2

Theory of Sets

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Objectives

After studying this chapter, you should be able to understand :

- *set, elements of a set, method of describing a set, types of sets. venn diagrams, operations on sets.*

- *algebra of sets.*
- *cartesian product.*
- *set relations and its properties*
- *binary relations, functions and mappings.*

2·0. INTRODUCTION

The statements in the first chapter were concerned with individual objects. In sets we deal with a group of objects which can be defined in terms of their distinctive characteristics, magnitudes, etc. However, both the logical statements and sets belong to the same class. In the case of logical statements, we had three Boolean operators, *viz.*, conjunction ($\wedge$), disjunction ($\vee$) and negation ($\sim$). In set theory these are called as intersection $\cap$, union $\cup$ and complementation $\{\quad\}^c$ respectively.

Both these play an important role in modern mathematics. The logical statements and truth tables help in designing the circuits to perform Boolean operations, the sets are of much wider application, especially they help in preparing the programme for feeding into the machine. In almost whole of the business mathematics the set theory is applied in one form or the other.

2.1. A SET

A set is a collection of well-defined and well-distinguished objects. From a set it is possible to tell whether a given object belongs to a set or not. The following are some illustrations of a set :

(*i*) The possible outcomes in the toss of a die

(*ii*) The integers from 1 to 100

(*iii*) The vowels in English alphabets.

The basic characteristic of a set is that it should be well-defined ; its objects or elements should be well-distinguished for easy recognition by description.

2·2. ELEMENTS OF A SET

The objects that make up a set are called the members or elements of the set. It is almost a convention to indicate sets by capital letters like A, B, C or $X. Y, Z$ while the elements in the set by smaller or lower case letters, *viz.*, $a. b, c$ or x, y, z. Now, to indicate that a particular element or object "belongs to a set" or "a member of the set" we use the Greek symbol capital epsilon $\in$. For example, if x is the member of a set A, we shall indicate it symbolically as :

$x \in A$, *i.e.*, x is a member or an element of the set A

When we want to say that an object does not belong to a particular set, or is not a member of that set, we use the symbol '$\in$'. Thus, 'x is not a member of the set A' is symbolically expressed as : $x \in A$.

2·3. METHODS OF DESCRIBING A SET

The expression of sets has to be compact and clear otherwise the basic quality of the set being well-defined and distinctive is lost. Broadly, there can be two approaches : (*i*) to list the elements called the extension method, or (*ii*) to indicate the nature or characteristics and the limits within which the elements lie. The latter method is rather unavoidable if the elements are too numerous, or not real but only conceptual. These two approaches have been named variously as :

(*i*) Tabular, Roster or Enumeration method.

(*ii*) Selector, Property builder or Rule method.

(*i*) **Tabular Method.** Under this method we enumerate or list all the elements of the set within braces. However, there is no rigidity about these, use of even parentheses ( ) or brackets [ ] has been there in many books.

(*a*) A set of vowels : $A=\{a, e, i, o, u\}$

(*b*) A set of odd natural numbers : $N=\{1, 3, 5, ...\}$

(*c*) A set of Prime Ministers : $P=$ {Nehru, Shastri, Indira Gandhi}

(*ii*) **Selector Method.** Under this method the elements are not listed but are indicated by description of their characteristics. We may state some characteristics which an object must possess in order to be an element in the set.

Here we choose the letter x to represent an arbitrary element of the set and write

$A=\{x \mid x$ is a vowel in English alphabet$\}$

$B=\{x \mid x$ is an odd natural number$\}$

$C=\{x \mid x$ is a Prime Minister of India$\}$

The vertical line " | " after x to be read as 'such that'. Sometimes we use ':' to denote 'such that', *e.g.*,

$A=\{x : x$ is a vowel in English alphabet$\}$

It will be clear from the above that the tabular method is particularly useful when the elements are few in number while the set-builder method is more suitable when the elements are numerous.

2·4. TYPES OF SETS

Sets may be of various types. We give below a few of them.

I. Finite Set. When the elements of a set can be counted by a finite number of elements then the set is called a finite set. The following are the examples of finite sets :

$A=\{1, 2, 3, 4, 5, 6\}$

$B=\{1, 2, 3, ..., 500\}$

$C=\{x \mid x$ is an even positive integer $\leqslant 100\}$

$D=\{x \mid x$ is a article clerk registered with the Institute of Chartered Accountants of India$\}$

In all the above sets the elements can be counted by a finite number. It should be denoted that a set containing a very large number of elements is also a finite set. Thus, the set of all human beings in India, the set of all integers between -1 crore and $+1$ crore are all finite sets.

II. Infinite Set. If the elements of a set cannot be counted in a finite number, the set is called an infinite set. The following are examples of infinite sets :

$A=\{1, 2, 3, \ldots\ldots\}$

$B=\{x \mid x \text{ is an odd integer}\}$

$C=\{x \mid x \text{ is a positive integer divisible by } 5\}$

In all the above sets, the process of counting the elements would be endless, hence these are infinite sets.

III. Singleton. A set containing only one element is called a singleton or a unit set. For example,

$A=\{a\}$

$B=\{x \mid x \text{ is an integer neither} +ve \text{ nor} -ve \mid$

$C=\{x \mid \text{ is a perfect square of all positive integers and } 60<x<70\}$.

IV. Empty, Null or Void Set. Any set which has no element in it is called an empty set, or a null set or a void set. The symbol used to denote an empty set is a Greek letter ϕ (read as phi), *i.e.*, zero with a slish through it. Here the rule or the property discribing a given set is such that no element can be included in the set.

The following are a few examples of empty sets :

(*i*) The set of people who have travelled from the earth to the sun is an empty set because none has travelled so far.

(*ii*) $A=\{x \mid x \text{ is a perfect square of an integer, } 26<x<35\}$.

Thus we may say that laying down an impossible condition for the formation of a set produces an empty set.

V. Equal Sets. Two sets A and B are said to be equal if every element of A is also an element of B, and every element of B is also an element of A, *i.e*,

$$A=B \text{ if and only if } \{\iota\in A \Leftrightarrow x\in B\}$$

This is also known as '*Axiom of Extension* or '*Axiom of Identity*' : For example,

(*i*) Let $A=\{3, 5, 5, 9\}$, $B=\{9, 5, 3\}$

Then $A=B$ since each of the elements of A belong to B and each of the elements of B also belong to A.

(*ii*) Let $A=\{2, 3\}$, $B=\{3, 2, 2, 3\}$ and $C=\{x : x^2-5x+6=0\}$

This $A=B=C$ since each element which belongs to any one of the sets also belongs to the other two sets.

(*iii*) $A=\{x \mid x \text{ is a letter in the word 'march'}\}$,

$B=\{x \mid x \text{ is a letter in the word 'charm'}\}$

$C=\{a, c, h, m, r\}$

We find that $A=B=C$, as each set contains the same elements namely a, c, h, m, r irrespective of their order. Hence the sets are equal.

It may be noted that the order of elements or the repetition of elements does not matter in set theory.

VI. Equivalent Sets. If the elements of one set can be put into one to one correspondence with the elements of another set, then the two sets are called equivalent sets.

The symbol $\equiv$ is used to denote equivalent sets. For example,

(*i*) $A=\{a, b, c, d, e\}$ and $B=\{1, 2, 3, 4, 5\}$.

Here the elements of A can be put into one-to-one correspondence with those of B, thus

$$\begin{array}{ccccc} a, & b, & c, & d, & e \\ 1, & 2, & 3, & 4, & 5, \end{array}$$

Hence $A\equiv B$

(*ii*) $A=\{x \mid x \text{ is a letter in the word 'good'}\}$

$B=\{y \mid y \text{ is a letter in the word 'sets'}\}$

$\therefore \qquad A\equiv B$

In short if the total number of elements in one set are equal to the total number of elements in another set, then the two sets are equivalent.

(*iii*) $A=\{a, b, c, d, e\}$, $B=\{1, 2, 3, 4, 5\}$ and $C=\{c, a, a, e, d, b, b, c, d\}$

Here A and C are equal sets while A and B are equivalent sets.

VII. Subsets. If every element of a set A is also an element of a set B then set A is called subset of set B. Smbolically we write this relationship as

$$A\subseteq B$$

and is read as 'A is a subset of B' or 'A is contained in B' or 'A is included in B'. Sometimes this relationship is written as $B\supseteq A$ and is read as 'B is a superset of A' or 'B contains A' or 'B includes A'.

It may be noted that $A \subseteq B$ means that every element of A is also an element of B and there is no restriction on set B other than it includes the set A. Thus set A may be smaller than set B, when it contains some (not all) the elements of B. Set A may be equal to set B, when it contains all the elements of B. But set A cannot be larger than the set B, since in that case every element of A will not be an element of B, *i.e.*, we conclude that every element of a subset is an element of the superset but the reverse is not necessarily true.

$$A\subseteq B \qquad \text{if} \qquad x \in A \Rightarrow x\in B$$

when we write $A\not\subseteq B$, it would mean that A is not a subset of B. In this case we have to show that there is at least one element x such that $x \in A$ and $x\notin B$. Similarly $B \not\supseteq A$ would mean that B is not a superset of A.

Illustrations of Subsets. (*i*) $\{2, 3, 4, 5\}$, $B=\{2, 3, 4, 5, 6, 7\}$.

Here all the elements of A are also the elements of B.

$\therefore \qquad A\subseteq B$

(*ii*) If A is a set of books on Algebra in library, and B is a set of books on Mathematics in library then A is a subset of B. as every book on Algebra is also a book on Mathematics. But B is not a subset of A, as every book on Mathematics is not a book on Algebra.

(*iii*) Let $A=\{x \mid x \text{ is a +ve power of } 3\}$

i.e. $A=\{3, 3^2, 3^3, \ldots\ldots\}=\{3, 9, 27, \ldots\ldots\}$

and $B=\{x \mid x \text{ is an odd +ve integer}\}$

i.e., $B=\{1, 3, 5, 7, 9, 11 \ldots\ldots\}$

Clearly $A \subseteq B$ as each member of A will be odd and shall belong to B.

VIII. Proper Subsets. Set A is called proper subset of superset B if each and every element of set A are the elements of the set B and at least one element of superset B is not an element of set A. Symbolically this is written a '$A \subset B$' and is read as 'A is a proper subset of superset B'.

For example,

(*i*) $A=\{1, 2, 3, 5, 9, 12\}$

$B=\{1, 2, 2, 3, 5, 9, 12, 12\}$

$C=\{1, 2, 2, 3, 3, 5, 9\}$

Here $C \subset A$, $A \subseteq B$, $B \subseteq A$ and $A=B$.

(*ii*) $A=\{x \mid x$ is Fellow of the Institute of Chartered Accountants of India.$\}$

$B=\{x \mid x$ is a Chartered Accountant$\}$

Here all Fellows are Chartered Accountants. Thus all elements of A are also elements of B.

$$A \subset B$$

However $B \not\subseteq A$ (B is not a subset of A) as all Chartered Accountants (elements of B) are not Fellows (elements of A). A proper subset can also be denoted : $A \subset B$ if $A \subseteq B$ and $A \neq B$.

Remarks. (*i*) The symbol "$\subset$" and "$\subseteq$" or "$\supset$" and "$\supseteq$" are inclusion symbols.

(*ii*) The larger set is always at the open end of the sign $\subset$ or $\supset$ and $\subseteq$ or $\supseteq$.

(*iii*) If A is a subset of B, then B is called a superset of A.

(*iv*) If A is not a subset of B then, there exists at least one element in A which is not a member of B.

(*v*) A set is always a subset of itself, *i.e.*, $A \subseteq A$.

(*vi*) The null set is a subset of every set.

(*vii*) If A is a subset of B and B is a subset of C then A is a subset of C, *i.e.*,

$$A \subseteq B \text{ and } B \subseteq C \Rightarrow A \subseteq C$$

(*viii*) If A is subset of B and B is subset of A then the sets are said be equal, *i.e.*,

$$A \subseteq B,\ B \subseteq A \Rightarrow A = B$$

(*ix*) If $A \subseteq \phi$, then $A = \phi$.

IX. Family of Sets. *If all the* elements of a set are sets themselves then it is called a 'set of sets' or better term is a '*family of sets*'.

For example, if $A = \{a, b\}$ then the set

$A = [\phi, \{a\}, \{b\}, \{a, b\}]$ is a family of sets whose elements are subsets of the set A.

X. Power Set. From a set containing n elements, 2^n subsets can be formed. The set consisting of these 2^n subsets is called a power set. In other words, if A be a given set then the family of sets each of whose number is a subset of the given set A is called the power set of set A and is denoted as $P(S)$. For example,

(*i*) If $A = \{a\}$, then its subsets are ϕ, $\{a\}$

$\therefore \quad P(S) = [\phi, \{a\}]$

(*ii*) If $A = \{a, b\}$, then its subsets are ϕ, $\{a\}$, $\{b\}$, $\{ab\}$

$\therefore \quad P(S) = [\phi, \{a\}, \{b\}, \{ab\}]$

(*iii*) If $A = \{a, b, c\}$ then 2^3 subsets are

$$\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}$$

$\therefore \quad P(S) = [\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}]$

XI. Universal Set. When analysing some particular situation we are never required to go beyond some particular well-defined limits. This particular well-defined set may be called the universal set for that particular situation. Now onwards, we will assume that we are well aware of the particular universal set under consideration. The universal set will generally be denoted by the symbol 'U'.

Examples of Universal Sets :

(*i*) A set of integers may be considered as a universal set for a set of odd or even integers.

(*ii*) A set of Chartered Accountants in India may be considered as a universal set for a set of Fellows or Associates of C. As.

(*iii*) A deck of cards may be a universal set for a set of spade.

2·5. VENN DIAGRAMS

The Venn diagrams are named after English logician John Venn (1834-1923) to present pictorial representation. The universal set, say, U or X is denoted by a region enclosed by a rectangle and one or more sets say, A, B, C are shown through circles or closed curves within these rectangles. These circles or closed curves intersect each other if there are any common elements amongst them, if there are no common elements then they are shown separately as disjoints. Several set relations can be easily shown by these diagrams. These are useful to illustrate the set relations, such as the subset, set relations, and the set-operations such as intersection, union, complementation, etc. by using regions in a plane to indicate sets. But Venn-diagrams [also known as Venn Euler diagrams] cannot be used to prove any statements regarding sets, just as geometric figures cannot be used to prove geometric theorems. They are mere aids for searching appropriate proofs.

2·6. OPERATIONS ON SETS

In the first chapter, we studied the compounding of statements with the help of certain connectives or Boolean operators. On the same pattern we will now have operations on sets chiefly with intersection $\cap$, union $\cup$ and complementation ($\sim$). The algebraic properties, as we will see later are similar in both logical statements and sets.

