Appendix: Calculus Review

The rules for some useful derivatives are found in Table A–1. The reader will notice in rule (*a*) that the rate of change of a constant k with respect to a change in x is zero since it is evident that a constant, by definition, does not change. As noted in rule (*d*), the base of the natural logarithm raised to the power x has the strange property of remaining unchanged on differentiation.

TABLE A–1. ***Rules of Differentiation***

Function	Derivative	Rule
$y = k$	$dy/dx = 0$	(*a*)
$y = kx$	$dy/dx = k$	(*b*)
$y = kx^n$	$dy/dx = knx^{n-1}$	(*c*)
$y = ke^x$	$dy/dx = ke^x$	(*d*)
$y = k \ln x$	$dy/dx = k/x$	(*e*)
$y = u + v + w$	$dy/dx = du/dx + dv/dx + dw/dx$	(*f*)
$y = uv$	$dy/dx = u(dv/dx) + v(du/dx)$	(*g*)
$y = u/v$	$dy/dx = \dfrac{v(du/dx) - u(dv/dx)}{v^2}$	(*h*)

Several examples are given here to illustrate the use of the rules of Table A–1.

Example A–1. Differentiate $y = 2x^5 - \dfrac{4}{\sqrt[3]{x^2}} + 4x - 6$. According to rule (*f*), the derivative of y with respect to x is the sum of the derivatives of the separate terms. The separate functions are differentiated by the application of rule (*c*). Hint: The second right-hand term may be written $-4x^{-2/3}$.

$$\frac{dy}{dx} = (2 \times 5)x^{5-1} - \left(-\frac{4 \times 2}{3}\right) x^{-(2/3)-3/3} + 4x^{1-1} + 0$$

$$\frac{dy}{dx} = 10x^4 + \frac{8}{3\sqrt[3]{x^5}} + 4$$

Example A–2. Differentiate the product

$$y = 3x^4(x^2 + 2)$$

Applying rule (*g*), in which, in this case, $u = 3x^4$ and $v = (x^2 + 2)$, one obtains

$$\frac{du}{dx} = 12x^3 \text{ and } \frac{dv}{dx} = 2x$$

then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 3x^4(2x) + (x^2 + 2)12x^3$$

$$\frac{dy}{dx} = 6x^5 + 12x^5 + 24x^3 = 18x^5 + 24x^3$$

This particular problem is actually solved more simply by first multiplying out the terms on the right-hand side of the equation to give

$$y = 3x^6 + 6x^4$$

and then differentiating by rule (*f*) in Table A–1 to obtain directly

$$\frac{dy}{dx} = 18x^5 + 24x^3$$

Example A–3. Differentiate the quotient $y = \dfrac{2 \ln x}{\ln x + 1}$. Using rules (*e*) and (*h*), one proceeds as follows. Let $u = 2 \ln x$ and $v = \ln x + 1$. Then

$$\frac{du}{dx} = \frac{2}{x} \text{ and } \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

$$= \frac{(\ln x + 1)\dfrac{2}{x} - (2 \ln x)\dfrac{1}{x}}{(\ln x + 1)^2}$$

and, upon simplifying,

$$\frac{dy}{dx} = \frac{2}{x(\ln x + 1)^2}$$

At times an expression may appear too complicated to differentiate directly. When y is some function not of x but of u, or $y = f(u)$, which in turn is a function of x, or $u = f(x)$, three steps are used in the differentiation. For example, if

$$y = (x^2 + 1)^2$$

this is the form $y = f(u)$, in which $u = (x^2 + 1)$ is in turn some function of x.

