Supplement

[GUIDELINES FOR BUSINESS MATHEMATICS OF B. COM. (HONS.)]

Note : *Problems and Theorems involving Trigonometrical ratios need not be done.*

1 **Calculus** : Concept of limits and continuity. Evaluation of limits, Simple application to Commerce such as Depreciation, etc.

Definition of differentiation. Theorems on differentiation. Sum and difference of functions. Multiplication and Division of functions. A function raised to power of a function. Function of a function.

Implicit function (Derivation of one function with respect to another function (Derivation of the formulae not necessary). Geometric and Economic interpretation of derivatives.

Successive differentiation. Simple standard forms (without LEIBNITZ RULE). Partial differentiation. Definition of Euler's Theorem (Ist order). Total differentiation of implicit functions.

Maxima and Minima. Cases of one variable involving second and higher order derivatives. Cases of two variables involving one constraint.

Integration. Standard forms. Reducible to standard forms. Method of substitution. Integration by parts. Use of partial fractions. Definite integration for finding areas in simple cases.

Application of Calculus to Business and Economics :

Knowledge of common forms of functions used in Business and Economics and shapes of their curves like demand function, supply function, cost function, revenue function, utility function, production function with one or more factors of production (Tracing of curve not to be done).

Mathematical interpretation of average, marginal and elasticity concepts. Derivation of their interrelations by using Calculus. Calculations of these values and from them (in simple forms) obtaining of original functions. Cross and Direct elasticities. Compound interest and rate of growth.

Problems involving one or two variables of optimum level of production under monopoly. Simple cases of duopoly. Equilibrium prices under Perfect Competition. Simple cases of inventory control. Consumers surplus and Producers surplus.

2. **Matrices** : Definition. Different types of matrices. Algebra of matrices. Transpose of a matrix. Elementary row transformations including method of finding inverse.

Determinants. Properties of determinants. Calculation of values and product of determinants up to third order.

Adjoint of a matrix and method of finding inverse. Solution of equation with the help of matrices and determinants.

Problems relating to Business and Economics based on solution of equation and matrix multiplication. Leontief input and output model.

3. **Linear Programming :** Graphical method. Problems relating to maximisation and minimisation involving two variables and inequalities of both types greater than and equal to and less than and equal to. Cases when no solution exist and multiple solution exist.

Simplex Method. Concept of slack variable. Solution of problems involving not more than three variables. Maximisation problems involving inequalities of types less than and equal to. Concept of Duality. Minimisation problems involving inequalities of type greater than and equal to, solving of them by reducing to a problem in maximisation.

4 **Probability** Concept of probability. Calculation of probability in simple cases from definition. Independent and mutually exclusive events. Addition rule for two or more mutually exclusive events. Form of the theorem when two events are not mutually exclusive. Multiplication rule for independent events. Conditional probability.

Expectation and Bayes' Theorem : Definition and simple problems relating to Business and Commerce situations.

SUGGESTED FURTHER READINGS (For Reference)

1. *A. Mizrahi & M. Sulbvan* ; Mathematics for Business and Social Sciences—John Wiley & Sons (Chapter III to XII).

2 *R.G.D. Allen* : Mathematical Analysis for Economist Macmillan St. Mertines Press—Chapter V to XII, XIV to XV.

3. *L.W.T. Shafford* : Business Mathematics—Relevant Chapters.

4. *Finite Mathematics with Business Applications* : Kemeny, Schleifer, Snell and Thompson. Prentice Hall of India—EEE Book Relevant Chapter.

5. *Linear Programming* : N. Paul Loomba. MacGraw Hill Co.

SECTION A

Applications to Commerce and Economics

FUNCTIONS

1. **Supply Functions and Demand Functions.** The *supply function* in economics is used to specify the amounts of a particular commodity that sellers have available to offer in the market at various prices. The *demand function* specifies the amounts of a particular commodity that buyers are willing to purchase at various prices. It is well known that an increase in price usually causes an increase in the supply, but a decrease in demand ; on the other hand, a decrease in prices brings about a decrease in supply but an increase in demand. The *market price* is defined as the price at which supply and demand are equal.

Let x denote the quantity of commodity demanded and p its price. x and p being variables we may write the demand function

$x=f(p)$ showing dependence of x on p or

$p=g(x)$ showing dependence of p on x

[These are the explicit forms of the implicit demand function, $F(x, p)=0$].

(*a*) The variables in the case of demand function, as in the case of other functions in economics, are hypothetical quantities and not actual observable quantities. Changes in the values of parameters cause shifts in the demand curve.

(*b*) The arguments given above apply to a supply function if x stands for the variable supply.

(*c*) (*i*) The slope of a demand curve is negative, *i.e.*, it slopes downwards from left to right indicating that demand under normal circumstances expands as price is lower.

(*ii*) The slope of a supply curve is positive, *i.e.*, supply curves normally rise from left to right.

(*d*) Examples of demand functions are :

(*i*) $Q_d=5-3p$, (*ii*) $Q_d=\frac{15}{p}$, (*iii*) $Q_d=-3p^2+p+65$,

(*iv*) $Q_d=5\sqrt{p}$ and so on.

Similarly the supply functions are :

(*i*) $Q_s=3p-3$, (*ii*) $Q_s=2p+p^2$, (*iii*) $Q_s=3p-3$, etc.

2. **Cost functions.** If x is the quantity produced of a certain good by a firm at total cost c, we write the total cost function $C=c(x)$ explicitly. We may write this in the implicit form :

$$g(C, x)=0$$

(*a*) It may be noted that the cost (C) of producing so much goods can be analysed into two parts : (*i*) fixed cost which is independent of x

(with certain limits) and (*ii*) variable cost depending on x. Thus we may have cost function of the type

$$C(x)=200+ax,$$

where a is a known constant.

(*b*) Average cost of production or cost per unit is obtained by dividing total cost by the quantity produced.

$$AC=\frac{C}{x} \quad \Rightarrow \quad C=AC.\,x$$

(*c*) Cost curves are obtained from the knowledge of production functions. Usually the cost curve is rising to the right as the cost of production generally increases with the output (x).

3. **Total Revenue Function.** Revenue is the amount of money derived from the sale of a product and depends upon the price of the product and the quantity of the product that is actually sold. If Q_d is the demand for the output of a firm at price p, then the total revenue (R) collected by the firm is

$$R(x)=pQ_d \quad \Rightarrow \quad p=\frac{R}{Q_d}$$

Thus the price p is also average revenue of the firm.

4. **Profit Function.** The revenue and cost function lead to the profit function of a firm. as the profit is the excess of revenue over the cost of production. The profit function of the firm is

$$P(x)=R(x)-C(x)$$

5. **Production Function.** Production of a firm cannot usually be expressed satisfactorily as a function of the single variable such as capital for the simple reason that production necessarily implies the coming together of several economic factors such as capital, labour, etc. The production function is written as

$$P=f(L, K)$$

where L, K are quantities of labour and capital respectively required to produce P.

In Economics the *Cobb-Douglas production function* defined as

$$P=cK^{\alpha}\ L^{\beta}\ , \qquad \alpha+\beta=1.$$

is most generally used.

$$P=100K^{0\cdot25}\ L^{0\cdot75}$$

6. **Utility Function.** If $U(x, y)$ denotes the satisfaction obtained by an individual when he buys quantities x and y of two commodities X and Y. then $U(x, y)$. the function of two variables x and y is called utility function or utility index of the individual.

(*a*) For a fixed value $U=U_0$. we get a curve $U(x, y)=U_0$. Since combinations (x, y) of the commodities X and Y which are on this curve. give the same satisfaction to the individual, he would be indifferent to the particular combination (x, y) that he buys. The curve is, therefore, known as indifference curve.

(*b*) It may be noted that for different values of U_0, we will get different indifference curves.

(*c*) If $U(x, y) = xy$, then the indifference curves are hyperbolas $xy = U_0$, where U_0 takes different values for different level of satisfaction *e.g.*, when

$$U = (x+3)(y+2)$$

$$y = \frac{U}{x+3} - 2 \text{ and } x = \frac{U}{y+2} - 3$$

7. **Overall Consumption Function.** If C is the total consumption of the community dependent on income Y and properity to consume c the aggregate consumption function is indicated by

$$C = a + cY$$

But since $Y = C + S$

$$S = Y - (a + cY)$$

This is the savings function of the community.

EQUILIBRIUM

Equilibrium price or quantity can be found by equating demand and supply functions or by calculating excess of demand over supply as shown below :

Example 1. *Find equilibrium price and quantity given the functions*

$$Q_d = 2 - 0{\cdot}02\ P$$

$$Q_s = 0{\cdot}2 + 0{\cdot}07\ P$$

Solution. Take $Q_d = Q_s$

$$\Rightarrow \quad 2 - 0{\cdot}02\ P = 0{\cdot}2 + 0{\cdot}07\ P$$

$$\Rightarrow \quad -0{\cdot}02\ P - 0{\cdot}07\ P = -2 + 0{\cdot}2$$

$$\Rightarrow \quad P = \frac{-1{\cdot}8}{-{\cdot}09} = 20$$

Aliter. Excess demand $= Q_d - Q_s$

$$\text{Excess Demand} = (2 - 0{\cdot}02\ P) - (0{\cdot}2 + 0{\cdot}07\ P)$$

$$= (2 - 0{\cdot}2) - (0{\cdot}02\ P + 0{\cdot}07\ P) = 1{\cdot}8 - 0{\cdot}09\ P$$

Equating excess demand to zero, we have

$$P = \frac{1{\cdot}8}{0{\cdot}09} = 20.$$

The equilibrium quantity is found by substituting the value of equilibrium price in any of the given demand or supply functions.

$$Q_d = 2 - 0{\cdot}02\ P$$

With $P = 20$, $Q_d = 2 - (0{\cdot}02 \times 20) = 2 - {\cdot}4 = 1{\cdot}6$.

Example 2. *Find equilibrium price by the method of excess demand given the functions* :

$$Q_d = 50 - \frac{8p}{7} \ ; \ Q_s = 10 + \frac{2p}{3}.$$

Solution. Excess demand $=Q_d-Q_s$

i.e., $$Q_d-Q_s=(50-\tfrac{8}{7}p)-(10+\tfrac{2}{3}p)$$
$$=50-\tfrac{8}{7}p-10-\tfrac{2}{3}p=40-1{\cdot}81\,p$$

Equating excess demand to zero, we have
$$1{\cdot}81\,p=40$$
$$\Rightarrow \quad p=\frac{40}{1{\cdot}81}=22{\cdot}1$$

Example 3. *Find equilibrium price given*
$$Q_d=\frac{8p}{p-2}\,;\; Q_s=p^2$$

Solution. Let $$Q_d=Q_s$$

i.e., $$\frac{8p}{p-2}=p^2 \;\Rightarrow\; 8p=p^2\,(p-2)$$

Dividing both sides by p, we get
$$8=p^2-2p$$
$$\Rightarrow \quad p^2-2p-8=0$$
$$\Rightarrow \quad p^2-4p+2p-8=0$$
$$\Rightarrow \quad p(p-4)+2\,(p-4)=0 \quad \text{or} \quad (p-4)(p+2)=0$$
$$\therefore \quad p=4 \quad \text{or} \quad p=-2.$$

Since price cannot be a negative figure, $p=4$.

Example 4. *Assume that for a closed economy, $E=C+I+G$, where E is total expenditure, C is expenditure on consumption goods, I is expenditure on investment goods and G is Government spending. For equilibrium, we must have $E\equiv Y$, where Y is total income received.*

For a certain economy, it is given that $C=15+0{\cdot}9\,Y$, $I=20+0{\cdot}05\,Y$, and $G=25$.

Find the equilibrium values of Y, C and I. How will these change if there is no Government spending ?