2·7. INTERSECTION OF SETS

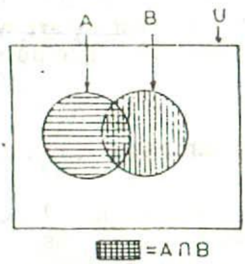
The intersecton of two sets A and B is the set consisting of all elements which belong to both A and B. The inter-section of A and B is denoted as $A \cap B$ which is read as "A cap B", or "A intersection B". Symbolically,

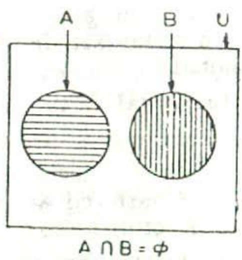
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

In other words

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

The following diagrams show how the intersection of sets can be expresed by Venn diagrams





Illustrations. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 6\}$

and $C = \{3, 4, 6, 8\}$, then

$$A \cap B = \{2, 4\}$$
$$(A \cap B) \cap C = \{4\}$$
$$B \cap C = \{4, 6\}$$
$$A \cap (B \cap C) = \{4\}$$

We notice that

$$A \cap (B \cap C) = (A \cap B) \cap C$$

2. If $A=\{x \mid x$ is an integer, $1 \leqslant x \leqslant 40\}$, and
$B=\{x \mid x$ is an integer, $21 \leqslant x \leqslant 100\}$, then
$A \cap B=\{x \mid x$ is an integer, $21 \leqslant x \leqslant 40\}$

Example 1. *Let* $A=\{0, 1, 2, 3, 4\}$, $B=\{2, 4, 5\}$, $C=\{0\}$ *and* $D=\phi$, *compute* $A \cap B$, $B \cap C$, $A \cap C$ *and* $C \cap D$.

Solution. $A \cap B=\{2, 4\}$, $B \cap C=\phi$, $A \cap C=\{0\}$ and $C \cap D=\phi$

The important properties of intersection are :

I. $A \cap B$ is the subset of both the set A and the set B. Symbolically
$(A \cap B) \subseteq A$ and $A \cap B \subseteq B$.

II. Intersection of any set with an empty set is the null set. Symbolically

$A \cap \phi=\phi$ for every set A

III. Intersection of a set with itself is the set itself. Symbolically
$A \cap A=A$ for every set A

IV. Intersection has commutative property, *i.e.*,
$A \cap B=B \cap A$

V. Intersection has associative property. For any three sets A, B and C

$(A \cap B) \cap C=A \cap(B \cap C)$.

Example 2. *Prove that*
$(A \cap B) \cap C=A \cap(B \cap C)$

Solution. In order to prove
$(A \cap B) \cap C=A \cap(B \cap C)$
we are required to prove :
$(A \cap B) \cap C \subseteq A \cap(B \cap C)$
and $A \cap(B \cap) \subseteq(A \cap B) \cap C$

(*i*) Let x be any arbitrary element of $(A \cap B) \cap C$

Then by definition of intersection,

$$\begin{aligned} x \in(A \cap B) \cap C &\Rightarrow x \in(A \cap B) \text{ and } x \in C \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C \\ &\Rightarrow x \in A \text{ and } x \in (B \cap C) \\ &\Rightarrow x \in A \cap (B \cap C) \end{aligned}$$

Thus every element x of $(A \cap B) \cap C$ is also an element of $A \cap(B \cap C)$

$\therefore \quad (A \cap B) \cap C \subseteq A \cap(B \cap C) \quad \ldots(1)$

(*ii*) Again, let y be any element of $A \cap(B \cap C)$

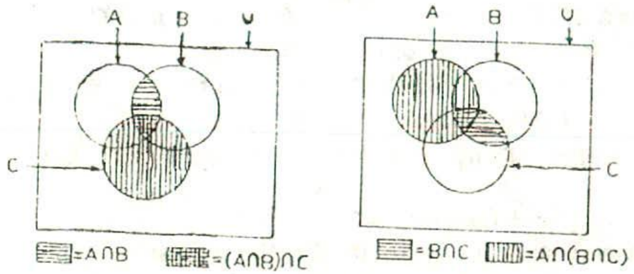
$$\begin{aligned} \text{If } y \in A \cap (B \cap C) &\Rightarrow y \in A \text{ and } y \in (B \cap C) \\ &\Rightarrow y \in A \text{ and } (y \in B \text{ and } y \in C) \\ &\Rightarrow (y \in A \text{ and } y \in B) \text{ and } y \in C \end{aligned}$$

$$\Rightarrow y \in (A \cap B) \text{ and } y \in C$$
$$\Rightarrow y \in (A \cap B) \cap C$$
$$\therefore \quad A \cap (B \cap C) \subseteq (A \cap B) \cap C \qquad ...(2)$$

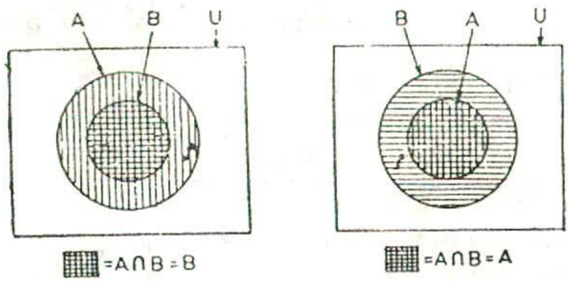
From (1) and (2), by the definition of equality of sets, we have

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Illustration by the help of Venn diagram



VI. If $B \subseteq A$ then $A \cap B = B$ and if $A \subseteq B$ then $A \cap B = A$. The above result can be illustrated by Venn diagram as shown below.



VII. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq (B \cap C)$

2·8. UNION OF SETS

The union of two sets A and B is the set consisting of all elements which belong to either A or B or both. The union of A and B is denoted as "$A \cap B$" read as "A cup B" or "A union B" Symbolically, we have

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or } x \in \text{ both } A \text{ and } B\}$$

In other words,

$$x \in A \cap B \Rightarrow x \in A \text{ or } x \in B$$

The union of two sets A and B is also called the logical sum of A and B.

Illustration 1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 6\}$ and

$$C = \{3, 4, 6, 8\}$$

then $\quad A\cup B=\{1, 2, 3, 4, 5, 6\}$, or $(A\cup B)\cup C=\{1, 2, 3, 4, 5, 6, 8\}$

$B\cup C=\{2, 3, 4, 5, 6. 8\}$ or $A\cup(B\cup C)=\{1, 2, 3, 4, 5, 6, 8\}$

$$A\cup(B\cup C)=(A\cup B)\cup C$$

2. If $A=\{1, 2, 3, 4\}$, $B=\{0\}$ and $C=\phi$, then

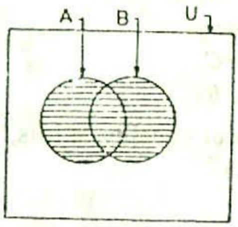
$A\cup B=\{0, 1, 2, 3, 4\}$, $A\cup C=\{1, 2, 3, 4\}$ and $B\cup C=\{0\}$

3. If $A=\{x \mid x$ is an integer, $1\leqslant x\leqslant 40\}$, and

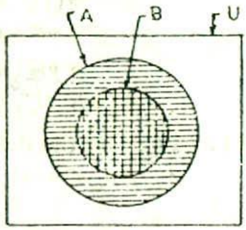
$B=\{x \mid x$ is an integer, $21\leqslant x\leqslant 100\}$, then

$A\cup B=\{x \mid x$ is an integer, $1\leqslant x\leqslant 100\}$

Union of two sets can be illustrated more clearly by Venn diagram as shown below :



=A∪B



=A∩B, Total shaded=A∪B

The total shaded region

$A\cup B=A,\ B\subseteq A$

Some important properties of the union are :

I. The individual sets composing a union are members of the union. In other words

$$A\subseteq(A\cup B) \text{ and } B\subseteq(A\cup B)$$

II. It has an identity property in an empty set. Therefore

$A\cup\phi=A$, for every set A

III. Union of a set with itself is the set itself, *i.e.*,

$A\cup A=A$, for every set A

IV. It has a commutative property, *i.e.*, for any two sets A and B

$$A\cup B=B\cup A$$

V. It has an associative property, *i.e.*, for any three sets A, B and C

$$(A\cup B)\cup C=A\cup(B\cup C)$$

Proof. For proving the associative property we are required to prove

(*i*) $(A\cup B)\cup C\subseteq A\cup(B\cup C)$ and

(*ii*) $A\cup(B\cup C)\subseteq(A\cup B)\subset C$

(*i*) Let x be any element of $(A\cup B)\cup C$. Then

$$x\in(A\cup B)\cup C \Rightarrow x\in(A\cup B) \text{ or } x\in C$$

$$\Rightarrow (x\in A \text{ or } x\in B) \text{ or } x\in C$$
$$\Rightarrow x\in A \text{ or } (x\in B \text{ or } x\in C)$$
$$\Rightarrow x\in A \text{ or } x\in(B\cup C)$$
$$\Rightarrow x\in A\cup(B\cup C)$$

i.e., every element of $(A\cup B)\cup C$ is also a member of $A\cup(B\cup C)$

$\therefore$ $$(A\cup B)\cup C\subseteq A\cup(B\cup C) \qquad \ldots(1)$$

(*ii*) Let y be any element of $A\cup(B\cup C)$

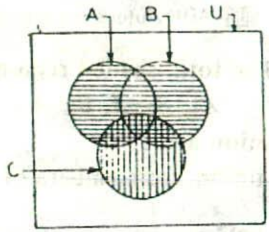
$\therefore$ $y\in A\cup(B\cup C) \Rightarrow y\in A$ or $y\in (B\cup C)$

$$\Rightarrow y\in A \text{ or } (y\in B \text{ or } y\in C)$$
$$\Rightarrow (y\in A \text{ or } y\in B) \text{ or } y\in C$$
$$\Rightarrow y\in(A\cup B) \text{ or } y\in C$$
$$\Rightarrow y\in(A\cup B)\cup C$$

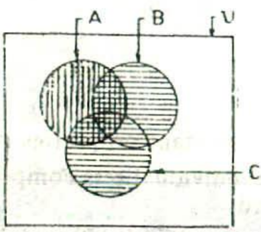
$\therefore$ $$A\cup(B\cup C)\subseteq (A\cup B)\cup C \qquad \ldots(2)$$

From (1) and (2), by the definition of equality of sets, we have

$$(A\cup B)\cup C=A\cup(B\cup C)$$



Total shaded region=(A U B)UC



Total shaded region=AU(B UC)

VI. If $B\subseteq A$, then $A\cup B=A$ and if $A\subseteq B$ then $A\cup B=B$.

VII. $A\cup B=\phi \Rightarrow A=\phi$ and $B=\phi$, in other words both are null sets.

VIII. $$A\cap B\subset A\subset A\cup B$$

To prove $A\cap B\subset A$, let us take

Proof $x\in A\cap B \Rightarrow x\in A$ and $x\in B$

So $x\in A$

Thus $A\cap B\subset A$

Again to show $A\subset A\cup B$, let us take

$y\in A$

$\Rightarrow$ $y\in A\cup B$

So $A\subset A\cup B$

Hence $A\cap B\subset A\subset A\cup B$

Distributive Laws o Unions and Intersections

I. In Algebra of sets, the union distributes over intersection which is not there in ordinary algebra.

Example 3. *Let A, B and C be any three sets, prove that*

$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$$

Also verify this result by means of a Venn diagram.

Solution. The law will stand proved if we prove the following results :

$$A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C)$$

and $$(A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C)$$

(*i*) Let x be any element of $A\cup(B\cap C)$ Then

$$\begin{aligned} x\in A\cup(B\cap C) &\Rightarrow x\in A \text{ or } x\in(B\cap C)\\ &\Rightarrow x\in A \text{ or } (x\in B \text{ and } (x\in C)\\ &\Rightarrow (x\in A \text{ or } x\in B) \text{ and } (x\in A \text{ or } x\in C)\\ &\Rightarrow x\in(A\cup B) \text{ and } x\in(A\cup C)\\ &\Rightarrow x\in(A\cup B)\cap(A\cup C)\end{aligned}$$

$\therefore$ $$A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C) \qquad ...(1)$$

(*ii*) Let y be any element of $(A\cup B)\cap(A\cup C)$.

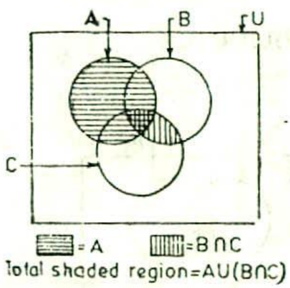
$$\begin{aligned}\therefore\ y\in(A\cup B)\cap(A\cup C) &\Rightarrow y\in(A\cup B) \text{ and } y\in(A\cup C)\\ &\Rightarrow (y\in A \text{ or } y\in B) \text{ and } (y\in A \text{ or } y\in C)\\ &\Rightarrow y\in A \text{ or } (y\in B \text{ and } y\in C)\\ &\Rightarrow y\in A\cup(B\cap C)\end{aligned}$$

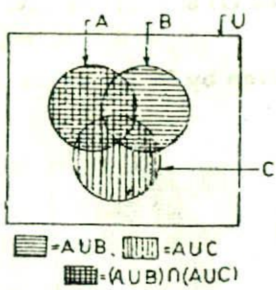
$$(A\cup B)\cap(A\cap C)\subseteq A\cup(B\cap C) \qquad ...(2)$$

From (1) and (2), we infer that

$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$$

Illustration by Venn diagram :





II. Intersection distributes over the union which is also there in ordinary algebra.

Example 4. Prove that : $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.
Also, verify this relation for the sets,
$A=\{1, 2, 3, 5\}$, $B=\{2, 3, 4, 6\}$ *and* $C=\{1, 2, 4, 5, 7\}$.

Solution. Here we have to prove that

$$A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C) \quad ...(1)$$

and

$$(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C) \quad ...(2)$$

New let us take

$$\begin{aligned} x\in A\cap(B\cup C) &\Rightarrow x\in A \text{ and } x\in(B\cup C)\\ &\Rightarrow x\in A \text{ and } (x\in B \text{ or } x\in C)\\ &\Rightarrow (x\in A \text{ and } x\in B) \text{ or } (x\in A \text{ and } x\in C)\\ &\Rightarrow x\in(A\cap B) \text{ or } x\ x\in(A\cap C)\\ &\Rightarrow x\in(A\cap B)\cup(A\cap C)\end{aligned}$$

Thus $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$

Again to show (2), let us take

$$\begin{aligned} y\in(A\cap B)\cup(A\cap C) &\Rightarrow y\in(A\cap B) \text{ or } y\in(A\cap C)\\ &\Rightarrow (y\in A \text{ and } y\in B) \text{ or } (y\in A \text{ and } y\in B)\\ &\Rightarrow y\in A \text{ and } (y\in B \text{ or } y\in C)\\ &\Rightarrow y\in A\cap(B\cup C)\end{aligned}$$

Thus $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$

Combining (1) and (2), we infer that

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$

$$B\cup C=\{1, 2, 3, 4, 5, 6, 7\}$$

$$\therefore \quad A\cap(B\cup C)=\{1, 2, 3, 5\} \quad ...(1)$$

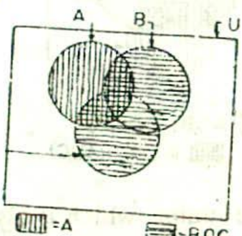
$$A\cap B=\{2, 3\}$$
$$A\cap C=\{1, 2, 5\}$$

$$(A\cap B)\cup(A\cap C)=\{1, 2, 3, 5\} \quad ...(2)$$

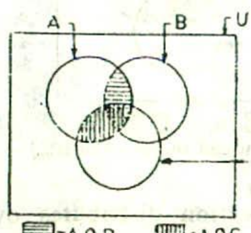
From (1) and (2), we have

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C).$$

Illustration by Venn diagram :



=A =B∩C
=A∩(B∩C)



=A∩B =A∩C
Total shaded region = (A∩B)∪(A∩C)

Example 5. *List the sets A, B, and C given that*

$A \cup B = \{p, q, r, s\}$; $A \cup C = \{q, r, s, t\}$;

$A \cap B = \{q, r\}$ *and* $A \cap C = \{q, s\}$.

