Relation between Linear and Angular Kinematics for a Particle Moving in a Plane

SUPPLEMENTARY TOPIC I

In Section 11–6 we discussed the relations between the linear and angular kinematic variables for a particle moving in a plane but confined to move in a circle about an axis at right angles to the plane. Such a particle might be any particle in a rigid body rotating about a fixed axis. Here we relax the restriction and allow the particle to move freely in the plane. A planet moving in an elliptical orbit about the sun is an example.

We start from Eq. 11–11, $\mathbf{r} = \mathbf{u}_r r$, in which, however, we now take *both* $\mathbf{u}_r$ and r to be variables; the particle is no longer confined to a circle of constant radius. We find the velocity by differentiation, or

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{u}_r \frac{dr}{dt} + r\frac{d\mathbf{u}_r}{dt}.$$

Equation 11–13 shows us that $d\mathbf{u}_r/dt = \mathbf{u}_\theta \omega$. Thus we can write

$$\mathbf{v} = \mathbf{u}_r \frac{dr}{dt} + \mathbf{u}_\theta \omega r, \tag{I–1}$$

which shows that $\mathbf{v}$ has two components, a radial component $v_r = dr/dt$ and a component at right angles, $v_\theta = \omega r$. If we hold r constant, then $dr/dt = 0$ and Eq. I–1 reduces to Eq. 11–14a as it must.

To find the acceleration we differentiate Eq. I–1, remembering that *all five* quantities on the right are variables. We obtain

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{u}_r \frac{d^2r}{dt^2} + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + (\mathbf{u}_\theta)\left(\omega\frac{dr}{dt} + r\frac{d\omega}{dt}\right) + (\omega r)\left(\frac{d\mathbf{u}_\theta}{dt}\right).$$

Now $d\mathbf{u}_r/dt = \mathbf{u}_\theta\omega$, $d\mathbf{u}_\theta/dt = -\mathbf{u}_r\omega$ (see Eq. 11–16), and $d\omega/dt = \alpha$. Substituting and rearranging leads us finally to

$$\mathbf{a} = \mathbf{u}_r\left(\frac{d^2r}{dt^2} - \omega^2 r\right) + \mathbf{u}_\theta\left(\alpha r + 2\omega\frac{dr}{dt}\right). \tag{I–2}$$

Once again, if r = a constant, then $dr/dt = d^2r/dt^2 = 0$ and Eq. I–2 reduces to Eq. 11–17, which we derived especially for this case.

The two new terms in Eq. I–2, $\mathbf{u}_r d^2r/dt^2$ and $\mathbf{u}_\theta 2\omega\, dr/dt$, need a little explanation. The first of these terms is simple and we can understand it by imagining that the particle moving in the plane is *not* rotating about the axis. If we put $\omega = \alpha = 0$ in Eq. I–2 this equation reduces to

$$\mathbf{a} = \mathbf{u}_r\frac{d^2r}{dt^2},$$

which is just the familiar acceleration of a particle moving along a straight line. Hence this term in Eq. I–2 gives the radial acceleration due to the change in the *magnitude* of $\mathbf{r}$, the other radial acceleration term arising from the changing *direction* of $\mathbf{r}$ as the particle rotates.

There are also two θ-directed acceleration terms. The first one, $\mathbf{u}_\theta\alpha r$, arises simply from the angular acceleration α of a particle in circular motion (r = constant) and is the tangential acceleration of Section 11–5. To understand the second term, $\mathbf{u}_\theta 2\omega\, dr/dt$, consider a man walking outward along a radial line painted on the floor of a merry-go-round. The merry-go-round is rotating with constant angular velocity ω so that its angular acceleration α is now zero. If the man were simply to stand still on the merry-go-round, ($dr/dt = 0$, and r = constant) his acceleration, as seen by an observer in a reference frame on the ground (see Eq. I–2), would be simply the familiar centripetal acceleration $-\mathbf{u}_r\omega^2 r$, directed radially inward. If he walks outward, however, $dr/dt \neq 0$ and then Eq. I–2 predicts that the ground observer would also measure a θ-directed acceleration given by $\mathbf{u}_\theta 2\omega v_r$, where $v_r = dr/dt$. This is called a *Coriolis acceleration.* It arises from the fact that even though the angular velocity of the man is constant his speed increases as r increases. Let us convince ourselves that this effect really exists.*

