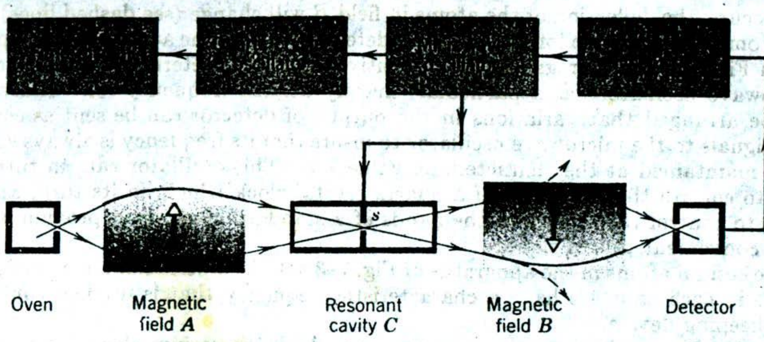


Fig. 1–3 A schematic diagram of a cesium atomic clock. The vertical arrows in the shaded magnetic field areas point from the strong field to the weak field region of the (nonuniform) magnetic field, as the variable shading also suggests.

well as secondary time standards. The best of these have kept time for a year with a maximum error of 0.02 sec.

One of the most common uses of a time standard is the determination of frequencies. In the radio range, frequency comparisons to a quartz clock can be made electronically to a precision of at least 1 part in 10^{10} and, indeed, many situations require such precision. However, this precision is about one hundred times greater than that with which a quartz clock itself can be calibrated by astronomical observations. To meet the need for a better time standard, *atomic clocks* have been developed in several countries, using periodic atomic vibrations as a standard.

A particular type of atomic clock, based on a characteristic frequency associated with the cesium atom, has been in continuous operation at the National Physical Laboratory in England since 1955. Figure 1–2 shows a similar clock at the U. S. Bureau of Standards.

A cesium atom behaves like a tiny magnet. It experiences a sideways deflection as it moves through a nonuniform magnetic field; the amount and the direction of the deflection depend on the strength of this magnet and on the orientation of the axis of the magnet in this field.

In a cesium atomic clock, the oven in Fig. 1–3 serves as a source of cesium atoms, which enter and are deflected by nonuniform magnetic field *A*. The atoms then pass through slit *S* located in the center of a resonating cavity *C* and enter nonuniform magnetic field *B*. If no change in the effective magnetic strength of the atoms occurs while they pass through the cavity, the field *B* just cancels out the deflections produced by field *A* and the moving atoms strike the detector.

If the cavity *C* is filled with radiation produced by a microwave oscillator and *if this radiation has a sharply defined critical frequency* ν_{cs}, the cesium atoms may change their effective magnetic strength as they pass through the cavity.* If

---

* An atom can exist in a number of discrete configurations, or stationary states, each with a well-defined energy. The atom can be induced to change from one of these states to another by irradiating it with, or by stimulating it to emit, light waves or other radiations with certain sharply defined frequencies. Radiations with frequencies that do not belong to this discrete set will, in general, have no effect. When such transitions between configurations occur, many properties of the atom, among them its effective magnetic strength, may change.

this occurs, the deflection of the atoms in field B will change (see dashed lines) and the atomic beam will no longer strike the detector. Thus the atomic beam apparatus in Fig. 1–3 can be regarded as a sensitive device for determining whether the microwave oscillator has a particular, sharply defined frequency ν_{Cs}. Indeed, it can be arranged that variations in the output of detector can be sent as correction signals to the microwave oscillator to insure that its frequency is always accurately maintained at the characteristic value ν_{Cs}. This oscillator can, in turn, be used to control the frequency of a quartz crystal clock which, in its turn, can be made to control the motion of the hands of a standard clock or to provide other, more convenient timing signals.

The cesium atoms in the apparatus of Fig. 1–3 act like a pendulum in a pendulum clock; in each case, we have a characteristic frequency that is used to control a time-keeping device.

The fundamental atomic frequency ν_{Cs} on which the cesium clock is based has been measured in terms of the standard second defined in terms of the earth's orbital motion as: $\nu_{Cs} = 9{,}192{,}631{,}700 \pm 20$ vibrations/sec of ephemeris time, the particular earth orbit being the tropical year 1957.

Figure 1–4 shows, by comparison with the cesium clock, variations in the rate of rotation of the earth over nearly a three-year period. Note that the earth's rotation rate is high in summer and low in winter (northern hemisphere) and exhibits a steady decrease from year to year. It is because of this variability of the earth's rotation, pointed up so sharply in Fig. 1–4 but also known from astronomical observations, that UT was replaced by ET for precise scientific work.

In connection with Fig. 1–4, it is legitimate to ask how we can be sure that the rotating earth and not the cesium clock is "at fault." There are two answers. (1) The relative simplicity of the atom compared to the earth leads us to ascribe any differences between the two as timekeepers to physical phenomena on the earth. Tidal friction between the water and the land, for example, causes a slowing down of the earth's rotation. Also the seasonal motion of the winds introduces a regular seasonal variation in the rotation. Other variations may be associated with the melting of polar icecaps and shifts of other earth masses. (2) The solar system contains other timekeepers, such as the orbiting planets and the orbiting moons of the planets; the rotation of the earth shows variations with respect to these, too, which are similar to, but less accurately observable than, the variation exhibited in Fig. 1–4.

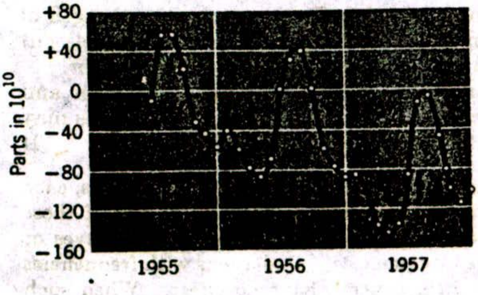


Fig. 1–4 Variation in the rate of rotation of the earth as revealed by comparison with a cesium clock. (Adapted from L. Essen, *Physics Today*, July 1960.)

The time standard can be made available at remote locations by radio transmission. Many countries maintain radio stations for this purpose. Station WWV, located in Beltsville, Maryland, and operated by the National Bureau of Standards, is one of these. It broadcasts on carrier frequencies of 2.5, 5, 10, 15, and 25×10^6 cycles/sec, stabilized to 1 part in 10^{10} by comparison to a cesium clock. At 5-min intervals, WWV alternately broadcasts an accurate 440-cycle/sec tone (concert A) and a 600-cycle/sec tone. Ten times per hour it broadcasts time signals, using a binary digit coding system; the signals are based on the earth's rotation, that is, they refer to universal time. Corrections are made for the wandering of the earth's axis and the annual variation in the earth's rotational speed.

In 1964, the second based on the cesium clock was *temporarily* adopted as an international standard by the Twelfth General Conference of Weights and Measures meeting in Paris. The action increases the accuracy of time measurements to 1 part in 10^{11}, an improvement over the accuracy associated with astronomical methods of about 200. If two cesium clocks are operated at this precision, and if there are no other sources of error, the clocks will differ by only one second after running 5000 years.

Atomic clocks are still in a phase of rapid development as of 1965 and it is for this reason that the "cesium second" was adopted only temporarily. For example, the hydrogen maser gives promise of producing a clock with an error of only one second in 33,000,000 years.

1–6 Systems of Units

As already pointed out, there is a certain amount of arbitrariness in the choice of the fundamental quantities.* For example, length, time, and mass can be chosen as fundamental quantities; all other mechanical quantities, such as force, torque, density, etc., can be expressed in terms of these fundamental quantities. However, we might equally well choose force instead of mass as a fundamental quantity. However, having picked the fundamental quantities and determined units for them, we thereby automatically determine the units of the derived quantities.

Three different systems of units are most commonly used in science and engineering. They are the *m*eter-*k*ilogram-*s*econd or mks system, the Gaussian system, in which the fundamental mechanical units are the *c*entimeter, the *g*ram, and the *s*econd (a cgs system), and the British engineering system (a *f*oot-*p*ound-*s*econd or fps system). The gram and kilogram are mass units, and the pound is a force unit; these will be defined and discussed in Chapter 5.

We shall use the mks system principally throughout the text, except in mechanics where the fps system will also be used. The metric system is used universally in scientific work and provides the common units of commerce in most countries of the world.

---

* See "Dimensions, Units, and Standards" by A. G. McNish, in *Physics Today*, April 1957.

Table 1–3

PREFIXES USED FOR MULTIPLES AND SUBMULTIPLES OF METRIC QUANTITIES

10^{-1}	deci-	deca-	10^{1}
10^{-2}	centi-	hecto-	10^{2}
10^{-3}	milli-	kilo-	10^{3}
10^{-6}	micro-	mega-	10^{6}
10^{-9}	nano-	giga-	10^{9}
10^{-12}	pico-	tera-	10^{12}

Some prefixes used to identify multiples and submultiples of metric quantities are shown in Table 1–3. Thus 1 millimeter = 10^{-3} meter, 1 nanosecond = 10^{-9} sec, 1 megavolt = 10^{6} volt, etc.

Much of the literature of physics is written in the Gaussian system. The student of physics must become familiar with several systems of units and must develop a facility for their manipulation. Appendix L shows how the equations of physics, given in this book in the form appropriate to the mks system, may be written in the form suitable to the Gaussian system; it also provides a Gaussian units table and gives their mks equivalents. The laws of physics, which express relations among observable physical quantities, are unchanged in physical content and significance, however, no matter what unit system is chosen to express them.

QUESTIONS

1. Do you think that a definition of a physical quantity for which no method of measurement is known or given has meaning?

2. According to operational philosophy, if we cannot prescribe a feasible operation for determining a physical quantity, the quantity is undetectable by physical means and should be given up as having no physical reality. Not all scientists accept this view. What are the merits and drawbacks of this point of view in your opinion?

3. What characteristics, other than accessibility and invariability, would you consider desirable for a physical standard?

4. If someone told you that every dimension of every object had shrunk to half its former value overnight, how could you refute his statement?

5. How would you criticize the following statement: "Once you have picked a physical standard, by the very meaning of standard it is invariable?"

6. What does an observer on the earth mean by "up" and "down"? Do all such observers use the same reference frame? How could one make the meaning clearly understood to *any* observer?

7. Why was it necessary to specify the temperature at which comparisons with the standard meter bar were to be made? Can length be called a fundamental quantity if another physical quantity, such as temperature, must be specified in choosing a standard?

8. Can length be measured along a curved line? If so, how?

9. Can you suggest a way to measure (a) the radius of the earth; (b) the distance between the sun and the earth; (c) the radius of the sun?

10. Can you suggest a way to measure (a) the thickness of a sheet of paper; (b) the thickness of a soap bubble film; (c) the diameter of an atom?

11. What criteria should a good clock satisfy?

12. Name several repetitive phenomena occurring in nature which could serve as reasonable time standards.

13. The time it takes the moon to return to a given position as seen against the background of the fixed stars is called a siderial month. The time interval between identical phases of the moon is called a lunar month. The lunar month is longer than a siderial month. Why?

14. When man colonizes other planets, what drawbacks would our present standards of length and time have? What drawbacks would atomic standards have?

15. Can you think of a way to define a length standard in terms of a time standard or vice versa? (Think about a pendulum clock.) If so, can length and time both be considered as fundamental quantities?

PROBLEMS

1. Express your height in the metric system of units.

2. In track meets both 100 yards and 100 meters are used as distances for dashes. Which is longer? By how many meters is it longer? By how many feet?

3. A rocket attained a height of 300 kilometers. What is this distance in miles?

4. Machine-tool men would like to have master gauges (1 in. long, say) good to 0.0000001 in. Show that the platinum-iridium meter is not measurable to this accuracy but that the krypton-86 meter is. Use data given in this chapter.

5. Assume that the average distance of the sun from the earth is 400 times the average distance of the moon from the earth. Now consider a total eclipse of the sun and state conclusions that can be drawn about (*a*) the relation between the sun's diameter and the moon's diameter; (*b*) the relative volumes of sun and moon. List the assumptions made in arriving at these answers. (*c*) Find the angle intercepted at the eye by a dime that just eclipses the full moon and from this experimental result and the given distance between sun and earth, estimate the diameter of the moon.

6. Astronomical distances are so large compared to terrestrial ones that much larger units of length are used for easy comprehension of the relative distances of astronomical objects. An *astronomical unit* (AU) is equal to the average distance from the earth to the sun, about 92.9×10^6 miles. A *parsec* is the distance at which one astronomical unit would subtend an angle of 1 sec of arc. A *light year* is the distance that light, traveling through a vacuum with a speed of 186,000 miles/sec, would cover in one year. (*a*) Express the distance from earth to sun in parsecs and in light years. (*b*) Express a light year and a parsec in miles.

7. Assuming that the length of the day uniformly increases by 0.001 sec in a century, calculate the cumulative effect on the measure of time over twenty centuries. Such a slowing down of the earth's rotation is indicated by observations of the occurrences of solar eclipses during this period.

8. (*a*) A unit of time sometimes used in microscopic physics is the *shake*. One shake equals 10^{-8} sec. Are there more shakes in a second than there are seconds in a year? (*b*) Mankind has existed for about 10^6 years, whereas the universe is about 10^{10} years old. If the age of the universe is taken to be one day, for how many seconds has mankind existed?

9. (*a*) The radius of the proton is about 10^{-15} meter; the radius of the observable universe is about 10^{28} cm. Identify a physically meaningful distance which is approximately halfway between these two extremes on a logarithmic scale. (*b*) The mean life of a neutral pion (an elementary particle) is about 2×10^{-16} sec. The age of the universe is about 4×10^9 years. Identify a physically meaningful time interval that is approximately halfway between these two extremes on a logarithmic scale.

10. From Fig. 1-4, calculate by what length of time the earth's rotation period in midsummer differs from that in the following spring.

11. A naval destroyer is testing five clocks. Exactly at noon, as determined by the WWV time signal, on the successive days of a week the clocks read as follows:

Clock	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
A	12:36:40	12:36:56	12:37:12	12:37:27	12:37:44	12:37:59	12:38:14
B	11:59:59	12:00:02	11:59:57	12:00:07	12:00:02	11:59:56	12:00:03
C	15:50:45	15:51:43	15:52:41	15:53:39	15:54:37	15:55:35	15:56:33
D	12:03:59	12:02:52	12:01:45	12:00:38	11:59:31	11:58:24	11:57:17
E	12:03:59	12:02:49	12:01:54	12:01:52	12:01:32	12:01:22	12:01:12

How would you arrange these five clocks in the order of their relative value as good timekeepers? Justify your choice.

Vectors

CHAPTER 2

2–1 Vectors and Scalars

A change of position of a particle is called a *displacement*. If a particle moves from position A to position B (Fig. 2–1a), we can represent its displacement by drawing a line from A to B; the direction of displacement can be shown by putting an arrowhead at B indicating that the displacement was *from* A *to* B. The path of the particle need not necessarily be a straight line from A to B; the arrow represents only the net effect of the motion, not the actual motion.

In Fig. 2–1b, for example, we plot a path followed by a particle from A to B. The path is not the same as the displacement AB. If we were to take snapshots of the particle when it was at A and, later, when it was at some intermediate position P, we could obtain the displacement vector AP, representing the net effect of the motion during this interval, even though we would not know the actual path taken between these points. Furthermore, a displacement such as $A'B'$ (Fig. 2–1a), which is parallel to AB, similarly directed, and equal in length to AB, represents the same *change* in position as AB. We make no distinction between these two displacements. A displacement is therefore characterized by a *length* and a *direction*.

In a similar way, we can represent a subsequent displacement from B to C (Fig. 2–1c). The net effect of the two displacements will be the same as a displacement from A to C. We speak then of AC as the *sum* or *resultant* of the displacements AB and BC. Notice that this sum is not an algebraic sum and that a number alone cannot uniquely specify it.

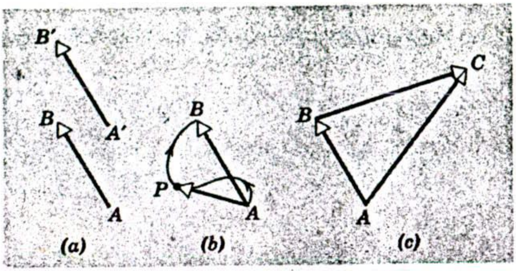


Fig. 2–1 Displacement vectors. (*a*) Vectors AB and $A'B'$ are identical since they have the same length and point in the same direction. (*b*) The actual *path* of the particle in moving from A to B may be the curve shown; the *displacement* remains the vector AB. At some intermediate point P the displacement from A is the vector AP. (*c*) After displacement AB the particle undergoes another displacement BC. The net effect of the two displacements is represented by the vector AC.

Quantities that behave like displacements are called *vectors.** Vectors, then, are quantities that have both magnitude and direction and combine according to certain rules of addition. These rules are stated below. The displacement vector can be considered as the prototype. Some other physical quantities which are vectors are force, velocity, acceleration, electric field strength, and magnetic induction. Many of the laws of physics can be expressed in compact form using vectors; derivations involving these laws are often greatly simplified if this is done.

Quantities that can be completely specified by a number and unit and that therefore have magnitude only are called *scalars*. Some physical quantities which are scalars are mass, length, time, density, energy, and temperature. Scalars can be manipulated by the rules of ordinary algebra.

2–2 Addition of Vectors, Geometrical Method

To represent a vector on a diagram we draw an arrow. We choose the length of the arrow proportional to the magnitude of the vector (that is, we choose a scale), and we choose the direction of the arrow to be the direction of the vector, with the arrowhead giving the sense of the direction. For example, a displacement of 40 ft north of east on a scale of 1.0 in. per 10 ft would be represented by an arrow 4.0 in. long, drawn at an angle of 45° to the horizontal direction with the arrowhead at the top right extreme. A vector such as this is represented conveniently in printing by a boldface symbol such as **d**. In handwriting it is convenient to put an arrow above the symbol to denote a vector quantity, such as $\vec{d}$.

* The word *vector* comes from the Latin and means *carrier*, which suggests a displacement. A good general reference on vectors is *Vector and Tensor Analysis* by G. E. Hay, Dover Publications, 1953.

Often we shall be interested only in the magnitude of the vector and not in its direction. The magnitude of **d** may be written as $|\mathbf{d}|$, called the absolute value of **d**; more frequently we represent the magnitude alone by the italic letter symbol, such as d. The boldface symbol is meant to signify both properties of the vector, magnitude and direction.

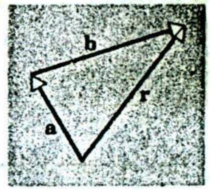


Fig. 2-2 The vector sum $\mathbf{a} + \mathbf{b} = \mathbf{r}$. Compare with Fig. 2-1*c*.

Consider now Fig. 2-2 in which we have redrawn and relabeled the vectors of Fig. 2-1*c*. The relation among these displacements (vectors) can be written as

$$\mathbf{a} + \mathbf{b} = \mathbf{r}. \tag{2-1}$$

The rules to be followed in performing this (vector) addition geometrically are these: On a diagram drawn to scale lay out the displacement vector **a**; then draw **b** with its tail at the head of **a**, and draw a line from the tail of **a** to the head of **b** to construct the vector sum **r**. This is a displacement equivalent in length and direction to the successive displacements **a** and **b**. This procedure can be generalized to obtain the sum of any number of successive displacements.

Since vectors are new quantities, we must expect new rules for their manipulation. The symbol "+" in Eq. 2-1 simply has a different meaning from arithmetic or ordinary algebra. It tells us to carry out a different set of operations.

Using Fig. 2-3 we can prove two important properties of vector addition:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \qquad \text{(commutative law)} \tag{2-2}$$

and

$$\mathbf{d} + (\mathbf{e} + \mathbf{f}) = (\mathbf{d} + \mathbf{e}) + \mathbf{f}. \qquad \text{(associative law)} \tag{2-3}$$

These laws assert that it makes no difference in what order or in what grouping we add vectors; the sum is the same. In this respect, vector addition and scalar addition follow the same rules.

The operation of subtraction can be included in our vector algebra by defining the negative of a vector to be another vector of equal magnitude

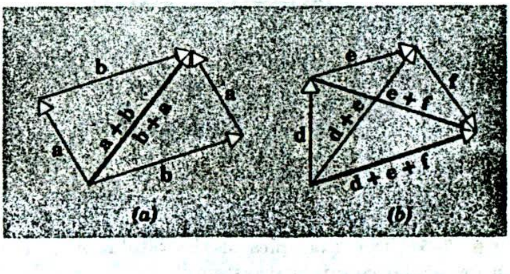


Fig. 2-3 (a) The commutative law for vector sums, which states that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. (b) The associative law, which states that $\mathbf{d} + \mathbf{e} + \mathbf{f}$.

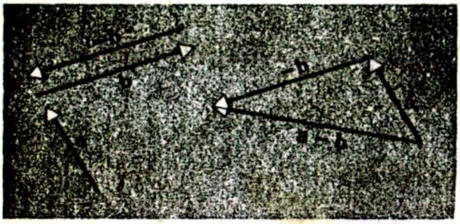


Fig. 2–4 The vector difference $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

but opposite direction. Then

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \tag{2–4}$$

as shown in Fig. 2–4.

Remember that, although we have used displacements to illustrate these operations, the rules apply to *all* vector quantities.

2–3 Resolution and Addition of Vectors, Analytic Method

The geometrical method of adding vectors is not very useful for vectors in three dimensions; often it is even inconvenient for the two-dimensional case. Another way of adding vectors is the analytical method, involving the resolution of a vector into components with respect to a particular coordinate system.

Figure 2–5*a* shows a vector **a** whose tail has been placed at the origin of a rectangular coordinate system. If we drop perpendicular lines from the head of **a** to the axes the quantities a_x and a_y so formed are called the *components* of the vector **a**. The process is called *resolving a vector into its components.* Figure 2–5 shows a two-dimensional case for convenience; the extension of our conclusions to three dimensions will be clear.

A vector may have many sets of components. For example, if we rotate the x-axis and y-axis in Fig. 2–5*a* by 10° counterclockwise, the components of **a** would be different. Furthermore, we may use a nonrectangular coordinate system, that is, the angle between the two axes need not be 90°. Thus the components of a vector are only uniquely specified if we specify

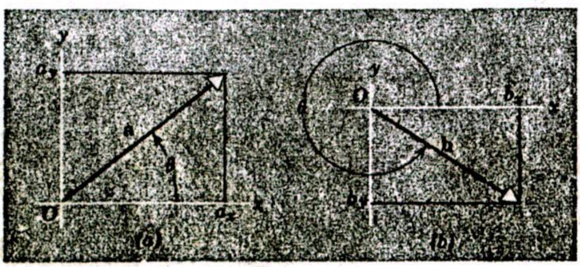


Fig. 2–5 Two examples of the resolution of a vector into its scalar components in a particular coordinate system.

the particular coordinate system being used. The vector need not be drawn with its tail at the origin of the coordinate system to find its components—although we have done so for convenience; the vector may be moved anywhere in the coordinate space and, as long as its angles with the coordinate directions are maintained, its components will be unchanged.

The components a_x and a_y in Fig. 2–5*a* are found readily from

$$a_x = a \cos \theta \qquad \text{and} \qquad a_y = a \sin \theta, \tag{2–5}$$

where θ is the angle that the vector **a** makes with the positive x-axis, measured counterclockwise from this axis. Note that, depending on the angle θ, a_x and a_y can be positive or negative. For example, in Fig. 2–5*b*, b_y is negative and b_x is positive. The components of a vector behave like scalar quantities because, in any particular coordinate system of a given reference frame, only a number, with an algebraic sign, is needed to specify them.

Once a vector is resolved into its components, the components themselves can be used to specify the vector. Instead of the two numbers a (magnitude of the vector) and θ (direction of the vector relative to the x-axis), we now have the two numbers a_x and a_y. We can pass back and forth between the description of a vector in terms of its components a_x, a_y and the equivalent description in terms of magnitude and direction a and θ. To obtain a and θ from a_x and a_y, we note from Fig. 2–5*a* that

$$a = \sqrt{a_x{}^2 + a_y{}^2} \tag{2–6a}$$

and

$$\tan \theta = a_y/a_x. \tag{2–6b}$$

The quadrant in which θ lies is determined from the sign of a_x and a_y.

When resolving a vector into components it is sometimes useful to introduce a vector of unit length in a given direction. Thus vector **a** in Fig. 2–6*a* may be written, for example, as

$$\mathbf{a} = \mathbf{u}_a a, \tag{2–7}$$

where $\mathbf{u}_a$ is a *unit vector* in the direction of **a**. Often it is convenient to draw unit vectors along the particular coordinate axes chosen. In the rectangular coordinate system the special symbols **i**, **j**, and **k** are usually

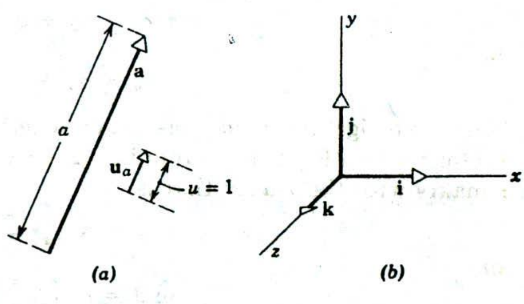


Fig. 2–6 (a) The vector **a** may be written as $\mathbf{u}_a a$ in which $\mathbf{u}_a$ is a unit vector in the direction of **a**. (b) The unit vectors **i**, **j**, and **k**, used to specify the positive x-, y-, and z-*directions respectively.*

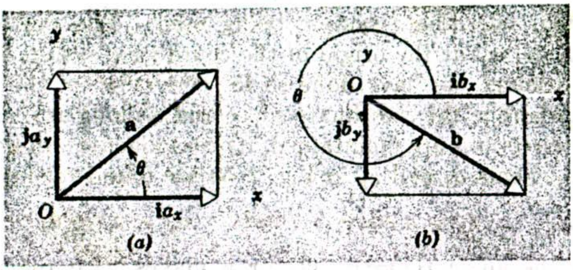


Fig. 2–7 Two examples of the resolution of a vector into its vector components in a particular coordinate system; compare with Fig. 2-5.

used for unit vectors in the positive x-, y-, and z-directions, respectively; see Fig. 2–6*b*. Note that **i**, **j**, and **k** need not be located at the origin. Like all vectors, they can be translated anywhere in the coordinate space as long as their directions with respect to the coordinate axes are not changed.

The vectors **a** and **b** of Fig. 2–5 may be written in terms of their components and the unit vectors as

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y \tag{2–8a}$$

and

$$\mathbf{b} = \mathbf{i}b_x + \mathbf{j}b_y; \tag{2–8b}$$

see Fig. 2–7. The vector relation Eq. 2–8*a* is equivalent to the scalar relation Eq. 2–6; each relates the vector (**a**, or a and θ) to its components (a_x and a_y). Sometimes we will call quantities such as $\mathbf{i}a_x$ and $\mathbf{j}a_y$ in Eq. 2–8*a* the *vector components* of **a**; they are drawn as vectors in Fig. 2–7*a*. The word *component* alone will continue to refer to the scalar quantities a_x and a_y.

We now consider the addition of vectors by the analytical method. Let **r** be the sum of the two vectors **a** and **b** lying in the x-y plane, so that

$$\mathbf{r} = \mathbf{a} + \mathbf{b}. \tag{2–9}$$

In a given coordinate system, two vectors such as **r** and **a** + **b** can only be equal if their corresponding components are equal, or

$$r_x = a_x + b_x \tag{2–10a}$$

and

$$r_y = a_y + b_y. \tag{2–10b}$$

These two algebraic equations, taken together, are equivalent to the single vector relation Eq. 2–9. From Eqs. 2–6 we may find r and the angle θ that **r** makes with the x-axis; that is,

$$r = \sqrt{r_x^2 + r_y^2}$$

and

$$\tan\theta = r_y/r_x.$$

Thus we have the following analytic rule for adding vectors: Resolve each vector into its components in a given coordinate system; the algebraic sum of the individual components along a particular axis is the component of the sum vector along that same axis; the sum vector can be reconstructed once its components are known. This method for adding vectors may be generalized to many vectors and to three dimensions (see Problems 6 and 11).

The advantage of the method of breaking up vectors into components, rather than adding directly with the use of suitable trigonometric relations, is that we always deal with right triangles and thus simplify the calculations.

In adding vectors by the analytical method, the choice of coordinate axes determines how simple the process will be. Sometimes the components of the vectors with respect to a particular set of axes are known to begin with, so that the choice of axes is obvious. Other times a judicious choice of axes can greatly simplify the job of resolution of the vectors into components. For example, the axes can be oriented so that at least one of the vectors lies parallel to an axis.

▶ **Example 1.** An airplane travels 130 miles on a straight course making an angle of 22.5° east of due north. How far north and how far east did the plane travel from its starting point?

We choose the positive x-direction to be east and the positive y-direction to be north. Next (Fig. 2-8) we draw a displacement vector from the origin (starting point), making an angle of 22.5° with the y-axis (north) inclined along the positive x-direction (east). The length of the vector is chosen to represent a magnitude of 130 miles. If we call this vector $\mathbf{d}$, then d_x gives the distance traveled east of the starting point and d_y gives the distance traveled north of the starting point. We have

$$\theta = 90.0° - 22.5° = 67.5°,$$

so that (see Eqs. 2-5)

$$d_x = d \cos \theta = (130 \text{ miles}) \cos 67.5° = 50 \text{ miles},$$

and

$$d_y = d \sin \theta = (130 \text{ miles}) \sin 67.5° = 120 \text{ miles}.$$

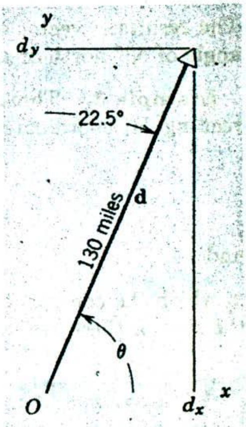


Fig. 2-8 Example 1.

Example 2. An automobile travels due east on a level road for 30 miles. It then turns due north at an intersection and travels 40 miles before stopping. Find the resultant displacement of the car.

We choose a reference frame fixed with respect to the earth, with the positive x-direction of our coordinate system pointing east and the positive y-direction pointing north. The two successive displacements, $\mathbf{a}$ and $\mathbf{b}$, are then drawn as shown in Fig. 2-9. The resultant displacement $\mathbf{r}$ is obtained from $\mathbf{r} = \mathbf{a} + \mathbf{b}$

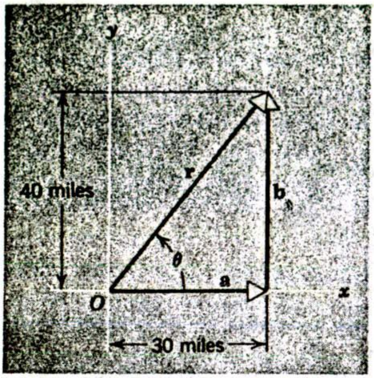


Fig. 2–9 Example 2.

Since **b** has no x-component and **a** has no y-component, we obtain (see Eqs. 2–10)

$$r_x = a_x + b_x = 30 \text{ miles} + 0 = 30 \text{ miles},$$

$$r_y = a_y + b_y = 0 + 40 \text{ miles} = 40 \text{ miles}.$$

The magnitude and direction of **r** are then (see Eqs. 2–6)

$$r = \sqrt{r_x^2 + r_y^2} = \sqrt{(30 \text{ miles})^2 + (40 \text{ miles})^2} = 50 \text{ miles},$$

$$\tan\theta = r_y/r_x = \frac{40 \text{ miles}}{30 \text{ miles}} = 1.33 \qquad \theta = \tan^{-1}(1.33) = 53°.$$

The resultant vector displacement **r** has a magnitude of 50 miles and makes an angle of 53° north of east.

Example 3. Three coplanar vectors are expressed, with respect to a certain rectangular coordinate system of a given reference frame, as

$$\mathbf{a} = 4\mathbf{i} - \mathbf{j},$$

$$\mathbf{b} = -3\mathbf{i} + 2\mathbf{j},$$

and

$$\mathbf{c} = -3\mathbf{j},$$

in which the components are given in arbitrary units. Find the vector **r** which is the sum of these vectors.

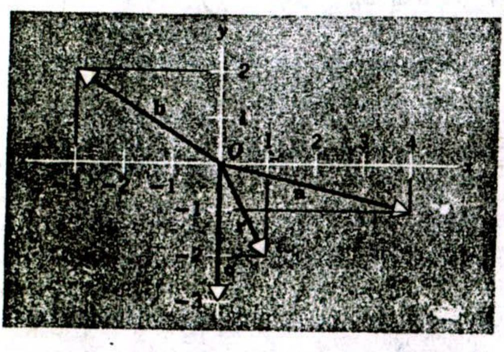


Fig. 2–10 Three vectors, **a**, **b**, and **c**, and their vector *sum* **r**.

From Eqs. 2-10 we have

$$r_x = a_x + b_x + c_x = 4 - 3 + 0 = 1,$$

and

$$r_y = a_y + b_y + c_y = -1 + 2 - 3 = -2.$$

Thus

$$\mathbf{r} = \mathbf{i}r_x + \mathbf{j}r_y$$
$$= \mathbf{i} - 2\mathbf{j}.$$

Figure 2-10 shows the four vectors. From Eqs. 2-6 we can calculate that the magnitude of $\mathbf{r}$ is $\sqrt{5}$ and that the angle that $\mathbf{r}$ makes with the positive x-axis, measured counterclockwise from that axis, is

$$\tan^{-1}(-2/1) = 297°.$$ ◀

2-4 Multiplication of Vectors*

We have assumed in the previous discussion that the vectors being added together are of like kind; that is, displacement vectors are added to displacement vectors, or velocity vectors are added to velocity vectors. Just as it would be meaningless to add together scalar quantities of different kinds, such as mass and temperature, so it would be meaningless to add together vector quantities of different kinds, such as displacement and electric field strength.

However, like scalars, vectors of different kinds can be multiplied by one another to generate quantities of new physical dimensions. Because vectors have direction as well as magnitude, vector multiplication cannot follow exactly the same rules as the algebraic rules of scalar multiplication. We must establish new rules of multiplication for vectors.

We find it useful to define three kinds of multiplication operations for vectors: (1) multiplication of a vector by a scalar, (2) multiplication of two vectors in such a way as to yield a scalar, and (3) multiplication of two vectors in such a way as to yield another vector. There are still other possibilities, but we shall not consider them here.

The multiplication of a vector by a scalar has a simple meaning: The product of a scalar k and a vector $\mathbf{a}$, written $k\mathbf{a}$, is defined to be a new vector whose magnitude is k times the magnitude of $\mathbf{a}$. The new vector has the same direction as $\mathbf{a}$ if k is positive and the opposite direction if k is negative. To divide a vector by a scalar we simply multiply the vector by the reciprocal of the scalar.

When we multiply a vector quantity by another vector quantity, we must distinguish between the *scalar* (or *dot*) *product* and the *vector* (or *cross*) *product*. The *scalar product* of two vectors $\mathbf{a}$ and $\mathbf{b}$, written as $\mathbf{a} \cdot \mathbf{b}$, is defined to be

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \phi, \tag{2-11}$$

* The material of this section will be used later in the text. The scalar product is used first in Chapter 7 and the vector product in Chapter 11. The instructor can postpone this section accordingly if he wishes. Its presentation here gives a unified treatment of vector algebra and serves as a convenient reference for later work.

where a is the magnitude of vector **a**, b is the magnitude of vector **b**, and $\cos \phi$ is the cosine of the angle ϕ between the two vectors* (see Fig. 2–11).

Since a and b are scalars and $\cos \phi$ is a pure number, the *scalar product of two vectors is a scalar.* The scalar product of two vectors can be regarded as the product of the magnitude of one vector and the component of the other vector in the direction of the first. Because of the notation $\mathbf{a} \cdot \mathbf{b}$ is also called the dot product of **a** and **b** and is spoken as "**a** dot **b**."

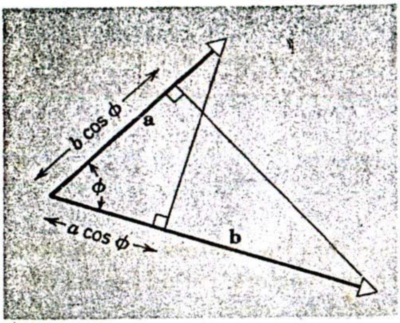


Fig. 2–11 The scalar product $\mathbf{a} \cdot \mathbf{b}$ $(= a\, b \cos \phi)$ is the product of the magnitude of either vector (a, say) by the component of the other vector in the direction of the first vector $b \cos \phi$, say).

We could have defined $\mathbf{a} \cdot \mathbf{b}$ to be any operation we want, for example, to be $a^{1/3}b^{1/3} \tan (\phi/2)$, but this would turn out to be of no use to us in physics. With our definition of the scalar product, a number of important physical quantities can be described as the scalar product of two vectors. Some of them are mechanical work, gravitational potential energy, electrical potential, electric power, and electromagnetic energy density. When such quantities are discussed later, their connection with the scalar product of vectors will be pointed out.

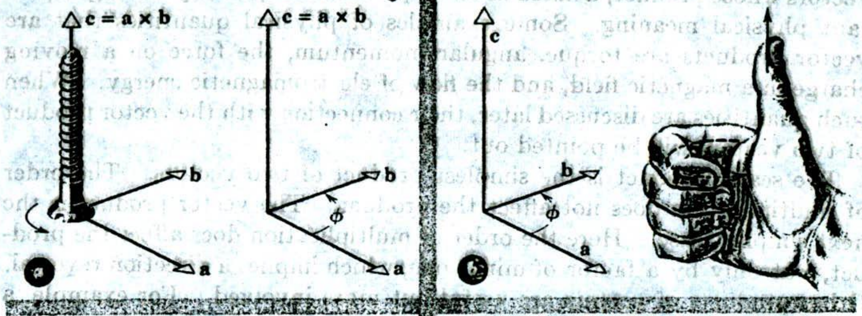
The *vector product* of two vectors **a** and **b** is written as $\mathbf{a} \times \mathbf{b}$ and is another vector **c**, where $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. The *magnitude* of **c** is defined by

$$c = ab \sin \phi, \tag{2–12}$$

where ϕ is the angle between **a** and **b**.

The *direction* of **c**, the vector product of **a** and **b**, is defined to be perpendicular to the plane formed by **a** and **b**. To specify the sense of the vector **c** we must refer to Fig. 2–12. Imagine rotating a right-handed screw whose axis is perpendicular to the plane formed by **a** and **b** so as to turn it *from* **a** *to* **b** through the angle ϕ between them. Then the direction of advance of the screw gives the direction of the vector product $\mathbf{a} \times \mathbf{b}$ (Fig. 2–12a). Another convenient way to obtain the direction of a vector product is the following. Imagine an axis perpendicular to the plane of **a** and **b** through their origin. Now wrap the fingers of the *right hand* around this axis and push the vector **a** into the vector **b** through the smaller angle between them with the fingertips, keeping the thumb erect; the direction of the erect thumb then gives the direction of the vector product $\mathbf{a} \times \mathbf{b}$

* There are two different angles between a pair of vectors, depending on the sense of rotation. We always choose the *smaller* of the two in vector multiplication.



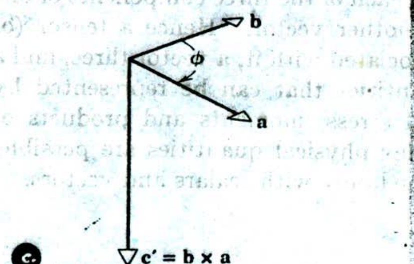


Fig. 2–12 The vector product. (*a*) In **c** = **a** × **b**, the direction of **c** is that in which a right-handed screw advances when turned from **a** to **b** through the smaller angle. (*b*) The direction of **c** can also be obtained from the "right-hand rule": If the right hand is held so that the curled fingers follow the rotation of **a** into **b**, the extended right thumb will point in the direction of **c**. (*c*) The vector product changes sign when the order of the factors is reversed: **a** × **b** = −**b** × **a**.

(Fig. 2–12*b*).* Because of the notation, **a** × **b** is also called the cross product of **a** and **b** and is spoken as "**a** cross **b**."

Notice that **b** × **a** is not the same vector as **a** × **b**, so that the order of factors in a vector product is important. This is not true for scalars because the order of factors in algebra or arithmetic does not affect the resulting product. Actually, **a** × **b** = −**b** × **a** (Fig. 2–12*c*). This can be deduced from the fact that the magnitude $ab \sin \phi$ equals the magnitude $ba \sin \phi$, but the direction of **a** × **b** is opposite to that of **b** × **a**; this is so because the right-handed screw advances in one direction when rotated from **a** to **b** through ϕ but advances in the opposite direction when rotated from **b** to **a**, through ϕ. The student can obtain the same result by applying the right-hand rule.

If ϕ is 90°, **a**, **b**, and **c** (= **a** × **b**) are all at right angles to one another and give the directions of a three-dimensional right-handed coordinate system.

The reason for defining the vector product in this way is that it proves to be useful in physics. We often encounter physical quantities that are

* The procedure described in Fig. 2–12 is a convention. Two vectors such as **a** and **b** form a plane and there are *two* directions that point away from any plane. The one selected (by convention) employs the right hand or a right-handed screw; the left hand or a left-handed screw would have led to the other choice for the direction of **a** × **b**.

vectors whose product, defined as above, is a vector quantity having important physical meaning. Some examples of physical quantities that are vector products are torque, angular momentum, the force on a moving charge in a magnetic field, and the flow of electromagnetic energy. When such quantities are discussed later, their connection with the vector product of two vectors will be pointed out.

The scalar product is the simplest product of two vectors. The order of multiplication does not affect the product. The vector product is the next simplest case. Here the order of multiplication does affect the product, but only by a factor of minus one, which implies a direction reversal. Other products of vectors are useful but more involved. For example, a tensor can be generated by multiplying each of the three components of one vector by the three components of another vector. Hence a tensor (of the second rank) has nine numbers associated with it, a vector three, and a scalar only one. Some physical quantities that can be represented by tensors are mechanical and electrical stress, moments and products of inertia, and strain. Still more complex physical quantities are possible. In this book, however, we are concerned only with scalars and vectors.

2–5 Vectors and the Laws of Physics

Vectors turn out to be very useful in physics. It will be helpful to look a little more deeply into why this is true. Suppose that we have three vectors **a**, **b**, and **r**, which have components a_x, a_y, a_z; b_x, b_y, b_z; and r_x, r_y, r_z, respectively in a particular coordinate system xyz of our reference frame. Let us suppose further that the three vectors are related so that

$$\mathbf{r} = \mathbf{a} + \mathbf{b}. \tag{2–13}$$

By a simple extension of Eqs. 2–10 this means that

$$r_x = a_x + b_x; \qquad r_y = a_y + b_y; \qquad \text{and} \qquad r_z = a_z + b_z. \tag{2–14}$$

Now consider another coordinate system $x'y'z'$ which has these properties: (1) its origin does not coincide with the origin of the first, or xyz, system and (2) its three axes are not parallel to the corresponding axes in the first system. In other words, the second set of coordinates has been both *translated* and *rotated* with respect to the first.

The components of the vectors **a**, **b**, and **r** in the new system would all prove, in general, to be different; we may represent them by $a_{x'}$, $a_{y'}$, $a_{z'}$; $b_{x'}$, $b_{y'}$, $b_{z'}$; and $r_{x'}$, $r_{y'}$, $r_{z'}$, respectively. These new components would be found, however, to be related (see Problem 34) in that

$$r_{x'} = a_{x'} + b_{x'}; \qquad r_{y'} = a_{y'} + b_{y'}; \qquad \text{and} \qquad r_{z'} = a_{z'} + b_{z'}. \tag{2–15}$$

That is, in the new system we would find once again (see Eq. 2–13) that

$$\mathbf{r} = \mathbf{a} + \mathbf{b}.$$

In more formal language: *Relations among vectors*, of which Eq. 2–13 is only one example, *are invariant* (that is, *are unchanged*) *with respect to*

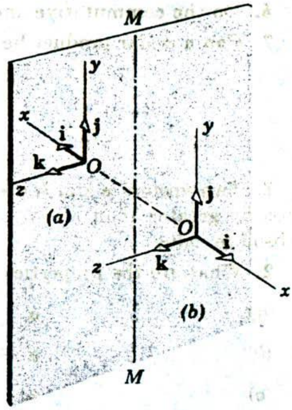


Fig. 2-13 Showing (*a*) a left-handed and (b) a right-handed coordinate system. Notice that (*a*) and (*b*) are related in that each may be viewed as the image of the other in mirror *MM*. The "handedness" of a coordinate system cannot be changed by rotating it. Note that in (*b*), $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, whereas in (*a*), $\mathbf{i} \times \mathbf{j} = -\mathbf{k}$.

translation or rotation of the coordinates. Now it is a fact of experience that the experiments on which the laws of physics are based and indeed the *laws of physics* themselves are similarly unchanged in form when we rotate or translate the reference system. Thus the language of vectors is an ideal one in which to express physical laws. If we can express a law in vector form, the invariance of the law for translation and rotation of the coordinate system is assured by this purely geometrical property of vectors.

It was thought until about 1956 that all laws of physics were invariant under another kind of transformation of coordinates, the substitution of a right-handed coordinate system for a left-handed one (see Fig. 2-13). In that year, however, some experiments involving the decay of certain elementary particles were studied in which the result of the experiment *did* turn out to depend on the "handedness" of the coordinate system used to express the results. In other words, the experiment and its image in a mirror would yield different results!* This surprising result led to a re-examination of the whole question of the symmetry of physical laws; as of 1965 these studies remain among the most challenging in modern physics.

QUESTIONS

1. Can two vectors of different magnitude be combined to give a zero resultant? Can three vectors?

2. Can a vector be zero if one of its components is not zero?

3. Does it make any sense to call a quantity a vector when its magnitude is zero?

4. Name several scalar quantities. Is the value of a scalar quantity dependent on the reference frame chosen?

5. We can order events in time. For example, event *b* may precede event *c* but follow event *a*, giving us a time order of events *a*, *b*, *c*. Hence there is a sense of time, distinguishing past, present, and future. Is time a vector therefore? If not, why not?

* C. N. Yang and T. D. Lee were awarded the Nobel prize in 1957 for their theoretical prediction that this would be the case.

6. Do the commutative and associative laws apply to vector subtraction?

7. Can a scalar product be a negative quantity?

PROBLEMS

1. Two vectors **a** and **b** are added. Show that the magnitude of the resultant cannot be greater than $a + b$ or smaller than $|a - b|$, where the vertical bars signify absolute value.

2. What are the properties of two vectors **a** and **b** such that

(*a*) $\mathbf{a} + \mathbf{b} = \mathbf{c}$ and $a + b = c$,

(*b*) $\mathbf{a} + \mathbf{b} = \mathbf{a} - \mathbf{b}$,

(*c*) $\mathbf{a} + \mathbf{b} = \mathbf{c}$ and $a^2 + b^2 = c^2$.

3. Consider two displacements, one of magnitude 3 meters and another of magnitude 4 meters. Show how the displacement vectors may be combined to get a resultant displacement of magnitude (*a*) 7 meters, (*b*) 1 meter, and (*c*) 5 meters.

4. Two vectors **a** and **b** have equal magnitudes, say 10 units. They are oriented as shown in Fig. 2–14 and their vector sum is **r**. Find (*a*) the x- and y-components of **r**; (*b*) the magnitude of **r**; and (*c*) the angle **r** makes with the x-axis.

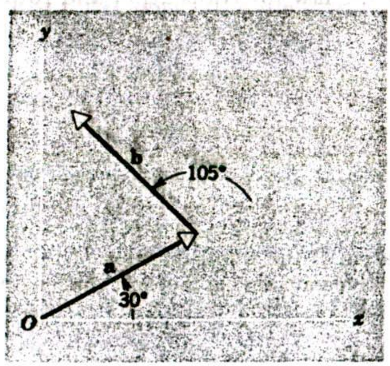


Fig. 2–14

5. Given two vectors $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} + 8\mathbf{j}$, find the magnitude and direction of **a**, of **b**, of $\mathbf{a} + \mathbf{b}$, of $\mathbf{b} - \mathbf{a}$, and of $\mathbf{a} - \mathbf{b}$.

6. Generalize the analytical method of resolution and addition to the case of *three or more vectors*.

7. A car is driven eastward for a distance of 50 miles, then northward for 30 miles, and then in a direction 30° east of north for 25 miles. Draw the vector diagram and determine the total displacement of the car from its starting point.

8. A golfer takes three strokes to get his ball into the hole once he is on the green. The first stroke displaces the ball 12 ft north, the second stroke 6.0 ft southeast, and the third stroke 3.0 ft southwest. What displacement was needed to get the ball into the hole on the first stroke?

9. A particle undergoes three successive displacements in a plane, as follows: 4.0 meters southwest, 5.0 meters east, 6.0 meters in a direction 60° north of east. Choose

the y-axis pointing north and the x-axis pointing east and find (*a*) the components of each displacement, (*b*) the components of the resultant displacement, (*c*) the magnitude and direction of the resultant displacement, and (*d*) the displacement that would be required to bring the particle back to the starting point.

10. Use a scale of 2 meters to the inch and add the displacements of Problem 9 graphically. Determine *from your graph* the magnitude and direction of the resultant.

11. Generalize the analytical method of resolving and adding two vectors to *three dimensions*.

12. (*a*) A man leaves his front door, walks 1000 ft east, 2000 ft north, and then takes a penny from his pocket and drops it from a cliff 500 ft high. Set up a coordinate system and write down an expression, using unit vectors, for the displacement of the penny. (*b*) The man then returns to his front door, following a different path on the return trip. What is his resultant displacement for the round trip?

13. Find the sum of the vector displacements **c** and **d** whose components in miles along three perpendicular directions are

$$c_x = 5.0,\ c_y = 0,\ c_z = -2.0;\ d_x = -3.0,\ d_y = 4.0,\ d_z = 6.0.$$

14. A vector **d** has a magnitude 2.5 meters and points due north. What are the magnitudes and directions of the vectors

$$(a)\ -\mathbf{d},\ (b)\ \mathbf{d}/2.0,\ (c)\ -2.5\mathbf{d}\quad \text{and}\quad (d)\ 4.0\mathbf{d}?$$

15. A room has the dimensions 10 ft × 12 ft × 14 ft. A fly starting at one corner ends up at a diametrically opposite corner. (*a*) What is the magnitude of its displacement? (*b*) Could the length of its path be less than this distance? Greater than this distance? Equal to this distance? (*c*) Choose a suitable coordinate system and find the components of the displacement vector in this frame.

16. In Problem 15, if the fly does not fly but crawls, what is the length of the shortest path it can take?

17. A man flies from Washington to Manila. Describe the displacement vector. What is its magnitude if the latitude and longitude of the two cities are 39° N, 77° W and 15° N, 121° E?

18. Two vectors of lengths **a** and **b** make an angle θ with each other when placed tail to tail. Prove, by taking components along two perpendicular axes, that the length of the resultant vector is

$$r = \sqrt{a^2 + b^2 + 2ab\cos\theta}.$$

19. Show for any vector **a** that $\mathbf{a}\cdot\mathbf{a} = a^2$ and that $\mathbf{a}\times\mathbf{a} = 0$.

20. Use the standard (right-hand) xyz system of coordinates. Given vector **a** in the $+x$-direction, vector **b** in the $+y$-direction, and the scalar quantity d: (*a*) What is the direction of $\mathbf{a}\times\mathbf{b}$? (*b*) What is the direction of $\mathbf{b}\times\mathbf{a}$? (*c*) What is the direction of $\mathbf{b}/d$? (*d*) What is the magnitude of $\mathbf{a}\cdot\mathbf{b}$?

21. A vector **a** of magnitude ten units and another vector **b** of magnitude six units point in directions differing by 60°. Find (*a*) the scalar product of the two vectors and (*b*) the vector product of the two vectors.

22. In the coordinate system of Fig. 2–6*b* show that

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

and

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

23. In the right-handed coordinate system of Fig. 2–6*b* show that

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$$

$$\mathbf{i}\times\mathbf{j} = \mathbf{k};\qquad \mathbf{k}\times\mathbf{i} = \mathbf{j};\qquad \mathbf{j}\times\mathbf{k} = \mathbf{i}.$$

24. (*a*) We have seen that the commutative law does *not* apply to vector products, that is, $\mathbf{a} \times \mathbf{b}$ does not equal $\mathbf{b} \times \mathbf{a}$. Show that the commutative law *does* apply to scalar products, that is, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. (*b*) Show that the distributive law applies to both scalar products and vector products, that is, show that

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \text{ and that } \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

(*c*) Does the associative law apply to vector products, i.e., does $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ equal $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$? Does it make any sense to talk about an associative law for scalar products?

25. *Scalar product in unit vector notation.* Let two vectors be represented in terms of their coordinates as

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z$$

and

$$\mathbf{b} = \mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z$$

Show analytically that

$$\mathbf{a} \cdot \mathbf{b} = a_xb_x + a_yb_y + a_zb_z.$$

(Hint: See Problem 22.)

26. Use the definition of scalar product $\mathbf{a} \cdot \mathbf{b} = ab \cos \phi$ and the fact that $\mathbf{a} \cdot \mathbf{b} = a_xb_x + a_yb_y + a_zb_z$ (see Problem 25) to obtain the angle between the two vectors given by $\mathbf{a} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

27. *Vector product in unit vector notation.* Show analytically that $\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_yb_z - a_zb_y) + \mathbf{j}(a_zb_x - a_xb_z) + \mathbf{k}(a_xb_y - a_yb_x)$. (Hint: See Problem 23.)

28. Show that the magnitude of a vector product gives numerically the area of the parallelogram formed with the two component vectors as sides (see Fig. 2–15). Does this suggest how an element of area oriented in space could be represented by a vector?

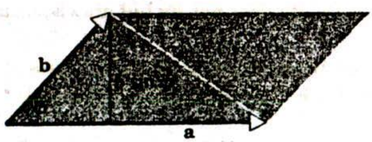


Fig. 2–15

29. Show that the area of the triangle contained between the vectors **a** and **b** is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$, where the vertical bars signify absolute value (see Fig. 2–15).

30. Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is equal in magnitude to the volume of the parallelepiped formed on the three vectors **a**, **b**, and **c**

31. Let **b** and **c** be the intersecting face diagonals of a cube of edge a, as shown in

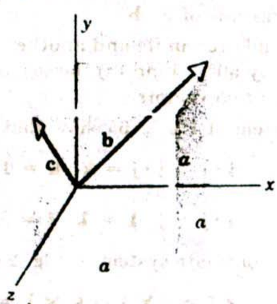


Fig. 2–16

Fig. 2–16. (*a*) Find the components of the vector **d**, where

$$\mathbf{d} = \mathbf{b} \times \mathbf{c}.$$

(*b*) Find the values of $\mathbf{b} \cdot \mathbf{c}$, of $\mathbf{d} \cdot \mathbf{c}$, and of $\mathbf{d} \cdot \mathbf{b}$.

32. Suppose **a**, **b**, and **c** are any three noncoplanar vectors. They are not necessarily mutually at right angles. (*a*) show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

(*b*) Let

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{v}, \qquad \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{v}, \qquad \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{v},$$

where $v = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Evaluate the dot product of each of **a**, **b**, **c** with each of **A**, **B**, **C**. (*c*) If **a**, **b**, **c** have dimensions of length, what are the dimensions of **A**, **B**, **C**?