Solution. Here we are given that $E=C+I+G$ ...(1)

and for equilibrium $E\equiv Y$ ...(2)

From (1) and (2), we have
$$Y=C+I+G \qquad \text{...(3)}$$

Substituting the given values of C, I and G in (3), we get
$$Y=(15+0{\cdot}9\,Y)+(20+0{\cdot}05\,Y)+25=60+0{\cdot}95\,Y$$
$$\Rightarrow \quad Y(1-0{\cdot}95)=60$$
$$\Rightarrow \quad Y=\frac{60}{0{\cdot}05}=1200. \qquad \text{...(4)}$$

Now $$C=15+0{\cdot}9Y=15+\frac{9}{10}\times 1200=1095 \qquad \text{...(5)}$$

and $$I=20+0{\cdot}05Y=20+\frac{5}{100}\times1200=80. \quad \ldots(6)$$

The required equilibrium values are given by (4), (5) and (6). If there is no government spending then $G=0$ and the equilibrium equation takes the form

$$Y=C+I \quad \ldots(7)$$

Substituting the given values of C and I in (7), we find

$$Y=(15+0{\cdot}9\ Y)+(20+0{\cdot}05\ Y)=35+0\ 95\ Y$$

$$\Rightarrow \quad Y(1-0{\cdot}95)=35$$

$$\Rightarrow \quad Y=\frac{35}{0{\cdot}05}=\frac{35\times100}{5}=7000 \quad \ldots(8)$$

Now $C=15+0{\cdot}9\ Y=15+\frac{9}{10}\times700=645 \quad \ldots(9)$

and $$I=20+0{\cdot}05\ Y=20+\frac{5}{10}\times700=55. \quad \ldots(10)$$

The changed values of Y, C and I, if there is no government spending, are respectively given by (8), (9) and (10).

ELASTICITY

Elasticity of the function $y=f(x)$ at the point x is defined as the rate of "proportional change in y or $f(x)$ per unit proportional change in x".

Price Elasticity of supply is the relative change in supply in response to a relative change in price. If now x stands for supply and the supply function is written as $x=g\ (p)$, the formula for elasticity of supply retains the same form as that of η_d.

$$\eta_s=\frac{p}{x}\quad\frac{dx}{dp}, \text{ where } x \text{ is supply function.}$$

Since the slope of the supply curve is positive, η_s is also positive.

Price Elasticity of Demand. The average price elasticity of demand is the proportionate change in quantity demanded to proportionate change in price. Precisely if the demand changes from x to $x+\delta x$ when the price changes from p to $p+\delta p$, then

$$\text{Average price elasticity of demand}=\frac{\delta x/x}{\delta p/p}=\frac{p}{x}\cdot\frac{\delta x}{\delta p}$$

The point elasticity of demand. It is the elasticity of demand at a particular price level say p, by definition, it is the limiting value of average price elasticity.

Point elasticity of demand at price 'p' is

$$\eta_d=\lim_{\delta p\to0}\left\{\frac{\delta x}{\delta p}\right\}\cdot\frac{x}{p}=\lim_{\delta p\to0}\left\{\frac{\delta x}{\delta p}\right\}\cdot\frac{p}{x}=\frac{dx}{dp}\cdot\frac{p}{x}\cdot$$

In general, the slope of demand curve is negative and hence η_d is negative.

$$\therefore \qquad \eta_d = -\frac{p}{x} \cdot \frac{dx}{dp} = -\frac{\text{Marginal quantity demanded}}{\text{Average quantity damanded}}.$$

(Usually we write η_d in the form $|\eta_d|$ which means that we only consider the absolute value of η_d irrespective of its sign.)

The crucial value of η_a is 1.

When $|\eta_d| > 1$, demand is *elastic.*
When $|\eta_d| < 1$, demand is *inelastic.*
When $|\eta_d| = 1$, *demand has unit elasticity.*

Working rule for finding Elasticity of demand :

If $x = f(p)$ is the demand function, then

(*i*) Marginal quantity demanded $= \dfrac{dx}{dp}$.

(*ii*) Average quantity demanded $= \dfrac{x}{p}$.

(*iii*) $\qquad |\eta_d| = \dfrac{dx/dp}{x/p}$

Illustration 1. *If the demand law is* $p = \dfrac{10}{(x+1)^2}$, *find the elasticity of demand in terms of* x.

Solution. The elasticity of demand is defined as

$$\eta_d = -\frac{p}{x} \times \frac{dx}{dp}. \qquad \ldots(*)$$

$$\text{Given : } p = \frac{10}{(x+1)^2} = 10\,(x+1)^{-2}$$

$$\frac{dp}{dx} = 10\,.\,(-2)(x+1)^{-3} = -\frac{20}{(x+1)^3}.$$

Substituting the values in (*), we get

$$\eta_d = -\frac{10}{(x+1)^2} \times \frac{1}{x} \times \left\{-\frac{(x+1)^3}{20}\right\} = \left(\frac{x+1}{2x}\right).$$

Illustration 2. *Find the elasticity of demand for the demand functoin* $x = \dfrac{27}{p^3}$, *where* x *is the demand of a good at a price* p.

Solution. Marginal quantity demanded

$$= \frac{dx}{dp} = -\frac{81}{p^4}.$$

Average quantity demanded

$$= \frac{x}{p} = \frac{27}{p^3} \cdot \frac{1}{p} = \frac{27}{p^4}.$$

Hence η_e = elasticity of demand $=\left|\dfrac{dx/dp}{x/p}\right|$

$$=\left|\frac{(-81/p^4)}{27/p^4}\right|=3.$$

Illustration 3. Find η_d when $p=5$, if the demand function is $x=50+p-p^2$ where x is the demand for the commodity at price p.

Marginal quantity demanded $=\left(\dfrac{dx}{dp}\right)=1+2p.$

Average quantity demanded $=\left(\dfrac{x}{p}\right)=(50+p+p^2)/p.$

$$|\eta_d|=\left|\frac{\text{Marginal quantity demanded}}{\text{Average quantity demanded}}\right|=\left|\frac{p(1+2p)}{50+p+p^2}\right|$$

$\therefore$ η_d when $p=5$ is given by

$\eta_e=\dfrac{5(1+2\times5)}{50+5+25}=\dfrac{55}{80}$ which is <1 shows that the demand is inelastic.

Income elasticity of demand. It is the elasticity of quantity demanded in response to change in income. It is defined as

$$\eta_y=\frac{y}{x}\cdot\frac{dx}{dy}$$

where x is the quantity demanded and y is the income per head in the relevant group of people.

If $\eta_y>1$, the goods are Luxury.

$0<\eta_y<1$, the goods are Necessity of life.

and $\eta_y<0$, goods are Inferior.

Remark. Elasticities can also be expressed in terms of logarithms. For example, let demand curve be

$$x=f(p)$$

then $\dfrac{d}{dp}(\log x)=\dfrac{1}{x}\cdot\dfrac{dx}{dp}$ and $\dfrac{d}{dp}(\log p)=\dfrac{1}{p}$

$$\Rightarrow\quad |\eta_d|=\frac{p}{x}\cdot\frac{dx}{dp}=\frac{\left[\frac{1}{x}\frac{dx}{dp}\right]}{1/p}=\frac{\frac{d}{dp}(\log x)}{\frac{d}{dp}(\log p)}=\frac{d(\log x)}{d(\log p)}$$

Example 5. *Find the elasticity of demand w.r.t. price for the following demand functions :*

(*a*) $p=\sqrt{a-bD}$, *a and b being constants.*

(*b*) $D=\dfrac{8}{p^{3/2}}$, (*c*) $D=p^a e^{-b(p+c)}$; *a, b and c are constants.*

Solution. $|\eta_d|=\left|\frac{p}{D}\times\frac{dD}{dp}\right|$

(*a*) $$p=(a-bD)^{\frac{1}{2}}$$

$$\therefore \quad \frac{dp}{dD}=\frac{1}{2}(a-bD)^{-\frac{1}{2}}\times(-b)=-\frac{b}{2(a-bD)^{1/2}}$$

$$\therefore \quad \frac{dD}{dp}=\frac{1}{(dp/dD)}=-\frac{2(ab-D)^{1/2}}{b}$$

$$\therefore \quad |\eta_d|=\left|\frac{(ab-D)^{1/2}}{D}\times\left[-\frac{2(a-bD)^{1/2}}{b}\right]\right|$$

$$=\frac{2}{bD}(a-bD)$$

(*b*) $$D=\frac{8}{p^{3/2}}=8p^{-3/2}.$$

$$\frac{dD}{dp}=8\times\left(-\frac{3}{2}\right)\times p^{-\frac{3}{2}-1}=-12p^{-5/2}$$

$$\therefore \quad |\eta_d|=\left|\frac{p}{8p^{-3/2}}\times\left[-12p^{-5/2}\right]\right|$$

$$=\frac{12}{8}\times p^{1-\frac{5}{2}+\frac{3}{2}}=\frac{3}{2}p^0=\frac{3}{2}$$

(*c*) $$D=p^a\,e^{-b(p+c)}$$

$$\therefore \quad \frac{dD}{dp}=a\,.\,p^{a-1}\,e^{-b(p+c)}+p^a\,.\,e^{-b(p+c)}\,(-b)$$

$$=p^a\,.\,e^{-b(p+c)}\left[\frac{a}{p}-b\right]$$

$$=p^{a-1}\,e^{-b(p+c)}[a-bp]$$

$$\therefore \quad |\eta_d|=\left|\frac{p}{p^a\,e^{-b(p+c)}}\times[p^{a-1}\,e^{-b(p+c)}\,(a-bp)]\right|$$

$$=(a-bp).$$

Example 6. *Given the demand function*

$$Q=\frac{20}{p+1}, \text{ find the elasticity at point } p=3.$$

Solution. $$Q=\frac{20}{(p+1)}=20(p+1)^{-1}$$

$$\therefore \quad \frac{dQ}{dp}=-20\,(p+1)^{-2}=-\frac{20}{(p+1)^2}$$

$$\therefore \qquad |\eta_d| = \left| -\frac{p(p+1)}{20} \times \frac{20}{(p+1)^2} \right| = \frac{p}{p+1}$$

$\therefore$ η_d when $p=3$ is given by

$$\eta_d = \frac{3}{3+1} = \frac{3}{4} = 0{\cdot}75$$

Example 7. *A demand function is given by $xp^n = k$, where n and k e constants. Calculate price elasticity of demand.*

Solution. Here $x = kp^{-n}$

$$\therefore \qquad \frac{dx}{dp} = -nkp^{-n-1}$$

Now
$$|\eta_d| = \left| \frac{p}{x} \cdot \frac{dx}{dp} \right|$$

$$= \left| \frac{p}{kp^{-n}} \times (-nkp^{-n-1}) \right| = n$$

Hence the demand curve $xp^n = k$ has elasticity equal to n at all level of prices.

Example 8. *Show that the elasticity of demand at all points on the curve $xy = \alpha^2$ will be numerically equal to one.*

Solution. Here $x = \alpha^2 y^{-1}$

$$\therefore \qquad \frac{dx}{dy} = -\alpha^2 y^{-2}$$

Now
$$|\eta_d| = \left| \frac{y}{x} \cdot \frac{dx}{dy} \right|$$

$$= \left| \frac{y}{\alpha^2 y^{-1}} \times (-\alpha^2 y^{-2}) \right| = 1.$$

Hence the elasticity of demand at all points on the curve is one.