Figure I–1*a* shows the walking man (point P) as he appears to the ground observer at times t and $t + \Delta t$. We show at time t his radially directed velocity $\mathbf{v}_r\, (= \mathbf{u}_r\, dr/dt)$ and also a θ-directed velocity caused by the rotation of the merry-go-round and given by $\mathbf{v}_\theta (= \mathbf{u}_\theta\omega r)$. At a time Δt later each of these velocities has changed. The radial velocity has changed in direction, although its magnitude remains dr/dt. The θ-directed velocity has not only changed direction (we have learned to account for this as a centripetal acceleration), but, because the man has moved outward to a point at which the floor is moving faster, its *magnitude* has also changed, from ωr to $\omega(r + \Delta r)$.

Figure I–1*b* shows the change in velocity caused by the change in direction of the radial line along which the man is walking. If $\Delta\theta$ in the triangle shown is small enough, we have

$$\Delta v_r = v_r\,\Delta\theta.$$

Dividing by Δt and letting Δt approach zero yields

$$a' = \frac{dv_r}{dt} = v_r\frac{d\theta}{dt} = v_r\omega.$$

The change in tangential velocity caused by the fact that the man is moving radially outward is

$$\Delta v_\theta = \omega(r + \Delta r) - \omega r = \omega\Delta r.$$

* See "The Coriolis Effect," James E. McDonald, *Scientific American,* May 1952.

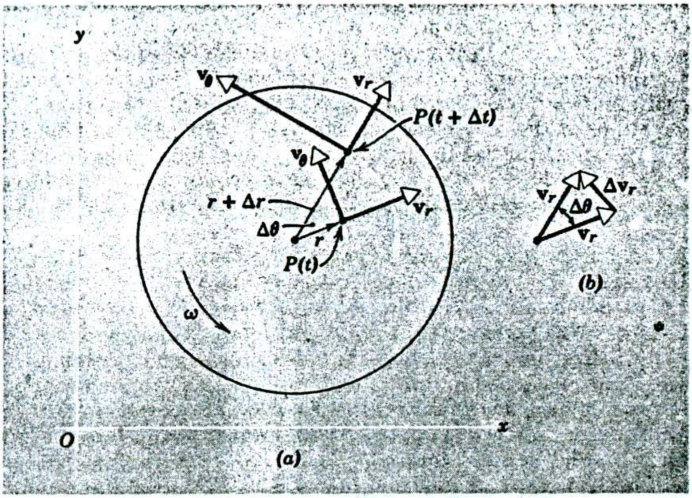

Fig. I–1 (*a*) A merry-go-round, rotating about a fixed axis, is observed by an observer in inertial reference frame x, y. A man walks along a radial line at constant speed v. In a time interval Δt this line, as seen by the ground observer, sweeps through an angle $\Delta\theta$ and the man moves between the positions shown. His r- and θ-directed velocities are shown for each position. (*b*) Showing the change $\Delta \mathbf{v}_r$ in the walking man's r-directed velocity. Note that, as $\Delta t \to 0$, $\Delta \mathbf{v}_r$ points in the θ-direction at P.

Dividing by Δt and letting Δt approach zero yields

$$a'' = \frac{dv_\theta}{dt} = \omega \frac{dr}{dt} = \omega v_r.$$

Now both a' and a'' are magnitudes of vectors that point in the same direction, namely the direction of increasing θ at point $P(t)$. The total acceleration in this direction is then

$$a' + a'' = v_r\omega + \omega v_r = 2\omega v_r,$$

which is just what we set out to prove.

If there is indeed an acceleration in the θ-direction in Fig. I–1, there must be a force in this direction. For a man walking outward along a radial line on a rotating merry-go-round this force can only be provided by the friction between his feet and the floor.

We remember that we can interpret classical mechanics most simply if we always view events from an inertial frame. If we do so we can always associate accelerations with forces exerted by bodies that we can point to in the environment. We can still apply classical mechanics, however, if we select a noninertial reference frame, such as a rotating frame. The small penalty that we must pay is that we must introduce *pseudo-forces*, that is, forces that we cannot associate with objects in the environment and which cannot be detected by an observer in an inertial frame. In Section 6–4 we saw that centrifugal force is such a pseudo-force.