Solution. Since $A \cap B = \{q, r\}$ and $A \cap C = \{q, s\}$,

$q, r, s \in A$; $q, r \in B$ and $q, s \in C$ ...(1)

Since $A \cup C = \{q, r, s, t\}$, $p \notin A$

Since $p \notin A$ and $A \cup B = \{p, q, r, s\}$, $p \in B$ ...(2)

Since $A \cup B = \{p, q, r, s\}$, $t \notin A$

Since $t \notin A$ and $A \cup C = \{q, r, s, t\}$, $t \in C$ ...(3)

Hence from (1), (2) and (3), we have

$A = \{q, r, s\}$

$B = \{p, q, r\}$

$C = \{q, s, t\}$

2·9 COMPLEMENT OF A SET

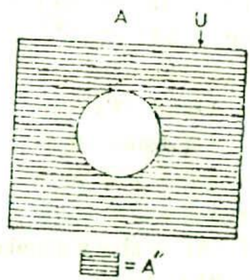
The complement of a set is the set of all those elements which do not belong to that set. In other words, if U be the universal set and A be any set then the complement of set A is the set $U - A$ and is denoted as A', A^c, $\bar{A}$ or $\sim A$. Symbolically,

$$A' = U - A = \{x : x \in U, x \notin A\} = \{x ; x \notin A\}$$

Illustration. *If* $U = \{1, 3, 5, 9, 10, 18\}$ *and* $A = \{3, 5, 10\}$

Then $U - A = \{1, 9, 18\}$

Venn diagram showing the complement of the set A in set U is given below :



The important properties of complementation are :

I. The intersection of a set A and its complement A' is a null set. In other words,

$$A \cap A' = \phi$$

II The union of a set A and its complement A' is the universal set. In other words,

$$A \cup A' = U$$

III. The complement of the universal set is the empty set and the complement of the empty set is the universal set. Symbolic expressions for the same are

$$U'=\phi \text{ and } \phi'=U$$

IV. The complement of the complement of a set is the set itself. Symbolic expression is

$$(A')'=A \text{ or } \sim(\sim A)=A,$$

V. If $A \subset B$ then $B' \subset A'$.

VI. Expansion or contraction of sets is possible by taking into account the complements of a set.

(*i*) $(A \cap B) \cup (A \cap B\ )=A$, (*ii*) $(A \cup B) \cap (A \cup B')=A$

Remark. $x \in A \cup B \Rightarrow x \in A$ or $x \in B$ (by def.)

But $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$

Proof. $\because$ $A \subseteq (A \cap B)$ and $B \subseteq (A \cap B)$

$\therefore$ $A \cap B$ is superset of both A and B.

$\therefore$ $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$

Conversely, if $x \notin A$ and $x \notin B$ then $x \notin A \cup B$.

2·10. DE-MORGAN 'S LAWS

I. *Complement of a union is the intersection of complements.*

Example 6. *Let A and B be any two sets, prove that*

$$(A \cup B)'=A' \cap B'$$

Solution. For proving the above law we have to prove the following two results :

(*i*) $(A \cup B)' \subset A' \cap B'$ and (*ii*) $A' \cap B' \subset (A \cup B)$.

(*i*) Let x be any element of $(A \cap B)'$

Then by def. of the complement,

$$\begin{aligned} x \in (A \cup B)' &\Rightarrow x \notin (A \cup B) \\ &\Rightarrow x \notin A \text{ and } x \notin B \\ &\Rightarrow x \in A' \text{ and } x \in B' \\ &\Rightarrow x \in A' \cap B' \end{aligned}$$

i.e., every member of $(A \cup B)'$ is also a member of $A' \cap B'$.

$$\therefore \quad (A \cup B) \subseteq A' \cap B \qquad (1)$$

(*ii*) Let y be any element of $A' \cap B'$

Then by def., of the intersection,

$$\begin{aligned} y \in A' \cap B' &\Rightarrow y \in A' \text{ and } y \in B' \\ &\Rightarrow y \notin A \text{ and } y \notin B \\ &\Rightarrow y \notin (A \cup B) \\ &\Rightarrow y \in (A \cup B)' \end{aligned}$$

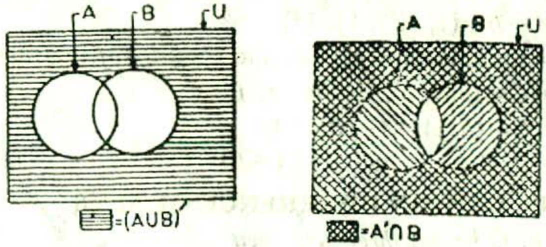
i.e., every member of $A' \cap B'$ is also a member of $(A \cup B)$

$$A' \cap B' \subseteq (A \cup B)' \quad \ldots(2)$$

From (1) and (2), we conclude that

$$(A \cup B)' = A' \cap B'$$

Illustration by Venr diagram :



II. *Complement of an intersection is the union of the complements,*

i.e., $(A \cap B)' = A' \cup B'$ for any two sets A and B.

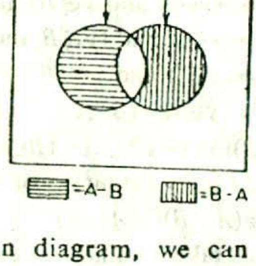
[Student is advised to do the proof independently on the same lines shown in the case of the proof for the preceding law.]

2·11. DIFFERENCE OF TWO SETS

The difference of two sets A and B is the set of all those elements which belong to A and not to B and is denoted by $A-B$ (also $A \sim B$) to be read as '*A difference B*'. Symbolically,

$A \quad B = \{x : x \in A$ and $x \notin B\}$ and $B - A = \{x : x \in B$ and $x \notin A\}$.

Difference of two sets can be shown by Venn diagram as follows :



From the above Venn diagram, we can conclude the following results :

(*i*) $A-B$ is the subset of A, *i.e.*, $A-B \subseteq A$ and $B-A$ is the subset of B, *i.e.*, $B-A \subseteq B$.

(*ii*) $A-B$, $A \cap B$ and $B-A$ are mutually disjoints.

(*iii*) $A-(A-B) = A \cap B$ and $B-(B-A) = A \cap B$.

Illustrations. (*i*) Let $A=\{a, b, c, d, e, f, g, h\}$, $B=\{a, e, i, o, u\}$ then

$$A-B=\{b, c, d, f, g, h\} \text{ and } B-A=\{i, o, u\}$$

Obviously $\quad A-B \neq B-A$

(*ii*) Let $A=\{1, 3, 5, 7, 9, 11, 13, 15\}$, $B=\{5, 9, 13, 17, 21\}$ then

$$A-B=\{1, 3, 7, 11, 15\}$$

(*iii*) If $A=\{a, b, c\}$, $B=\{b, c, d\}$ and $C=\{d, e, f, g\}$, then

$$A-C=\{a, b, c\},\ C-B=\{e, f, g\},$$
$$B-C=\{b, c\},\ A-(B-C)=\{a\}$$
$$A-B=\{a\}, \quad \text{and} \quad (A-B)-C=\{a\}$$

DE-MORGAN'S LAW ON DIFFERENCE OF SETS

I. If A, B, C *be any three sets, then*

$$A-(B\cup C)=(A-B)\cap(A-C)$$

Proof. To prove the law, we are required to prove

$$A-(B\cup C)\subseteq(A-B)\cap(A-C)$$

and $\quad (A-B)\cap(A-C)\subseteq A-(A\cup C)$

(*i*) Let x be an element of $A-(B\cap C)$. Then

$$\begin{aligned} x\in A-(B\cap C) &\Rightarrow x\in A \text{ and } x\notin(B\cap C)\\ &\Rightarrow x\in A \text{ and } (x\notin B \text{ and } x\notin C)\\ &\Rightarrow (x\in A \text{ and } x\notin B) \text{ and } (x\in A \text{ and } x\notin C)\\ &\Rightarrow x\in(A-B) \text{ and } x\in(A-C)\\ &\Rightarrow x\in(A-B)\cap(A-C)\end{aligned}$$

$\therefore \quad A-(B\cup C)\subseteq(A-B)\cap(A-C) \quad \ldots(1)$

(*ii*) Let y be an element of $(A-B)\cap(A-C)$

$$\begin{aligned}\therefore \quad y\in(A-B)\cap(A-C) &\Rightarrow y\in(A-B) \text{ and } y\in(A-C)\\ &\Rightarrow (y\in A \text{ and } y\notin B) \text{ and } (y\in A \text{ and } y\notin C)\\ &\Rightarrow y\in A \text{ and } (y\notin B \text{ and } y\notin C)\\ &\Rightarrow y\in A \text{ and } y\notin(B\cap C)\\ &\Rightarrow y\in A-(B\cap C)\end{aligned}$$

$(A-B)\cap(A-C)\subseteq A-(B\cup C) \quad \ldots(2)$

Hence from (1) and (2), by definition of equality of sets, we get

$$A-(B\cup C)=(A-B)\cap(A-C)$$

Illustration. Let us take a numerical example :

$$A=\{1, 2, 3, 4, 5, 6, 7, 8, 9\},\ B=\{3, 5, 7\}, \text{ and } C=\{2, 4, 6\}$$

Now $\quad B\cup C=\{2, 3, 4, 5, 6, 7\}$

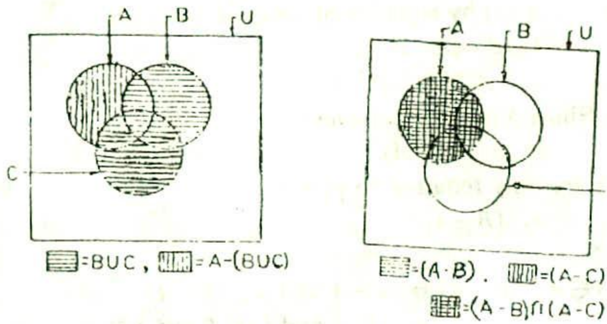
Therefore $A-(B\cup C)=\{1, 8, 9\}$

But $\quad A-B=\{1, 2, 4, 6, 8, 9\}$ and $A-C=\{1, 3, 5, 7, 8, 9\}$

Thus $\quad (A-B)\cap(A-C)=\{1, 8, 9\}$

$\therefore \qquad A-(B\cup C)=(A-B)\cap(A-C)$

The above law can also be shown through Venn diagrams as follows :



II. *If A, B, C be any three sets, then*

$$A-(B\cap C)=(A-B)\cup(A-C)$$

[Student is advised to do the proof independently on the same lines shown in the case of the proof for the preceding law.]

Some Useful Results on Difference, Union and Intersection.

I. *If A, B be any two sets, then*

$$A\cup B=(A-B)\cup B$$

Proof. To prove this result, we have to show that

$$A\cup B\subseteq(A-B)\cup B \text{ and } (A-B)\cup B\subseteq A\cup B$$

(*i*) Let x be any element of $A\cup B$. Then

$x\in A\cup B \Rightarrow x\in A$ or $x\in B$

$\Rightarrow x\in B$ or $x\in A$

$\Rightarrow (x\in B$ or $x\in A)$ and $(x\in B$ or $x\notin B)$ (*Note this step*)

$\Rightarrow x\in B$ or $(x\in A$ and $x\notin B)$

$\Rightarrow x\in B$ or $x\in(A-B)$

$\Rightarrow x\in(A-B)$ or $x\in B$

$\Rightarrow x\in(A-B)\cup B$

$\therefore \qquad A\cup B\subseteq(A-B)\cup B \qquad \ldots(1)$

(*ii*) Let y be any element of $(A-B)\cup B$. Then

$y\in(A-B)\cup B \Rightarrow y\in(A-B)$ or $y\in B$

$\Rightarrow y\in B$ or $y\in(A-B)$

$\Rightarrow y\in B$ or $(y\in A$ and $y\notin B)$

$\Rightarrow (y\in B$ or $y\in A)$ and $(y\in B$ or $y\notin B)$

$\Rightarrow (y\in A$ or $y\in B)$ and $(y\in B$ or $y\notin B)$

$$\Rightarrow y \in A \text{ or } y \in B$$
$$\Rightarrow y \in (A \cup B)$$

$\therefore$ $$(A-B) \cup B \subseteq (A \cup B) \qquad \ldots(2)$$

From (1) and (2) by equality of sets, we conclude

$$A \cup B = (A-B) \cup B$$

II. $$A \cap (B-A) = \phi$$

Proof. Since ϕ is subset of every set

$$\phi \subset A \cap (B-A) \qquad \ldots(1)$$

and hence we are only required to prove that

$$A \cap (B-A) \subset \phi$$

Now

$$\begin{aligned} x \in A \cap (B-A) &\Rightarrow x \in A \text{ and } x \in (B-A) \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \notin A) \\ &\Rightarrow x \in A \text{ and } (x \notin A \text{ and } x \in B) \\ &\Rightarrow (x \in A \text{ and } x \notin A) \text{ and } x \in B \\ &\Rightarrow x \in \phi \text{ and } x \in B \\ &\Rightarrow x \in \phi \cap B \\ &\Rightarrow x \in \phi \end{aligned}$$

$\therefore$ $$A \cap (B-A) \subseteq \phi \qquad \ldots(2)$$

From (1) and (2), we conclude that

$$A \cap (B-A) = \phi$$

III. $A-(A-B) = A \cap B$

IV. $A-B = A \Rightarrow A \cap B = \phi$

V. $A \cap (B-A) = \phi$,

VI. $A-B = A \cap -B$

Proof. To verify this we have to show that

$$A-B \subseteq A \cap -B \qquad \ldots(1)$$

and $$A \cap -B \subseteq A-B \qquad \ldots(2)$$

To prove (1) let

$$\begin{aligned} x \in A-B &\Rightarrow x \in A \text{ and } x \notin B \\ &\Rightarrow x \in A \text{ and } x \in -B \\ &\Rightarrow x \in A \cap -B \end{aligned}$$

[where $-B = U-B$, U being the universal set

Thus $$A-B \subseteq A \cap -B$$

Again to show the (2) let

$$\begin{aligned} y \in A \cap -B &\Rightarrow y \in A \text{ and } y \notin B \\ &\Rightarrow y \in A-B, \text{ whence} \end{aligned}$$
$$A \cap -B \subseteq A-B$$

Hence $\quad A-B=A\cap -B$

VII. $\quad A'-B'=B-A$

Proof. To prove the above, we have to show that

$$A'-B'\subseteq B-A \quad ...(1)$$

and $$B-A\subset A'-B' \quad ...(2)$$

Now let us take

$$x\in A'-B' \Rightarrow x\in A' \text{ and } x\notin B'$$
$$\Rightarrow x\notin A \text{ and } x\in B$$
$$\Rightarrow x\in B-A$$

implying thereby $\quad A'-B'\subseteq B-A$

To prove the second part let us take

$$y\in B-A \Rightarrow y\in B \text{ and } y\notin A$$
$$\Rightarrow y\notin B' \text{ and } y\in A'$$
$$\Rightarrow y\in A'-B', \text{ whence}$$
$$B-A\subseteq A'-B'$$