Example 9. *Find the elasticities of demand and supply at equilibrium price for demand function $p = \sqrt{100 - x}$ and supply function $x = 2p - 10$, where p is price and x is quantity.* [*Delhi Univ. B. Com.* (*Hons.*), *1992*]

Solution. Equilibrium conditions are determined by equating demand and supply laws.

$$\therefore \qquad \sqrt{100 - x^2} = \frac{x+10}{2}$$

$$\Rightarrow \qquad 4(100 - x^2) = x^2 + 20x + 100$$

$$\Rightarrow \qquad x^2 + 4x - 60 = 0$$

$$\Rightarrow \qquad (x+10)(x-6) = 0$$

$$\therefore \qquad x = 6 \quad \text{or} \quad x = -10$$

$x = -10$ is not admissible as quantity cannot be negative.

$$\therefore \qquad x = 6$$

$$\therefore \qquad p = \frac{x+10}{2} = \frac{6+10}{2} = 8.$$

$$\therefore \qquad \eta_d = -\frac{p}{x} \cdot \frac{dx}{dp} \text{ and } \eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$$

Calculation of η_d.

$$\overline{100-x^2}$$

$$\therefore \qquad \frac{dp}{dx} = \frac{1}{2} \cdot (100-x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{100-x^2}}$$

$$\therefore \qquad \eta_d = -\frac{p}{x} \cdot \frac{dx}{dp} = -\frac{8}{6} \left(-\frac{\sqrt{100-6^2}}{6}\right) = \frac{16}{9},$$

Calculation of η_s.

$$x = 2p - 10$$

$$\therefore \qquad \frac{dx}{dp} = 2$$

$$\therefore \qquad \eta_s = \frac{p}{x} \cdot \frac{dx}{dp} = \frac{8}{6} \cdot 2 = \frac{8}{3}.$$

Marginal Revenue and Elasticity of Demand

We know

Total Revenue = Price × Quantity sold

or $\qquad R = p \times x$

$$\text{Average revenue } (AR) = \frac{R}{x} = p$$

$$\text{Marginal revenue } (MR) = \frac{dR}{dx} = p + x \frac{dp}{dx}$$

$$= p\left(1 + \frac{x}{p} \cdot \frac{dp}{dx}\right)$$

$$= p\left(1 - \frac{1}{\eta_d}\right)\left[\text{Since } \eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}\right]$$

$$\therefore \qquad MR = p\left(1 - \left|\frac{1}{\eta_d}\right|\right)$$

or $$MR = AR\left(1 - \left|\frac{1}{\eta_d}\right|\right) \quad \text{or} \quad AR = MR \cdot \frac{|\eta_d|}{|\eta_d| - 1}$$

It follows from this that when

(*i*) $|\eta_d| = 1$, *TR* remains constant with a fall in price

(*ii*) $|\eta_d| > 1$, *TR* rises with a fall in price

(*iii*) $|\eta_d| < 1$, *TR* falls with a fall in price.

Example 10. *Verify the relationship*

$$MR=p\left(1-\frac{1}{\eta_d}\right)$$

for the demand function $p=(12-x)^{1/2}$, $0\leqslant x\leqslant 12$.

Solution. We have $p=(12-x)^{1/2}$ ...(1)

$$\therefore \quad \frac{dp}{dx}=(\tfrac{1}{2})(12-x)^{-1/2}(-1)$$

$$\therefore \quad \frac{dx}{dp}=\frac{1}{(dp/dx)}=-2(12-x)^{1/2}$$

$$\therefore \quad \eta_d=-\frac{p}{x}\cdot\frac{dx}{dp}=\left\{-\frac{(12-x)^{1/2}}{x}\right\}\left\{-2(12-x)^{1/2}\right\}$$

$$=\frac{2(12-x)}{x} \qquad \text{...(2)}$$

The total revenue is

$$R=px=x(12-x)^{1/2}$$

$$\therefore \quad MR=(12-x)^{1/2}-\frac{x}{2}(12-x)^{-1/2}$$

$$=(12-x)^{1/2}\left[1-\frac{x}{2(12-x)}\right]$$

$$=p\left(1-\frac{1}{\eta_d}\right) \text{[From (1) and (2)]}$$

Hence $MR=p\left(1-\frac{1}{\eta_d}\right)$

Example 11. *If AR and MR denote the average and marginal revenue at any output, show that elasticity of demand is equal to* $\frac{AR}{AR-MR}$. *Verify this law for the linear demand law* $p=a+bx$.

Solution. Total revenue : $R=px$

$$AR=\frac{R}{x}=p, \text{ whereas } MR=\frac{dR}{dx}=p+x\cdot\frac{dp}{dx}$$

$$\text{Now } \frac{AR}{AR-MR}=\frac{p}{p-\left(p+x\frac{dp}{dx}\right)}=\frac{p}{x}\cdot\frac{1}{\frac{dp}{dx}}$$

$$=\frac{p}{x}\cdot\frac{dx}{dp}=|\eta_d|$$

For $p=a+bx$,

$$R=px=ax+bx^2$$

So
$$AR=\frac{R}{x}=a+bx \text{ and } MR=\frac{dR}{dx}=a+2bx$$

Also
$$|\eta_d|=\frac{p}{x}\cdot\frac{dx}{dp}=\left(\frac{a+bx}{x}\right)\cdot\frac{1}{\frac{dp}{dx}}$$

$$=\left(\frac{a+bx}{x}\right)\cdot\frac{1}{b}=\frac{a+bx}{bx}=\frac{(a+bx)}{(a+bx)-(a+2bx)}$$

$$=\frac{AR}{AR-MR}$$

EXERCISE (I)

1. What do you understand by market equilibrium ? State its uses. Explain your answer graphically also.

Find the market equilibrium of prices and quantities if the demand laws for two commodities are :

$$x=5-p+q, \qquad y=10-p+q$$

and the supply laws are

$$x=-5+p+q, \qquad y=-2-p+2q$$

where p and q represent the price per unit of commodities x and y respectively.

2. Explain what you understand by market equilibrium Show graphically or otherwise that no price other than the equilibrium price can last longer in the market.

Find the market equilibrium price and quantities if the demand laws for two commodities are

$$p=24-x-2y, \qquad q=-27-x-3y$$

and supply laws are

$$x=-6+2p-q, \qquad y=-3-p+8q$$

where p and q represent the price per unit of commodities x and y respectively.

3. Find the equilibrium prices and quantities for the two commodity market models :

$$x_{d_1}=-2-p+q, \qquad x_{s_1}=-2-q$$
$$x_{d_2}=-3-p-q, \qquad x_{s_2}=-9+p+q$$

where p is price and q is quantity.

[**Hint.** At equilibrium, $x_{1d}=x_{s_1}$, and $x_{d_2}=x_{s_2}$]

4. (*a*) Explain (*i*) Demand function and Supply function. (*ii*) Market equilibrium.

(*b*) The price p of a certain commodity is partly constant and partly varies as the reciprocal of the quantity demanded d. The supply function is $S=\alpha+\beta p$ where α and β are constants. The demand and supply curves were drawn on the same graph taking the quantity on x-axis and price on y-axis. The equilibrium point is (4, 6) and at price 5 units, the quantity demanded and the quantity supplied are 5 and 3 units respectively. Determine the demand and supply function and find the price when (*i*) the quantity demanded is 8 units and (*ii*) the quantity supplied is 10 units.

5. Explain what you understand by Demand and Supply function and Market equilibrium.

The demand law of a commodity is $p=m\sqrt{x}+n$. If the price is one unit, the demand is 100 units and if the demand is 16 units, the total revenue is 144 units. Find the constant m and n.

If $p=2$ units, what is the total revenue ?

6. Explain the effect of taxation on market equilibrium.

The demand law is $3p+2x=27$ and supply law is $6p-2x=9$

(*a*) If the tax of $\frac{3}{4}$ per unit is imposed, find the equilibrium price and quantity and the total government revenue.

(*b*) If a subsidy of 1 per unit is granted, find the new price and quantity and total government expenditure.

7. Explain what you understand by demand and supply functions. State their uses. State reasons for the change in demand and supply of a commodity.

When the price of sweets was Rs. 3 per kg. its demand was 12 thousand kg. and when the price was Rs. 5 per kg. its demand was 8 thousand kg. If the demand function is $p=\sqrt{a-bx}$. find the values of the constants a and b. What will be demand when the price is Rs. 7 per kg.? Which of these three prices of sweets will give more benefit ?

8. The demand curve and the supply curve of a commodity are given by $D=19 \quad 3p-p^2$ and $S=5p-1$. Find the equilibrium price and the quantity.

[**Hint**. For equilibrium, we have $D=S$

$$\Rightarrow \qquad 19-3p-p^2=5p-1$$

$$\Rightarrow \qquad p^2+8p-20=0$$

$$\Rightarrow \qquad (p+10)(p-2)=0, \text{ i.e., } p=2 \text{ and } p=-10$$

We reject the value $p=-10$, since price cannot be negative. Hence equilibrium price is $p=2$ and substituting it in the demand or supply curve, we get

$$D=S=9.]$$

9. The demand functions of two commodities A and B are

$$D_A=10-p_A-2p_B, \quad D_B=6-p_A-p_B$$

and the corresponding supply functions are

$$S_A=-3+p_A+p_B, \quad S_B=-2+p_B$$

where p_A and p_B denote the prices of A and B respectively. Find (*i*) The equilibrium prices, and

(*ii*) The equilibrium quantities exchanged in the market.

[**Hint.** For equilibrium, we have

$$D_A=S_A \quad \text{and} \quad D_B=S_B$$

$$\Rightarrow \quad 10-p_A-2p_B=-3+p_A+p_B \text{ and } 6-p_A-p_B=-2+p_B$$

$$\Rightarrow \quad 2p_A+3p_B-13=0 \quad \text{and} \quad p_A+2p_B-8=0$$

Solving, we get the equilibrium prices as

$$p_A=2 \quad \text{and} \quad p_B=3$$

Substituting in demand function or supply function, the equilibrium quantities are given by

$$D_A=S_A=2 \quad \text{and} \quad D_B=S_B=1.]$$

10. The demand y for a commodity when its price is x, is given by $y=\frac{x+2}{x-1}$; find the elasticity of demand when the price is 3 units.

11. Define elasticity of demand. Interpret $\eta=\frac{1}{2}$, $\eta=\frac{1}{3}$.

12. Define demand elasticity η for a given demand law and interpret the cases when $\eta>1$, $\eta=1$ and $\eta<1$

If AR and MR be the average and marginal revenue at any output show that $\eta=\frac{AR}{AR-MR}$ at this output. Verify this relation for the demand law $p=a-bx$.

13. Define elasticity of a function. Hence or otherwise explain in particular the elasticity of demand and supply.

If t is the elasticity of $f(x)$, then find the elasticities of $xf(x)$ and $\frac{f(x)}{x}$.

14. The supply of certain goods is given by $x_s=a\sqrt{p-b}$, when p is price and a and b are positive constants $(p>b)$, find an expression for elasticity of supply e_s. Show that e_s decreases as price and supply increases and becomes unity at the price=2b.

15. Express the elasticities of demand in terms of q for the following demand laws :

(*a*) $p=(a-bq)^2$

(*b*) $p=\sqrt{a-bq}$

(*c*) $p=\frac{a}{q+b}-c$

16. Determine the price elasticities of demand for the following :

(*a*) $p=qe^q$, (*b*) $p=qe^{-q}$

(c) $a=qe^{\frac{1}{q^2}}$ (d) $q=b\,p^{-a}$, (e) $q=\frac{b}{p}$

17. If the demand function is $p=4-5x^2$, for what value of x, the elasticity of demand will be unity ?

[**Hint.** $p=4-5x^2$

Differentiating w.r.t. p, we get

$$1=-10x\times\frac{dx}{dp} \quad\Rightarrow\quad \frac{dx}{dp}=-\frac{1}{10x}$$

$$\therefore\quad \eta=-\frac{p}{x}\cdot\frac{dx}{dp}=\frac{4-5x^2}{10x^2}.$$

Elasticity of demand will be unity if

$$\frac{4-5x^2}{10x^2}=1 \quad\Rightarrow\quad 15x^2=4 \quad\text{or}\quad x=\frac{2}{\sqrt{15}}.\Big]$$

18. If the demand curve is of the form

$$p=a\,e^{-kx}$$

where p is the price and x is the demand, prove that the elasticity of demand is $\frac{1}{kx}$. Hence deduce the elasticity of demand for the curve

$$p=10\,e^{-x/2}$$

[**Hint.** We have $p=a\,e^{-kx}$

$$\therefore\quad 1=-ak\,e^{-kx}.\,\frac{dx}{dp}$$

$$\therefore\quad |\eta_d|=\left|-\frac{p}{x}\cdot\frac{dx}{dx}\right|=\left|\frac{ae^{\quad kx}}{x}\cdot\frac{1}{ake^{-kx}}\right|$$

$$=\frac{1}{kx}.$$

For the curve $p\quad 10e^{-x/2}$, we have

$$a=10,\ k-\frac{1}{2}$$

$\therefore$ The elasticity of demand for the curve

$$\eta\quad 10e^{\ x/2}$$

is given by

$$|\eta_d|-\frac{2}{x}\Big|$$

19. Define elasticity of a function. Hence or otherwise explain in particular the elasticity of demand and supply. State the uses of elasticity in Economics.