Consider an observer on the rotating merry-go-round watching a man walk along a radial line at a constant speed $v_r = dr/dt$. He would say that the man is in equilibrium because he has no acceleration. Yet the floor is exerting a (very real) frictional force on the soles of the man's feet. This force has one component

$(-\mathbf{u}_r F_r)$ that points radially inward and one $(\mathbf{u}_\theta F_\theta)$ that points in the θ-direction, that is, in the direction of rotation.

From the point of view of the ground observer these forces are understandable and, indeed, quite necessary. F_r is associated with the centripetal acceleration $\omega^2 r$ and F_θ with the Coriolis acceleration $2\omega v_r$. The observer on the merry-go-round does not see either of these accelerations however; to him the walking man is in equilibrium. How can this be, in view of the frictional forces that act on the soles of the walking man's shoes? The man himself is well aware of these forces; if he did not lean to compensate for their turning effect, they would knock him off his feet!

The observer on the merry-go-round saves the situation by declaring that two pseudo-forces act on the walking man, just canceling the (real) frictional forces. One of these pseudo-forces, called the *centrifugal force*, has magnitude F_r and acts radially *outward*. The other, called the *Coriolis force*, has magnitude F_θ and acts in the negative θ-direction, that is, *opposite* to the direction of rotation. By introducing these forces, which seem quite "real" to him although he cannot point to any body in the environment that is causing them, the observer in the rotating (noninertial) reference frame can apply classical mechanics in the usual way. The ground observer, who is in an inertial frame, cannot detect these pseudo-forces. Indeed there is no need for them—and no room for them—in his applications of classical mechanics.

Equations I–1 and I–2 are general kinematical descriptions for the motion of a particle in two dimensions. An obvious extension, which we will not attempt here, is to derive corresponding descriptions for motion in three dimensions; this will require us to introduce a third unit vector to define the third dimension.*

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* See, for example, *Mechanics*, Section 3–5, by Keith R. Symon, Addison-Wesley Publishing Co., 2nd ed., 1960.

Polar Vectors and Axial Vectors

SUPPLEMENTARY TOPIC II

Some vectors called *axial vectors*, such as **ω**, **α**, **τ**, and **l**, differ in a rather important way from other vectors called *polar vectors*, of which **r**, **v**, **a**, **F**, and **p** are examples. Although we shall not need to take this difference into account in this book, it may prove to be instructive and interesting to the student to examine briefly what the difference is.

Consider a typical polar vector such as **r**. If a student leaves his dormitory and goes to a classroom, his displacement vector **r** points *from* the dormitory *to* the classroom; there is no question as to our choice of direction. This direction is both "physical" and "natural." Similar remarks apply to the other typical polar vectors listed, namely, **v**, **a**, **F**, and **p**.

If a student sees a wheel rotating about a fixed axis, he can assign an angular velocity **ω** to the wheel and can give direction to **ω** by the right-hand rule (see Section 11–4). This direction, however, *is a convention only*, based on this arbitrary rule. A left-hand rule would have given the opposite direction. The things that are "physical" and "natural" about the wheel are the axis of rotation and the sense of rotation, that is, is it going clockwise or counterclockwise as the student looks at it from a particular end of the axis? Whether **ω** is chosen to point in one way or the other along the axis does not really matter as long as we are consistent. The same remarks apply to the angular acceleration **α** and to the other axial vectors listed, namely $\boldsymbol{\tau}(= \mathbf{r} \times \mathbf{F})$ and $\mathbf{l}(= \mathbf{r} \times \mathbf{p})$. It is for this reason that we sometimes find it more comfortable to say "torque *around* an axis" than "torque *along* an axis" although they mean the same thing. All vectors defined as the vector product of two *polar* vectors are axial vectors because they all depend for their direction assignment on the (arbitrary) right-hand rule.