Hence $\quad A'-B'=B-A$

VIII. $\quad A\cap(B-C)=(A\cap B)-(A\cap C)$

Proof. Let $\quad x\in A\cap(B-C)\Rightarrow x\in A$ and $x\in(B-C)$

$$\Rightarrow x\in A \text{ and } (x\in B \text{ but } x\notin C)$$
$$\Rightarrow x\in(A\cap B) \text{ but } x\notin(A\cap C)$$
$$(\because\ x\notin C \Rightarrow x\notin A\cap C)$$
$$\Rightarrow x\in(A\cap B)-(A\cap C)$$

$\therefore$ $$A\cap(B-C)\subseteq(A\cap B)-(A\cap C) \quad ...(1)$$

Again let $y\in\{(A\cap B)-(A\cap B)\} \Rightarrow y\in A$ and $y\in A$ but $y\notin A\cap C$

$$\Rightarrow (y\in A) \text{ and } (y\in B \text{ but } y\notin A\cap C)$$
$$\Rightarrow y\in A \text{ and } (y\in B \text{ but } y\notin C)$$
$$\Rightarrow y\in A \text{ and } y\in(B-B)\ (\because\ y\notin A\cap C\Rightarrow y\notin C)$$
$$\Rightarrow y\subset A\cap(B-C)$$

$\therefore$ $$\{(A\cap B)-(A\cap C)\}\subseteq A\cap(B-C) \quad ...(2)$$

From (1) and (2) we conclude that

$$A\cap(B-C)=(A\cap B)-(A\cap C)$$

IX. $\quad A\cap(B-C)=(A\cap B)-C$

Proof. To prove the above we will prove that

$$A\cap(B-C)\subseteq(A\cap B)-C \quad ..(1)$$

and $$(A\cap B)-C\subseteq A\cap(B-C) \quad ...(2)$$

To prove the first part let

$$x\in(A\cap(B-C)\Rightarrow x\in A \text{ and } x\in(B-C)$$

$$\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \notin C)$$
$$\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \notin C$$
$$\Rightarrow x \in A \cap B \text{ and } x \notin C$$
$$\Rightarrow x \in (A \cap B) - C,$$

whence $$A \cap (B-C) \subseteq (A \cap B) - C$$

To prove the second part, let

$$y \in (A \cap B) - C \Rightarrow y \in A \cap B \text{ and } y \notin C$$
$$\Rightarrow (y \in A \text{ and } y \in B) \text{ and } y \notin C$$
$$\Rightarrow y \in A \text{ and } (y \in B \text{ and } y \notin C)$$
$$\Rightarrow y \in A \cap (B-C), \text{ which shows that}$$
$$\{(A \cap B) - C\} \subseteq A \cap (B-C)$$

Hence $$A \cap (B-C) = (A \cap B) - C$$

X. $$A \cup (B-C) \neq (A \cup B) - (A \cup C)$$

Proof. To prove the above let us take

$$x \in A \cup (B-C) \Rightarrow x \in A \text{ or } x \in (B-C)$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \notin C)$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \notin C)$$
$$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$$
$$\Rightarrow x \notin (A \cup B) - (A \cup C)$$

Hence $$A \cup (B-C) \neq (A \cup B) - (A \cup C)$$

XI. $$(A-B) \cap B = \phi$$

Proof. To prove the above let us suppose that

$$(A-B) \cap B \neq \phi$$

So there is at least one element, say, x, such that

$$x \in A - B \text{ and } x \in B$$
$$\Rightarrow (x \in A \text{ and } x \notin B \text{ and } x \in B) \text{ which is absurd}$$

Hence $$(A-B) \cap B = \phi$$

XII. $$(A-B) \cup (B-A) = (A \cup B) - (A \cap B)$$

Example 7. *Prove that :*

$$(A-B) \cup (B-A) = (A \cup B) - (A \cap B).$$

Solution. Let x be any arbitrary chosen element of the set $(A-B) \cup (B-A)$. Then

$$x \in (A-B) \cup (B-A) \Rightarrow x \in (A-B) \text{ or } x \in (B-A)$$
$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B)$$
$$\Rightarrow x \in (A \cup B) \text{ and } x \notin (A \cap B)$$
$$\Rightarrow x \in (A \cup B) - (A \cap B)$$

$$\therefore \quad (A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B) \quad \ldots(1)$$

Let y be any arbitrary element of the set $(A\cup B)-(A\cap B)$. Then

$$y\in(A\cup B)-(A\cap B)\Rightarrow y\in(A\cup B) \text{ and } y\notin(A\cap B)$$
$$\Rightarrow(y\in A \text{ or } y\in B) \text{ and } (y\notin A \text{ or } y\notin B)$$
$$\Rightarrow(y\in A \text{ and } y\notin B) \text{ or } (y\in B \text{ and } y\notin A)$$
$$\Rightarrow y\in(A-B) \text{ or } y\in(B-A)$$
$$\Rightarrow y\in(A-B)\cup(B-A)$$

$$(A\cup B)-(A\cap B)\subseteq(A-B)\cup(B-A) \quad \ldots(2)$$

From (1) and (2), we get

$$(A-B)\cup(B-A)=(A\cup B)-(A\cap B).$$

2·12. SYMMETRIC DIFFERENCE

A difference set is called a symmetric difference of two sets if it contains all those elements which are in set A and not in set B or those which are in set B and not in set A. For example, the symmetric difference of two sets A and B will be denoted by

$$A\triangle B=(A-B)\cup(B-A)$$

For example :

$$A=\{1, 2, 3, 4, 5, 6\},\ B=\{6, 7, 8\}$$

Then
$$A\triangle B=(A-B)\cup(B-A)$$
$$=\{1, 2, 3, 4, 5\}\cup\{7, 8\}$$
$$=\{1, 2, 3, 4. 5, 7, 8\}$$

thus element common to both A, B only is excluded.

The following are some important results :

I. $A\triangle B=B\triangle A$, $(A\triangle B)\triangle C=A\triangle(B\triangle C)$

II. $A\triangle A=\phi$, $A\triangle\phi=A$

III. $A\triangle(A\cap B)=(A-B)$

IV. $(A\triangle B)\cup(A\cap B)=A\cup B$

V. $A\triangle B=A\cup B-(A\cap B)$

It may be noted that it is like the exclusive connective OR, $\vee$ which does include $p\vee q$, $q\vee p$ but not $p\wedge q$.

2·13. ALGEBRA OF SETS

It is mere recapitulation of certain properties or laws governing operations on sets and in what way they differ from ordinary algebraic properties with which we are conversant.

Commutative law. Additions and multiplications of real numbers are commutative, *i.e.*,

$$a+b=b+a \quad \text{and} \quad a\times b=b\times a$$

Union and intersection of sets are also commutative. Let A and B be two sets, then

$$A\cup B=B\cup A \text{ and } A\cap B=B\cap A$$

Associative law. Additions and multiplications of ordinary numbers are associative indicated as follows :

(*i*) $(a+b)+c=a+(b+c)$

(*ii*) $(a\times b)\times c=a\times(b\times c)$

Union and intersection are also associative. Let A, B and C be three sets, then

$$(A\cup B)\cup C=A\cup(B\cup C) \text{ and } (A\cap B)\cap C=A\cap(B\cap C)$$

Distributive Law. Only one distributive law operates in ordinary algebra, *viz.*,

$$a\times(b+c)=(a\times b)+(a\times c)$$

It is not distributive across multiplication as shown below :

$$a+(b\times c)\neq(a+b)\times(a+c)$$

In the algebra of sets, we have

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$

as well as $$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$$

That is, in the algebra of sets, union distributes across intersection, also.

Idempotent law :

Whereas in ordinary algebra $a+a=2a$ and $a\times a=a^2$. But if A is a set then

$$A\cup A=A$$
$$A\cap A=A$$

This follow from the basic character of the sets in which we take into account the character of the elements and not the number of each type of elements. Two sets are equal if the type of elements are the same though the number of each might be different.

Identity law :

In ordinary algebra 0 was an identity element with respect to addition because zero added to any number gave the same number, *i.e.*, $a+0=a$. In the algebra of sets too the union of any set A and a null set ϕ is the same set A, *i.e.*,

$$A\cup\phi=A$$

Further in ordinary algebra "1" is an identity element with respect to multiplication, since multiplication of any number a by 1 yielded the same number as product, *i.e.*

$$a\times 1=a$$

In set algebra, the intersection of any set and the universal set U is the same set A, *i.e.*,

$$A\cap U=A$$

Because of these similarities we call $A\cup B$ as the logical sum and $A\cap B$ as the logical product.

Complement law :

For every subset A of any universal set U, there is one and only one complement of A, namely A'.

In ordinary algebra of numbers if 'a' is a fraction say 1/4, the complement of it would be 3/4 where

$$(1/4)+(3/4)=1 \text{ but } (1/4)\times(3/4)=3/16.$$

Thus the multiplication of a fraction by its complement does not result in zero.

2·14. Duality ;

It states that union and intersection are dual operations and that if we have established the validity of one law involving them then its dual, which is obtained through replacing $\cup$ by $\cap$ and $\cap$ by $\cup$ is also true. It is also possible to interchange U and ϕ when they appear with $\cup$ and $\cap$.

For example, the dual of

(*i*) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ is $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ and (*ii*) $(A\cap\phi \cup(U\cap A)=A$, has $(A\cup U)\cap(\phi\cup A)=A$ as its dual.

2·15. PARTITION OF A SET

Under partition of a set, a universal set U is sub divided into subsets which are disjoint but make into a union U, we can say

$$U=A_1\cup A_2\cup\ldots\ldots\cup A_n$$

such that if $a\in U$, it belongs to one and only one of the subsets. Therefore

$$A_i\cap A_j=\phi$$

The following diagram shows the partition of set U into four subsets A, B, C and D none of which has common elements.

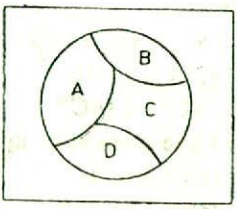


Illustration.

Let $U=\{1, 2, 3, 4, 5, 6, 7\}$

Now

(*i*) $[\{1, 2, 3\}, \{2,4\} \{5, 6, 7\}]$ is not a partition because 2 is in both the first and the second subsets.

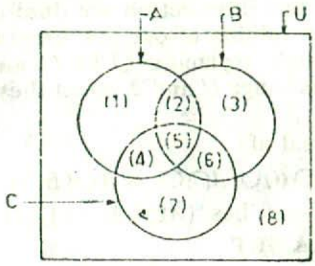
(*ii*) $[\{1, 5\}, \{3, 7\}, \{2, 4, 6\}]$ is a partition where no element is common and the union of the three subsets makes the set.

(*iii*) The following are two sets :

$$E=\{x : x\in I \text{ is even}\}$$
$$O=\{x : x\in I \text{ is odd}\}.$$

Now, every element of I is either in E or O but not in both. This relation brings about a partition of the set of integers.

However, if we include those elements in a set which do not form part of any of the subsets, then they form into a partition. In case of three subsets of a universal set, there are eight partitions, *i.e.*, 2^n or $2^3=8$ as shown in the diagram below.



The following are the various intersections :

(1) $A\cap B'\cap C'$ (2) $A\cap B\cap C'$ (3) $A'\cap B\cap C'$

(4) $A\cap B'\cap C$ (5) $A\cap B\cap C$ (6) $A'\cap B\cap C$

(7) $A'\cap B'\cap C$ (8) $A'\cap B'\cap C'$

Splitting up of Sets

The basic sets of A, B or C can be slit up into intermediate and ultimate sub-groups as follows :

$$\{A\}=\{AB\}+\{AB'\} \quad \ldots(1)$$

These can be further split-up into

$$=\{ABC\}+\{ABC'\}+\{AB'C\}+\{AB'C'\} \quad \ldots(2)$$

(1) above could also have been split up into

$$\{A\}=\{AC\}+\{AC'\} \quad \ldots(3)$$

$$=\{ACB\}+\{ACB'\}+\{AC'B\}+\{AC'B'\} \quad \ldots(4)$$

Similar, splitting-up can be of the basic sets of B and C. After excluding repeating sets and counting them only once, the ultimate subsets of the union of three basic sets, which are not disjoints, will be

$$\{A\cup B\cup C\}=\{AB'C'\}+\{ABC'\}+\{BA'C'\}+\{AB'C'\}+\{ABC\}+\{A'BC\}+\{A'B'C\} \quad \ldots(5)$$

The residual subset is $\{A'B'C'\}$ only.

2 16. REGROUPING OF THE SETS

The ultimate sub-sets can be regrouped into intermediate or the basic sets. The use of the splitting up or the regrouping process will depend on the nature of the problem, what is given and what is wanted. The process of regrouping has been illustrated below :

$$\{ABC'\}+\{ABC\}=\{AB\} \quad \ldots(1)$$

$$\{ACB'\}+\{ACB\}=\{AC\} \quad ...(2)$$
$$\{BCA\}+\{BCA'\}=\{BC\} \quad ...(3)$$
$$\{AB\}+\{AB'\}=\{A\} \quad ...(4)$$
$$\{BC\}+\{BC'\}=\{B\} \quad ...(5)$$
$$\{CA\}+\{CA'\}=\{C\} \quad ...(6)$$

The previous manipulations were simple but sometimes more involved ones are necessary. These are illustrated below :

$$\{ABC'\}=\{AB\}-\{ABC\} \quad ...(7)$$
$$\{AB'C\}=\{AC\}-\{ACB\} \quad ...(8)$$
$$\{A'BC\}=\{BC\}-\{ABC\} \quad ...(9)$$
$$\{AB'C'\}=\{A\}-\{AB\}-\{AC\}+\{ABC\} \quad ...(10)$$

In the above case (10) to have $\{AB'C'\}$ we have to remove from $\{A\}$ elements common to $\{AB\}$ and $\{AC\}$ but in doing so we remove the elements common to $\{ABC\}$ twice which is restored back. Hence

$$\{AB'C'\}=\{A\}-\{AB\}-\{AC\}+\{ABC\}$$

We will be able to appreciate this better if you look at the diagram on page 49 particlarly the 4 sub-sets of A. Likewise we have

$$\{BC'A'\}=\{B\}-\{BC\}-\{BA\}+\{BCA\}$$
$$\{CA'B'\}=\{C\}-\{CA\}-\{CB\}+\{CAB\}$$

In these we look at the 4 subsets, each of the set B and the set C respectively.

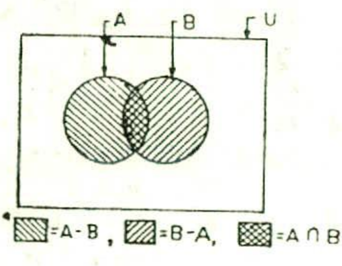
2·17. NUMBER OF ELEMENTS IN A FINITE SET

From operations on abstract sets we now switch over to the numbers attached to a set which is of great practical utility in finding out the values of new sets formed through operations on some basic sets. Therefore, we find it convenient to introduce a symbol "$n(A)$" to denote the number of elements in a set A. In this section, we derive a formula for $n(A \cup B)$ in terms of $n(A)$, $n(B)$ and $n(A \cap B)$. First we observe that if A and B are disjoint, *i.e.*, if $A \cap B=\phi$, then

$$n(A \cup B)=n(A)+n(B) \quad ...(1)$$

Next we take the case of the union of two finite sets which are not mutually disjoint meaning thereby that they have some elements common to them.