33. Let N be an integer greater than one; then

$$\cos 0 + \cos \frac{2\pi}{N} + \cos \frac{4\pi}{N} + \cdots + \cos (N-1) \frac{2\pi}{N} = 0;$$

that is,

$$\sum_{n=0}^{n=N-1} \cos \frac{2\pi n}{N} = 0.$$

Also

$$\sum_{n=0}^{n=N-1} \sin \frac{2\pi n}{N} = 0.$$

Prove these two statements by considering the sum of N vectors of equal length, each vector making an angle of $2\pi/N$ with that preceding.

34. *Invariance of vector addition under rotation of the coordinate system.* Figure 2–17 shows two vectors **a** and **b** and two systems of coordinates which differ in that the x and x' axes and the y and y' axes each make an angle ϕ with each other. Prove analytically that $\mathbf{a} + \mathbf{b}$ has the same magnitude and direction no matter which system is used to carry out the analysis.

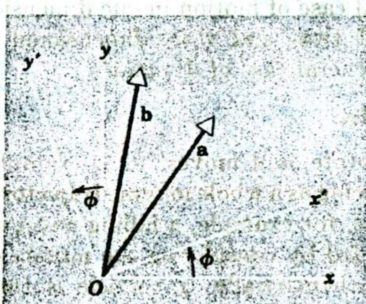


Fig. 2–17

Motion in One Dimension

CHAPTER 3

3–1 Mechanics

Mechanics, the oldest of the physical sciences, is the study of the motion of objects. The calculation of the path of an artillery shell or of a space probe sent from Earth to Mars are among its problems. So too is the analysis of tracks formed in bubble chambers, representing the collisions, decay, and interactions of elementary particles (see Fig. 10–11 and Appendix E).

When we describe motion we are dealing with that part of mechanics called *kinematics.* When we relate motion to the forces associated with it and to the properties of the moving objects, we are dealing with *dynamics.* In this chapter we shall define some kinematical quantities and study them in detail for the special case of motion in one dimension. In Chapter 4 we discuss some cases of two- and three-dimensional motion. Chapter 5 deals with the more general case of dynamics.

3–2 Particle Kinematics

A real object can rotate as it moves. For example, a baseball may be spinning while it is moving as a whole in some trajectory. Also, a body may vibrate as it moves, as, for example, a falling water droplet. These complications can be avoided by considering the motion of a very small body called a *particle.* Mathematically, a particle is treated as a point, an object without extent, so that rotational and vibrational considerations are not involved.

Actually, there is no such thing in nature as an object without extent. The concept of "particle" is nevertheless very useful because real objects

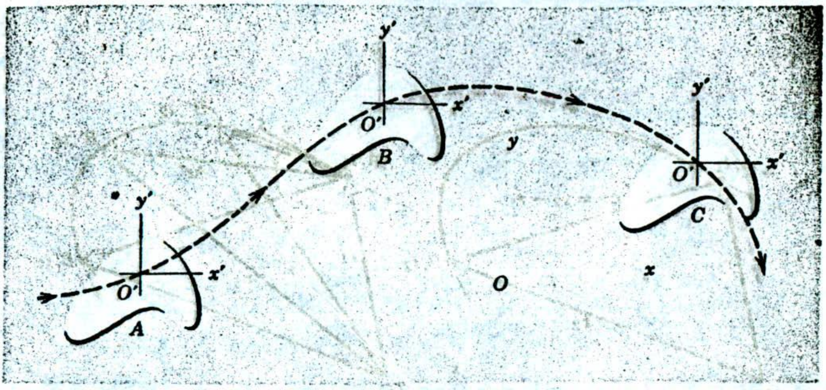


Fig. 3-1 Translational motion of an object. Translation can occur in three dimensions, but only two are shown for simplicity.

often behave to a very good approximation as though they were particles. A body need not be "small" in the usual sense of the word in order to be treated as a particle. For example, if we consider the distance from the earth to the sun, with respect to this distance the earth and the sun can usually be considered to be particles. We can find out a great deal about the motion of the sun and planets, without appreciable error, by treating these bodies as particles. Baseballs, molecules, protons, and electrons can be often treated as particles. Even if a body is too large to be considered a particle for a particular problem, it can always be thought of as made up of a large number of particles, and the results of particle motion may be useful in analyzing the problem. As a simplification, therefore, we confine our present treatment to the motion of a particle.

Bodies that have only motion of translation behave like particles. An observer will call motion *translational* if the axes of a reference frame which is imagined rigidly attached to the object, say x', y', and z', always remain parallel to the axes of his own reference frame, say x, y, and z. In Fig. 3-1, for example, we show the translational motion of an object moving from positions A to B to C. Notice that the path taken is not necessarily a straight line. Notice too that throughout the motion every point of the body undergoes the same displacements as every other point. We can assume the body to be a particle because in describing the motion of one point on the body we have described the motion of the body as a whole.

3-3 Average Velocity

The velocity of a particle is the rate at which its position changes with time. The position of a particle in a particular reference frame is given by a *position vector* drawn from the origin of that frame to the particle. At time t_1, let a particle be at point A in Fig. 3-2a, its position in the x-y plane

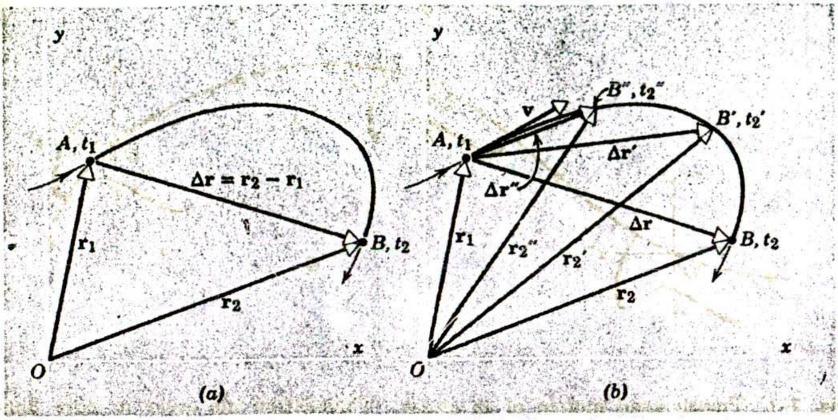


Fig. 3–2 (*a*) A particle moves from A to B in time Δt $(= t_2 - t_1)$ undergoing a displacement $\Delta \mathbf{r}$ $(= \mathbf{r}_2 - \mathbf{r}_1)$. *The average velocity* $\bar{\mathbf{v}}$ between A and B is in the direction of $\Delta \mathbf{r}$. (*b*) As B moves closer to A the average velocity approaches the *instantaneous velocity* $\mathbf{v}$ at A; $\mathbf{v}$ is tangent to the path at A.

being described by position vector $\mathbf{r}_1$. For simplicity we treat motion in two dimensions only; the extension to three dimensions will not be difficult.

At a later time t_2 let the particle be at point B, described by position vector $\mathbf{r}_2$. The *displacement vector* describing the *change* in position of the particle as it moves from A to B is $\Delta \mathbf{r}$ $(= \mathbf{r}_2 - \mathbf{r}_1)$ and the elapsed time for the motion between these points is Δt $(= t_2 - t_1)$. The *average velocity* for the particle during this interval is defined by

$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\text{displacement (a vector)}}{\text{elapsed time (a scalar)}} \tag{3–1}$$

A bar above a symbol indicates an average value for the quantity in question.

The quantity $\bar{\mathbf{v}}$ is a vector, for it is obtained by dividing the vector $\Delta \mathbf{r}$ by the scalar Δt. Velocity, therefore, involves both direction and magnitude. Its direction is the direction of $\Delta \mathbf{r}$ and its magnitude is $|\Delta \mathbf{r}/\Delta t|$. The magnitude is expressed in distance units divided by time units, as, for example, meters per second or miles per hour.

The velocity defined by Eq. 3–1 is called an *average* velocity because the measurement of the net displacement and the elapsed time does not tell us anything at all about the motion between A and B. The path may have been curved or straight; the motion may have been steady or erratic. The average velocity involves simply the total displacement and the total elapsed time. For example, suppose a man leaves his house and goes on an automobile trip, returning to his house in a time Δt after he left it. His average velocity for the trip is zero because his displacement *for this particular time interval* Δt is zero.

If we were to measure the time of arrival of the particle at each of many points along the actual path between A and B in Fig. 3-2a, we could describe the motion in more detail. If the average velocity turned out to be the same (in magnitude and direction) between any two points along the path, we would conclude that the particle moved with *constant velocity*, that is, along a straight line (constant direction) at a uniform rate (constant magnitude).

3-4 Instantaneous Velocity

Suppose that a particle is moving in such a way that its average velocity, measured for a number of different time intervals, does *not* turn out to be constant. This particle is said to move with variable velocity. Then we must seek to determine a velocity of the particle at any given instant of time, called the *instantaneous velocity*.

Velocity can vary by a change in magnitude, by a change in direction, or both. For the motion portrayed in Fig. 3-2a, the average velocity during the time interval $t_2 - t_1$ may differ both in magnitude and direction from the average velocity obtained during another time interval $t_2' - t_1$. In Fig. 3-2b we illustrate this by choosing the point B to be successively closer to point A. Points B' and B'' show two intermediate positions of the particle corresponding to the times t_2' and t_2'' and described by position vectors $\mathbf{r}_2'$ and $\mathbf{r}_2''$, respectively. The vector displacements $\Delta\mathbf{r}$, $\Delta\mathbf{r}'$, and $\Delta\mathbf{r}''$ differ in direction and become successively smaller. Likewise, the corresponding time intervals Δt $(= t_2 - t_1)$, $\Delta t'$ $(= t_2' - t_1)$, and $\Delta t''$ $(= t_2'' - t_1)$ become successively smaller.

As we continue this process, letting B approach A, we find that the ratio of displacement to elapsed time approaches a definite limiting value. Although the displacement in this process becomes extremely small, the time interval by which we divide it becomes small also and the ratio is not necessarily a small quantity. Similarly, while growing smaller, the displacement vector approaches a limiting direction, that of the tangent to the path of the particle at A. This limiting value of $\Delta\mathbf{r}/\Delta t$ is called the *instantaneous* velocity at the point A, or the velocity of the particle at the instant t_1.

If $\Delta\mathbf{r}$ is the displacement in a small interval of time Δt, following the time t, the velocity at the time t is the limiting value approached by $\Delta\mathbf{r}/\Delta t$ as both $\Delta\mathbf{r}$ and Δt approach zero. That is, if we let $\mathbf{v}$ represent the instantaneous velocity,

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t}.$$

The direction of $\mathbf{v}$ is the limiting direction that $\Delta\mathbf{r}$ takes as B approaches A or as Δt approaches zero. As we have seen, this limiting direction is that of the tangent to the path of the particle at point A.

In the notation of the calculus, the limiting value of $\Delta\mathbf{r}/\Delta t$ as Δt approaches zero is written $d\mathbf{r}/dt$ and is called the *derivative* of $\mathbf{r}$ with respect to t. We have then

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}. \tag{3-2}$$

The magnitude v of the instantaneous velocity is called the *speed* and is simply the absolute value of $\mathbf{v}$. That is,

$$v = |\mathbf{v}| = |d\mathbf{r}/dt|. \tag{3-3}$$

Speed, being the magnitude of a vector, is intrinsically positive.

Just as a particle is a physical concept making use of the mathematical concept of a point, so here velocity is a physical concept using the mathematical concept of differentiation. In fact, the calculus was invented originally by Isaac Newton (1642–1727) in order to have a proper mathematical tool for treating fundamental mechanical problems.

In the next section we shall examine the concept of instantaneous velocity in detail for the special case of motion in one dimension, sometimes called rectilinear motion.

3–5 One-Dimensional Motion—Variable Velocity

Figure 3–3 shows a particle moving along a path in the x-y plane. At time t its position with respect to the origin is described by position vector $\mathbf{r}$ (see Fig. 3–3a) and it has a velocity $\mathbf{v}$ (see Fig. 3–3b) tangent to its path as shown. We can write (see Eq. 2–8)

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y, \tag{3-4}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the positive x- and y-directions, respectively, and x and y are the (scalar) components of vector $\mathbf{r}$. Because $\mathbf{i}$ and $\mathbf{j}$ are constant vectors, we have, on combining Eqs. 3–2 and 3–4,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt},$$

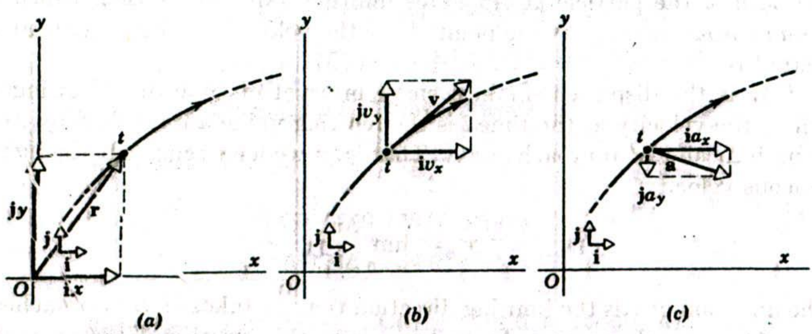


Fig. 3–3 A particle at time t has (a) a position described by $\mathbf{r}$, (b) an instantaneous velocity $\mathbf{v}$, and (c) an instantaneous acceleration $\mathbf{a}$. The vector components $\mathbf{i}x$ and $\mathbf{j}y$ of Eq. 3-4, $\mathbf{i}v_x$ and $\mathbf{j}v_y$ of Eq. 3-5, and $\mathbf{i}a_x$ and $\mathbf{j}a_y$ of Eq. 3-10 are also shown, as are the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

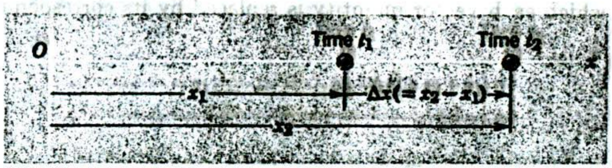


Fig. 3-4 A particle is moving to the right along the x-axis.

which we can express as

$$\mathbf{v} = \mathbf{i}v_x + \mathbf{j}v_y \quad \text{(two-dimensional motion)}, \tag{3-5}$$

where v_x $(= dx/dt)$ and v_y $(= dy/dt)$ are the (scalar) components of the vector $\mathbf{v}$.

We now consider motion in one dimension only, chosen for convenience to be the x-axis. We must then have $v_y = 0$ so that Eq. 3-5 reduces to

$$\mathbf{v} = \mathbf{i}v_x \quad \text{(one-dimensional motion)}. \tag{3-6}$$

Since $\mathbf{i}$ points in the positive x-direction, v_x will be positive (and equal to $+v$) when $\mathbf{v}$ points in that direction, and negative (and equal to $-v$) when it points in the other direction. Since, in one-dimensional motion, there are only two choices as to the direction of $\mathbf{v}$, the full power of the vector method is not needed; we can work with the velocity component v_x alone.

▶ **Example 1.** *The limiting process.* As an illustration of the limiting process in one dimension, consider the following table of data taken for motion along the x-axis. The first four columns are experimental data. The symbols refer to Fig. 3-4 in which the particle is moving from left to right, that is, in the positive x-direction. The particle was at position x_1 (100 cm from the origin) at time t_1 (1.00 sec). It was at position x_2 at time t_2. As we consider different values for x_2, and different corresponding times t_2, we find

x_1, cm	t_1, sec	x_2, cm	t_2, sec	$x_2 - x_1 = \Delta x$, cm	$t_2 - t_1 = \Delta t$, sec	$\Delta x/\Delta t$, cm/sec
100.0	1.00	200.0	11.00	100.0	10.00	10.0
100.0	1.00	180.0	9.60	80.0	8.60	9.3
100.0	1.00	160.0	7.90	60.0	6.90	8.7
100.0	1.00	140.0	5.90	40.0	4.90	8.2
100.0	1.00	120.0	3.56	20.0	2.56	7.8
100.0	1.00	110.0	2.33	10.0	1.33	7.5
100.0	1.00	105.0	1.69	5.0	0.69	7.3
100.0	1.00	103.0	1.42	3.0	0.42	7.1
100.0	1.00	101.0	1.14	1.0	0.14	7.1

Equation 3-2, which holds for the general case of motion in three dimensions, is

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}.$$

For one-dimensional motion along the x-axis we have a similar relation, scalar in character, in which each vector quantity is replaced by its corresponding component or

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \tag{3-7}$$

It is clear from the table that as we select values of x_2 closer to x_1, Δt approaches zero and the ratio $\Delta x/\Delta t$ approaches the apparent limiting value $+7.1$ cm/sec. At time t_1, therefore, $v_x = +7.1$ cm/sec, as closely as we are able to determine from the data. Since v_x is positive, the velocity $\mathbf{v}$ ($= \mathbf{i}v_x$; see Eq. 3–6) points to the right in Fig. 3–4. This is tangent to the path in the direction of motion, as it must be.

Example 2. Figure 3-5a shows six successive "snapshots" of a particle moving along the x-axis with variable velocity. At $t = 0$ it is at position $x = +1.00$ ft to the right of the origin; at $t = 2.5$ sec it has come to rest at $x = +5.00$ ft; at $t = 4.0$ sec it has returned to $x = +1.40$ ft. Figure 3–5b is a plot of position x versus time t for this motion. The *average velocity* for the entire 4.0-sec interval is the net displacement or change of position (+0.40 ft) divided by the elapsed time (4.0 sec) or $\bar{v}_x = +0.10$ ft/sec. (We call $\bar{v}_x$ average velocity and v_x velocity, for one-dimensional motion, even though velocity is a vector and not a scalar. This conforms to common usage and should cause no misunderstandings. These quantities are not speeds because they may be negative, whereas speed is intrinsically positive.) The average velocity vector $\bar{\mathbf{v}}$ points in the positive x-direction (that is, to the right in Fig. 3–5a) because the net displacement points in this direction. The quantity $\bar{v}_x$ can be obtained directly from the slope of the dashed line af in Fig. 3–5b, where by slope we mean the ratio of the net displacement gf to the elapsed time ga. (The slope is *not* the tangent of the angle fag measured on the graph with a protracter. This angle is arbitrary because it depends on the scales we choose for x and t.)

The velocity v_x at any instant is found from the slope of the curve of Fig. 3–5b at that instant. Equation 3–7 is in fact the relation by which the slope of the curve is defined in the calculus. In our example the slope at b, which is the value of v_x at b, is $+1.7$ ft/sec; the slope at d is zero and the slope at f is -6.2 ft/sec. When we determine the slope dx/dt at each instant t, we can make a plot of v_x versus t, as in Fig. 3–5c. Note that for the interval $0 < t < 2.5$ sec, v_x is positive so that the velocity vector $\mathbf{v}$ points to the right in Fig. 3–5a; for the interval 2.5 sec $< t <$ 4.0 sec v_x is negative so that $\mathbf{v}$ points to the left in Fig. 3–5a. ◀

3–6 Acceleration

Often the velocity of a moving body changes either in magnitude, in direction, or both as the motion proceeds. The body is then said to have an acceleration. *The acceleration of a particle is the rate of change of its velocity with time.* Suppose that at the instant t_1 a particle, as in Fig. 3–6, is at point A and is moving in the x-y plane with the instantaneous velocity $\mathbf{v}_1$, and at a later instant t_2 it is at point B and moving with the instantaneous velocity $\mathbf{v}_2$. The *average acceleration* $\bar{\mathbf{a}}$ during the motion from A to B is defined to be the *change of velocity* divided by the time interval, or

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \frac{\Delta \mathbf{v}}{\Delta t}. \tag{3-8}$$

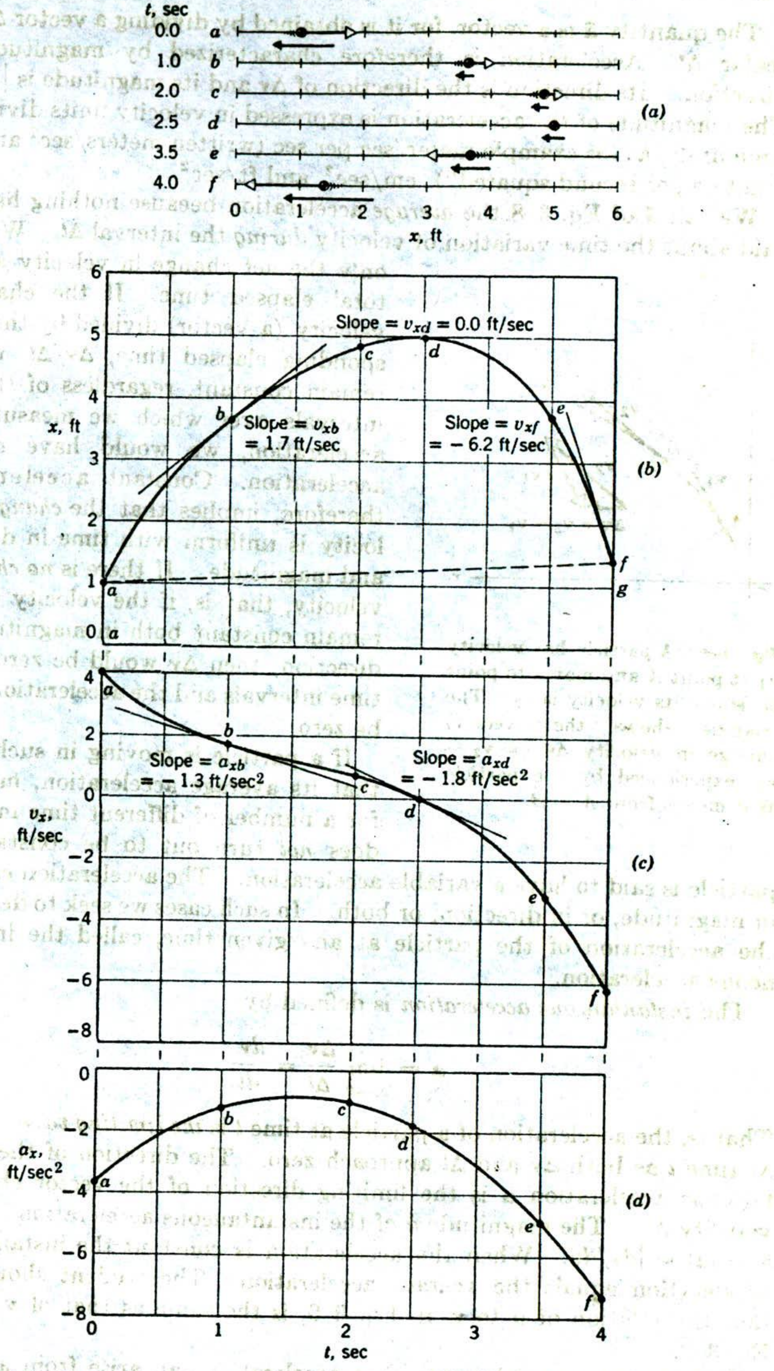


Fig. 3–5 (*a*) Six consecutive "snapshots" of a particle moving along the x-axis. The vector joined to the particle is its instantaneous velocity; that below the particle is its instantaneous acceleration. (*b*) A plot of x versus t for the motion of the

The quantity $\bar{\mathbf{a}}$ is a vector, for it is obtained by dividing a vector $\Delta\mathbf{v}$ by a scalar Δt. Acceleration is therefore characterized by magnitude and direction. Its direction is the direction of $\Delta\mathbf{v}$ and its magnitude is $|\Delta\mathbf{v}/\Delta t|$. The magnitude of the acceleration is expressed in velocity units divided by time units, as for example meter/sec per sec (written meters/sec^2 and read "meters per second squared"), cm/sec^2, and ft/sec^2.

We call $\bar{\mathbf{a}}$ of Eq. 3–8 the *average* acceleration because nothing has been said about the time variation of velocity *during* the interval Δt. We know only the net change in velocity and the total elapsed time. If the change in velocity (a vector) divided by the corresponding elapsed time, $\Delta\mathbf{v}/\Delta t$, were to remain constant, regardless of the time intervals over which we measured the acceleration, we would have *constant* acceleration. Constant acceleration, therefore, implies that the *change* in velocity is uniform with time in direction and magnitude. If there is *no change* in velocity, that is, if the velocity were to remain constant both in magnitude and direction, then $\Delta\mathbf{v}$ would be zero for all time intervals and the acceleration would be zero.

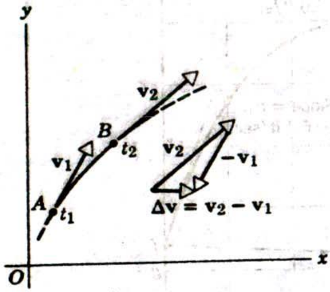


Fig. 3–6 A particle has velocity $\mathbf{v}_1$ at point A and moves to point B, where its velocity is $\mathbf{v}_2$. The triangle shows the (vector) change in velocity $\Delta\mathbf{v}$ $(= \mathbf{v}_2 - \mathbf{v}_1)$ experienced by the particle as it moves from A to B.

If a particle is moving in such a way that its average acceleration, measured for a number of different time intervals, does *not* turn out to be constant, the particle is said to have a variable acceleration. The acceleration can vary in magnitude, or in direction, or both. In such cases we seek to determine the acceleration of the particle at any given time, called the instantaneous acceleration.

The *instantaneous acceleration* is defined by

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}. \tag{3–9}$$

That is, the acceleration of a particle at time t is *the limiting value of* $\Delta\mathbf{v}/\Delta t$ at time t as both $\Delta\mathbf{v}$ and Δt approach zero. The direction of the instantaneous acceleration $\mathbf{a}$ is the limiting direction of the vector change in velocity $\Delta\mathbf{v}$. The magnitude a of the instantaneous acceleration is simply $a = |\mathbf{a}| = |d\mathbf{v}/dt|$. When the acceleration is constant the instantaneous acceleration equals the average acceleration. The student should note that the relation of $\mathbf{a}$ to $\mathbf{v}$, in Eq. 3–9, is the same as that of $\mathbf{v}$ to $\mathbf{r}$, in Eq. 3–2.

Two special cases illustrate that acceleration can arise from a change in either the magnitude or the direction of the velocity. In one case we have motion along a straight line with uniformly changing speed (as in Section 3–8). Here the velocity does not change in direction but its magnitude changes uniformly with time. This is a case of constant acceleration. In the second case we have motion in a circle at constant speed (Section 4–4). Here the velocity vector changes continuously in direction but its magnitude remains constant. This, too, is accelerated motion, though the direction of the acceleration vector is not constant. Later we will encounter other important cases of accelerated motion.

3–7 One-Dimensional Motion—Variable Acceleration

From Eqs. 3–5 and 3–9 we can write, for motion in two dimensions as in Fig. 3–3,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{i}\,\frac{dv_x}{dt} + \mathbf{j}\,\frac{dv_y}{dt}$$

or

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y, \tag{3–10}$$

where a_x ($= dv_x/dt$) and a_y ($= dv_y/dt$) are the (scalar) components of the acceleration vector $\mathbf{a}$ (see Fig. 3–3c).

We again restrict ourselves to motion in one dimension only, chosen for convenience to be the x-axis. Since v_y for such motion does not change with time (and is, in fact, zero), a_y, which is dv_y/dt, must also be zero so that

$$\mathbf{a} = \mathbf{i}a_x. \tag{3–11}$$

Since $\mathbf{i}$ points in the positive x-direction, a_x will be positive when $\mathbf{a}$ points in this direction and negative when it points in the other direction.

► **Example 3.** The motion of Fig. 3-5a is one of variable acceleration along the x-axis. To find the acceleration* a_x at each instant we must determine dv_x/dt at each instant. This is simply the slope of the curve of v_x versus t at that instant. The slope of Fig. 3–5c at point b is -1.3 ft/sec² and that at point d is -1.8 ft/sec², as shown in the figure. The result of calculating the slope for all points is shown in Fig. 3–5d. Notice that a_x is negative at all instants, which means that the acceleration vector $\mathbf{a}$ points in the negative x-direction. This means that v_x is always decreasing with time, as is clearly seen from Fig. 3–5c. The motion is one in which the acceleration vector has a constant direction but varies in magnitude (see Fig. 3–5a). ◄

3–8 One-Dimensional Motion—Constant Acceleration

Let us now further restrict our considerations to motion which not only occurs in one dimension (the x-axis) but for which a_x = a constant. For such *constant acceleration* the *average* acceleration for any time interval is equal to the (constant) instantaneous acceleration a_x. Let $t_1 = 0$ and let

* As for velocity, we commonly call a_x for one-dimensional motion the acceleration even though acceleration is a vector and a_x is correctly an acceleration component. For one-dimensional motion there is only one component if the axis is chosen along the line of the motion.

t_2 be any arbitrary time t. Let v_{x0} be the value of v_x at $t = 0$ and let v_x be its value at the arbitrary time t. With this notation we find a_x (see Eq. 3–8) from

$$a_x = \frac{\Delta v}{\Delta t} = \frac{v_x - v_{x0}}{t - 0}$$

or

$$v_x = v_{x0} + a_x t. \tag{3–12}$$

This equation states that the velocity v_x at time t is the sum of its value v_{x0} at time $t = 0$ plus the change in velocity during time t, which is $a_x t$.

Figure 3–7*c* shows a graph of v_x versus t for constant acceleration; it is a graph of Eq. 3–12. Notice that the slope of the velocity curve is constant, as it must be because the acceleration a_x $(= dv_x/dt)$ has been assumed to be constant, as Fig. 3–7*d* shows.

When the velocity v_x changes uniformly with time, its average value over any time interval equals one-half the sum of the values of v_x at the beginning and at the end of the interval. That is, the average velocity $\overline{v_x}$ between $t = 0$ and $t = t$ is

$$\overline{v_x} = \tfrac{1}{2}(v_{x0} + v_x). \tag{3–13}$$

This relation would not be true if the acceleration were not constant, for then the curve of v_x versus t would not be a straight line.

If the position of the particle at $t = 0$ is x_0, the position x at $t = t$ can be found from

$$x = x_0 + \overline{v_x}t$$

which can be combined with Eq. 3–13 to yield

$$x = x_0 + \tfrac{1}{2}(v_{x0} + v_x)t. \tag{3–14}$$

The displacement due to the motion in time t is $x - x_0$. Often the origin is chosen so that $x_0 = 0$.

Notice that aside from initial conditions of the motion, that is, the values of x and v_x at $t = 0$ (taken here as $x = x_0$ and $v_x = v_{x0}$), there are four parameters of the motion. These are x, the displacement; v_x, the velocity; a_x, the acceleration; and t, the elapsed time. If we know only that the acceleration is constant, but not necessarily its value, from any two of these parameters we can obtain the other two. For example, if a_x and t are known, Eq. 3–12 gives v_x, and having obtained v_x, we find x from Eq. 3–14.

In most problems in uniformly accelerated motion, two parameters are known and a third is sought. It is convenient, therefore, to obtain relations between any three of the four parameters. Equation 3–12 contains v_x, a_x, and t, but *not* x; Eq. 3–14 contains x, v_x, and t but *not* a_x. To complete our system of equations we need two more relations, one containing x, a_x, and t but *not* v_x and another containing x, v_x, and a_x but *not* t. These are easily obtained by combining Eqs. 3–12 and 3–14.

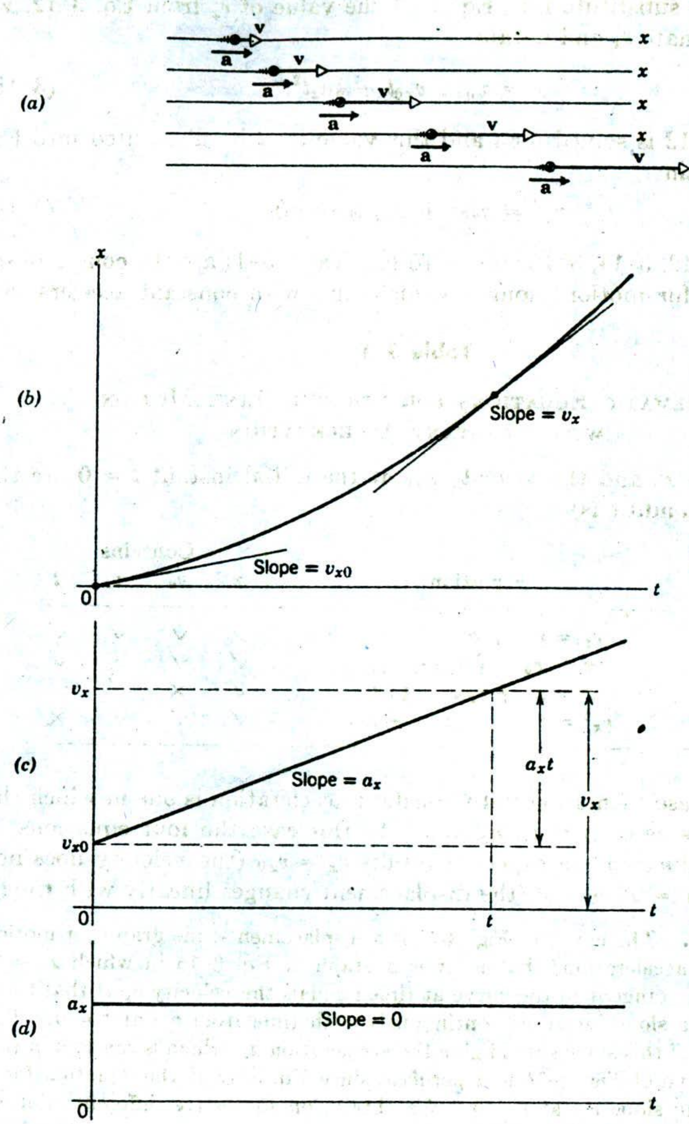


Fig. 3–7 (*a*) Five successive "snapshots" of rectilinear motion with constant acceleration. The arrows on the spheres represent **v**; those below represent **a**. (*b*) The displacement increases quadratically according to $x = v_{x0}t + \frac{1}{2}a_x t^2$. Its slope increases uniformly and at each instant has the value v_x, the velocity. (*c*) The velocity v_x increases uniformly according to $v_x = v_{x0} + a_x t$. Its slope is constant and at each instant has the value a_x, the acceleration. (*d*) The acceleration a_x has a constant value; its slope is zero. Figure 3-5 shows similar plots for one-dimensional motion in which the acceleration is *not* constant.

Thus, if we substitute into Eq. 3–14 the value of v_x from Eq. 3–12, we thereby eliminate v_x and obtain

$$x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2. \tag{3–15}$$

When Eq. 3–12 is solved for t and this value for t is substituted into Eq. 3–14, we obtain

$$v_x{}^2 = v_{x0}{}^2 + 2a_x(x - x_0). \tag{3–16}$$

Equations 3–12, 3–14, 3–15, and 3–16 (see Table 3–1) are the complete set of equations for motion along a straight line with constant acceleration.

Table 3–1

KINEMATIC EQUATIONS FOR STRAIGHT LINE MOTION WITH CONSTANT ACCELERATION

(The position x_0 and the velocity v_{x0} at the initial instant $t = 0$ are the given initial conditions)

Equation Number	Equation	Contains x	Contains v_x	Contains a_x	Contains t
3–12	$v_x = v_{x0} + a_x t$	×	✓	✓	✓
3–14	$x = x_0 + \frac{1}{2}(v_{x0} + v_x)t$	✓	✓	×	✓
3–15	$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$	✓	×	✓	✓
3–16	$v_x{}^2 = v_{x0}{}^2 + 2a_x(x - x_0)$	✓	✓	✓	×

A special case of motion with constant acceleration is one in which the acceleration is zero, that is, $a_x = 0$. In this case the four equations in Table 3–1 reduce to the expected results $v_x = v_{x0}$ (the velocity does not change) and $x = x_0 + v_{x0}t$ (the displacement changes linearly with time).

▶ **Example 4.** The curve of Fig. 3-7*b* is a displacement-time graph for motion with constant acceleration; that is, it is a graph of Eq. 3–15 in which $x_0 = 0$. The slope of the tangent to the curve at time t equals the velocity v_x at that time. Notice that the slope increases continuously with time from v_{x0} at $t = 0$. The *rate of increase* of this slope should give the acceleration a_x, which is *constant* in this case. The curve of Fig. 3–7*b* is a *parabola* since Eq. 3–15 is the equation for a parabola having slope v_{x0} at $t = 0$. We obtain, on successive differentiation of Eq. 3–15,

$$x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2$$

$$dx/dt = v_{x0} + a_x t \quad \text{or} \quad v_x = v_{x0} + a_x t,$$

which gives the velocity v_x at time t (compare Eq. 3–12), and

$$dv_x/dt = a_x,$$

the constant acceleration. The displacement-time graph for uniformly accelerated rectilinear motion will therefore always be parabolic. ◀

3–9 Consistency of Units and Dimensions

The student should not feel compelled to memorize relations such as those of Table 3–1. The important thing is to be able to follow the line of reasoning used to obtain the results. These relations will be recalled automatically after the student has used them repeatedly to solve problems, partly as a result of the familiarity acquired but chiefly as a result of the better understanding obtained through application.

We can use any convenient *units* of time and distance in these equations. If we choose to express time in seconds and distance in feet, for self-consistency we must express velocity in ft/sec and acceleration in ft/sec^2. If we are given data in which the units of one quantity, as velocity, are not consistent with the units of another quantity, as acceleration, then before using the data in our equations we should transform both quantities to units that are consistent with one another. Having chosen the units of our fundamental quantities, we automatically determine the units of our derived quantities consistent with them. In carrying out any calculation, always remember to attach the proper units to the final result, for the result is meaningless without this label.

▶ **Example 5.** Suppose we wish to find the speed of a particle which has a uniform acceleration of 5.00 cm/sec^2 for an interval of $\frac{1}{2}$ hr if the particle has a speed of 10.0 ft/sec at the beginning of this interval. We decide to choose the foot as our length unit and the second as our time unit. Then

$$a_x = 5.00\ \text{cm/sec}^2 = 5.00\ \text{cm/sec}^2 \times \left(\frac{1\ \cancel{\text{in.}}}{2.54\ \cancel{\text{cm}}}\right) \times \left(\frac{1\ \text{ft}}{12\ \cancel{\text{in.}}}\right)$$

$$= \frac{5.00}{30.5}\ \text{ft/sec}^2 = 0.164\ \text{ft/sec}^2.$$

The time interval

$$\Delta t = t - t_0 = \frac{1}{2}\ \cancel{\text{hr}} \times \left(\frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hr}}}\right) \times \left(\frac{60\ \text{sec}}{1\ \cancel{\text{min}}}\right) = 1800\ \text{sec}.$$

Note that the conversion factors in large parentheses are equal to unity. Taking the initial time $t_0 = 0$, as in Eq. 3–12, we then have

$$v_x = v_{x0} + a_x t = 10.0\ \text{ft/sec} + (0.164\ \text{ft/sec}^2)(1800\ \text{sec})$$

$$= 305\ \text{ft/sec.}$$ ◀

One way to spot an erroneous equation is to check the *dimensions* of all its terms. The dimensions of any physical quantity can always be expressed as some combination of the fundamental quantities, such as mass, length, and time, from which they are derived. The dimensions of velocity are length (L) divided by time (T); the dimensions of acceleration are length divided by time squared, etc. *In any legitimate physical equation the dimensions of all the terms must be the same.* This means, for example, that we cannot equate a term whose total dimension is a velocity to one whose total dimension is an acceleration. The dimensional labels attached to various quantities may be treated just like algebraic quantities and may be combined, canceled, etc., just as if they were factors in the equation. For example, to check Eq. 3–15, $x = x_0 + v_{x0}t + \frac{1}{2}a_xt^2$, dimensionally, we note that x and x_0 have the dimension of a length. Therefore the two remaining terms must also have the dimension of a length. The dimension of the term $v_{x0}t$ is

$$\frac{\text{length}}{\text{time}} \times \text{time} = \text{length} \qquad \text{or} \qquad \frac{L}{T} \times T = L,$$

and that of $\frac{1}{2}a_xt^2$ is

$$\frac{\text{length}}{\text{time}^2} \times \text{time}^2 = \text{length} \qquad \text{or} \qquad \frac{L}{T^2} \times T^2 = L.$$

The equation is therefore *dimensionally correct*. The student should check the dimensions of all the equations he uses.

▶ **Example 6.** The speed of an automobile traveling due east is uniformly reduced from 45.0 miles/hr to 30.0 miles/hr in a distance of 264 ft.

(*a*) What is the magnitude and direction of the constant acceleration?

We choose, arbitrarily, the direction from west to east to be the positive x-direction. We are given x and v_x and we seek a_x. The time is not involved. Equation 3–16 is therefore appropriate (see Table 3–1). We have $v_x = +30.0$ miles/hr, $v_{x0} = +45.0$ miles/hr, $x - x_0 = +264$ ft $= 0.0500$ mile. From Eq. 3–16, $v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$, we obtain

$$a_x = \frac{v_x^2 - v_{x0}^2}{2(x - x_0)},$$

or

$$a_x = \frac{(30.0 \text{ miles/hr})^2 - (45.0 \text{ miles/hr})^2}{2(0.0500 \text{ mile})} = -1.13 \times 10^4 \text{ miles/hr}^2$$

$$= -4.58 \text{ ft/sec}^2.$$

The direction of the acceleration **a** is due west, that is, in the negative x-direction because a_x is negative. The car is slowing down as it moves eastward, as it must do if it is being accelerated toward the west. When the speed of a body is decreasing, we often say that it is decelerating.

(*b*) How much time has elapsed during this deceleration?

If we use only the original data, Table 3–1 shows that Eq. 3–14 is appropriate. From Eq. 3–14, $x = x_0 + \frac{1}{2}(v_{x0} + v_x)t$, we obtain

$$t = \frac{2(x - x_0)}{v_{x0} + v_x},$$

or

$$t = \frac{(2)(0.0500 \text{ mile})}{(45.0 + 30.0) \text{ miles/hr}} = \frac{1}{750} \text{ hr} = 4.80 \text{ sec}.$$

If we use the derived data of part (*a*), Eq. 3–12 is appropriate. This gives us a check. From Eq. 3–12, $v_x = v_{x0} + a_x t$, we have

$$t = \frac{v_x - v_{x0}}{a_x}$$

or

$$t = \frac{(30.0 - 45.0)\ \text{miles/hr}}{-1.13 \times 10^4\ \text{miles/hr}^2} = 1.33 \times 10^{-3}\ \text{hr} = 4.80\ \text{sec.}$$

(*c*) If one assumes that the car continues to decelerate at the same rate, how much time would elapse in bringing it to rest from 45.0 miles/hr?

Equation 3–12 is useful here. We have $v_{x0} = 45.0$ miles/hr, $a_x = -1.13 \times 10^4$ miles/hr², and the final velocity $v_x = 0$. Then from Eq. 3–12, $v_x = v_{x0} + a_x t$, we obtain

$$t = \frac{v_x - v_{x0}}{a_x},$$

or

$$t = \frac{(0 - 45.0)\ \text{miles/hr}}{-1.13 \times 10^4\ \text{miles/hr}^2} = 4.00 \times 10^{-3}\ \text{hr} = 14.4\ \text{sec.}$$

(*d*) What total distance is required to bring the car to rest from 45.0 miles/hr? Equation 3–15 is appropriate here. We have $v_{x0} = 45.0$ miles/hr, $a_x = -1.13 \times 10^4$ miles/hr², $t = 4.00 \times 10^{-3}$ hr. From Eq. 3–15, $x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$, we obtain

$$\begin{aligned} x - x_0 &= v_{x0}t + \tfrac{1}{2}a_x t^2 \\ &= (45.0\ \text{miles/hr})(4.00 \times 10^{-3}\ \text{hr}) \\ &\qquad + \tfrac{1}{2}(-1.13 \times 10^4\ \text{miles/hr}^2)\ (4.00 \times 10^{-3}\ \text{hr})^2 \\ &= 0.0900\ \text{mile} = 475\ \text{ft.} \end{aligned}$$

Example 7. The nucleus of a helium atom (alpha-particle) travels along the inside of a straight hollow tube 2.0 meters long which forms part of a particle accelerator. (*a*) If one assumes uniform acceleration, how long is the particle in the tube if it enters at a speed of 1000 meters/sec and leaves at 9000 meters/sec? (*b*) What is its acceleration during this interval?

(*a*) We choose an x-axis parallel to the tube, its positive direction being that in which the particle is moving and its origin at the tube entrance. We are given x and v_x and we seek t. The acceleration a_x is not involved. Hence we use Eq. 3–14, $x = x_0 + \frac{1}{2}(v_{x0} + v_x)\,t$ with $x_0 = 0$ or

$$t = \frac{2x}{v_{x0} + v_x},$$

$$t = \frac{(2)(2.0\ \text{meters})}{(1000 + 9000)\ \text{meters/sec}} = 4.0 \times 10^{-4}\ \text{sec},$$

or 400 microseconds.

(*b*) The acceleration follows from Eq. 3–12, $v_x = v_{x0} + a_x t$, or

$$a_x = \frac{v_x - v_{x0}}{t} = \frac{(9000 - 1000)\ \text{meters/sec}}{4.0 \times 10^{-4}\ \text{sec}} = +2.0 \times 10^7\ \text{meters/sec}^2,$$

or 20 million meters per second per second! Although this acceleration is enormous by standards of the previous example, it occurs over an extremely short time. The acceleration **a** is in the positive x-direction, that is, in the direction in which the particle is moving, because a_x is positive. ◀

3–10 Freely Falling Bodies

The most common example of motion with (nearly) constant acceleration is that of a body falling toward the earth. In the absence of air resistance it is found that all bodies, regardless of their size, weight, or composition, fall with the same acceleration at the same point of the earth's surface, and if the distance covered is not too great, the acceleration remains constant throughout the fall. This ideal motion, in which air resistance and the small change in acceleration with altitude are neglected, is called "free fall."

The acceleration of a freely falling body is called the acceleration due to gravity and is denoted by the symbol **g**. Near the earth's surface its magnitude is approximately 32 ft/sec^2, 9.8 meters/sec^2, or 980 cm/sec^2, and it is directed down toward the center of the earth. The variation of the exact value with latitude and altitude will be discussed later (Chapter 16).

The nature of the motion of a falling object was long ago a subject of interest in natural philosophy. Aristotle had asserted that "the downward movement . . . of any body endowed with weight is quicker in proportion to its size." It was not until many centuries later when Galileo Galilei (1564–1642), an Italian scientist of the Renaissance, appealed to experiment to discover the truth, and then publicly proclaimed it, that Aristotle's authority on the matter was seriously challenged. In the later years of his life, Galileo wrote the treatise entitled *Dialogues Concerning Two New Sciences* in which he detailed his studies of motion.* This treatise may be considered as marking the beginning of the science of dynamics.

Aristotle's belief that a heavier object will fall faster is a commonly held view. It appears to receive support from a well-known lecture demonstration in which a ball and a sheet of paper are dropped at the same instant, the ball reaching the floor much sooner than the paper. However, when the lecturer first crumples the paper tightly and then repeats the demonstration, both ball and paper strike the floor at essentially the same time. In the former case, it is the effect of greater resistance of the air which makes the paper fall more slowly than the ball. In the latter case, the effect of air resistance on the paper is reduced and is about the same for both bodies, so that they fall at about the same rate. Of course, a direct test can be made by dropping bodies in vacuum. Even in easily obtainable partial vacuums we can show that a feather and a ball of lead thousands of times heavier drop at rates that are practically indistinguishable.

In Galileo's time, however, there was no effective way to obtain a partial vacuum, nor did equipment exist to time freely falling bodies with sufficient precision to obtain reliable numerical data. Nevertheless, Galileo proved his result by showing

---

* Galileo made noteworthy contributions to astronomy by the application of his telescope. His strong evidence in favor of the Copernican hypothesis of the solar system served to refute the Ptolemaic system and on this account raised strong feelings against him in the minds of the leaders of the Church. Twice he was brought before the Inquisition. He was ordered not to publish anything in support of the Copernican system and was compelled to publicly disclaim his belief in it. It was during a period of fear and uncertainty that he wrote his dialogue on motion, not published until after his death.

first that the character of the motion of a ball rolling down an incline was the same as that of a ball in free fall. The incline merely served to reduce the effective acceleration of gravity and to slow the motion thereby. Time intervals measured by the volume of water discharged from a tank could then be used to test the speed and acceleration of this motion. Galileo showed that if the acceleration along the incline is constant, the acceleration due to gravity must also be constant; for the acceleration along the incline is simply a component of the vertical acceleration of gravity, and along an incline of constant slope the ratio of the two accelerations remains fixed.

He found from his experiments that the distances covered in consecutive time intervals were proportional to the odd numbers 1, 3, 5, 7, . . . , etc. Total distances for consecutive intervals thus were proportional to $1 + 3$, $1 + 3 + 5$, $1 + 3 + 5 + 7$, etc., that is, to the squares of the integers 1, 2, 3, 4, etc. But if the distance covered is proportional to the square of the elapsed time, velocity acquired is proportional to the elapsed time, a result which is true only if motion is uniformly accelerated. He found that the same results held regardless of the mass of the ball used.

3–11 Equations of Motion in Free Fall

We shall select a reference frame rigidly attached to the earth. The y-axis will be taken as positive vertically upward. Then the acceleration due to gravity **g** will be a vector pointing vertically down (toward the center of the earth) in the negative y-direction. (This choice is arbitrary. In other problems it may be convenient to choose down as positive.) Our equations for constant acceleration are applicable here. We simply replace x by y and set $y_0 = 0$ in Eqs. 3–12, 3–14, 3–15, and 3–16, obtaining

$$\begin{aligned} v_y &= v_{y0} + a_y t, \\ y &= \tfrac{1}{2}(v_{y0} + v_y)t, \\ y &= v_{y0}t + \tfrac{1}{2}a_y t^2, \\ v_y{}^2 &= v_{y0}{}^2 + 2a_y y, \end{aligned} \tag{3–17}$$

and, for problems in free fall, we set $a_y = -g$. Notice that we have chosen the initial position as the origin, that is we have chosen $y_0 = 0$ at $t = 0$. Note also that g is the magnitude of the acceleration due to gravity.

▶ **Example 8.** A body is dropped from rest and falls freely. Determine the position and speed of the body after 1.0, 2.0, 3.0, and 4.0 sec have elapsed.

We choose the starting point as the origin. We know the initial speed and the acceleration and we are given the time. To find the position we use

$$y = v_{y0}t - \tfrac{1}{2}gt^2.$$

Then, $v_{y0} = 0$ and $g = 32\ \text{ft/sec}^2$, and with $t = 1.0$ sec we obtain

$$y = 0 - \tfrac{1}{2}(32\ \text{ft/sec}^2)(1.0\ \text{sec})^2 = -16\ \text{ft}.$$

To find the speed with $t = 1.0$ sec, we use

$$v_y = v_{y0} - gt$$

and obtain $$v_y = 0 - (32\ \text{ft/sec}^2)(1.0\ \text{sec}) = -32\ \text{ft/sec}.$$

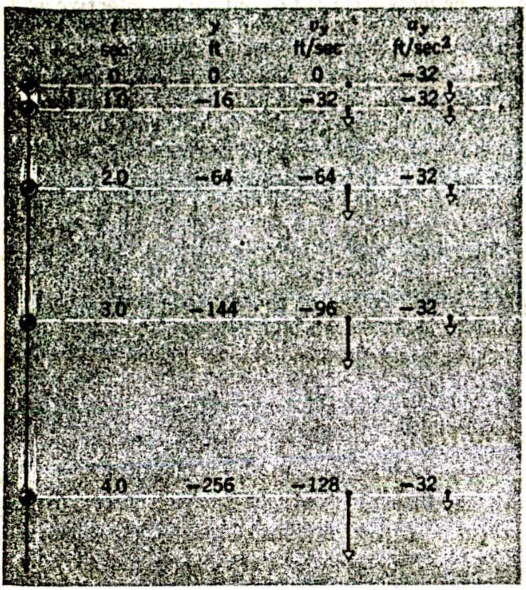


Fig. 3–8 A body in free fall; showing y, v_y, and a_y at particular times t.

After 1.0 sec of falling from rest, the body is 16 ft below its starting point and has a velocity directed downward whose magnitude is 32 ft/sec; the negative signs for y and v_y show that the associated vectors each point in the negative y-direction, that is, downward.

The student should now show that the values of y, v_y, and a_y obtained at times $t = 2.0$, 3.0, and 4.0 sec are those shown in Fig. 3–8.

Example 9. A ball is thrown vertically upward from the ground with a speed of 80 ft/sec.

(*a*) How long does it take to reach its highest point?

At its highest point, $v_y = 0$, and we have $v_{y0} = +80$ ft/sec. To obtain the time t we use $v_y = v_{y0} - gt$, or

$$t = \frac{v_{y0} - v_y}{g}$$

$$t = \frac{(80 - 0)\ \text{ft/sec}}{32\ \text{ft/sec}^2} = 2.5\ \text{sec}.$$

(*b*) How high does the ball rise? Using only the original data, we choose the relation $v_y{}^2 = v_{y0}{}^2 - 2gy$, or

$$y = \frac{v_{y0}{}^2 - v_y{}^2}{2g},$$

$$= \frac{(80\ \text{ft/sec})^2 - 0}{2 \times 32\ \text{ft/sec}^2} = +100\ \text{ft}.$$

(c) At what times will the ball be 96 ft above the ground? Using $y = v_{y0}t - \frac{1}{2}gt^2$, we have

$$\tfrac{1}{2}gt^2 - v_{y0}t + y = 0,$$

$$\tfrac{1}{2}(32\ \text{ft/sec}^2)t^2 - (80\ \text{ft/sec})t + 96\ \text{ft} = 0,$$

or

$$t^2 - 5.0t + 6.0 = 0,$$

which yields $t = 2.0$ sec and $t = 3.0$ sec.

At $t = 2.0$ sec, the ball is moving upward with a speed of 16 ft/sec, for

$$v_y = v_{y0} - gt = 80\ \text{ft/sec} - (32\ \text{ft/sec}^2)(2.0\ \text{sec}) = +16\ \text{ft/sec}.$$

At $t = 3.0$ sec, the ball is moving downward with the same speed, for

$$v_y = v_{y0} - gt = 80\ \text{ft/sec} - (32\ \text{ft/sec}^2)(3.0\ \text{sec}) = -16\ \text{ft/sec}.$$

Notice that in this 1.0-sec interval the velocity changed by -32 ft/sec, corresponding to an acceleration of $-32\ \text{ft/sec}^2$.

The student should be able to convince himself that in the absence of air resistance the ball will take as long to rise as to fall the same distance, and that it will have the same speed going down at each point as it had going up. ◀

QUESTIONS

1. Can you think of physical phenomena involving the earth in which the earth cannot be treated as a particle?

2. Each second a rabbit moves half the remaining distance from his nose to a head of lettuce. Does he ever get to the lettuce? What is the limiting value of his average velocity? Draw graphs showing his velocity and position as time increases.

3. Average speed can mean the magnitude of the average velocity vector. Another meaning given to it is that average speed is the total length of path traveled divided by the elapsed time. Are these meanings different? If so, give an example.

4. When the velocity is constant, does the average velocity over any time interval differ from the instantaneous velocity at any instant?

5. Is the average velocity of a particle moving along the x-axis $\frac{1}{2}(v_{x0} + v_x)$ when the acceleration is not uniform? Prove your answer with the use of graphs.