(*1*) Let $u = x^2 + 1$, and first differentiate y with respect to u: $y = u^2$;

$$dy/du = 2u = 2(x^2 + 1)$$

(*2*) Then differentiate u with respect to x:

$$u = x^2 + 1;\ du/dx = 2x$$

(*3*) Now, it is observed that when the differential equations of steps (*1*) and (*2*) are multiplied together, du is eliminated and dy/dx is obtained:

$$dy/dx = dy/du \times du/dx = 2u \times 2x$$

$$dy/dx = 2(x^2 + 1)2x = 4x(x^2 + 1)$$

Example A- [illegible] $y = \ln(x^2 + 5)$, find dy/dx. First, let $u = x^2 + 5$.

$$y = \ln u;\ dy/du = \frac{1}{u};\ du/dx = 2x$$

Therefore, $dy/dx = dy/du \times du/dx = 2x/(x^2 + 5)$

Example A–5. If $y = e^{ax}$, find dy/dx.

Let $u = ax$, then $y = e^u$ and $dy/du = e^u = e^{ax}$, $du/dx = a$ and $dy/dx = ae^{ax}$.

SUCCESSIVE DIFFERENTIATION

In addition to serving as a necessary tool in integral calculus, differentiation allows one to compute the rate of change of the dependent variable, for example, distance in a falling body problem, with respect to the independent variable, for example, time. It is also useful for computing maxima and minima of various functions. These two applications are illustrated in the examples of the following sections.

We know from physics that the derivative of distance with respect to time ds/dt gives the velocity v of a body. It will also be recalled from physics that acceleration a is defined as the rate of change of velocity with time, v/t, or distance divided by the square of time, s/t^2. In incremental notation, the *average acceleration* over the distance Δs can be written as

$$a_{\text{aver.}} = \frac{\Delta v}{\Delta t} \equiv \frac{\Delta s}{(\Delta t)^2} \qquad \text{(A–1)}$$

The *instantaneous acceleration* at any time during the fall of a body is expressed by writing the limit of the ratio of the increments in equation (A–1). The change in velocity with respect to time dv/dt may be expressed as ds/dt taken a second time with respect to time, or $d(ds/dt)/dt$. This is known as the *second derivative with respect to time* and is written in the short-hand symbol d^2s/dt^2. Hence,

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \equiv \frac{dv}{dt} \equiv \frac{d^2s}{dt^2}$$

In general, beginning with the function $y = f(x)$ and differentiating y with respect to x gives the *first derivative* dy/dx. Differentiating again with respect to x gives the *second derivative*, d^2y/dx^2, and successive differentiations give the *third derivative* d^3y/dx^3, the *fourth derivative* d^4y/dx^4, and so on.

Example A–6. The equation from general physics relating the distance and time of fall of a body beginning at rest is $s = \frac{1}{2}gt^2$. The velocity after 4 seconds is given by

$$v = \frac{ds}{dt} = gt = 981 \times 4 = 3924 \text{ cm/sec.}$$

in which g is the acceleration of gravity, 981 cm/sec². What is the acceleration a at this time?

Taking the second derivative of distance with respect to time simply involves differentiating the result just given a second time:

$$a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = g$$

$$a = 981 \text{ cm/sec}^2$$

This result expresses the fact that the acceleration is constant at 981 cm/sec² throughout the fall. Taking the third derivative of distance with respect to time yields

$$\frac{d^3s}{dt^3} = 0$$

which shows that the rate of change of acceleration with time is zero. This is another way of saying that the acceleration is constant. See Problem 14 in Chapter 1.

From a geometric point of view, on a graph of y plotted against x, the first derivative dy/dx gives the slope of the line at any point, and the second derivative d^2y/dx^2 gives the rate of change of the slope with respect to x. The slope of the curve is equal to two times the value of x at any point, since $dy/dx = 2x$. The rate of change of the slope of the curve is constant at a value of 2 since $d^2y/dx^2 = 2$.

Maxima and Minima. Within the region where a curve slopes up to the right, dy/dx is positive. Where it slopes up to the left, dy/dx is negative. Where the curve is flat, exhibiting a maximum, a minimum, or a horizontal point of inflection, dy/dx is zero, and the tangent to the curve is a horizontal line.