Consider the following Venn diagram.



We observe that

$$A \cup B = (A-B) \cup (B-A)(AB) \qquad \ldots(2)$$

and also we observe that

$$(A-B) \cap (B-A) = \phi,$$
$$(A-B) \cap (A \cap B) = \phi,$$
$$(B-A) \cap (A \cap B) = \phi,$$

i.e., $A \cup B$ is the disjoint union of $A-B$, $A \cap B$ and $B-A$.

$$n(A \cup B) = n(A-B) + n(A \cap B) + n(B-A) \qquad \ldots(3)$$

Since A and B are finite sets, let us assume that

$$n(A) = p,\ n(B) = q,\ n(A \cap B) = r$$

Then, we have

$$n(A-B) = p-r,\ n(B-A) = q \quad r$$

Substituting these values in (2), we get

$$n(A \cup B) = (p-r) + (q-r) + r = p + q - r$$
$$= n(A) + n(B) - n(A \cap B) \qquad \ldots(4)$$

Now for the union of any three sets A, B and C, which are not mutually disjoint, we have

$$n(A \cup B \cup C) = n[A \cup (B \cup C)]$$
$$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$$
$$= n(A) + [n(B) + n(C) - n(B \cap C)]$$
$$- n[(A \cap B) \cup (A \cap C)] \quad \text{(using Distributive Law)}$$
$$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

If the sets A, B and C are mutually disjoint, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C)$$

Example 8. *A company studies the product preferences of 20,000 consumers. It was found that each of the products A, B, C was liked by 7020, 6230, and 5980 respectively and all the products were liked by 1500 ; products A and B were liked by 2580, products A and C were liked by 1200 and products B and C were liked by 1950. Prove that the study results are not correct.*

Solution. Let A, B, C denote the set of people who like products A, B, C respectively.

The given data means

$$n(A) = 7420, \quad n(A \cap B) = 2580, \quad n(A \cap B \cap C) = 1500$$
$$n(B) = 6230, \quad n(A \cap C) = 1200, \quad n(A \cup B \cup C) = 20{,}000$$
$$n(C) = 5980, \quad n(B \cap C) = 1950.$$

We also know that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$$
$$- n(B \cap C) + n(A \cap B \cap C)$$

$$=7020+6230+5980-2580-1200-1950+1500$$
$$=15{,}000\neq 20{,}000$$

This shows that the data is not consistent.

Example 9. *In a class of 25 students, 12 students have taken economics : 8 have taken economics but not politics. Find the number of students who have taken economics and politics and those who have taken politics but not economics.*

Solution. Total number of students $=25$

$n(A)=$ Number of students taking economics $=12$

$n(B)=$ Number of students taking politics.

We have to find $n(A\cap B)$ and $n(B\cap A')$

Now $\qquad n(A)=n(A\cap B')+n(A\cap B) \quad [\quad A=(A\cap B')\cup(A\cap B)]$

$\Rightarrow \qquad 12=8+n(A\cap B)$

$\Rightarrow \qquad n(A\cap B)=12-8=4$

Also $\qquad n(A\cup B)=n(A)+n(B)-n(A\cap B)$

$\Rightarrow \qquad 25=12+n(B)-4$

$\qquad n(B)=17$

Again $\qquad n(B)=n(A\cap B)+n(A'\cap B)$

$\Rightarrow \qquad 17=4+n(A'\cap B)$

$\therefore \qquad n(A'\cup B)=17-4=13$

Example 10. *Out of 880 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball : of the total 64 played both basketball and hockey ; 80 played cricket and basketball and 40 played cricket and hockey : 24 boys played all the three games. How many boys did not play any game, and how many played only one game ?*

Solution. Let the C, H and B denote the sets of player playing Cricket, Hockey and Basket-ball respectively. Now we are given

$$n(C)=224, \quad n(H)=240, \quad n(B)=336,$$
$$n(H\cap B)=64, \quad n(C\cap B)=80, \quad n(C\cap H)=40$$

and $\qquad n(C\cap H\cap B)=24, \qquad$ and $\qquad n(S)=880$

(*i*) Number of players who played at least one game is given by

$$n(C\cup H\cup B)=n(C)+n(H)+n(B)-n(C\cap H)-n(H\cap B)-n(C\cap B)+n(C\cap H\cap C)$$
$$=224+240+336-40-64-80+24=824-184=640$$

$\therefore$ Number of boys who did not play any game

$$=n(S)-n(C\cup H\cup B)=880-640=240$$

(*ii*) Now $n(C\cap H)=n(C\cap H\cap B)+n(C\cap H\cap B')$

$\Rightarrow \qquad 40=24+n(C\cap H\cap B')$

$\Rightarrow \qquad n(C\cap H\cap B')=40-24=16$

Similarly

$n(C \cap B \cap H') = 80 - 24 = 56$, and $n(B \cap H \cap C') = 64 - 24 = 40$

Number of boys who played only basket ball can be obtained from

$$n(B) = n(B \cap H' \cap C') + n(B \cap H \cap C) + n(B \cap H \cap C') + n(B \cap H \cap C)$$

$$\Rightarrow \quad 336 = n(B \cap H' \cap C') + 56 + 40 + 24$$

$$n(B \cap H' \cap C') = 216$$

Hence number of boys who played only basket ball are 216.

Also
$$n(H) = n(H \cap B' \cap C') + n(H \cap B \cap C') + n(H \cap B' \cap C) + n(H \cap B \cap C)$$

$$\Rightarrow \quad 240 = n(H \cap B' \cap C') + 40 + 16 + 24$$

$$\Rightarrow \quad n(H \cap B' \cap C') = 160$$

Similarly
$$n(C) = n(C \cap B' \cap H') + n(C \cap B \cap H') + n(C \cap B' \cap H) + n(C \cap B \cap H)$$

$$\Rightarrow \quad 224 = n(C \cap B' \cap H') + 56 + 16 + 24$$

$$\Rightarrow \quad n(C \cap B' \cap H') = 128$$

Hence the number of students who play only hockey and cricket are 160 and 128 respectively.

$\therefore$ Number of boys who played only one game

$$= n(B \cap H' \cap C') + n(H \cap B' \cap C') + n(C \cap B' \cap H') = 216 + 160 + 128$$
$$= 504$$

Example 11. *An inquiry into 1,000 candidates who failed at ICWA Final Examination revealed the following data :*

658 failed in the aggregate	*166 failed in the aggregate and in group I*
372 in group I	*434 failed in the aggregate and in group II*
590 in group II	*126 failed in both groups*

You have to find out how many candidates failed in :

(*a*) *all the three,*

(*b*) *in aggregate but not in group II,*

(*c*) *group I but not in the aggregate,*

(*d*) *group II but not in group I,*

(*e*) *aggregate or group II but not in group I,*

(*f*) *aggregate but not in group I and II.*

Solution. Let $n(A)$ denote the students who fail in the aggregate $n(B)$, those who fail in the group I and $n(C)$ those who fail in the group II.

Therefore, those who fail in all three will be represented by $n(A \cap B \cap C)$ and those who fail in A or B or C by $n(A \cup B \cup C)$. We are using + for union, − for 'not' and we are not using sign of intersection (in the diagram). The number of elements in a set are shown here by putting n before the braces indicating a given set. We know that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$$
$$- n(A \cap C) + n(A \cap B \cap C)$$

Now by substituting values in the above, we get

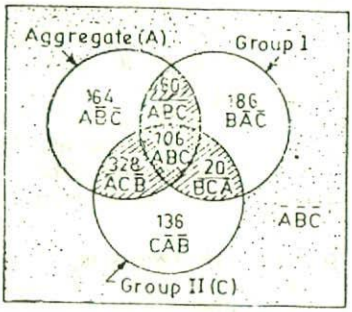
(*a*) $1{,}000 = 658 + 372 + 590 - 166 - 434 - 126 + n(A \cap B \cap C).$

$n(A \cap B \cap C) = 106$

(*b*) Now we have to find out the value of $n(A \cap \bar{C})$, *viz.*,

$n(A \cap \bar{C}) = n(A) - n(A \cap C) = 658 - 434 = 224$

This we can very easily be verified by use of the following figure also :



(*c*) In this case we have to find the value of $n(B \cap \bar{A})$ which can obtained as follows :

$$n(B \cap \bar{A}) = n(B) - n(B \cap A) = 372 - 166 = 206$$

Alternatively,

$$n(B \cap \bar{A}) = n(B \cap \bar{A} \cap C) + n(B \cap \bar{A} \cap \bar{C}) = 20 + 186 = 206$$

(*d*) Here we have to find the value of $n(C \cap \bar{B})$, which we can obtain as follows :

$$n(C \cap \bar{B}) = n(C \cap \bar{B} \cap A) + n(C \cap \bar{B} \cap \bar{A}) = 328 + 136 = 464$$

(*e*) Here $n[(A \cup C) - B] = n[(A - B) \cup (C - B)]$

$$= n(A \cap \bar{B}) + n(C \cap \bar{B}) - n(A \cap \bar{B} \cap C)$$
$$= n(A \cap \bar{B} \cap C) + n(A \cap \bar{B} \cap \bar{C}) + n(C \cap \bar{B} \cap A) + n(C \cap \bar{B} \cap \bar{A})$$
$$- n(A \cap \bar{B} \cap C)$$
$$= n(A \cap \bar{B} \cap \bar{C}) + n(A \cap \bar{B} \cap \bar{C}) + n(C \cap \bar{B} \cap \bar{A})$$
$$= 164 + 328 + 136 = 628$$

(*f*) We have the value of $n(A \cap \bar{B} \cap \bar{C})$ equal to 164.

2·18. ORDERED PAIR

An ordered pair of objects consists of two elememnts *a* and *b* written in parentheses (*a*, *b*) such that one of them, say, *a* is designated as the first member and *b* as the second member.

Illustration. (*i*) The natural numbers and their squares can be represented by ordered pairs in the following manner :

$$(1, 1) ; (2, 4) ; (3, 9) ; (4, 16); \ldots\ldots$$

(*ii*) The points in plane can be represented by an ordered pair (x, y) where x is the first coordinate called abscissa and y the second coordinate called ordinate. Thus a point represents an ordered pair.

(*iii*) The order of occurrence of members is of prime importance in an ordered pair. For example an ordered pair (3, 5) is not the same as an ordered pair (5, 3).

(*iv*) Two ordered pairs (a, b) and (c, d) will be equal if and only if

$$a=c \quad \text{and} \quad b=d$$

In other words

$$(a, b)=(c, d) \Rightarrow a=c, b=d$$

2·19. CARTESIAN PRODUCTS

Definition. *If A and B be any two sets then the set of all ordered pairs whose first member belongs to set A and second member belongs to set B is called the cartesian product of A and B in that order* and is denoted by $A \times B$, to be read as 'A cross B'.

In other words, if A, B are two sets, then the set of all ordered pairs of the form (x, y), where $x \in A$ and $y \in B$ is called the cartesian product of the sets A and B. Symbolically

$$A \times B=\{(x, y) ; x \in A \text{ and } y \in B\}$$

Illustration. Let $A=\{1, 2, 3, 4, 5\}$, $B=\{a, b, c, d\}$

$$A \times B=\{(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d), (3, a), (3, b), (3, c), (3, d), (4, a), (4, b), (4, c), (4, d), (5, a), (5, b), (5, c), (5, d)\}.$$

Similarly we can write down $B \times A$.

Example 12. *If* $A=\{1, 2, 3)$ *and* $B=\{2, 3\}$, *prove that*

$$A \times B \neq B \times A.$$

Solution. The cartesian product of sets are :

$$A \times B=\{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

and $$B \times A=\{2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

We notice that (1, 2) and (1, 3) which are the elements of $A \times B$ are not elements of $B \times A$.

$$\therefore \quad A \times B \neq B \times A$$

Example 13. $A=\{1, 4)$, $B=\{2, 3\}$, $C=\{3, 5\}$, *prove that*

$$A \times B \neq B \times A.$$

Also find $(A \times B) \cap (A \times C)$.

Solution. $A \times B=\{(1, 2), (1, 3), (4, 2), (4,3)\}$, and

$$B \times A=\{(2, 1), (2, 4), (3, 1), (3, 4)\}$$

Clearly $A \times B \neq B \times A$

Moreover, $A \times C = \{(1, 3), (1, 5), (4, 3), (4, 5)\}$

Thus $(A \times B) \cap (A \times C) = \{(1, 3), (4, 3)\}$

If R be a set of real numbers then $R \times R = R^2$ denotes the cartesian plane as

$$R \times R = \{(x, y) : x \in R, y \in R\}$$

each element of this set represents cartesian coordinates of a point in plane.

Example 14. *Let* $A = \{a, b\}$, $B = \{p, q\}$ and $C = \{q, r\}$

Find (*a*) $A \times (B \cup C)$

(*b*) $(A \times B) \cup (A \times C)$

(*c*) $A \times (B \cap C)$

(*d*) $(A \times B) \cap (A \times C)$

Solution. (*a*) *If* $B \cup C = (p, q, r)$

Then $A \times (B \cup C) = \{a, b\} \times \{p, q, r\}$

$= \{(a, p), (a, q), (a, r), (b, p), (b, q), (b, r)\}$

(*b*) Since $A \times B = \{(a, p), (a, q), (b, p), (b, q)\}$ and

$A \times C = \{(a, q), (a, r), (b, q), (b, r)\}$

Then

$$(A \times B) \cup (A \times C) = \{(a, p), (a, q), (b, p), (b, q), (a, r), (b, r)\}$$

From (*a*) and (*b*), we get

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

(*c*) $B \cap C = \{q\}$

$A \times (B \cap C) = \{a, b\} \times \{q\} = \{(a, q), (b, q)\}$

(*d*) $(A \times B) \cap (A \times C) = \{(a, q), (b, q)\}$

From (*c*) and (*d*), we find that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Example 15. *If* $A = \{1, 4\}$; $B = \{4, 5\}$; $C = \{5, 7\}$, verify that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

(*C.A. Entrance, June 1984*]

Solution We have

$B \cap C = \{5\}$

$A \times (B \cap C) = \{(1, 5), (4, 5)\}$ (1)

$A \times B = \{1, 4), (1, 5), (4, 4), (4, 5)\}$

$A \times C = \{(1, 5), (1, 7), (1. 5), (4, 7)\}$

$\therefore$ $(A \times B) \cap (A \times C) = \{(1, 4), (4, 5)\}$ (2)

From (1) and (2), we have

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Some Important Results of Cartesian Product

I. $A \times B$ and $B \times A$ have the same number of elements but $A \times B \neq B \times A$, unless $A = B$. Thus, the cartesian product of two sets is commutative if the two sets are equal.