6. Does the speedometer on an automobile register speed as we defined it?

7. (a) Can a body have zero velocity and still be accelerating? (b) Can a body have a constant speed and still have a varying velocity? (c) Can a body have a constant velocity and still have a varying speed?

8. Can an object have an eastward velocity while experiencing a westward acceleration?

9. Can the direction of the velocity of a body change when its acceleration is constant?

10. Devise a scheme for keeping time with a "water clock" such as Galileo used. Can you avoid repetitive operations and still keep accurate time?

11. If a particle is released from rest ($v_{y0} = 0$) at $y_0 = 0$ at the time $t = 0$, Eq. 3-17 for constant acceleration says that it is at position y at two different times, namely, $+\sqrt{2y/a_y}$ and $-\sqrt{2y/a_y}$. What is the meaning of the negative root of this quadratic equation?

12. What happens to our kinematic equations under the operation of time reversal, that is, replacing t by $-t$? Explain.

13. Consider a ball thrown vertically up. Taking air resistance into account, would you expect the time during which the ball rises to be longer or shorter than the time during which it falls?

14. Can there be motion in two dimensions with an acceleration in only one dimension?

15. A man standing on the edge of a cliff at some height above the ground below throws one ball straight up with initial speed u and then throws another ball straight down with the same initial speed. Which ball, if either, has the larger speed when it hits the ground? Neglect air resistance.

16. From what you know about angular measure, what *dimensions* would you assign to an angle? Can a quantity have units without having dimensions?

17. If m is a light stone and M is a heavy one, according to Aristotle M should fall faster than m. Galileo attempted to show that Aristotle's belief was logically inconsistent by the following argument. Tie m and M together to form a double stone. Then, in falling, m should retard M, since it tends to fall more slowly, and the combination would fall faster than m but more slowly than M; but according to Aristotle the double body $(M + m)$ is heavier than M and hence should fall faster than M.

If you accept Galileo's reasoning as correct, can you conclude that M and m must fall at the same rate? What need is there for experiment in that case?

If you believe Galileo's reasoning is incorrect, explain why.

PROBLEMS

1. Compare your average speed in the following two cases. (*a*) You walk 240 ft at a speed of 4.0 ft/sec and then run 240 ft at a speed of 10 ft/sec along a straight track. (*b*) You walk for 1.0 min at a speed of 4.0 ft/sec and then run for 1.0 min at 10 ft/sec along a straight track.

2. A train moving at an essentially constant speed of 60 miles/hr moves eastward for 40 min, then in a direction 45° east of north for 20 min, and finally westward for 50 min. What is the average velocity of the train during this run?

3. Two trains, each having a speed of 30 miles/hr, are headed at each other on the same straight track. A bird that can fly 60 miles/hr flies off one train when they are 60 miles apart and heads directly for the other train. On reaching the other train it flies directly back to the first train, and so forth. (*a*) How many trips can the bird make from one train to the other before they crash? (*b*) What is the total distance the bird travels?

4. A particle moving along a horizontal line has the following positions at various instants of time:

x(meters) =	0.080	0.050	0.040	0.050	0.080	0.13	0.68
t(sec) =	0.0	1.0	2.0	3.0	4.0	5.0	10

(*a*) Plot displacement (not position) versus time. (*b*) Find the average velocity of the particle in the intervals 0.0 to 1.0 sec, 0.0 to 2.0 sec, 0.0 to 3.0 sec, 0.0 to 4.0 sec. (*c*) Find the slope of the curve drawn in part *a* at the points t = 1.0, 2.0, 3.0, 4.0, and 5.0. (*d*) Plot the slope (units?) versus time. (*e*) From the curve of part *d* determine the acceleration of the particle at times t = 2.0, 3.0 and 4.0 sec.

5. A tennis ball is dropped onto the floor from a height of 4.0 ft. It rebounds to a height of 3.0 ft. If the ball was in contact with the floor for 0.010 sec, what was its average acceleration during contact?

6. The graph of x versus t (see Fig. 3–9a) is for a particle in straight line motion. State for each interval whether the velocity v_x is $+$, $-$, or 0, and whether the acceleration a_x is $+$, $-$, or 0. The intervals are OA, AB, BC, and CD. From the curve is there any *interval* over which the acceleration is obviously *not constant*? (Ignore the behavior at the end points of the intervals.)

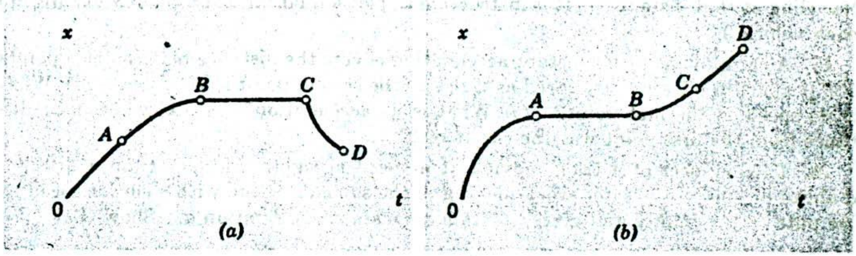


Fig. 3–9

7. Answer the previous questions for the motion described by the graph of Fig. 3–9b.

8. An arrow while being shot from a bow was accelerated over a distance of 2.0 ft. If its speed at the moment it left the bow was 200 ft/sec what was the average acceleration imparted by the bow? Justify any assumptions you need to make.

9. An electron with initial velocity $v_{x0} = 1.0 \times 10^4$ meters/sec enters a region where it is electrically accelerated (Fig. 3–10). It emerges with a velocity $v_x = 4.0 \times 10^6$ meters/sec. What was its acceleration, assumed constant? (Such a process occurs in the electron gun in a cathode-ray tube, used in television receivers and oscilloscopes.)

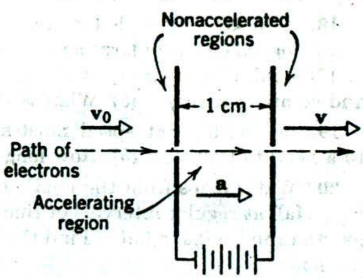


Fig. 3–10

10. Suppose that you were called upon to give some advice to a lawyer concerning the physics involved in one of his cases. The question is whether a driver was exceeding a 30 miles/hr speed limit before he made an emergency stop, brakes locked and wheels sliding. The length of skid marks on the road was 19.2 ft. The policeman made the reasonable assumption that the maximum deceleration of the car would not exceed the acceleration of a freely falling body and arrested the driver for speeding. Was he speeding? Explain.

11. A meson is shot with constant speed 5.00×10^6 meters/sec into a region where an electric field produces an acceleration on the meson of magnitude 1.25×10^{14} meters/sec^2 directed opposite to the initial velocity. How far does the meson travel before coming to rest? How long does the meson remain at rest?

12. A rocketship in free space moves with constant acceleration equal to 32 ft/sec^2. (a) If it starts from rest, how long will it take to acquire a speed one-tenth that of light? (b) How far will it travel in so doing?

13. A train started from rest and moved with constant acceleration. At one time it was traveling 30 ft/sec, and 160 ft farther on it was traveling 50 ft/sec. Calculate (a) the acceleration, (b) the time required to travel the 160 ft mentioned, (c) the time

required to attain the speed of 30 ft/sec, (*d*) the distance moved from rest to the time the train had a speed of 30 ft/sec.

14. At the instant the traffic light turns green, an automobile starts with a constant acceleration a_x of 6.0 ft/sec^2. At the same instant a truck, traveling with a constant speed of 30 ft/sec, overtakes and passes the automobile. (*a*) How far beyond the starting point will the automobile overtake the truck? (*b*) How fast will the car be traveling at that instant? (It is instructive to plot a qualitative graph of x versus t for each vehicle.)

15. A car moving with constant acceleration covers the distance between two points 180 ft apart in 6.0 sec. Its speed as it passes the second point is 45 ft/sec. (*a*) What is its speed at the first point? (*b*) What is its acceleration? (*c*) At what prior distance from the first point was the car at rest?

16. The engineer of a train moving at a speed v_1 sights a freight train a distance d ahead of him on the same track moving in the same direction with a slower speed v_2. He puts on the brakes and gives his train a constant deceleration a. Show that

$$\text{if} \quad d > \frac{(v_1 - v_2)^2}{2a}, \quad \text{there will be no collision;}$$

$$\text{if} \quad d < \frac{(v_1 - v_2)^2}{2a}, \quad \text{there will be a collision.}$$

(It is instructive to plot a qualitative graph of x versus t for each train.)

17. Two trains, one traveling at 60 miles/hr and the other at 80 miles/hr, are headed toward one another along a straight level track. When they are 2.0 miles apart, both engineers simultaneously see the other's train and apply their brakes. If the brakes decelerate each train at the rate of 3.0 ft/sec^2, determine whether there is a collision.

18. A rocket-driven sled running on a straight level track is used to investigate the physiological effects of large accelerations on humans. One such sled can attain a speed of 1000 miles/hr in 1.8 sec starting from rest. (*a*) Assume the acceleration is constant and compare it to g. (*b*) What is the distance traveled in this time?

19. (*a*) With what speed must a ball be thrown vertically upward in order to rise to a height of 50 ft? (*b*) How long will it be in the air?

20. Water drips from the nozzle of a shower onto the stall floor 81 in. below. The drops fall at regular intervals of time, the first drop striking the floor at the instant the fourth drop begins to fall. Find the location of the individual drops when a drop strikes the floor.

21. If a body travels half its total path in the last second of its fall from rest, find the time and height of its fall. Explain the physically unacceptable solution of the quadratic time equation.

22. An artillery shell is fired directly up from a gun; a rocket, propelled by burning fuel, takes off vertically from a launching area. Plot qualitatively (numbers not required) *possible* graphs of a_y versus t, of v_y versus t, and of y versus t for each. Take $t = 0$ at the instant the shell leaves the gun barrel or the rocket leaves the ground. Continue the plots until the rocket and the shell fall back to earth; neglect air resistance; assume that up is positive and down is negative.

23. A rocket is fired vertically and ascends with a constant vertical acceleration of 64 ft/sec^2 for 1.0 min. Its fuel is then all used and it continues as a free particle. (*a*) What is the maximum altitude reached? (*b*) What is the total time elapsed from take-off until the rocket strikes the earth?

24. A lead ball is dropped into a lake from a diving board 16 ft above the water. It hits the water with a certain velocity and then sinks to the bottom with this same constant velocity. It reaches the bottom 5.0 sec after it is dropped. (*a*) How deep is the lake? (*b*) What is the average velocity of the ball? (*c*) Suppose all the water is drained from the lake. The ball is thrown from the diving board so that it again reaches the bottom in 5.0 sec. What is the initial velocity of the ball?

25. A stone is dropped into the water from a bridge 144 ft above the water. Another stone is thrown vertically down 1.0 sec after the first is dropped. Both stones strike the water at the same time. (*a*) What was the initial speed of the second stone? (*b*) Plot speed versus time on a graph for each stone, taking zero time as the instant the first stone was released.

26. A steel ball bearing is dropped from the roof of a building (the initial velocity of the ball is zero). An observer standing in front of a window 4.0 ft high notes that the ball takes $\frac{1}{8}$ sec to fall from the top to the bottom of the window. The ball bearing continues to fall, makes a completely elastic collision with a horizontal sidewalk, and reappears at the bottom of the window 2.0 sec after passing it on the way down. How tall is the building? (The ball will have the same speed at a point going up as it had going down after a completely elastic collision.)

27. A dog sees a flowerpot sail up and then back down past a window 5.0 ft high. If the total time the pot is in sight is 1.0 sec, find the height above the window that the pot rises.

28. A balloon is ascending at the rate of 12 meters/sec at a height 80 meters above the ground when a package is dropped. How long does it take the package to reach the ground?

29. A parachutist after bailing out falls 50 meters without friction. When the parachute opens, he decelerates downward 2.0 meters/sec^2. He reaches the ground with a speed of 3.0 meters/sec. (*a*) How long is the parachutist in the air? (*b*) At what height did he bail out?

30. An elevator ascends with an upward acceleration of 4.0 ft/sec^2. At the instant its upward speed is 8.0 ft/sec, a loose bolt drops from the ceiling of the elevator 9.0 ft from the floor. Calculate (*a*) the time of flight of the bolt from ceiling to floor and (*b*) the distance it has fallen relative to the elevator shaft.

31. The position of a particle moving along the x-axis depends on the time according to the equation

$$x = at^2 - bt^3,$$

where x is in feet and t in seconds. (*a*) What dimensions and units must a and b have? For the following, let their numerical values be 3.0 and 1.0, respectively. (*b*) At what time does the particle reach its maximum positive x-position? (*c*) What total length of path does the particle cover in the first 4.0 sec? (*d*) What is its displacement during the first 4.0 sec? (*e*) What is the particle's speed at the end of each of the first four seconds? (*f*) What is the particle's acceleration at the end of each of the first four seconds?

32. An electron, starting from rest, has an acceleration that increases linearly with time, that is, $a = kt$, the change in acceleration being $k = (1.5 \text{ meters/sec}^2)/\text{sec}$. (*a*) Plot a versus t during the first 10-sec interval. (*b*) From the curve of part (*a*) plot the corresponding v versus t curve and estimate the electron's velocity 5.0 sec after its motion starts. (*c*) From the v versus t curve of part (*b*) plot the corresponding x versus t curve and estimate how far the electron moved during the first 5.0 sec of its motion.

33. The position of a particle moving along the x-axis depends on the time according to the relation

$$x = \frac{v_{x0}}{k}(1 - e^{-kt})$$

in which v_{x0} and k are constants. (*a*) Plot a curve of x versus t. Notice that $x = 0$ at $t = 0$ and that $x = v_{x0}/k$ at $t = \infty$; that is, the total distance through which the particle moves is v_{x0}/k. (*b*) Show that the velocity v_x is given by

$$v_x = v_{x0}e^{-kt}$$

so that the velocity decreases exponentially with time from its initial value of v_{x0}, coming to rest only in infinite time. (*c*) Show that the acceleration a_x is given by

$$a_x = -kv_x$$

so that the acceleration is directed opposite to the velocity and has a magnitude proportional to the speed. (*d*) This particular motion is one with variable acceleration. Give a plausible physical argument explaining how it can take an infinite time to bring to rest a particle that travels a finite distance.

Motion in a Plane

CHAPTER 4

4–1 Displacement, Velocity, and Acceleration

In this chapter we return to a consideration of motion in two dimensions taken, for convenience, to be the x-y plane. Figure 4–1 shows a particle at time t moving along a curved path in this plane. Its *position*, or displacement from the origin, is measured by the vector $\mathbf{r}$; its *velocity* is indicated by the vector $\mathbf{v}$ which, as we have seen in Section 3–4, must be tangent to the path of the particle. The *acceleration* is indicated by the vector $\mathbf{a}$; the direction of $\mathbf{a}$, as we shall see more explicitly later, does not bear any unique relationship to the path of the particle but depends rather on the rate at which the velocity $\mathbf{v}$ changes with time as the particle moves along its path.

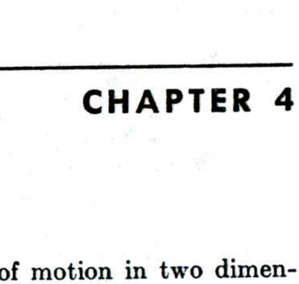


Fig. 4–1 A particle moves along a curved path in the x-y plane. Its position $\mathbf{r}$, velocity $\mathbf{v}$, and acceleration $\mathbf{a}$ are shown at time t, along with the vector components of those vectors. Note that x, y, v_x, v_y, and a_x are positive but that a_y is negative. Compare to Fig. 3-3.

The vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ are interrelated (see Eqs. 3–4, 3–5, and 3–10) and can be expressed in terms of their components, using unit vector notation, as

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y, \tag{4–1}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}v_x + \mathbf{j}v_y, \tag{4–2}$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{i}a_x + \mathbf{j}a_y. \tag{4–3}$$

These equations can easily be extended to three dimensions by adding to them the terms $\mathbf{k}z$, $\mathbf{k}v_z$, and $\mathbf{k}a_z$, respectively in which $\mathbf{k}$ is a unit vector in the z-direction.

In Chapter 3 we considered the special case in which the particle moved in one dimension only, say along the x-axis, where the vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ were directed along this axis, either in the positive x-direction or the negative x-direction. The components y, v_y, and a_y were zero and we described the motion in terms of equations relating the scalar quantities x, v_x, and a_x. Or, when the particle moved along the y-axis, the components x, v_x, and a_x were zero and the motion was described in terms of equations relating the scalar quantities y, v_y, and a_y. In this chapter we consider motion in the x-y plane so that, in general, both sets of components have nonzero values.

4–2 Motion in a Plane with Constant Acceleration

Let us consider first the special case of motion in a plane with *constant* acceleration. Here, as the particle moves, the acceleration $\mathbf{a}$ does not vary either in magnitude or in direction. Hence the components of $\mathbf{a}$ in any particular reference frame also will not vary, that is, a_x = constant and a_y = constant. We then have a situation which can be described as the sum of two component motions occurring simultaneously with constant acceleration along each of two perpendicular directions. The particle will move, in general, along a curved path in the plane. This may be so even if one component of the acceleration, say a_x, is zero, for then the corresponding component of the velocity, say v_x, may have a constant, nonzero value. An example of this latter situation is the motion of an artillery shell which follows a curved path in a vertical plane and, neglecting the effects of air resistance, is subject to a constant acceleration $\mathbf{g}$ directed down along the y-axis only.

We can obtain the general equations for plane motion with constant $\mathbf{a}$ simply by setting

$$a_x = \text{constant} \qquad \text{and} \quad a_y = \text{constant}.$$

The equations for constant acceleration, summarized in Table 3–1, then apply to both the x- and y-components of the position vector $\mathbf{r}$, the velocity vector $\mathbf{v}$, and the acceleration vector $\mathbf{a}$ (see Table 4–1).

The two sets of equations in Table 4–1 are related in that the time parameter t is the same for each, since t represents the time at which the particle, moving in a curved path in the x-y plane, occupied a position described by the position components x and y.

The equations of motion in Table 4–1 may also be expressed in vector form. For example, substituting Eqs. 4–4a, 4a' into Eq. 4–2 yields

$$\begin{aligned}\mathbf{v} &= \mathbf{i}v_x + \mathbf{j}v_y\\ &= \mathbf{i}(v_{x0} + a_xt) + \mathbf{j}(v_{y0} + a_yt)\\ &= (\mathbf{i}v_{x0} + \mathbf{j}v_{y0}) + (\mathbf{i}a_x + \mathbf{j}a_y)t.\end{aligned}$$

Table 4–1

MOTION WITH CONSTANT ACCELERATION IN THE x-y PLANE

Equation No.	x-Motion Equations	Equation No.	y-Motion Equations
4–4a	$v_x = v_{x0} + a_x t$	4–4a'	$v_y = v_{y0} + a_y t$
4–4b	$x = x_0 + \frac{1}{2}(v_{x0} + v_x)t$	4–4b'	$y = y_0 + \frac{1}{2}(v_{y0} + v_y)t$
4–4c	$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$	4–4c'	$y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2$
4–4d	$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$	4–4d'	$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$

The first quantity in parentheses is the initial velocity vector $\mathbf{v}_0$ (see Eq. 4–2) and the second is the (constant) acceleration vector $\mathbf{a}$ (see Eq. 4–3). Thus the vector relation

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t \tag{4–5a}$$

is equivalent to the two scalar relations Eqs. 4–4a, a' in Table 4–1. It shows simply and compactly that the velocity $\mathbf{v}$ at time t is the sum of the initial velocity $\mathbf{v}_0$ which the particle would have in the absence of acceleration plus the (vector) change in velocity, $\mathbf{a}t$, acquired during the time t under the constant acceleration $\mathbf{a}$. Similarly, the scalar equations 4–4c, c' are equivalent to the single vector equation

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2, \tag{4–5b}$$

which is also easily interpreted. The proof of this and other relations is left to Problem 17.

4–3 Projectile Motion

An example of curved motion with constant acceleration is projectile motion. This is the two-dimensional motion of a particle thrown obliquely into the air. The ideal motion of a baseball, a golf ball, or a bullet is an example of projectile motion.* We assume that the effect the air itself would have on their motions can be neglected.

The motion of a projectile is one of constant acceleration $\mathbf{g}$, directed downward, and thus should be described by the equations in Table 4–1. There is no horizontal component of acceleration. If we choose a reference frame with the positive y-axis vertically upward, we may put $a_y = -g$ and $a_x = 0$ in these equations.

Let us further choose the origin of our reference frame to be the point at which the projectile begins its flight (see Fig. 4–2). Hence the origin will be the point at which the ball leaves the thrower's hand or the fuel in the rocket burns out, for example. In Table 4–1 this choice of origin implies

* See Galileo Galilei, *Dialogues Concerning Two New Sciences*, the "Fourth Day," for a fascinating discussion of Galileo's research on projectiles.

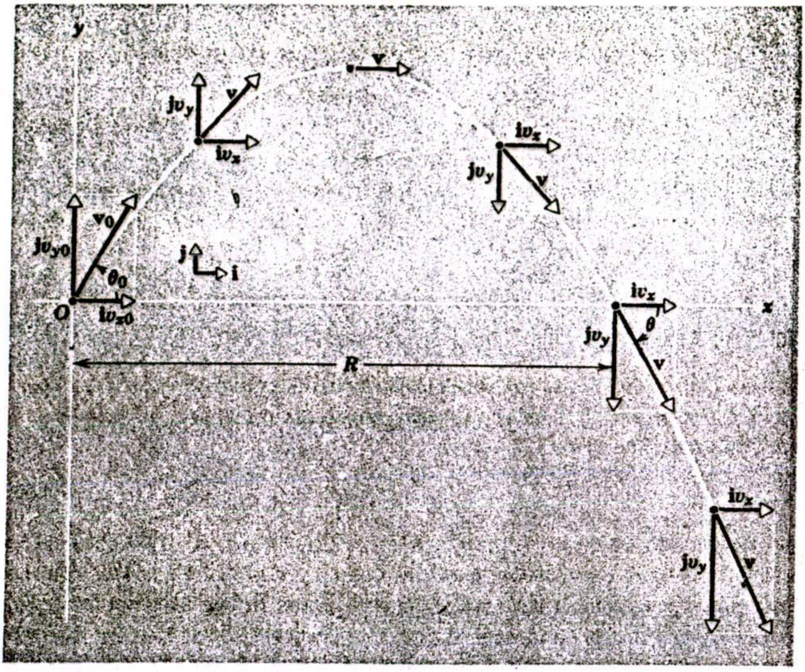


Fig. 4–2 The trajectory of a projectile, showing the initial velocity $\mathbf{v}_0$ and its vector components and also the velocity $\mathbf{v}$ and its vector components at five later times. Note that $v_x = v_{x0}$ throughout the flight. The distance R is the horizontal range.

that $x_0 = y_0 = 0$. The velocity at $t = 0$, the instant the projectile begins its flight, is $\mathbf{v}_0$, which makes an angle θ_0 with the positive x-direction. The x- and y-components of $\mathbf{v}_0$ (see Fig. 4–2) are then

$$v_{x0} = v_0 \cos \theta_0 \qquad \text{and} \qquad v_{y0} = v_0 \sin \theta_0.$$

Because there is no horizontal component of acceleration, the horizontal component of the velocity will be constant. In Eq. 4–4a we set $a_x = 0$ and $v_{x0} = v_0 \cos \theta_0$, so that

$$v_x = v_0 \cos \theta_0. \tag{4–6a}$$

The horizontal velocity component retains its initial value throughout the flight.

The vertical component of the velocity will change with time in accordance with vertical motion with constant downward acceleration. In Eq. 4–4a' we set

$$a_y = -g \qquad \text{and} \qquad v_{y0} = v_0 \sin \theta_0,$$

so that

$$v_y = v_0 \sin \theta_0 - gt. \tag{4–6a$'$}$$

The vertical velocity component is that of free fall. Indeed, if we view the motion of Fig. 4-2 from a reference frame that moves to the right with a speed v_{x0}, the motion will be that of an object thrown vertically upward with an initial speed $v_0 \sin \theta_0$.

The *magnitude* of the resultant velocity vector at any instant is

$$v = \sqrt{v_x^2 + v_y^2}. \tag{4-7}$$

The angle θ that the velocity vector makes with the horizontal at that instant is given by

$$\tan \theta = \frac{v_y}{v_x}.$$

The velocity vector is tangent to the path of the particle at every point, as shown in Fig. 4-2.

The x-coordinate of the particle's position at any time, obtained from Eq. 4-4c with $x_0 = 0$, $a_x = 0$, and $v_{x0} = v_0 \cos \theta_0$, is

$$x = (v_0 \cos \theta_0)t. \tag{4-6c}$$

The y-coordinate, obtained from Eq. 4-4c' with $y_0 = 0$, $a_y = -g$, and $v_{y0} = v_0 \sin \theta_0$, is

$$y = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2. \tag{4-6c$'$}$$

Equations 4-6c, c' give us x and y as functions of the common parameter t, the time in flight. By combining and eliminating t from them, we obtain

$$y = (\tan \theta_0)x - \frac{g}{2(v_0 \cos \theta_0)^2} x^2, \tag{4-8}$$

which relates y to x and is the equation of the trajectory of the projectile. Since v_0, θ_0, and g are constants, this equation has the form

$$y = bx - cx^2,$$

the equation of a parabola. Hence the trajectory of a projectile is parabolic.

▶ **Example 1.** A bomber is flying at a constant horizontal velocity of 820 miles/hr at an elevation of 52,000 ft toward a point directly above its target. At what angle of sight ϕ should a bomb be released to strike the target (Fig. 4-3)?

We choose a reference frame fixed with respect to the earth, its origin O being the bomb release point. The motion of the bomb at the instant of release is the same as that of the bomber. Hence the initial projectile velocity $\mathbf{v}_0$ is horizontal and its magnitude is 820 miles/hr or 1200 ft/sec. The angle of projection θ_0 is zero.

The time of fall is obtained from Eq. 4-6c'. With $\theta_0 = 0$ and $y = -52{,}000$ ft, this gives

$$t = \sqrt{-\frac{2y}{g}} = \sqrt{-\frac{2(-52{,}000)\text{ft}}{32\text{ ft/sec}^2}} = 57 \text{ sec}.$$

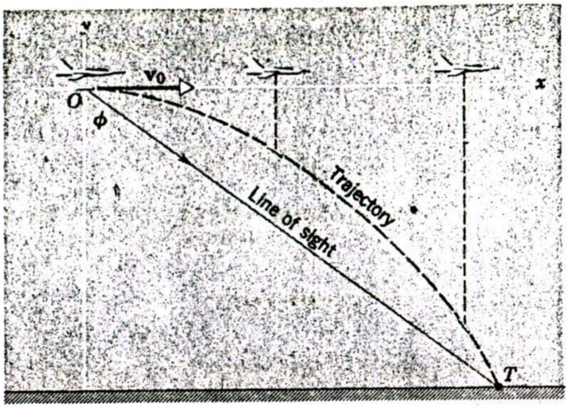


Fig. 4–3 Example 1. A bomb is released from an airplane with horizontal velocity v_0.

Note that the time of fall of the bomb does *not* depend on the speed of the plane for a *horizontal* projection. (See, however, Problem 10.)

The horizontal distance traveled by the bomb in this time is given by Eq. 4–6c, $x = (v_0 \cos \theta_0)t$, or $x = (1200 \text{ ft/sec})(57 \text{ sec}) = 68{,}000$ ft so that the angle of sight (Fig. 4–3) should be

$$\phi = \tan^{-1} \frac{x}{|y|} = \tan^{-1} \frac{68{,}000}{52{,}000} = 53°.$$

Does the motion of the bomb appear to be parabolic when viewed from a reference frame fixed with respect to the bomber?

Example 2. A soccer player kicks a ball at an angle of 37° from the horizontal with an initial speed of 50 ft/sec. (A right triangle, one of whose angles is 37°, has sides in the ratio 3:4:5, or 6:8:10.) Assuming that the ball moves in a vertical plane:

(*a*) Find the time t_1 at which the ball reaches the highest point of its trajectory. At the highest point, the vertical component of velocity v_y is zero. Solving Eq. 4–6*a′* for *t*, we obtain

$$t = \frac{v_0 \sin \theta_0 - v_y}{g}.$$

With

$$v_y = 0, \qquad v_0 = 50 \text{ ft/sec}, \qquad \theta_0 = 37°, \qquad g = 32 \text{ ft/sec}^2,$$

we have

$$t_1 = \frac{[50(\tfrac{6}{10}) - 0] \text{ ft/sec}}{32 \text{ ft/sec}^2} = \frac{15}{16} \text{ sec}.$$

(*b*) How high does the ball go? The maximum height is reached at $t = 15/16$ sec. By using Eq. 4–6*c′*,

$$y = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2,$$

we have

$$y_{max} = (50 \text{ ft/sec})(\tfrac{6}{10})(\tfrac{15}{16} \text{ sec}) - \tfrac{1}{2}(32 \text{ ft/sec}^2)(\tfrac{15}{16})^2 \text{ sec}^2 = 14 \text{ ft}$$

(*c*) What is the horizontal range of the ball and how long is it in the air?

The horizontal distance from the starting point at which the ball returns to its original elevation (ground level) is the *range* R. We set $y = 0$ in Eq. 4–6*c*′ and find the time t_2 required to traverse this range. We obtain

$$t_2 = \frac{2v_0 \sin \theta_0}{g} = \frac{2(50 \text{ ft/sec})(\frac{6}{10})}{32 \text{ ft/sec}^2} = \frac{15}{8} \text{ sec.}$$

Notice that $t_2 = 2t_1$. This corresponds to the fact that the same time is required for the ball to go up (reach its maximum height from ground) as is required for the ball to come down (reach the ground from its maximum height).

The range R can then be obtained by inserting this value t_2 for t in Eq. 4–6*c*. We obtain, from $x = (v_0 \cos \theta_0)t$,

$$R = (v_0 \cos \theta_0)t_2 = (50 \text{ ft/sec})(\tfrac{8}{10})(\tfrac{15}{8} \text{ sec}) = 75 \text{ ft.}$$

(*d*) What is the velocity of the ball as it strikes the ground? From Eq. 4–6*a* we obtain

$$v_x = v_0 \cos \theta_0 = (50 \text{ ft/sec})(\tfrac{8}{10}) = 40 \text{ ft/sec.}$$

From Eq. 4–6*a*′ we obtain for $t = t_2 = \frac{15}{8}$ sec,

$$v_y = v_0 \sin \theta_0 - gt = (50 \text{ ft/sec})(\tfrac{6}{10}) - (32 \text{ ft/sec}^2)(\tfrac{15}{8} \text{ sec}) = -30 \text{ ft/sec.}$$

Hence, from Eq. 4–7,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(40 \text{ ft/sec})^2 + (-30 \text{ ft/sec})^2} = 50 \text{ ft/sec,}$$

and

$$\tan \theta = v_y/v_x = -\tfrac{30}{40},$$

so that $\theta = -37°$, or 37° *clockwise* from the x-axis. Notice that $\theta = -\theta_0$, as we expect from symmetry (Fig. 4–2).

Example 3. In a favorite lecture demonstration a gun is sighted at an elevated target which is released in free fall by a trip mechanism as the bullet leaves the muzzle. No matter what the initial speed of the bullet, it always hits the falling target.

The simplest way to understand this is the following. If there were no acceleration due to gravity, the target would not fall and the bullet would move along the line of sight directly into the target (Fig. 4–4). The effect of gravity is to cause each body to accelerate down at the same rate from the position it would otherwise have had. Therefore, in the time t, the bullet will fall a distance $\frac{1}{2}gt^2$ from the position it would have had along the line of sight and the target will fall the same distance from its starting point. When the bullet reaches the line of fall of the target, it will be the same distance below the target's initial position as the target is and hence the collision. If the bullet moves faster than shown in the figure (v_0 larger), it will have a greater range and will cross the line of fall at a higher point; but since it gets there sooner, the target will fall a correspondingly smaller distance in the same time and collide with it. A similar argument holds for slower speeds.

For an equivalent analysis, let us use Eq. 4–5*b*

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2$$

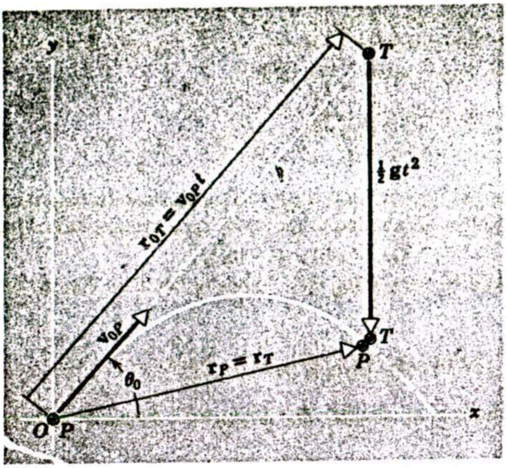


Fig. 4–4 Example 3. In the motion of a projectile, its displacement from the origin at any time t can be thought of as the sum of two vectors: $\mathbf{v}_{0P}t$, directed along $\mathbf{v}_{0P}$, and $\mathbf{g}t^2/2$, directed downward.

to describe the positions of the projectile and the target at any time t. For the projectile P, $\mathbf{r}_0 = 0$ and $\mathbf{a} = \mathbf{g}$, and we have

$$\mathbf{r}_P = \mathbf{v}_{0P}t + \tfrac{1}{2}\mathbf{g}t^2.$$

For the target T, $\mathbf{r}_0 = \mathbf{r}_{0T}$, $\mathbf{v}_0 = 0$, and $\mathbf{a} = \mathbf{g}$, leading to

$$\mathbf{r}_T = \mathbf{r}_{0T} + \tfrac{1}{2}\mathbf{g}t^2.$$

If there is a collision, we must have $\mathbf{r}_P = \mathbf{r}_T$. Inspection shows that this will always occur at a time t given by $\mathbf{r}_{0T} = \mathbf{v}_{0P}t$, that is, in the time t $(= r_{0T}/v_{0P})$ required for the projectile to travel to the target position along the line of sight, assuming that its initial velocity remains unchanged. ◀

4–4 Uniform Circular Motion

In Section 3–6 we saw that acceleration arises from a change in velocity. In the simple case of free fall the velocity changed in magnitude only, but not in direction. In a particle moving in a circle with constant speed, called uniform circular motion, the velocity vector changes continuously in direction but not in magnitude. We seek now to obtain the acceleration in uniform circular motion.

The situation is shown in Fig. 4–5a. Let P be the position of the particle at the time t and P' its position at the time $t + \Delta t$. The velocity at P is $\mathbf{v}$, a vector tangent to the curve at P. The velocity at P' is $\mathbf{v}'$, a vector tangent to the curve at P'. Vectors $\mathbf{v}$ and $\mathbf{v}'$ are equal in magnitude, the speed being constant, but their directions are different. The length of path traversed during Δt is the arc length PP', which is equal to $v\,\Delta t$, v being the constant speed.

Now redraw the vectors $\mathbf{v}$ and $\mathbf{v}'$, as in Fig. 4–5b, so that they originate at a common point. We are free to do this as long as the magnitude and direction of each vector are the same as in Fig. 4–5*a*. This diagram (Fig. 4–5*b*) enables us to see clearly the *change in velocity* as the particle moved from P to P'. This change, $\mathbf{v}' - \mathbf{v} = \Delta\mathbf{v}$, is the vector which must be added to $\mathbf{v}$ to get $\mathbf{v}'$. Notice that it points inward, approximately toward the center of the circle.

Now the triangle OQQ' formed by $\mathbf{v}$, $\mathbf{v}'$, and $\Delta\mathbf{v}$ is similar to the triangle CPP' formed by the chord PP' and the radii CP and CP'. This is so because both are isosceles triangles having the same vertex angle; the angle θ between $\mathbf{v}$ and $\mathbf{v}'$ is the same as the angle PCP' because $\mathbf{v}$ is perpendicular to CP and $\mathbf{v}'$ is perpendicular to CP'. We can therefore write

$$\frac{\Delta v}{v} = \frac{v\,\Delta t}{r}, \qquad \text{approximately,}$$

the chord PP' being taken equal to the arc length PP'. This relation becomes more nearly exact as Δt is diminished, since the chord and the arc then approach each other. Notice also that $\Delta\mathbf{v}$ approaches closer and closer to a direction perpendicular to $\mathbf{v}$ and $\mathbf{v}'$ as Δt is diminished and therefore approaches closer and closer to a direction pointing to the exact center of the circle. It follows from this relation that

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}, \qquad \text{approximately,}$$

and in the limit when $\Delta t \to 0$ this expression becomes exact. We therefore obtain

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v^2}{r} \tag{4-9}$$

as the magnitude of the acceleration. The direction of $\mathbf{a}$ is instantaneously along a radius inward toward the center of the circle.

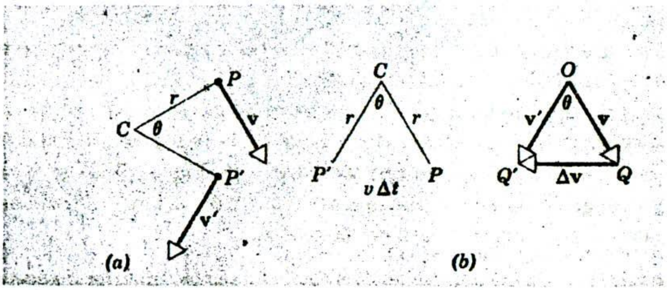


Fig. 4–5 Uniform circular motion. The particle travels around a circle at constant speed. Its velocity at two points P and P' is shown. Its change in velocity in going from P to P' is $\Delta\mathbf{v}$.

Figure 4–6 shows the instantaneous relation between **v** and **a** at various points of the motion. The magnitude of **v** is constant, but its direction changes continuously. This gives rise to an acceleration **a** which is also constant in magnitude (but not zero) but continuously changing in direction. The velocity **v** is always tangent to the circle in the direction of motion; the acceleration **a** is always directed radially inward. Because of this, **a** is called a *radial,* or *centripetal,* acceleration. *Centripetal* means "seeking a center."

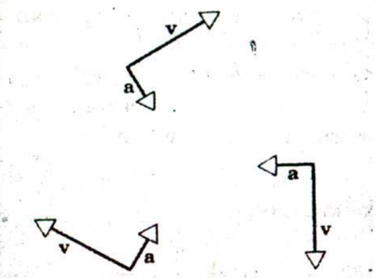


Fig. 4–6 In uniform circular motion the acceleration **a** is always directed toward the center of the circle and hence is perpendicular to **v**.

Both in free fall and in projectile motion **a** is constant in direction and magnitude and we can use the equations developed for constant acceleration (see Table 4–1). We cannot use these equations for uniform circular motion because **a** varies in direction and is therefore not constant.

The units of centripetal acceleration are the same as those of an acceleration resulting from a change in the magnitude of a velocity. Dimensionally, we have

$$\frac{v^2}{r} = \left(\frac{\text{length}}{\text{time}}\right)^2 \bigg/ \text{length} = \frac{\text{length}}{\text{time}^2} \qquad \text{or} \qquad \frac{L}{T^2},$$

which are the dimensions of acceleration. The units therefore may be ft/sec^2, meters/sec^2, among others.

The acceleration resulting from a change in direction of a velocity is just as real and just as much an acceleration in every sense as that arising from a change in magnitude of a velocity. By definition, acceleration is the time rate of change of velocity, and velocity, being a vector, can change in direction as well as magnitude. If a physical quantity is a vector, its directional aspects cannot be ignored, for their effects will prove to be every bit as important and real as those produced by changes in magnitude.

It is worth emphasizing at this point that there need not be any motion in the direction of an acceleration and that there is no fixed relation in general between the directions of **a** and **v**. In Fig. 4–7 we give examples in which the angle between **v** and **a** varies from 0 to 180°. Only in one case, $\theta = 0°$, is the motion in the direction of **a**.

▶ **Example 4.** The moon revolves about the earth, making a complete revolution in 27.3 days. Assume that the orbit is circular and has a radius of 239,000 miles. What is the magnitude of the acceleration of the moon toward the earth?

We have r = 239,000 miles = 3.85×10^8 meters. The time for one complete

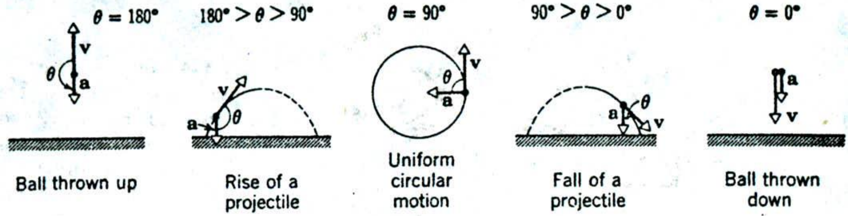


Fig. 4–7 Showing the relation between **v** and **a** for various motions.

revolution, called the period, is $T = 27.3$ days $= 2.36 \times 10^6$ sec. The speed of the moon (assumed constant) is therefore

$$v = 2\pi r/T = 1020 \text{ meters/sec.}$$

The centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{(1020 \text{ meters/sec})^2}{3.85 \times 10^8 \text{ meters}} = 0.00273 \text{ meter/sec}^2, \qquad \text{or} \quad \text{only } 2.78 \times 10^{-4}g.$$

Example 5. Calculate the speed of an artificial earth satellite, assuming that it is traveling at an altitude h of 140 miles above the surface of the earth where $g = 30$ ft/sec^2. The radius of the earth R is 3960 miles.

Like any free object near the earth's surface the satellite has an acceleration g toward the earth's center. It is this acceleration that causes it to follow the circular path. Hence the centripetal acceleration is g, and from Eq. 4–9, $a = v^2/r$, we have

$$g = v^2/(R + h),$$

or

$$v = \sqrt{(R + h)g} = \sqrt{(3960 \text{ miles} + 140 \text{ miles})(5280 \text{ ft/mile})\ (30 \text{ ft/sec}^2)}$$
$$= 2.55 \times 10^4 \text{ ft/sec} = 17{,}400 \text{ miles/hr.}$$ ◀

Let us now derive Eq. 4–9 using vector methods. Figure 4–8*a* shows a particle in uniform circular motion about the origin O of a reference frame. For this motion the polar coordinates r, θ are more useful than the rectangular coordinates x, y because r remains constant throughout the motion and θ increases in a simple linear way with time; the behavior of x and y during such motion is more complex. The two sets of coordinates are related by

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1} y/x \tag{4–10a}$$

or by the reciprocal relations

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta. \tag{4–10b}$$

In rectangular reference frames we used the unit vectors **i** and **j** to describe motion in the x-y plane. Here we find it more convenient to introduce two new unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$. These, like **i** and **j**, have unit length and are dimensionless; they designate direction only.

The unit vector $\mathbf{u}_r$ at any point is in the direction of increasing r at that point; it is directed radially outward from the origin. The unit vector $\mathbf{u}_\theta$ at any point is in the direction of increasing θ at that point; it is always tangent to a circle

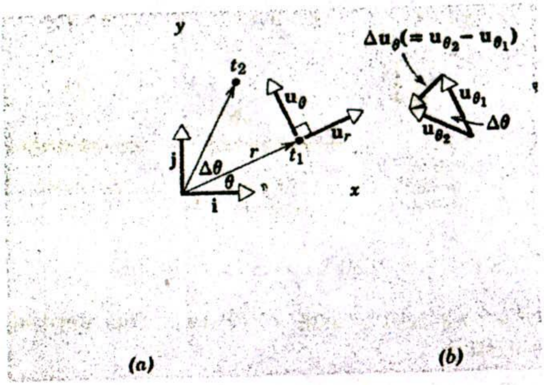


Fig. 4–8 (a) A particle moving counterclockwise in a circle of radius r. (b) The unit vectors $\mathbf{u}_{\theta_1}$ and $\mathbf{u}_{\theta_2}$ at times t_1 and t_2 respectively, and the change $\Delta\mathbf{u}_\theta$ $(= \mathbf{u}_{\theta_2} - \mathbf{u}_{\theta_1})$.

through the point in a counterclockwise direction. As Fig. 4–8*a* shows, $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are at right angles to each other. The unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ differ from the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in that the directions of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ vary from point to point in the plane; the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are thus *not constant* vectors.

In terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ the motion of a particle moving counterclockwise at uniform speed v in a circle about the origin in Fig. 4–8*a* can be described by the vector equation

$$\mathbf{v} = \mathbf{u}_\theta v. \tag{4–11}$$

This relation tells us, correctly, that the direction of $\mathbf{v}$ (which is the same as the direction of $\mathbf{u}_\theta$) is tangent to the circle and that the magnitude of $\mathbf{v}$ is the constant quantity v (because the magnitude of $\mathbf{u}_\theta$ is unity).

To find the acceleration we combine Eqs. 4-3 and 4–11, yielding

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{u}_\theta}{dt}v. \tag{4–12}$$

Note that v in Eq. 4–11 is a constant, but $\mathbf{u}_\theta$ is not since its direction changes as the particle moves. To evaluate $d\mathbf{u}_\theta/dt$, consider Fig. 4–8*b* which shows the unit vectors $\mathbf{u}_{\theta_1}$ and $\mathbf{u}_{\theta_2}$ corresponding to an elapsed time Δt $(= t_2 - t_1)$ for the moving particle. The vector $\Delta\mathbf{u}_\theta$ $(= \mathbf{u}_{\theta_2} - \mathbf{u}_{\theta_1})$ points radially inward toward the origin in the limiting case as $\Delta t \to 0$. In other words, $d\mathbf{u}_\theta$ at any point has the direction of $-\mathbf{u}_r$. The angle between $\mathbf{u}_{\theta_2}$ and $\mathbf{u}_{\theta_1}$ in the figure is $\Delta\theta$, which is the angle swept out by a radial line from the origin to the particle in time Δt. The magnitude of $\Delta\mathbf{u}_\theta$ is simply $\Delta\theta$; bear in mind that the vectors $\mathbf{u}_{\theta_1}$ and $\mathbf{u}_{\theta_2}$ in Fig. 4–8*b* have the magnitude unity. Thus

$$\frac{d\mathbf{u}_\theta}{dt} = -\mathbf{u}_r \lim_{\Delta t \to 0} \frac{\Delta\theta}{dt} = -\mathbf{u}_r \frac{d\theta}{dt}$$

and, from Eq. 4–12,

$$\mathbf{a} = \frac{d\mathbf{u}_\theta}{dt}v = -\mathbf{u}_r \frac{d\theta}{dt}v. \tag{4–13}$$

Now, $d\theta/dt$ is the uniform angular rotation rate of the particle and is given by

$$\frac{d\theta}{dt} = \frac{2\pi \text{ radians}}{\text{time for one revolution}} = \frac{2\pi}{2\pi r/v} = \frac{v}{r}.$$

Putting this into Eq. 4-13 leads us finally to

$$\mathbf{a} = -\mathbf{u}_r \frac{v^2}{r} \tag{4-14}$$

which tells us that the acceleration in uniform circular motion has a magnitude v^2/r (see Eq. 4-9) and points radially inward (note the factor $-\mathbf{u}_r$). The vector relation Eq. 4-14 thus tells us *both* the magnitude *and* the direction of the centripetal acceleration **a**. Note that, as expected, **a** has a constant magnitude but changes continually in direction because $\mathbf{u}_r$ changes continually in direction.

4-5 Tangential Acceleration in Circular Motion

We now consider the more general case of circular motion in which the speed v of the moving particle is *not* constant. We shall use vector methods in polar coordinates.

As before, the velocity is given by Eq. 4-11, or

$$\mathbf{v} = \mathbf{u}_\theta v$$

except that, in this case, not only $\mathbf{u}_\theta$ but also v varies with time. Recalling the formula for the derivative of a product, one obtains for the acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{u}_\theta \frac{dv}{dt} + v\frac{d\mathbf{u}_\theta}{dt} \tag{4-15}$$

In Eq. 4-12 the first term in this equation was not present because, v being there assumed to be constant, dv/dt was zero. The last term in Eq. 4-15 reduces, as we saw in the last section, to $-\mathbf{u}_r(v^2/r)$. We can now write Eq. 4-15 as

$$\mathbf{a} = \mathbf{u}_\theta a_T - \mathbf{u}_r a_R, \tag{4-16}$$

in which $a_T = dv/dt$ and $a_R = v^2/r$ The first term, $\mathbf{u}_\theta a_T$, is the vector component of **a** that is tangent to the path of the particle and arises from a change in the magnitude of the velocity in circular motion (see Fig. 4-9). This term and a_T are called the *tangential acceleration*. The second term $-\mathbf{u}_r a_R$ is the vector component of **a** directed radially in toward the center of the circle and arises from a

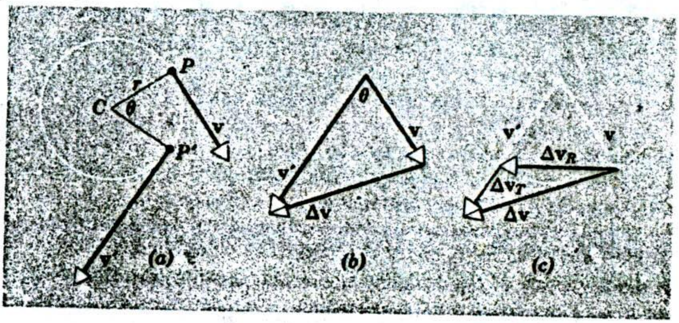


Fig. 4-9 In nonuniform circular motion the speed is variable. The change in velocity $\Delta\mathbf{v}$ in going from P to P' is made up of two parts: $\Delta\mathbf{v}_R$ caused by the change in direction of **v**, and $\Delta\mathbf{v}_T$ caused by the change in magnitude of **v**. In the limit as $\Delta t \to 0$, $\Delta\mathbf{v}_R$ points toward the center C of the circle and $\Delta\mathbf{v}_T$ is tangent to the circular path.

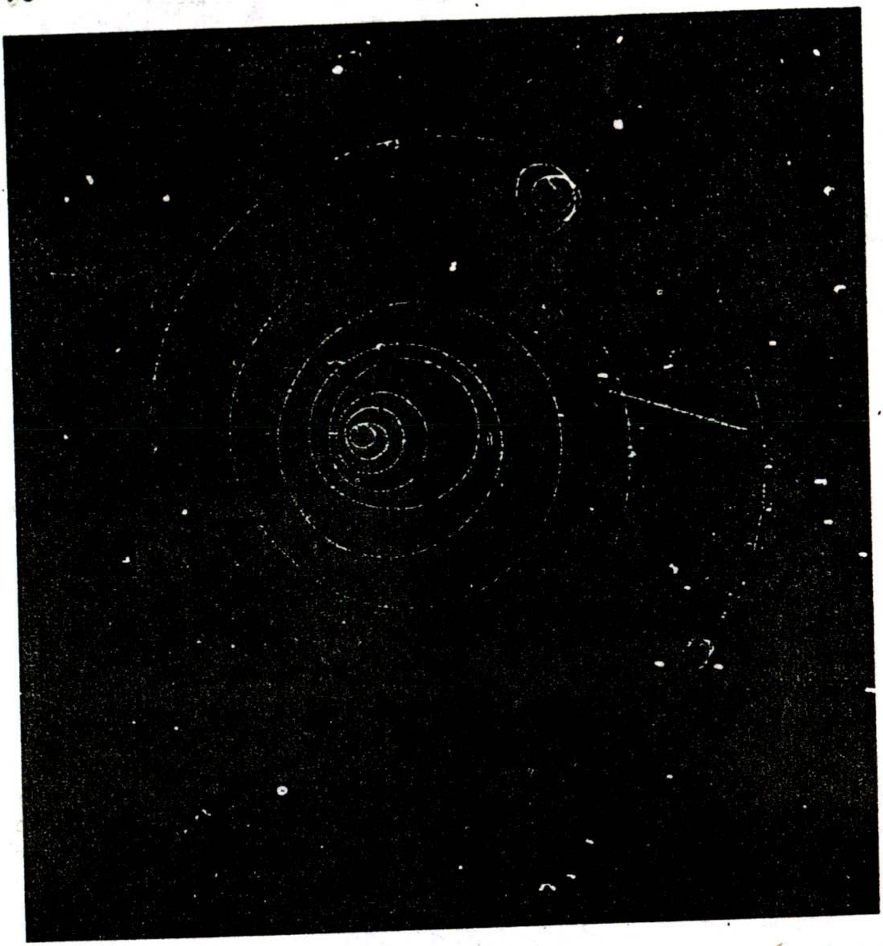


Fig. 4–10 A track left in a 10-in. liquid-hydrogen-filled bubble chamber by an energetic spiralling electron. (Courtesy Lawrence Radiation Laboratory.) This picture is one of a number in a collection prepared for easy stereoscopic viewing and published, with explanatory material, as *Introduction to the Detection of Nuclear Particles in a Bubble Chamber,* The Ealing Press, Cambridge 40, Massachusetts (1964). When viewed stereoscopically the electron is seen to be moving toward the reader as it moves in along the spiral. Its velocity vector at any point, thus, does not lie in the plane of the figure, but tilts up out of it; its motion is thus three-dimensional, rather than two-dimensional as we assumed for other examples in this chapter.

change in the direction of the velocity in circular motion (see Fig. 4–9). This term and a_R are called the *centripetal acceleration.*

The magnitude of the instantaneous acceleration is

$$a = \sqrt{a_T^2 + a_R^2} \tag{4–17}$$

If the speed is constant, then $a_T = dv/dt = 0$ and Eq. 4–16 reduces to Eq. 4–14. When the speed v is not constant, a_T is not zero and a_R varies from point to point. If the speed changes at a rate that is not constant, then a_T will also vary from point to point.

If the motion is not circular, the formulas for a_T ($= dv/dt$) and for a_R ($= v^2/r$) can still be applied if instead of using for r the magnitude of the radius vector from the origin we substitute the radius of curvature of the path at the instantaneous position of the particle. Then a_T gives the component of acceleration tangent to the curve at that position, and a_R gives the component of acceleration normal to the curve at that position. Figure 4–10 shows the track left in a liquid-hydrogen-filled bubble chamber by an energetic electron that spirals inward. The electron loses energy as it traverses the liquid in the chamber so that its speed v is being reduced steadily. Thus there is at every point a *tangential acceleration* a_T given by dv/dt. The *centripetal acceleration* a_R at any point is given by v^2/r, where r is the radius of curvature of the track at the point in question; both v and r become smaller as the particle loses energy. The force causing the electron to spiral is produced by a magnetic field present in the bubble chamber and at right angles to the plane of Fig. 4–10 (see Chapter 33).

4–6 Relative Velocity and Acceleration

In earlier sections we considered the addition of velocities in a particular reference frame. Let us now consider the relation between the velocity of an object as determined by one observer S (= reference frame S) and the velocity of the same object as determined by another observer S' (= reference frame S') who is moving with respect to the first.

Consider observer S fixed to the earth, so that his reference frame is the earth. The other observer S' is moving on the earth—for example, a passenger sitting on a moving train—so that his reference frame is the train. They each follow the motion of the same object, say an automobile on a road or a man walking through the train. Each observer will record a displacement, a velocity, and an acceleration for this object measured *relative to his reference frame*. How will these measurements compare? In this section we consider only the case in which the second frame is in motion with respect to the first with a *constant* velocity $\mathbf{u}$.