If the demand curve is given by

$$x = p^{a}\, e^{-b(p+c)}$$

show that the demand increases as the price decreases, becoming large as the price approaches the value a/b. Find the effect of any price greater than a/b, on the elasticity of demand.

20. Compare the elasticities of the demand curves

$$x = \frac{a}{p-c} - b \text{ and } p = \left(\frac{b}{x-c}\right)^{1/a}$$

at a price p ; a, b, c are positive constants, x is the quantity demanded, and p is the price ?

Total, Average and Marginal Cost

Total cost (C) is represented as a function of output x, *i.e.*,

$$C = f(x)$$

Remark. Some books use the notation $C = f(Q)$ where C is the cost and Q is the output.

$$\therefore \quad \text{Average cost} = \frac{C}{x} \text{ or } \frac{f(x)}{x}$$

The average cost (AC) represents the cost per unit of production.

The term marginal cost represents the change in the total cost for each additional unit of production. MC is the first derivative of the total cost function, *i.e.*,

$$\text{Marginal cost } (MC) = \frac{dC}{dx} \cdot$$

Let us now generalise the total cost function :

Total cost $(TC) = f(x) + b$ (b is fixed cost)

From this total cost function, other cost functions can be derived as follow :

$$\text{Average cost } (AC) = \frac{f(x)+b}{x}$$

$$\left.\begin{array}{ll} \text{Average variable cost } (AVC) = \dfrac{f(x)}{x} & \\ \text{Average fixed cost} & (AFC) = \dfrac{b}{x} \\ \text{Marginal cost} & (MC) = \dfrac{dC}{dx} \end{array}\right\} \begin{array}{l} ATC = AVC + AFC \\ \quad = \dfrac{f(x)+b}{x} \end{array}$$

Relation between Average and Marginal Cost Curves

Although cost functions may assume many different shapes under different circumstances, yet usually under natural economic limitations, we assume average and marginal costs to have U-shapes. The relation between them is established as follows :

We know $AC=\frac{C}{x}$, the slope is given by

$$\frac{d}{dx}\left(\frac{C}{x}\right)=\frac{x\,\frac{dC}{dx}-C}{x^2}=\frac{1}{x}\left(\frac{dC}{dx}-\frac{C}{x}\right)$$

$$=\frac{1}{x}\ (MC-AC)$$

Case I. *When average cost curve slopes downwards, i.e., when AC is declining,* its slope will be negative. In other words,

$$\frac{d}{dx}\left(\frac{C}{x}\right)<0$$

$$\Rightarrow \qquad (MC-AC)<0$$

$$\Rightarrow \qquad MC<AC$$

Thus when AC curve slopes downwards MC curve will lie below AC curve.

Case II. *When AC curve reaches a minimum point,* its slope becomes zero, *i.e.,*

$$\frac{d}{dx}\left(\frac{C}{x}\right)=0$$

$$\Rightarrow \qquad MC-AC=0$$

$$\Rightarrow \qquad MC=AC$$

Thus MC curve and AC curve intersect at the point of minimum average cost.

Case III. *When average cost curve rises upwards, its slope is positive.* In other words,

$$\frac{d}{dx}\left(\frac{C}{x}\right)>0 \qquad \Rightarrow \qquad MC>AC$$

Thus when AC curve slopes upwards, MC curve will be above AC curve.

Example 12. *The total cost C for output x is given by*

$$C=\frac{2}{3}\ x+\frac{35}{2}$$

Find (i) Cost when output is 4 units,

(ii) Average cost of output of 10 units.

(iii) Marginal cost when output is 3 units.

Solution. (*i*) $C=\frac{2}{3}\ x+\frac{35}{2}$

$$\therefore \qquad C \text{ for 4 units} = \frac{2}{3}\ (4) + \frac{35}{2} = 20{\cdot}16 \text{ units}$$

(*ii*) $$C \text{ for 10 units} = \frac{2}{3}\ (10) + \frac{35}{2} = \frac{145}{6} = 24{\cdot}16 \text{ units}$$

$$AC = \frac{145}{6} \times \frac{1}{10} = \frac{29}{12} = 2{\cdot}42 \text{ units}$$

(*iii*) $$MC = \frac{d}{dx}\left(\frac{2}{3}\ x + \frac{35}{2}\right) = \frac{2}{3} = 0{\cdot}67 \text{ units}$$

(MC is constant here

Example 13. *The average cost function* (AC) *for a commodity is given by*

$$AC = x + 5 + \frac{36}{x}$$

in terms of the output x. Find the outputs for which AC is increasing and the outputs for which AC is decreasing, with increasing output.

Also, find the total cost C and the marginal cost (MC) *as function of x.*

Solution. Slope of $AC = \frac{d}{dx}\left(x + 5 + \frac{36}{x}\right) = 1 - \frac{36}{x^2}$

AC is increasing if $1 - \frac{36}{x^2} > 0$, *i.e.*, if $x^2 > 36$, or $x > 6$

and decreasing if $1 - \frac{36}{x^2} < 0$, *i.e.*, if $x < 6$.

Now $$AC = x + 5 + \frac{36}{x} = \frac{x^2 + 5x + 36}{x}$$

$\Rightarrow$ Total cost $(C) = x.\ AC = x^2 + 5x + 36$

Marginal cost $(MC) = \frac{dC}{dx} = 2x + 5.$

Example 14. *The total cost function of a firm is given by*

$$C = 0{\cdot}04\ q^3 - 0{\cdot}9\ q^2 + 10\ q + 10$$

Find (*a*) *Average cost* (AC).

(*b*) *Marginal cost* (MC).

(*c*) *Slope of AC.*

(*d*) *Slope of MC.*

(*e*) *Value of q at which average variable cost is minimum.*

Solution. (*a*) $AC = \frac{C}{q} = 0{\cdot}04\ q^2 - 0{\cdot}9\ q + 10 + \frac{10}{q}$

(*b*) Marginal cost $(MC) = \frac{dC}{dq} = 0{\cdot}12\ q^2 - 1{\cdot}8\ q + 10$

(c) Slope of $AC=\frac{d}{dq}\left(\frac{C}{q}\right)$

$$=\frac{d}{dq}\left(0{\cdot}04\ q^2-0{\cdot}9\ q+10+\frac{10}{q}\right)$$

$$=\left(0{\cdot}08\ q-0{\cdot}9-\frac{10}{q^2}\right)$$

$$=\frac{1}{q}\left(0{\cdot}08\ q^2-0{\cdot}9\ q-\frac{10}{q}\right)$$

$$=\frac{1}{q}\left[(0{\cdot}12\ q^2-1{\cdot}8\ q+10)-\left(0{\cdot}04\ q^2-0{\cdot}9\ q+10+\frac{10}{q}\right)\right]$$

$$=\frac{1}{q}\ [MC-AC]$$

(d) Slope of $MC=\frac{d}{dq}\left(\frac{dMC}{dq}\right)=0{\cdot}24\ q-1{\cdot}8$

(e) When AVC is minimum, the slope of AVC curve is zero, *i.e.*,

$$\frac{d}{dq}\ (AVC)=0 \qquad \text{or} \quad \frac{d}{dq}\ (0{\cdot}04\ q^2-0{\cdot}9\ q+10)=0$$

$$\Rightarrow \qquad 0{\cdot}08\ q-0{\cdot}9=0 \qquad \text{or} \quad q=\frac{0{\cdot}9}{0{\cdot}08}=11{\cdot}25$$

Example 15. *Let the cost function of a firm be given by the following equation :*

$C=300x-10x^2+\frac{1}{3}\ x^3$, *where C stands for cost and x for output.*

Calculate (i) Output, at which marginal cost is minimum.

(ii) Output, at which average cost is minimum.

(iii) Output, at which average cost is equal to marginal cost.

[*I.C.W.A., June 1991*]

Solution. (*i*) $C=300x-10x^2+\frac{1}{3}\ x^3$

$$\therefore \qquad MC=\frac{dC}{dx}=300-10\ (2x)+\frac{1}{3}\ .\ 3x^2$$

$$=300-20x+x^2$$

Differentiating w.r.t. x and equating to zero, we have

$$\frac{d(MC)}{dx}=-20+2x=0$$

or $\quad x=10$ is the necessary condition for marginal cost minimisation.

To get the sufficient condition, we have

$\frac{d^2(MC)}{dx^2}=2$, a positive quantity which means that marginal cost is minimum at $x=10$.

(*ii*) Average Cost $(AC)=\frac{C}{x}=\frac{300x-10x^2+\frac{1}{3}x^3}{x}$

$$=300-10\,x+\frac{1}{3}\,x^2$$

Now to find output at which average cost is minimum, we have to differentiate the AC and equating it to zero.

$$\therefore \qquad \frac{d\,(AC)}{dx}=0-10+\tfrac{1}{3}\,.\,2x=0$$

or $$x=15$$

Also $\frac{d^2\,(AC)}{dx^2}=\frac{d}{dx}\left(-10+\frac{2}{3}\,x\right)=\frac{2}{3}$, a positive quantity.

$\therefore$ Second condition is also satisfied. Hence the output at which AC is minimum is given by $x=15$.

(*iii*) Now $\qquad AC=MC$

$$\Rightarrow \qquad 300-10x+\tfrac{1}{3}x^2=300-20x+x^2$$

$$\Rightarrow \qquad \frac{3x^2}{3}=10x \qquad \text{or} \qquad x=15$$

Hence for $x=15$, average cost is equal to marginal cost.

Example 16. *The total variable cost of a monthly output x tons by a firm producing a variable metal is Rs.* $\frac{1}{10}x^3-3x^2+5x$ *and the fixed cost is Rs. 300 per month. Draw the average cost curve when cost includes* (*i*) *variable cost only*, (*ii*) *all costs. Find the output for minimum average cost in each case.* [*Delhi Univ. B.A.* (*Hons.*) *Econ., 1991*

Solution. We have

$$TC=\text{Total cost}=\frac{1}{10}x^3-3x^2+5x+300$$

and $$TVC=\text{Total variable cost}-\frac{1}{10}\,x^3-3x^2+5x$$

(*i*) *When cost includes variable cost only* :

$$AVC=\text{Average variable cost}=\frac{TVC}{x}=\frac{1}{10}\,x^2-3x+5$$

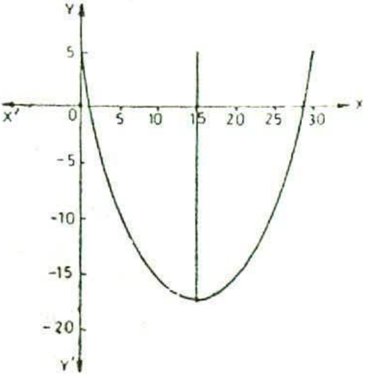
It is a parabola with vertex at $(15, -17\cdot5)$ and the axis of the parabola is $x=15$. The graph of the curve is shown in the figure below

$$\frac{d(AVC)}{dx}=\frac{1}{5}x-3$$

and $$\frac{d^2(AVC)}{dx^2}=\frac{1}{5}>0$$

$$\frac{d(AVC)}{dx}=0 \Rightarrow \frac{1}{5}x-3=0$$

or $$x=15$$



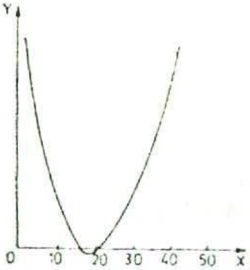
Hence average cost is minimum when the output is 15 tons.