The second derivative expresses the difference between these three possibilities. If the second derivative is positive for the value of x at the *critical value*, that is, at the point in qustion, the point represents a minimum; if the second derivative is negative, the point represents a maximum; and if the second derivative is zero, there may be a point of inflection (or there may be an unusually flat maximum or minimum or none of these).

Example A–7. Does the curve of the equation $y = x^2$ show a minimum or a maximum, and if so, what is its value?

The problem is solved by taking the first derivative and setting the result equal to zero.

$$\frac{dy}{dx} = 2x$$

$$2x = 0$$

Hence the curve shows a critical value at $x = 0$. The second derivative $d^2y/dx^2 = 2$ is positive, so that y has a *minimum* value at $x = 0$. By such a calculation, it can be shown that the buffer capacity of an acid buffer exhibits a maximum at $pH = pK_a$ where the concentrations of the salt and the acid are equivalent, as seen on page 174.

The Differential. Before considering the topic of partial differentiation, it is necessary to introduce the *differential*. The differential of y is dy and the differential of x is dx. If $y = \ln x$, $dy/dx = 1/x$ or, written as the differential of y,

$$dy = \frac{1}{x}\,dx$$

Beginning with the function, $y = 3x^2 + 4x - 3$, the differential of y is written,

$$dy = 6x\,dx + 4\,dx$$

Partial Differentiation. In these examples and all previous discussions, y has depended on only one variable, x. When y is a function of several variables, for example, $y = f(u,v,w)$, the total change in y, that is, the total differential dy, is the sum of the individual changes in y with respect to each of the variables. In partial differentiation, v and w are held constant while u is allowed to change, u and w are held constant while v changes, and u and v are held constant while w changes.

The symbol ∂ is used for partial differentiation, and the fundamental equation for the *total differential* of y is written,

$$dy = \left(\frac{\partial y}{\partial u}\right) du + \left(\frac{\partial y}{\partial v}\right) dv + \left(\frac{\partial y}{\partial w}\right) dw \qquad (A-2)$$

in which $\partial y/\partial u$, $\partial y/\partial v$, and $\partial y/\partial w$ are the partial derivatives of y with respect to the three variables.

Example A-8. Find the total differential of y for the function, $y = 2u^3 - 3uv^2 + 4w$, in which y is a function of the variables u, v, and w.

$$\frac{\partial y}{\partial u} = 6u^2 + 3v^2$$

$$\frac{\partial y}{\partial v} = 6uv$$

$$\frac{\partial y}{\partial w} = 4$$

$$\therefore dy = (6u^2 + 3v^2)\, du + 6uv\, dv + 4\, dw$$

Equation (A-2) is used in thermodynamics and may be applied here to the relationship between the volume V, temperature T, and pressure P of a gas. According to the ideal gas equation, the volume of a gas depends on the temperature and pressure, that is, $V = f(T,P)$, and the change of volume is given by the equation

$$dV = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T dP \qquad (A-3)$$

which expresses the fact that an infinitesimal change dV in V is obtained from the temperature coefficient of volume $\partial V/\partial T$ multiplied by the change in temperature dT plus the pressure coefficient of volume $\partial V/\partial P$ multiplied by the change in pressure dP. The partial derivatives are written as $(\partial V/\partial T)_P$ and $(\partial V/\partial P)_T$ to show that P is held constant while differentiating V with respect to T, and T is held constant while differentiating V with respect to P.

Figure A-1*a* is meant to be an infinitesimally small section of a surface representing the function, $V = f(T,P)$. The projection of this three-dimensional diagram to a V vs. T and a V vs. P plot are shown in Figures A-1*b* and A-1*c*. The general shapes of the curves in Figure A-1 agree with the ideal gas laws, that is, the volume increases linearly with temperature, and volume decreases as the pressure increases. The slope of the line *ab* or *cd* in Figure A-1*b* is the temperature coefficient of volume $(\partial V/\partial T)_P$, and the change in volume in going from a to b, $(V_b - V_a)$, is equal to the slope multiplied by the change in temperature dT, or $(\partial V/\partial T)_P\, dT$. The slope of the line *bd* or *ac* in Figure A-1*c* is $(\partial V/\partial P)_T$, and the change in volume in going from b to d, $(V_d - V_b)$, is $(\partial V/\partial P)_T\, dP$.