In Fig. 4–11 the reference frame S represented by the x- and y-axes can

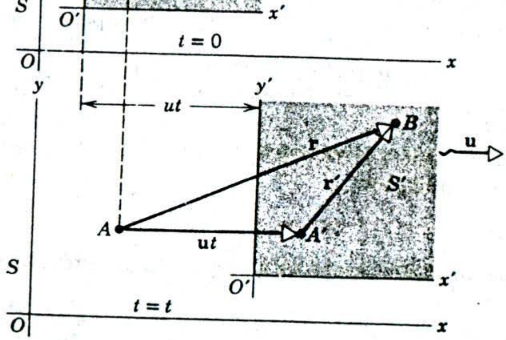


Fig. 4–11 Two reference frames, S ($= x, y$) and S' ($= x', y'$); S' moves to the right, relative to S, with speed u.

be thought of as fixed to the earth. The shaded region indicates another reference frame S', represented by x'- and y'-axes, which moves along the x-axis with a constant velocity $\mathbf{u}$, as measured in the S-system; it can be thought of as drawn on the floor of a railroad flatcar.

Initially, a particle (say a ball on the flatcar) is at a position called A in the S-frame and called A' in the S'-frame. At a time t later the flatcar and its S' reference frame have moved a distance ut to the right and the particle has moved to B. The *displacement* of the particle from its initial position *in the S-frame* is the vector $\mathbf{r}$ from A to B. The *displacement* of the particle from its initial position *in the S′-frame* is the vector $\mathbf{r}'$ from A' to B. These are different vectors because the reference point A' of the moving frame has been displaced a distance ut along the x-axis during the motion. From the figure we see that $\mathbf{r}$ is the vector sum of $\mathbf{r}'$ and $\mathbf{u}t$:

$$\mathbf{r} = \mathbf{r}' + \mathbf{u}t. \tag{4–18}$$

Differentiating Eq. 4–18 leads to

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \mathbf{u}.$$

But $d\mathbf{r}/dt = \mathbf{v}$, the instantaneous velocity of the particle measured in the S-frame, and $d\mathbf{r}'/dt = \mathbf{v}'$, the instantaneous velocity of the same particle measured in the S' frame, so that

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}. \tag{4–19}$$

Hence the velocity of the particle relative to the S-frame, $\mathbf{v}$, is the vector sum of the velocity of the particle relative to the S'-frame, $\mathbf{v}'$, and the velocity $\mathbf{u}$ of the S'-frame relative to the S-frame.

▶ **Example 6.** (*a*) The compass of an airplane indicates that it is heading due east. Ground information indicates a wind blowing due north. Show on a diagram the velocity of the plane with respect to the ground.

The object is the airplane. The earth is one reference frame (S) and the air is the other reference frame (S') moving with respect to the first. Then

$\mathbf{u}$ is the velocity of the air with respect to the ground.
$\mathbf{v}'$ is the velocity of the plane with respect to the air.
$\mathbf{v}$ is the velocity of the plane with respect to the ground.

In this case $\mathbf{u}$ points north and $\mathbf{v}'$ points east. Then the relation $\mathbf{v} = \mathbf{v}' + \mathbf{u}$ determines the velocity of the plane with respect to the ground, as shown in Fig. 4–12*a*.

The angle α is the angle N of E of the plane's course with respect to the ground and is given by

$$\tan \alpha = u/v'.$$

The airplane's speed with respect to the ground is given by

$$v = \sqrt{(v')^2 + u^2}.$$

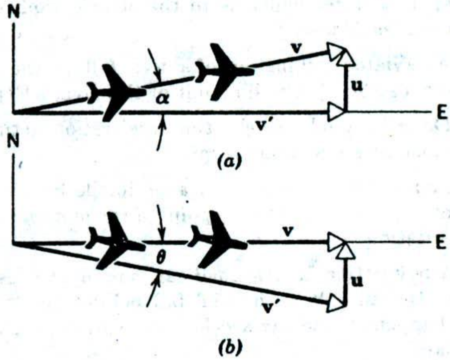


Fig. 4–12 Example 6.

For example, if the air-speed indicator shows that the plane is moving relative to the air at a speed of 200 miles/hr, and if the speed of the wind with respect to the ground is 40.0 miles/hr, then

$$v = \sqrt{(200)^2 + (40.0)^2}\ \text{miles/hr} = 204\ \text{miles/hr}$$

is the ground speed of the plane and

$$\alpha = \tan^{-1}\frac{40.0}{200} = 11^\circ\ 20'$$

gives the course of the plane N of E.

(*b*) Now draw the vector diagram showing the direction the pilot must steer the plane through the air for the plane to travel due east with respect to the ground.

He would naturally head partly into the wind. His speed relative to the earth will therefore be less than before. The vector diagram is shown in Fig. 4–12*b*. The student should calculate θ and v, using the previous data for **u** and **v**′. ◀

We have seen that different velocities are assigned to a particle by different observers when the observers are in relative motion. These velocities *always differ by* the relative velocity of the two observers, which here is *a constant velocity*. It follows that when the particle velocity changes, the *change will be the same* for both observers. Hence they each measure the *same acceleration* for the particle. The acceleration of a particle is the same in *all* reference frames moving relative to one another with constant velocity; that is, $\mathbf{a} = \mathbf{a}'$. This result follows in a formal way if we differentiate Eq. 4–19. Thus $d\mathbf{v}/dt = d\mathbf{v}'/dt + d\mathbf{u}/dt$; but $d\mathbf{u}/dt = 0$ when $\mathbf{u}$ is constant, so that $\mathbf{a} = \mathbf{a}'$.

QUESTIONS

1. In projectile motion when air resistance is negligible, is it ever necessary to consider three-dimensional motion rather than two-dimensional?

2. In broad jumping does it matter how high you jump? What factors determine the span of the jump?

3. Why doesn't the electron in the beam from an electron gun fall as much because of gravity as a water molecule in the stream from a hose? Assume horizontal motion initially in each case.

4. An aviator, pulling out of a dive, follows the arc of a circle. He was said to have "experienced $3g$'s" in pulling out of the dive. Explain what this statement means.

5. Describe qualitatively the acceleration acting on a bead which moves inward with constant speed along a spiral.

6. Could the acceleration of a projectile be represented in terms of a radial and a tangential component at each point of the motion? If so, is there any advantage to this representation?

7. A boy sitting in a railroad car moving at constant velocity throws a ball straight up into the air. Will the ball fall behind him? In front of him? Into his hand? What happens if the car accelerates forward or goes around a curve while the ball is in the air?

8. A man on the observation platform of a train moving with constant velocity drops a coin while leaning over the rail. Describe the path of the coin as seen by (*a*) the man on the train, (*b*) a person standing on the ground near the track, and (*c*) a person in a second train moving in the opposite direction to the first train on a parallel track.

9. A bus with a vertical windshield moves along in a rainstorm at speed v_b. The raindrops fall vertically with a terminal speed v_r. At what angle do the raindrops strike the windshield?

10. Drops are falling vertically in a steady rain. In order to go through the rain from one place to another in such a way as to encounter the least number of raindrops, should you move with the greatest possible speed, the least possible speed, or some intermediate speed?

11. An elevator is descending at a constant speed. A passenger takes a coin from his pocket and drops it to the floor. What accelerations would (*a*) the passenger and (*b*) a person at rest with respect to the elevator shaft observe for the falling coin?

PROBLEMS

1. Prove that for a vector **a** defined by

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z$$

the scaler components are given by

$$a_x = \mathbf{i} \cdot \mathbf{a}, \qquad a_y = \mathbf{j} \cdot \mathbf{a}, \qquad \text{and} \qquad a_z = \mathbf{k} \cdot \mathbf{a}.$$

2. A ball rolls off the edge of a horizontal table top 4.0 ft high. If it strikes the floor at a point 5.0 ft horizontally away from the edge of the table, what was its speed at the instant it left the table?

3. A ball rolls off the top of a stairway with a horizontal velocity of magnitude 5.0 ft/sec. The steps are 8.0 in. high and 8.0 in. wide. Which step will the ball hit first?

4. A shell is fired horizontally from a powerful gun located 144 ft above a horizontal plane with a muzzle speed of 800 ft/sec. (*a*) How long does the shell remain in the air? (*b*) What is its range? (*c*) What is the magnitude of the vertical component of its velocity as it strikes the target?

5. Show that the maximum height reached by a projectile is $y_{\max} = (v_0 \sin \theta_0)^2/2g$.

6. Show that the horizontal range of a projectile having an initial speed v_0 and angle of projection θ_0 is $R = (v_0^2/g) \sin 2\theta_0$. Then show that a projection angle of 45° gives the maximum horizontal range (Fig. 4-13).

7. Find the angle of projection at which the horizontal range and the maximum height of a projectile are equal.

8. In Galileo's *Two New Sciences* the author states that "for elevations (angles of projection) which exceed or fall short of 45° by equal amounts, the ranges are equal . . . ". Prove this statement.

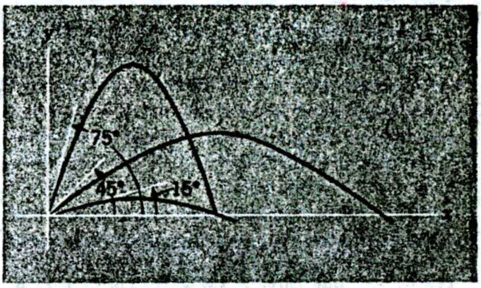


Fig. 4-13

9. A rifle with a muzzle velocity of 1500 ft/sec shoots a bullet at a small target 150 ft away. How high above the target must the gun be aimed so that the bullet will hit the target?

10. A dive bomber, diving at an angle of 53° with the vertical, releases a bomb at an altitude of 2400 ft. The bomb hits the ground 5.0 sec after being released. (*a*) What is the speed of the bomber? (*b*) How far did the bomb travel horizontally during its flight? (*c*) What were the horizontal and vertical components of its velocity just before striking the ground?

11. A batter hits a pitched ball at a height 4.0 ft above ground so that its angle of projection is 45° and its horizontal range is 350 ft. The ball is fair down the left field line where a 24-ft-high fence is located 320 ft from home plate. Will the ball clear the fence?

12. A football is kicked off with an initial speed of 64 ft/sec at a projection angle of 45°. A receiver on the goal line 60 yd away in the direction of the kick starts running to meet the ball at that instant. What must his speed be if he is to catch the ball before it hits the ground?

13. In a cathode-ray tube a beam of electrons is projected horizontally with a speed of 1.0×10^9 cm/sec into the region between a pair of horizontal plates 2.0 cm long. An electric field between the plates exerts a constant downward acceleration on the electrons of magnitude 1.0×10^{17} cm/sec^2. Find (*a*) the vertical displacement of the beam in passing through the plates and (*b*) the velocity of the beam (direction and magnitude) as it emerges from the plates.

14. (*a*) Show that if the acceleration of gravity changes by an amount dg, the range of a projectile (see Problem 6) of given initial speed v_0 and angle of projection θ_0 changes by dR where $dR/R = -dg/g$. (*b*) If the acceleration of gravity changes by a small amount Δg (say by going from one place to another), the range for a given projectile system will change as well. Let the change in range be ΔR. If Δg, ΔR are small enough, we may write $\Delta R/R = -\Delta g/g$. In 1936, Jesse Owens (United States) established a world's running broad jump record of 8.09 meters at the Olympic games at Berlin ($g = 9.8128$ meters/sec^2). By how much would his record have differed if he had competed instead in 1956 at Melbourne ($g = 9.7999$ meters/sec^2)? (In this connection see "Bad Physics in Athletic Measurements," by P. Kirkpatrick, *American Journal of Physics*, February 1944.)

15. Electrons, nuclei, atoms and molecules, like all forms of matter, will fall under the influence of gravity. Consider separately a beam of electrons, of nuclei, of atoms, and of molecules traveling a horizontal distance of 1.0 meter. Let the average speed be for an electron 3.0×10^7 meters/sec, for a thermal neutron 2.2×10^3 meters/sec, for a neon atom 5.8×10^2 meters/sec, and for an oxygen molecule 4.6×10^2 meters/sec. Let the beams move through vacuum with initial horizontal velocities and find by how much their paths deviate from a straight line (vertical displacement in 1.0 meter) due to gravity. How do these results compare to that for a beam of golf balls (use reasonable data)? What is the controlling factor here?

16. A radar observer on the ground is "watching" an approaching projectile. At a certain instant he has the following information: (*a*) the projectile has reached maximum altitude and is moving horizontally with a speed v; (*b*) the straight-line distance to the projectile is l; (*c*) the line of sight to the projectile is an angle θ above the horizontal. Find the distance D between the observer and the point of impact of the projectile. Does the projectile pass over his head or strike the ground before reaching him? D is to be expressed in terms of the observed quantities v, l, and θ and the known value of g. Assume a flat earth; assume also that the observer lies in the plane of the projectile's trajectory.

17. Show that Eqs. 4–4*d*, *d'* in Table 4–1 can be expressed in vector form as

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{v}_0 \cdot \mathbf{v}_0 + 2\mathbf{a} \cdot (\mathbf{r} + \mathbf{r}_0),$$

that Eqs. 4–4*b*, *b'* can be expressed as

$$\mathbf{r} = \mathbf{r}_0 + \tfrac{1}{2}(\mathbf{v}_0 + \mathbf{v})t,$$

and Eqs. 4–4*c*, *c'* as

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2.$$

18. Projectiles are hurled at a horizontal distance R from the edge of a cliff of height h in such a way as to land a horizontal distance x from the bottom of the cliff. If you want x to be as small as possible, how would you adjust θ_0 and v_0, assuming that v_0 can be varied from zero to some maximum finite value and that θ_0 can be varied continuously? Only one collision with the ground is allowed (see Fig. 4–14).

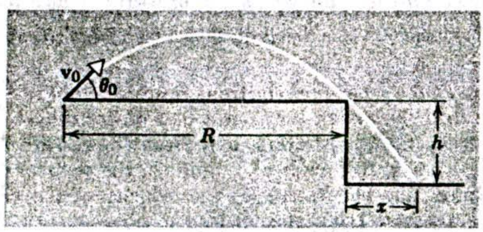


Fig. 4–14

19. Consider a projectile at the top of its trajectory. (*a*) What is its speed in terms of v_0 and θ_0? (*b*) What is its acceleration? (*c*) How is the direction of its acceleration related to that of its velocity? (*d*) Over a short distance a circular arc is a good approximation to a parabola. What then is the radius of the circular arc approximating the projectile's motion near the top of its path?

20. A particle rests on the top of a hemisphere of radius R. Find the smallest horizontal velocity that must be imparted to the particle if it is to leave the hemisphere without sliding down it.

21. A magnetic field will deflect a charged particle perpendicular to its direction of motion. An electron experiences a radial acceleration of 3.0×10^{14} meters/sec^2 in one such field. What is its speed if the radius of its curved path is 0.15 meter?

22. In Bohr's model of the hydrogen atom an electron revolves around a proton in a circular orbit of radius 5.28×10^{-11} meter with a speed of 2.18×10^6 meters/sec. What is the acceleration of the electron in the hydrogen atom?

23. Find the magnitude of the centripetal acceleration of a particle on the tip of a fan blade, 0.30 meter in diameter, rotating at 1200 rev/min.

24. By what factor would the speed of the earth's rotation have to increase for a body on the equator to require a centripetal acceleration of g to keep it on the earth? Such a body now requires a centripetal acceleration of only about 3.0 cm/sec^2.

25. A particle travels with constant speed on a circle of radius 3.0 meters and completes one revolution in 20 sec (Fig. 4–15). Starting from the origin O, find (*a*) the magnitude and direction of the displacement vectors 5.0 sec, 7.5 sec, and 10 sec later; (*b*) the magnitude and direction of the displacement in the 5.0-sec interval from the fifth to the tenth second; (*c*) the average velocity vector in this interval; (*d*) the instantaneous velocity vector at the beginning and at the end of this interval; (*e*) the average acceleration vector in this interval; and (*f*) the instantaneous acceleration vector at the beginning and at the end of this interval.

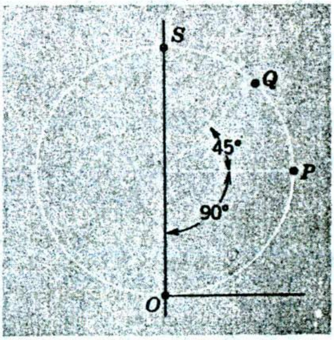


Fig. 4–15

26. An earth satellite moves in a circular orbit 400 miles above the earth's surface. The time for one revolution (the period) is found to be 98 min. Find the acceleration of gravity at the orbit from these data.

27. The earth revolves about the sun in a (nearly) circular orbit with a (nearly) constant speed of 30 km/sec. What is the acceleration of the earth toward the sun?

28. A particle moves in a plane according to

$$x = R \sin \omega t + \omega R t,$$

$$y = R \cos \omega t + R,$$

where ω and R are constants. This curve, called a *cycloid*, is the path traced out by a point on the rim of a wheel which rolls without slipping along the x-axis. (*a*) Sketch the path. (*b*) Calculate the instantaneous velocity and acceleration when the particle is at its maximum and minimum value of y.

29. (*a*) Write an expression for the position vector $\mathbf{r}$ for a particle describing uniform circular motion, using rectangular coordinates and the unit vectors $\mathbf{i}$ and $\mathbf{j}$. (*b*) From (*a*) derive vector expressions for the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$. (*c*) Prove that the acceleration is directed toward the center of the circular motion.

30. Write an expression, using the unit vectors $\mathbf{u}_\theta$ and $\mathbf{u}_r$, for the position vector $\mathbf{r}$ for a particle describing uniform circular motion and from it derive Eq. 4–11, $\mathbf{v} = \mathbf{u}_\theta v$.

31. Express the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$ in terms of $\mathbf{i}$, $\mathbf{j}$, and the angle θ in Fig. 4–8.

32. A person walks up a stalled escalator in 90 sec. When standing on the same escalator, now moving, he is carried up in 60 sec. How much time would it take him to walk up the moving escalator?

33. Find the speed of two objects if, when they move uniformly toward each other, they get 4.0 meters closer each second, and, when they move uniformly in the same direction with the original speeds, they get 4.0 meters closer each 10 sec.

34. A man can row a boat 4.0 miles/hr in still water. (*a*) If he is crossing a river where the current is 2.0 miles/hr, in what direction will his boat be headed if he wants to reach a point directly opposite from his starting point? (*b*) If the river is 4.0 miles wide, how long will it take him to cross the river? (*c*) How long will it take him to row 2.0 miles *down* the river and then back to his starting point? (*d*) How long will it take him to row 2.0 miles *up* the river and then back to his starting point? (*e*) In what direction should he head the boat if he wants to cross in the smallest possible time?

35. A man wants to cross a river 500 meters wide. His rowing speed (relative to the water) is 3000 meters/hr. The river flows at a speed of 2000 meters/hr. If the man's walking speed on shore is 5000 meters/hr, (*a*) find the path (combined rowing and walking) he should take to get to the point directly opposite his starting point in the shortest time. (*b*) How long does it take?

36. A train travels due south at 88.2 ft/sec (relative to ground) in a rain that is blown toward the south by the wind. The path of each raindrop makes the angle 21.6° with the vertical, as measured by an observer stationary on the earth. An observer seated in the train, however, sees perfectly vertical tracks of rain on the windowpane. Determine the speed of each raindrop relative to the earth.

37. An airplane has a speed of 135 miles/hr in still air. It is flying straight north so that it is at all times directly above a north-south highway. A ground observer tells the pilot by radio that a 70-miles/hr wind is blowing, but neglects to tell him the wind direction. The pilot observes that in spite of the wind he can travel 135 miles *along* the highway in one hour. In other words, his ground speed is the same as if there were no wind. (*a*) What is the direction of the wind? (*b*) What is the heading of the plane, that is, the angle between its axis and the highway?.

38. A pilot is supposed to fly due east from A to B and then back again to A due west. The velocity of the plane in air is $\mathbf{v}'$ and the velocity of the air with respect to the ground is $\mathbf{u}$. The distance between A and B is l and the plane's air speed v' is constant. (*a*) If $u = 0$ (still air), show that the time for the round trip is $t_0 = 2l/v'$. (*b*) Suppose that the air velocity is due east (or west). Show that the time for a round trip is then

$$t_{\mathrm{E}} = \frac{t_0}{1 - u^2/(v')^2}.$$

(*c*) Suppose that the air velocity is due north (or south). Show that the time for a round trip is then

$$t_{\mathrm{N}} = \frac{t_0}{\sqrt{1 - u^2/(v')^2}}.$$

(*d*) In parts (*b*) and (*c*) one must assume that $u < v'$. Why?

Particle Dynamics—I

CHAPTER 5

5–1 Classical Mechanics

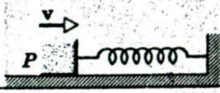
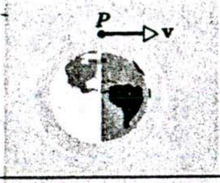
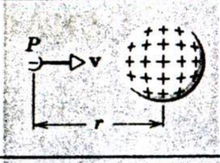
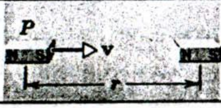
In Chapters 3 and 4, we studied the motion of a particle, with emphasis on motion along a straight line or in a plane. We did not ask what "caused" the motion; we simply described it in terms of the vectors **r**, **v**, and **a**. Our discussion was thus largely geometrical. In this chapter and the next we discuss the causes of motion, an aspect of mechanics called *dynamics*. As before, bodies will be treated as though they were single particles. Later in the book we shall treat groups of particles and extended bodies as well.

The motion of a given particle is determined by the nature and the arrangement of the other bodies that form its *environment*. In general, only nearby objects need to be included in the environment, the effects of more distant objects usually being negligible. Table 5–1 shows some "particles" and possible environments for them.

In what follows, we limit ourselves to the very important special case of gross objects moving at speeds that are small compared to c, the speed of light; this is the realm of *classical mechanics*. Specifically, we shall not inquire here into such questions as the motion of an electron in a uranium atom or the collision of two protons whose speeds are, say, $0.90c$. The first inquiry would involve us with the quantum theory and the second with the theory of relativity. We leave consideration of these theories, of which classical mechanics is a special case (see Section 6–4), to later.

The central problem of classical particle mechanics is this; (1) We are given a particle whose characteristics (mass, charge, magnetic dipole

Table 5-1

	System	The "Particle"	The Environment
1.		A block	The spring; the rough surface
2.		A golf ball	The earth
3.		An artificial satellite	The earth
4.		An electron	A large uniformly charged sphere
5.		A bar magnet	A second bar magnet

moment, etc.) we know. (2) We place this particle, with a known initial velocity, in an environment of which we have a complete description. (3) Problem: what is the subsequent motion of the particle?

This problem was solved, at least for a large variety of environments, by Isaac Newton (1642–1727) when he put forward his laws of motion and formulated his law of universal gravitation. The program for solving this problem, in terms of our present understanding of classical mechanics,* is: (1) We introduce the concept of *force* $\mathbf{F}$ and define it in terms of the acceleration $\mathbf{a}$ experienced by a particular standard body. (2) We develop a procedure for assigning a *mass* m to a body so that we may understand the fact that different particles of the same kind experience different accelerations in the same environment. (3) Finally, we try to find ways of calculating the forces that act on particles from the properties of the particle and of its environment; that is, we look for *force laws*. Force, which is at

* See "Presentation of Newtonian Mechanics" by Norman Austern, *American Journal of Physics*, September 1961, "On the Classical Laws of Motion" by Leonard Eisenbud, *American Journal of Physics*, March 1958, and "The Laws of Classical Motion: What's F? What's m? What's a?" by Robert Weinstock, *American Journal of Physics*, October 1961, for expositions of the laws of classical mechanics as we now view them, almost 300 years after Newton.

root a technique for relating the environment to the motion of the particle, appears both in the laws of motion (which tell us what acceleration a given body will experience under the action of a given force) and in the force laws (which tell us how to calculate the force that will act on a given body in a given environment). The laws of motion and the force laws, taken together, constitute the laws of mechanics.

The program of mechanics cannot be tested piecemeal. We must view it as a unit and we shall judge it to be successful if we can say "yes" to these two questions. (1) Does the program yield results that agree with experiment? (2) Are the force laws simple in form? It is the crowning glory of Newtonian mechanics that we can indeed answer each of these questions in the affirmative.

In this section we have used the terms force and mass rather unprecisely, having identified force with the influence of the environment, and mass with the resistance of a body to be accelerated when a force acts on it, a property often called inertia. In later sections we shall refine these primitive ideas about force and mass.

5-2 Newton's First Law

For centuries the problem of motion and its causes was a central theme of natural philosophy. It was not until the time of Galileo and Newton, however, that dramatic progress was made. Isaac Newton, born in England in the year of Galileo's death, is the principal architect of classical mechanics.* He carried to full fruition the ideas of Galileo and others who preceded him. His three laws of motion were first presented (in 1686) in his *Principia Mathematica Philosophiae Naturalis.*

Before Galileo's time most philosophers thought that some influence or "force" was needed to keep a body moving. They thought that a body was in its "natural state" when it was at rest. For a body to move in a straight line at constant speed, for example, they believed that some external agent had to continually propel it; otherwise it would "naturally" stop moving.

If we wanted to test these ideas experimentally, we would first have to find a way to free a body from all influences of its environment or from all forces. This is hard to do, but in certain cases we can make the forces very small. If we study the motions as we make the forces smaller and smaller, we shall have some idea of what the motion would be like if the external forces were truly zero.

Let us place our test body, say a block, on a rigid horizontal plane. If we let the block slide along this plane, we notice that it gradually slows down and stops. This observation was used, in fact, to support the idea that motion stopped when the external force, in this case the hand initially

* Newton also invented the (fluxional) calculus, conceived the idea of universal gravitation and formulated its law, and discovered the composition of white light. He was a skillful experimenter, a mathematician of first rank, and a biblical scholar as well as what today would be called theoretical physicist.

pushing the block, was removed. Galileo argued against this idea, however, reasoning as follows: Let us repeat our experiment, now using a smoother block and a smoother plane and providing a lubricant. We notice that the velocity decreases more slowly than before. Let us use still smoother blocks and surfaces and better lubricants. We find that the block decreases in velocity at a slower and slower rate and travels farther each time before coming to rest.* We can now extrapolate and say that if all friction could be eliminated, the body would continue indefinitely in a straight line with constant speed. This was Galileo's conclusion. Galileo asserted that some external force was necessary to *change* the velocity of a body but that no external force was necessary to *maintain* the velocity of a body. Our hand, for example, exerts a force on the block when it sets it in motion. The rough plane exerts a force on it when it slows it down. Both of these forces produce a change in the velocity, that is, they produce an acceleration.

This principle of Galileo was adopted by Newton as the first of his three laws of motion. Newton stated his first law in these words: "*Every body persists in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed on it.*"

Newton's first law is really a statement about reference frames. For, in general, the acceleration of a body depends on the reference frame relative to which it is measured. The first law tells us that, if there are no nearby objects (and by this we mean that there are no forces because every force must be associated with an object in the environment) then it is possible to find a family of reference frames in which a particle has no acceleration. The fact that bodies stay at rest or retain their uniform linear motion in the absence of applied forces is often described by assigning a property to matter called inertia. Newton's first law is often called the law of inertia and the reference frames to which it applies are therefore called inertial frames. Such frames are either fixed with respect to the distant stars or moving at uniform velocity with respect to them.

In nearly all cases in this book we will apply the laws of classical mechanics from the point of view of an observer in an inertial frame. It is possible to solve problems in mechanics using a noninertial frame, such as a frame rotating with respect to the fixed stars, but to do so we have to introduce forces (often called *pseudo-forces*) that cannot be associated with objects in the environment. We will discuss this in Chapters 6, 11, and 16. A reference frame attached to the earth can be considered to be an inertial frame for most practical purposes. We shall see in Chapter 16 how good an approximation this is.

Notice that there is no distinction in the first law between a body at rest and one moving with a constant velocity. Both motions are "natural" in the absence of forces. That this is so becomes clear when a body at rest

* The student may have experimented in the laboratory with a "dry ice puck." This is a device which can be made to move over a smooth horizontal surface, floating on a layer of CO_2 gas. The friction between the puck and the surface is very low indeed and it is hard to measure any reduction in speed for path lengths of practical dimensions.

in one inertial frame is viewed from a second inertial frame, that is, a frame moving with constant velocity with respect to the first. An observer in the first frame finds the body to be at rest; an observer in the second frame finds the same body to be moving with uniform velocity. Both observers find the body to have no acceleration, that is, no change in velocity, and both may conclude from the first law that no force acts on the body.

Notice, too, that by implication there is no distinction in the first law between the absence of all forces and the presence of forces whose resultant is zero. For example, if the push of our hand on the book exactly counteracts the force of friction on it, the book will move with uniform velocity. Hence another way of stating the first law is: *If no net force acts on a body its acceleration* **a** *is zero.*

If there *is* an interaction between the body and objects present in the environment, the effect may be to change the "natural" state of the body's motion. To investigate this we must now examine carefully the concept of force.

Fig. 5-1 The national standard kilogram No. 4, kept at the United States National Bureau of Standards. It is an accurate copy of the International standard kept at the International Bureau of Weights and Measures near Paris. The standard kilogram is the platinum cylinder housed under the double bell-jar.

5-3 Force

Let us refine our concept of force by defining it operationally. In our everyday language force is associated with a push or a pull, perhaps exerted by our muscles. In physics, however, we need a more precise definition. We define force here in terms of the acceleration that a given standard body experiences when placed in a suitable environment.

As a standard body we find it convenient to use (or rather to imagine that we use!) a particular platinum cylinder carefully preserved at the International Bureau of Weights and Measures near Paris, and called the *standard kilogram* (see Fig. 5-1). For use in later sections we state here that this body has been selected as our standard of *mass* and has been assigned, by definition, a mass m_0 of exactly 1 kg. Later we will describe how masses are assigned to other bodies.

As for an environment we place the standard body on a horizontal table having negligible friction and we attach a spring to it. We hold the other end of the spring in our hand, as in Fig. 5-2*a*. Now we pull the spring horizontally to the right so that by trial and error the standard body experiences a measured uniform acceleration of 1.00 meter/sec^2. We then declare, *as a matter of definition*, that the spring (which is the significant

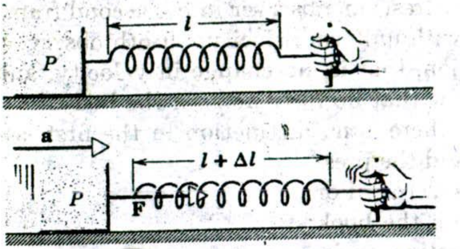


Fig. 5–2 (*a*) A "particle" *P* (the standard kilogram) at rest on a horizontal frictionless surface. (*b*) The body is accelerated by pulling the spring to the right.

body in the environment) is exerting a constant force whose magnitude we will call "1.00 newton" on the standard body. We note that, in imparting this force, the spring is kept stretched an amount Δl beyond its normal unextended length, as Fig. 5–2*b* shows.

We can repeat the experiment, either stretching the spring more or using a stiffer spring, so that we measure an acceleration of 2.00 meters/sec^2 for the standard body. We now declare that the spring is exerting a force of 2.00 newtons on the standard body. In general, if we observe this particular standard body to have an acceleration a in a particular environment, we then say that the environment is exerting a force F on the standard body, where F (in newtons) is numerically equal to a (in meters/sec^2).

Now let us see whether force, as we have defined it, is a *vector* quantity. In Fig. 5–2*b* we assigned a magnitude to the force F, and it is a simple matter to assign a direction to it as well, namely, the direction of the acceleration that the force produces. However, to be a vector it is not enough for a quantity to have magnitude and direction; it must also obey the laws of vector addition described in Chapter 2. We can learn only from experiment whether forces, as we defined them, do indeed obey these laws.

Let us arrange to exert a 4.00-newton force along the x-axis and a 3.00-newton force along the y-axis and let us apply these forces *simultaneously* to the standard body placed, as before, on a horizontal, frictionless surface. What will be the acceleration of the standard body? We would find by experiment that it was 5.00 meters/sec^2, directed along a line that makes an angle of 37° with the x-axis. In other words, we would say that the standard body was experiencing a force of 5.00 newtons in this same direction. This same result can be obtained by adding the 4.00-newton and 3.00-newton forces vectorially according to the parallelogram method. Experiments of this kind show conclusively that forces are vectors; they have magnitude; they have direction; they add according to the parallelogram law.

The result of experiments of this general type is often stated as follows: ***When several forces act on a body, each produces its own acceleration independently. The resulting acceleration is the vector sum of the several independent accelerations.***

5-4 Mass; Newton's Second Law

In Section 5-3 we considered only the accelerations given to one particular object, the standard kilogram. We were able thereby to define forces quantitatively. What effect would these forces have on other objects? Since our standard body was chosen arbitrarily in the first place, we know that for any given object the acceleration will be directly proportional to the force applied. The significant question remaining then is: What effect will the *same force* have on *different objects?* Everyday experience gives us a qualitative answer. The same force will produce different accelerations on different bodies. A baseball will be accelerated more by a given force than will an automobile. In order to obtain a quantitative answer to this question we need a method to measure mass, the property of a body which determines its resistance to a change in its motion.

Let us attach a spring to our standard body (the standard kilogram, to which we have arbitrarily assigned a mass $m_0 = 1.00$ kg, exactly) and arrange to give it an acceleration a_0 of, say 2.00 meters/sec^2, using the method of Fig. 5-2b. Let us measure carefully the extension Δl of the spring associated with the force that the spring is exerting on the block.

Now we remove the standard kilogram and substitute an arbitrary body, whose mass we label m_1. We apply the *same force* (the one that accelerated the standard kilogram 2.00 meters/sec^2) to the arbitrary body (by stretching the spring by the same amount) and we measure an acceleration a_1 of, say, 0.50 meter/sec^2.

We *define* the ratio of the masses of the two bodies to be the inverse ratio of the accelerations given to these bodies by the same force, or

$$m_1/m_0 = a_0/a_1 \qquad \text{(same force } \mathbf{F} \text{ acting).}$$

In this example we have, numerically,

$$\begin{aligned} m_1 &= m_0(a_0/a_1) = 1.00 \text{ kg } [(2.00 \text{ meters/sec}^2)/(0.50 \text{ meters/sec}^2)] \\ &= 4.00 \text{ kg.} \end{aligned}$$

The second body, which has only one-fourth the acceleration of the first body when the same force acts on it, has, by definition, four times the mass of the first body. Hence mass may be regarded as a quantitative measure of inertia.

If we repeat the preceding experiment with a different common force acting, we find the ratio of the accelerations, a_0'/a_1', to be the same as in the previous experiment, or

$$m_1/m_0 = a_0/a_1 = a_0'/a_1'.$$

The ratio of the masses of two bodies is thus independent of the common force used.

Furthermore, experiment shows that we can consistently assign masses to any body by this procedure. For example, let us compare a second arbitrary body with the standard body, and thus determine its mass, say m_2. We can now compare the two arbitrary bodies, m_2 and m_1, directly, obtaining accelerations a_2'' and a_1'' when the same force is applied. The mass ratio, defined as usual from

$$m_2/m_1 = a_1''/a_2'', \qquad \text{(same force acting)}$$

turns out to have the same value that we obtain by using the masses m_2 and m_1 determined previously by direct comparison with the standard.

We can show, in still another experiment of this type, that if objects of mass m_1 and m_2 are fastened together they behave mechanically as a single object of mass $(m_1 + m_2)$. In other words, masses add like (and are) scalar quantities.

Table 5–2 shows the range of values over which masses can be determined, using various techniques.

Table 5–2

SOME MEASURED MASSES

Object	Mass (kg)
Our galaxy	2.2×10^{41}
The sun	2.0×10^{30}
The earth	6.0×10^{24}
The moon	7.4×10^{22}
Mass of all the water in the oceans	1.4×10^{21}
An ocean liner	7.2×10^{7}
An elephant	4.5×10^{3}
A man	7.3×10^{1}
A grape	3.0×10^{-3}
A tobacco mosaic virus	6.7×10^{-10}
A speck of dust	2.3×10^{-13}
A penicillin molecule	5.0×10^{-17}
A uranium atom	4.0×10^{-25}
A proton	1.7×10^{-27}
An electron	9.1×10^{-31}

We can now summarize all the experiments and definitions described above in one equation, the fundamental equation of classical mechanics,

$$\mathbf{F} = m\mathbf{a}. \tag{5–1}$$

In this equation **F** is the (vector) *sum* of *all* the forces acting *on* the body, m is the mass of the body, and **a** is its (vector) acceleration. Equation 5–1 may be taken as a statement of Newton's second law. If we write it in the form $\mathbf{a} = \mathbf{F}/m$, we can see easily that the acceleration of the body is directly proportional to the resultant force acting on it and parallel in direction to this force and that the acceleration, for a given force, is inversely proportional to the mass of the body.

Notice that the first law of motion is contained in the second law as a special case, for if $\mathbf{F} = 0$, then $\mathbf{a} = 0$. In other words, if the resultant force on a body is zero, the acceleration of the body is zero. Therefore in the absence of applied forces a body will move with constant velocity or be at rest (zero velocity), which is what the first law of motion says. Therefore of Newton's three laws of motion only two are independent, the second and the third (Section 5-5). The division of translational particle dynamics that includes only systems for which the resultant force $\mathbf{F}$ is zero is called *statics*.

Equation 5-1 is a vector equation. We can write this single vector equation as three scalar equations,

$$F_x = ma_x, \qquad F_y = ma_y, \qquad \text{and} \quad F_z = ma_z, \tag{5-2}$$

relating the x, y, and z components of the resultant force (F_x, F_y, and F_z) to the x, y, and z components of acceleration (a_x,a_y, and a_z) for the mass m. It should be emphasized that F_x is the *sum* of the x-components of *all* the forces, F_y is the sum of the y-components of *all* the forces, and F_z is the *sum* of the z-components of *all* the forces acting on m.

5-5 Newton's Third Law of Motion

Forces acting on a body originate in other bodies that make up its environment. Any single force is only one aspect of a mutual interaction between *two* bodies. We find by experiment that when one body exerts a force on a second body, the second body always exerts a force on the first. Furthermore, we find that these forces are equal in magnitude but opposite in direction. A single isolated force is therefore an impossibility.

If one of the two forces involved in the interaction between two bodies is called an "action" force, the other is called the "reaction" force. *Either* force may be considered the "action" and the other the "reaction." Cause and effect is *not* implied here, but a mutual simultaneous interaction *is* implied.

This property of forces was first stated by Newton in his third law of motion: "*To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*"

In other words, if body A exerts a force on body B, body B exerts an equal but oppositely directed force on body A; and furthermore the forces lie along the line joining the bodies. Notice that the action and reaction forces, which always occur in pairs, act on *different* bodies. If they were to act on the same body, we could never have accelerated motion because the resultant force on every body would always be zero.

Imagine a boy kicking open a door. The force exerted by the boy B on the door D accelerates the door (it flies open); at the same time, the door D exerts an equal but opposite force on the boy B, which decelerates the

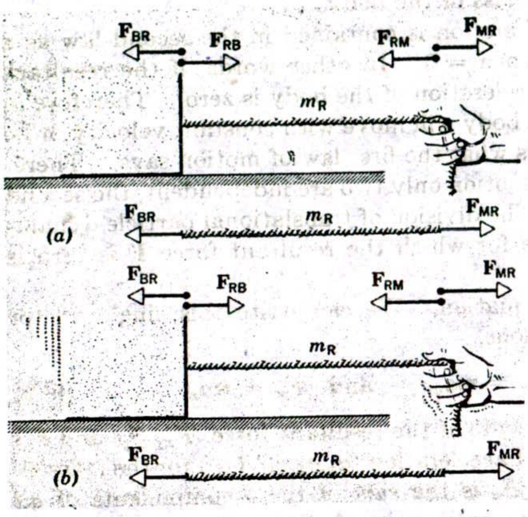


Fig. 5–3 Example 1. A man pulls on a rope attached to a block. (*a*) The forces exerted on the rope by the block and by the man are equal and opposite. Thus the resultant horizontal force on the rope is zero, as is shown in the free-body diagram. The rope does not accelerate. (*b*) The force exerted on the rope by the man exceeds that exerted by the block. The net horizontal force has magnitude $F_{MR} - F_{BR}$ and points to the right. Thus the rope is accelerated to the right. The block is also acted upon by a frictional force not shown here.

boy (his foot loses forward velocity). The boy will be painfully aware of the "reaction" force to his "action," particularly if his foot is bare.

The following examples illustrate the application of the third law and clarify its meaning.

▶ **Example 1.** Consider a man pulling horizontally on a rope attached to a block on a horizontal table as in Fig. 5–3. The man pulls on the rope with a force $\mathbf{F}_{MR}$. The rope exerts a reaction force $\mathbf{F}_{RM}$ on the man. According to Newton's third law, $\mathbf{F}_{MR} = -\mathbf{F}_{RM}$. Also, the rope exerts a force $\mathbf{F}_{RB}$ on the block, and the block exerts a reaction force $\mathbf{F}_{BR}$ on the rope. Again according to the third law, $\mathbf{F}_{RB} = -\mathbf{F}_{BR}$.

Suppose that the rope has a mass m_R. Then, in order to start the block and rope moving from rest, we must have an acceleration, say $\mathbf{a}$. The only forces acting *on the rope* are $\mathbf{F}_{MR}$ and $\mathbf{F}_{BR}$, so that the resultant force on it is $\mathbf{F}_{MR} + \mathbf{F}_{BR}$, and this must be different from zero if the rope is to accelerate. In fact, from the second law we have

$$\mathbf{F}_{MR} + \mathbf{F}_{BR} = m_R\mathbf{a}$$

Since the forces and the acceleration are along the same line, we can drop the vector notation and write the relation between the *magnitudes* of the vectors, namely

$$F_{MR} - F_{BR} = m_R a.$$

We see therefore that in general $\mathbf{F}_{MR}$ does not have the same magnitude as $\mathbf{F}_{BR}$ (Fig. 5–3*b*). These two forces act on the *same* body and are *not* action and reaction pairs.

According to Newton's third law the magnitude of $\mathbf{F}_{MR}$ always equals the magnitude of $\mathbf{F}_{RM}$, and the magnitude of $\mathbf{F}_{RB}$ always equals the magnitude of $\mathbf{F}_{BR}$. However, only if the acceleration $\mathbf{a}$ of the system is zero will we have the pair of forces $\mathbf{F}_{MR}$ and $\mathbf{F}_{RM}$ equal in magnitude to the pair of forces $\mathbf{F}_{RB}$ and $\mathbf{F}_{BR}$ (Fig. 5-3*a*). In this special case only, we could imagine that the rope merely transmits the force exerted by the man to the block without change. This same result holds in principle if $m_R = 0$. In practice, we never find a massless rope. However, we can often neglect the mass of a rope, in which case the rope is assumed to transmit a force unchanged. The force exerted at any point in the rope is called the *tension* at that point. We may measure the tension at any point in the rope by cutting a suitable length from it and inserting a spring scale; the tension is the reading of the scale. The tension is the same at all points in the rope only if the rope is unaccelerated or assumed to be massless.

Example 2. Consider a spring attached to the ceiling and at the other end holding a block at rest (Fig. 5-4*a*). Since no body is accelerating, all the forces on any body will add vectorially to zero. For example, the forces on the suspended block are **T**, the tension in the stretched spring, pulling vertically up on the mass, and **W**, the pull of the earth acting vertically down on the body, called its weight. These are drawn in Fig. 5-4*b*, where we show only the block for clarity. There are no other forces on the block.

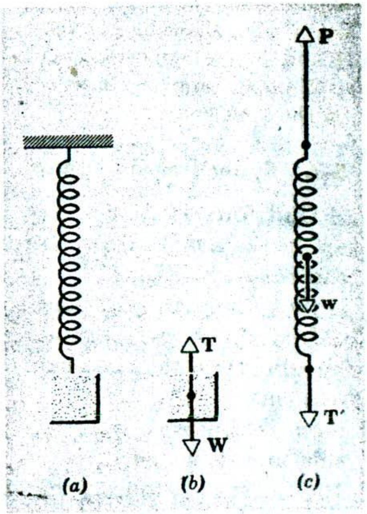


Fig. 5-4 Example 2. (*a*) A block is suspended by a spring. (*b*) A free-body diagram showing all the vertical forces exerted on the block. (*c*) A similar diagram for the vertical forces on the spring.

In Newton's second law, **F** represents the *sum* of *all* the forces acting *on* a body, so that for the block

$$\mathbf{F} = \mathbf{T} + \mathbf{W}.$$

The block is at rest so that its acceleration is zero, or $\mathbf{a} = 0$. Hence, from the relation $\mathbf{F} = m\mathbf{a}$, we obtain $\mathbf{T} + \mathbf{W} = 0$, or

$$\mathbf{T} = -\mathbf{W}.$$

The forces act along the same line, so that their magnitudes are equal, or

$$T = W.$$

Therefore the tension in the spring is an exact measure of the weight of the block. We shall use this result later in presenting a static procedure for measuring forces.

It is instructive to examine the forces exerted on the spring; they are shown in Fig. 5-4*c*. $\mathbf{T}'$ is the pull of the block on the spring and is the reaction force of the action force **T**. $\mathbf{T}'$ therefore has the same magnitude as **T**, which is W. **P** is the upward pull of the ceiling on the spring, and **w** is the weight of the spring, that is, the pull of the earth on it. Since the spring is at rest and all forces act along the same line, we have

$$\mathbf{P} + \mathbf{T}' + \mathbf{w} = 0,$$

or

$$P = W + w.$$

The ceiling therefore pulls up on the spring with a force whose magnitude is the sum of the weights of the block and spring.

From the third law of motion, the force exerted by the spring on the ceiling, $\mathbf{P}'$, must be equal in magnitude to $\mathbf{P}$, which is the reaction force to the action force $\mathbf{P}'$. $\mathbf{P}'$ therefore has a magnitude $W + w$.

In general, the spring exerts different forces on the bodies attached at its different ends, for $P' \neq T$. In the special case in which the weight of the spring is negligible, $w = 0$ and $P' = W = T$. Therefore a *weightless* spring (or cord) may be considered to transmit a force from one end to the other without change.

It is instructive to classify all the forces in this problem according to action and reaction pairs. The reaction to $\mathbf{W}$, a force exerted by the earth on the block, must be a force exerted by the block on the earth. Similarly, the reaction to $\mathbf{w}$ is a force exerted by the spring on the earth. Because the earth is so massive, we do not expect these forces to impart a noticeable acceleration to the earth. Since the earth is not shown in our diagrams, these forces have not been shown. The forces $\mathbf{T}$ and $\mathbf{T}'$ are action-reaction pairs, as are $\mathbf{P}$ and $\mathbf{P}'$. Notice that although $\mathbf{T} = -\mathbf{W}$ in our problem, these forces are *not* an action-reaction pair because they act on the *same* body. ◀

5–6 Systems of Mechanical Units

Unit force is defined as a force that causes a unit of acceleration when applied to a unit mass. The mks (meter, kilogram, second) unit of mass is the *kilogram* (Fig. 5–1). The cgs (centimeter, gram, second) unit of mass is the *gram*, defined as one-thousandth of the kilogram mass.

In the mks system unit force is the force that will accelerate a one-kilogram mass at one meter/sec^2; we have seen that this unit is called the *newton*. In the cgs system, which includes the Gaussian system, unit force is the force that will accelerate a one-gram mass at one cm/sec^2; this unit is called the *dyne*. Since 1 kg $= 10^3$ gm and 1 meter/sec^2 $= 10^2$ cm/sec^2, it follows that 1 nt $= 10^5$ dynes.

In each of our systems of units we have chosen mass, length, and time as our fundamental quantities. Standards were adopted for these fundamental quantities and units defined in terms of these standards. Force appears as a derived quantity, determined from the relation $\mathbf{F} = m\mathbf{a}$.

In the British engineering system of units, however, *force*, length, and time are chosen as the fundamental quantities and mass is a derived quantity. In this system, mass is determined from the relation $m = F/a$. The standard and unit of force in this system is the *pound*. Actually, the pound of force was originally defined to be the pull of the earth on a certain standard body at a certain place on the earth. We can get this force in an operational way by hanging the standard body from a spring at the particular point where the earth's pull on it is defined to be 1 lb of force. If the body is at rest, the earth's pull on the body, its weight W, is balanced by the tension in the spring. Therefore $T = W = 1$ lb, in this instance. We can now use this spring (or any other one thus calibrated) to exert a force of 1 lb on any other body; to do this we simply attach the spring to another body and stretch it the same amount as the pound force had stretched it. The standard body can be compared to the kilogram and it is found to have the mass 0.45359237 kg. The acceleration due to gravity at the certain place on the earth is found to be 32.1740 ft/sec^2. The pound of force can therefore be defined from $F = ma$ as the force that accelerates a mass of 0.45359237 kg at the rate of 32.1740 ft/sec^2.

This procedure enables us to compare the pound-force with the newton. Using the fact that 32.1740 ft/sec^2 equals 9.8066 meters/sec^2, we find that

$$\begin{aligned} 1 \text{ lb} &= (0.45359237 \text{ kg})(32.1740 \text{ ft/sec}^2) \\ &= (0.45359237 \text{ kg})(9.8066 \text{ meters/sec}^2) \\ &\cong 4.45 \text{ nt}. \end{aligned}$$

The unit of mass in the British engineering system can now be derived. It is defined as the mass of a body whose acceleration is 1 ft/sec^2 when the force on it is 1 lb; this mass is called the *slug*. Thus, in this system

$$F[\text{lb}] = m\,[\text{slugs}] \times a\,[\text{ft/sec}^2].$$

Legally the pound is a unit of mass but in engineering practice the pound is treated as a unit of force or weight. This has given rise to the terms pound-mass and pound-force. The pound-mass is a body of mass 0.45359237 kg; no standard block of metal is preserved as the pound-mass, but, like the yard, it is defined in terms of the mks standard. The pound-force is the force that gives a standard pound an acceleration equal to the standard acceleration of gravity, 32.1740 ft/sec^2. As we shall see later, the acceleration of gravity varies with distance from the center of the earth, and this "standard acceleration" is, therefore, the value at a particular distance from the center of the earth. (Any point at sea level and 45°N latitude is a good approximation.)

In this book only forces will be measured in pounds. Thus the corresponding unit of mass is the slug. The units of force, mass, and acceleration in the three systems are summarized in Table 5–3.

The *dimensions* of force are the same as those of mass times acceleration. In a system in which mass, length, and time are the fundamental quantities,

Table 5–3

UNITS IN $F = ma$

Systems of Units	Force	Mass	Acceleration
Mks	newton (nt)	kilogram (kg)	meter/sec^2
Cgs (Gaussian)	dyne	gram (gm)	cm/sec^2
Engineering	pound (lb)	slug	ft/sec^2

the dimensions of force are, therefore, mass × length/time², or MLT^{-2}. We shall arbitrarily adopt mass, length, and time as our fundamental mechanical quantities.

Recalling that our length and time standards are atomic standards, some have speculated that the standard kilogram may some day be replaced by an atomic standard of mass. This new standard might consist of a specification of a number of atoms of a certain type whose collective mass under suitable circumstances is 1 kg. At the present time, however, the accuracy with which masses can be compared, as on a balance, exceeds the accuracy with which we can determine the exact number of atoms that make up a given mass.

5–7 The Force Laws

The three laws of motion that we have described are only part of the program of mechanics that we outlined in Section 5–1. It remains to investigate the *force laws*, which are the procedures by which we calculate the force acting on a given body in terms of the properties of the body and its environment. Newton's second law

$$\mathbf{F} = m\mathbf{a} \tag{5–3}$$

is essentially not a law of nature but a definition of force. We need to identify various functions of the type:

$$\mathbf{F} = \text{a function of the properties of the particle and of the environment} \tag{5–4}$$

so that we can, in effect, eliminate **F** between Eqs. 5–3 and 5–4, thus obtaining an equation that will let us calculate the acceleration of a particle in terms of the properties of the particle and its environment. We see here clearly that force is a concept that connects the acceleration of the particle on the one hand with the properties of the particle and its environment on the other. We indicated earlier that one criterion for declaring the program of mechanics to be successful would be the discovery that *simple* laws of the type of Eq. 5–4 do indeed exist. This turns out to be the case, and this fact constitutes the essential reason that we "believe" the laws of classical mechanics. If the force laws had turned out to be very complicated, we would not be left with the feeling that we had gained much insight into the workings of nature.

The number of possible environments for an accelerated particle is so great that a detailed discussion of all the force laws is not feasible in this chapter. We shall, however, indicate in Table 5–4 the force laws that apply to the five particle-plus-environment situations of Table 5–1. At appropriate places throughout the text we will discuss these and other force laws in detail; several of the laws in Table 5–4 are approximations or special cases.

Table 5-4

THE FORCE LAWS FOR THE SYSTEMS OF TABLE 5-1

System	Force Law
1. A block propelled by a stretched spring over a rough horizontal surface	(a) Spring force: $F = -kx$, where x is the extension of the spring and k is a constant that describes the spring; **F** points to the right; see Chapter 15 (b) Friction force: $F = \mu mg$, where μ is the *coefficient of friction* and mg is the weight of the block; **F** points to the left; see Chapter 6
2. A golf ball in flight	$F = mg$; **F** points down (see Section 5-8)
3. An artificial satellite	$F = GmM/r^2$, where G is the *gravitational constant*, M the mass of the earth, and r the orbit radius; **F** points toward the center of the earth; see Chapter 16. This is *Newton's law of universal gravitation*
4. An electron near a charged sphere	$F = (1/4\pi\epsilon_0)eQ/r^2$, where ϵ_0 is a constant, e is the electron charge, Q is the charge on the sphere, and r is the distance from the electron to the center of the sphere; **F** points to the right; see Chapter 26. This is *Coulomb's law of electrostatics*
5. Two bar magnets	$F = (3\mu_0/2\pi)\mu^2/r^4$, where μ_0 is a constant, μ is the *magnetic dipole* moment of each magnet, and r is the center-to-center separation of the magnets; we assume that $r \gg l$, where l is the length of each magnet; **F** points to the right

5-8 Weight and Mass

The *weight* of a body is the gravitational force exerted on it by the earth. Weight, being a force, is a vector quantity. The direction of this vector is the direction of the gravitational force, that is, toward the center of the earth. The magnitude of the weight is expressed in force units, such as pounds or newtons.

When a body of mass m is allowed to fall freely, its acceleration is that of gravity $\mathbf{g}$ and the force acting on it is its weight $\mathbf{W}$. Newton's second law, $\mathbf{F} = m\mathbf{a}$, when applied to a freely falling body, gives us $\mathbf{W} = m\mathbf{g}$. Both $\mathbf{W}$ and $\mathbf{g}$ are vectors directed toward the center of the earth. We can therefore write

$$W = mg, \tag{5-5}$$

where W and g are the magnitudes of the weight and acceleration vectors. To keep an object from falling we have to exert on it an upward force equal in magnitude to W, so as to make the net force zero. In Fig. 5-4*a* the tension in the spring supplies this force.