(ii) *When cost includes all costs :*

$$AC=\text{Average cost}=\frac{TC}{x}$$

$$=\frac{1}{10}x^2-3x+\frac{300}{x}$$

The graph of the curve is shown in the adjoining figure.



$$\frac{d(AC)}{dx}=\frac{1}{5}x-3-\frac{300}{x^2}$$

and $$\frac{d^2(AC)}{dx^2}=\frac{1}{5}+\frac{600}{x^3}>0$$

$$\frac{d(AC)}{dx}=0 \Rightarrow \frac{1}{5}x-3-\frac{300}{x^2}=0$$

which gives $$x=19\cdot1$$

Hence average cost is minimum when the output is 19·1 tons.

Conditions for Profit Maximization

We know that if $y=f(x)$ then for y to be maximum,

$$\frac{dy}{dx}=f'(x)=0 \text{ and } \frac{d^2y}{dx^2}=f''(x)<0$$

Now assuming that we are given the total cost function along with the total revenue function—both in terms of output, *i.e.*, given functions are :

Total cost function $C=f(x)$
Total revenue function : $R=\phi(x)$
Total profit : $P=R-C=\phi(x)-f(x)$

For P to be maximum, the conditions are :

First order condition :

$$\frac{dP}{dx}=\frac{dR}{dx}-\frac{dC}{dx}=0$$

$$\Rightarrow \qquad \frac{dR}{dx}-\frac{dC}{dx}=0$$

$$\Rightarrow \qquad MR=MC$$

Thus, the profit P is maximized at that quantity x for which marginal revenue equals marginal cost.

Remark. It may be noted that $MR=MC$ means that slope of total revenue function = slope of total cost function.

Second order condition :

$$\frac{d^2P}{dx^2}=\frac{d^2R}{dx^2}-\frac{d^2C}{dx^2}<0$$

$$\Rightarrow \qquad \frac{d^2R}{dx^2}<\frac{d^2C}{dx^2}$$

$$\Rightarrow \qquad \frac{d}{dx}\left(\frac{dR}{dx}\right)<\frac{d}{dx}\left(\frac{dC}{dx}\right)$$

$$\Rightarrow \qquad \frac{d}{dx}(MR)<\frac{d}{dx}(MC)$$

Thus rate of change of MR (slope of MR) should be less than the rate of change of MC (*i.e.*, slope of MC) at the profit maximising output level.

We shall now discuss the problem of maximization of the profits of a firm under various market conditions :

(*a*) **Perfect competition.** Under perfect competition, the price p is constant. The profit maximization conditions obtained above, *viz.*,

$MR=MC$, which is the condition for equilibrium of a firm and

$\frac{d(MR)}{dx}$ (*i.e.*, the rate of change of MR) should be less than $\frac{d(MC)}{dx}$ (*i.e.*, the rate of change of MC) at the equilibrium output.

(*b*) **Monopoly.** Under monopoly, the monopolist fixes the output leaving price to be determined by demand conditions.

The profit maximization conditions, obtained above, apply to this case also.

Example 17. *Find the profit maximising output given the following revenue and cost functions* :

$$R(Q)=1000\,Q-2Q^2$$

$$C(Q)=Q^3-59Q^2+1315Q+2000.$$

[*Delhi Univ., B.A.* (*Hons.*) *Econ., 1991*]

Solution. We have

$$P = \text{Profit} = R(Q) - C(Q)$$

$$= (1000Q - 2Q^2) - (Q^3 - 59Q^2 + 1351Q + 2000)$$

$$= -Q^3 + 57Q^2 - 315O - 2000$$

First order condition ;

$$\frac{dP}{dQ} = 0$$

$\therefore$ $$\frac{dP}{dQ} = -3Q^2 + 114Q - 315$$

$$\frac{dP}{dQ} = 0 \quad \Rightarrow \quad -3Q^2 + 114Q - 315 = 0$$

or $$Q^2 - 38Q + 105 = 0$$

or $$(Q-3)(Q-35) = 0$$

$\therefore$ $$Q = 3 \quad \text{or} \quad Q = 35$$

Second order condition :

$$\frac{d^2P}{dQ^2} < 0$$

$$\frac{d^2P}{dQ^2} = -6Q + 114$$

$$\left.\frac{d^2P}{dQ^2}\right|_{Q=3} = -18 + 114 = 96 > 0$$

$$\left.\frac{d^2P}{dQ^2}\right|_{Q=35} = -210 + 114 = -96 < 0$$

Hence the profit maximising output is given by $Q = 35$.

Example 18. *A radio manufacturer produces x sets per week at a total cost of Rs.* $(x^2 + 78x + 2500)$. *He is a monopolist and the demand function for his product is* $x = \frac{600 - p}{8}$ *when the price is Rs. p per set. Show that maximum net revenue (i.e., profit) is obtained when 29 sets are produced per week. What is the monopoly price ?*

Solution. Total cost $(C) = x^2 + 78x + 2500$

Marginal $(MC) = \frac{dC}{dx} = 2x + 78$

Demand function is $x = \frac{600 - p}{8}$

$\Rightarrow$ $$8x = 600 - p$$

$\Rightarrow$ $$p = 600 - 8x$$

Now total revenue for x sets is

$$R=p\times x=(600-8x)\ x=600x-8x^3$$

Marginal revenue $(MR)=\frac{dR}{dx}=\frac{d}{dx}\ (600x-8x^2)=600-16x \quad ...(**)$

Net revenue will be maximum at the level of output, where $MR=MC$.

$$\therefore \qquad 2x+78=600-16x$$

$$\Rightarrow \qquad 18x=522$$

$$\Rightarrow \qquad x=\frac{522}{18}=29$$

Hence in order to maximise his profit, the manufacturer should manufacture 29 sets per week. Also the monopoly price is given by

$$p=600-8x=600-8\times 29=\text{Rs. } 368.$$

Aliter. We know : Net revenue = Total revenue − Total cost

or $\qquad \pi=px-C=x\ (600-8x)-(x^2+78x+2500)$

For maxima and minima : $\frac{dP}{dx}=0 \Rightarrow 600-16x-2x-78=0$

or $\qquad x=29$

[**Remark.** Also examine whether second order condition is satisfied at output level.]

Example 19. *The total revenue function of a firm is given as* $R=21q-q^2$ *and its total cost function as* $C=\frac{1}{3}\ q^3-3q^2-7q+16$, *where* q *is the output. Find*

(*i*) *the output at which the total revenue is maximum, and*

(*ii*) *the output at which the total cost is minimum.*

Solution. (*i*) $R=21\ q-q^2$

Differentiating w.r.t. q and equating to zero, we have

$$\frac{dR}{dq}=21-2q=0$$

or $\qquad q=\frac{21}{2}=10{\cdot}5$ is the necessary condition for revenue maximisation.

To get the sufficient condition, we have

$\frac{d^2R}{dq^2}=-2$, a negative quantity, which means the revenue is maximum at $p=10{\cdot}5$.

(*ii*) $$C=\frac{1}{3}\ q^3-3q^2-7q+16$$

Differentiating w.r.t. q and equating to zero, we have

$$\frac{dC}{dq}=\frac{1}{3}\ .\ 3q^2-3\times 2q-7=0$$

$$\Rightarrow \qquad q^2-6q-7=0$$

$$\Rightarrow \qquad (q-7)\ (q+1)=0$$

$\Rightarrow$ $q=7$ or $q=-1$ is the necessary condition for cost maximisation or minimisation. $q=-1$ is not admissible as output cannot be negative.

To get the sufficient condition, we have

$$\frac{d^2C}{dq^2}=2q-6$$

$\left[\frac{d^2C}{dq^2}\right]_{q=7}=2\times 7-6=8$, a positive quantity which means that cost is minimum at $q=7$.

Example 20. *The unit demand function is* $x=\frac{1}{3}\ (25-2p)$, *where* x *is the number of units and* p *is the price. Let the average cost per unit be Rs. 40. Find*

(*a*) *the revenue function* R *in terms of price* p,

(*b*) *the cost function* C,

(*c*) *the profit function* P,

(*d*) *the price per unit that maximizes the profit function, and*

(*e*) *the maximum profit.*

Solution. (*a*) $R(x)=xp=\frac{1}{3}\ (25-2p)\ p=\frac{1}{3}(25p-2p^2)$.

(*b*) $C(x)=40x=40\ .\ \frac{1}{3}(25-2p)=\frac{40}{3}\ (25-2p)$

(*c*) $P(x)=R(x)-C(x)$

$$=\frac{1}{3}\ (25p-2p^2)-\frac{40}{3}\ (25-2p)$$

$$=\frac{25p}{3}-\frac{2p^2}{3}-\frac{1000}{3}+\frac{80p}{3}$$

$$=\frac{1}{3}\ [-2p^2+105p-1000]$$

(*d*) The derivative of $P(x)$ is

$$P'(x)=\frac{1}{3}\ (-4p+105)$$

Solving the equation $P'(x)=0$ we find that

$$p=\frac{105}{4}=26{\cdot}25$$

Using second derivative test, we have

$$P''(x)=-\frac{4}{3}<0$$

$\therefore$ Maximum profit is found when $p=26{\cdot}25$

(e) Maximum profit is

$$P(x)=\frac{1}{3}\left[-2\left(\frac{105}{4}\right)^2+105\left(\frac{105}{4}\right)-1000.\right]=126{\cdot}04$$

Example 21. *The demand function faced by a firm is $p=500-0{\cdot}2x$ and its cost function is $C=25x+10{,}000$ (p=price, x=output and C=cost). Find the output at which the profits of the firm are maximum. Also find the price it will charge.*

Solution. Revenue, $R(x)=p.x=x(500-0{\cdot}2x)=500x-0{\cdot}2x^2$

$$\text{Profit}=\text{Revenue}-\text{Cost}$$

$$\Rightarrow \quad P(x)=R(x)-C(x)=500x-0{\cdot}2x^2-(25x+10{,}000)$$
$$=475x-10{,}000-0{\cdot}2x^2$$

For maximum or minimum :

$$\frac{dP}{dx}=475-0{\cdot}2\times 2x=475-0{\cdot}4x=0$$

$$\Rightarrow \quad x=\frac{475}{0{\cdot}4}=1187{\cdot}50.$$

Also $$\frac{d^2P}{dx^2}=-0{\cdot}4<0.$$

Hence the profit is maximum when the output $(x)=1187{\cdot}50$. At this level, the price is given by

$$p=500-0{\cdot}2x$$
$$=500-0{\cdot}2\,(1187{\cdot}50)=262{\cdot}50.$$

Example 22. *ABC Co. Ltd. is planning to market a new model of shaving razor. Rather than set the selling price of the razor based only on production cost estimates, management polls the retailers of the razors to see how many razors they would buy for various prices. From this survey it is determined that the unit demand function (the relationship between the amount x each retailer would buy and the price p he is willing to pay) is*

$$x=-1500\,p+30{,}000$$

The fixed costs to the company for production of the razors are found to be Rs. 28,000 and the cost for material and labour to produce each razor is estimated to be Rs. 8·00 per unit. What price should the company charge retailers in order to obtain a maximum profit ?

Solution. Let x denote the number of units produced, and C denote the cost of production to the company, and let p denote the price per unit (in rupees).

Then the cost C is given by $C \Rightarrow$ Rs. $8x +$ Rs. $28,900$

and the unit demand is $x = -1500\ p + 30,000$

Substituting, we find that the cost function $C(x)$ in terms of the price p per unit is

$$C(x) = 8 \cdot (-1500\ p + 30,000) + 28,000$$

$$\Rightarrow \qquad C(x) = -12,000p + 2,68,000.$$

The money derived from the sales of the shaving razors as a function of the price p per unit is the product of the number sold by the price per unit, *i.e.*, the revenue function $R(x)$ is

$$R(x) = (-1500\ p + 30,000) \cdot p$$

$$= -1500\ p^2 + 30,000\ p.$$

The profit P to the company is merely the difference between revenue (money derived from sales) and total cost, *i.e.*, the profit function is

$$P(x) = R(x) - C(x)$$

$$= (-1500\ p^2 + 30,000\ p) - (-12,000\ p + 2,68,000)$$

$$= -1500\ p^2 + 42,000\ p - 2,68,000.$$

The derivative of $P(x)$ is

$$P'(x) = -3000\ p + 42000.$$

Solving the equation, $P'(x) = 0$, we find that

$$x = 14.$$

Using second derivative test, we have

$$P''(x) = -3000 < 0$$

$\therefore$ Maximum profit is found when $p = 14$.