We have stressed that the laws of physics remain the same no matter how we change the inertial reference frame in which they are expressed. In Section 2–5 we discussed this for translations and rotations of the reference frame and noted that laws expressed in vector form remained unchanged (that is, *invariant*) under such transformations. We also noted that something special may occur when we change the reference frame in another way, namely, by substituting a left-handed

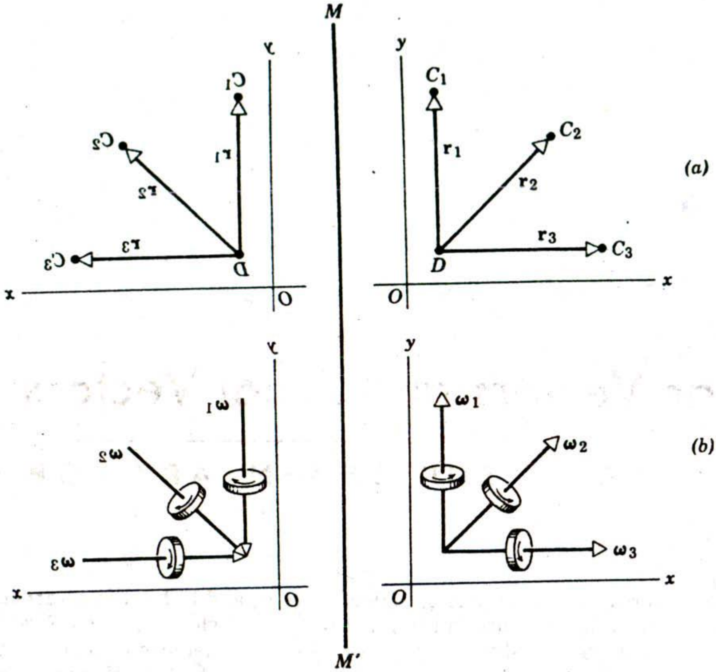

Fig. II–1 (*a*) *Polar vectors,* showing, on the right, the displacements $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ between a dormitory D and three classrooms C_1, C_2, and C_3. On the left we have the mirror images of D, C_1, C_2, and C_3, along with the corresponding displacements. (*b*) *Axial vectors,* showing, on the right, the angular velocities $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_3$ of three wheels rotating as shown. On the left we have the mirror images of these wheels, along with the angular velocities assigned using the usual right hand rule.

frame for a right-handed one. There is an easy way to make such a transformation: Build a right-handed frame and look at its image in a mirror; it will be converted to a left-handed frame (see Fig. II–1) because of the well-known property of a mirror to reverse right and left.

Figure II–1*a* shows the vector displacement of a student from his dormitory to each of three classrooms. In the mirror each displacement is *still* from the dormitory D to a classroom C. In Fig. II–1*b*, however, we show a rotating wheel in three orientations. If we establish the directions of $\boldsymbol{\omega}$ for both the wheels and their mirror images by the right-hand rule, we see that the image vectors are reversed in comparison to the corresponding image vectors in Fig. II–1*a* (toward the origin rather than away from the origin). Polar vectors and axial vectors behave differently when we transform reference frames by mirror reflection! This behavior of axial vectors under mirror reflection is not hard to understand. If we imagine ourselves physically applying the right-hand rule to a real rotating wheel, in the mirror, we shall *seem* to be applying a left-hand rule because the image of our right hand is our left hand. A left-hand rule, of course, will give us the opposite direction for $\boldsymbol{\omega}$.

Hence an axial vector is a vector whose sense of direction depends on the handedness of the reference frame. It is sometimes called a *pseudovector*. A polar vector is a vector that has a direction independent of the reference frame. We mention these facts (1) to stress the arbitrary character of the direction assigned to axial vectors and (2) to stress the importance of testing experiments and physical laws for invariance under translation, rotation, and mirror reflection of the inertial reference frame. In Section 2–5 we referred briefly to some experiments that were *not* invariant under a reflection transformation. This fact, which constituted a violation under certain circumstances of a law of physics previously thought to be well founded (the law of *conservation of parity*), has posed some challenging problems and is leading us to an understanding of the physical world at a deeper level.*

* See "The Overthrow of Parity," by Philip Morrison, *Scientific American*, April 1957.

The Wave Equation for a Stretched String

SUPPLEMENTARY TOPIC III

Figure III-1 shows a section of a long string which is under tension F. The string has been pulled transversely in the y-direction so that a displacement wave travels along the string in the x-direction. We consider a differential element of the string dx and apply Newton's second law of motion to it in order to find how the wave moves along the string.