We stated previously that g is found experimentally to have the same value for all objects *at the same place*. From this it follows that the ratio of the weights of two objects must be equal to the ratio of their masses. Therefore a chemical balance, which actually is an instrument for comparing two downward forces, can be used in practice to compare masses. If a sample of salt in one pan of a balance is pulling down on that pan with the same force as is a standard one gram-mass on the other pan, we know* that the mass of salt is equal to one gram. We are likely to say that the salt "weighs" one gram, although a gram is a unit of mass, not weight. However, it is always important to distinguish carefully between weight and mass.

We have seen that the weight of a body, the downward pull of the earth on that body, is a vector quantity. The mass of a body is a scalar quantity. The quantitative relation between weight and mass is given by $\mathbf{W} = m\mathbf{g}$. Because $\mathbf{g}$ varies from point to point on the earth, $\mathbf{W}$, the weight of a body of mass m, is different in different localities. Thus, the weight of a one kg-mass in a locality where g is 9.80 meters/sec^2 is 9.80 nt; in a locality where g is 9.78 meters/sec^2, the same one kg-mass weighs 9.78 nt. If these weights were determined by measuring the amount of stretch required in a spring to balance them, the difference in weight of the same one kg-mass at the two different localities would be evident in the slightly different stretch of the spring at these two localities. Hence, unlike the mass of a body, which is an intrinsic property of the body, the weight of a body depends on its location relative to the center of the earth. Spring scales read differently, balances the same, at different parts of the earth.

We shall generalize the concept of weight in Chapter 16 in which we discuss universal gravitation. There we shall see that the weight of a body is zero in regions of space where the gravitational effects are nil, although the inertial effects, and hence the mass of the body, remain unchanged from those on earth. In a space ship free from the influence of gravity it is a simple matter to lift a large block of lead ($\mathbf{W} = 0$), but the astronaut would still stub his toe if he were to kick the block ($m \neq 0$).

It takes the same force to accelerate a body in gravity-free space as it does to accelerate it along a horizontal frictionless surface on earth, for its mass is the same in each place. But it takes a greater force to hold the body up against the pull of the earth on the earth's surface than it does high up in space, for its weight is different in each place.

Often, instead of being given the mass, we are given the weight of a body on which forces are exerted. The acceleration $\mathbf{a}$ produced by the force $\mathbf{F}$ acting on a body whose weight has a magnitude W can be obtained by combining Eq. 5–3 and Eq. 5–5. Thus from $\mathbf{F} = m\mathbf{a}$ and $W = mg$ we obtain

$$m = W/g, \qquad \text{so that} \qquad \mathbf{F} = (W/g)\mathbf{a}. \tag{5–6}$$

The quantity W/g plays the role of m in the equation $F = ma$ and is, in fact, the mass of a body whose weight has the magnitude W. For example, a man whose weight is 160 lb at a point where $g = 32.0$ ft/sec^2 has a

* Corrections for buoyancy, owing to the different volumes of air displaced by the salt and the standard, must be made. These are discussed in Chapter 17.

mass $m = W/g = (160 \text{ lb})/(32.0 \text{ ft/sec}^2) = 5.00$ slugs. Notice that his weight at another point where $g = 32.2 \text{ ft/sec}^2$ is $W = mg = (5.00 \text{ slugs})(32.2 \text{ ft/sec}^2) = 161$ lb.

5-9 A Static Procedure for Measuring Forces

In Section 5-3 we defined force by measuring the acceleration imparted to a standard body by pulling on it with a stretched spring. That may be called a dynamic method for measuring force. Although convenient for the purposes of definition, it is not a particularly practical procedure for the measurement of forces. Another method for measuring forces is based on measuring the change in shape or size of a body (a spring, say) on which the force is applied when the body is unaccelerated. This may be called the static method of measuring forces.

The idea of the static method is to use the fact that when a body, under the action of several forces, has zero acceleration, the vector sum of all the forces acting on the body must be zero. This is, of course, just the content of the first law of motion. A single force acting on a body would produce an acceleration; this acceleration can be made zero if we apply another force to the body equal in magnitude but oppositely directed. In practice we seek to keep the body at rest. If now we choose some force as our unit force, we are in a position to measure forces. The pull of the earth on a standard body at a particular point can be taken as the unit force, for example.

The instrument most commonly used to measure forces in this way is the spring balance. It consists of a coiled spring having a pointer at one end that moves over a scale. A force exerted on the balance changes the length of the spring. If a body weighing 1.00 lb is hung from the spring, the spring stretches until the pull of the spring on the body is equal in magnitude but opposite in direction to its weight. A mark can be made on the scale next to the pointer and labeled "1.00-lb force." Similarly, 2.00-lb, 3.00-lb, etc., weights may be hung from the spring and corresponding marks can be made on the scale next to the pointer in each case. In this way the spring is calibrated. We assume that the force exerted on the spring is always the same when the pointer stands at the same position. The calibrated balance can now be used to measure any suitable unknown force, not merely the pull of the earth on some body.

The third law is tacitly used in our static procedure because we assume that the force exerted by the spring on the body is the same in magnitude as the force exerted by the body on the spring. This latter force is the force we wish to measure. The first law is used too, because we assume $\mathbf{F}$ is zero when $\mathbf{a}$ is zero. It is worth noting again here that if the acceleration were not zero, the body of weight W would not stretch the spring to the same length as it did with $\mathbf{a} = 0$. In fact, if the spring and attached body of weight W were to fall freely under gravity so that $\mathbf{a} = \mathbf{g}$, the spring would not stretch at all and its tension would be zero.

5–10 Some Applications of Newton's Laws of Motion

It will be helpful to write down some procedures for solving problems in classical mechanics and to illustrate them by several examples. Newton's second law states that the vector sum of all the forces acting on a body is equal to its mass times its acceleration. The first step in problem solving is therefore: (1) Identify the body to whose motion the problem refers. As obvious as this seems, lack of clarity on the point as to what has been or should be picked as "the body" is a major source of mistakes. (2) Having selected "the body," we next turn our attention to the objects in "the environment" because these objects (inclined planes, springs, cords, the earth, etc.) exert forces on the body. We must be clear as to the nature of these forces. (3) The next step is to select a suitable (inertial) reference frame. We should position the origin and orient the coordinate axes so as to simplify the task of our next step as much as possible. (4) We now make a separate diagram of the body alone, showing the reference frame and *all* of the forces acting *on* the body. This is called a *free-body* diagram. (5) Finally we apply Newton's second law, in the form of Eq. 5–2, to each component of force and acceleration.

The following examples illustrate the method of analysis used in applying Newton's laws of motion. Each body is treated as if it were a particle of definite mass, so that the forces acting on it may be assumed to act at a point. Strings and pulleys are considered to have negligible mass. Although some of the situations picked for analysis may seem simple and artificial, they are the prototypes for many interesting real situations; but, more important, the method of analysis—which is the chief thing to understand—is applicable to all the modern and sophisticated situations of classical mechanics, even sending a spaceship to Mars.

▶ **Example 3.** Figure 5-5*a* shows a weight W hung by strings. Consider the knot at the junction of the three strings to be "the body." The body remains at rest under the action of the three forces shown in Fig. 5–5*b*. Suppose we are given

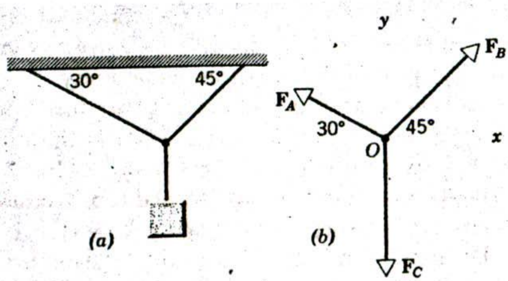


Fig. 5–5 Example 3. (*a*) A mass is suspended by strings. (*b*) A free-body diagram showing all the forces acting on the knot. The strings are assumed to be weightless.

the magnitude of one of these forces. How can we find the magnitude of the other forces?

$\mathbf{F}_A$, $\mathbf{F}_B$, and $\mathbf{F}_C$ are *all* the forces acting *on* the body. Since the body is unaccelerated (actually at rest), $\mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C = 0$. Choosing the x- and y-axes as shown, we can write this vector equation as three scalar equations:

$$F_{Ax} + F_{Bx} = 0,$$

$$F_{Ay} + F_{By} + F_{Cy} = 0,$$

using Eq. 5-2. The third scalar equation for the z-axis is simply

$$F_{Az} = F_{Bz} = F_{Cz} = 0.$$

That is, the vectors all lie in the x-y plane so that they have no z-components.

From the figure we see that

$$F_{Ax} = -F_A \cos 30° = -0.866F_A,$$

$$F_{Ay} = F_A \sin 30° = 0.500F_A,$$

and

$$F_{Bx} = F_B \cos 45° = 0.707F_B,$$

$$F_{By} = F_B \sin 45° = 0.707F_B.$$

Also,

$$F_{Cy} = -F_C = -W,$$

because the string C merely serves to transmit the force on one end to the junction at its other end. Substituting these results into our original equations, we obtain

$$-0.866F_A + 0.707F_B = 0,$$

$$0.500F_A + 0.707F_B - W = 0.$$

If we are given the magnitude of any one of these three forces, we can solve these equations for the other two. For example, if $W = 100$ lb, we obtain $F_A = 73.3$ lb and $F_B = 89.6$ lb.

Exampl 4. We wish to analyze the motion of a block on a smooth incline. (*a*) *Static case.* Figure 5-6*a* shows a block of mass m kept at rest on a smooth plane, inclined at an angle θ with the horizontal, by means of a string attached to the vertical wall. The forces acting *on* the block are shown in Fig. 5-6*b*. $\mathbf{F}_1$ is the force exerted *on* the block by the string; mg is the force exerted *on* the block by the earth, that is, its weight; and $\mathbf{F}_2$ is the force exerted *on* the block by the inclined surface. $\mathbf{F}_2$, called the normal force, is normal to the surface of contact because there is no frictional force between the surfaces.* If there were a frictional force, $\mathbf{F}_2$ would have a component parallel to the incline. Because we wish to analyze the motion of *the block*, we choose *ALL the forces acting ON the block.* The student will note that the block will exert forces on other bodies in its environment (the string, the earth, the surface of the incline) in accordance with the action-reaction

* The normal force is an example of a constraining force, one which limits the freedom of movement a body might otherwise have. It is an elastic force arising from small deformations of the bodies in contact, which are never perfectly rigid as we often tacitly assume.

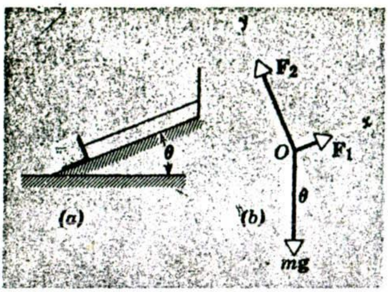


Fig. 5–6 Example 4. (*a*) A block is held on a smooth inclined plane by a string. (*b*) A free-body diagram showing all the forces acting on the block.

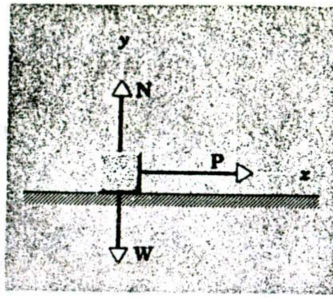


Fig. 5–7 Example 5. A block is being pulled along a smooth table. The forces acting on the block are shown.

principle; these forces, however, are not needed to determine the motion of the block because they do not act on the block.

Suppose θ and m are given. How do we find F_1 and F_2? Since the block is unaccelerated, we obtain

$$\mathbf{F}_1 + \mathbf{F}_2 + m\mathbf{g} = 0.$$

It is convenient to choose the x-axis of our reference frame to be along the incline and the y-axis to be normal to the incline (Fig. 5–6*b*). With this choice of coordinates, only one force, $m\mathbf{g}$, must be resolved into components in solving the problem. The two scalar equations obtained by resolving $m\mathbf{g}$ along the x- and y-axes are

$$F_1 - mg \sin \theta = 0, \qquad \text{and} \qquad F_2 - mg \cos \theta = 0,$$

from which F_1 and F_2 can be obtained if θ and m are given.

(*b*) *Dynamic case.* Now suppose that the string is cut. Then the force $\mathbf{F}_1$, the pull of the string on the block, will be removed. The resultant force on the block will no longer be zero, and the block will accelerate. What is its acceleration?

From Eq. 5–2 we have $F_x = ma_x$ and $F_y = ma_y$. Using these relations we obtain

$$F_2 - mg \cos \theta = ma_y = 0,$$

and

$$-mg \sin \theta = ma_x,$$

which yield

$$a_y = 0, \qquad a_x = -g \sin \theta.$$

The acceleration is directed down the incline with a magnitude of $g \sin \theta$.

Example 5. Consider a block of mass m pulled along a smooth horizontal surface by a horizontal force $\mathbf{P}$, as shown in Fig. 5–7. $\mathbf{N}$ is the normal force exerted on the block by the frictionless surface and $\mathbf{W}$ is the weight of the block.

(*a*) If the block has a mass of 2.0 kg, what is the normal force?

From the second law of motion with $a_y = 0$, we obtain

$$F_y = ma_y \qquad \text{or} \qquad N - W = 0.$$

Hence, $N = W = mg = (2.0 \text{ kg})(9.8 \text{ meters/sec}^2) = 20 \text{ nt}.$

(b) What force P is required to give the block a horizontal velocity of 4.0 meters/sec in 2.0 sec starting from rest?

The acceleration a_x follows from

$$a_x = \frac{v_x - v_{x0}}{t} = \frac{4.0 \text{ meters/sec} - 0}{2.0 \text{ sec}} = 2.0 \text{ meters/sec}^2.$$

From the second law, $F_x = ma_x$ or $P = ma_x$. The force P is then

$$P = ma_x = (2.0 \text{ kg})(2.0 \text{ meters/sec}^2) = 4.0 \text{ nt}.$$

Example 6. Figure 5–8*a* shows a block of mass m_1 on a smooth horizontal surface pulled by a string which is attached to a block of mass m_2 hanging over a pulley. We assume that the pulley has no mass and is frictionless and that it merely serves to change the direction of the tension in the string at that point. Find the acceleration of the system and the tension in the string.

Suppose we choose the block of mass m_1 as the body whose motion we investigate. The forces on this block, taken to be a particle, are shown in Fig. 5–8*b*. **T**, the tension in the string, pulls on the block to the right; $m_1\mathbf{g}$ is the downward pull of the earth on the block and **N** is the vertical force exerted on the block by the smooth table. The block will accelerate in the x-direction only, so that $a_{1y} = 0$. We, therefore, can write

$$N - m_1 g = 0 = m_1 a_{1y}, \tag{5–7}$$

and $$T = m_1 a_{1x}.$$

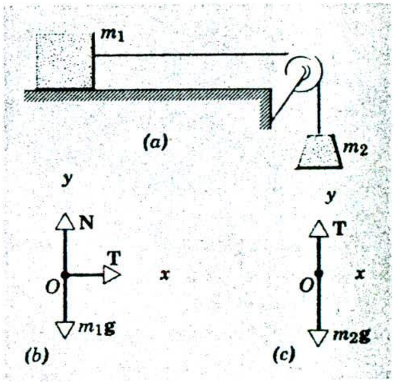


Fig. 5–8 Example 6. (*a*) Two masses are connected by a string; m_1 lies on a smooth table, m_2 hangs freely. (*b*) A free-body diagram showing all the forces acting on m_1. (*c*) A similar diagram for m_2.

From these equations we conclude that $N = m_1 g$. We do not know T, so we cannot solve for a_{1x}.

To determine T we must consider the motion of the block m_2. The forces acting on m_2 are shown in Fig. 5–8*c*. Because the string and block are accelerating, we cannot conclude that T equals $m_2 g$. In fact, if T were to equal $m_2 g$, the resultant force on m_2 would be zero, a condition holding only if the system is not accelerated.

The equation of motion for the suspended block is

$$m_2 g - T = m_2 a_{2y}. \tag{5–8}$$

The direction of the tension in the string changes at the pulley and, because the string has a fixed length, it is clear that

$$a_{2y} = a_{1x},$$

so that we can represent the acceleration of the system as simply a. We then obtain from Eqs. 5–7 and 5–8

$$m_2g - T = m_2a, \tag{5-9}$$

and

$$T = m_1a.$$

These yield

$$m_2g = (m_1 + m_2)a, \tag{5-10}$$

or

$$a = \frac{m_2}{m_1 + m_2}g,$$

and

$$T = \frac{m_1m_2}{m_1 + m_2}g, \tag{5-11}$$

which gives us the acceleration of the system a and the tension in the string T.

Notice that the tension in the string is always less than m_2g. This is clear from Eq. 5–11, which can be written

$$T = m_2g\frac{m_1}{m_1 + m_2}.$$

Notice also that a is always less than g, the acceleration due to gravity. Only when m_1 equals zero, which means that there is no block at all on the table, do we obtain $a = g$ (from Eq. 5–10). In this case $T = 0$ (from Eq. 5–9).

We can interpret Eq. 5–10 in a simple way. The net unbalanced force on the system of mass $m_1 + m_2$ is represented by m_2g. Hence, from $F = ma$, we obtain Eq. 5–10 directly.

To make the example specific, suppose $m_1 = 2.0$ kg and $m_2 = 1.0$ kg. Then

$$a = \frac{m_2}{m_1 + m_2}g = \tfrac{1}{3}g = 3.3 \text{ meters/sec}^2,$$

and

$$T = \frac{m_1m_2}{m_1 + m_2}g = (\tfrac{2}{3})(9.8) \text{ kg-m/sec}^2 = 6.5 \text{ nt}.$$

Example 7 Consider two unequal masses connected by a string which passes over a frictionless and massless pulley, as shown in Fig. 5–9a. Let m_2 be greater than m_1. Find the tension in the string and the acceleration of the masses.

We consider an *upward* acceleration *positive*. If the acceleration of m_1 is a, the acceleration of m_2 must be $-a$. The forces acting on m_1 and on m_2 are shown in Fig. 5–9b in which T represents the tension in the string.

The equation of motion for m_1 is

$$T - m_1g = m_1a$$

and for m_2 is

$$T - m_2g = -m_2a.$$

Combining these equations, we obtain

$$a = \frac{m_2 - m_1}{m_2 + m_1}g, \tag{5-12}$$

and

$$T = \frac{2m_1m_2}{m_1 + m_2}g.$$

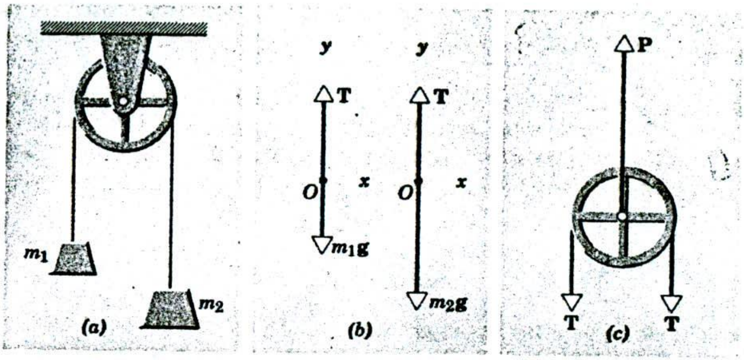


Fig. 5–9 Example 7. (*a*) Two unequal masses are suspended by a string from a pulley (Atwood's machine). (*b*) Free-body diagrams for m_1 and m_2. (*c*) Free-body diagram for the pulley, assumed massless.

For example, if $m_2 = 2.0$ slugs and $m_1 = 1.0$ slug,

$$a = (32/3.0)\ \text{ft/sec}^2 = g/3,$$

$$T = (\tfrac{4}{3})(32)\ \text{slug-ft/sec}^2 = 43\ \text{lb}.$$

Notice that the magnitude of T is always intermediate between the weight of the mass m_1 (32 lb in our example) and the weight of the mass m_2 (64 lb in our example). This is to be expected, since T must exceed m_1g to give m_1 an upward acceleration, and m_2g must exceed T to give m_2 a downward acceleration. In the special case when $m_1 = m_2$, we obtain $a = 0$ and $T = m_1g = m_2g$, which is the static result to be expected.

Figure 5–9*c* shows the forces acting on the massless pulley. If we treat the pulley as a particle, all the forces can be taken to act through its center. P is the upward pull of the support on the pulley and T is the downward pull of each segment of the string on the pulley. Since the pulley has no translational motion,

$$P = T + T = 2T.$$

If we were to drop our assumption of a massless pulley and assign a mass m to it, we would then be required to include a downward force mg on the support. Also, as we shall see later, the rotational motion of the pulley results in a different tension in each segment of the string. Friction in the bearings also affects the rotational motion of the pulley and the tension in the strings.

Example 8. Consider an elevator moving vertically with an acceleration **a**. We wish to find the force exerted by a passenger on the floor of the elevator.

Acceleration will be taken *positive upward* and *negative downward*. Thus positive acceleration in this case means that the elevator is either moving upward with increasing speed or is moving downward with decreasing speed. Negative acceleration means that the elevator is moving upward with decreasing speed or downward with increasing speed.

From Newton's third law the force exerted by the passenger on the floor will always be equal in magnitude but opposite in direction to the force exerted by the floor on the passenger. We can therefore calculate either the action force or the reaction force. When the forces acting on the passenger are used, we solve for the latter force. When the forces acting on the floor are used, we solve for the former force.

The situation is shown in Fig. 5–10: The passenger's true weight is **W** and the force exerted on him by the floor, called **P**, is his *apparent* weight in the accelerating elevator. The resultant force acting on him is **P** + **W**. Forces will be taken as positive when directed upward. From the second law of motion we have

$$F = ma,$$

or

$$P - W = ma, \tag{5–13}$$

where m is the mass of the passenger and a is his (and the elevator's) acceleration.

Suppose, for example, that the passenger weighs 160 lb and the acceleration is 2.0 ft/sec² upward. We have

$$m = \frac{W}{g} = \frac{160 \text{ lb}}{32 \text{ ft/sec}^2} = 5.0 \text{ slugs},$$

and from Eq. 5–13,

$$P - 160 \text{ lb} = (5.0 \text{ slugs})(2.0 \text{ ft/sec}^2)$$

or

$$P = \text{apparent weight} = 170 \text{ lb}.$$

If we were to measure this force directly by having the passenger stand on a spring scale fixed to the elevator floor (or suspended from the ceiling), we would find the scale reading to be 170 lb for a man whose weight is 160 lb. The passenger feels himself pressing down on the floor with greater force (the floor is pressing upward on him with greater force) than when he and the elevator are at rest. Everyone experiences this feeling when an elevator starts upward from rest.

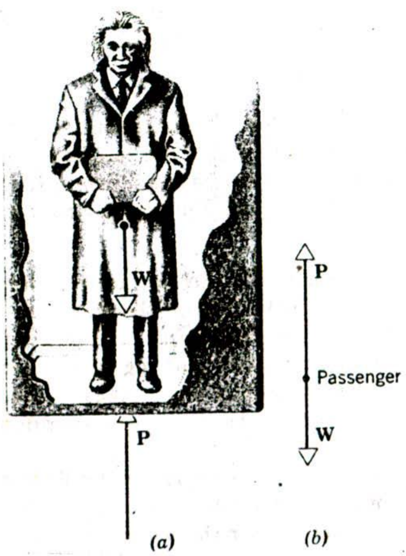


Fig. 5–10 Example 8. (*a*) A passenger stands on the floor of an elevator. (*b*) A free-body diagram for the passenger.

If the acceleration were taken as 2.0 ft/sec^2 downward, then $a = -2.0$ ft/sec^2 and $P = 150$ lb for the passenger. The passenger who weighs 160 lb feels himself pressing down on the floor with less force than when he and the elevator are at rest.

If the elevator cable were to break and the elevator were to fall freely with an acceleration $a = -g$, then P would equal $W + (W/g)(-g) = 0$. Then the passenger and floor would exert no forces on each other. The passenger's apparent weight, as indicated by the spring scale on the floor, would be zero. ◀

QUESTIONS

1. What is your mass in slugs? Your weight in newtons?

2. Why do you fall forward when a moving train decelerates to a stop and fall backward when a train accelerates from rest? What would happen if the train rounded a curve at constant speed?

3. A horse is urged to pull a wagon. The horse refuses to try, citing Newton's third law as his defense: "'The pull of the horse on the wagon is equal but opposite to the pull of the wagon on the horse.' If I can never exert a greater force on the wagon than it exerts on me, how can I ever start the wagon moving?" asks the horse. How would you reply?

4. A block of mass m is supported by a cord C from the ceiling, and another cord D is attached to the bottom of the block (Fig. 5–11). Explain this: If you give a sudden jerk to D it will break, but if you pull on D steadily, C will break.

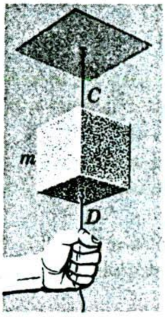


Fig. 5–11

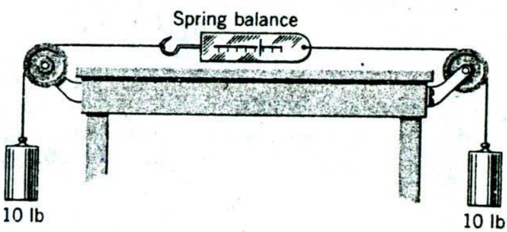


Fig. 5–12

5. Two 10-lb weights are attached to a spring scale as shown in Fig. 5–12. Does the scale read 0 lb, 10 lb, 20 lb, or give some other reading?

6. Criticize the statement, often made, that the mass of a body is a measure of the "quantity of matter" in it.

7. Using force, length, and time as fundamental quantities, what are the dimensions of mass?

8. Is the definition of mass that we have given limited to objects initially at rest?

9. Is the current standard of mass accessible, invariable, reproducible, indestructible? Does it have simplicity for comparison purposes? Would an atomic standard be better in any respect?

10. Suppose the carbon atom was chosen as the standard of mass. What information would be needed to express the mass of the standard kilogram in terms of this atomic standard? How could this information be obtained?

11. Suggest practical ways by which one could determine the masses of the various objects listed in Table 5–2.

12. In a tug of war, three men pull on a rope to the left at A and three men pull to the right at B with forces of equal magnitude. Now a weight of 5.0 lb is hung vertically from the center of the rope. (*a*) Can the men get the rope AB to be horizontal? (*b*) If not, explain. If so, determine the magnitude of the forces required at A and B to do this.

13. Both the following statements are true; explain them. Two teams having a tug of war must always pull equally hard on one another. The team that pushes harder against the ground wins.

14. Under what circumstances would your weight be zero? Does your answer depend on the choice of a reference system?

15. Two objects of equal mass rest on opposite pans of a trip scale. Does the scale remain balanced when it is accelerated up or down in an elevator?

16. A massless rope is strung over a frictionless pulley. A monkey holds onto one end of the rope and a mirror, having the same weight as the monkey, is attached to the other end of the rope at the monkey's level. Can the monkey get away from his image seen in the mirror (*a*) by climbing up the rope, (*b*) by climbing down the rope, (*c*) by releasing the rope?

17. A student standing on the large platform of a spring scale notes his weight. He then takes a step on this platform and notices that the scale reads less than his weight at the beginning of the step and more than his weight at the end of the step. Explain.

PROBLEMS

1. Two blocks, mass m_1 and m_2 are connected by a light spring on a horizontal frictionless table. Find the ratio of their accelerations a_1 and a_2 after they are pulled apart and then released.

2. Let the only forces acting on two bodies be their mutual interactions. If both bodies start from rest, show that the distances traveled by each are inversely proportional to the respective masses of the bodies.

3. A body of mass m is acted on by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, as shown in Fig. 5–13. If $m = 5.0$ kg, $F_1 = 3.0$ nt, and $F_2 = 4.0$ nt, find the vector acceleration of the body.

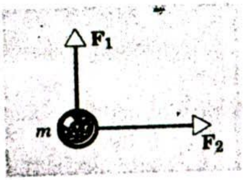


Fig. 5–13

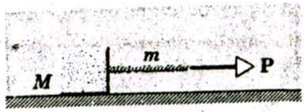


Fig. 5–14

4. A block of mass M is pulled along a horizontal frictionless surface by a rope of mass m, as in Fig. 5–14. A force **P** is applied to one end of the rope. (*a*) Find the acceleration of the block and the rope. (*b*) Find the force that the rope exerts on the block M in terms of $\mathbf{P}$, M, and m.

5. A car moving initially at a speed of 50 miles/hr and weighing 3000 lb is brought to a stop in a distance of 200 ft. (*a*) Find the braking force and the time required to stop. (*b*) Assuming the same braking force, find the distance and time required to stop if the car were going 25 miles/hr initially.

6. An electron travels in a straight line from the cathode of a vacuum tube to its anode, which is exactly 1.0 cm away. It starts with zero speed and reaches the anode with a speed of 6.0×10^6 meters/sec. (*a*) Assume constant acceleration and compute the force on the electron. Take the electron's mass to be 9.1×10^{-31} kg. This force is electrical in origin. (*b*) Compare it with the gravitational force on the electron, which we neglected when we assumed straight-line motion. Is this assumption valid?

7. A body of mass 2.0 slugs is acted on by the downward force of gravity and a horizontal force of 130 lb. Find its acceleration and its velocity as a function of time, assuming it starts from rest.

8. An electron is projected horizontally from an electron gun at a speed of 1.2×10^7 meters/sec into an electric field which exerts a constant vertical force of 4.5×10^{-15} nt on it. The mass of the electron can be taken to be 9.1×10^{-31} kg. Determine the vertical distance the electron is deflected during the time it has moved forward 3.0 cm horizontally.

9. A space traveler whose mass is 75 kg leaves the earth. Compute his weight (*a*) on the earth, (*b*) 400 miles above the earth (where $g = 8.1$ meters/sec^2), and (*c*) in interplanetary space. What is his mass at each of these locations?

10. Two blocks are in contact on a frictionless table. A horizontal force is applied to one block, as shown in Fig. 5–15. (*a*) If $m_1 = 2.0$ kg, $m_2 = 1.0$ kg, and $F = 3.0$ nt, find the force of contact between the two blocks. (*b*) Show that if the same force is applied to m_2 rather than to m_1, the force of contact between the blocks is 2.0 nt, which is not the same as the value derived in (*a*). Explain.

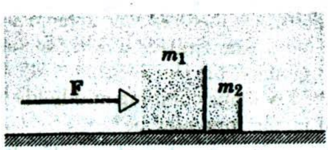


Fig. 5–15

11. Three blocks are connected, as shown in Fig. 5–16, on a horizontal frictionless table and pulled to the right with a force $T_3 = 60$ nt. If $m_1 = 10$ kg, $m_2 = 20$ kg, and $m_3 = 30$ kg, find the tensions T_1 and T_2. Draw an analogy to bodies being pulled in tandem, such as an engine pulling a train of coupled cars.

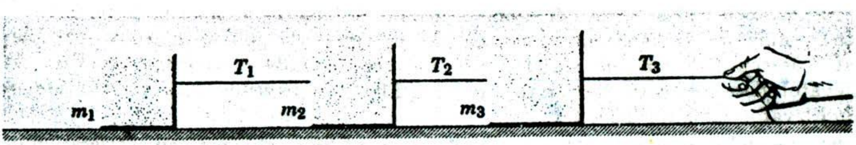


Fig. 5–16

12. A charged sphere of mass 3.0×10^{-4} kg is suspended from a string. An electric force acts horizontally on the sphere so that the string makes an angle of 37° with the vertical when at rest (Fig. 5–17). Find (*a*) the magnitude of the electric force and (*b*) the tension in the string.

13. How could a 100-lb object be lowered from a roof using a cord with a breaking strength of 87 lb without breaking the rope?

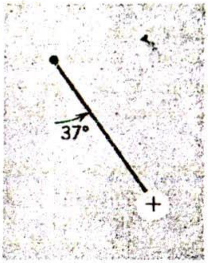


Fig. 5–17

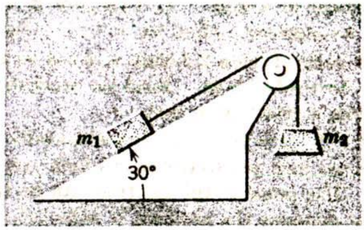


Fig. 5–18

14. Compute the initial upward acceleration of a rocket of mass 1.3×10^4 kg if the initial upward thrust of its engine is 2.6×10^5 nt. Can you neglect the weight of the rocket (the downward pull of the earth on it)?

15. A block of mass $m_1 = 3.0$ slugs on a smooth inclined plane of angle 30° is connected by a cord over a small frictionless pulley to a second block of mass $m_2 = 2.0$ slugs hanging vertically (Fig. 5–18). (*a*) What is the acceleration of each body? (*b*) What is the tension in the cord?

16. A 10-kg monkey is climbing a massless rope attached to a 15-kg mass over a (frictionless!) tree limb. (*a*) Explain quantitatively how the monkey can climb up the rope so that he can raise the 15-kg mass off the ground. (*b*) If, after the mass has been raised off the ground, the monkey stops climbing and holds on to the rope, what will his acceleration and the tension in the rope now be?

17. A block is released from rest at the top of a frictionless inclined plane 16 meters long. It reaches the bottom 4.0 sec later. A second block is projected up the plane from the bottom at the instant the first block is released in such a way that it returns to the bottom simultaneously with the first block. (*a*) Find the acceleration of each block on the incline. (*b*) What is the initial velocity of the second block? (*c*) How far up the inclined plane does it travel?

18. A block is projected up a frictionless inclined plane with a speed v_0. The angle of incline is θ. (*a*) How far up the plane does it go? (*b*) How long does it take to get there? (*c*) What is its speed when it gets back to the bottom? Find numerical answers for $\theta = 30°$ and $v_0 = 8.0$ ft/sec.

19. A block slides down a frictionless incline making an angle θ with an elevator floor. Find its acceleration relative to the incline in the following cases. (*a*) Elevator descends at constant speed v. (*b*) Elevator ascends at constant speed v. (*c*) Elevator descends with acceleration a. (*d*) Elevator descends with deceleration a. (*e*) Elevator cable breaks.

20. An elevator weighing 6000 lb is pulled upward by a cable with an acceleration of 4.0 ft/sec². (*a*) What is the tension in the cable? (*b*) What is the tension when the elevator is accelerating downward at 4.0 ft/sec²?

21. A lamp hangs vertically from a cord in a descending elevator. The elevator has a deceleration of 8.0 ft/sec^2 before coming to a stop. (*a*) If the tension in the cord is 20 lb, what is the mass of the lamp? (*b*) What is the tension in the cord when the elevator ascends with an acceleration of 8.0 ft/sec^2?

22. A plumb bob hanging from the ceiling of a railroad car acts as an accelerometer. (*a*) Derive the general expression relating the horizontal acceleration a of the car to the angle θ made by the bob with the vertical. (*b*) Find a when $\theta = 20°$. Find θ when $a = 5.0$ ft/sec^2.

23. Refer to Fig. 5–6. Let the mass of the block be 2.0 slugs and the angle θ equal 30°. (*a*) Find the tension in the string and the normal force acting on the block. (*b*) If the string is cut, find the acceleration of the block. Neglect friction.

24. Refer to Fig. 5–8*a*. Let $m_1 = 4.0$ slugs and $m_2 = 2.0$ slugs. Find the tension in the string and the acceleration of the two blocks.

25. Refer to Fig. 5–9*a*. Let $m_1 = 0.50$ kg and $m_2 = 1.0$ kg. Find the acceleration of the two blocks and the tension in the string.

26. A uniform flexible chain of length l, with weight per unit length λ, passes over a small, frictionless, massless pulley. It is released from a rest position with a length of chain x hanging from one side and a length $l - x$ from the other side. (*a*) Under what circumstances will it accelerate? (*b*) Assuming these circumstances are met, find the acceleration a as a function of x.

27. A triangular block of mass M with angles 30°, 60°, and 90° rests with its 30°-90° side on a horizontal table. A cubical block of mass m rests on the 60°-30° side (Fig. 5–19). (*a*) What horizontal acceleration a must M have relative to the table to keep m stationary relative to the triangular block, assuming frictionless contacts? (*b*) What horizontal force F must be applied to the system to achieve this result, assuming a frictionless table top? (*c*) Suppose no force is supplied to M and both surfaces are frictionless. Describe the resulting motion.

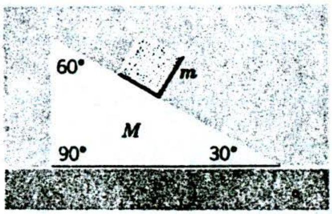


Fig. 5–19

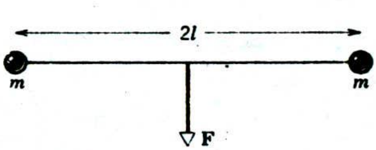


Fig. 5–20

28. Two particles, each of mass m, are connected by a light string of length $2l$, as shown in Fig. 5-20. A continuous force **F** is applied at the midpoint of the string ($x = 0$) at right angles to the initial position of the string. Show that the acceleration of m in the direction at right angles to **F** is given by

$$a_x = \frac{F}{2m} \frac{x}{\sqrt{l^2 - x^2}}$$

in which x is the perpendicular distance of one of the particles from the line of action of **F**. Discuss the situation when $x = l$.

29. *Terminal velocity.* The resistance of the air to the motion of bodies in free fall depends on many factors, such as the size of the body and its shape, the density and temperature of the air, and the velocity of the body through the air. A useful assumption, only approximately true, is that the resisting force $\mathbf{f}_R$ is proportional to the velocity and oppositely directed; that is, $\mathbf{f}_R = -k\mathbf{v}$, where k is a constant whose value in any particular case is determined by factors other than velocity.

Consider free fall of an object from rest through the air.

(*a*) Show that Newton's second law gives

$$mg - kv = ma \qquad \text{or} \qquad mg - k\frac{dy}{dt} = m\frac{d^2y}{dt^2}.$$

(*b*) Show that the body ceases to accelerate when it reaches a velocity $v_T = mg/k$, called the *terminal velocity*.

(*c*) Prove, by substituting it in the equation of motion of part (*a*), that the velocity varies with time as

$$v = v_T(1 - e^{-kt/m})$$

and plot v versus t.

(*d*) Sketch qualitatively curves of y versus t and a versus t for this motion, noting that the initial acceleration is g and the final acceleration is zero.

Particle Dynamics—II

CHAPTER 6

6–1 Introduction

In Chapter 5 we considered particle dynamics for bodies subject to a force that was constant in both magnitude and direction. The forces that we dealt with were exerted by the earth or by taut cords, that is, they were either gravitational or elastic in nature. In this chapter we consider another kind of force, that resulting from friction.

We shall also discuss the dynamics of uniform circular motion, in which the force, although constant in magnitude, changes in direction with time. In Chapter 10 we shall consider problems in which the force, although constant in direction, changes in magnitude with time, as when one body exerts a transient force on another during a collision. Finally, in Chapter 15, we shall consider problems in which the force changes in *both* magnitude *and* direction with time, such as the force exerted by a spring on an oscillating mass suspended from it.

6–2 Frictional Forces*

If we project a block of mass m with initial velocity $\mathbf{v}_0$ along a long horizontal table, it eventually comes to rest. This means that, while it is moving, it experiences an average acceleration $\bar{\mathbf{a}}$ that points in the direction opposite to its motion. If (in an inertial frame) we see that a body is being accelerated, we always associate a force, defined from Newton's

* See "The Friction of Solids" by E. H. Freitag, in *Contemporary Physics*, Vol. 2, 1961, p. 198, for a good general reference; see also the article "Friction" in the Encyclopedia Brittanica.

second law, with the motion. In this case we declare that the table exerts a *force of friction*, whose average value is $m\bar{a}$, on the sliding block.

Actually, whenever the surface of one body slides over that of another, each body exerts a frictional force on the other, parallel to the surfaces. The frictional force on each body is in a direction opposite to its motion relative to the other body. Frictional forces automatically oppose the motion and never aid it. Even when there is no relative motion, frictional forces may exist between surfaces.

Although we have ignored its effects up to now, friction is very important in our daily lives. Left to act alone it brings every rotating shaft to a halt. In an automobile, about 20% of the engine power is used to counteract frictional forces (only 1 or 2% in a turbojet engine, however). Friction causes wear and seizing of moving parts and many engineering man-hours are devoted to reducing it. On the other hand, without friction we could not walk as we now do; we could not hold a pencil in our hand and if we could it would not write; wheeled transport as we know it would not be possible.

We want to know how to express frictional forces in terms of the properties of the body and its environment; that is, we want to know the *force law* for frictional forces. In what follows we consider the sliding (not rolling) of one dry (unlubricated) surface over another. As we shall see later, friction, viewed at the microscopic level, is a very complicated phenomenon* and the force laws for dry, sliding friction are empirical in character and approximate in their predictions. They do not have the elegant simplicity and accuracy that we find for the gravitational force law (Chapter 16) or for the electrostatic force law (Chapter 26). It is remarkable, however, considering the enormous diversity of surfaces one encounters, that many aspects of frictional behavior can be understood qualitatively on the basis of a few simple mechanisms.

Consider a block at rest on a horizontal table as in Fig. 6–1. Attach a spring to it to measure the force required to set the block in motion. We find that the block will not move even though we apply a small force. We say that our applied force is balanced by an opposite frictional force exerted on the block by the table, acting along the surface of contact. As we increase the applied force we find some definite force at which the block just begins to move. Once motion has started, this same force produces accelerated motion. By reducing the force once motion has started, we find that it is possible to keep the block in uniform motion without acceleration; this force may be small, but it is never zero.

The frictional forces acting between surfaces at rest with respect to each other are called forces of *static friction*. The maximum force of static friction will be the same as the smallest force necessary to start motion. Once motion is started, the frictional forces acting between the surfaces

* See, for example, "Stick and Slip" by Ernest Rabinowicz, in *Scientific American*, May 1956.

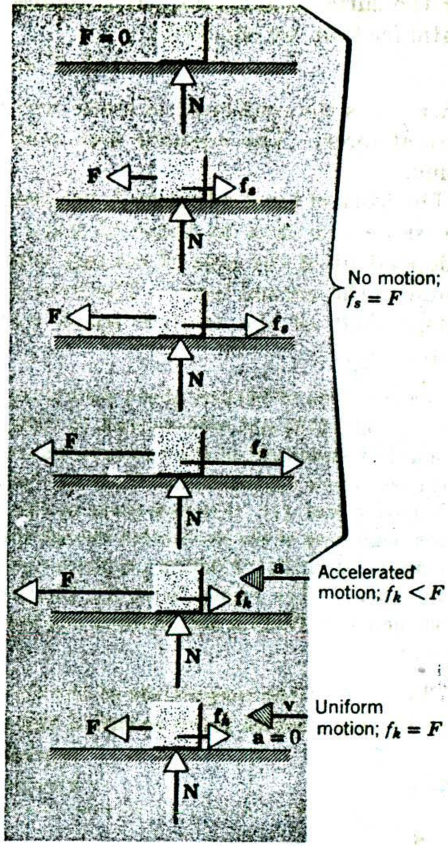


Fig. 6-1 A block being put into motion as applied force **F** overcomes frictional forces. In the first four drawings the applied force is gradually increased from zero to magnitude $\mu_s N$. No motion occurs until this point because the frictional force always just balances the applied force. The instant F becomes greater than $\mu_s N$, the block goes into motion, as is shown in the fifth drawing. In general, $\mu_k N < \mu_s N$; this leaves an unbalanced force to the left and the block accelerates. In the last drawing F has been reduced to equal $\mu_k N$. The net force is zero, and the block continues with constant velocity.

usually decrease so that a smaller force is necessary to maintain uniform motion. The forces acting between surfaces in relative motion are called forces of *kinetic friction*.

The maximum force of static friction between any pair of dry unlubricated surfaces follows these two empirical laws. (1) It is approximately independent of the area of contact, over wide limits and (2) it is proportional to the normal force. The normal force, sometimes called the loading force, is the one which either body exerts on the other at right angles to their mutual interface. It arises from the elastic deformation of the bodies in contact, such bodies never really being entirely rigid. For a block resting on a horizontal table or sliding along it, the normal force is equal in magnitude to the weight of the block. Because the block has no vertical acceleration, the table must be exerting a force on the block that is directed upward and is equal in magnitude to the downward pull of the earth on the block, that is, equal to the block's weight.

The ratio of the magnitude of the maximum force of static friction to the magnitude of the normal force is called the *coefficient of static friction* for the surfaces involved. If f_s represents the magnitude of the force of static friction, we can write

$$f_s \leq \mu_s N, \tag{6-1}$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force. The equality sign holds only when f_s has its maximum value.

The force of kinetic friction f_k between dry, unlubricated surfaces follows the same two laws as those of static friction. (1) It is approximately independent of the area of contact over wide limits and (2) it is proportional to the normal force. The force of kinetic friction is also reasonably independent of the relative speed with which the surfaces move over each other.

The two laws of friction above were discovered experimentally by Leonardo da Vinci (1452–1519) and rediscovered, in 1699, by the French engineer G. Amontons. Leonardo's statement of the two laws was remarkable, coming as it did about two centuries before the concept of force was fully developed by Newton. Leonardo's formulation was: (1) "Friction made by the same weight will be of equal resistance at the beginning of the movement though the contact may be of different breadths or lengths" and (2) "Friction produces double the amount of effort if the weight be doubled." The French scientist, Charles A. Coulomb, (1736–1806) did many experiments on friction and pointed out the difference between static and kinetic friction.

The ratio of the magnitude of the force of kinetic friction to the magnitude of this normal force is called the *coefficient of kinetic friction*. If f_k represents the magnitude of the force of kinetic friction,

$$f_k = \mu_k N, \tag{6-2}$$

where μ_k is the coefficient of kinetic friction.

Both μ_s and μ_k are dimensionless constants, each being the ratio of (the magnitudes of) two forces. Usually, for a given pair of surfaces $\mu_s > \mu_k$. The actual values of μ_s and μ_k depend on the nature of both the surfaces in contact. Both μ_s and μ_k can exceed unity, although commonly they are less than one. Notice that Eqs. 6–1 and 6–2 are relations between the *magnitudes only* of the normal and frictional forces. These forces are always directed perpendicularly to one another.

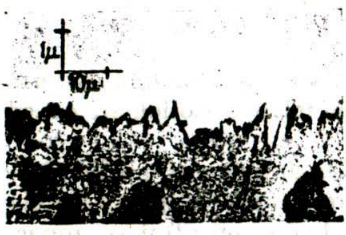


Fig. 6–2 A highly magnified view of a section of a finely polished steel surface. The section was cut at an angle so that vertical distances are exaggerated by a factor of ten with respect to horizontal distances. The surface irregularities are several thousand atomic diameters high. From *Friction and Lubrication of Solids*, by F. P. Bowden and D. Tabor, Clarendon Press, 1950.

On the atomic scale even the most finely polished surface is far from plane. Figure 6-2,

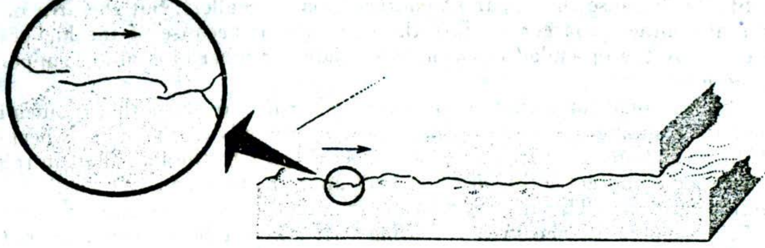


Fig. 6-3 Sliding friction. (*a*) The upper body is sliding to the right over the lower body in this enlarged diagram. (*b*) A further enlarged view showing two spots where surface adhesion has occurred. Force is required to break these welds apart and maintain the motion.

for example, shows an actual profile, highly magnified, of a steel surface that would be considered to be highly polished. One can readily believe that when two bodies are placed in contact, the actual microscopic area of contact is much less than apparent macroscopic area of contact; in a particular case these areas can be easily in the ratio of 1 to 10^4.

The actual (microscopic) area of contact is proportional to the normal force, because the contact points deform plastically under the great stresses that develop at these points. Many contact points actually become "cold-welded" together. This phenomenon, *surface adhesion*, occurs because at the contact points the molecules on opposite sides of the surface are so close together that they exert strong intermolecular forces on each other.

When one body (a metal, say) is pulled across another, the frictional resistance is associated with the rupturing of these thousands of tiny welds, which continually reform as new chance contacts are made (see Fig. 6-3). Radioactive tracer experiments have shown that, in the rupturing process, small fragments of one metallic surface may be sheared off and adhere to the other surface. If the relative speed of the two surfaces is great enough, there may be local melting at certain contact areas even though the surface as a whole may feel only moderately warm.

The coefficient of friction depends on many variables, such as the nature of the materials, surface finish, surface films, temperature, and extent of contamination. For example, if two carefully cleaned metal surfaces are placed in a highly evacuated chamber so that surface oxide films do not form, the coefficient of friction rises to enormous values and the surfaces actually become firmly "welded" together. The admission of a small amount of air to the chamber so that oxide films may form on the opposing surfaces reduces the coefficient of friction to its "normal" value.

With these complications it is not surprising that there is no exact theory of dry friction and that the laws of friction are empirical. The surface adhesion theory of friction for metals leads to a ready understanding of the two laws of friction mentioned above however. (1) The microscopic contact area, which determines the frictional force f_k, is proportional to the normal force N and thus f_k is proportional to N, as Eq. 6-2 shows. (2) The fact that the frictional force is independent of the apparent area of contact means, for example, that the force required to drag a metal "brick" along a metal table is the same no matter which face of the brick is in contact with the table. We can understand this only if the microscopic area of contact is the same for all positions of the brick, and this is indeed the case. With the largest face down, there are a relatively large number of relatively small area contacts supporting the load; with the smallest face down there are fewer contacts (because the apparent contact area is smaller), but the area of an individual contact is larger by just the same factor because of the higher pressure exerted by the up-ended brick on this smaller number of contacts supporting the same load.

The frictional force that opposes one body *rolling* over another is much less than that for a sliding motion and this, indeed, is the advantage of the wheel over the sledge. This reduced friction is due in large part to the fact that, in rolling, the microscopic contact welds are "peeled" apart rather than "sheared" apart as in sliding friction. This may reduce the frictional force by as much as 1000-fold.

Frictional resistance in dry, sliding, friction can be considerably reduced by lubrication. A mural in a grotto in Egypt dating back to 1900 B.C. shows a large stone statue being pulled on a sledge while a man in front of the sledge pours lubricating oil in its path. A still more effective technique is to introduce a layer of gas between the sliding surfaces; the "dry ice puck" mentioned on page 82 and the gas-supported bearing are two examples. Friction can be reduced still further by suspending a rotating object in an evacuated space by means of magnetic forces. J. W. Beams, for example, has spun a 30-lb rotor of this type at 1000 rev/sec; when the drive was cut off, the rotor lost speed at the rate of only 1 rev/sec in a day.*

Examples of the application of the empirical force law for friction follow. The coefficients of friction given are assumed to be constant. Actually μ_k can be regarded as a good average value that is not greatly different from the value at any particular speed in the range.

▶ **Example 1.** A block is at rest on an inclined plane making an angle θ with the horizontal, as in Fig. 6–4*a*. As the angle of incline is raised, it is found that slipping just begins at an angle of inclination θ_s. What is the coefficient of static friction between block and incline?

The forces acting on the block, considered to be a particle, are shown in Fig. 6–4*b*. $\mathbf{W}$ is the weight of the block, $\mathbf{N}$ the normal force exerted on the block by the inclined surface, and $\mathbf{f}_s$ the tangential force of friction exerted by the inclined surface on the block. Notice that the resultant force exerted by the inclined surface on the block, $\mathbf{N} + \mathbf{f}_s$, is no longer perpendicular to the surface of contact, as was true for smooth surfaces ($\mathbf{f}_s = 0$). The block is at rest, so that

$$\mathbf{N} + \mathbf{f}_s + \mathbf{W} = 0.$$

Resolving our forces into x- and y-components, along the plane and the normal to

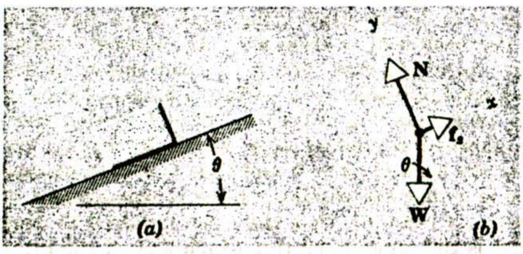


Fig. 6–4 Example 1. (*a*) A block at rest on a rough inclined plane. (*b*) A free-body force diagram for the block.

* See "Ultrahigh—Speed Rotation," Jesse W. Beams in *Scientific American*, April 1961.

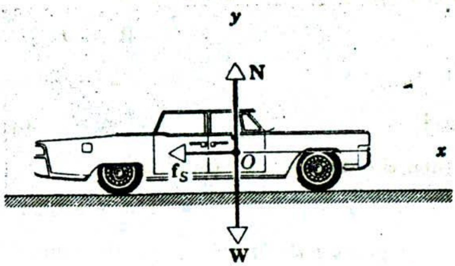


Fig. 6–5 Example 2. The forces on a decelerating automobile.

the plane, respectively, we obtain

$$N - W\cos\theta = 0,$$
$$f_s - W\sin\theta = 0. \tag{6–3}$$

However, $f_s \leqq \mu_s N$. If we increase the angle of incline slowly until slipping just begins, then for that angle, $\theta = \theta_s$ and we can use $f_s = \mu_s N$. Substituting this into Eqs. 6–3, we obtain

$$N = W\cos\theta_s$$

and
$$\mu_s N = W\sin\theta_s,$$

so that
$$\mu_s = \tan\theta_s.$$

Hence measurement of the angle of inclination at which slipping just starts provides a simple experimental method for determining the coefficient of static friction between two surfaces.

The student can make use of similar arguments to show that the angle of inclination θ_k required to maintain a *constant speed* for the block as it slides down the plane, once it has been started by tapping, is given by

$$\mu_k = \tan\theta_k,$$

where $\theta_k < \theta_s$. With the aid of a ruler the student can now determine μ_s and μ_k for a coin sliding down his textbook.

Example 2. Consider an automobile moving along a straight horizontal road with a speed v_0. If the coefficient of static friction between the tires and the road is μ_s, what is the shortest distance in which the automobile can be stopped?

The forces acting on the automobile, considered to be a particle, are shown in Fig. 6–5. The car is assumed to be moving in the positive x-direction. If we assume that f_s is a constant force, we have uniformly decelerated motion.

From the relation (see Eq. 3–16)

$$v^2 = v_0^2 + 2ax,$$

with the final speed $v = 0$, we obtain

$$x = -v_0^2/2a,$$

where the minus sign means that **a** points in the negative x-direction.

To determine a, apply the second law of motion to the x-component of the motion:

$$-f_s = ma = (W/g)a \qquad \text{or} \qquad a = -g(f_s/W).$$

From the y components we obtain

$$N - W = 0 \qquad \text{or} \qquad N = W,$$

so that

$$\mu_s = f_s/N = f_s/W$$

and

$$a = -\mu_s g.$$

Then the distance of stopping is

$$x = -v_0^2/2a = v_0^2/2g\mu_s. \tag{6-4}$$

The greater the initial speed, the longer the distance required to come to a stop; in fact, this distance varies as the square of the initial velocity. Also, the greater the coefficient of static friction between the surfaces, the less the distance required to come to a stop.