It is equally proper to carry out the pressure change first, proceeding in Figure A-1*a* from a to c, and then bring about the temperature change, passing from c to d. The total change in volume in going from a to d, viz., $V_d - V_a \equiv dV$, may be represented by a diagonal curve on the surface of the figure, A-1*a*. According to the method of partial differentiation, the total change is obtained by the two-step process:

$$V_d - V_a = (V_b - V_a) + (V_d - V_b)$$

or

$$dV = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T dP$$

The path by which we arrive at V_d from V_a has no influence on the value of dV. The differential depends only on the initial and final values V_a and V_d; thus, dV is said to be an *exact differential* and V is referred to as a *thermodynamic property*. These terms are used in the chapter on thermodynamics. Since V, T, and P are all independent variables, the change in any one can be

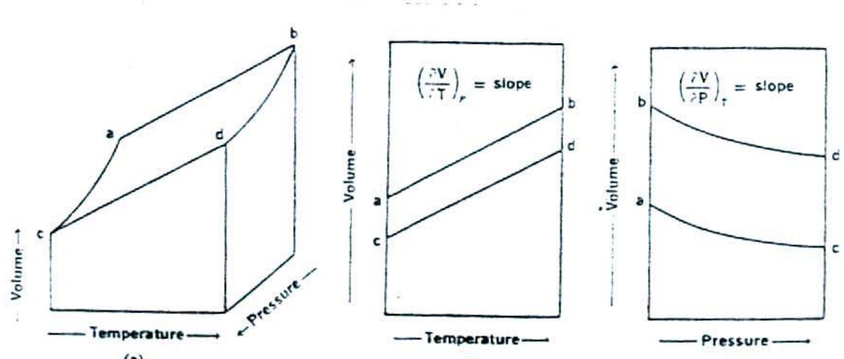

Fig. A-1. Graphical demonstration of the fundamental equation of partial differentiation. (After F. Daniels, *Mathematical Preparation for Physical Chemistry*, McGraw-Hill, New York, 1928, p. 179.)

obtained in terms of the others. For example, $P = f(T, V)$ and thus the infinitesimal change in P with changes in temperature and volume is

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \qquad (A-4)$$

INTEGRAL CALCULUS

Integration. Integration can be considered as the summation of infinitesimal elements, such as dy. The symbol for integration is an elongated s, written $\int$, and $\int dy = y$ means that the summation of the infinitesimal elements of y gives the whole value y. Integration is also considered as the reverse of differentiation in the same sense that division is the reverse of multiplication. When we divide 15 by 3, we obtain the answer by thinking of the number by which 3 is multiplied to yield 15. Similarly in calculus when we are asked to integrate $2x$, we attempt to recall the value that, when differentiated, yielded $2x$. The function $y = x^2$ and the more general expression $y = x^2 + C$ come to mind as possible answers, since, in either case,

$$\frac{dy}{dx} = 2x \qquad (A-5)$$

The symbol C stands for a constant. Equation (A–5) may also be written in the differential form for the purpose of integration where dy can be summed to give y:

$$dy = 2x\,dx$$

and we signify our intention to integrate by adding the integral sign to both sides of the equation.

$$\int dy = \int 2x\,dx$$

Both *integral signs must always be followed by differentials*, this being the reason for separating dy and dx in the previous step. The final result after integration is written:

$$y = x^2 + C \qquad (A-6)$$

This process of finding the function when the differential is given is known as *integration*. Thus, if the rate of change of y with x is known, it is possible by integration to obtain the functional relationship between the variables y and x. The constant of integration C has been added to "play safe," since it is quite possible that the value we are seeking in the integration process contained a constant that dropped out on differentiation.