The profit for this price is

$$P = -1,500\ (14)^2 + 42,000 \times 14 - 2,68,000$$

$$= \text{Rs. } 26,000.$$

The number of units sold at this price p is

$$x = (-1500)(14) + 30,000 = 9,000.$$

Example 23. *A company making a single product has manufacturing and distribution divisions. Stock of finished goods are not held, all production being to order.*

The average net revenue per unit, allowing for quantity discounts, is Rs. $(100 - 0{\cdot}01\ Q)$ where Q is the quantity sold.

The average variable costs per unit for the two divisions are :

Manufacturing = Rs. 10 + Rs. 0·015 Q

Distribution = Rs. 2 + Rs. 0·001 Q.

The fixed costs per annum are :

Manufacturing = Rs. 40,000

Distribution = Rs. 20,000

You are required to calculate :

(a) the optimum annual production quantity to maximise the profit of the company ;

(b) the profit of the company at the level of activity in (a) above ;

(c) the annual production quantity to maximise the manufacturing division's profit, if it has been instructed to transfer the product to the distribution division at Rs. 73 per unit.

(d) the profit of the company, showing the results of the two divisions, at the level of activity in (c) above.

Solution. (*a*) Profit = Revenue − Variable costs − Fixed costs

$$P = Q(100 - 0{\cdot}01\ Q) - Q(10 + 0{\cdot}015\ Q) - Q(2 + 0{\cdot}001\ Q) - (40{,}000 + 20{,}000)$$

$$\Rightarrow \qquad P = 88\ Q - 0{\cdot}026\ Q^2 - 60{,}000.$$

For maximisation, we have

$$\frac{dP}{dQ} = 88 - 0{\cdot}052\ Q = 0 \qquad \Rightarrow \quad Q = 1692 \text{ units.}$$

Also $$\frac{d^2P}{dQ^2} = -0{\cdot}052 < 0.$$

∴ Profit maximisation output is when $Q = 1692$ units.

(*b*) When $Q = 1692$,

Profit $= 88\ (1692) - 0{\cdot}026\ (1692)^2 - 60{,}000 =$ Rs. 14,461·54

(*c*) If the manufacturing division are to transfer out at Rs. 73 per unit, we can express their profit as :

Profit = 73 × (Production quantity) − Manufacturing variable costs − Manufacturing fixed costs

$$\therefore \qquad P = 73\ Q - Q\ (10 + 0{\cdot}015\ Q) - 40{,}000$$

$$\Rightarrow \qquad P = 63\ Q - 0{\cdot}015\ Q^2 - 40{,}000.$$

For maximisation,

$$\frac{dP}{dQ} = 63 - 0{\cdot}03\ Q = 0 \qquad \Rightarrow \quad Q = 2100 \text{ units.}$$

Also $$\frac{d^2P}{dQ^2} = -0{\cdot}03 < 0.$$

(*d*) Company Profit (Part *a*) $= 88\ Q - 0{\cdot}026\ Q^2 - 60{,}000$.

if $\quad Q = 2{,}100$; Profit $= 88\ (2{,}100) - 0{\cdot}026\ (2{,}100)^2 - 60{,}000 =$ Rs. 10,140.

Manufacturing Profit (part *c*) $= 63\ Q - 0{\cdot}015\ Q^2 - 40{,}000$

if $\quad Q = 2100$, Profit $= 63\ (2{,}100) - 0{\cdot}015\ (2{,}100)^2 - 40{,}000$
$=$ Rs. 26,150.

Distribution profit = Revenue − Transfer cost and Department cost

$$= Q\ (100 - 0{\cdot}01\ Q) - \{73\ Q + Q\ (2 + 0{\cdot}001\ Q) + 20{,}000\}$$
$$= 100\ Q - 0{\cdot}01\ Q^2 - 73\ Q - 2\ Q - 0{\cdot}001\ Q^2 - 20{,}000$$
$$= -0{\cdot}011\ Q^2 + 25\ Q - 20{,}000$$

if $\quad Q = 2{,}100$; Profit $= -0{\cdot}011\ (2{,}100)^2 + 25\ (2{,}100) - 20{,}000$
$= -$ Rs. 16,010.

This shows that there is a loss of Rs. 16,010 from the distribution unit.

Example 24. *It is given that a demand curve is convex from below* $\left(\frac{d^2p}{dx^2} > 0\right)$ *at all points. Show that the marginal revenue curve is also convex from below either if* $\frac{d^3p}{dx^3}$ *is positive or if* $\frac{d^3p}{dx^3}$ *is negative and is numerically less than* $\frac{3}{x}\ \frac{d^2p}{dx^2}$ *If the demand curve is always concave from below, does a similar property hold of marginal revenue curve ?*
[*Delhi Univ. B.A.* (*Hons.*) *Economics 1991*]

Solution. Let the demand curve be

$$p = f(x)$$

$$\therefore \qquad \frac{dp}{dx} = -\text{ve and } \frac{d^2p}{dx^2} = +\text{ve}.$$

Then, we have $TR = p\ .\ x$

$$\frac{d(TR)}{dx} = MR = p\ .\ 1 + x\ .\ \frac{dp}{dx}$$

$$\therefore \qquad \frac{d(MR)}{dx} = \frac{dp}{dx} + \frac{dp}{dx} + x\ .\ \frac{d^2p}{dx^2}$$

$$= 2\ .\ \frac{dp}{dx} + x\ .\ \frac{d^2p}{dx^2}$$

and
$$\frac{d^2(MR)}{dx^2} = 2\ .\ \frac{d^2p}{dx^2} + \frac{d^2p}{dx^2} + x\ .\ \frac{d^3p}{dx^3}$$

$$= 3\ .\ \frac{d^2p}{dx^2} + x\ .\ \frac{d^3p}{dx^3}$$

For MR to be convex from below

$$3 \cdot \frac{d^2p}{dx^2} + x \cdot \frac{d^3p}{dx^3} > 0$$

But $\frac{d^2p}{dx^2} > 0$ (given).

So for MR to be convex from below either

$\frac{d^3p}{dx^3} > 0$ or if $\frac{d^3p}{dx^3}$ is negative then it should be numerically less than $\frac{3}{x} \cdot \frac{d^2p}{dx^2}$ so that $\frac{d^2(MR)}{dx^2} > 0$.

For concave demand curve $\frac{d^2p}{dx^2}$ will be negative, so for MR to be concave from below we should have either $\frac{d^3p}{dx^3}$ negative or if $\frac{d^3p}{dx^3}$ is positive then it is numerically less than $\frac{3}{x} \cdot \frac{d^2p}{dx^2}$.

Example 25. *The production function of a commodity is given by*

$$Q = 40F + 3F^2 - \frac{F^3}{3}$$

where Q is the total output and F is the units of input.

(*i*) *Find the number of units of input required to give maximum output.*

(*ii*) *Find the maximum value of marginal product.*

(*iii*) *Verify that when the average product is maximum, it is equal to marginal product.*

Solution. (*i*) $\frac{dQ}{dF} = 40 + 6F - \frac{3F^2}{3} = 40 + 6F - F^2$.

(*First order condition*)

For maximum or minimum :

$$40 + 6F - F^2 = 0$$

$$\Rightarrow \quad (F+10)(F-4) = 0$$

$$\Rightarrow \quad F = -10 \quad \text{or} \quad F = 4$$

$F = -10$ is not admissible as input cannot be negative.

$$\frac{d^2Q}{dF^2} = 6 - 2F \qquad (\textit{Second order condition})$$

$$\left[\frac{d^2Q}{dF^2}\right]_{F=4} = 6 - 2\,(4) = -2 < 0.$$

Hence output is maximum when 4 units of input are used.

(*ii*) $$MP=\frac{dQ}{dF}=40+6F-F^2$$

For maximum or minimum : $\dfrac{d(MP)}{dF}=6-2F=0$

$$\Rightarrow \qquad F=3.$$

Also $$\frac{d^2(MP)}{dF^2}=-2<0.$$

Hence maximum value of marginal product is when input is 3 units.

$\therefore$ Value of marginal product $=40+6\times3-3^2=49$ units.

(*iii*) Average product $(AP)=\dfrac{Q}{F}$

$$=\frac{40F+3F^2-\frac{1}{3}F^3}{F}=40+3F-\frac{F^2}{3}$$

For maximum or minimum :

$$\frac{d(AP)}{dF}=3-\frac{2F}{3}=0$$

$$\Rightarrow \qquad F=\frac{9}{2}=4{\cdot}5$$

Also $$\frac{d^2(AP)}{dF^2}=-\frac{2}{3}<0.$$

$\therefore$ Average product is maximum when $F=\dfrac{9}{2}=4{\cdot}5$

Average product $\left(\text{when } F=\dfrac{9}{2}=4{\cdot}5\right)$

$$=40+3\left(\frac{9}{2}\right)-\frac{81}{2}=\frac{187}{4}=46{\cdot}75$$

Marginal product $\left(\text{when } AP \text{ is maximum, } i.e., F=\dfrac{9}{2}=4{\cdot}5\right)$

$$=40+27-\frac{81}{4}=\frac{187}{4}=46{\cdot}75.$$

Example 26. *The quantity sold q and the price p are related by*

$$q=ae^{-bp}$$

The production cost is given by $C(q)=l+mq$; a, b, l and m are positive constants. Find the optimal price which maximises the profit ?

Solution. Profit $P=\text{Revenue}-\text{Cost}=pq-(l+mq)$

$$=pae^{-bp}-(l+mae^{-bp})=ae^{-bp}(p-m)-l \qquad \ldots(*)$$

Differentiating (*) w.r.t. p., we get

$$\frac{dP}{dp}=-ab\ e^{-bp}\ (p-m)+ae^{-bp}.$$

$$=ae^{-bp}\ (-bp+bm+1) \qquad ...(**)$$

$$\frac{dP}{dp}=0 \text{ gives } ae^{-bp}\ (-bp+bm+1)=0.$$

For any finite value of p, $e^{-bp}\neq 0$.

$\therefore \quad -bp+bm+1=0.$

$$\therefore \qquad p=\frac{1+bm}{b}=\frac{1}{b}+m.$$

Differentiating (**) with respect to p, we get

$$\frac{d^2P}{dp^2}=ab^2\ e^{-bp}\ (p-m)-abe^{-bp}-abe^{-bp}$$

$$=ab\ e^{-bp}\ \{b(p-m)-2\}.$$

When $p=\frac{1}{b}+m$,

$$\frac{d^2P2}{dp^2}=abe^{-b\left(\frac{1}{b}+m\right)}\left[\ b\left(\frac{1}{b}+m-m\right)-2\ \right]$$

$$=ab\ e^{-(1+bm)}\ (-1)$$

$$=-ab\ e^{-(1+bm)}<0, \text{ since } a,\ b>0.$$

$$\therefore \qquad p=\frac{1}{b}+m \text{ maximises the profit } P.$$

$$P=ae^{-bp}\ (p-m)-l$$

$$\therefore \qquad P=_{max}=ae^{-b\left(\frac{1}{b}+m\right)}\left(\frac{1}{b}+m-m\right)-l.$$

$$=\frac{a}{b}\ e^{-(1+bm)}-l.$$

Example 27. *A monopolist firm has the following total cost and demand functions :*

$$C=ax^2+bx+c,\ p=\beta-\alpha x.$$

What is the profit maximising level of output when :

(*i*) *The firm is assumed to fix the output ;*

(*ii*) *The firm is assumed to fix the price ?*

Solution. *When firm fixes the output level :*

Revenue $(R)=px=x\ (\beta-\alpha x)=\beta x-\alpha x^2$

$$MR=\frac{dR}{dx}=\beta-2\alpha x$$

Total Cost $(C)=ax^2+bx+c$

$$MC=\frac{dC}{dx}=2ax+b$$

Now condition for profit maximising output level is

$$MR=MC$$

i.e., $$\beta-2\alpha x=2ax+b$$

$$\Rightarrow \quad \beta-b=2ax+2\alpha x=2x\,(a+\alpha)$$

$$\Rightarrow \quad x=\frac{\beta-b}{2\,(a+\alpha)}$$

which is the profit maximising level of output.