We have used the coefficient of static friction in this problem, rather than the coefficient of sliding friction, because we assume there is no sliding between the tires and the road. We have neglected rolling friction. Furthermore, we have assumed that the maximum force of static friction ($f_s = \mu_s N$) operates because the problem seeks the shortest distance for stopping. With a smaller static frictional force the distance for stopping would obviously be greater. The correct braking technique required here is to keep the car just on the verge of skidding. If the surface is smooth and the brakes are applied fully, sliding may occur. In this case μ_k replaces μ_s, and the distance required to stop is seen to increase from Eq. 6–4.

As a specific example, if v_0 = 60 miles/hr = 88 ft/sec, and $\mu_s = 0.60$ (a typical value), we obtain

$$x = \frac{v_0^2}{2\mu_s g} = \frac{(88\text{ ft/sec})^2}{2(0.60)(32\text{ ft/sec}^2)} = 200\text{ ft}.$$

Notice that the mass of the car does not appear in Eq. 6–4. How can you explain the practice of "weighing down" a car in order to increase safety in driving on icy roads?

The student should now investigate how, in principle, forces of friction would modify the results of the examples of Section 5–10. ◀

6–3 The Dynamics of Uniform Circular Motion

In Section 4–4 we pointed out that if a body is moving at uniform speed v in a circle of radius r, it experiences a centripetal acceleration $\mathbf{a}$ whose magnitude is v^2/r. The direction of $\mathbf{a}$ is always radially inward toward the center of rotation. Thus $\mathbf{a}$ is a variable vector because, even though its magnitude remains constant, its direction changes continuously as the motion progresses.

Recall that there need not be any motion in the direction of an acceleration. In general, there is no fixed relation between the directions of the acceleration $\mathbf{a}$ and the velocity $\mathbf{v}$ of a particle, as Fig. 4–7 shows. For a particle in uniform circular motion the acceleration $\mathbf{a}$ and velocity $\mathbf{v}$ are always at right angles to each other.

Every accelerated body must have a force $\mathbf{F}$ acting on it, defined by Newton's second law ($\mathbf{F} = m\mathbf{a}$). Thus (assuming that we are in an inertial frame), if we see a body undergoing uniform circular motion, we can be

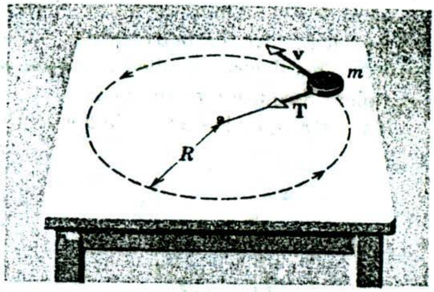


Fig. 6-6 A disk m moves with constant speed in a circular path on a horizontal frictionless surface. The only horizontal force acting on m is the centripetal force **T** with which the string pulls on the body.

certain that a net force **F**, given in magnitude by

$$F = ma = mv^2/r$$

must be acting on the body; the body is *not* in equilibrium. The direction of **F** at any instant must be the direction of **a** at that instant, namely, radially inward. We must always be able to account for this force by pointing to a particular object in the environment that is exerting the force on the accelerating, circulating body.

If the body in uniform circular motion is a disc on the end of a string moving in a circle on a frictionless horizontal table as in Fig. 6-6, the force **F** on the disc is provided by the tension **T** in the string. This force **T** is the net force acting on the disc. It accelerates the disc by constantly changing the direction of its velocity so that the disc moves in a circle. **T** is always directed toward the pin at the center and its magnitude is mv^2/R. If the string were to be cut where it joins the disc, there would be no net force exerted on the disc. The disc would then move with constant speed in a straight line along the direction of the tangent to the circle at the point at which the string was cut. Hence, to keep the disc moving in a circle, a force must be supplied to it pulling it *inward* toward the center.

Forces responsible for uniform circular motion are called *centripetal* forces because they are directed "toward the center" of the circular motion. To label a force as "centripetal," however, simply means that it always points radially inward; the name tells us nothing about the nature of the force or about the body that is exerting it. Thus, for the revolving disc of Fig. 6-6, the centripetal force is an elastic force provided by the string; for the moon revolving around the earth (in an approximately circular orbit) the centripetal force is the gravitational pull of the earth on the moon; for an electron circulating about an atomic nucleus the centripetal force is electrostatic. A centripetal force is *not* a new kind of force but simply a way of describing the behavior with time of forces that are attributable to specific bodies in the environment. Thus a force can be centripetal *and* elastic, centripetal *and* gravitational, or centripetal *and* electrostatic, among other possibilities.

Let us consider some examples of forces that act centripetally.

▶ **Example 3.** *The Conical Pendulum.* Figure 6–7a represents a small body of mass m revolving in a horizontal circle with constant speed v at the end of a string of length L. As the body swings around, the string sweeps over the surface of a cone. This device is called a *conical pendulum.* Find the time required for one complete revolution of the body.

If the string makes an angle θ with the vertical, the radius of the circular path is $R = L \sin \theta$. The forces acting on the body of mass m are $\mathbf{W}$, its weight, and $\mathbf{T}$, the pull of the string, as shown in Fig. 6–7b. It is clear that $\mathbf{T} + \mathbf{W} \neq 0$. Hence, the resultant force acting on the body is nonzero, which is as it should be because a force is required to keep the body moving in a circle with constant speed.

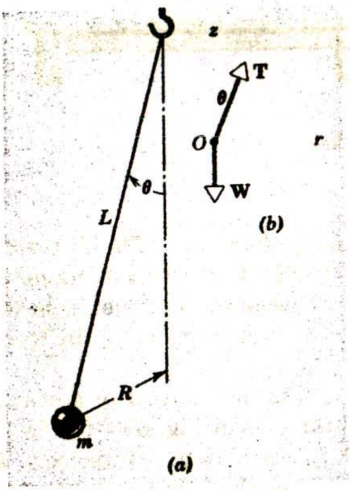


Fig. 6–7 Example 3. (a) A mass m suspended from a string of length L swings so as to describe a circle. The string describes a right circular cone of semiangle θ. (b) A free-body force diagram for m.

We can resolve $\mathbf{T}$ at any instant into a radial and a vertical component

$$T_r = T \sin \theta \qquad \text{and} \qquad T_z = T \cos \theta.$$

Since the body has no vertical acceleration,

$$T_z - W = 0.$$

But

$$T_z = T \cos \theta \qquad \text{and} \qquad W = mg,$$

so that

$$T \cos \theta = mg.$$

The radial acceleration is v^2/R. This acceleration is supplied by T_r, the radial component of $\mathbf{T}$, which is the centripetal force acting on m. Hence

$$T_r = T \sin \theta = mv^2/R.$$

Dividing this equation by the preceding one, we obtain

$$\tan \theta = v^2/Rg, \qquad \text{or} \qquad v^2 = Rg \tan \theta,$$

which gives the constant speed of the bob. If we let τ represent the time for one complete revolution of the body, then

$$v = \frac{2\pi R}{\tau} = \sqrt{Rg \tan \theta}$$

or

$$\tau = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{Rg \tan \theta}} = 2\pi \sqrt{R/(g \tan \theta)}.$$

But $R = L \sin \theta$, so that

$$\tau = 2\pi \sqrt{(L \cos \theta)/g}.$$

This equation gives the relation between τ, L, and θ. Notice that τ, called the *period* of motion, does not depend on m.

If $L = 3.0$ ft and $\theta = 30°$, what is the period of the motion? We have

$$\tau = 2\pi\sqrt{\frac{(3.0\text{ ft})(0.866)}{32\text{ ft/sec}^2}} = 1.8\text{ sec.}$$

Example 4. *The Rotor.* In many amusement parks we find a device called the *rotor.* The rotor is a hollow cylindrical room which can be set rotating about the central vertical axis of the cylinder. A person enters the rotor, closes the door, and stands up against the wall. The rotor gradually increases its rotational speed from rest until, at a predetermined speed, the floor below the person is opened downward, revealing a deep pit. The passenger does not fall but remains "pinned up" against the wall of the rotor. Find the coefficient of friction necessary to prevent falling.

The forces acting on the passenger are shown in Fig. 6–8. **W** is the passenger's weight, $\mathbf{f}_s$ is the force of static friction between passenger and rotor wall, and **P** is the centripetal force exerted by the wall on the passenger necessary to keep him moving in a circle. Let the radius of the rotor be R and the final speed of the passenger be v. Since the passenger does not move vertically, but experiences a radial acceleration v^2/R at any instant, we have

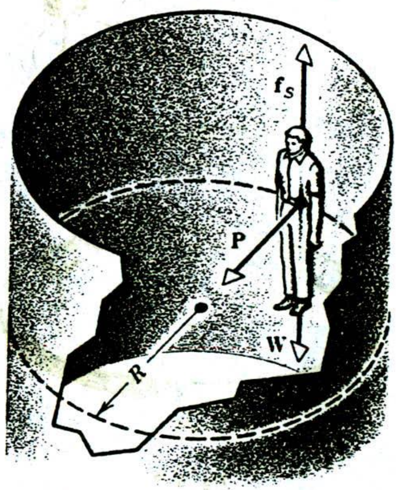


Fig. 6–8 The forces on a person in a "rotor" of radius R.

$$f_s - W = 0$$

and

$$P(=ma) = (W/g)(v^2/R).$$

If μ_s is the coefficient of static friction between passenger and wall necessary to prevent slipping, then $f_s = \mu_s P$ and

$$f_s = W = \mu_s P$$

or

$$\mu_s = \frac{W}{P} = \frac{gR}{v^2}.$$

This equation gives the minimum coefficient of friction necessary to prevent slipping for a rotor of radius R when a particle on its wall has a speed v. Notice that the result does not depend on the passenger's weight.

As a practical matter the coefficient of friction between the textile material of clothing and a typical rotor wall (canvas) is about 0.40. For a typical rotor the radius is 7.0 ft, so that v must be about 24 ft/sec or 16 miles/hr or more.

Example 5. Let the block in Fig. 6–9*a* represent an automobile or railway car moving at constant speed v on a *level* road-bed around a curve having a radius of curvature R. In addition to two vertical forces, namely the force of gravity **W** and a normal force **N**, a horizontal centripetal force **P** acts on the car. In the case of the automobile this centripetal force is supplied by a sidewise frictional force exerted by the road on the tires; in the case of the railway car the centripetal force is supplied by the rails exerting a sidewise force on the inner rims of the car's wheels. Neither of these sidewise forces can be safely relied upon to be large enough at all times and both cause unnecessary wear. Hence, the roadbed is *banked* on curves, as shown in Fig. 6–9*b*. In this case, the normal force **N** has not only a vertical component, as before, but also a horizontal component which supplies the centripetal force necessary for uniform circular motion; no additional sidewise forces are needed, therefore, with a properly banked roadbed.

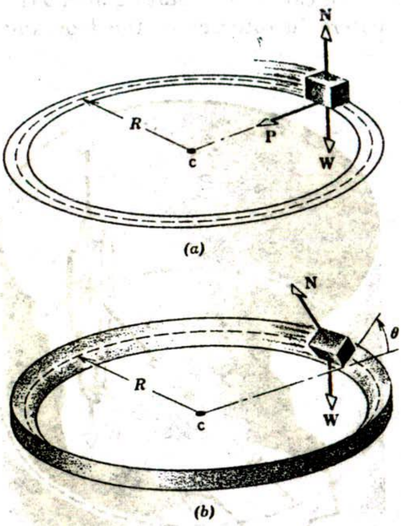


Fig. 6–9

The correct angle θ of banking can be obtained as follows. There is no vertical acceleration, so that

$$N \cos \theta = W.$$

The centripetal force is $N \sin \theta$, so that $N \sin \theta = mv^2/R$. Dividing the latter equation by the former and setting $W = mg$, we obtain

$$\tan \theta = v^2/Rg$$

Notice that the proper angle of banking depends upon the speed of the car and the curvature of the road. For a given curvature, the road is banked at an angle corresponding to an expected average speed. Often curves are marked by signs giving the proper speed for which the road was banked.

The student should check the banking formula for the limiting cases $v = 0$; $R \to \infty$; v large; and R small. He should also note the great similarity between Fig. 6–7 of Example 3 and Fig. 6–9*b* of this example. ◄

6–4 Forces and Pseudo-Forces

All forces in nature can be classified under three headings, each with a different relative strength: (1) gravitational forces, which are relatively very weak, (2) electromagnetic forces, which are of intermediate strength, and (3) nuclear forces. Nuclear forces are of two types, those which bind neutrons and protons in the nucleus (very strong) and those responsible for beta decay (weak). These forces are "real" in the sense that we can associate them with specific objects in the environment. Such forces as the tension in a rope, the force of friction, the force that we exert on a wall by pushing on it, or the force exerted by a compressed spring are electromagnetic forces; all are macroscopic manifestations of the (electromagnetic) attractions and repulsions between atoms.

In our treatment of classical mechanics so far we have assumed that our measurements and observations were made from an inertial frame. This, we recall, is a reference frame that is either at rest or is moving at constant velocity with respect to the fixed stars; it is the set of reference frames defined by Newton's first law, namely, that set of frames in which a body will not be accelerated ($\mathbf{a} = 0$) if there are no identifiable force-producing bodies in its environment ($\mathbf{F} = 0$). The choice of a reference frame is always ours to make, so that if we choose to select only inertial frames, we do not restrict in any way our ability to apply classical mechanics to natural phenomena.

Nevertheless we can, if we find it convenient, apply classical mechanics from the point of view of an observer in a *noninertial frame.* Such a frame might be one that is rotating (and therefore accelerating) with respect to the fixed stars. We sometimes choose an accelerating reference frame when we consider, for example, the separation of liquids of different density in a spinning centrifuge, the global circulation of the winds on the rotating earth, or the experiences of an astronaut in an orbiting satellite.

We can apply classical mechanics in noninertial frames if we introduce forces called *pseudo-forces* (or *inertial forces*). They are so named because, unlike the forces that we have examined so far, we cannot associate them with any particular body in the environment of the particle on which they act; we cannot classify them into any of the categories listed in the first paragraph of this section. Finally, if we view the particle from an inertial frame, the pseudo-forces disappear. These forces are, then, simply a technique that permits us to apply classical mechanics in the normal way to events if we insist on viewing the events from an accelerating reference frame.

Consider a rotating merry-go-round on which a marble is lodged against a raised rim at the outer edge. An observer on the merry-go-round is in a noninertial system. As he kneels down and examines the marble he sees that, with respect to him, it is not moving; if he pulls it away a bit from the rim toward the center of rotation, he observes that it moves back again, as if under the influence of a force directed radially outward. He would declare the marble to be in equilibrium under the action of this outward force (a *pseudo-force* called, in this case, a *centrifugal force*) and the radially inward force exerted by the rim.

An observer on the ground (an inertial frame) watching the marble would describe it differently. He would declare the marble to be in uniform circular motion, accelerated radially inward with $a = v^2/R$. The inward force $\mathbf{F}$ exerted by the rim on the marble accounts for this acceleration from Newton's second law, or $F = ma = mv^2/R$. The marble is definitely *not* in equilibrium from the point of view of this observer or of an observer in any inertial frame. Only if the rim were *not* exerting this inward force would the marble move with uniform speed in a straight line and be in equilibrium. This observer would find no trace of a force directed radially outward (the pseudo-force) and, indeed, there is no room for such a force in his analysis of the motion.

It is clear from this simple example that the radially outward pseudo-force (or centrifugal force) noted by the observer on the rotating merry-go-round must have a magnitude mv^2/R. Thus the magnitude of the pseudo-force depends on the speed of the particle *as seen from another reference frame,* namely, the ground; the speed of the particle in its own (rotating) reference frame is zero.

In mechanical problems, then, we have two choices: (1) select an *inertial frame* as a reference frame and consider only "real" forces, that is, forces that we can associate with definite bodies in the environment or (2) select a *noninertial frame* as a reference frame and consider not only the "real" forces but suitably defined pseudo-forces. Although we usually choose the first alternative, we sometimes choose the second; both are completely equivalent and the choice is a matter of convenience. We shall discuss noninertial frames and pseudo-forces further in Chapters 11 and 16.

6–5 Classical Mechanics, Relativistic Mechanics, and Quantum Mechanics

In these first chapters we have laid the groundwork of classical mechanics. We have presented the laws of motion and have given several examples of the force laws. In later chapters we shall discuss other kinds of forces and shall continue to develop the structure of the theory. Here we want to point out where classical mechanics stands in the framework of modern physics.

Physics is not a static body of doctrine but a developing science. Historically there have been long periods of deep concern with a certain class of problem, culminating, often rather suddenly and in unexpected ways, in a "breakthrough" in the form of a new, more comprehensive theory. This occured about 1670 (Newtonian mechanics), about 1870 (Maxwell's theory of electromagnetism), 1905 (Einstein's theory of relativity), and about 1925 (quantum mechanics). Some physicists believe that our present concern for problems in the area of elementary particles (see Appendix E) will lead us eventually to another major "breakthrough."

As physics has evolved, many things have changed, such as the problems to be solved and the tools we use to investigate them. But through it all the general method of inquiry or process of solution remains basically the same. Thus earlier theories of physics are found to have limited ranges of validity and to be special cases of more comprehensive theories, which in turn are found to have limitations, and so on. However, independent of any particular area or problem in physics, we always demand that theory meet the test of experiment, we search for quantities that are invariant, we are guided by a belief in the simplicity and symmetry of nature, and we seek and use analogies and models. Major unifying concepts arise which are valid in all domains of physics, such as the conservation laws. All this is important to understand for its own sake, independent of mastery of any particular special topic, and is exemplified throughout the book. If, in addition to mastering classical mechanics, the student comes to understand this process, he will find it much easier to understand and master such theories as relativity theory and quantum theory, wherein the same method of inquiry applies but whose areas of application, unlike those of classical mechanics, are not a familiar part of his daily life experience.

Classical mechanics, like all theories in physics, is based on observations of things that happen in nature. It will help to point out how limited are our normal experiences of natural phenomena. This is particularly true during our formative years which is the period during which we develop our intuitive notions (often false!) of what is "common sense" in natural events and what is not.

For example, the highest speed that can be used to transmit signals from one point to another is the speed of light (c = 186,000 miles/sec = 3.00×10^8 meters/sec) and this seems to set an upper limit to the speeds of material objects. However, gross objects, even the fastest of them, such as jet planes or earth satellites, have speeds v that are very much less than c. For an earth satellite moving at 17,000 miles/hr, v/c is only 0.00025. Classical mechanics was built up over several centuries on a body of observations of relatively slow-moving objects such as planets, balls rolling down inclined planes, and falling bodies. Our experience with moving objects has indeed been limited, until the last few decades, to a tiny fraction of the range of possible speeds.

During these last decades it has become possible to make measurements on small particles, of potentially high speed, such as electrons, protons, and other fundamental particles. A proton accelerated in the 30-billion electron volt accelerator at the Brookhaven National Laboratories has, for example, $v/c = 0.98$. Are we to expect that the laws of classical mechanics, which work so beautifully when $v/c \ll 1$, will also describe correctly the collisions, decays, and interactions of these elementary particles moving at such high speeds? This is the grossest kind of extrapolation and indeed we find by experiment that it simply does not work; classical mechanics gives answers that do not agree with experiment if the speeds of the objects involved are appreciable compared to the speed of light. This does not make us think less of classical mechanics, which serves so well in the region of low speed, precisely the very important region of our daily experiences. We are led, however, to view classical mechanics as a special case of a more general theory which would hold for all speeds up to the speed of light.

Einstein, in 1905, first proposed this more general theory, the *special theory of relativity*. We shall discuss it in depth later but will state here its fundamental postulate. This is that the speed of light c is the *same* for all observers in inertial frames, no matter what the motion of the light source may be. In other words, if a light source is moving directly toward you at a speed v, you would measure the same value for c, if you observed a light pulse passing you, no matter what the value of v; you would also obtain speed c for the light pulse if the source were rushing away from you at speed v. If this basic assumption seems to violate "common sense," we must realize that our intuitive feelings are based on "common sense at low speeds." We have no direct experience in our daily activities about what really happens in nature at high speeds. Furthermore, all of Einstein's predictions (1) agree with experiment and (2) reduce to the predictions of classical mechanics at low speeds.

We list here just one of the predictions of the theory of relativity that is at variance with classical mechanics. If two observers watch an object moving parallel to the common $x - x'$-axis in Fig. 4-11 they will find, from Eq. 4-19,

$$v = v' + u, \tag{6-5}$$

where v' is the speed as measured by observer S', v is that measured by observer S, and u is the relative speed of separation of the two reference frames. Note that there is nothing in Eq. 6-5 to prevent v from exceeding c if v' and u are large enough.

The theory of relativity predicts that Eq. 6-5 is a special case of a more general formula, namely,

$$v = \frac{v' + u}{1 + v'u/c^2}. \tag{6-6}$$

Note that for $v' \ll c$ and $u \ll c$ Eq. 6-6 does indeed reduce to Eq. 6-5. Also, if $v' < c$ and $u < c$, then v cannot exceed c. If $v' = u = 0.8\ c$, for example, Eq. 6-6 yields $v = 0.975\ c$; Eq. 6-5, on the other hand, yields $v = 1.6\ c$, which is contrary to experience.

For gross objects, Eqs. 6-5 and 6-6 give the same results within experimental error, so that we naturally use the simpler, Eq. 6-5. If two satellites moving in opposite directions have speeds $v' = u = 17{,}000$ miles/hr, the denominator in Eq. 6-6 has the value 1.0000000007, so that the speed v of one satellite as seen from the other differs very slightly indeed from the value $v' + u$ predicted by Eq. 6-5. It would take speeds almost 3000 *times* as great as above, nearly 50 million miles/hr, generally achievable only in the subatomic domain, to obtain a difference as great as one-half of one percent in the two formulas.

We point out a second way in which our daily experiences are limited, namely, that all the objects that we normally deal with have masses that greatly exceed, for example, the electron mass ($m = 9.11 \times 10^{-31}$ kg). This turns out to have an interesting consequence, closely related to the very concept of "particle" on which classical mechanics is based. We have not hesitated to assign a mass m, a position x, and a velocity v_x to a particle, assumed to be moving along the x-axis.* If we

---

* We assume $v_x \ll c$ so that considerations of relativity do not enter this new discussion.

are asked within what accuracy Δx and Δv_x we could measure the position x and the velocity v_x respectively, we would be inclined to say that, although there might be limits in practice there are none in principle and, with sufficient attention to methods of measurement, we can specify x and v_x as closely as we wish. Experiment seems to confirm this view for large objects like golf balls or rifle bullets.

When we deal with objects of very small mass, however, such as electrons, we learn that the very procedures of measurement introduce fundamental uncertainties and that, in fact, the more precise our knowledge of x becomes the less precise is our knowledge of v_x and conversely. We can express this in terms of the famous Heisenberg uncertainty relation, which we write as

$$\Delta x \cong \frac{h}{m\,\Delta v_x} \tag{6–7}$$

in which h (Planck's constant) is a fundamental constant of nature and has the value $h = 6.63 \times 10^{-34}$ kg meter^2/sec. Equation 6–7 shows clearly that if Δv_x is very small (which means that we know v_x very precisely), then Δx must be relatively large (which means that we do not know x very precisely). Thus it does not seem possible to measure *both* the position *and* the velocity of a particle to any given precision at the same time. If we cannot do this, then our whole concept of a particle as a mass point following a trajectory, which is a basic concept of classical mechanics, is open to question.

Just as for relativity theory, these considerations of quantum mechanics simply do not make any difference for the gross objects of our daily experience. Consider a bullet with a speed of 10^3 meters/sec and a mass of 1.0 gm ($= 10^{-3}$ kg). Let us assume that we know the speed to be accurate to 0.1%, which means that $\Delta v_x = 0.001 \times 10^3 = 1$ meter/sec. The uncertainty in the position of the bullet is now given by Eq. 6–7 as

$$\Delta x \cong \frac{6.63 \times 10^{-34}\text{ kg meter}^2/\text{sec}}{(10^{-3}\text{ kg})(1\text{ meter/sec})} \cong 7 \times 10^{-31}\text{ meter}$$

This is such a small distance (being 10^{-15} times smaller than an atomic nucleus!) that we could not possibly detect any limitation on the measurement of x set by Eq. 6–7.

Consider, however, not a bullet but an electron ($m = 9.11 \times 10^{-31}$ kg) whose velocity is measured to be 2×10^6 meters/sec, which is about the speed of an electron in a hydrogen atom. If we assume that we know this velocity to be accurate to, say, 1%, then $\Delta v_x = 0.01 \times 2 \times 10^6$ meters/sec $= 2 \times 10^4$ meters/sec. The uncertainty in position predicted by Eq. 6–7 is then

$$\Delta x \cong \frac{6.63 \times 10^{-34}\text{ kg meter}^2/\text{sec}}{(9.11 \times 10^{-31}\text{ kg})(2 \times 10^{-4}\text{ meter/sec})} = 3 \times 10^{-8}\text{ meter.}$$

Since the radius of a hydrogen atom is about 5×10^{-11} meter we see that the uncertainty with which we can locate the electron in the hydrogen atom, assuming that we have measured its speed as accurately as we claim, is 600 times the radius of the atom! The concept of "particle" does not mean much under these circumstances. This simply means that we cannot use classical mechanics to describe the motions of electrons in atoms; we need quantum mechanics.

The situation is very much like that of relativity theory. Ideas that we find acceptable in a certain region of experience (bullets) fall down when we apply them to a region outside our direct normal experience (electrons in atoms). Once again the solution is the same: Classical mechanics turns out to be an important special case of a more general theory. In this case the general theory is that of quantum mechanics developed about 1925 to 1926 by Heisenberg, Schrödinger, Born, and others. Once again, quantum mechanics does not detract from the merit of classical mechanics, which continues to give results that agree admirably with experiment for particles of relatively large mass.

The situation most remote from our daily experience deals with particles that have *both* small mass *and* high speed. Here we must use a still more general theory, *relativistic quantum mechanics*, which combines both relativity theory and quantum mechanics; such a theory was first developed by Dirac in 1927.

In the rest of our treatment of mechanics we return to the familiar special case of our daily experience, that of relatively massive and relatively slow-moving objects (classical mechanics). From time to time we will point out parenthetically how the predictions of classical mechanics must be modified when we depart from this region of experience.

QUESTIONS

1. There is a limit beyond which further polishing of a surface *increases* rather than decreases frictional resistance. Can you explain this?

2. Is it unreasonable to expect a coefficient of friction to exceed unity?

3. How could a person who is at rest on completely frictionless ice covering a pond reach shore? Could he do this by walking, rolling, swinging his arms, or kicking his feet? How could a person be placed in such a position in the first place?

4. Explain how the range of your car's headlights limits the safe driving speed at night.

5. Your car skids across the center line on an icy highway. Should you turn the front wheels in the direction of skid or in the opposite direction (*a*) when you want to avoid a collision with an oncoming car, (*b*) when no other car is near but you want to regain control of the steering?

6. If you want to stop the car in the shortest distance on an icy road, should you (*a*) push hard on the brakes to lock the wheels, (*b*) push just hard enough to prevent slipping, or (*c*) "pump" the brakes?

7. A cube of weight W rests on a rough inclined plane which makes an angle θ with the horizontal. Compare the minimum force necessary to start the cube moving down the plane with that necessary to start the cube moving up the plane. How do these compare with the minimum *horizontal* force (transverse to the slope) that will cause the cube to move down the plane?

8. Why are the train roadbeds and highways banked on curves?

9. How does the earth's rotation affect the apparent weight of a body at the equator?

10. A car is riding on a country road that resembles a roller coaster track. If the car travels with uniform speed, compare the force it exerts on a horizontal section of the road to the force it exerts on the road at the top of a hill and at the bottom of a hill. Explain.

11. Suppose you need to measure whether a table top in a train is truly horizontal. If you use a spirit level can you determine this when the train is moving down or up a grade? When the train is moving along a curve? (Hint: there are two horizontal components.)

12. In the conical pendulum of Example 3 what happens to the period τ and the speed v when $\theta = 90°$? Why is this angle not achievable physically? Discuss the case for $\theta = 0°$.

13. A coin is put on a phonograph turntable. The motor is started, but before the final speed of rotation is reached, the coin flies off. Explain.

14. A passenger in the front seat of a car finds himself sliding toward the door as the driver makes a sudden left turn. Describe the forces on the passenger and on the car at this instant if (*a*) the motion is viewed from a reference frame attached to the earth and (*b*) if attached to the car.

15. What is the distinction between inertial reference frames and those differing only by a translation or rotation of the axes?

PROBLEMS

1. A fireman weighing 160 lb slides down a vertical pole with an average acceleration of 10 ft/sec^2. What is the average vertical force he exerts on the pole?

2. A railroad flatcar is loaded with crates having a coefficient of static friction 0.25 with the floor. If the train is moving at 30 miles/hr, in how short a distance can the train be stopped without letting the crates slide?

3. Frictional heat generated by the moving ski is the chief factor promoting sliding in skiing. The ski sticks at the start, but once in motion will melt the snow beneath it. Waxing the ski makes it water repellent and reduces friction with the film of water. A magazine reports that a new type of plastic ski is even more water repellent and that on a gentle 700-ft slope in the Alps, a skier reduced his time from 61 to 42 sec with new skis. (*a*) Determine the average accelerations for each pair of skis. (*b*) Assuming a 3°-slope compute the coefficient of kinetic friction for each case.

4. A student wants to determine the coefficients of static friction and kinetic friction between a box and a plank. He places the box on the plank and gradually raises the plank. When the angle of inclination with the horizontal reaches 30°, the box starts to slip and slides 4.0 meters down the plank in 4.0 sec. Show how he can determine the coefficients from these observations.

5. A hockey puck weighing 0.25 lb slides on the ice for 50 ft before it stops. (*a*) If its initial speed was 20 ft/sec, what is the force of friction between puck and ice? (*b*) What is the coefficient of kinetic friction?

6. A 10-lb block of steel is at rest on a horizontal table. The coefficient of static friction between block and table is 0.50. (*a*) What is the magnitude of the horizontal force that will just start the block moving? (*b*) What is the magnitude of a force acting upward 60° from the horizontal that will just start the block moving? (*c*) If the force acts down at 60° from the horizontal, how large can it be without causing the block to move?

7. A piece of ice slides down a 45°-incline in twice the time it takes to slide down a frictionless 45°-incline. What is the coefficient of kinetic friction between the ice and the incline?

8. A horizontal force F of 12 lb pushes a block weighing 5.0 lb against a vertical wall (Fig. 6–10). The coefficient of static friction between the wall and the block is 0.60

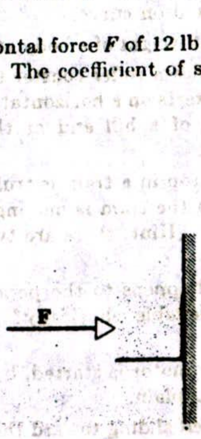


Fig. 6–10

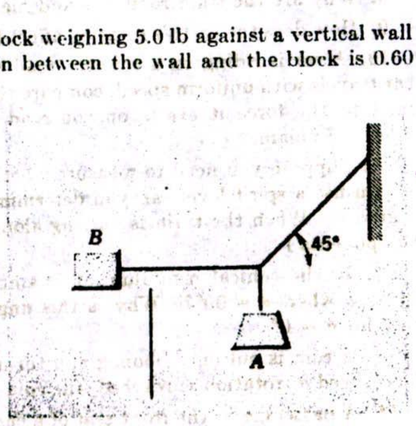


Fig. 6–11

and the coefficient of kinetic friction is 0.40. Assume the block is not moving initially. (*a*) Will the block start moving? (*b*) What is the force exerted on the block by the wall?

9. Block B in Fig. 6–11 weighs 160 lb. The coefficient of static friction between block and table is 0.25. Find the maximum weight of block A for which the system will be in equilibrium.

10. A 4.0-kg block is put on top of a 5.0-kg block. In order to cause the top block to slip on the bottom one, a horizontal force of 12 nt must be applied to the top block (Fig. 6–12). Assume a frictionless table and find (*a*) the maximum horizontal force F which can be applied to the lower block so that the blocks will move together and (*b*) the resulting acceleration of the blocks.

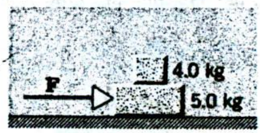


Fig. 6–12

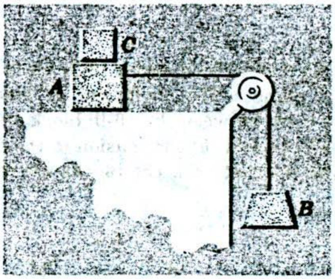


Fig. 6–13

11. In Fig. 6–13, A is a 10-lb block and B is a 5.0-lb block. (*a*) Determine the minimum weight (block C) which must be placed on A to keep it from sliding, if μ_s between A and the table is 0.20. (*b*) The block C is suddenly lifted off A. What is the acceleration of block A, if μ_k between A and the table is 0.20?

12. The handle of a floor mop of mass m makes an angle θ with the vertical direction. Let μ_k be the coefficient of kinetic friction between mop and floor, and μ_s be the coefficient of static friction between mop and floor. Neglect the mass of the handle. (*a*) Find the magnitude of the force F directed along the handle required to slide the mop with uniform velocity across the floor. (*b*) Show that if θ is smaller than a certain angle θ_0, the mop cannot be made to slide across the floor no matter how great a force is directed along the handle. (*c*) What is the angle θ_0?

13. A block slides down an inclined plane of slope angle ϕ with constant velocity. It is then projected up the same plane with an initial speed v_0. How far up the incline will it move before coming to rest? Will it slide down again?

14. Body B weighs 100 lb and body A weighs 32 lb (Fig. 6–14). Given $\mu_s = 0.56$ and $\mu_k = 0.25$, (*a*) find the acceleration of the system if B is initially at rest and (*b*) find the acceleration if B is moving initially.

15. Two masses, $m_1 = 1.65$ kg and $m_2 = 3.30$ kg, attached by a massless rod parallel to the incline on which both slide, as shown in Fig. 6–15, travel down along the plane with m_1 trailing m_2. The angle of incline is $\theta = 30°$. The coefficient of kinetic friction between m_1 and the incline is $\mu_1 = 0.226$; between m_2 and the incline the corresponding coefficient is $\mu_2 = 0.113$. Compute (*a*) the tension in the rod linking m_1 and m_2 and (*b*) the common acceleration of the two masses. (*c*) Would the answers to (*a*) and (*b*) be changed if m_2 trails m_1?

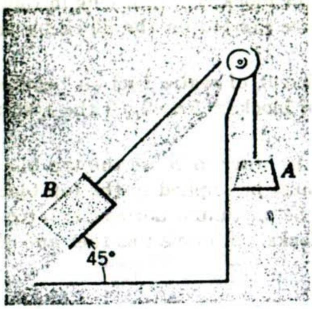


Fig. 6–14

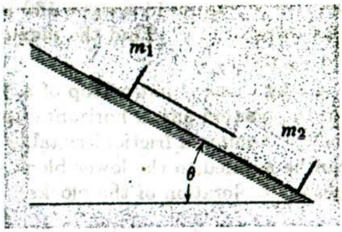


Fig. 6–15

16. An 8.0-lb block and a 16-lb block connected together by a string slide down a 30° inclined plane. The coefficient of kinetic friction between the 8.0-lb block and the plane is 0.10; between the 16-lb block and the plane it is 0.20. Find (*a*) the acceleration of the blocks and (*b*) the tension in the string, assuming that the 8.0-lb block leads. (*c*) Describe the motion if the blocks are reversed.

Fig. 6–16

17. A block of mass m slides in an inclined right-angled trough as in Fig. 6–16. If the coefficient of kinetic friction between the block and the material composing the trough is μ_k, find the acceleration of the block.

18. A block of mass m at the end of a string is whirled around in a vertical circle of radius R. Find the critical speed below which the string would become slack at the highest point.

19. A 5000-lb airplane loops the loop at a speed of 200 miles/hr. Find (*a*) the radius of the largest circular loop possible and (*b*) the force on the plane at the bottom of this loop.

20. In the Bohr model of the hydrogen atom, the electron revolves in a circular orbit around the nucleus. If the radius of the orbit is 5.3×10^{-11} meter and the electron makes 6.6×10^{15} rev/sec, find (*a*) the acceleration (magnitude and direction) of the electron and (*b*) the centripetal force acting on the electron. (This force is due to the attraction between the positively charged nucleus and the negatively charged electron.) The mass of the electron is 9.1×10^{-31} kg.

21. Assume that the standard kilogram would weigh exactly 9.80 nt at sea level on the earth's equator if the earth did not rotate about its axis. Then take into account the fact that the earth does rotate so that this mass moves in a circle of radius 6.40×10^6 meters (earth's radius) at a constant speed of 465 meters/sec. (*a*) Determine the centripetal force needed to keep the standard moving in its circular path. (*b*) Determine the force exerted by the standard kilogram on a spring balance from which it is suspended at the equator (its weight).

22. Because of the rotation of the earth, a plumb bob may not hang exactly along the direction of the earth's gravitational pull (its weight) but deviates slightly from this direction. Calculate the deviation (*a*) at 40° latitude, (*b*) at the poles, and (*c*) at the equator.

23. A circular curve of highway is designed for traffic moving at 40 miles/hr. (*a*) If the radius of the curve is 400 ft, what is the correct angle of banking of the road? (*b*) If the curve is not banked, what is the minimum coefficient of friction between tires and road that would keep traffic from skidding at this speed?

24. An old streetcar rounds a corner on unbanked tracks. If the radius of the tracks is 30 ft and the car's speed is 10 miles/hr, what angle with the vertical will be made by the loosely hanging hand straps? Is there a force acting on these straps? If so, is it a centripetal or a centrifugal force? Do your answers depend on what reference frame you choose?

25. A mass m on a frictionless table is attached to a hanging mass M by a cord through a hole in the table (Fig. 6–17). Find the conditions (v and r) with which m must spin for M to stay at rest.

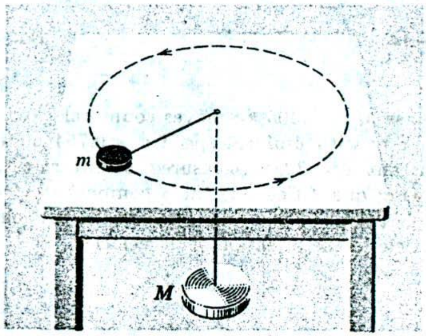


Fig. 6–17

26. Imagine that the disk of Fig. 6–6 is attached to a spring rather than a string. The unstretched length of the spring is l_0 and the tension in the spring increases in direct proportion to its elongation, the tension per unit elongation being k. If the disk rotates with a frequency f (revolutions per unit time), show that (*a*) the radius R of the uniform circular motion is $kl_0/(k - 4\pi^2 mf^2)$ and (*b*) the tension T in the spring is $4\pi^2 mkl_0 f^2/(k - 4\pi^2 mf^2)$.

27. (*a*) What is the smallest radius of a circle at which a bicyclist can travel if his speed is 18 mi/hr and the coefficient of static friction between the tires and the road is 0.32? (*b*) Under these conditions what is the largest angle of inclination to the vertical at which the bicyclist can ride without falling?

28. A small coin is placed on a flat, horizontal turntable. The turntable is observed to make three revolutions in 3.14 sec. (*a*) What is the speed of the coin when it rides without slipping at a distance 5.0 cm from the center of the turntable? (*b*) What is the acceleration (magnitude and direction) of the coin in part (*a*)? (*c*) What is the frictional-force acting on the coin in part (*a*) if the coin has a mass m? (*d*) What is the coefficient of static friction between the coin and the turntable if the coin is observed to slide off the turntable when it is greater than 10 cm from the center of the turntable?

29. A very small cube of mass m is placed on the inside of a funnel (Fig. 6–18) rotating about a vertical axis at a constant rate of ν rev/sec. The wall of the funnel makes an angle θ with the horizontal. If the coefficient of static friction between the cube and the funnel is μ and the center of the cube is a distance r from the axis of rotation, what are the largest and smallest values of ν for which the block will not move with respect to the funnel?

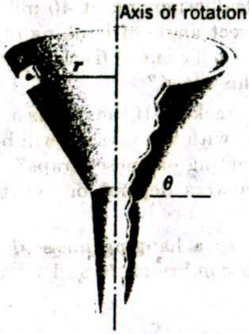


Fig. 6–18

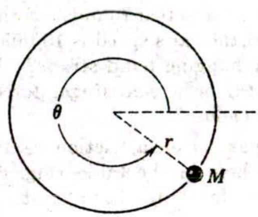


Fig. 6–19

30. A particle of mass $M = 0.305$ kg moves counterclockwise in a horizontal circle of radius $r = 2.63$ meters with uniform speed $v = 0.754$ meter/sec as in Fig. 6–19. Determine at the instant $\theta = 322°$ (measured counterclockwise from the positive x-direction) the following quantities: (*a*) the x-component of the velocity; (*b*) the y-component of the acceleration; (*c*) the total force on the particle; (*d*) the component of total force on the particle in the direction of its velocity.

Work and Energy

CHAPTER 7

7–1 Introduction

A fundamental problem of particle dynamics is to find how a particle will move when we know the forces that act on it. By "how a particle will move" we mean how its position varies with time. If the motion is one-dimensional, the problem is to find x as a function of time, $x(t)$. In the previous two chapters we solved this problem for the special case of a constant force. The method used is this. We find the resultant force $\mathbf{F}$ acting on the particle from the appropriate force law. We then substitute $\mathbf{F}$ and the particle mass m into Newton's second law of motion. This gives us the acceleration $\mathbf{a}$ of the particle; or

$$\mathbf{a} = \mathbf{F}/m.$$

If the force $\mathbf{F}$ and the mass m are constant, the acceleration $\mathbf{a}$ must be constant. Let us choose the x-axis to be along the direction of this constant acceleration. We can then find the speed of the particle from Eq. 3–12,

$$v = v_0 + at,$$

and the position of the particle from Eq. 3–15 (with $x_0 = 0$), or

$$x = v_0t + \tfrac{1}{2}at^2;$$

note that, for simplicity and convenience, we have dropped the subscript x in these equations. The last equation gives us directly what we usually want to know, namely $x(t)$, the position of the particle as a function of time.

The problem is more difficult, however, when the force acting on a particle is *not constant.* In such a case we still obtain the acceleration of the particle, as before, from Newton's second law of motion. However, in order to get the speed or position of the particle, we can no longer use the formulas previously developed for constant acceleration because the acceleration now is *not* constant. To solve such problems, we use the mathematical process of integration, which we consider in this chapter.

We confine our attention to forces that vary with the position of the particle in its environment. This type of force is common in physics. Some examples are the gravitational forces between bodies, such as the sun and earth or earth and moon, and the force exerted by a stretched spring on a body to which it is attached. The procedure used to determine the motion of a particle subject to such a force leads us to the concepts of work and kinetic energy and to the development of the work-energy theorem, which is the central feature of this chapter. In Chapter 8 we consider a broader view of energy, embodied in the law of conservation of energy, a concept which has played a major role in the development of physics.

7–2 Work Done by a Constant Force

Consider a particle acted on by a force. In the simplest case the force $\mathbf{F}$ is constant and the motion takes place in a straight line in the direction of the force. In such a situation we define the *work done by the force on the particle* as the product of the magnitude of the force F and the distance d through which the particle moves. We write this as

$$W = Fd.$$

However, the constant force acting on a particle may not act in the direction in which the particle moves. In this case we define the *work done by the force on the particle* as the product of the component of the force along the line of motion by the distance d the body moves along that line. In Fig. 7–1 a constant force $\mathbf{F}$ makes an angle ϕ with the x-axis and acts on a particle whose displacement along the x-axis is $\mathbf{d}$. If W represents the work done by $\mathbf{F}$ during this displacement, then according to our definition

$$W = (F \cos \phi)d. \tag{7–1}$$

Of course, other forces must act on a particle that moves in this way (its

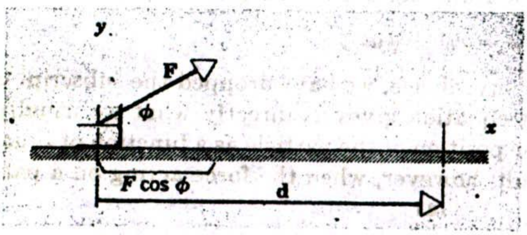


Fig. 7–1 A force $\mathbf{F}$ makes the block undergo a displacement $\mathbf{d}$. The component of $\mathbf{F}$ that does the work has magnitude $F \cos \phi$; the work done is $Fd \cos \phi$ ($= \mathbf{F} \cdot \mathbf{d}$).

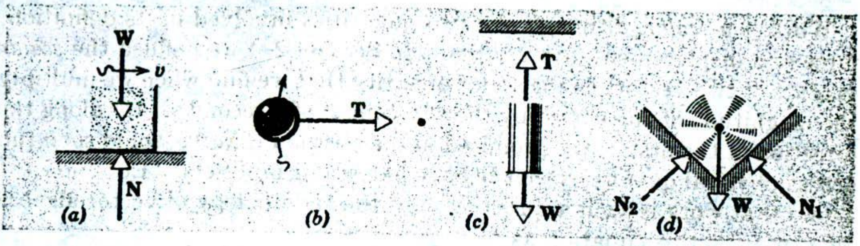


Fig. 7–2 Work is not always done by a force that is applied to a body. (*a*) The block is moving to the right at constant speed v over a frictionless surface. Work is not done by either the weight $\mathbf{W}$ or the normal force $\mathbf{N}$. (*b*) The ball moves in a circle under the influence of a centripetal force $\mathbf{T}$. There is a centripetal acceleration $\mathbf{a}$ but no work is done by $\mathbf{T}$. In both (*a*) and (*b*) the forces being considered ($\mathbf{W}$, $\mathbf{N}$, and $\mathbf{T}$) are at right angles to the displacement so that $W = \mathbf{F} \cdot \mathbf{d} = Fd \cos \phi = Fd \cos 90° = 0$. (*c*) A cylinder hangs from a cord. No work is done either by $\mathbf{T}$, the tension in the cord, or by $\mathbf{W}$ the weight of the cylinder. (*d*) A cylinder rests in a groove; no work is done by $\mathbf{W}$, $\mathbf{N}_1$ or $\mathbf{N}_2$. In both (*c*) and (*d*) the work done by the individual forces is zero because the displacement is zero.

weight and the frictional force exerted by the plane, to name two). A particle acted on by only a single force may have a displacement in a direction other than that of this single force, as in projectile motion. But it cannot move in a straight line unless the line has the same direction as that of the single force applied to it. *Equation 7–1 refers only to the work done on the particle by the particular force* **F**. *The work done on the particle by the other forces must be calculated separately.* The total work done on the particle is the sum of the works done by the separate forces.

When ϕ is zero, the work done by $\mathbf{F}$ is simply Fd, in agreement with our previous equation. Thus, when a horizontal force draws a body horizontally, or when a vertical force lifts a body vertically, the work done by the force is the product of the magnitude of the force by the distance moved. When ϕ is 90°, the force has no component in the direction of motion. That force then does no work on the body. For instance, the vertical force holding a body a fixed distance off the ground does no work on the body, even if the body is moved horizontally over the ground. Also, the centripetal force acting on a body in motion does no work on that body because the force is always at right angles to the direction in which the body is moving. Of course, a force does no work on a body that does not move, for its displacement is then zero. In Fig. 7–2 we illustrate common examples in which a force applied to a body does no work on that body.

Notice that we can write Eq. 7–1 either as $(F \cos \phi)d$ or $F(d \cos \phi)$. This suggests that the work can be calculated in two different ways: Either we multiply the magnitude of the displacement by the component of the force in the direction of the displacement or we multiply the magnitude of the force by the component of the displacement in the direction of the force. These two methods always give the same result.

Work is a *scalar*, although the two quantities involved in its definition, force and displacement, are vectors. In Section 2–4 we defined the *scalar product* of two vectors as the scalar quantity that we find when we multiply the magnitude of one vector by the component of a second vector along the direction of the first. We promised in that section that we would soon run across physical quantities that behave like scalar products. Equation 7–1 shows that work is such a quantity. In the terminology of vector algebra we can write this equation as

$$W = \mathbf{F} \cdot \mathbf{d}, \tag{7–2}$$

where the dot indicates a scalar (or dot) product. Equation 7–2 for **F** and **d** corresponds to Eq. 2–11 for **a** and **b**.

Work can be either positive or negative. If the particle on which a force acts has a component of motion opposite to the direction of the force, the work done by that force is negative. This corresponds to an obtuse angle between the force and displacement vectors. For example, when a person lowers an object to the floor, the work done on the object by the upward force of his hand holding the object is negative. In this case ϕ is 180°, for **F** points up and **d** points down.

Work as we have defined it (Eq. 7–2) proves to be a very useful concept in physics. Our special definition of the word *work* does not correspond to the colloquial usage of the term. This may be confusing. A person holding a heavy weight at rest in the air may say that he is doing hard work—and he may work hard in the physiological sense—but from the point of view of physics we say that he is not doing any work. We say this because the applied force causes no displacement. The word *work* is used only in the strict sense of Eq. 7–2. In many scientific fields words are borrowed from our everyday language and are used to name a very specific concept. The words *basic* and *cell*, for example, mean quite different things in chemistry and biology than in everyday language.

The *unit* of work is the work done by a unit force in moving a body a unit distance in the direction of the force. In the mks system the unit of work is 1 *newton-meter*, called 1 *joule*. In the British engineering system the unit of work is the *foot-pound*. In cgs systems the unit of work is 1 *dyne-centimeter*, called 1 *erg*. Using the relations between the newton, the dyne and the pound, and the meter, the centimeter, and foot, we obtain 1 joule = 10^7 ergs = 0.7376 ft-lb.

▶ **Example 1.** A block of mass 10.0 kg is to be raised from the bottom to the top of an incline 5.00 meters long and 3.00 meters off the ground at the top. Assuming frictionless surfaces, how much work must be done by a force parallel to the incline pushing the block up at *constant speed* at a place where g = 9.80 meters/sec^2.

The situation is shown in Fig. 7–3*a*. The forces acting on the block are shown in Fig. 7–3*b*. We must first find P, the magnitude of the force pushing the block up the incline. Because the motion is not accelerated, the resultant force parallel to the incline.

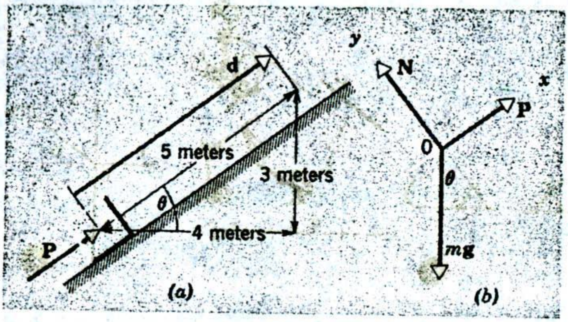


Fig. 7-3 Example 1. (*a*) A force **P** displaces a block a distance **d** up an inclined plane which makes an angle θ with the horizontal. (*b*) A free-body force diagram for the block.

the plane must be zero. Thus

$$P - mg \sin \theta = 0,$$

or

$$P = mg \sin \theta = (10.0 \text{ kg})(9.80 \text{ meters/sec}^2)(\tfrac{3}{5}) = 58.8 \text{ nt}.$$

Then the work done by **P**, from Eq. 7-1 with $\phi = 0°$, is

$$W = \mathbf{P} \cdot \mathbf{d} = Pd \cos 0° = Pd = (58.8 \text{ nt})(5.00 \text{ meters}) = 294 \text{ joules}.$$

If a man were to raise the block vertically without using the incline, the work he would do would be the vertical force mg times the vertical distance or

$$(98.0 \text{ nt})(3.00 \text{ meters}) = 294 \text{ joules},$$

the same as before. The only difference is that with the incline he could apply a smaller force ($P = 58.8$ nt) to raise the block than is required without the incline ($mg = 98.0$ nt); on the other hand, he had to push the block a greater distance (5.00 meters) up the incline than he had to raise the block directly (3.00 meters).

Example 2. A boy pulls a 10-lb sled 30 ft along a horizontal surface at a *constant speed.* What work does he do on the sled if the coefficient of kinetic friction is 0.20 and his pull makes an angle of 45° with the horizontal?

The situation is shown in Fig. 7-4*a* and the forces acting on the sled are shown in Fig. 7-4*b*. **P** is the boy's pull, **w** the sled's weight, **f** the frictional force, and **N** the normal force exerted by the surface on the sled. The work done by the boy on the sled is

$$W = \mathbf{P} \cdot \mathbf{d} = Pd \cos \phi.$$

To evaluate this we first must determine P, whose value has not been given. To obtain P we refer to the force diagram.

The sled is unaccelerated, so that from the second law of motion we obtain

$$P \cos \phi - f = 0,$$

and

$$P \sin \phi + N - w = 0.$$

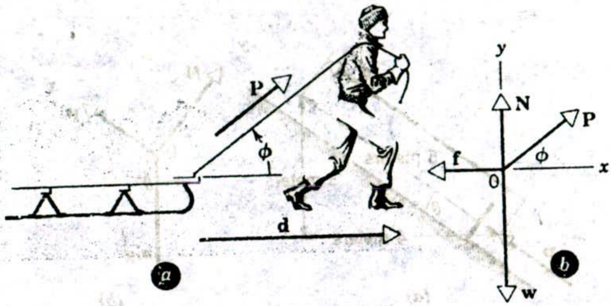


Fig. 7–4 Example 2. (a) A boy displaces a sled an amount **d** by pulling with a force **P** on a rope that makes an angle ϕ with the horizontal. (b) A free-body force diagram for the sled.

We know also that f and N are related by

$$f = \mu_k N.$$

These three equations contain three unknown quantities, P, f, and N. To find P we eliminate f and N from these equations and solve the remaining equation for P. The student should verify that

$$P = \mu_k w/(\cos\phi + \mu_k \sin\phi).$$

With $\mu_k = 0.20$, $w = 10$ lb, and $\phi = 45°$ we obtain

$$P = (0.20)(10 \text{ lb})/(0.707 + 0.141) = 2.4 \text{ lb}.$$

Then with $d = 30$ ft, the work done by the boy on the sled is

$$W = Pd\cos\phi = (2.4 \text{ lb})(30 \text{ ft})(0.707) = 51 \text{ ft-lb}.$$

The vertical component of the boy's pull **P** does no work on the sled. Notice, however, that it reduces the normal force between the sled and the surface ($N = w - P\sin\phi$) and thereby reduces the magnitude of the force of friction ($f = \mu_k N$).

Would the boy do more work, less work, or the same amount of work on the sled if he pulled horizontally instead of at 45° from the horizontal? Do any of the other forces acting on the sled do work on it? ◀

7–3 Work Done by a Variable Force—One Dimensional Case

Let us now consider the work done by a force that is not constant. We consider first a force that varies in magnitude only. Let the force be given as a function of position $F(x)$ and assume that the force acts in the x-direction. Suppose a body is moved along the x-direction by this force. What is the work done by this variable force in moving the body from x_1 to x_2?