II. In the product set $B \times A$, the first component of ordered pairs are taken from B and the second from A.

III. If A and B are disjoint sets, then $A \times B$ and $B \times A$ are also disjoint.

IV. If the set A consists of m elements $a_1, a_2, \ldots\ldots, a_m$ and B consists of the n elements $b_1, b_2, \ldots\ldots, b_n$ then the product set $A \times B$ consists of mn elements.

V. If either A or B is null then the set $A \times B$ is also a null set.

VI. If either A or B is infinite and other is a non-empty set, then $A \times B$ is also an infinite set.

Some Further Results of Cartesian Product

VII. *If $A \subset B$, then $A \times C \subset B \times C$*

VIII. *If $A \subset B$ and $C \subset D$, then*

$$A \times C \subset B \times D$$

IX. *If $A \subseteq B$ then*

$$A \times B \Rightarrow (A \times B) \cap (B \times A)$$

Proof. Let (x, y) be the element of $A \times B$ then

$$\begin{aligned}(x, y) \in (A \times B) &\Rightarrow x \in A \text{ and } y \in B\\ &\Rightarrow x \in A \text{ and } (x \in A \text{ and } y \in B)\\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in B \text{ and } y \in A)\\ (\because\ & A \subseteq B, y \in A \Rightarrow y \in B \text{ and } x \in A \text{ and } x \in B).\\ &\Rightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (B \times A)\\ &\Rightarrow (x, y) \in (A \times B) \cap (B \times A)\end{aligned}$$

$$\therefore \quad A \times B \Rightarrow (A \times B) \cap (B \times A), \text{ provided } A \subseteq B$$

X *If A, B and C be any three sets, then*

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Proof. Let (x, y) be any element of $A \times (B \cap C)$. Then

$$\begin{aligned}(x, y) \in A \times (B \cap C) &\Rightarrow x \in A \text{ and } y \in (B \cap C)\\ &\Rightarrow x \in A \text{ and } (y \in B \text{ and } y \in C)\\ &\Rightarrow (x \in A) \text{ and } (y \in B) \text{ and } (x \in A \text{ and } y \in C)\\ &\Rightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C)\\ &\Rightarrow (x, y) \in (A \times B) \cap (A \times C)\end{aligned}$$

$$\therefore \quad A \times (B \cap C) \subseteq (A \times B) \cap (A \times C) \quad \ldots(1)$$

Again, let (u, v) be any element of $(A \times B) \cap (A \times C)$.

$$(u, v) \in (A \times B) \cap (A \times C) \Rightarrow (u, v) \in (A \times B) \text{ and } (u, v) \in (A \times C)$$

$\Rightarrow (u \in A \text{ and } v \in B) \text{ and } (u \in A \text{ and } v \in C)$

$\Rightarrow u \in A \text{ and } (v \in B \text{ and } v \in C)$

$\Rightarrow u \in A \text{ and } v \in (B \cap C)$

$\Rightarrow (u, v) \in A \times (B \cap C)$

$\therefore \quad (A \times B) \cap (A \times C) \subseteq A \times (B \cap C) \qquad (2)$

From (1) and (2), by equality of sets, we have

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

XI. *If A, B, C, be any three sets, then*

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Proof. It is exactly similar to part X.

XII. $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$

Proof. Let (x, y) be any element of $(A \times B) \cap (S \times T)$. Then

$(x, y) \in (A \times B) \cap (S \times T) \Rightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (S \times T)$

$\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in S \text{ and } y \in T)$

$\Rightarrow (x \in A \text{ and } x \in S) \text{ and } (y \in B \text{ and } y \in T)$

$\Rightarrow x \in A \cap S \text{ and } y \in B \cap T$

$\Rightarrow (x, y) \in (A \cap S) \times (B \cap T)$

$\therefore \quad (A \times B) \cap (S \times T) \subseteq (A \cap S) \times (B \cap T) \qquad \text{.. (1)}$

Again, let (u, v) be any element of $(A \cap S) \times (B \cap T)$ Then

$(u, v) \in (A \cap S) \times (B \cap T) \Rightarrow u \in (A \cap S) \text{ and } v \in (B \cap T)$

$\Rightarrow (u \in A \text{ and } u \in S) \text{ and } (v \in B \text{ and } v \in T)$

$\Rightarrow (u \in A \text{ and } v \in B) \text{ and } (u \in S \text{ and } v \in T)$

$\Rightarrow (u, v) \in (A \times B) \text{ and } (u, v) \in (S \times T)$

$\Rightarrow (u, v) \in (A \times B) \cap (S \times T)$

$\therefore \quad (A \cap S) \times (B \cap T) \subseteq (A \times B) \cap (S \times T) \qquad \text{...(2)}$

From (1) and (2), we conclude that

$$(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$$

Example 16. *If* $A = \{1, 2, 3\}$; $B = \{2, 3, 4\}$; $S = \{1, 3, 4\}$; $T = \{2, 4, 5\}$, *verify that* $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$.

Solution. $A \times B = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

$S \times T = \{(1, 2), (1, 4), (1, 5), (3, 2), (3, 4), (3, 5), (4, 2), (4, 4), (4, 5)\}$

$\therefore \quad (A \times B) \cap (S \times T) = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\} \cap \{(1, 2), (1, 4), (1, 5), (3, 2), (3, 4), (3, 5), (4, 2), (4, 4), (4, 5)\}$

$= \{(1, 2), (1, 4), (3, 2), (3, 4)\} \qquad \text{...(1)}$

Also $A \cap S = \{1, 3\}, B \cap T = \{2, 4\}$

$\therefore \quad (A \cap S) \times (B \cap T) = \{1, 3\}, \times \{2, 4\}$

$$= \{(1, 2), (1, 4), (3, 2), (3, 4)\} \quad \ldots(2)$$

Hence from (1) and (2), we conclude that

$$(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$$

2·20. SET RELATIONS

The mathematical concept of relation deals with the way the numbers or variables are related or paired. A relation in its ordinary meaning signifies some ties. In mathematics, the relevant expressions are "is parallel to", "is greater than", "is equal to", etc. When these relations are between two numbers or objects, they are called binary relations and will be expressed as $s\ R\ t$ or $a\ R\ b$. Similarly, relations between three variables will be expressed as $s\ R\ t\ R\ p$. The higher order relations will accordingly be expressed with multiplication signs. However, binary relations are the most important for our purpose and we would confine mostly to these relations. The following are some statements which have been expressed in the form $x\ R\ y$:

5 is less than 9

9 is the square root of 81

The negative of R will be denoted by $\bar{R}$(R bar) in the form $x\ \overline{R}\ y$ indicating that "x is not related to y".

The converse of R denoted by R' (R dash) so that $y\ R'\ x$ if and only if $x\ R\ y$ is true. The respective converse of the relations given above are

9 is more than 5 and

81 is the square of 9

Such relations are also called inverse relations. For example, the inverse relation of $x\ R\ y$ will be $y\ R'\ x$. Formally

$$y\ R'\ x \leftrightarrow x\ R\ y$$

Thus if $x\ R\ y$ is a subset of the Cartesian product set $A \times B$, $y\ R'\ x$ will be a subset of the Cartesian product set $B \times A$.

2·21. PROPERTIES OF RELATIONS

Reflexive. When in a relation x is related to itself, *i.e.*, $x\ R\ x$ for all x. For example, if $x\ R\ y$ stands for statements like :

x has the same experience as y

y's selection synchronised with z's

They indicate relations which are reflexive.

Symmetric. When for every $x\ R\ y$, we have $y\ R\ x$, so that we can make statement :

$$x\ R\ y \rightarrow y\ R\ x$$

For example, a statement x is a relation of y or y is 100 metres away from x signify symmetric relations. But the above statements if modified as "x is the father of y" or "x is 100 metres north of y" then they cease to be symmetric ; in other words, they are asymmetric. If, however, the statements are such that nothing can be said whether $x R y \rightarrow y R x$, then it would be asymmetric. Anti-symmetric is one which has the property : xRy and $yRx \rightarrow x=y$.

Transitive. This is a relation in between three or more elements, *e.g.*,

$$x R y \text{ and } y R z \rightarrow x R z \text{ for all } x, y, z$$

For example, if a relation is expressed in the form that "x is the parent of y" and "y is the parent of z", the relation is transitive because in that case the statement "x is the parent of z" is true. If, however, the statement is that "x is the father of y" and "y is father of z" the relation is not transitive because "x cannot be father of z". The following are some relations which are transitive :

$$x \parallel y \wedge y \parallel z \rightarrow x \parallel z$$

Spoken as ' if x is parallel to y" and "y is parallel to z" then x is parallel to z. Similarly, if

$$x > y \wedge y > z \rightarrow x > z$$

Spoken as ,'If x is greater than y" and "y is greater than z" then x is greater than z.

Equivalence. A relation which is reflexive; symmetric and transitive is called an equivalence relation. The following are some examples :

(*i*) x is parallel to y.

(*ii*) x has the same price as y.

Order. A relation which is non-reflexive, non-symmetric but transitive is called an order relation. For example, the relations ' is less than" and ' is more than" are order relations. On the other hand, the relations "is less than or equal to" and "is more than or equal to" are not order relations because although they are transitive, they are reflexive as well ; the reflexivity goes against the order relation.

2·22. BINARY RELATIONS

A binary relation is a set, all of whose members (elements) are ordered pairs. Since every sub-set of a certesian product $A \times B$ is, by definition a set of ordered pairs, it follows that every sub-set of $A \times B$ is a relation.

A relation $R \subset A \times B$ is called a relation from A to B (or relation between A and B), and a relation $R \subset A \times A$ is called a relation in (or on) the set A. If R is a relation then $(x, y) \in R$ is sometimes written as xRy.

The domain D of the relation R is the set of all first elements of the ordered pairs which belong to R. Symbolically,

$$D = \{x : (x, y) \in R\}$$

The range E or the relation R is the set of all second elements of the ordered pairs which belong to R. Symbolically,

$$E = \{y : (x, y) \in R\}$$

Inverse Relation. Let R be a relation from A to B. The inverse relation of R is denoted by R^{-1} and it is defined as a relation from B to A.

(*i*) Let us have order relation R of the type $<$ (less than) from set $A=\{1, 2, 3\}$ to set $B=\{1, 3, 5\}$ as follows :

$$R=\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}.$$

The inverse R^{-1} is the relation $>$ (greater than) as follows :

$$R^{-1}=\{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3)\}$$

(*ii*) Let us now find the relation R of the type "x divides y" from set $C=\{2, 3, 4\}$ to $D=\{3, 6, 7\}$ as follows :

$$R=\{(2, 6), (3, 3), (3, 6)\}$$

Its inverse R^{-1} will be

$$R^{-1}=\{(6, 2), (3, 3), (6, 3)\}$$

(*iii*) We can have relation R of set $E=\{a, b, c, d\}$ in itself as follows :

$$R=\{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$$
$$R^{-1}=\{(b, a), (c, a), (d, a), (c, b), (d, b), (d, c)\}$$

Remarks. 1. Let R be a relation in a set A, *i. e.*, R be a sub-set of $A\times A$, then R is called reflexive relation if $(a, a)\in R$ for all $a\in A$. Thus R is reflexive if every element in A is related to itself.

2. Let R be a relation in a set A and R be a subset of $A\times A$. Then R is said to be symmetric relation if for all $(a, b)\in R$, $(b, a)\in R$ *i.e.*, whenever a is related to b and then b is also related as a, *i.e.*, $aRb \Rightarrow bRa$.

3. The relation R is said to be a transitive relation if $(a, b)\in R$ and $(b, c)\in R$ implies $(a, c)\in R$, *i.e.*, if a Rb and bRc then aRc, then R is transitive

Let us consider

(*i*) Let R be a relation $\leqslant$ (less than or equal to) in $F=\{1, 2, 3, 4...\}$, such that $(a, b) \in R$ iff $a\leqslant b$.

Now

(*a*) R is reflexive because $a\leqslant b$ for every $a\in F$.

(*b*) R is not symmetric because $2\leqslant 5$ but $5<2$.

We can also say $(2, 5) \in R$ but $(5, 2) \notin R$.

(*c*) R is transitive because $a\leqslant b$ and $b\leqslant c$ so that $a\leqslant c$.

(*d*) R is not equivalence relation because $a\leqslant b$ but $b \nleqslant a$.

(*ii*) Let there be a set T of all congruent triangles. Show that they have an equivalence relations :

(*a*) The relation is reflexive $a\ R\ a$ for all $a \in T$

(*b*) The relation is symmetric : $a\ R\ b$, $b\ R\ a$

(*c*) The relation is transitive : $a\ R\ b$, $b\ R\ c$ and $a\ R\ c$.

2·23. FUNCTIONS OR MAPPINGS

Let us consider the following relations which relate

(*i*) each given circle to its radius,

(*ii*) each given natural number to its square, (*iii*) each given printed book to its title.

(*iv*) each given firm of Charatered Accountants to the number of its audit clerks

(*v*) each given examinee of C.A. (Inter) examination and roll number allotted to him.

Each of these relations relates each given element of an appropriate first non-empty set to one and only one (unique) element of an appropriate second non-empty set. Each of these relations is called a function or mapping from an appropriate first set to an appropriate second set, according to the following definition.

Let A and B be two non-empty sets, not necessarily disjoint. Then any relation or correspondence J, which relates or associates each given element, say, a, of set A to one and only one (i e., a unique) element, b, of set B is called a function (or mapping) of (from) the set A into the set B.

f is a mapping of A into B is symbolised as

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

It immediately follows that

$$f : A \to B \text{ if and only if}$$

(*i*) $f \subset A \times B$

(*ii*) For each given $x \in A$, there exists at least one $y \in B$ such that $(x, y) \in f$.

Remarks. 1. A set of ordered pairs specifies a function, from the set of all the first elements of the ordered pairs to the set of all the second elements of the ordered pairs, if and only if, no two distinct (*i.e.*, unequal) ordered pairs have the same first element.

2. The set A is called the domain of the mapping (or function) f, and the set B is called the co-domain of the mapping (or function) f.

3. The element of B which the mapping f associates any element a of A, will be denoted by $f(a)$ (which reads 'f of a') and is called the f image of a or the value of the function for a. Also the element a is called the pre-image of A. It should be remembered that each element of B need not appear as the image of an element in A. We find the range of f to consist of those elements in B which appear as image of at least one element in A. There can be more than one elements of A which have the same image in B. The image set of $f(A)$ is called the range of f.

4. Clearly the mapping is well defined if

(*i*) each element of A has an image in B,

(*ii*) an element $x \in A$ has only one image of $f(x) \in B$, but two or more elements of A may have the same image in B.

2·24. TYPES OF MAPPINGS

I. Injective Mapping. If to each element of set A there corresponds one element of set B, but there are some elements of set B which do not correspond to any of the elements of A, this type of mapping is called injective or one-one into mapping.

Def. A mapping f of a set A into a set B is called injective or one-one mapping of A into B if two different elements of A necessarily correspond to two different images in B.

Symbolically,

$f : A \rightarrow B$ is one-one mapping of A into B if

$$f(a)=f(b) \Rightarrow a=b$$

or

$$a \neq b \Rightarrow f(a) \neq f(b)$$

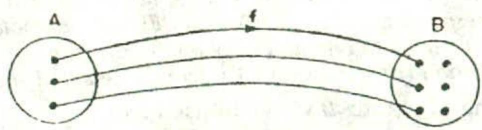


Illustration. Let A be the set of students sitting on chairs in a class-room and suppose B is the set of chairs in the class-room. Again let f be the correspondence which associates to each student and the chair occupied by him, *i.e.,*, $f : A \rightarrow B$.