The constant can be evaluated from the *boundary conditions* of the problem, that is, from the values of y for known values of x. If the function does not contain a constant term, the value of C will turn out to be zero, and no harm has been done by its inclusion. If, in the example just given, the boundary condition is given as $y = 5$ when $x = 3$, one obtains the following result by substituting the boundary condition into equation (A–6):

$$5 = 9 + C$$

or

$$C = 5 - 9 = -4$$

and substituting this value of C in equation (A–6) the final result becomes

$$y = x^2 - 4$$

Rules of Integration. The rules of integration are not obtained by mathematical derivations as were those of differentiation. Instead, the integration rules follow from a consideration of examples, such as the following

(*a*) If $y = x^5 + C$, $\dfrac{dy}{dx} = 5x^4$, or

$$dy = 5x^4\,dx$$

then

$$\int 5x^4\,dx = x^5 + C$$

(*b*) If $y = \dfrac{x^3}{3} + C$, $\dfrac{dy}{dx} = \dfrac{3x^2}{3} = x^2$, or

$$dy = x^2\,dx;$$

then

$$\int x^2\,dx = \frac{x^3}{3} + C$$

(*c*) If $y = x^{(n+1)} + C$, $\dfrac{dy}{dx} = (n + 1)x^n$

then

$$\int (n + 1)x^n\,dx = x^{(n+1)} + C$$

or in general $$\int x^n\,dx = \frac{x^{(n+1)}}{n + 1} + C$$

The integral $x^{-1}\,dx$ is a special case to which the general formula in (*c*) does not apply. This case is treated in (*d*)

(*d*) If $y = \ln x + C$, $\dfrac{dy}{dx} = \dfrac{1}{x}$ from Table A–1,

$$dy = dx/x;$$

then

$$y = \int \frac{dx}{x} = \ln x + C$$

The student should take particular note of the integral under (*d*), since it is frequently used in science. The

TABLE A–2. ***Summary of Several Important Derivatives and Integrals***

Function	Derivative	Integral
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C;\ n \neq -1$
$y = e^x$	$\frac{dy}{dx} = e^x$	$\int e^x\,dx = e^x + C$
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$	$\int \ln x\,dx = x(\ln x - 1) + C^*$
$y = \frac{1}{x}$	$\frac{dy}{dx} = -\frac{1}{x^2}$	$\int \frac{1}{x}\,dx = \ln x + C$

*This integration is done by the method illustrated in *Example A–11*.

most important derivatives and integrals are summarized in Table A–2. The integration of sums and differences and the treatment of constants, not shown in these examples, are best learned by studying the following problems.

Example A–9. Find y when the *differential equation*, as it is called, $dy/dx = 4x^3 + 6x^2 - 3$, is given.

$$\int dy = \int (4x^3 + 6x^2 - 3)\,dx$$

$$= 4\int x^3\,dx + 6\int x^2\,dx - 3\int dx$$

The constants are taken outside the integral signs since they are not changed by integration. Applying (*c*) to each term, the integration results in

$$y = 4\frac{x^4}{4} + 6\frac{x^3}{3} - 3x + C = x^4 + 2x^3 - 3x + C$$

Example A–10. Integrate $dy/dx = -2/x^{1/2}$ with $y = 2$ when $x = 4$

$$y = -2\int \frac{dx}{x^{1/2}} = -2\int x^{-1/2}\,dx$$

and by use of the general formula (*c*), above,

$$y = -2\int x^{-1/2}\,dx = \frac{-2x^{(-1/2+2/2)}}{(-\frac{1}{2}+\frac{2}{2})} + C = -4x^{1/2} + C$$