(*ii*) *When firm fixes the price* : In this case the total revenue and cost are put in terms of price p.

Now $$p=\beta-\alpha x \quad \Rightarrow \quad x=\frac{\beta-p}{\alpha}$$

$$R=px=p\left(\frac{\beta-p}{\alpha}\right)=\frac{\beta p-p^2}{\alpha}$$

$$MR=\frac{dR}{dp}=\frac{1}{\alpha}\,(\beta-2p) \qquad \ldots(*)$$

Also $$C=ax^2+bx+c$$

$$\Rightarrow \quad C=a\left(\frac{\beta-p}{\alpha}\right)^2+b\left(\frac{\beta-p}{\alpha}\right)+c$$

$$MC=\frac{dC}{dp}=\left[\frac{-2a\beta+2ap}{\alpha^2}-\frac{b}{\alpha}\right]$$

For profit maximisation : $MR=MC$

$$\Rightarrow \quad \frac{1}{\alpha}\,(\beta-2p)=\left[\frac{-2a\beta+2ap}{\alpha^2}-\frac{b}{\alpha}\right]$$

$$\Rightarrow \quad \frac{\beta}{\alpha}-\frac{2p}{\alpha}=-\frac{2a\beta}{\alpha^2}+\frac{2ap}{\alpha^2}-\frac{b}{\alpha}$$

$$\Rightarrow \quad \frac{\beta}{\alpha}+\frac{2a\beta}{\alpha^2}+\frac{b}{\alpha}=\frac{2p}{\alpha}+\frac{2ap}{\alpha^2}$$

$$\Rightarrow \quad \frac{\alpha\beta+2a\beta+\alpha b}{\alpha^2}=\frac{2p\alpha+2ap}{\alpha^2}=\frac{2p(\alpha+a)}{\alpha^2}$$

$$\Rightarrow \quad p=\frac{\alpha\beta+2a\beta+\alpha b}{2\,(\alpha+a)}$$

Now the demand function is

$$p=\beta-\alpha x$$

$$\Rightarrow \qquad \frac{\alpha\beta+2a\beta+\alpha b}{2(\alpha+a)}=\beta-\alpha x$$

$$\Rightarrow \qquad \alpha x=\beta-\frac{\alpha\beta+2a\beta+\alpha b}{2(\alpha+a)}=\frac{2\beta\alpha+2a\beta-\alpha\beta-\alpha b}{2(\alpha+a)}$$

$$\Rightarrow \qquad x=\frac{\alpha\beta-\alpha b}{2(\alpha+a)\alpha}=\frac{\beta-b}{2(\alpha+a)}$$

which gives the same level of output when the firm assumed to fix the output level.

Example 28. *A monopolist has total cost function : $C=ax^2+bx+c$ and if demand law is $p=\beta-\alpha x^2$, show that the output for maximum revenue is*

$$x=\frac{\sqrt{a^2+3\alpha(\beta-b)}-a}{3\alpha}$$

Solution. Total revenue$=px=\beta x-\alpha x^3$

Net revenue$=$Total revenue$-$Total cost

$$R=(\beta x-\alpha x^3)-(ax^2+bx+c)$$

For maximum or minimum :

$$\frac{dR}{dx}=\beta-3\alpha x^2-2ax-b=0$$

or $$3\alpha x^2+2ax-(\beta-b)=0$$

or $$x=\frac{-2a\pm\sqrt{4a^2+4\times 3\alpha(\beta-b)}}{6\alpha}$$

$$=\frac{-a\pm\sqrt{a^2+3\alpha(\beta-b)}}{3\alpha}$$

$\therefore$ $$x=\frac{\sqrt{a^2+3\alpha(\beta-b)}-a}{3\alpha}$$

or $x=\dfrac{-a-\sqrt{a^2+3\alpha(\beta-b)}}{3\alpha}$, this value of x is not admissible as output cannot be negative.

Also $$\frac{d^2R}{dx^2}=-6\alpha x-2a$$

When $x=\dfrac{\sqrt{a^2+3\alpha(\beta-b)}-a}{3\alpha}$, we have

$$\frac{d^2R}{dx^2}=-2\left(\sqrt{a^2+3\alpha(\beta-b)}-a\right)-2a$$

$$= -2\sqrt{a^3+3\alpha(\beta-b)} < 0.$$

Hence the net revenue is maximum when the output is given by

$$x = \frac{\sqrt{a^2+3\alpha\,(\beta-b)}-b}{3\alpha}$$

Example 29. *The total cost function of a firm is*

$$C=\tfrac{1}{3}x^3-5x^2+28x+10$$

where C is total cost and x is output. A tax at the rate of Rs. 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by

$$p=2530-5x,$$

where Rs. p is the price per unit of output, find the profit maximising output and price.

[*C.A. Intermediate, May 1990*]

Solution. Total revenue function, $TR=(2530-5x)\ x$

$$=2530-5x^2$$

After imposition of a tax of Rs. 2 per unit,

Total cost function, $TC=\tfrac{1}{3}x^3-5x^2+28x+10+2x$

$$=\tfrac{1}{3}x^3-5x^2+30x+10$$

Now Profit, $P=TR-TC$

$$=(2530x-5x^2)-(\tfrac{1}{3}x^3-5x^2+30x+10)$$

For maximisation, we find

$$\frac{dP}{dx}=(2530-10x)-(x^2-10x+30)=0$$

$$\Rightarrow \qquad x^2=2500$$

$$\Rightarrow \qquad x=\pm 50$$

But $x=-50$ is not admissible as output can not be negative.

and $$\frac{d^2P}{dx^2}=-2x<0.$$

∴ Profit maximising output is 50 units.

∴ Price when $x=50$ is $p=2530-5\times 50=2280$.

Example 30. *Suppose the demand and total cost functions of a monopolist are $p=20-4x$ and $TC=4x+2$ respectively where p is price and x is quantity. If the government imposes tax at the rate of 20% of sales, determine the total tax revenue that the government will be able to collect.*

[*Delhi Univ., B. Com. (Hons), 1992*]

Solution. We are given that

$$p=20-4x \quad \text{and} \quad TC=4x+2$$

∴ Total Revenue $=TR=px=20x-4x^2$

$$\text{Tax}=20\% \text{ of } TR=\frac{1}{5}(20x-4x^2)$$

$$\text{Total new cost}=TC+\text{Tax}=4x+2+\frac{1}{5}(20x-4x^2)$$

$$=-\frac{4}{5}x^2+8x+2$$

Now, profit P = Total Revenue − Total new cost

$$=(20x-4x^2)-\left(-\frac{4}{5}x^2+8x+2\right)$$

$$=-\frac{16}{5}x^2+12x-2$$

$$\therefore \quad \frac{dP}{dx}=-\frac{32}{5}x+12 \text{ and } \frac{d^2P}{dx^2}=-\frac{32}{5}<0$$

$$\frac{dP}{dx}=0 \text{ gives } -\frac{32}{5}x+12=0 \text{ or } x=\frac{12\times 5}{32}=\frac{15}{8}$$

$\therefore \quad x=\frac{15}{8}$ will give maximum profit.

Also, $x=\frac{15}{8}$ will yield the maximum tax

$\therefore$ Tax when $x=\frac{15}{8}$ is given by

$$\frac{1}{5}\left[30\times\frac{15}{8}-4\times\left(\frac{15}{8}\right)^2\right]$$

$$=\frac{75}{16}.$$

Hence the government will be able to collect $\frac{75}{16}$ as tax revenue.

Example 31. *Given the demand and cost functions :*

$$p=20-4x$$
$$C=4x$$

(*a*) *Find the optimum quantity, price and the profit on this level.*

(*b*) *What will be the new equilibrium after a tax of Rs. 0·50 is imposed ?*

(*c*) *Determine the tax rate that will maximise tax revenue and determine that tax revenue.*

(*d*) *Find the total tax revenue if in addition 10% sales tax is also imposed.*

Solution. (*a*) $TR=20x-4x^3$, $MR=20-8x$

$$C=4x,\ MC=4$$

For optimum level, $MR=MC$

$$\Rightarrow \qquad 20-8x=4,\ i.e.,\ x=2$$

$$\therefore \qquad p=12$$

(*b*) After Tax, $C=4x+0{\cdot}5x$

$$MC=4{\cdot}5$$

At the optimum level $MC=MR$

$$\Rightarrow \qquad 20-8x=4{\cdot}5,\ i.e.,\ \text{when}\ \ x=31/16=1{\cdot}94$$

$$\therefore \qquad p=12{\cdot}25$$

(*c*) Tax revenue is maximum where

$$MR=MC\ (\text{after tax } t)$$

$$20-8x=4+t \quad \Rightarrow \quad x=(16-t)/8$$

New price after tax is

$$p=20-4\left(2-\frac{t}{8}\right)=\left(12+\frac{t}{2}\right)$$

Thus, the increase in price is half of the tax imposed and profit after tax is

$$\text{Profit } (P)=TR-TC$$

$$=(20x-4x^2)-(4x+tx)=x(16-4x-t)$$

Substituting $x=\dfrac{16-t}{8}$, we obtain

$$\text{Maximum profit}=\frac{16-t}{8}\left[16-t-\frac{16-t}{2}\right]=\frac{(16-t)^2}{16}$$

$$\text{Tax revenue}=tx=\frac{16t-t^2}{8}$$

T will be maximum where $\dfrac{dT}{dt}=0$ and $\dfrac{d^2T}{dt^2}<0$

$$\therefore \qquad \frac{16-2t}{8}=0 \qquad \Rightarrow \quad t=8$$

$$\text{Maximum tax}=tx=8\left(\frac{16-8}{8}\right)=8$$

(*d*) With sales tax of 10% the net TR is

$$TR=0{\cdot}90\,(20-4x)\,x$$

$$\therefore \qquad MR=0{\cdot}90\,(20-8x)$$

At optimum level, $MR=MC$

$$0{\cdot}90\,(20-8x)=4 \quad \Rightarrow \quad x=140/72=1{\cdot}94.$$

Example 32. *XYZ Company, as a result of past experience and estimates for the future, has decided that the cost of production of their sold product, P, an advanced process machine, is :*

$$C=1064+5x+0{\cdot}04\,x^2,$$

where $C=$ *total in cost '000 Rs.*

$x=$ *quantity produced (and sold)*

The marketing department has estimated that the price of the product is related to the quantity produced and sold by the equation :

$$P=157-3x,$$

where $P=$ *Price per unit in '000 Rs.*

$x=$ *quantity sold*

The government has proposed a tax of Rs. t,000 per unit on product P but it is not expected that this will have any effect on the costs incurred in making P or on the demand price relationship. Find :

(*a*) *the price and quantity that will maximise profit when there was no tax ;*

(*b*) *the price and quantity that will maximise profit if the proposed tax is introduced ;*

(*c*) *how much of the tax t per unit is passed on to the customer ;*

(*d*) *the effect on the profit of the company if t was fixed at Rs. 4,000 per unit.*

Solution. (*a*) Profit $(Y)=$ Revenue $(R)-$ Cost (C)

Revenue $(R)=$ Price $(P)\times$ Quantity (x)

$$=(157-3x)\,x=157x-3x^2$$

Cost $(C)=1064+5x+0{\cdot}04x^2$

$$Y=157x-3x^2-(1064+5x+0{\cdot}04x^2)$$
$$=-3{\cdot}04x^2+152x-1064$$

Differentiating Y *w.r.t.* x, we have

$$\frac{dY}{dx}=152-6{\cdot}08x=0$$

$$\Rightarrow \qquad x=25 \text{ units}$$

$$\frac{d^2Y}{dx^2}=-6{\cdot}08<0$$

$\therefore$ Profit is maximum when 25 units are produced.