If all the students occupy different chairs (one-one) but some chairs are still left vacant (into), it will be one one into mapping.

II. Bijective Mapping or one-one onto mapping. If to each element of A there corresponds one and only one element of B and every element of B have one and only one image in A, the mapping is called bijective or one-one onto mapping or one-to-one correspondence.

Def. A mapping f, of a set A into set B is said to be a one one mapping of A onto B if.

(*i*) f is an onto mapping, and (*ii*) is one-one mapping.

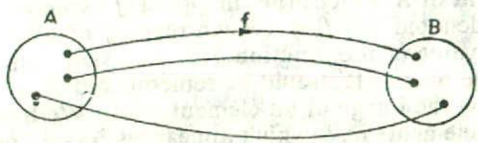


Illustration. If all the students occupy different chairs (one-one) and no chair is left vacant (onto) then it will be one-one onto mapping.

III. Surjection or Onto Mapping. If to each element of A there corresponds one element of B but one element of B have more than one image in A, this type of mapping is called *surjective or many-one mapping* or *onto mapping*.

Def. Let A and B be the two sets. If the mapping f be such that each element of B is the f-image of at least one element of A, then f is said to be a mapping of A onto B, or f is said to be a function from A onto B or surjection mapping.

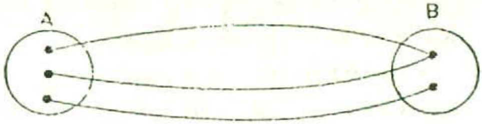


Illustration. If two or more students occupy same chairs (many-one) and no chair is left vacant (onto) then it will be many-one onto mapping

EXERCISES

1. Define the following and give an example of each ;

(*i*) Subset of a set. (*ii*) Complement of a set. (*iii*) Union of two sets. (*iv*) Intersection of two sets. (*v*) Disjoint set.

2. Write, in words, the following set notations :

$A \subset B$; $x \notin A$; $A \supset B$; $\{0\}$; $A \not\subset B$; $A = \phi$.

3. Represent the following sets in set notation :

(*i*) Set of all alphabets in English language,

(*ii*) the set of all odd integers less than 25,

(*iii*) the set of all odd integers,

(*iv*) the set of positive integers x satisfying the equation

$$x^2+5x+7=0$$

4. Rewrite the following sets in a set-builder form :

(*i*) $A=\{a, e, i, o, u\}$ (*ii*) $B=\{1, 2, 3, 4, \ldots\ldots\}$

(*iii*) C is a set of integers between -15 and $+15$.

5. Let $V=\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $X=\{0, 2, 4, 6, 8\}$

$Y=\{3, 5, 7\}$ and $Z=\{3, 7\}$

Find (*a*) (*i*) $Y \cup Z$ (*ii*) $(V \cup Y) \cap X$

(*iii*) $(X \cup Z) \cup V$, (*iv*) $(X \cup Y) \cap Z$,

(*v*) $(\phi \cup V) \cap \phi$.

(*b*) $V=\{x : x+2=0\}$, $R=\{x : x^2+2x=0\}$

$S=\{x : x^2+x-2=0\}$; state if V, R and S are equal.

6. (*a*) Given $A=\{2, 3, 4\}$ and $B=\{4, 5\}$ which of the following statements are not correct and why ?

(*i*) $5 \in A$. (*ii*) $\{5\} \subset A$. (*iii*) $4 \in A$

(*b*) If $A=\{2, 3, 4\}$ and $U=\{0, 1, 2, 3, 4\}$, which of the following statements are correct or incorrect. Give reasons.

(*i*) $\{O\} \in A'$, (*ii*) $\phi \in A'$. (*iii*) $\{O\} \subset A'$,

(*iv*) $O \in A'$, (*v*) $O \subset A'$.

7. What is the relationship between the following sets ;

$A=\{x : x$ is a letter in the word flower$\}$,

$B=\{x : x$ is a letter in the word flow$\}$,

$C=\{x : x$ is a letter in the word wolf$\}$,

$D=\{x : x$ is a letter in the word follow$\}$.

8. State whether each of these statements is correct or incorrect :

(i) $\{a, b, c\}=\{c, b, a\}$, (ii) $\{a, c, a, d, c, d\}\subseteq\{a, c, d\}$,
(iii) $\{b\}\in\{\{b\}\}$, (iv) $\{b\}\subset\{\{b\}\}$ and (v) $\phi\subset\{\{b\}\}$.

9. Let $A=\{a, b, c\}$, $B\{a, b\}$, $C=\{a, b, d\}$, $D=\{c, d\}$, $E=\{d\}$. State which of the following statements are correct and give reasons :

(i) $B\subset A$, (ii) $D\neq C$, (iii) $C\supset E$, (iv) $D\not\supset E$, (v) $D\subset B$, (vi) $D=A$, (vii) $B\not\subset C$, (viii) $E\subset A$, (ix) $E\not\subset B$, (x) $a\in A$, (xi) $a\subset A$, (xii) $\{a\}\in A$, (xiii) $\{a\}\subset A$.

10. Let $A=\{0\}$, $B=\{0, 1\}$, $C=\phi$, $D=\{\phi\}$,

$E=\{x \mid x$ is human being 200 years old$\}$

$F=\{x \mid x\in$ A and $x\in B\}$.

State which of the following are true and which are false.

(a) (i) $A\subset B$, (ii) $B=F$, (iii) $C\subset D$, (iv) $C=E$, (v) $A=F$, (vi) $F=1$. and (viii) $E=C=D$.

(b) If $A=\{0, 1\}$, state whether the following statements are true or false :

(i) $\{1\}\subset A$, (ii) $\{1\}\in A$, (iii) $\phi\in A$. (iv) $0\in A$, (v) $1\subset A$, (vi) $\{0\}\in A$, (vii) $\phi\subset A$.

(c) State whether the following sets are finite, infite, or empty :

(i) $X=\{1, 2, 3, \ldots\ldots, 500\}$, (ii) $Y=\{y : y=a^2$; a is an integer$\}$,

(iii) $A=\{x : x$ is a positive integer multiple of 2$\}$,

(iv) $B=\{x : x$ is an integer which is a perfect root of $26<x<35\}$.

11. Give an eaxmple of three sets A, B and C such that $A\cap B\neq\phi$, $B\cap C\neq\phi$ and $A\cap C\neq\phi$ but $A\cap B\cap C=\phi$.

12. Let $A=\{x \mid x$ is a letter in English alphabet$\}$ be the universal set.

$V=\{x \mid x$ is a vowel$\}$, $C=\{x \mid x$ is a consonant$\}$,

$N=\{x \mid x$ is a letter in your full name$\}$.

(a) Describe the four sets by listing the elements of each set.

(b) List the elements of the following sets :

(i) $N\cup V$, (ii) $N\cap C$, (iii) $V\cup C$ (iv) $N\cap V'$
(v) $C\cap N'$, (vi) C'.

13. If the universal set is $X=\{x : x \in N, 1 \leqslant x \leqslant 12\}$ and $A=\{1, 9, 10\}$, $B=\{3, 4, 6, 11, 12\}$ and $C=\{2, 5, 6\}$ are the subsets of X, find the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

14. The following table gives the distribution of radio sets by income groups of a sample of 1,172 families.

No. of Radio sets	*Income Groups*				
	Less than Rs. 6,000	*Rs. 6,000 to Rs. 10,999*	*Rs. 11,000 to Rs. 15,999*	*Rs. 16.000 and above*	*Total*
Two or more	10	174	84	94	362
One	152	308	114	46	620
None	70	50	20	50	190
TOTAL	232	532	218	190	1,172

Let universal set U be the set of families in the sample and let the following subsets of U be defined :

$A=\{x \mid x$ is the family owning two or more radio sets$\}$,

$B=\{x \mid x$ is a family with one radio set$\}$,

$C=\{x \mid x$ is a family with less than Rs. 6,000 income$\}$,

$D=\{x \mid x$ is a family with Rs. 6,000 to Rs. 10,999 income$\}$,

$E=\{x \mid x$ is a family with Rs. 11,000 to Rs. 15,999 income$\}$.

(*a*) Find the number of families in each of the following sets :

(*i*) $C \cap B$, (*ii*) $A \cup E$, (*iii*) $(A \cup B)' \cap E$,

(*iv*) $(C \cup D \cup E) \cap (A \cup B)'$.

(*b*) A number of sets are specified below. Express each of them in set notation using the basic operations of $\cap$, $\cup$ and $'$.

(*i*) $\{x \mid x$ is family with one radio set and an income of less than Rs. 11,000$\}$,

(*ii*) $\{x \mid x$ is a family with no radio set and more than Rs. 16,000 income$\}$,

(*iii*) $\{x \mid x$ is a family with two or more radio sets or an income of Rs. 11,000 to Rs. 15,999$\}$,

(*ix*) $\{x \mid x$ is the family with no radio set$\}$.

15. Let $A=\{a, b, c, d\}$, where a, b, c, d represent the members of a decision-making body, say a committee.

(*i*) List the elements of power set $P(A)$.

(*ii*) Each element of $P(A)$ can represent a voting *coalition* casting votes for a specific measure. If each committee member has one vote and if three votes are needed to carry a motion, a set of any three committee members represents a *winning coalition*, while a set of any one member is a *losing* coalition and the set of any two members a *blocking* coalition. Specify the winning, the losing and the blocking coalitions of the set $P(A)$.

(*iii*) Let the four members a, b, c, d of the decision-making body represent the shareholders of a corporation with 100 shares and let 'a' own 50 shares, 'b' 20 shares, and 'c' and 'd' 15 shares each. . If each share has one vote and an absolute majority, *i.e.*, 51 votes can carry a motion, list the winning, losing and blocking coalitions of the set $P(A)$.

16. If $A=\{a, b, c, d, e, f\}$, $B=\{a, e, i, o, u\}$,

and $C=\{m, n, o, p, q, r, s, t, u\}$

Compute the following :

(*i*) $A\cup B$, (*ii*) $A\cup C$, (*iii*) $B\cup C$, (*iv*) $A-B$, (*v*) $A\cap B$, (*vi*) $B\cap C$, (*vii*) $A\cup(B-C)$, (*viii*) $A\cup B\cup C$, and (*ix*) $A\cap B\cap C$.

17. (*a*) If $A=\{5, 6, 7, 8, 9\}$, $B=\{2, 4, 6, 8, 10, 12\}$,

$C=\{3, 6, 9, 12\}$,

Verify that $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

(*b*) $U=\{a, b, c, d, e, f, x, y, z, w\}$, $A=\{a, b, c, d, e\}$,

$B=\{b, d, x, y, z\}$.

If $A-B$ is defined as $A\cap B'$, verify that $(A-B)'=A'\cup B$.

18. If the universal set $U=\{x \mid x \text{ is a +ve integer} <25\}$.

$A=\{2. 6, 8, 14, 22\}$ $B=\{4, 8, 10, 14\}$, $C=\{6, 10, 12, 14, 18, 20\}$, verify the relations

(*i*) $(A\cap B)'=A'\cup B'$ (*ii*) $(B'\cap C)\cup(A'\cap C)=C\cap(A'\cup B')$,

19. If $A=\{1, 2, 3\}$; $B=\{2, 3, 4, 5\}$ and $C=\{2, 4, 6, 8\}$, verify that

(*i*) $A\cup B=(A-B)\cup B$, (*ii*) $A-(A-B)=A\cap B$,

(*iii*) $A\cap(B-C)=(A\cap B)-(B\cap C)$.

20. Let the universal set

$U=\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $A=\{3, 4, 5, 6\}$.

$B=\{3, 7, 9, 5\}$, and $C=\{6, 8, 10, 12, 7\}$.

Write down the following sets :

(*i*) A', (*ii*) B', (*iii*) C' (*iv*) $(A')'$, (*v*) $(B')'$, (*vi*) $(A\cup B)'$, (*vii*) $(A\cap B)'$, and (*viii*) $A'\cup C'$.

21. $A=\{1, 2, 3, 4,\ldots\ldots, 8, 9\}$, $B=\{2, 4, 6, 8\}$, $C=\{1, 3, 5, 7, 9\}$, $D=\{3, 4, 5\}$, $E=\{3, 5\}$.

What is the S if

(*i*) $S\subset D$ and $S\not\subset A$. (*ii*) $S\subset B$ and $S\not\subset C$.

22. Write down all the subsets of the sets :

$B=\{6, 8, 11\}$.

23. $A=\{1, 3, a, \{1\}, \{1, a\}\}$. State whether the following statements are true or false :

(i) $1 \in A$, (ii) $\{1\} \in A$, (iii) $\{1\} \subset A$. (iv) $\phi \in A$, (v) $\phi \subset A$, (vi) $\{1, a\} \subset A$, (vii) $\{3, a\} \subset A$. (viii) $\{1, a\} \in A$.

24. Let $A=\{1, 3, 5, 7, 9\}$, $B=\{2, 4, 6, 8, 10\}$,

$C=\{3, 4, 7, 8, 11, 12\}$.

Show that

(i) $(A \cup B) \cup C = A \cup (B \cup C)$, (ii) $(A \cap B) \cap C = A \cap (B \cap C)$,

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

25. If $U=\{1, 2, 3, \ldots\ldots\ldots, 8, 9\}$ be the universal set

$A=\{1, 2, 3, 4\}$ and $B=\{2, 4, 6, 8\}$

Write down the following sets :

(i) $A \cup B$, (ii) $A \cap B$, (iii) A', (iv) $(A \cup B)'$, (v) $(A \cap B)'$.

26. State with reasons whether the following statements are true or false :

(i) If $A \cup B = B \cup C$ then $A = C$, (ii) If $A \subset B$ then $A \cup B = B$.

27. Show than $A \cap B = \phi$ if and only if $A - B = A$.

28. If A and B are two sets, then

(i) $A - B \subseteq A$, (ii) $A \cap (B - A) = \phi$, (iii) $A \cup B = (A - B) \cup B$,

(iv) $A - B = A \cap B' = B' - A$.

29. Prove that

(i) $(A \cup B) \cap (A \cup B') = A$ and (ii) $(A \cap B) \cup (A \cap B') = A$.

30. Prove that $(B - A') = B \cap A$.

31. Prove that $A \cup B = (A - B) \cup B$.

32. Let A, B and C be three subsets of the universal set U. Prove each of the following :

(i) $A \cap (A \cup B) = A$, (ii) $(A \cap B) \cup (A \cap B\ ) = A$,

(iii) $(A \cup B) = (A' \cap B')'$, (iv) $(A \cup B) \cap (A \cup B') = A$,

(v) $[A' \cup (B \cup C)]' = A \cap B' \cap C'$,

(vi) $A \cup B = (A \cap B) \cup (A \cap B') \cup (A' \cap B)$,

(vii) $[A' \cup (A \cap B')]' = A \cap B$. (viii) $B \cup (A \cup B) = (A \cup B)$.

(ix) $(A' \cup B')' \cup (A' \cup B)' = A$.

33. Let $A=\{a, b, c, d, e, f\}$, $B=\{x : x \text{ is a vowel}\}$, $C\{x, y, z\}$.

Give the following Cartesian product sets and count the number of elements in each.