Let μ be the mass per unit length of the string, so that the mass of element dx is $\mu\, dx$. The net force in the y-direction acting on this element is

$$F \sin \theta_{x+dx} - F \sin \theta_x.$$

We consider only small transverse displacements of the string, so that the restoring force will vary linearly with displacement and the principle of superposition will hold (see Section 19–4). This means that θ in Fig. III–1 will be small, so that we may replace $\sin \theta$ by $\tan \theta$. Now $\tan \theta$ is simply the slope of the string, that is, it equals $\partial y/\partial x$. We must use partial derivatives because the transverse displacement y depends not only on x but also on t. The net force in the y-direction is then

$$F\left(\frac{\partial y}{\partial x}\right)_{x+dx} - F\left(\frac{\partial y}{\partial x}\right)_x,$$

which we may write as

$$F\,\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) dx$$

or

$$F\,\frac{\partial^2 y}{\partial x^2}\, dx.$$

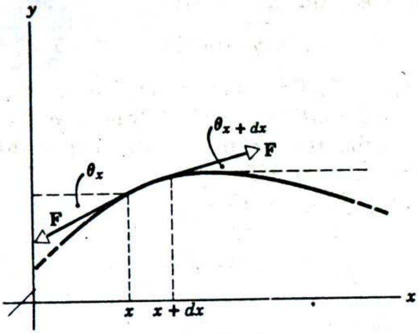

Fig. III–1

The mass of the element of the string is $\mu\, dx$ and its transverse acceleration is simply $\partial^2 y/\partial t^2$. Hence, Newton's second law, applied to the transverse motion of the string, is

$$F \frac{\partial^2 y}{\partial x^2}\, dx = (\mu\, dx) \frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F} \frac{\partial^2 y}{\partial t^2}. \tag{III–1}$$

Equation III–1, called the *wave equation*, is the differential equation that describes wave propagation in a string of mass per unit length μ and tension F.

To prove this we show that Eqs. 19–2 and 19–3

$$y = f(x \pm vt), \tag{III–2}$$

which is the general equation representing a wave of any shape traveling along x, is a solution of Eq. III–1. Recall that v in Eq. III–2 is the speed of the wave disturbance and f is *any* function of $(x \pm vt)$.

Let us see whether Eq. III–2 is indeed a solution of Eq. III–1 by substituting the former equation into the latter. To do so we note that the two second partial derivatives of y are

$$\frac{\partial^2 y}{\partial x^2} = f'' \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = v^2 f''$$

in which f'' is the second derivative of the function f of Eq. III–2 with respect to $(x \pm vt)$. Substitution of these derivatives into Eq. III–1 yields

$$f'' = \frac{\mu}{F} v^2 f'',$$

which we may write as (see Eq. 19–12)

$$v = \sqrt{\frac{F}{\mu}}. \tag{III–3}$$

Thus we conclude that Eq. III–2 is indeed a solution of the partial differential equation Eq. III–1 if the speed of the wave disturbance described by this equation is given by Eq. III–3.

In particular, let us check that Eq. 19–10

$$y = y_m \sin (kx \pm \omega t) \tag{19–10}$$

is a solution of Eq. III–1. We know that it must be because Eq. 19–10 is simply a special case of the general relation Eq. III–2, which we have just shown to be a solution. Even so it is instructive to test this important specific function of $(x \pm vt)$ by substitution into Eq. III–1.

The second derivatives of Eq. 19–10 are

$$\frac{\partial^2 y}{\partial x^2} = -k^2 y_m \sin (kx \pm \omega t)$$

and

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y_m \sin (kx \pm \omega t).$$

Substitution into Eq. III–1 yields

$$-k^2 y_m \sin (kx \pm \omega t) = \left(\frac{\mu}{F}\right) [-\omega^2 y_m \sin (kx \pm \omega t)]$$

or

$$\frac{\omega}{k} = \sqrt{\frac{F}{\mu}}.$$

Since $\omega/k = v$ (see Eq. 19–11), this relation is identical with Eq. III–3, and Eq. 19–10, as we expect, is indeed a solution of Eq. III–1.

Derivation of Maxwell's Speed Distribution Law

SUPPLEMENTARY TOPIC IV

Boltzmann, in 1876, derived the Maxwell speed distribution law from this line of argument: Let a uniform gravitational field **g** act on an ideal gas maintained at a fixed temperature T. The number of molecules per unit volume n will then decrease with altitude z according to the law of atmospheres (see Example 1, Chapter 17). From what we know about the statistical-mechanical interpretation of temperature, however, the speed distribution law—whose form we assume that we do not yet know—must remain the same at all altitudes because it depends only on the temperature. However this law determines the rate at which molecules move vertically in the atmosphere at any altitude and must thus be intimately related to the decrease of n with z. By exploring this relationship in detail we can, in fact, deduce the speed distribution law.