Then, employing the boundary conditions: $2 = -4 \times (4)^{1/2} + C$

$$C = 2 + 8 = 10$$

and finally

$$y = -4x^{1/2} + 10$$

If one is not sure of the answer, he or she should check it by differentiating the result:

$$y = -4x^{1/2} + 10$$

$$\frac{dy}{dx} = -\frac{4}{2}x^{(1/2-2/2)} = -2x^{-1/2} = -\frac{2}{x^{1/2}}$$

Example A–11. Integrate

$$dy = (3 + x^4)^{-1/2}x^3\,dx$$

Problems of this type are solved by introducing the function u. Let

$$u = 3 + x^4$$

then

$$du = 4x^3\,dx$$

It would therefore be possible to substitute

$$dy = u^{-1/2}\,du$$

for this function, except for the fact that

$$u^{-1/2}\,du = (3 + x^4)^{-1/2}4x^3\,dx$$

which differs from the original differential by a factor of 4. Therefore, we write

$$dy = \tfrac{1}{4}u^{-1/2}\,du = (3 + x^4)^{-1/2}x^3\,dx$$

The left-hand side is easily integrated to give

$$y = \frac{1}{4}\frac{u^{1/2}}{1/2} + C = \frac{(3 + x^4)^{1/2}}{2} + C$$

and this is obviously the solution of the original problem.

Exercises. Integrate the following expressions.

1. $dy = (5 - x^2)x\,dx$; with $y = 9$ when $x = 2$

Answer: $y = \frac{5}{2}x^2 - \frac{1}{4}x^4 + 3$

2. $dy = \frac{5 - x}{x^2}\,dx$

Answer: $y = C - \frac{5}{x} - \ln x$

[Hint: Write as

$$y = \int (5 - x)x^{-2}\,dx = \int (5x^{-2} - x^{-1})\,dx]$$

3. $dy = (3 + x^2)^3x\,dx$

Answer: $y = \frac{(3 + x^2)^4}{8} + C$

[Hint: Let $u = (3 + x^2)$; $du = 2x\,dx$. Then $(3 + x^2)^3 2x\,dx = u^3\,du$ and $\frac{1}{2}u^3\,du = (3 + x^2)^3x\,dx$.]

The Definite Integral. All previous formulas of integration have involved an arbitrary constant C, and these are known as *indefinite integrals*. When the integration is carried out between two definite values of x, the integration constant drops out of the result and the integral is called a *definite integral*. The process by which the definite integral is obtained is known as *integration between limits*. The definite integral of $f(x)\,dx$ is written

$$\int_a^b f(x)dx \qquad \text{(A–7)}$$

in which a and b represent the limits between which the integration is carried out. The process is described as follows. After $f(x)\,dx$ is integrated in the usual way, the limits b and a are substituted successively for x in the result, and the second quantity is subtracted from the first. The constant of integration disappears when the subtraction is carried out. The details of the method are illustrated in *Example A–12*. Notice that y is integrated between limits in the same manner as x.

Example A–12. Find the solution of the *differential equation*, as it is called, $dy/dx = 2x$, given that $y = 0$ when $x = 2$ and $y = a$ when $x = 3$.

$$\int_0^a dy = 2\int_2^3 x\,dx$$

$$[y]_0^a = [x^2 + C]_2^3$$

$$a - 0 = (9 + C) - (4 + C)$$

$$a = 5$$

Example A–13. The velocity of a body falling freely from rest is expressed by the equation $v = gt$. What is the distance (in cm) that the body has fallen between the third and the fourth second? The problem is solved by integration.

$$v = \frac{ds}{dt} = gt$$

The distance at $t = 3$ sec is given the symbol s_3 and the corresponding distance traveled at $t = 4$ sec is written s_4. The distance traveled between the third and fourth second is therefore $s_4 - s_3 = \Delta s$. It is solved by integration as follows.