(i) $A \times B$, (ii) $B \times A$, (iii) $C \times B$, (iv) $(A \times B) \times C$, (v) $A \times (B \times C)$.

34. Let $P=\{1, 2, x\}$, $Q=\{a, x, y\}$. $R=\{x, y, z\}$

Find :

(i) $P \times Q$, (ii) $P \times R$ (iii) $Q \times R$, (iv) $(P \times Q) \cap (P \times R)$,
(v) $(R \times Q) \cap (R \times P)$, (vi) $(P \times Q) \cup (R \times P)$.

35. (a) If P has three elements, Q four and R two, how many elements does the Cartesian product set $P \times Q \times R$ will have ?

(b) Identify the elements of B, if set

$Q = \{1, 2, 3\}$ and $B \times Q = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), 5, 3)$
$(6, 1), (6, 2), (6, 3)\}$

36. Given $A = \{2, 3\}$, $B = \{4, 5\}$, $C = \{5, 6\}$, find
$A \times (B \cup C)$, $A \times (B \cap C)$, $(A \times B) \cup (B \times C)$.

37. If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 5\}$ and
$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Compute the following :

(i) $A \cap (B - C)$, (ii) $A \cup (B \cap C)$, (iii $A' \cup (B - C)$,
(iv) $A' \cap (B - C)$, (v) $A - (B' - C')$, (vi) $A' \cap (B' \cap C')$,
(vii) $A' \cup (B' - C')$. (viii) $(A' - B') \cap (B' - C')$, (ix) $(A' \cup B') \cap C'$,
(x) $A' - (B \cap C)'$, (xi) $A \times B$, (xii) $A \times (B \cup C)$,
(xiii) $(A \cup B) - (B \cup C)$, (xiv) $(A \cap B) \times (B \cap C)$, (xv) $(A - B) \times (B - C)$
(xvi) $(A' - B') \times (B - C)'$, (xvii) $A \times (B \cup C)'$, (xviii) $A \times (B \times C)$
(xix) $A \triangle B$, and (xx) $(A \triangle B) \triangle C$.

38. Let A, B, C, D be any four sets, show that

(i) $A \subset B \rightarrow A \times C \subset B \times C$.

(ii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(iii) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

(vi) $(A \subset B) \wedge (C \subset D) \rightarrow A \times C \subset B \times D$.

39. If A has 32 elements, B has 42 elements and $A \cup B$ has 62 elements, indicate the number of elements in $A \cap B$.

40. A town has a total population of 50,000. Out of it 28,000 read Patriot and 23,000 read Times of India while 4,000 read, both the papers. Indicate how many read neither Patriot nor Times of India ?

41. In a survey conducted of 2,000 clerks in an office it was found that 48% preferred coffee (C), 54% liked Tea (T) and 64% used to smoke (S). Of the total 28% used C and T, 32% used T and S and 30% preferred C and S. Only 6% did none of these. Find (i) the number having all the three, (ii) T and S but not C, and (iii) only C.

42. Complaints about work canteen fell into three categories. Complaints about (i) Mess, (M) (ii) Food, (F) (iii) Service (S). Total complaints 173 were received as follows :

$n(M) = 110$, $n(F) = 55$, $n(S) = 67$, $n(M \cap F \cap S') = 20$, $n(M \cap S \cap F') = 11$, $n(F \cap S \cap M') = 16$. Determine the complaints about (i) all the three, (ii) about two or more than two.

43. Out of the total 150 students who appeared for ICWA Examination from a centre, 45 failed in Accounts, 50 failed in Maths, and 30 in

Costing. Those who failed both in Accounts and Maths were 30, those who failed both in Maths and Costing were 32 and those who failed both in Accounts and Costing were 35. The students who failed in all the three subjects were 25. Find out the number who failed at least in any one of the subjects.

44. A survey of 400 recently qualified Chartered Accountants revealed that 112 joined industry, 120 started practice and 160 joined the firms of practising chartered accountants as paid assistants. There were 32 who joined service and also did practice ; 40 had both practice and assistantship and 20 had both job in industry and assistantship. There were 12 who did all the three. Indicate how many could not get any of these and how many did only one of these.

45. A market research team interviews 100 people, asking each whether he smokes any or all of the items, A : cigarettes, B ; cigars, C- pipe tobacco. The team returns the following data :

Category	Number	Category	Number
ABC	3	A	42
AB	7	B	17
BC	13	C	27
AC	18		
		Total	100

Are the returns consistent ?

46. (*i*) In a survey of 100 students it was found that 50 used the college library, 40 had their own and 30 borrowed books, 20 used both college library and their own, 15 borrowed books and used their own books, whereas 10 used borrowed books and college library. Assuming that all students use either college library books or their own or borrowed books, find the number of students using all the three sources.

(*ii*) If the number of students using no book at all is 10, and the number of students using all the three is 20, show that the information is inconsistent.

47. A class of 60 students appreared for an examination of Mercantile law, Statistics and Accountancy. 25 students failed in Mercantile law, 24 failed in Statistics, 32 failed in Accountancy, 9 failed in Mercantile law alone, 6 failed in Statistics alone ; 5 failed in Accountancy and Statistics only and 3 failed in Mercantile law and Statistics only. Find

(*i*) how many failed in all-three subjects.

(*ii*) how many passed in all the three subjects.

48. Asked if you will vote for Congress, the following responses are recorded :

	Yes	*No*	*Don't know*
Adult Male	10	20	
Adult Female	20	15	5
Youth just to enter adulthood	10	5	10

Write A = set of adults, C = set of women and children,

Y = set of 'Yes' answers, N = set of 'No' answers.

Find (*i*) $n(A')$, (*ii*) $n(A \cap C)$, (*iii*) $n(Y \cup N)'$, (*iv*) $n[A \cap (Y \cap V)']$ (*v*) $C' \cap Y'$.

49. In a market survey, a manufacturer obtained the following data :

Did you use our brand	*Percentage answering yes*
1. April	59
2. May	62
3. June	62
4. April and May	35
5. May and June	33
6. April and June	31
7. April, May and June	22

Is this correct ?

50. The report of the inspector of an assembly line showed the following 100 units :

Item	*Defect*	*No. of pieces*
1	Strength defect (S)	35
2	Flexibility defect (F)	40
3	Radius defect (R)	18
4	S and F	7
5	S and R	11
6	F and R	12
7	S, F and R	3

The report was returned. Why ?

51. In a survey of 1,000 customers, the number of people that buy various grades of coffee seeds were found to be as follows :

A grade only	180	A grade but not B grade	230
A grade and C grade	 80	A grade	260
C grade	480	C grade and B grade	80

None of the three grades......240

(*a*) How many buy B grade coffee seeds ?

(*b*) How many buy C grade, if and only if they do not buy B grade ? and

(*c*) How many buy C and B grades but not the A grade ?

52. The workers in a factory were classified according to skill, number of years of service in the factory and whether they performed direct or indirect labour If they had less than three years of service they were considered as short term workers, if they served 10 years or more they were considered long term workers, and all others were classified medium-term workers. Consider the following data :

	Skilled and direct	*Unskilled and direct*	*Skilled and indirect*	*Unskilled and indirect*
Short-term	6	8	10	20
Medium-term	7	10	16	9
Long-term	3	2	8	0

If S, M, L, Sk, I denote short, medium, long term, skilled and indirect respectively,

(*i*) Determine the number of workers in the following classes :

(*a*) M, (*b*) $L \cap I$, (*c*) $S \cap Sk \cap I$,

(*d*) $(M \cup L) \cap (Sk \cup I)$, (*e*) $S' \cup (S' \cap I)'$.

(*ii*) Which set of the following pairs has more workers as its members

(*a*) $(S \cup M)'$ or L

(*b*) $(I \cap Sk)'$ or S $(I \cap S')$

ANSWERS

1. See text.

2. A is a proper subset of B ; x is not an element of A ; A contains B ; singleton with an only element zero ; A is not contained in B ; A is a null set ϕ.

3. (*i*) $A = \{x : x$ is an alphabet in English language$\}$,
(*ii*) $I = \{x : x$ is an odd integer < 25 ,
(*iii*) $I = \{1, 3, 5, 7, \ldots\ldots\}$
(*iv*) $I = \{x : x^2 + 5x + 7 = 0\}$,

4. $V = \{x : x$ is a vowel$\}$,
$B = \{x : x$ is a natural number$\}$
$C = \{x : -15 < x < 15 \wedge x$ is a whole number$\}$,

5. (*a*) (*i*) $\{3, 5, 7\}$, (*ii*) $\{0, 2, 4, 6, 8\}$,
(*iii*) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
(*iv*) $\{3, 7\}$, (*v*) ϕ.

(*b*) V, R and S are equal when $x = -2$.

6. (*a*) All are incorrect.
(*b*) Only (*v*) is not corect.

7. $B = C = D$, and all these are subsets of the set A.

8. Only (*iv*) is incorrect.

9. (*i*), (*ii*), (*iii*), (*ix*), (*x*), (*xiii*) only are correct.

10. (*a*) (*i*), (*iii*), (*iv*) and (*v*) only are true.
(*b*) (*i*), (*iv*) and (*vii*) only are true.
(*c*) (*i*) finite, (*ii*) infinite, (*iii*) infinite, (*iv*) empty,

11. $A=\{1, 2, 3, 4\}$, $B=\{2, 3, 7, 9\}$, $C=\{1, 4, 7, 9\}$,
12. Try it yourself.
13. $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)=\{1, 6, 9, 10\}$
14 (*a*) (*i*) 152, (*ii*) 402, (*iii*) 20, (*iv*) 140
 (*b*) (*i*) $(C\cup D)\cap B$, (*ii*) $(A\cup B)'\cap(C'\cup D'\cup E')$
 (*iii*) $A\cup E$, (*iv*) $(A\cup B)'$.
15. (*i*) $P(A)=[\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a. b, c, d\}]$
 (*ii*) See (*i*) for the answer.
 (*iii*) The winning coalitions are :
 $\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}$. The blocking coalition is $\{b, c, d\}$. Rest are all losing coalitions.
15. (*i*) $A\cup B=\{a, b, c, d, e, f, i, o, u\}$
 (*ii*) $A\cup C=\{a, b, c, d, e, f, m, n, o, p, q, r, s, t, u\}$
 (*iii*) $B\cup C=\{a, e, i, o, u, m, n, p, q, r, s, t\}$,
 (*iv*) $A-B=\{b, c, d, f\}$ (*v*) $A\cap B=\{a, e\}$, (*vi*) $B\cap C=\{o, u\}$,
 (*vii*) $A\cup(B-C)=\{a, b, c, d, e, f, i\}$
 (*viii*) $A\cup B\cup C=\{a, b, c, d, e, f, i, o, u, m, n, p, q, r, s, t\}$,
 (*ix*) $A\cap B\cap C=\phi$.
20. (*i*) $A'=\{7, 8, 9, 10, 11, 12, 13\}$,
 (*ii*) $B'=\{4, 6, 8, 10, 11, 12, 13\}$,
 (*iii*) $C'=\{3, 4, 5, 9, 11, 13\}$,
 (*iv*) $(A')'=A$, (*v*) $(B')'=B$,
 (*vi*) $(A\cup B)'=A'\cap B'=\{8, 10, 11, 12, 13\}$,
 (*vii*) $(A\cap B)'=A'\cup B'=\{4, 6, 7, 8, 9, 10, 11, 12, 13\}$,
 (*viii*) $A'\cup C'=\{3, 4, 5, 7, 8, 9, 10, 11, 12, 13\}$,
22. $\phi, \{6\}, \{8\}, \{11\}, \{6, 8\}, \{8, 11\}, \{6, 11\}, \{6, 8, 11\}$,
23. Except (*iv*) all are true.
25. (*i*) $\{1, 2, 3, 4, 6, 8\}$, (*ii*) $\{2, 4\}$, (*iii*) $\{5, 6, 7, 8, 9\}$,
 (*iv*) $\{5, 7, 9)$, (*v*) $\{1, 3, 5, 6, 7, 9\}$.
26. (*i*) false, (*ii*) true.
27. **Hint.** $A\cap B=(A-B)\cap B$ if $A=A-B$
 $=(A\cap B')\cap B=A\cap(B'\cap B)=A\cap\phi=\phi$.
33. Left to the reader.
34. (*i*) $P\times Q=\{(1, a), (1, x), (1, y) ; (2, a), (2, x), (2, y) ; (x, a), (x, x) (x, y)\}$
 (*ii*) $P\times R=\{(1, x) ; (1, y) ; (1, z) ; (2, x) ; (2, y) ; (2, z) ; (x, x), (x, y), (x, z)\}$
 (*iii*) $Q\times R=\{(a, x), (a, y), (a, z) ; (x, x), (x, y), (x, z); (y, x), (y, y), (y, z)\}$,

(*iv*) $(P\times Q)\cap(P\times R)=\{(1, x), (1, y), (2, x), (2, y), (x, x), (x, y)\}$

(*v*) $(R\times Q)\cap(R\times P)=\{(x, x), (y, x), (z, x)\}$,

(*iv*) $(P\times Q)\cup(R\times P)=\{(1, a), (1, x), (1, y), (2, a), (2, x),$
$(2, y), (x, a), (x, x), (x, y), (x, 1),$
$(x, 2), (y, 1), (y, 2), (y, x), (z, 1),$
$(z, 2), (z, z)\}$,

[**Hints.** For (*v*) and (*vi*) above find out $R\times P$ and $P\times Q$].

35. (*a*) 24, (*b*) $B=\{4, 5, 6\}$.

36. $A\times(B\cup C)=\{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

$A\times(B\cap C)=\{(2, 5), (3, 5)\}$

$(A\times B)\cup(B\times C)=\{(2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (4, 6), (5, 5), (5, 6)\}$

39. **Hints.** $n(U)=n(A)+n(B)-n(A\cap B)$, *i e.*, $62=32+42-n(A\cap B)$

$\therefore$ $n(A\cap B)=74-62=12$.

40. **Hint.** $U=50{,}000$ $A\cap B=4{,}000$

$A\cup B=28{,}000+23{,}000-4{,}000=47{,}000$

$(A\cup B)'=50{,}000-47{,}000=3{,}000$

41. (*i*) 360, (*ii*) 280, (*iii*) 160.

42. (*i*) 6, (*ii*) 53.

43. $n(A)+n(M)+n(C)-n(A\cap M)-n(M\cap C)-n(A\cap C)$
$+n(A\cap M\cap C)=53$.

44. 88,244.

45. Inconsistent :

$$n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C)\neq n(A\cup B\cup C)$$

$$42+17+27-7-19-13+3\neq 50$$

46. (*i*) 25, (*ii*) see answer to 45. **47.** (*i*) 10. (*ii*) $60-50=10$

48. (*i*) 25, (*ii*) 40, (*iii*) 20, (*iv*) 20. **49.** No.

50. The number of items with radius defect alone was—2 which was impossible.

51. (*a*) 180 (*b*) 400, (*c*) 50.

52. (*i*) (*a*) 42, (*b*) 8, (*c*) 10, (*d*) 43, (*e*) 99.

(*ii*) (*a*) Both have equal number of workers.

(*b*) $L\cap(Sk)'$ has more workers.