The weight of gas per unit area between the levels z and $z + dz$ in Fig. IV–1 is $nmg\, dz$ in which m is the mass of a single molecule. For equilibrium, this weight per unit area must equal the difference in pressure between z and $z + dz$, or

$$nmg\, dz = -dp \tag{IV–1}$$

in which we have inserted a minus sign because p decreases as z increases.

We can write the equation of state of an ideal gas, $pV = \mu RT$, as

$$p = nkT \tag{IV–2}$$

because $\mu = nV/N_0$, where N_0 $(= R/k)$ is Avogadro's number, the number of molecules per mole, and k is Boltzmann's constant. Combining Eqs. IV–1 and IV–2 yields

$$\frac{dp}{p} = \frac{dn}{n} = -\frac{mg}{kT}\,dz.$$

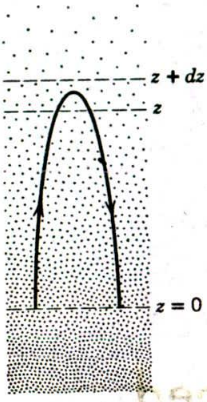

Fig. IV–1

For a constant temperature, we can integrate this relation to yield

$$n = \text{constant}\; e^{-mgz/kT} \tag{IV–3}$$

which, in view of Eq. IV–2, agrees with the result of Example 1, Chapter 17.

We can find the change in n as we go from z to $z + dz$ by differentiating Eq. IV–3, or

$$dn = -\,\text{constant}\; e^{-mgz/kT}\,dz. \tag{IV–4}$$

We associate this decrease in n over the interval dz with the fact that, at $z = 0$ (which can be any level we choose) there are some upward-directed molecules—we call them "special molecules" temporarily for convenience—whose vertical velocity components lie in a particular range v_z to $v_z + dv_z$ such that (neglecting collisions; see below) they can rise as high as z but not as high as $z + dz$. Such molecules pass upward through the level z, reverse their direction and pass downward again, as Fig. IV–1 shows. At this point we see more clearly the relationship between Eq. IV–3 and the speed distribution law. Molecules that pass through the interval dz (from above or below) or molecules that never reach the interval cannot contribute to the decrease dn of Eq. IV–4.

The rate per unit area at which "special molecules" leave level $z = 0$ (and arrive at level z) is $v_z n(v_z)\,dv_z$. Here $n(v_z)\,dv_z$ is the number of molecules per unit volume whose vertical velocity components lie between v_z and $v_z + dv_z$;

Now the rate per unit area at which the "special molecules" arrive at level z, but not as high as level $z + dz$, is proportional to the magnitude of the density difference dn between z and $z + dz$, or, from Eq. IV–4,

$$v_z n(v_z)\,dv_z = \text{constant}\; e^{-mgz/kT}\,dz, \tag{IV–5}$$

in which the constant is independent of z. Equation IV–5, which requires that the change dn be accounted for by the "special molecules" is, in fact, the defining equation for $n(v_z)$.

From conservation of energy the special molecules have the property that*

$$\tfrac{1}{2}mv_z^2 = mgz$$

or

$$mv_z\,dv_z = mg\,dz.$$

We use these two relations to eliminate z and dz from Eq. IV–5, obtaining, as the student should verify,

$$n(v_z)\,dv_z = \text{constant}\; e^{-mv_z^2/2kT}\,dv_z \tag{IV–6a}$$

* If we consider collisions this result is still true *on the average* for the many molecules that start at $z = 0$ with a given value of v_z and move to the interval z to $z + dz$, having $v_z = 0$ there, even though such molecules would follow very erratic paths because of the collisions.

in which $n(v_z)\,dv_z$ is the number of molecules per unit volume whose vertical velocity components lie between v_z and $v_z + dv_z$. Note that Eq. IV–6*a* does not contain g or z. The gravitational field of Fig. IV–1, introduced to allow us to calculate the speed distribution, has served its purpose. We may apply Eq. IV–6*a* to a gas for which $g = 0$ or in which gravitational effects are negligible. In such a case the vertical direction, which we have identified as the z-direction, no longer has any special meaning. That is, the speed distribution for one component of velocity should be the same for another component of velocity since there is no special or preferred direction in a gas in equilibrium free of external forces. Thus we can write

$$n(v_x)\,dv_x = \text{constant}\; e^{-mv_x^2/2kT}\,dv_x \qquad \text{(IV–6}b\text{)}$$

and

$$n(v_y)\,dv_y = \text{constant}\; e^{-mv_y^2/2kT}\,dv_y, \qquad \text{(IV–6}c\text{)}$$

for the other two velocity components.