$$\int_{s_3}^{s_4} ds = g\int_3^4 t\,dt$$

$$[s]_{s_3}^{s_4} = \tfrac{1}{2}g[(t^2)]_3^4 = \tfrac{1}{2}g(16 - 9) = \tfrac{7}{2}g$$

$$s_4 - s_3 = \Delta s = \tfrac{7}{2} \times 981 = 3434 \text{ cm}$$

Applications. The rate of disintegration of a radioactive element may be expressed as

$$\text{Rate} = \lambda N$$

$$-\frac{dN}{dt} = \lambda N$$

in which λ is the specific reaction rate and N the number of atoms remaining undecomposed at time t. The rate of decrease of radioactive atoms with time is written as $-dN/dt$, the negative sign being included because the number of atoms is decreasing with increasing time. Radioactive disintegration is one case of what is called *first-order* decomposition. The general expression for a first-order rate, which will be discussed in Chapter 12, is ordinarily written in the form of a differential equation

$$-\frac{dc}{dt} = kc \qquad (A-8)$$

in which c is the concentration of the substance decomposing at any time. It is desirable to integrate this equation so that k can be computed conveniently. The limits of the definite integrals are obtained by writing the initial concentration, i.e., the concentration at $t = 0$, as c_0 and the concentration at some other time t as c. Equation (A–8) is put into a convenient form for integration by separating the variables, that is, by collecting c and dc on one side of the equation and dt on the other side. The boundary limits are added, and the equation is ready for integration:

$$-\int_{c_0}^{c}\frac{dc}{c} = k\int_0^t dt$$

$$-[(\ln c)]_{c_0}^{c} = k[(t)]_0^t$$

$$(-\ln c) - (-\ln c_0) = k(t - 0)$$

$$kt = \ln\frac{c_0}{c} = 2.303\log\frac{c_0}{c}$$

$$k = \frac{2.303}{t}\log\frac{c_0}{c}$$

Based on the rules of logarithms, the solution may also be written

$$c = c_0e^{-kt}$$

or

$$c = c_0 10^{-kt/2.303}$$

The rate of growth of bacteria frequently may be expressed by a similar equation, $dN/dt = \alpha N$, in which N is the number of cells present at any moment t, and α is the *rate* constant. On integration between the limits N_0 at $t = L$ and N at time t, the equation is written,

$$\alpha = \frac{2.303}{(t - L)}\log\frac{N}{N_0}$$

in which L is the lag or induction period before the bacteria begin to follow the logarithmic growth law.

As an illustration of both differentiation and integration, we may consider a derivation of the ideal gas law. It follows from Boyle's law and Charles' law that the volume of one mole of an ideal gas is a function of the pressure and the temperature

$$V = f(P, T) \qquad (A-9)$$

and the total differential can be written

$$dV = \left(\frac{\partial V}{\partial P}\right)_T dP + \left(\frac{\partial V}{\partial T}\right)_P dT \qquad (A-10)$$

Now, at a fixed temperature, according to Boyle's law,

$$V = \frac{k_1}{P};\ k_1 = PV \qquad (A-11)$$

and at a fixed pressure, according to Charles' law,

$$V = k_2T;\ k_2 = \frac{V}{T} \qquad (A-12)$$

The partial derivative of equation (A–11) is given by the expression

$$\left(\frac{\partial V}{\partial P}\right)_T = -\frac{k_1}{P^2} = -\frac{PV}{P^2} = -\frac{V}{P} \qquad (A-13)$$

and the partial derivative of (A–12) is

$$\left(\frac{\partial V}{\partial T}\right)_P = k_2 = \frac{V}{T} \qquad \text{(A–14)}$$

Substituting these values into equation (A–10) gives

$$dV = -\left(\frac{V}{P}\right) dP + \left(\frac{V}{T}\right) dT$$

or by factoring V from the right-hand terms and dividing both sides by V, we have

$$\frac{dV}{V} + \frac{dP}{P} = \frac{dT}{T} \qquad \text{(A–15)}$$

Integration of equation (A–15) then yields

$$\ln V + \ln P = \ln T - \ln R \qquad \text{(A–16)}$$

The $\ln R$ term has been written in place of C, the integration constant. R is the molar gas constant.

The *equation of state*, that is, the equation relating P, V, and T for an ideal gas, is finally obtained by taking the antilogarithms of the terms in equation (A–16),

$$PV = RT$$

which, for n moles of gas, becomes

$$PV = nRT \qquad \text{(A–17)}$$