In Fig. 7–5 we plot F versus x. Let us divide the total displacement into a large number of small equal intervals Δx (Fig. 7–5a). Consider the small displacement Δx from x_1 to $x_1 + \Delta x$. During this small displacement the force F has a nearly constant value and the work it does, ΔW, is approximately

$$\Delta W = F\,\Delta x, \qquad (7\text{-}3)$$

where F is the value of the force at x_1. Likewise, during the small displacement from $x_1 + \Delta x$ to $x_1 + 2\Delta x$, the force F has a nearly constant value and the work it does is approximately $\Delta W = F\,\Delta x$, where F is the value of the force at $x_1 + \Delta x$. The total work done by F in displacing the body from x_1 to x_2, W_{12}, is approximately the sum of a large number of terms like that of Eq. 7-3, in which F has a different value for each term. Hence

$$W_{12} = \sum_{x_1}^{x_2} F\,\Delta x, \qquad (7\text{-}4)$$

where the Greek letter sigma (Σ) stands for sum over all intervals from x_1 to x_2.

To make a better approximation we can divide the total displacement from x_1 to x_2 into a larger number of equal intervals, as in Fig. 7-5*b*, so that Δx is smaller and the value of F at the beginning of each interval is more typical of its values within the interval. It is clear that we can obtain better and better approximations by taking Δx smaller and smaller so as to have a larger and larger number of intervals. We can obtain an exact result for the work done by F if we let Δx go to zero and the number of intervals go to infinity. Hence the exact result is

$$W_{12} = \lim_{\Delta x \to 0} \sum_{x_1}^{x_2} F\,\Delta x. \qquad (7\text{-}5)$$

The relation

$$\lim_{\Delta x \to 0} \sum_{x_1}^{x_2} F\,\Delta x = \int_{x_1}^{x_2} F\,dx,$$

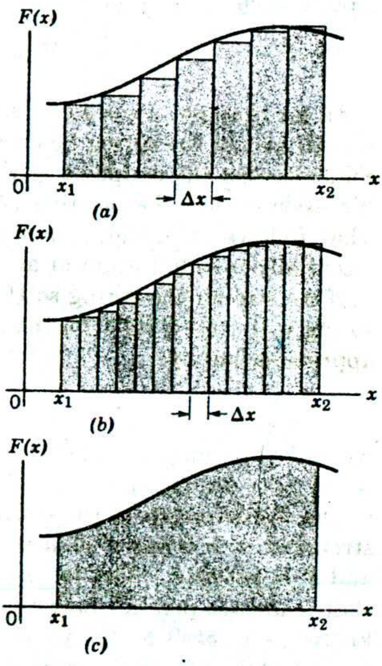


Fig. 7-5 Computing $\int_{x_1}^{x_2} F(x)\,dx$ amounts to finding the area under the curve $F(x)$ between the limits x_1 and x_2. This can be done approximately as in the top drawing (*a*) by dividing the area into a few strips, each of width Δx. The areas of the rectangles are then summed to give a rough value of the area. In the middle drawing (*b*) the strips are narrower and the value for the area becomes more exact as the errors at the tops of the rectangles become smaller. In the bottom drawing (*c*) the strips are only infinitesimal in width. The measurement of area is exact, since the errors at the tops of the rectangles go to zero as the strip width dx goes to zero.

as the student may have learned in his calculus course, *defines the integral* of F with respect to x from x_1 to x_2. Numerically, this quantity is exactly equal to the area between the force curve and the x-axis between the limits x_1 and x_2 (Fig. 7–5c). Hence, graphically an integral can be interpreted as an area. The symbol $\int$ is a distorted S (for *sum*) and symbolizes the integration process. We can write the total work done by F in displacing a body from x_1 to x_2 as

$$W_{12} = \int_{x_1}^{x_2} F(x)\,dx. \tag{7–6}$$

As an example, consider a spring attached to a wall. Let the (horizontal) axis of the spring be chosen as an x-axis, and let the origin, $x = 0$, coincide with the endpoint of the spring in its normal, unstretched state. We assume that the positive x-direction points away from the wall. In what follows we imagine that we stretch the spring so slowly that it is essentially in equilibrium at all times ($\mathbf{a} = 0$).

If we stretch the spring so that its endpoint moves to a position x, the spring will exert a force on the agent doing the stretching given to a good approximation by

$$F = -kx, \tag{7–7}$$

where k is a constant called the *force constant* of the spring. Equation 7–7 is the *force law* for springs. The direction of the force is always opposite to the displacement of the endpoint from the origin. When the spring is stretched, $x > 0$ and F is negative; when the spring is compressed, $x < 0$ and F is positive. The force exerted by the spring is a *restoring force* in that it always points toward the origin. Real springs will obey Eq. 7–7, known as *Hooke's law*, if we do not stretch them beyond a limited range. We can think of k as the magnitude of the force per unit elongation. Thus very stiff springs have large values of k.

To stretch a spring we must exert a force F' on it equal but opposite to the force F exerted by the spring on us. The applied force* is therefore $F' = kx$ and the work done by the applied force in stretching the spring so that its endpoint moves from x_1 to x_2 is†

$$W_{12} = \int_{x_1}^{x_2} F'(x)\,dx = \int_{x_1}^{x_2} (kx)\,dx = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2.$$

---

* If the applied force were different from $F' = kx$, we would have a net unbalanced force acting on the spring and its motion would be accelerated. To compute the work done we would have to specify exactly what the applied force is at each point. No matter what the force turned out to be, the work done would always be the same for the same displacement x_1 to x_2, providing the spring has the same speed initially and finally. However, it is much easier to use the simple force $F' = kx$ in calculating the work done. Such an applied force leads to unaccelerated motion. It is in order to be able to use this simple force that we specified unaccelerated motion in the first place.

† The student just becoming familiar with calculus should consult the list of integrals in Appendix I.

If we let $x_1 = 0$ and $x_2 = x$, we obtain

$$W = \int_0^x (kx)\, dx = \tfrac{1}{2}kx^2. \tag{7-8}$$

This is the work done in stretching a spring so that its endpoint moves from its unstretched position to x. Note that the work to *compress* a spring by x is the same as that to stretch it by x because the displacement x is squared in Eq. 7-8; either sign for x gives a positive value for W.

We can also evaluate this integral by computing the area under the force-displacement curve and the x-axis from $x = 0$ to $x = x$. This is drawn as the white area in Fig. 7-6. The area is a triangle of base x and altitude kx. The white area is therefore

$$\tfrac{1}{2}(x)(kx) = \tfrac{1}{2}kx^2,$$

in agreement with Eq. 7-8.

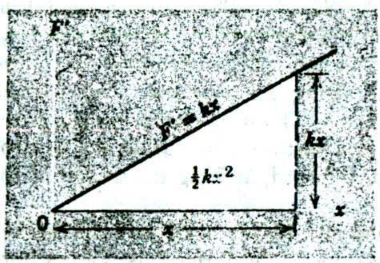


Fig. 7-6 The force exerted in stretching a spring is $F' = kx$. The area under the force curve is the work done in stretching the spring a distance x and can be found by integrating or by using the formula for the area of a triangle.

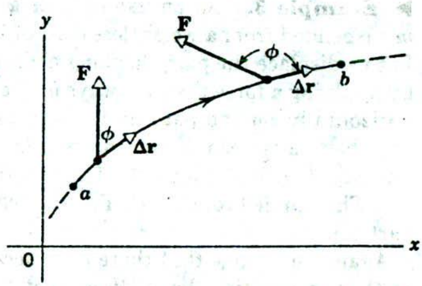


Fig. 7-7 How **F** and ϕ might change along a path. As $\Delta\mathbf{r} \to 0$ we may replace it by the differential $d\mathbf{r}$, which always points in the direction of the velocity of the moving object, since $\mathbf{v} = d\mathbf{r}/dt$, and hence is tangent to the path at all points.

7-4 Work Done by a Variable Force—Two-Dimensional Case

The force **F** acting on a particle may vary in direction as well as in magnitude, and the particle may move along a curved path. To compute the work in this general case we divide the path up into a large number of small displacements $\Delta\mathbf{r}$, each pointing along the path in the direction of motion. Figure 7-7 shows two selected displacements for a particular situation; it also shows the value of **F** and the angle ϕ between **F** and $\Delta\mathbf{r}$ at each location. We can find the amount of work done on the particle during a displacement $\Delta\mathbf{r}$ from

$$dW = \mathbf{F} \cdot \Delta\mathbf{r} = F \cos\phi\, \Delta r \tag{7-9}$$

The work done by the variable force **F** on the particle as the particle moves, say, from a to b in Fig. 7-7 is found very closely by adding up (summing) the elements of work done over each of the line segments that make it up. As the line segments $\Delta\mathbf{r}$ become smaller they may be replaced by differentials $d\mathbf{r}$ and the sum over the line segments may be replaced by an integral, as in Eq. 7–6. The work is then found from

$$W_{ab} = \int_a^b \mathbf{F} \cdot d\mathbf{r} = \int_a^b F \cos \phi \, dr. \tag{7–10a}$$

We cannot evaluate this integral until we are able to say how F and ϕ in Eq. 7–10a vary from point to point along the path; both are functions of the x- and y-coordinates of the particle in Fig. 7–7.

We can obtain another equivalent expression for Eq. 7–10a by expressing $\mathbf{F}$ and $d\mathbf{r}$ in terms of their components. Thus $\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y$ and $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy$, so that $\mathbf{F} \cdot d\mathbf{r} = F_x\,dx + F_y\,dy$. In this evaluation recall (see Problem 22, Chapter 2) that $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$. Substituting this result into Eq. 7–10a, we obtain

$$W_{ab} = \int_a^b (F_x\,dx + F_y\,dy) \tag{7–10b}$$

Integrals such as those in Eqs. 7-10a and 7-10b are called *line integrals.*

▶ **Example 3.** As an example of a variable force consider a particle of mass m suspended from a weightless cord of length l. This is called a simple pendulum. Let us displace the particle along a circular path of radius l from $\phi = 0$ to $\phi = \phi_0$ by applying a force that is always *horizontal*. We can apply such a force by pulling horizontally on the particle with an attached string, for example. The particle will then have been displaced a vertical distance h. Figure 7–8a shows the situation and Fig. 7–8b shows the forces acting on the particle in the arbitrary position ϕ. The applied force is $\mathbf{F}$, $\mathbf{T}$ is the tension in the cord, and $m\mathbf{g}$ the weight of the particle.

Again we assume that there is no acceleration (the reason is the same as before), so that in practice the motion must be very slow. The force $\mathbf{F}$ is always horizontal, but the displacement $d\mathbf{r}$ is along the arc. The direction of $d\mathbf{r}$ depends on the value of ϕ and is tangent to the circle at each point. $\mathbf{F}$ will vary in magnitude in such a way as to balance the horizontal component of the tension. Notice that the angle between $\mathbf{F}$ and $d\mathbf{r}$ is equal to the angular displacement ϕ in this case.

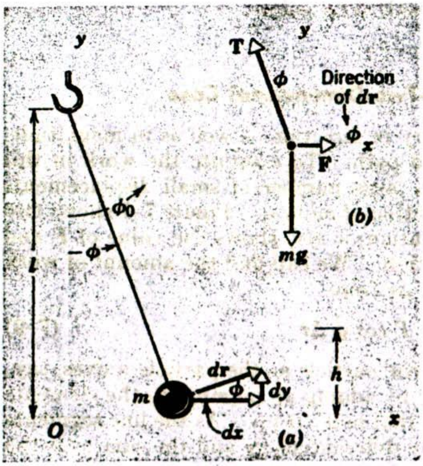


Fig. 7–8 (*a*) A simple pendulum. A mass point m is suspended on a string of length l. Its maximum displacement is ϕ_0. (*b*) A free-body force diagram for the mass subjected to an applied horizontal force.

The work done as the mass m moves from $\phi = 0$ to $\phi = \phi_0$ under the action of the force $\mathbf{F}$ is

$$W = \int_{\phi=0}^{\phi=\phi_0} \mathbf{F} \cdot d\mathbf{r} = \int_{\phi=0}^{\phi=\phi_0} F \cos \phi \, dr \qquad (7\text{–}10a)$$

or

$$W = \int_{x=0,y=0}^{x=(l-h)\tan\phi_0, y=h} (F_x \, dx + F_y \, dy). \qquad (7\text{–}10b)$$

Let us evaluate Eq. 7–10*b*.

Note that, from Newton's first law (see Fig. 7–8*b*)

$$F_x = T \sin \phi \qquad \text{and} \qquad mg = T \cos \phi.$$

Eliminating T between these relations gives us

$$F_x = mg \tan \phi.$$

We also note in Fig. 7–8*b* that $F_y = 0$. Substituting these values for F_x and F_y into Eq. 7–10*b* yields

$$W = \int_{x=0,y=0}^{x=(l-h)\tan\phi_0, y=h} mg \tan \phi \, dx.$$

Now from Fig. 7–8*a* we see that

$$\tan \phi = dy/dx \qquad \text{or} \qquad \tan \phi \, dx = dy.$$

Making this substitution and noting that the integral depends only on the variable y, we obtain finally

$$W = \int_{y=0}^{y=h} (mg) \, dy = mg \int_0^h dy = mgh.$$

The student should now try to compute the work done in displacing the particle along the arc with constant speed by applying a force that is always directed along the arc. Here it will be simpler to work with Eq. 7–10*a*, using the tangential force and taking $dr = l\,d\phi$. The result will be the same as before, $W = mgh$. Notice that both these results are the same as the work that would be done in raising a mass m vertically through a height h.

What work has been done on the particle by the tension T in the string? ◀

7–5 Kinetic Energy and the Work-Energy Theorem

In our previous examples of work done by forces, we dealt with *unaccelerated* objects. In such cases the *resultant force* acting on the object is zero. Let us suppose now that the *resultant force* acting on an object is *not zero*, so that the object is *accelerated*. The conditions are the same in all respects to those that exist when a single unbalanced force acts on the object.

The simplest situation to consider is that of a *constant resultant force* $\mathbf{F}$. Such a force, acting on a particle of mass m, will produce a constant acceleration $\mathbf{a}$. Let us choose the x-axis to be in the common direction of $\mathbf{F}$ and $\mathbf{a}$. What is the work done by this force on the particle in causing a displacement x? We have (for constant acceleration) the relations

$$a = \frac{v - v_0}{t}$$

and

$$x = \frac{v + v_0}{2} \cdot t,$$

which are Eqs. 3–12 and 3–14 respectively (in which we have dropped the subscript x, for convenience, and chosen $x_0 = 0$ in the last equation). Here v_0 is the particle's speed at $t = 0$ and v its speed at the time t. Then the work done is

$$W = Fx = max$$

$$= m\left(\frac{v - v_0}{t}\right)\left(\frac{v + v_0}{2}\right)t = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2. \qquad (7\text{–}11)$$

We call one-half the product of the mass of a body and the square of its speed the kinetic energy of the body. If we represent kinetic energy by the symbol K, then

$$K = \tfrac{1}{2}mv^2. \qquad (7\text{–}12)$$

We may then state Eq. 7–11 in this way: *The work done by the* resultant *force acting on a particle is equal to the change in the kinetic energy of the particle.*

Although we have proved this result for a constant force only, it holds whether the resultant force is constant or variable. Let the resultant force vary in magnitude (but not in direction), for example. Take the displacement to be in the direction of the force. Let this direction be the x-axis. The work done by the resultant force in displacing the particle from x_0 to x is

$$W = \int \mathbf{F} \cdot d\mathbf{r} = \int_{x_0}^{x} F\,dx.$$

But from Newton's second law we have $F = ma$, and the acceleration a can be written as

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx}v = v\frac{dv}{dx}.$$

Hence

$$W = \int_{x_0}^{x} F\,dx = \int_{x_0}^{x} mv\frac{dv}{dx}\,dx = \int_{v_0}^{v} mv\,dv = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2. \qquad (7\text{–}13)$$

A more general case is that in which the force varies both in direction and magnitude and the motion is along a curved path, as in Fig. 7–7. (See Problem 7.) Once again we find that the work done on a particle by the resultant force is equal to the change in the kinetic energy of the particle.

The work done *on* a particle by the *resultant* force is *always* equal to the change in the kinetic energy of the particle:

$$W \text{ (of the } \textit{resultant} \text{ force)} = K - K_0 = \Delta K. \qquad (7\text{–}14)$$

Equation 7–14 is known as the *work-energy theorem* for a particle.

Notice that when the speed of the particle is constant, there is no change in kinetic energy and the work done by the resultant force is zero. With uniform circular motion, for example, the speed of the particle is constant and the centripetal force does no work on the particle. A force at right angles to the direction of motion merely changes the *direction* of the velocity and not its magnitude. Only when the *resultant* force has a component along the direction of motion does it change the speed of the particle or its kinetic energy. Work is done on a particle only by that component of the resultant force along the line of motion. This agrees with our definition of work in terms of a scalar product, for in $\mathbf{F} \cdot d\mathbf{r}$ only the component of $\mathbf{F}$ along $d\mathbf{r}$ contributes to the product.

If the kinetic energy of a particle decreases, the work done on it by the resultant force is negative. The displacement and the component of the resultant force along the line of motion are oppositely directed. The work done *on* the particle by the force is the negative of the work done *by* the particle on whatever produced the force. This is a consequence of Newton's third law of motion. Hence Eq. 7–14 can be interpreted to say that the kinetic energy of a particle *decreases* by an amount just equal to the amount of work which the particle *does*. A body is said to have energy stored in it because of its motion; as it does work it slows down and loses some of this energy. Therefore, *the kinetic energy of a body in motion is equal to the work it can do in being brought to rest.* This result holds whether the applied forces are constant or variable.

The units of kinetic energy and of work are the same. Kinetic energy, like work, is a scalar quantity. The kinetic energy of a group of particles is simply the (scalar) sum of the kinetic energies of the individual particles in the group.

▶ **Example 4.** A neutron, one of the constituents of a nucleus, is found to pass two points 6.0 meters apart in a time interval of 1.8×10^{-4} sec. Assuming its speed was constant, find its kinetic energy. The mass of a neutron is 1.7×10^{-27} kg.

The speed is obtained from

$$v = \frac{d}{t} = \frac{6.0 \text{ meters}}{1.8 \times 10^{-4} \text{ sec}} = 3.3 \times 10^4 \text{ meters/sec}.$$

The kinetic energy is

$$K = \tfrac{1}{2}mv^2 = (\tfrac{1}{2})(1.7 \times 10^{-27} \text{ kg})(3.3 \times 10^4 \text{ meters/sec})^2 = 9.3 \times 10^{-19} \text{ joule}.$$

For purposes of nuclear physics the joule is a very large energy unit. A unit more commonly used is the electron volt (ev), which is equal to 1.60×10^{-19} joule. The kinetic energy of the neutron in our example can then be expressed as

$$K = (9.3 \times 10^{-19} \text{ joule}) \left(\frac{1 \text{ ev}}{1.60 \times 10^{-19} \text{ joule}} \right) = 5.8 \text{ ev}.$$

Example 5. Assume the force of gravity to be constant for small distances above the surface of the earth. A body is dropped from rest at a height h above the earth's surface. What will its kinetic energy be just before it strikes the ground?

The gain in kinetic energy is equal to the work done by the resultant force, which here is the force of gravity. This force is constant and directed along the line of motion, so that the work done by gravity is

$$W = \mathbf{F} \cdot \mathbf{d} = mgh.$$

Initially the body has a speed $v_0 = 0$ and finally a speed v. The gain in kinetic energy of the body is

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 - 0.$$

Equating these two equivalent terms we obtain

$$K = \tfrac{1}{2}mv^2 = mgh$$

as the kinetic energy of the body just before it strikes the ground.

The speed of the body is then

$$v = \sqrt{2gh}.$$

The student should show that in falling from a height h_1 to a height h_2 a body will increase its kinetic energy from $\frac{1}{2}mv_1^2$ to $\frac{1}{2}mv_2^2$, where

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = mg(h_1 - h_2).$$

In this example we are dealing with a constant force and a constant acceleration. The methods developed in previous chapters should be useful here too. Can you show how the results obtained by energy considerations could be obtained directly from the laws of motion for uniformly accelerated bodies?

Example 6. A block weighing 8.0 lb slides on a horizontal frictionless table with a speed of 4.0 ft/sec. It is brought to rest in compressing a spring in its path. By how much is the spring compressed if its force constant is 0.25 lb/ft?

The kinetic energy of the block is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(w/g)v^2.$$

This kinetic energy is equal to the work W that the block can do before it is brought to rest. The work done in compressing the spring a distance x beyond its unstretched length is

$$W = \tfrac{1}{2}kx^2,$$

so that

$$\tfrac{1}{2}kx^2 = \tfrac{1}{2}(w/g)v^2$$

or

$$x = \sqrt{\frac{w}{gk}}\,v = \sqrt{\frac{8.0}{(32)(0.25)}}\,4.0 \text{ ft} = 4.0 \text{ ft}. \qquad \blacktriangleleft$$

7–6 Significance of the Work-Energy Theorem

The work-energy theorem does *not* represent a new, independent law of classical mechanics. We have simply *defined* work and kinetic energy and *derived* the relation between them directly from Newton's second law. The work-energy theorem is useful, however, for solving problems in which the work done by the resultant force is easily computed and in which we are interested in finding the particle's speed at certain positions. Of greater significance, perhaps, is the fact that the work-energy theorem is the starting point for a sweeping generalization in physics. It has been emphasized that the work-energy theorem is valid when W is interpreted as the work done by the *resultant* force acting on the particle. However, it is helpful in many problems to compute separately the work done by certain types of force and give special names to the work done by each type. This leads to the concepts of different types of energy and the principle of the conservation of energy, which is the subject of the next chapter.

7-7 Power

Let us now consider the time involved in doing work. The same amount of work is done in raising a given body through a given height whether it takes one second or one year to do so. However, the *rate at which work is done* is often more interesting to us than the total work performed.

We define *power* as the time rate at which work is done. The average power delivered by an agent is the total work done by the agent divided by the total time interval, or

$$\bar{P} = W/t.$$

The instantaneous power delivered by an agent is

$$P = dW/dt. \tag{7-15}$$

If the power is constant in time, then $P = \bar{P}$ and

$$W = Pt.$$

In the mks system the unit of power is 1 joule/sec, which is called 1 *watt*. This unit of power is named in honor of James Watt whose steam engine is the predecessor of today's more powerful engines. In the British engineering system, the unit of power is 1 ft-lb/sec. Because this unit is quite small for practical purposes, a larger unit, called the *horsepower*, has been adopted. Actually Watt himself suggested as a unit of power the power delivered by a horse as an engine. One horsepower was chosen to equal 550 ft-lb/sec. One horsepower is equal to about 746 watts or about three-fourths of a kilowatt. A horse would not last very long at that rate.

Work can also be expressed in units of power × time. This is the origin of the term *kilowatt-hour*, for example. One kilowatt-hour is the work done in 1 hr by an agent working at a constant rate of 1 kw.

▶ **Example 7.** An automobile uses 100 hp and moves at a uniform speed of 60 miles/hr (= 88 ft/sec). What is the forward thrust exerted by the engine on the car?

$$P = \frac{W}{t} = \frac{\mathbf{F} \cdot \mathbf{d}}{t} = \mathbf{F} \cdot \mathbf{v}.$$

The forward thrust **F** is in the direction of motion given by **v**, so that

$$P = Fv,$$

and

$$F = \frac{P}{v} = \left(\frac{100 \text{ hp}}{88 \text{ ft/sec}}\right)\left(\frac{550 \text{ ft-lb/sec}}{1 \text{ hp}}\right) = 630 \text{ lb}.$$

Why doesn't the car accelerate? ◀

QUESTIONS

1. Can you think of other words like "work" whose colloquial meanings are often different from their scientific meanings?

2. In a tug of war one team is slowly giving way to the other. What work is being done and by whom?

3. The inclined plane (see Example 1) is a simple machine which enables us to do work with the application of a smaller force than is otherwise necessary. The same statement applies to a wedge, a lever, a screw, a gear wheel, and a pulley. Do such machines save us work?

4. Springs A and B are identical except that A is stiffer than B, that is, $k_A > k_B$. On which spring is more work expended if (*a*) they are stretched by the same amount, (*b*) they are stretched by the same force?

5. A man rowing a boat upstream is at rest with respect to the shore. (*a*) Is he doing any work? (*b*) If he stops rowing and moves down with the stream, is any work being done on him?

6. The work done by the resultant force is always equal to the change in kinetic energy. Can it happen that the work done by one of the component forces alone will be greater than the change in kinetic energy? If so, give examples.

7. When two children play catch on a train, does the kinetic energy of the ball depend on the speed of the train? Does the reference frame chosen affect your answer? If so, would you call kinetic energy a scalar quantity? (See Problem 19.)

8. Does the work done in raising a box onto a platform depend on how fast it is raised?

PROBLEMS

1. A 100-lb block of ice slides down an incline 5.0 ft long and 3.0 ft high. A man pushes up on the ice parallel to the incline so that it slides down at constant speed. The coefficient of friction between the ice and the incline is 0.10. Find (*a*) the force exerted by the man, (*b*) the work done by the man on the block, (*c*) the work done by gravity on the block, (*d*) the work done by the surface of the incline on the block, (*e*) the work done by the resultant force on the block, and (*f*) the change in kinetic energy of the block.

2. A man pushes a 60-lb block 30 ft along a level floor at constant speed with a force directed 45° below the horizontal. If the coefficient of kinetic friction is 0.20, how much work does the man do on the block?

3. A crate weighing 500 lb is suspended from the end of a rope 40 ft long. The crate is then pushed aside 4.0 ft from the vertical and held there. (*a*) What is the force needed to keep the crate in this position? (*b*) Is work being done in holding it there? (*c*) Was work done in moving it aside? If so, how much? (*d*) Does the tension in the rope perform any work on the crate?

4. A cord is used to lower vertically a block of mass M a distance d at a constant downward acceleration of $g/4$. Find the work done by the cord on the block.

5. A block of mass $m = 3.57$ kg is drawn at constant speed a distance $d = 4.06$ meters along a horizontal floor by a rope exerting a constant force of magnitude $F = 7.68$ nt making an angle $\theta = 15.0°$ with the horizontal. Compute (*a*) the total work done on the block; (*b*) the work done by the rope on the block; (*c*) the work done by friction on the block; (*d*) the coefficient of kinetic friction between block and floor.

6. (*a*) Estimate the work done by the force shown on the graph (Fig. 7–9) in displacing a particle from $x = 1$ to $x = 3$ meters. Refine your method to see how close you can come to the exact answer of 6 joules. (*b*) The curve is given analytically by $F = a/x^2$ where $a = 9$ nt-meters^2. Show how to get the work done by the rules of integration.

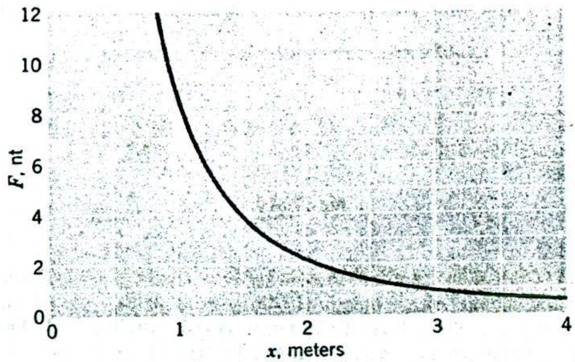


Fig. 7–9

7. When the force **F** varies both in direction and magnitude and the motion is along a curved path the work done by **F** is obtained from $dW = \mathbf{F} \cdot d\mathbf{r}$, the subsequent integration being taken along the curved path. Notice that both F and ϕ, the angle between **F** and $d\mathbf{r}$, may vary from point to point (see Fig. 7–7). Using Example 3 as a guide, show that for two-dimensional motion

$$W = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2,$$

where v is the final speed and v_0 the initial speed.

8. Generalize the results of the previous problem to three dimensions.

9. A running man has half the kinetic energy that a boy of half his mass has. The man speeds up by 1.0 meter/sec and then has the same kinetic energy as the boy. What were the original speeds of man and boy?

10. From what height would an automobile have to fall to gain the kinetic energy equivalent to what it would have when going 60 miles/hr?

11. A proton (nucleus of the hydrogen atom) is being accelerated in a linear accelerator. In each stage of such an accelerator the proton is accelerated along a straight line by 3.6×10^{15} meters/sec^2. If a proton enters such a stage moving initially with a speed of 2.4×10^7 meters/sec and the stage is 3.5 cm long, compute (*a*) its speed at the end of the stage and (*b*) the gain in kinetic energy resulting from the acceleration. Take the mass of the proton to be 1.67×10^{-27} kg and express the energy in electron volts.

12. A 30-gm bullet initially traveling 500 meters/sec penetrates 12 cm into a wooden block. What average force does it exert?

13. Show *from considerations of work and kinetic energy* that the minimum stopping distance for a car of mass m moving with speed v along a level road is $v^2/2\mu_s g$, where μ_s is the coefficient of static friction between tires and road. (See Example 2, Chapter 6.)

14. A single force acts on a body in rectilinear motion. A plot of velocity versus time for the body is shown in Fig. 7–10. Find the sign (positive or negative) of the work done *by* the force *on* the body in each of the intervals AB, BC, CD, and DE.

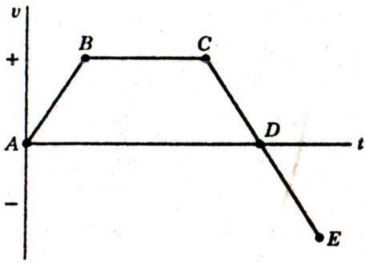


Fig. 7–10

15. (*a*) A mass of 0.675 kg on a frictionless table is attached to a string which passes through a hole in the table at the center of the horizontal circle in which the mass moves with constant speed. If the radius of the circle is 0.500 meter and the speed is 10.0 meters/sec, compute the tension in the string. (*b*) It is found that drawing an additional 0.200 meter of the string down through the hole, thereby reducing the radius of the circle to 0.300 meter, has the effect of multiplying the original tension in the string by 4.63. Compute the total work done by the string on the revolving mass during the reduction of the radius.

16. A proton starting from rest is accelerated in a cyclotron to a final speed of 3.0×10^7 meters/sec (about one-tenth the speed of light). How much work, in electron volts, is done on the proton by the electrical force of the cyclotron which accelerates it?

17. An outfielder throws a baseball with an initial speed of 60 ft/sec. An infielder at the same level catches the ball when its speed is reduced to 40 ft/sec. What work was done in overcoming the resistance of the air? The weight of a baseball is 9.0 oz.

18. The block of mass M shown in Fig. 7–11 initially has a velocity v_0 to the right and its position is such that the spring exerts no force on it, i.e., the spring is not stretched or compressed. The block moves to the right a distance l before stopping in the dotted position shown. The spring constant is k and the coefficient of kinetic friction between block and table is μ_k. As the block moves the distance l, (*a*) what is the work done on it by the friction force? (*b*) What is the work done on it by the spring force? (*c*) Are there other forces acting on the block, and, if so, what work do they do? (*d*) What is the total work done on the block? (*e*) Use the work-energy theorem to find the value of l in terms of M, v_0, μ_k, g, and k.

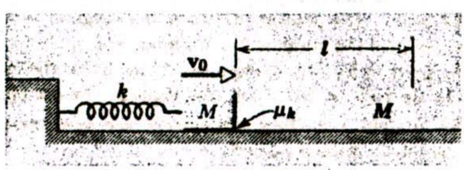


Fig. 7–11

19. *Work and Kinetic Energy In Moving Reference Frames.* Consider two observers, one whose frame is attached to the ground and another whose frame is attached, say,

to a train moving with uniform velocity u with respect to the ground. Each observes that a particle, initially at rest with respect to the train, is accelerated by a constant force applied to it for time t in the forward direction.

(*a*) Show that for each observer the work done by the force is equal to the gain in kinetic energy of the particle, but that one observer measures these quantities to be $\frac{1}{2}ma^2t^2$, whereas the other observer measures them to be $\frac{1}{2}ma^2t^2 + maut$. Here a is the common acceleration of the particle of mass m.

(*b*) Explain the differences in work done by the same force in terms of the different distances through which the observers measure the force to act during the time t. Explain the different final kinetic energies measured by each observer in terms of the work the particle could do in being brought to rest relative to each observer's frame.

20. A net force of 5.0 nt acts on a 15-kg body initially at rest. Compute the work done by the force in the first, second, and third second and the instantaneous power exerted by the force at the end of the third second.

21. A satellite rocket weighing 100,000 lb acquires a speed of 4000 miles/hr in 1.0 min after launching. (*a*) What is its kinetic energy at the end of the first minute? (*b*) What is the average power expended during this time, neglecting frictional and gravitational forces?

22. A truck can move up a road having a grade of 1.0-ft rise every 50 ft with a speed of 15 miles/hr. The resisting force is equal to one-twenty-fifth the weight of the truck. How fast will the same truck move down the hill with the same horsepower?

23. A horse pulls a wagon with a force of 40 lb at an angle of 30° with the horizontal and moves along at a speed of 6.0 miles/hr. (*a*) How much work does the horse do in 10 min? (*b*) What is the power output of the horse?

24. The force required to tow a boat at constant velocity is proportional to the velocity. If it takes 10 hp to tow a certain boat at a speed of 2.5 miles/hr, how much horsepower does it take to tow it at a speed of 7.5 miles/hr?

25. What power is developed by a grinding machine whose wheel has a radius of 8.0 in and runs at 2.5 rev/sec when the tool to be sharpened is held against the wheel with a force of 40 lb? The coefficient of friction between the tool and the wheel is 0.32.

26. A boy whose mass is 51.0 kg climbs, with constant speed, a vertical rope 6.00 meters long in 10.0 sec. (*a*) How much work does the boy perform? (*b*) What is the boy's power output during the climb?

27. A body of mass m accelerates uniformly from rest to a speed v_f in time t_f. (*a*) Show that the work done on the body as a function of time t, in terms of v_f and t_f, is

$$\frac{1}{2} m \frac{v_f^2}{t_f^2} t^2.$$

(*b*) As a *function of time t*, what is the instantaneous power delivered to the body? (*c*) What is the instantaneous power at the end of 10 sec delivered to a 3200-lb body which accelerates to 60 miles/hr in 10 sec?

The Conservation of Energy

CHAPTER 8

8–1 Introduction

In Chapter 7 we derived the *work-energy theorem* from Newton's second law of motion. This theorem says that the work W done by the resultant force $\mathbf{F}$ acting on a particle as it moves from one point to another is equal to the change ΔK in the kinetic energy of the particle, or

$$W = \Delta K. \tag{8–1}$$

Often several forces act on a particle, the resultant force $\mathbf{F}$ being their vector sum, that is, $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots \mathbf{F}_n$, in which we assume that n forces act. The work done by the resultant force $\mathbf{F}$ is the algebraic sum of the work done by these individual forces, or $W = W_1 + W_2 + \cdots W_n$. Thus we can write the work-energy theorem (Eq. 8–1) as

$$W_1 + W_2 + \cdots + W_n = \Delta K. \tag{8–2}$$

In this chapter we shall consider systems in which a single particle is acted upon by various kinds of forces and we shall compute W_1, W_2, etc., for these forces; this will lead us to define different kinds of energy such as potential energy and heat energy. The process culminates in the formulation of one of the great principles of science, the *conservation of energy principle*.

8–2 Conservative Forces

Let us first distinguish between two types of forces, *conservative* and *nonconservative*. We shall consider an example of each type and we discuss each example from several different, but related, points of view.

Imagine a spring fastened at one end to a rigid wall as in Fig. 8-1. Let us slide a block of mass m with velocity $\mathbf{v}$ directly toward the spring; we assume that the horizontal plane is frictionless and that the spring is ideal, that is, that it obeys Hooke's law (Eq. 7-7)

$$F = -kx, \tag{8-3}$$

where F is the force exerted by the spring when its free end is displaced through a distance x; we assume further that the mass of the spring is so small compared to that of the block that we can neglect the kinetic energy of the spring. Thus, in the system (mass + spring), all the kinetic energy is concentrated in the mass.

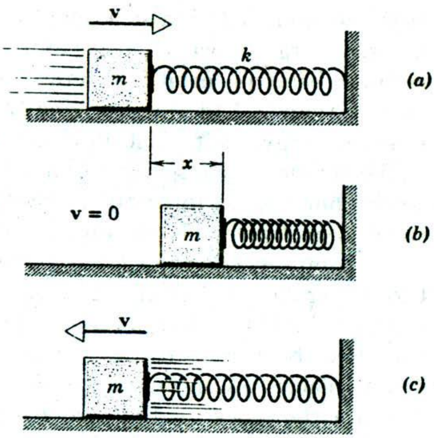


Fig. 8-1 (*a*) A block of mass m is projected with speed v against a spring. (*b*) The block is brought to rest by the action of the spring force. (*c*) The block has regained its initial speed v as it returns to its starting point.

After the block touches the spring, the speed and hence the kinetic energy of the block decrease until finally the block is brought to rest by the action of the spring force, as in Fig. 8-1*b*. The block now reverses its motion as the compressed spring expands. It gains speed and kinetic energy and, when it comes once again to its position of initial contact with the spring, we find that it has the same speed and kinetic energy as it had originally; only the direction of motion has changed. The block loses kinetic energy during one part of its motion but gains it all back during the other part of its motion as it returns to its starting point (Fig. 8-1*c*).

We have interpreted the kinetic energy of a body as its ability to do work by virtue of its motion. It is clear that at the completion of a round trip the ability of the block in Fig. 8-1 to do work remains the same; it has been *conserved*. The elastic force exerted by an ideal spring, and other forces that act in this same way, are called *conservative*. The force of gravity is also conservative; if we throw a ball vertically upward, it will (if we assume air resistance to be negligible) return to our hand with the same kinetic energy that it had when it left our hand.

If, however, a particle on which one or more forces act returns to its initial position with either more or less kinetic energy than it had initially, then in a round trip its ability to do work has been changed. In this case the ability to do work has *not* been conserved and at least one of the forces acting is labeled *nonconservative*.

To illustrate a nonconservative force let us assume that the surfaces of the block and the plane in Fig. 8–1 are not frictionless but rather that a force of friction **f** is exerted by the plane on the block. The frictional force opposes the motion of the block no matter which way the block is moving and we find that the block returns to its starting point with *less* kinetic energy than it had initially. Since we showed in our first experiment that the spring force was conservative, we must attribute this new result to the action of the friction force.* We say that this force, and other forces that act in this same way, are *nonconservative*. The induction force in a betatron (Section 35–6) is also a nonconservative force. Instead of dissipating kinetic energy, however, it generates it, so that an electron moving in the circular betatron orbit will return to its initial position with *more* kinetic energy than it had there originally. In a round trip the electron gains kinetic energy, as it must do if the betatron is to be effective.

We can define conservative force from another point of view, that of the work done by the force on the particle. In our first example above, the work done by the elastic spring force on the block while the spring was being compressed was negative, because the force exerted on the block by the spring (to the left in Fig. 8–1a) was directed opposite to the displacement of the block (to the right in Fig. 8–1a). While the spring was being extended the work that the spring force did on the block was positive (force and displacement in the same direction). In our first example the net work done on the block by the spring force during a complete cycle, or round trip, is zero.

In our second example we considered the effect of the frictional force. The work done on the block by this force was negative for each portion of the cycle because the frictional force always opposed the motion. Hence the work done by friction in a round trip cannot be zero. In general, then: *A force is conservative if the work done by the force on a particle that moves through any round trip is zero. A force is nonconservative if the work done by the force on a particle that moves through any round trip is not zero.*

The work-energy theorem shows that this second way of defining conservative and nonconservative forces is fully equivalent to our first definition. If there is no change in the kinetic energy of a particle moving through any round trip then $\Delta K = 0$ and, from Eq. 8–1, $W = 0$ and the resultant force acting must be conservative. Similarly, if $\Delta K \neq 0$ then, from Eq. 8–1, $W \neq 0$ and at least one of the forces acting must be nonconservative.

We can look into this matter in a little more detail. When friction is present in the system of Fig. 8–1, four forces act on the block, the resultant force being

$$\mathbf{F} = \mathbf{F}_s + \mathbf{W} + \mathbf{N} + \mathbf{f}$$

in which the forces are the spring force $\mathbf{F}_s$, the weight of the block **W**, the normal

* Actually two other forces act on the block in Fig. 8–1, its weight **W** and the normal force **N** exerted by the plane. Since these act at right angles to the motion, they cannot change the kinetic energy of the block and hence do not enter into this discussion.

force exerted on the block by the plane **N**, and the frictional force **f**. We can write Eq. 8-2, the work-energy theorem, as

$$W_s + W_W + W_N + W_f = \Delta K,$$

where the terms on the left are the work done on the block by the four forces above. We have seen that for a round trip $W_s = 0$. Similarly, $W_W = W_N = 0$ because the corresponding forces are at right angles to the displacement of the block. Thus the change in kinetic energy is due entirely to W_f, the work done by the frictional force.

We can consider the difference between conservative and nonconservative forces in still a third way. Suppose a particle goes from a to b along path 1 and back from b to a along path 2 as in Fig. 8-2a. Several forces may act on the particle during this round trip; we consider each force separately. If the force being considered is conservative, the work done on the particle by that particular force for the round trip is zero, or

$$W_{ab,1} + W_{ba,2} = 0,$$

which we can write as

$$W_{ab,1} = -W_{ba,2}.$$

(a) (b)

Fig. 8-2

That is, the work in going from a to b along path 1 is the negative of the work in going from b to a along path 2. However, if we cause the particle to go from a to b along path 2, as shown in Fig. 8-2b, we merely reverse the direction of the previous motion along 2, so that

$$W_{ab,2} = -W_{ba,2}.$$

Hence

$$W_{ab,1} = W_{ab,2},$$

which tells us that the work done on the particle by a conservative force in going from a to b is the same for either path.

Paths 1 and 2 can be any paths at all as long as they go from a to b; and a and b can be chosen to be any two points at all. We always find the same result if the force is conservative. Hence, we have another equivalent definition of conservative and nonconservative forces: *A force is conservative if the work done by it on a particle that moves between two points depends only on these points and not on the path followed. A force is nonconservative if the work done by that force on a particle that moves between two points depends on the path taken between those points.*

To illustrate this third (equivalent) definition of conservative forces, let us consider a second kind of conservative force, that due to gravity. Suppose that we take a stone of mass m in our hand and raise it to a height h above the ground, going from a to b by several different paths as in Fig. 8-3. We already know that in a round trip the total work done by a con-

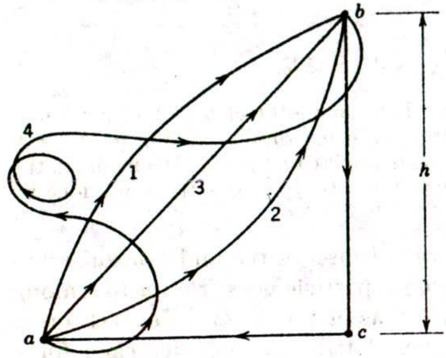


Fig. 8–3 A stone is raised from a to b via various paths 1, 2, 3, and 4.

servative force is zero and that the gravitational force is conservative. The work done *on* the stone by gravity along the return path bca is simply mgh. Hence, because gravity is a conservative force, the work done by gravity *on* the stone along any of the paths from a to b must be $-mgh$, for only if this is true can the total work done by gravity in a round trip be zero. This means that gravity does negative work on the stone as it moves from a to b, or, to put it another way, work must be done *against* gravity along any of the paths ab. The student can compute directly the result that the work done by gravity along any path ab equals $-mgh$. For any of these paths can be decomposed into infinitesimal displacements which are alternately horizontal and vertical; no work is done by gravity in horizontal displacements, and the net vertical displacement is the same in all cases. Hence the work done by gravity on the stone moving from a to b depends only on the positions of a and b and not at all on the path taken.

For a nonconservative force, such as friction, the work done is *not* independent of the path taken between two fixed points. We need only point out that as we push a block over a (rough) table between any two points a and b by various paths, the distance traversed varies and so does the work done by the frictional force. It depends on the path.

The definitions of conservative force which we have given are equivalent to one another. Which one we use depends only on convenience. The round-trip approach shows clearly that kinetic energy is conserved when conservative forces act. To develop the idea of potential energy, however, the path independence statement is preferable.

8–3 Potential Energy

In this section we shall focus attention not on the moving block of Fig. 8–1 but rather on the (isolated) system (block + spring). Instead of saying that the block is moving we prefer, from this point of view, to say that the configuration of the system is changing. We measure both the position of the block and the configuration of the system at any instant by the same parameter x, namely, the displacement of the free end of the spring from its normal position, corresponding to an unstretched spring. The kinetic energy of the system is the same as that of the block because we have assumed the spring to be massless.

We have seen that the kinetic energy of the system of Fig. 8–1 decreases during the first half of the motion, becomes zero, and then increases during the second half of the motion. If there is no friction, the kinetic energy of the system when it has regained its initial configuration returns to its initial value.

Under these circumstances (conservative forces acting) it makes sense to introduce the concept of *energy of configuration*, or *potential energy* U, and to say that if K for the system changes by ΔK as the configuration of the system changes (that is, as the block moves in the system of Fig. 8–1), then U for the system must change by an equal but opposite amount so that the sum of the two changes is zero, or

$$\Delta K + \Delta U = 0. \tag{8–4a}$$

Alternatively, we can say that any change in kinetic energy K of the system is compensated for by an equal but opposite change in the potential energy U of the system so that their sum remains constant throughout the motion, or

$$K + U = \text{a constant}. \tag{8–4b}$$

The potential energy of a system represents a form of stored energy which can be fully recovered and converted into kinetic energy. We cannot associate a potential energy with a nonconservative force such as the force of friction because the kinetic energy of a system in which such forces act does *not* return to its initial value when the system returns to its initial configuration.

Equations 8–4 apply to a closed system of interacting objects, such as the (mass + spring) system of Fig. 8–1. In this example, because we have taken the spring to be effectively massless, the kinetic energy may be associated with the moving mass alone. The block slows down (or speeds up) because a force is exerted on it *by the spring;* it is appropriate, then, to associate the potential energy of the system with this force, that is to say, with the spring. Thus in this simple case we say that kinetic energy, localized in the mass, decreases during the first part of the motion while potential energy, localized in the spring, increases during this same time.*

Equations 8–4 are essentially bookkeeping statements about energy. They, and the concept of potential energy, have no real meaning, however, until we have shown how to calculate U as a function of the configuration of the system within which the conservative forces act; in the example of Fig. 8–1 this means that we must be able to calculate $U(x)$, where x is the spring displacement.

---

* Just as we assumed the spring to be effectively massless we also assume the block to be rigid, that is, effectively "springless." In a more general system, kinetic and potential energy could each be present in various portions of the system, in varying proportions as the system configuration changed.

To refine our concept of potential energy U let us consider the work-energy theorem, $W = \Delta K$, in which W is the work done by the resultant force on a particle as it moves from a to b. For simplicity let us assume that only a single force **F** is acting on the particle; this is effectively true in the system of Fig. 8–1. If **F** is conservative we can combine the work-energy theorem (Eq. 8–1) with Eq. 8–4a, obtaining

$$W = \Delta K = -\Delta U. \tag{8-5a}$$

The work W done by a conservative force depends only on the starting and the end points of the motion and not on the path followed between them. Such a force can depend only on the position of a particle; it does not depend on the velocity of the particle or on the time, for example.

For motion in one dimension, Eq. 8–5a becomes

$$\Delta U = -W = -\int_{x_0}^{x} F(x)\,dx, \tag{8-5b}$$

the particle moving from x_0 to x. Equation 8–5b shows how to calculate the change in potential energy ΔU when a particle, acted on by a conservative force $F(x)$, moves from point a, described by x_0, to point b, described by x. The equation shows that we can only calculate ΔU if the force **F** depends only on the position of the particle (that is, on the system configuration), which is equivalent to saying that potential energy has meaning only for conservative forces.

Now that we know that the potential energy U depends on the position of the particle only, we can write Eq. 8–4b as

$$\tfrac{1}{2}mv^2 + U(x) = E \qquad \text{(one-dimension)} \tag{8-6a}$$

in which E, which remains constant as the particle moves, is called the *total mechanical energy*. Suppose that the particle moves from point a (where its position is x_0 and its speed is v_0) to point b (where its position is x and its speed is v); the total mechanical energy E must be the same for each system configuration when the force is conservative, or, from Eq. 8–6a,

$$\tfrac{1}{2}mv^2 + U(x) = \tfrac{1}{2}mv_0{}^2 + U(x_0). \tag{8-6b}$$

The quantity on the right depends only on the initial position x_0 and the initial speed v_0, which have definite values; it is, therefore, *constant during the motion*. This is the constant total mechanical energy E. Notice that force and acceleration do not appear in this equation, only position and speed. Equations 8–6 are often called the *law of conservation of mechanical energy* for conservative forces.

In many problems we find that although some of the individual forces are not conservative, they are so small that we can neglect them. In such cases we can use Eqs. 8–6 as a good approximation. For example, air resistance may be present but may have so little effect on the motion that we can ignore it.

Notice that, instead of starting with Newton's laws, we can simplify problem solving when conservative forces alone are involved by starting with Eqs. 8-6. This relation is derived from Newton's laws, of course, but it is one step closer to the solution (the so-called first integral of the motion). We often solve problems without analyzing the forces or writing down Newton's laws by looking instead for something in the motion that is constant; here the mechanical energy is constant and we can write down Eqs. 8-6 as the first step.

For one-dimensional motion we can also write the relation between force and potential energy (Eq. 8-5b) as

$$F(x) = -\frac{dU(x)}{dx}. \tag{8-7}$$

To show this, substitute this expression for $F(x)$ into Eq. 8-5b and observe that you get an identity. Equation 8-7 gives us another way of looking at potential energy. *The potential energy is a function of position whose negative derivative gives the force.*

The student may have noticed that we wrote down the quantity $U(x)$ in Eqs. 8-6 although we are only able to calculate *changes* in U (from Eq. 8-5b) and not U itself. Let us imagine that a particle moves from a to b along the x-axis and that a single conservative force $F(x)$ acts on it. To assign a value to U_b, the potential energy at point b, let us write

$$\Delta U = U_b - U_a,$$

or (see Eq. 8-5b),

$$U_b = \Delta U + U_a = -\int_{x_a}^{x_b} F(x)\,dx + U_a. \tag{8-8}$$

We cannot assign a value to U_b until we have assigned one to U_a. If point b is any arbitrary position x, so that $U_b = U(x)$, we give meaning to $U(x)$ by choosing point a to be some convenient reference position, described by $x_a = x_0$, and by arbitrarily assigning a value to the potential energy $U_a = U(x_0)$ when the body is at that point. Thus Eq. 8-8 becomes

$$U(x) = -\int_{x_0}^{x} F(x)\,dx + U(x_0). \tag{8-9}$$

The potential energy when the body is at the reference position, that is, $U(x_0)$, is usually given the arbitrary value zero.

It is often convenient to choose the reference position x_0 to be that at which the force acting on the particle is zero. Thus the force exerted by a spring is zero when the spring has its normal unstretched length; we usually say that the potential energy is also zero for this condition. Also, the attraction of the earth on a body decreases as the body moves away from the earth, becoming zero at an infinite distance. We usually take infinity as our reference position and assign the value zero to the potential energy associated with the gravitational force at that position (see Chapter 16). So far, however, we have been more concerned with the gravitational pull on bodies such as baseballs, etc., which, in comparison to the earth's radius, never move very far from the earth's surface. Here the gravitational force ($= mg$) is essentially constant and we find it convenient to take the zero of potential energy to be, not at infinity, but at the surface of the earth.

The effect of changing the coordinate of the standard reference position x_0, or of the arbitrary value assigned to $U(x_0)$, is simply to change the value of $U(x)$ by an added constant. The presence of an arbitrary added constant in the potential energy expression (Eq. 8–9) makes no difference to the equations that we have written so far. This simply adds the same constant term to each side of Eq. 8–6*b*, for example, leaving that equation unchanged. Furthermore, changing $U(x)$ by an added constant does not change the force calculated from Eq. 8–7 because the derivative of a constant is zero. All this simply means that the choice of a reference point for potential energy is immaterial because we are always concerned with *differences* in potential energy, rather than with any absolute value of potential energy at a given point.

There is a certain arbitrariness in specifying kinetic energy also. In order to determine speed, and hence kinetic energy, we must specify a reference frame. The speed of a man sitting on a train is zero if we take the train as a reference frame, but it is not zero to an observer on the ground who sees the man move by with uniform velocity. The value of the kinetic energy depends on the reference frame used by the observer. Hence the important thing about mechanical energy E, which is the sum of the kinetic and the potential energies, is *not* its actual value during a given motion (this depends on the observer) but the fact that this value *does not change* during the motion for any particular observer when the forces are conservative.

8–4 One-Dimensional Conservative Systems

Let us now calculate the potential energy in one-dimensional motion for two examples of conservative forces, the force of gravity for motions near the earth's surface and the elastic restoring force of an (ideal) stretched spring.

For the force of gravity we take the one-dimensional motion to be vertical, along the y-axis. We take the positive direction of the y-axis to be upward; the force of gravity is then in the negative y-direction, or downward. We have $F(y) = -mg$, a constant. The potential energy at position y is found from Eq. 8–9, or

$$U(y) = -\int_0^y F(y)\,dy + U(0) = -\int_0^y (-mg)\,dy + U(0) = mgy + U(0).$$

The potential energy can be taken as zero where $y = 0$, so that $U(0) = 0$, and

$$U(y) = mgy. \tag{8–10}$$

The gravitational potential energy is then mgy. The relation $F(y) =$

$-dU/dy$ (Eq. 8-7) is satisfied, for $-d(mgy)/dy = -mg$. We choose $y = 0$ to be at the surface of the earth for convenience, so that the gravitational potential energy is zero at the earth's surface and increases linearly with altitude y.

If we compare points y and $y = 0$, the conservation of kinetic plus potential energy, Eq. 8-6b, gives us the relation

$$\tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}mv_0^2.$$

This is equivalent mathematically to the well-known result (see Eq. 3-17),

$$v^2 = v_0^2 - 2gy.$$

If our particle moves from a height h_1 to a height h_2, we can use Eq. 8-6b to obtain

$$\tfrac{1}{2}mv_1^2 + mgh_1 = \tfrac{1}{2}mv_2^2 + mgh_2.$$

This result is equivalent to that of Example 5, Chapter 7. The total mechanical energy E is constant and is conserved during the motion, even though the kinetic energy and the potential energy vary as the configuration of the system (particle + earth) changes.

A second example of a conservative force is that exerted by an elastic spring on a body of mass m attached to it moving on a horizontal frictionless surface. If we take $x_0 = 0$ as the position of the end of the spring when unextended, the force exerted *on* the mass when the spring is stretched a distance x from its unextended length is $F = -kx$. The potential energy is obtained from Eq. 8-9,

$$U(x) = -\int_0^x F(x)\,dx + U(0) = -\int_0^x (-kx)\,dx + U(0).$$

If we choose $U(0) = 0$, the potential energy, as well as the force, is zero when the spring is unextended, and

$$U(x) = -\int_0^x (-kx)\,dx = \tfrac{1}{2}kx^2.$$

The result is the same whether we stretch or compress the spring, that is, whether x is plus or minus.