We now seek to find Maxwell's speed distribution (Eq. 24–2); it is expressed in terms of the speed v, rather than in terms of the separate components v_x, v_y, and v_z. We are not concerned here with the direction of **v**, because we assume it to be completely random. We can represent any velocity **v** as a vector extending from the origin in Fig. IV–2; the projections of the vector in the x-y- and z-directions are v_x, v_y, and v_z, respectively. We commonly say that the axes of Fig. IV–2 define a "velocity space," which has many formal similarities to ordinary (or coordinate) space, in which the axes are x, y, and z.

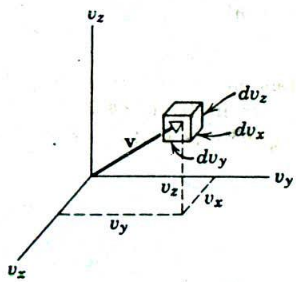

Fig. IV–2

We also show in Fig. IV-2*a* small "volume" element, whose sides are dv_x, dv_y, and dv_z; we say that this element has a volume $dv_x\,dv_y\,dv_z$ in velocity space. A point in this element corresponds to a particle whose velocity components lie between v_x and $v_x + dv_x$; v_y and $v_y + dv_y$; and v_z and $v_z + dv_z$. We can regard $n(v_z)$ in Eq. IV-6*a* as giving the *probability* that a given molecule will have a velocity component in the specified range v_z to $v_z + dv_z$, with similar interpretations for $n(v_x)$ and $n(v_y)$. The probability that a given molecule will have *all three* of its velocity components in the specified ranges, that is, the probability that the tip of the velocity vector **v** will lie inside the volume element of Fig. IV–2, is the *product* of the three (independent) probabilities given in Eq. IV–6, or

$$\text{constant}\; e^{-mv_x^2/2kT} e^{-mv_y^2/2kT} e^{-mv_z^2/2kT}\,dv_x\,dv_y\,dv_z$$

which, since

$$v^2 = v_x^2 + v_y^2 + v_z^2,$$

we may write as

$$\text{constant}\; e^{-mv^2/2kT}(dv_x\,dv_y\,dv_z). \qquad \text{(IV–7)}$$

The quantity in parentheses above is a volume element in velocity space. Since in Maxwell's speed distribution law we are not concerned with the direction of molecular velocities but only with their speeds, it is more convenient to substitute a different volume element for the above, namely one corresponding to all molecules whose speeds lie between v and $v + dv$, *regardless of direction*. This volume element is not a "cube" but is the space between two concentric spheres, one of radius v and one of radius $v + dv$. The volume of this element in velocity space is $(4\pi v^2)(dv)$. Substituting this for the quantity enclosed in parentheses in Eq. IV–7 yields for the number of molecules per unit volume whose speeds lie between v and $v + dv$,

$$n(v)\,dv = \text{constant}\; e^{-mv^2/2kT}(4\pi v^2\,dv)$$

or

$$n(v) = Cv^2 e^{-mv^2/2kT}$$

in which C is a constant. If we sum up over all possible speeds we simply obtain the total number of molecules per unit volume, regardless of speed. Hence, we can find C by requiring that

$$\int_0^\infty n(v)\,dv = n,$$

where n is the total number of particles per unit volume, regardless of speed. The student, guided by the methods of Example 3 (Chapter 24), should show that

$$C = 4\pi n(m/2\pi kT)^{3/2}$$

so that

$$n(v) = 4\pi n(m/2\pi kT)^{3/2} v^2 e^{-mv^2/2kT}. \tag{IV–8}$$

Let us consider a finite number N of molecules contained in a box of volume V. If we multiply each side of the above equation by V, we can replace nV on the right by N and $n(v)V$ on the left by $N(v)$, which gives us Eq. 24–2.