The relation $F(x) = -dU/dx$ (Eq. 8-7) is satisfied, for $-d(\frac{1}{2}kx^2)/dx = -kx$. The elastic potential energy of the spring is then

$$U(x) = \tfrac{1}{2}kx^2. \tag{8-11}$$

The body of mass m will undergo a motion in which the total energy E is conserved (Fig. 8-4). From Eq. 8-6b we have

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}mv_0^2.$$

Here v_0 is the speed of the particle for $x = 0$. Physically we achieve such a result by stretching the spring with an applied force to some position, x_m, and then releasing the spring. Notice that at $x = 0$ the energy of the system (particle + spring) is all kinetic. At $x = x_m$ (the maximum value

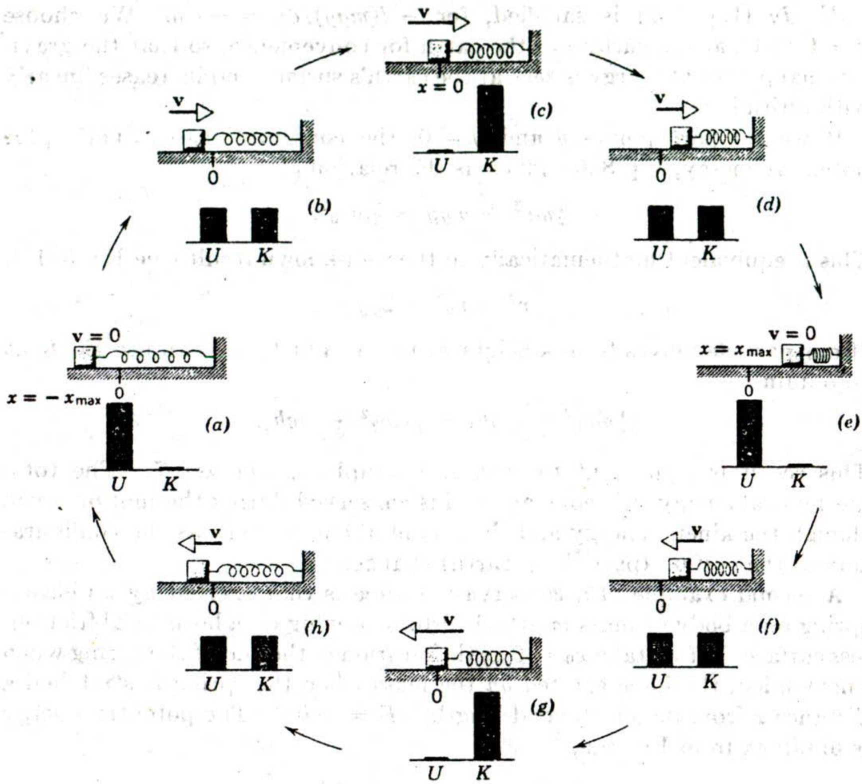


Fig. 8–4 A mass attached to a spring slides back and forth on a frictionless surface. The system is called a harmonic oscillator. The motion of the mass through one cycle is illustrated. Starting at the left (9 o'clock) the mass is in its extreme left position and momentarily at rest: $K = 0$. The spring is extended to its maximum length: $U = U_{\max}$. (K and U are illustrated in the bar graphs below each sketch.) An eighth-cycle later (next drawing) the mass has gained kinetic energy, but the spring is no longer so elongated; K and U have here the same value, $K = U = U_{\max}/2$. At the top the spring is neither elongated nor compressed and the speed is a maximum: $U = 0$, $K = K_{\max} = U_{\max}$. The cycle continues, with the total energy $E = K + U$ always the same: $E = K_{\max} = U_{\max}$. The harmonic oscillator will be analyzed more closely in Chapter 15.

of x), v must be zero, so that here the system energy is all potential. At $x = x_m$, we have

$$\tfrac{1}{2}kx_m^2 = \tfrac{1}{2}mv_0^2$$

or

$$x_m = \sqrt{m/k}\, v_0.$$

For positions between x_1 and x_2, Eq. 8–6b gives

$$\tfrac{1}{2}kx_1^2 + \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}kx_2^2 + \tfrac{1}{2}mv_2^2.$$

We have seen that *the kinetic energy of a body is the work that a body can do by virtue of its motion.* We express the kinetic energy by the formula $K = \frac{1}{2}mv^2$. We cannot give a similar universal formula by which potential energy can be expressed. *The potential energy of a system of bodies is the work that the system of bodies can do by virtue of the relative position of its parts, that is, by virtue of its configuration.* In each case we must determine how much work the system can do in passing from one configuration to another and then take this as the difference in potential energy of the system between these two configurations.

The potential energy of the spring depends on the relative position of the parts of the spring. Work can be obtained by allowing the spring to return from its extended to its unextended length, during which time it exerts a force through a distance. If a mass is attached to the spring, as in our example, the mass will be accelerated by this force and the potential energy will be converted to kinetic energy. In the gravitational case an object occupies a position with respect to the earth. The potential energy is a property of the object and the earth, considered as a system of bodies. It is the relative position of the parts of this system that determines its potential energy. The potential energy is greater when the parts are far apart than when they are close together. The loss of potential energy is equal to the work done in this process. This work is converted into kinetic energy of the bodies. In our example we ignored the kinetic energy acquired by the earth itself as an object fell toward it. In principle, this object exerts a force on the earth and causes it to acquire an acceleration, relative to some inertial frame. The resulting change in speed, however, is extremely small, and in spite of the enormous mass of the earth, its additional kinetic energy is negligible compared to that acquired by the falling object. This will be proved in a later chapter. In other cases, such as in planetary motion where the masses of the objects in our system may be comparable, we cannot ignore any part of the system. In general, *potential energy* is not assigned to either body separately but *is considered a joint property of the system.*

▶ **Example 1.** What is the change in gravitational potential energy when a 1600-lb elevator moves from street level to the top of the Empire State Building, 1250 ft above street level?

The gravitational potential energy of the system (elevator + earth) is $U = mgy$. Then

$$\Delta U = U_2 - U_1 = mg(y_2 - y_1).$$

But $mg = W = 1600 \text{ lb}$ and $y_2 - y_1 = 1250 \text{ ft}$,

so that $\Delta U = 1600 \times 1250 \text{ ft-lb} = 2.00 \times 10^6 \text{ ft-lb}.$

Example 2. As an example of the simplicity and usefulness of the energy method of solving dynamical problems, consider the problem illustrated in Fig. 8-5. A block of mass m slides down a curved frictionless surface. The force exerted by the surface on the block is always normal to the surface and to the direction of the motion of the block, so that this force does no work. Only the gravitational force

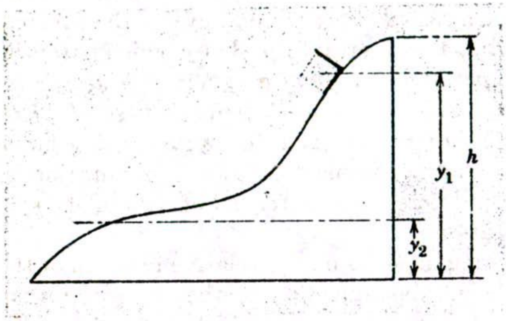


Fig. 8–5 A block sliding down a frictionless curved surface.

does work on the block and that force is conservative. The mechanical energy E is, therefore, conserved and we can write at once

$$\tfrac{1}{2}mv_1^2 + mgy_1 = \tfrac{1}{2}mv_2^2 + mgy_2.$$

This gives

$$v_2^2 = v_1^2 + 2g(y_1 - y_2).$$

The speed at the bottom of the curved surface depends only on the initial speed and the change in vertical height but does not depend at all on the shape of the surface. In fact, if the block is initially at rest at $y_1 = h$, and if we set $y_2 = 0$, we obtain

$$v_2 = \sqrt{2gh}.$$

At this point the student should recall the independence of path feature of work done by conservative forces and should be able to justify applying the ideas developed for one-dimensional motion to this two-dimensional example.

In this problem the value of the force depends on the slope of the surface at each point. Hence, the acceleration is not constant but is a function of position. To obtain the speed by starting with Newton's laws we would first have to determine the acceleration at each point and then integrate the acceleration over the path. We avoid all this labor by starting at once from the fact that the mechanical energy is constant throughout the motion.

Example 3. The spring in a spring gun has a force constant of 4.0 lb/in. It is compressed 2.0 in. from its natural length, and a ball weighing 0.030 lb is put into the barrel against it. Assuming no friction and a horizontal gun barrel, with what speed will the ball leave the gun when released?

The force is conservative so that mechanical energy is conserved in the process. The initial mechanical energy is the elastic potential energy of the spring, $\frac{1}{2}kx^2$, and the final mechanical energy is the kinetic energy of the ball, $\frac{1}{2}mv^2$. Hence,

$$\tfrac{1}{2}kx^2 = \tfrac{1}{2}mv^2$$

or

$$v = \sqrt{\frac{k}{m}}\,x = \sqrt{\frac{48\ \text{lb/ft}}{(0.030\ \text{lb})/(32\ \text{ft/sec}^2)}}\,(\tfrac{1}{6}\text{ft}) = 38\ \text{ft/sec}.$$ ◀

8–5 The Complete Solution of the Problem for One-Dimensional Forces Depending on Position Only

Equation 8–6a gives the relation between coordinate and speed for one-dimensional motion when the force depends on position only. The force and the acceleration have been eliminated in arriving at this equation. To complete the solution of the dynamical problem we must eliminate the speed and determine position as a function of time.

We can do this in a formal way, as follows. From Eq. 8–6a we have

$$\tfrac{1}{2}mv^2 + U(x) = E.$$

Solving for v, we obtain

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}\,[E - U(x)]}, \tag{8–12}$$

or

$$\frac{dx}{\sqrt{\frac{2}{m}\,[E - U(x)]}} = dt$$

Then the function $x(t)$ may be found by solving for x the equation

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}\,[E - U(x)]}} = \int_{t_0}^{t} dt = t - t_0. \tag{8–13}$$

Here the particle is taken to be at x_0 at the time t_0 and E is the constant total energy. In applying this equation, the sign of the square root taken corresponds to whether **v** points in the positive or in the negative x-direction. When **v** changes direction during the motion it may be necessary to carry out the integration separately for each part of the motion.

Even when this integral cannot be evaluated or when the resulting equation cannot be solved to give an explicit solution for $x(t)$, the equation of energy conservation gives us useful information about the solution. For example, for a given total energy E, Eq. 8–12 tells us that the particle is restricted to those regions on the x-axis where $E > U(x)$. We cannot have an imaginary speed or a negative kinetic energy physically, so that $E - U(x)$ must be zero or greater. Furthermore, we can obtain a good qualitative description of the types of motion possible by plotting $U(x)$ versus x. This description depends on the fact that the speed is proportional to the square root of the difference between E and U.

For example, consider the potential energy function shown in Fig. 8–6. This could be thought of as an actual profile of a frictionless roller coaster, but in general

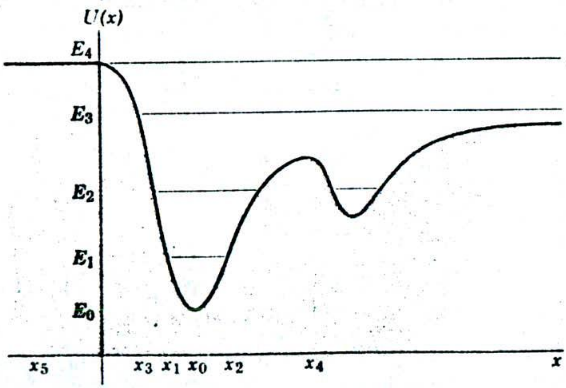


Fig. 8–6 A potential energy curve.

it can represent the potential energy of a nongravitational system. Since we must have $E \geq U(x)$ for real motion, the lowest total energy possible is E_0. At this value of the total energy, $E_0 = U$ and the kinetic energy must be zero. The particle must be at rest at the point x_0. At a slightly higher energy E_1 the particle can move between x_1 and x_2 only. As it moves from x_0 its speed decreases on approaching either x_1 or x_2. At x_1 or x_2 the particle stops and reverses its direction. These points x_1 and x_2 are, therefore, called *turning points* of the motion. At a total energy E_2 there are four turning points, and the particle can oscillate in either one of the two potential valleys. At the total energy E_3 there is only one turning point of the motion, at x_3. If the particle is initially moving in the negative x-direction, it will stop at x_3 and then move in the positive x-direction. It will speed up as U decreases and slow down as U increases. At energies above E_4 there are no turning points, and the particle will not reverse direction. Its speed will change according to the value of the potential at each point.

At a point where $U(x)$ has a minimum value, such as at $x = x_0$, the slope of the curve is zero so that the force is zero, that is, $F(x_0) = -(dU/dx)_{x=x_0} = 0$. A particle at rest at this point will remain at rest. Furthermore, if the particle is displaced slightly in either direction, the force, $F(x) = -dU/dx$, will tend to return it, and it will oscillate about the equilibrium point. This equilibrium point is, therefore, called a point of *stable equilibrium*.

At a point where $U(x)$ has a maximum value, such as at $x = x_4$, the slope of the curve is zero so that the force is again zero, that is, $F(x_4) = -(dU/dx)_{x=x_4} = 0$. A particle at rest at this point will remain at rest. However, if the particle is displaced even the slightest distance from this point, the force, $F(x) = -dU/dx$, will tend to push it farther away from the equilibrium position. Such an equilibrium point is, therefore, called a point of *unstable equilibrium*.

In an interval in which $U(x)$ is constant, such as near $x = x_5$, the slope of the curve is zero so that the force is zero, that is, $F(x_5) = -(dU/dx)_{x=x_5} = 0$. Such an interval is called one of neutral equilibrium, since a particle can be displaced slightly without experiencing either a repelling or a restoring force.

From this it is clear that if we know the potential energy function for the region of x in which the body moves, we know a great deal about the motion of the body.

▶ **Example 4.** The potential energy function for the force between two atoms in a diatomic molecule can be expressed approximately as follows:

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$$

where a and b are positive constants and x is the distance between atoms.

(*a*) At what values of x is $U(x)$ equal to zero? At what value of x is $U(x)$ a minimum?

In Fig. 8–7*a* we show $U(x)$ versus x. The values of x at which $U(x)$ equals zero are found from

$$\frac{a}{x^{12}} - \frac{b}{x^6} = 0.$$

Hence

$$x^6 = \frac{a}{b} \qquad x = \sqrt[6]{\frac{a}{b}}.$$

$U(x)$ also becomes zero as $x \to \infty$ [see figure or put $x = \infty$ into equation for $U(x)$], so that $x = \infty$ is also a solution.

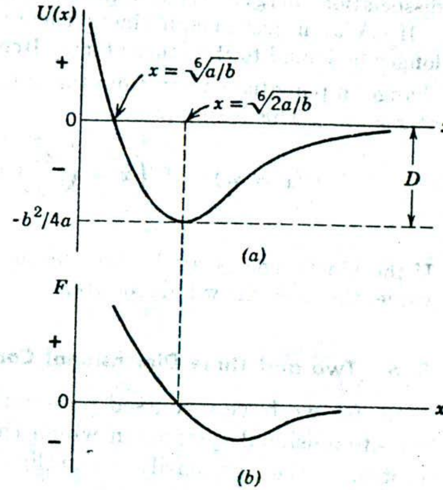


Fig. 8–7 Example 4. (*a*) The potential energy and (*b*) the force between two atoms in a diatomic molecule as a function of the distance x between atoms.

The value of x at which $U(x)$ is a minimum is found from

$$\frac{d}{dx}U(x) = 0.$$

That is,

$$\frac{-12a}{x^{13}} + \frac{6b}{x^7} = 0$$

or

$$x^6 = \frac{2a}{b} \qquad x = \sqrt[6]{\frac{2a}{b}}.$$

(*b*) Determine the force between the atoms.
From Eq. 8–7

$$F(x) = -\frac{d}{dx}U(x),$$

$$F = \frac{-d}{dx}\left(\frac{a}{x^{12}} - \frac{b}{x^6}\right) = \frac{12a}{x^{13}} - \frac{6b}{x^7}.$$

We plot the force as a function of the separation between the atoms in Fig. 8–7*b*. When the force is positive (from $x = 0$ to $x = \sqrt[6]{2a/b}$), the atoms are repelled from one another (force directed toward increasing x). When the force is negative (from $x = \sqrt[6]{2a/b}$ to $x = \infty$), the atoms are attracted to one another (force directed toward decreasing x). At $x = \sqrt[6]{2a/b}$ the force is zero; this is the equilibrium point and is a point of stable equilibrium.

(*c*) Assume that one of the atoms remains at rest and that the other moves along x. Describe the possible motions.

From the analysis of this section it is clear that the atom oscillates about the equilibrium separation at $x = \sqrt[6]{2a/b}$, much as a particle sliding up and down the frictionless hills of the potential valley.

(*d*) The energy needed to break up the molecule into separate atoms is called the dissociation energy. What is the dissociation energy of the molecule?

If one atom has enough kinetic energy to get over the potential hill, it will no longer be bound to the other atom. Hence, the dissociation energy D equals the change in potential energy from the minimum value at $x = \sqrt[6]{2a/b}$ to the value at $x = \infty$. This is simply

$$U(x = \infty) - U\left(x = \sqrt[6]{\frac{2a}{b}}\right) = 0 - \left(\frac{a}{4a^2/b^2} - \frac{b}{2a/b}\right) = \frac{b^2}{4a}.$$

If the kinetic energy at the equilibrium position is equal to or greater than this value, the molecule will dissociate. ◀

8–6 Two and Three-Dimensional Conservative Systems

So far we have discussed potential energy and energy conservation for one-dimensional systems in which the force was directed along the line of motion. We can easily generalize the discussion to three-dimensional motion.

If the work done by the force $\mathbf{F}$ depends only on the end points of the motion and is independent of the path taken between these points, the force is conservative. We define the potential energy U by analogy with the one-dimensional system and find that it is a function of three space coordinates, that is, $U = U(x,y,z)$. Again we obtain an expression for the conservation of mechanical energy.

The generalization of Eq. 8–5*b* to motion in three dimensions is

$$\Delta U = -\int_{x_0}^{x} F_x\,dx - \int_{y_0}^{y} F_y\,dy - \int_{z_0}^{z} F_z\,dz \qquad (8\text{–}5c)$$

or, more compactly in vector notation,

$$\Delta U = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \qquad (8\text{–}5d)$$

in which ΔU is the change in potential energy for the system as the particle moves from the point (x_0,y_0,z_0), described by the position vector $\mathbf{r}_0$, to the point (x,y,z), described by the position vector $\mathbf{r}$. F_x, F_y, and F_z are the components of the conservative force $\mathbf{F}(\mathbf{r}) = \mathbf{F}(x,y,z)$.

The generalization of Eq. 8–6*b* to three-dimensional motion is

$$\tfrac{1}{2}mv^2 + U(x,y,z) = \tfrac{1}{2}mv_0^2 + U(x_0,y_0,z_0) \qquad (8\text{–}6c)$$

which can be written in vector notation as

$$\tfrac{1}{2}m\mathbf{v} \cdot \mathbf{v} + U(\mathbf{r}) = \tfrac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0 + U(\mathbf{r}_0) \qquad (8\text{–}6d)$$

in which $\mathbf{v} \cdot \mathbf{v} = v_x^2 + v_y^2 + v_z^2 = v^2$ and $\mathbf{v}_0 \cdot \mathbf{v}_0 = v_{0x}^2 + v_{0y}^2 + v_{0z}^2 = v_0^2$. Likewise Eq. 8–6*a* becomes

$$\tfrac{1}{2}mv^2 + U(x,y,z) = E$$

in three dimensions, E being the constant total mechanical energy.

Finally, the generalization of Eq. 8–7 to three dimensions is

$$\mathbf{F}(\mathbf{r}) = -\mathbf{i}\frac{\partial U}{\partial x} - \mathbf{j}\frac{\partial U}{\partial y} - \mathbf{k}\frac{\partial U}{\partial z}$$

If we substitute this expression for **F** into Eq. 8–5*d* we again obtain an identity. In vector language the conservative force **F** is said to be the negative of the *gradient* of the potential energy $U(x,y,z)$.

The student can show that all these expressions reduce to the correct one-dimensional equations for motion along the x-axis.

▶ **Example 5.** Consider the simple pendulum, Section 7–4, Fig. 7–8*a*. The motion of the system is in the x–y plane, that is, it is a two-dimensional motion. The tension in the cord is always at right angles to the motion of the suspended particle, so that this force does no work on the particle. If the pendulum is displaced through some angle and is then released, only the gravitational force of attraction exerted on the particle by the earth does work on it. Since this force is conservative, we can use the equation of energy conservation in two dimensions,

$$\tfrac{1}{2}mv^2 + U(x,y) = E.$$

But $U(x,y)$ equals mgy, where y is taken as zero at the lowest point of the arc ($\phi = 0°$). Then,

$$\tfrac{1}{2}mv^2 + mgy = E.$$

The particle is pulled through an angle ϕ_0 before being released. The potential energy there is mgh, corresponding to a height $y = h$ above the reference point. At the release point ($\phi = \phi_0$) the speed and the kinetic energy are zero so that the potential energy equals the total mechanical energy at that point.

Hence,

$$E = mgh$$

and

$$\tfrac{1}{2}mv^2 + mgy = mgh,$$

or

$$\tfrac{1}{2}mv^2 = mg(h - y).$$

The maximum speed occurs at $y = 0$, where $v = \sqrt{2gh}$.
The minimum speed occurs at $y = h$, where $v = 0$.
At $y = 0$ the energy is entirely kinetic, the potential energy being zero.
At $y = h$ the energy is entirely potential, the kinetic energy being zero.
At intermediate positions the energy is partly kinetic and partly potential.

Notice that $U \leq E$ at all points of the motion; the pendulum cannot rise higher than $y = h$, its initial release point. ◀

8–7 Nonconservative Forces

So far we have considered only the action of a single conservative force on a particle. Starting from the work-energy theorem, or

$$W_1 + W_2 + \cdots + W_n = \Delta K \tag{8–2}$$

we saw that, if only one force, say $\mathbf{F}_1$, was acting and if it was conservative, then we could represent the work W_1 that it did on the particle as a decrease in potential energy ΔU_1 of the system (see Eq. 8–5*a*), or

$$W_1 = -\Delta U_1.$$

Combining this with Eq. 8–2 yielded

$$\Delta K + \Delta U_1 = 0.$$

If several conservative forces such as gravity, an elastic spring force, an electrostatic force, etc., are acting, we can easily extend these two equations to

$$\Sigma W_c = -\Sigma \Delta U \tag{8–14a}$$

and

$$\Delta K + \Sigma \Delta U = 0 \tag{8–14b}$$

in which ΣW_c is the sum of the work done by the various (conservative) forces and the ΔU's are the changes in the potential energy of the system associated with these forces. The quantity on the left of Eq. 8–14*b* is simply ΔE, the change in the total mechanical energy, for the case in which several conservative forces are acting on a particle. We can write this equation then as

$$\Delta E = 0 \qquad \text{(conservative forces),} \tag{8–15}$$

which tells us that, as the system configuration changes the total mechanical energy E for the system remains constant.

Let us now suppose that, in addition to the several conservative forces, a single nonconservative force due to friction acts on the particle. We can then write Eq. 8–2 as

$$W_f + \Sigma W_c = \Delta K,$$

where ΣW_c is again the sum of the work done by the conservative forces and W_f is the work done by friction. We can recast this (see Eq. 8–14*b*) as

$$\Delta K + \Sigma \Delta U = W_f. \tag{8–16}$$

Equation 8–16 shows that, if a frictional force acts, the total mechanical energy is *not* constant, but changes by the amount of work done by the frictional force. We can write Eq. 8–16 as

$$\Delta E = E - E_0 = W_f. \tag{8–17}$$

Since W_f, the work done by friction *on* the particle, is always negative, it follows from Eq. 8–17 that the final mechanical energy $E\ (= K + \Sigma U)$ is less than the initial mechanical energy $E_0\ (= K_0 + \Sigma U_0)$.

Friction is an example of a dissipative force, one which does negative work on a body and tends to diminish the total mechanical energy of the system. If we had used another nonconservative force, then W_f in Eqs. 8–16 and 8–17 would be replaced by a term W_{nc}, showing again that the total mechanical energy E of the system is *not* constant, but changes by the amount of work done by the nonconservative force. Hence, *only when there are no nonconservative forces, or when we can neglect the work they do, can we assume conservation of mechanical energy.*

What happened to the "lost" mechanical energy in the case of friction? It is transformed into heat. Heat is developed when surfaces are rubbed together, for example. The heat energy developed is exactly equal to the mechanical energy dissipated. We shall have much more to say about heat energy in later chapters.

Just as the work done by a conservative force *on* an object is the negative of the potential energy gain, so the work done by a frictional force *on* an object is the negative of the heat energy gained. In other words, the heat energy produced is equal to the work done *by* the object. Then we can replace W_f in Eq. 8–17 by $-Q$, in which Q is the heat energy produced, or

$$\Delta E + Q = 0. \tag{8–18}$$

This asserts that there is no change in the sum of the mechanical and heat energy of the system when only conservative and frictional forces act on the system. Writing this equation as $Q = -\Delta E$ we see that the loss of mechanical energy equals the gain in heat energy.

▶ **Example 6.** A 10-lb block is thrust up a 30° inclined plane with an initial speed of 16 ft/sec. It is found to travel 5.0 ft along the plane, stop, and slide back to the bottom. Compute the force of friction **f** (assumed to have a constant magnitude) acting on the block and find the speed v of the block when it returns to the bottom of the inclined plane.

Consider first the upward motion. At the top, where this motion ends,

$$E = K + U = 0 + (10 \text{ lb})(5.0 \text{ ft})(\sin 30°) = 25 \text{ ft-lb}.$$

At the bottom, where this motion begins,

$$E_0 = K_0 + U_0 = \frac{1}{2}\left(\frac{10 \text{ lb}}{32 \text{ ft/sec}^2}\right)(16 \text{ ft/sec})^2 + 0 = 40 \text{ ft-lb},$$

But

$$W_f = -fs = -f(5.0 \text{ ft}).$$

and

$$E - E_0 = W_f,$$

so that

$$25 \text{ ft-lb} - 40 \text{ ft-lb} = -f(5.0 \text{ ft})$$

and

$$f = 3.0 \text{ lb}.$$

Now consider the downward motion. The block returns to the bottom of the inclined plane with a speed v. Then, at the bottom, where this motion ends,

$$E = K + U = \frac{1}{2}\left(\frac{10 \text{ lb}}{32 \text{ ft/sec}^2}\right)v^2 + 0 = (\tfrac{5}{32} \text{ lb sec}^2/\text{ft})v^2.$$

At the top, where this motion begins,

But
$$E_0 = K_0 + U_0 = 0 + (10 \text{ lb})(5.0 \text{ ft})(\sin 30°) = 25 \text{ ft-lb},$$

$$W_f = -(3.0 \text{ lb})(5.0 \text{ ft}) = -15 \text{ ft-lb}.$$

and

$$E - E_0 = W_f,$$

so that

$$(\tfrac{5}{32}\text{ lb sec}^2/\text{ft})v^2 - 25\text{ ft-lb} = -15\text{ ft-lb},$$

and

$$v = 8.0\text{ ft/sec}.$$

◀

8–8 The Conservation of Energy

We can extend the discussion of the previous section by considering not only conservative forces and the force of friction but also other, nonfrictional, nonconservative forces. We can regroup the work-energy theorem (Eq. 8–2)

$$W_1 + W_2 + \cdots + W_n = \Delta K$$

as

$$\Sigma W_c + W_f + \Sigma W_{nc} = \Delta K \tag{8–19}$$

in which ΣW_c is the total work done on the particle by conservative forces, W_f is the work done by friction, and ΣW_{nc} is the total work done by nonconservative forces other than friction. We have seen that each conservative force can be associated with a potential energy and that friction is associated with heat energy, or

$$\Sigma W_c = -\Sigma \Delta U$$

and

$$W_f = -Q,$$

so that Eq. 8–19 becomes

$$\Sigma W_{nc} = \Delta K + \Sigma \Delta U + Q.$$

Now whatever the W_{nc} are, it has always been possible to find new forms of energy which corresponds to this work. We can then represent ΣW_{nc} by another change of energy term on the right-hand side of the equation, with the result that we can always write the work-energy theorem as

$$0 = \Delta K + \Sigma \Delta U + Q + (\text{change in other forms of energy}).$$

In other words, the total energy—kinetic plus potential plus heat plus all other forms—does not change. *Energy may be transformed from one kind to another, but it cannot be created or destroyed; the total energy is constant.*

This statement is a generalization from our experience, so far not contradicted by observation of nature. It is called the *principle of the conservation of energy.* Often in the history of physics this principle seemed to fail. But its apparent failure stimulated the search for the reasons. Experimentalists searched for physical phenomena besides motion that accompany the forces of interaction between bodies. Such phenomena have always been found. With work done against friction, heat is produced; in other interactions energy in the form of sound, light, electricity, etc., may be produced. Hence the concept of energy was generalized to include forms other than kinetic and potential energy of directly observable bodies. This procedure, which relates the mechanics of bodies observed to be in motion to phenomena which are not mechanical or in which motion is not directly detected, has linked mechanics to all other areas of physics. The energy concept now permeates all of physical science and has become one of the unifying ideas of physics.*

In subsequent chapters we shall study various transformations of energy —from mechanical to heat, mechanical to electrical, nuclear to heat, etc. It is during such transformations that we measure the energy changes in terms of work, for it is during these transformations that forces arise and do work.

Although the principle of the conservation of kinetic plus potential energy is often useful, we see that it is a restricted case of the more general principle of the conservation of energy. Kinetic and potential energy is conserved only when conservative forces act. Total energy is *always* conserved.

8-9 Mass and Energy

One of the great conservation laws of science has been the law of conservation of matter. From a philosophical point of view an early statement of this general principle was given by the Roman poet Lucretius, a contemporary of Julius Caesar, in his celebrated work *De Rerum Natura.* Lucretius wrote "Things cannot be born from nothing, cannot when begotten be brought back to nothing." It was a long time before this concept was established as a firm scientific principle. The principal experimental contribution was made by Antoine Lavoisier (1743–1794), regarded by many as the father of modern chemistry. He wrote in 1789 "We must lay it down as an incontestable axiom, that in all the operations of art and nature, nothing is created; an equal quantity of matter exists both before and after the experiment . . . . and nothing takes place beyond changes and modifications in the combinations of these elements."

This principle, subsequently called the conservation of mass, proved extremely fruitful in chemistry and physics. Serious doubts as to the general validity of this principle were raised by Albert Einstein in his papers introducing the theory of relativity. Subsequent experiments on fast-moving electrons and on nuclear matter confirmed his conclusions.

Einstein's findings suggested that, if certain physical laws were to be retained, the mass of a particle had to be redefined as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}. \tag{8-20}$$

Here m_0 is the mass of the particle when at rest with respect to the observer, called the *rest mass;* m is the mass of the particle measured as it moves at a speed v relative to the observer; and c is the speed of light, having a constant value of approximately 3×10^8 meters/sec. Experimental checks of this equation can be made, for example, by deflecting high-speed elec-

* See for example, "Concept of Energy in Mechanics," by R. B. Lindsay in *The Scientific Monthly,* October 1957.

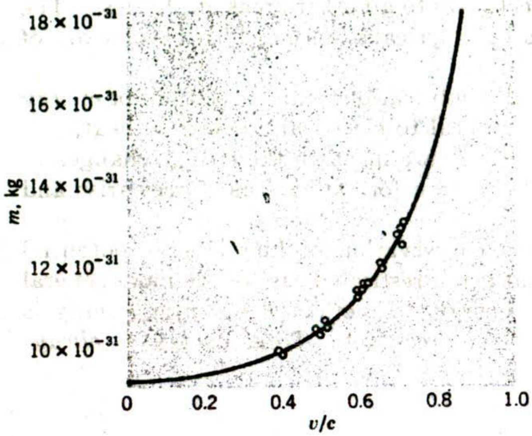


Fig. 8–8 The way an electron's mass increases as its speed relative to the observer increases. The solid line is a plot of $m = m_0(1 - v^2/c^2)^{-1/2}$, and the circles are adapted from experimental values obtained by Bucherer and Neumann in 1914. The curve tends toward infinity as $v \to c$.

trons in magnetic fields and measuring the radii of curvature of their path. The paths are circular and the magnetic force a centripetal one ($F = mv^2/r$, F and v being known). At ordinary speeds the difference between m and m_0 is too small to be detectable. Electrons, however, can be emitted from radioactive nuclei with speeds greater than nine-tenths that of light. In such cases the results (Fig. 8–8) confirm Eq. 8–20.

It is convenient to let the ratio v/c be represented by β. Then Eq. 8–20 becomes

$$m = m_0(1 - \beta^2)^{-1/2}.$$

To find the kinetic energy of a body, we compute the work done by the resultant force in setting the body in motion. In Section 7–5 we obtained

$$K = \int_0^v \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} m_0 v^2$$

for kinetic energy, when we assumed a constant mass m_0. Suppose now instead we take into account the variation of mass with speed and use $m = m_0(1 - \beta^2)^{-1/2}$ in our previous equation. We find (Problem 29, Chapter 9) that the kinetic energy is no longer given by $\frac{1}{2}m_0v^2$ but instead is

$$K = mc^2 - m_0c^2 = (m - m_0)c^2 = \Delta mc^2. \tag{8–21}$$

The kinetic energy of a particle is, therefore, the product of c^2 and the increase in mass Δm resulting from the motion.

Now, at small speeds we expect the relativistic result to agree with the classical result. By the binomial theorem we can expand $(1 - \beta^2)^{-1/2}$ as

$$(1 - \beta^2)^{-1/2} = 1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \cdots.$$

At small speeds $\beta = v/c \ll 1$ so that all terms beyond β^2 are negligible.

Then

$$K = (m - m_0)c^2 = m_0c^2[(1 - \beta^2)^{-1/2} - 1]$$
$$= m_0c^2(1 + \tfrac{1}{2}\beta^2 + \cdots - 1) \cong \tfrac{1}{2}m_0c^2\beta^2 = \tfrac{1}{2}m_0v^2,$$

which is the classical result. Notice also that when K equals zero, $m = m_0$ as expected.

The basic idea that energy is equivalent to mass can be extended to include energies other than kinetic. For example, when we compress a spring and give it elastic potential energy U, its mass increases from m_0 to $m_0 + U/c^2$. When we add heat in amount Q to an object, its mass increases by an amount Δm, where Δm is Q/c^2. We arrive at a principle of *equivalence of mass and energy:* For every unit of energy E of *any* kind supplied to a material object, the mass of the object increases by an amount

$$\Delta m = E/c^2.$$

This is the famous Einstein formula

$$E = \Delta mc^2. \tag{8-22}$$

In fact, since mass itself is just one form of energy, we can now assert that a body at rest has an energy m_0c^2 by virtue of its rest mass. This is called its rest energy. If we now consider a closed system, the principle of the conservation of energy, as generalized by Einstein, becomes

$$\Sigma\,(m_0c^2 + \mathcal{E}) = \text{constant}$$

or

$$\Delta(\Sigma\, m_0c^2 + \Sigma\,\mathcal{E}) = 0,$$

where $\Sigma\, m_0c^2$ is the total rest energy and $\Sigma\,\mathcal{E}$ is the total energy of *all other* kinds. As Einstein wrote, "Pre-relativity physics contains two conservation laws of fundamental importance, namely the law of conservation of energy and the law of conservation of mass; these two appear there as completely independent of each other. Through relativity theory they melt together into *one* principle."

Because the factor c^2 is so large, we would not expect to be able to detect changes in mass in ordinary mechanical experiments. A change in mass of 1 gm would require an energy of 9×10^{13} joules. But when the mass of a particle is quite small to begin with and high energies can be imparted to it, the relative change in mass may be readily noticeable. This is true in nuclear phenomena, and it is in this realm that classical mechanics breaks down and relativistic mechanics receives its most striking verification.

A beautiful example of exchange of energy between mass and other forms is given by the phenomenon of pair annihilation or pair production. In this phenomenon an electron and a positron, elementary material particles differing only in the sign of their electric charge, can combine and literally disappear. In their place we find high-energy radiation, called γ-radiation, whose radiant energy is exactly equal to the rest mass plus kinetic energies of the disappearing particles. The process is reversible, so that a materialization of mass from radiant energy can occur when a high enough energy γ-ray, under proper conditions, disappears; in its place appears a positron-electron pair whose total energy (rest mass + kinetic) is equal to the radiant energy lost.

▶ **Example 7.** Consider a quantitative example. On the atomic mass scale the unit of mass is 1.66×10^{-27} kg approximately. On this scale the mass of the proton (the nucleus of a hydrogen atom) is 1.00731 and the mass of the neutron (a neutral particle, one of the constituents of all nuclei except hydrogen) is 1.00867. A deuteron (the nucleus of heavy hydrogen) is known to consist of a neutron and a proton; the mass of the deuteron is found to be 2.01360. The mass of the deuteron is *less than* the combined masses of neutron and proton by 0.00238 atomic mass units. The discrepancy is equivalent in energy to

$$E = \Delta mc^2 = (0.00238 \times 1.66 \times 10^{-27}\text{ kg})(3.00 \times 10^8\text{ meters/sec})^2$$
$$= 3.57 \times 10^{-13}\text{ joules} = 2.22 \times 10^6\text{ ev.}$$

When a neutron and a proton combine to form a deuteron, this exact amount of energy is given off in the form of γ-radiation. Similarly, it is found that the same amount of energy must be *added* to the deuteron to break it up into a proton and a neutron. This energy is therefore called the *binding energy* of the deuteron. ◀

QUESTIONS

1. Mountain roads rarely go straight up the slope but wind up gradually. Explain why.

2. Is any work being done on a car moving with constant speed along a straight level road?

3. What happens to the potential energy an elevator loses in coming down from the top of a building to a stop at the ground floor?

4. In Example 2 (see Fig. 8–5) we asserted that the speed at the bottom does not depend at all on the *shape* of the surface. Would this still be true if friction were present?

5. Give physical examples of unstable equilibrium. Of neutral equilibrium. Of stable equilibrium.

6. Explain, using work and energy ideas, how a child pumps a swing up to large amplitudes from a rest position.

7. Two disks are connected by a stiff spring. Can one press the upper disk down enough so that when it is released it will spring back and raise the lower disk off the table (see Fig. 8–9)? Can mechanical energy be conserved in such a case?

Fig. 8–9

8. In the case of work done against friction, the amount of heat generated is independent of the velocity (or inertial reference frame) of the observer. That is, different observers would assign the same quantity of mechanical energy transformed into heat due to friction. How can this be explained, considering that such observers measure different quantities of total work done and different changes in kinetic energy in general (see Problem 19, Chapter 7)?

9. An object is dropped and observed to bounce to one and one-half times its original height. What conclusion can you draw from this observation?

10. The driver of an automobile traveling at speed v suddenly sees a brick wall at a distance d directly in front of him. To avoid crashing, is it better for him to slam on the brakes or to turn the car sharply away from the wall? (Hint: consider the force required in each case.)

11. A spring is kept compressed by tying its ends together tightly. It is then placed in acid and dissolves. What happened to its stored potential energy?

PROBLEMS

1. Show that for the same initial speed v_0, the speed v of a projectile will be the same at all points at the same elevation, regardless of the angle of projection.

2. The string in Fig. 8–10 has a length l = 4.0 ft. When the ball is released, it will swing down the dotted arc. How fast will it be going when it reaches the lowest point in its swing?

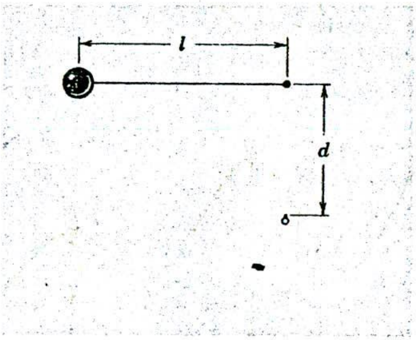


Fig. 8–10

3. The nail in Fig. 8–10 is located a distance d below the point of suspension. Show that d must be at least $0.6l$ if the ball is to swing completely around in a circle centered on the nail.

4. Suppose that the string of Fig. 8–10 is very elastic, made of rubber, say, and that the string is unextended at length l when the ball is released. (*a*) Explain why you would expect the ball to reach a low point greater than a distance l below the point of suspension. (*b*) Show, using dynamic and energy considerations, that if Δl is small compared to l the string will stretch by an amount $\Delta l = 3\,mg/k$, where k is the assumed force constant of the string. Notice that the larger k is, the smaller Δl is, and the better the approximation $\Delta l \ll l$. (*c*) Show, under these circumstances, that the speed of the ball at the bottom is $v = \sqrt{2g(l - 3mg/2k)}$, *less* than it would be for an inelastic string ($k = \infty$). Give a physical explanation for this result using energy considerations.

5. (*a*) A light rigid rod of length l has a mass m attached to its end, forming a simple pendulum. It is inverted and then released. What is its speed v at the lowest point and what is the tension T in the suspension at that instant? (*b*) The same pendulum is next put in a horizontal position and released from rest. At what angle from the vertical will the tension in the suspension equal the weight in magnitude?

6. A simple pendulum of length l, the mass of whose bob is m, is observed to have a speed v_0 when the cord makes the angle θ_0 with the vertical $(0 < \theta_0 < \pi/2)$, as in Fig. 8–11. In terms of g and the foregoing given quantities, determine (*a*) the total mechanical energy of the system; (*b*) the speed v_1 of the bob when it is at its lowest position; (*c*) the least value v_2 that v_0 could have if the cord is to achieve a horizontal position during the motion; (*d*) the speed v_3 such that if $v_0 > v_3$ the pendulum will not oscillate but rather will continue to move around in a vertical circle.

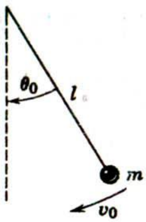


Fig. 8–11

7. An object is attached to a vertical spring and slowly lowered to its equilibrium position. This stretches the spring by an amount d. If the same object is attached to the same vertical spring but permitted to fall instead, through what distance does it stretch the spring?

8. A 2.0-kg block is dropped from a height of 0.40 meter onto a spring of force constant $k = 1960$ nt/meter. Find the maximum distance the spring will be compressed (neglect friction).

9. A frictionless roller coaster of mass m starts at point A with speed v_0, as in Fig. 8–12. Assume that the roller coaster can be considered as a point particle and that it always remains on the track. (*a*) What will be the speed of the roller coaster at points B and C? (*b*) What constant deceleration is required to stop it at point E if the brakes are applied at point D? (*c*) Suppose $v_0 = 0$; how long will it take the roller coaster to reach point B?

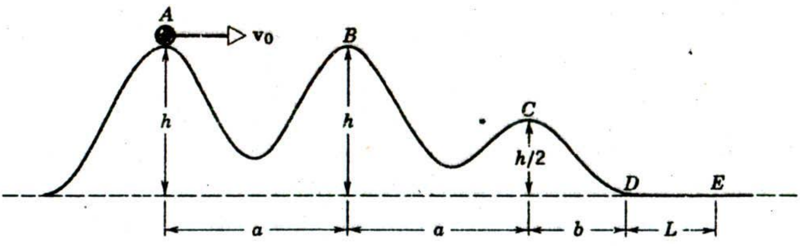


Fig. 8–12

10. A small block of mass m slides along the frictionless loop-the-loop track shown in Fig. 8–13. (*a*) If it starts from rest at P, what is the resultant force acting on it at Q? (*b*) At what height above the bottom of the loop should the block be released so that the force it exerts against the track at the top of the loop is equal to its weight?

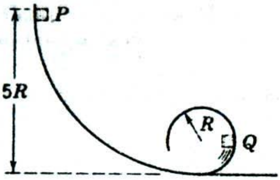


Fig. 8–13

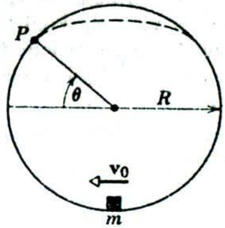


Fig. 8–14

11. The particle m in Fig. 8–14 is moving in a vertical circle of radius R inside a track. There is no friction. When m is at its lowest position, its speed is v_0. (*a*) What is the minimum value v_m of v_0 for which m will go completely around the circle without losing contact with the track? (*b*) Suppose v_0 is $0.775v_m$. The particle will move up the track to some point at P at which it will lose contact with the track and travel along a path shown roughly by the dashed line. Find the angular position θ of point P.

12. A point mass m starts from rest and slides down the surface of a frictionless solid sphere of radius r as in Fig. 8–15. Measure angles from the vertical and potential energy from the top. Find (*a*) the change in potential energy of the mass with angle; (*b*) the kinetic energy as a function of angle; (*c*) the radial and tangential accelerations as a function of angle; (*d*) the angle at which the mass flies off the sphere. (*e*) If there is friction between the mass and the sphere, does the mass fly off at a greater or lesser angle than in part (*d*)?

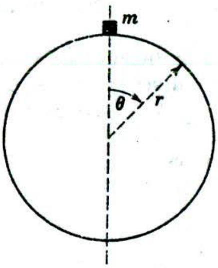


Fig. 8–15

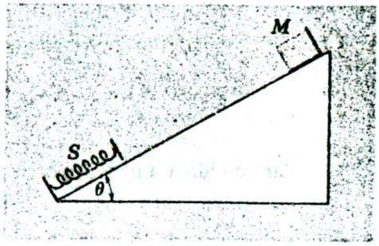


Fig. 8–16

13. An ideal massless spring S can be compressed 1.0 meter by a force of 100 nt. This same spring is placed at the bottom of a frictionless inclined plane which makes an angle of $\theta = 30°$ with the horizontal (see Fig. 8–16). A 10-kg mass M is released from rest at the top of the incline and is brought to rest momentarily after compressing the spring 2.0 meters. (*a*) Through what distance does the mass slide before coming to rest? (*b*) What is the speed of the mass just before it reaches the spring?

14. A body moving along the x-axis is subject to a force repelling it from the origin, given by $F = kx$. (a) Find the potential energy function $U(x)$ for the motion and write down the conservation of energy condition. (b) Describe the motion of the system and show that this is the kind of motion we would expect near a point of unstable equilibrium.

15. If the magnitude of the force of attraction between a particle of mass m_1 and one of mass m_2 is given by

$$F = k\frac{m_1 m_2}{x^2}$$

where k is a constant and x is the distance between the particles, find (a) the potential energy function and (b) the work required to increase the separation of the masses from $x = x_1$ to $x = x_1 + d$.

16. The magnitude of the force of attraction between the positively charged nucleus and the negatively charged electron in the hydrogen atom is given by

$$F = k\frac{e^2}{r^2}$$

where e is the charge of the electron, k is a constant, and r is the separation between electron and nucleus. Assume that the nucleus is fixed. The electron, initially moving in a circle of radius R_1 about the nucleus, jumps suddenly into a circular orbit of smaller radius R_2. (a) Calculate the change in kinetic energy of the electron, using Newton's second law. (b) Using the relation between force and potential energy, calculate the change in potential energy of the atom. (c) Show by how much the total energy of the atom has changed in this process. (The total energy will prove to have decreased; this energy is given off in the form of radiation.)

17. The potential energy corresponding to a certain two-dimensional force field is given by $U(x,y) = \frac{1}{2}k(x^2 + y^2)$. (a) Derive F_x and F_y and describe the vector force at each point in terms of its Cartesian coordinates x and y. (b) Derive F_r and F_θ and describe the vector force at each point in terms of the polar coordinates r and θ of the point. (c) Can you think of a physical model of such a force?

18. The so-called Yukawa potential

$$U(r) = -\frac{r_0}{r}U_0 e^{-r/r_0}$$

gives a fairly accurate description of the interaction between nucleons (that is, neutrons and protons, the constituents of the nucleus). The constant r_0 is about 1.5×10^{-15} meter and the constant U_0 is about 50 Mev. (a) Find the corresponding expression for the force of attraction. (b) To show the short range of this force, compute the ratio of the force at $r = 2r_0$, $4r_0$, and $10r_0$ to the force at $r = r_0$.

19. An α-particle (helium atom nucleus) in a large nucleus is bound by a potential like that shown in Fig. 8–17. (a) Construct a function of x, which has this general shape, with a minimum value U_0 at $x = 0$ and a maximum value U_1 at $x = x_1$ and

$x = -x_1$. (*b*) Determine the force between the α-particle and the nucleus as a function of x. (*c*) Describe the possible motions.

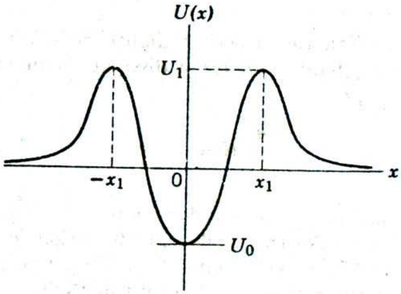


Fig. 8–17

20. A particle moves along a line in a region in which its potential energy varies as in Fig. 8–18. (*a*) Sketch, with the same scale on the abscissa, the force $F(x)$ acting on the particle. Indicate on the graph the approximate numerical scale for $F(x)$. (*b*) If the particle has a constant total energy of 4.0 joules, sketch the graph of its kinetic energy. Indicate the numerical scale on the $K(x)$ axis.

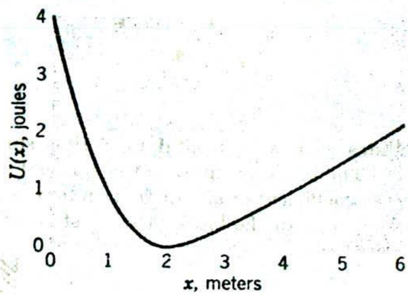


Fig. 8–18

21. A certain peculiar spring is found *not* to conform to Hooke's law. The force (in newtons) it exerts when stretched a distance x (in meters) is found to have magnitude $52.8x + 38.4x^2$ in the direction opposing the stretch. (*a*) Compute the total work

required to stretch the spring from $x = 0.50$ to $x = 1.00$ meter. (*b*) With one end of the spring fixed, a particle of mass 2.17 kg is attached to the other end of the spring when it is extended by an amount $x = 1.00$ meter. If the particle is then released from rest, compute its *speed* at the instant the spring has returned to the configuration in which the extension is $x = 0.50$ meter. (*c*) Is the force exerted by the spring conservative or nonconservative? Explain.

22. Show that when friction is present in an otherwise conservative mechanical system, the rate at which mechanical energy is dissipated equals the frictional force times the speed at that instant, or

$$\frac{d}{dt}(K + U) = -fv$$

23. A body of mass m starts from rest down a plane of length l inclined at an angle θ with the horizontal. (*a*) Take the coefficient of friction to be μ and find the body's speed at the bottom. (*b*) How far, d, will it slide horizontally on a similar surface after reaching the bottom of the incline. Solve by using energy methods and solve again using Newton's laws directly.

24. A particle slides along a track with elevated ends and a flat central part, as shown in Fig. 8–19. The flat part has a length $l = 2.0$ meters. The curved portions of the track are frictionless. For the flat part the coefficient of kinetic friction is $\mu_k = 0.20$. The particle is released at point A which is a height $h = 1.0$ meter above the flat part of the track. Where does the particle finally come to rest?

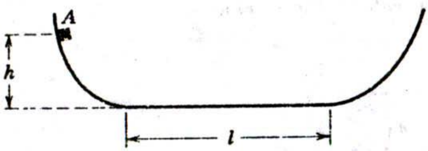


Fig. 8–19

25. A 1.0-kg block collides with a horizontal weightless spring of force constant 2.0 nt/meters (Fig. 8–20). The block compresses the spring 4.0 meters from the rest position. Assuming that the coefficient of kinetic friction between block and horizontal surface is 0.25, what was the speed of the block at the instant of collision?

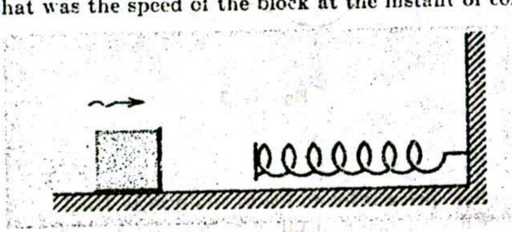


Fig. 8–20

26. The cable of a 4000-lb elevator in Fig. 8–21 snaps when the elevator is at rest at the first floor so that the bottom is a distance $d = 12$ ft above a cushioning spring whose spring constant is $k = 10{,}000$ lb/ft. A safety device clamps the guide rails so that a constant friction force of 1000 lb opposes the motion of the elevator. (*a*) Find the speed of the elevator just before it hits the spring. (*b*) Find the distance s that the spring is compressed. (*c*) Find the distance that the elevator will "bounce" back up the shaft. (*d*) Using the conservation of energy principle, find the total distance that the elevator will move before coming to rest.

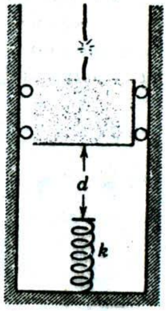


Fig. 8–21

27. A 40-lb body is pushed up a frictionless 30° inclined plane 10 ft long by a horizontal force **F**. (*a*) If the speed at the bottom is 2.0 ft/sec and at the top is 10 ft/sec, how much work is done by **F**? (*b*) Suppose the plane is not frictionless and that $\mu_k = 0.15$. What work will this same force do? How far up the plane does the body go?

28. A chain is held on a frictionless table with one-fifth of its length hanging over the edge. If the chain has a length l and a mass m, how much work is required to pull the hanging part back on the table?

29. An escalator joins one floor with another one 25 ft above. The escalator is 40 ft long and moves along its length at 2.0 ft/sec. (*a*) What power must its motor deliver if it is required to carry a maximum of 100 persons per minute, of average mass 5.0 slugs? (*b*) A 160-lb man walks up the escalator in 10 sec. How much work does the motor do on him? (*c*) If this man turned around at the middle and walked down the escalator so as to stay at the same level in space, would the motor do work on him? If so, what power does it deliver for this purpose? (*d*) Is there any (other?) way the man could walk along the escalator without consuming power from the motor?

30. Show that mc^2 has the dimensions of energy.

31. An electron (rest mass 9.1×10^{-31} kg) is moving with a speed $0.99c$. (*a*) What is its total energy? (*b*) Find the ratio of the Newtonian kinetic energy to the relativistic kinetic energy in this case.

32. What is the speed of an electron with a kinetic energy of (*a*) 100,000 ev, (*b*) 1,000,000 ev?

33. (*a*) The rest mass of a body is 0.010 kg. What is its mass when it moves at a speed of 3.0×10^7 meters/sec relative to the observer? At 2.7×10^8 meters/sec? (*b*) Compare the classical and relativistic kinetic energies for these cases. (*c*) What if the observer, or measuring apparatus, is riding on the body?

34. The United States consumed about 10^{15} watt-hr of electrical energy in 1960. How many kilograms of matter would have to be completely destroyed to yield this energy?

35. It is believed that the sun obtains its energy by a fusion process in which four hydrogen atoms are transformed into a helium atom with the emission of energy in various forms of radiation. If a hydrogen atom has a rest mass of 1.0081 atomic mass units (see Example 7) and a helium atom has a rest mass of 4.0039 atomic mass units, calculate the energy released in each fusion process.

36. A vacuum diode consists of a cylindrical anode enclosing a cylindrical cathode. An electron with a potential energy relative to the anode of 4.8×10^{-16} joule leaves the surface of the cathode with zero initial speed. Assume that the electron does not collide with any air molecules and that the gravitational force is negligible. (*a*) What kinetic energy would the electron have when it strikes the anode? (*b*) Take 9.1×10^{-31} kg as the mass of the electron and find its final speed. (*c*) Were you justified in using classical relations for kinetic energy and mass rather than the relativistic ones?