Now $\qquad P=157-3\times25$ in '000 Rs.

$=$ Rs. 82,000 per unit

and $\qquad Y=152\times25-3{\cdot}04\times(25)^2-1064$

$=$ Rs. 836 in '000 Rs.

$=$ Rs. 8,36,000.

(*b*) When a tax (t) is introduced,

$$Y = R - C \text{ becomes}$$

$$Y = (157 - 3x)x - (1064 + 5x + 0{\cdot}04x^2 + tx)$$

$$= 152x - 3{\cdot}04x^2 - 1064 - tx$$

Differentiating Y with respect to x and setting to zero, we have

$$\frac{dY}{dx} = 152 - 6{\cdot}08x - t = 0$$

$$\Rightarrow \qquad x = \frac{152 - t}{6{\cdot}08}$$

$$\frac{d^2Y}{dx^2} = -6{\cdot}08 < 0, \quad \text{when } x = \frac{152 - t}{6{\cdot}08}$$

$\therefore$ For maximum profit,

$$\text{Quantity } (x) = 25 - \frac{t}{6{\cdot}08} \text{ units}$$

Substituting for x in the price equation, we have

$$P = 157 - 3\left(25 - \frac{t}{6{\cdot}08}\right) = 82 + \frac{3t}{6{\cdot}08}$$

(*c*) The amount of tax passed on to the customer is $\frac{3t}{6{\cdot}08}$ or approximately 49·34%.

(*d*) When the tax per unit is Rs. 4,000, then $t = 4$.

$$\therefore \qquad x = 25 - \frac{4}{6{\cdot}08} = 24{\cdot}34 \text{ or } 24 \text{ in whole units.}$$

Now
$$Y = 152x - 3{\cdot}04x^2 - 1064 - tx$$

$$= (152 \times 24) - 3{\cdot}04 \times (24)^2 - 1064 - 4 \times 24$$

$$= 736{\cdot}96 \text{ in '000 Rs.}$$

$$= \text{Rs. } 7{,}36{,}960$$

The profit without tax in (*a*) above = Rs. 8,36,000

Profit with tax of Rs. 4,000 = Rs. 7,36,960

$\therefore$ Difference to profit = Rs. 99,040.

Example 33. *A monopolist's total cost is* $TC = ax^2 + bx + c$ *and the demand function is* $p = \beta - \alpha x$, *where* x *and* p *denote the units of output and price respectively and* a, b, c, α *and* β *are positive constants. If the government imposes tax at the rate of* t *per unit of output, show that the total tax is maximum when* $t = (\beta - b)/2$. [*Delhi Univ. B.Com.* (*Hons.*), *1991*]

Solution. After the imposition of tax, t per unit, the total cost function, TC, is given by

$$TC = ax^2 + bx + c + tx$$

$$\text{Revenue function} = R = px = (\beta - \alpha x)\, x = \beta x - \alpha x^2$$

∴ Profit function $= P = (\beta x - \alpha x^2) - (ax^2 + bx + c + tx)$

For P to be maximum,

First order condition :

$$\frac{dP}{dx} = 0, \text{ i.e., } (\beta - 2\alpha x) - (2ax + b + t) = 0$$

$$x = \frac{\beta - b - t}{2(\alpha + a)}$$

Second order condition :

$$\frac{d^2P}{dx^2} < 0$$

$\frac{d^2P}{dx^2} = -2\alpha - 2a = -2\ (\alpha + a) < 0$ as α and a are positive constants.

Therefore, the level of output that maximises the profit is

$$x = \frac{\beta - b - t}{2(\alpha + a)}$$

The total tax revenue for this level of output is

$$T = tx = \frac{\beta t - bt - t^2}{2(\alpha + a)}$$

For T to be maximum,

First order condition :

$$\frac{dT}{dx} = 0, \text{ i.e., } \frac{\beta - b - 2t}{2(\alpha + a)} = 0$$

or
$$t = \frac{1}{2}\ (\beta - b)$$

Second order condition :

$$\frac{d^2T}{dx^2} < 0$$

$\frac{d^2T}{dx^2} = -\frac{1}{(\alpha + a)} < 0$ as α and a are positive constants.

Hence the tax rate t that maximises the total tax revenue is

$$t = \tfrac{1}{2}(\beta - b).$$

Example 34. *There are two duopolists manufacturing equal and identical bicycles. The total cost of an output of x bicycles per month is Rs.* $\left(\frac{x^2}{25} + 3x + 100\right)$ *in each case. When the price is Rs. p per bicycle the market demand is* $x = 75 - 3p$ *bicycle per month. Find the total equilibrium output per month.*

Solution. Let x_1 and x_2, denote the output per week of the two duopolists. Then $x = x_1 + x_2$, is the total output.

Demand function

$$p=25-\frac{x}{3}=25-\frac{(x_1+x_2)}{3}$$

$\therefore$ Net revenue for the first firm

$$\pi_1=R_1--C_1=\left[25x_1-\frac{(x_1+x_2)x_1}{3}\right]-\left[\frac{x_1^2}{25}+3x_1+100\right]$$

$$\therefore \qquad \pi_1=22x_1-\frac{28x_1^2}{75}-\frac{x_1x_2}{3}-100$$

For maximum net revenue,

$$0=\frac{d\pi_1}{dx_1}=22-\frac{56}{75}x_1-\frac{x_1}{3}\cdot\frac{dx_2}{dx_1}-\frac{x_2}{3}.$$

But the conjectural variation $\frac{dx_2}{dx_1}=0$

$$0=22-\frac{56x_1}{75}-\frac{x_2}{3} \qquad \text{...(1)}$$

Similarly, we can show, by considering the net revenue for the second firm, that for maximum net revenue, with conjectural variation zero.

$$0=22-\frac{56x_2}{75}-\frac{x_1}{3} \qquad \text{...(2)}$$

The equilibrium output of the two firms in duopoly are the simultaneous solutions of (1) and (2). They are

$$x_1=\frac{51150}{2511}=x_2$$

i.e., $x_1=x_2=20{\cdot}37$, approx.

$\therefore$ Total output per week is $2(20{\cdot}37)=41$ (approx.)

EXERCISE (II)

1. A man producing very fine earthenware lampstands found that he could sell on an average of 4 stands per day at a price of Rs. 18 each. When he increased his output to an average of 4·5 per day he could only obtain Rs. 17·5 each, if he were to sell all his output.

Assume that he maintains no inventories, so that he sells all he produces, and that the appropriate demand function is linear and is of the form :

$$x=a+bp$$

where a and b are constants, x is the average number sold per day and p is the price. An accurate survey into his total daily production costs produced the relationship :

$$C=\tfrac{1}{2}x^2-\tfrac{1}{2}x+54$$

between the total production cost, C, and the average daily production x.

Required : (*a*) Determine the demand function giving the average number sold per day, x, in terms of the price, p.

(*b*) Find an expression for the gross profit per day in terms of the average number of stands produced and sold.

(*c*) Find the profit when 6 stands are produced and sold.

(*d*) What is the average number that must be produced and sold for maximum profit ?

[**Hint.** (*a*) Demand function : $x=a+bp$

When price $p=$ Rs. 18, demand $=4$ per day on average

$$\therefore \qquad 4=a+18b. \qquad \ldots(*)$$

When $p=$ Rs. 17·5, demand $=4·5$ per day on average

$$4·5=a+17·5b \qquad \ldots(**)$$

Solving (*) and (**), we get

$$a=22,\ b=-1$$

$\therefore$ The demand function is $x=22-p$.

(*b*) Profit :
$$P=x(22-x)-(\tfrac{1}{2}\,x^2-\tfrac{1}{2}\,x+54)$$
$$=-\tfrac{3}{2}\,x^2+22\tfrac{1}{2}\,x-54$$

(*c*) Gross profit when 6 stands are produced and sold is
$$=-\tfrac{3}{2}\times(6)^2+22\tfrac{1}{2}\times 6-54=\text{Rs. } 27$$

(*d*) To maximise gross profit :

$$\frac{dP}{dx}=0 \quad \text{and} \quad \frac{d^2P}{dx^2}<0$$

i.e., if $x=7\tfrac{1}{2}$ then the maximum gross profit would be

$$-\tfrac{3}{2}\,(7\tfrac{1}{2})^2+22\tfrac{1}{2}\times 7\tfrac{1}{2}-54=\text{Rs } 30·38 \text{ per day.}]$$

2. Let the unit demand function be

$$x=ap+b$$

and the cost function be

$$c=ex+f$$

where
$x=$ sales (in units)
$p=$ price (in rupees)
$f=$ fixed cost (in rupees)
$e=$ variable cost
$b=$ demand when $p=0$
$a=$ slope of unit demand function

(*a*) Find the cost C as a function of p.

(*b*) Find the revenue function $R(x)$.

(*c*) Find the profit function $P(x)$.

3. (*a*) A man derives Rs. x from his business this year and Rs. y next year. By alternative use of his resources he can very x and y according to the following relationship,

$$y=1000-\frac{x^2}{250}$$

What is the income this year if he plans for zero income next year ? Derive $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. What is the shape of the businessman's transformation curve ?

(*b*) A sugar mill has total cost function given by

$$\tfrac{1}{10}(x^2+5x+200),$$

where x tons of sugar are produced per week. If the market price is Rs. p per ton, what is the supply function of the firm ? What is the average fixed cost ?

4. (*a*) A business produces an income of Rs. x this year and Rs. p next year, where these values can be varied according to the relation $y=100-\frac{x^2}{250}$. Explain how $\left\{\left(-\frac{dy}{dx}\right)-1\right\}$ can be interpreted as the marginal rate of return over cost. Show that the value of his marginal rate is $\frac{x-125}{125}$ when this year's income is Rs. x.

(*b*) It is given that a demand curve is convex from below $\left(\frac{d^2p}{dx^2}>0\right)$ at all points. Show that the marginal revenue curve is also convex from below either if $\frac{d^2p}{dx^2}$ is positive or if $\frac{d^3p}{dx^3}$ is negative and numerically less than $\frac{3}{x}\cdot\frac{d^2p}{dx^2}$. If the demand curve is always concave from below, does a similar property hold of the marginal revenue curve ?

5. Show that the demand curves

$$p=\frac{a}{a+c}-c \text{ and } p=(a-bx)^2$$

are each downward sloping and convex from below. Do the same properties hold for the MR curves ? Show further that, for each of the demand laws $p=\sqrt{a-bx}$ and $p=a-bx^2$, the demand and MR curves are downward sloping and concave from below. Assume that a, b, c are positive.

[**Hint.** A curve $y=f(x)$ is downward sloping if $\frac{dy}{dx}<0$; and convex from below (or concave from below) if

$$\frac{d^2y}{dx^3}>0\left(\text{or } \frac{d^2y}{dx^2}<0\right)\Bigg].$$

6. (*a*) For a unit demand function of $p=24-8x$, where x is the number of units in thousands and p is the price in rupees, find the sales function. If the average cost per unit is Rs. 8, find

(*a*) The profit function.

(*b*) The number of units that maximize the profit function.

(*c*) Graph the cost and revenue functions.

[**Hint.** $P(x)=R(x)-C(x)=(24-8x)x-8x]$

(*b*) If the total cost function of a firm is

$$C=\tfrac{1}{3}x^3-5x^2+30x+10.$$

where C is the total cost and x is the output, and price under perfect competition is given as 6. find for what value of x the profit will be maximised. Examine both first and second ordet conditions.

7. If the demand function for a commodity is given by $p=12e^{-(q/4)}$ where p is the price per unit and q is the number of units demanded. Determine the price and the quantity for which the revenue is maximum.

[**Hint.** Revenue function is given by

$$R=pq=12qe^{-(q/4)}$$

For R to be maximum

$$\frac{dR}{dq}=0 \text{ and } \frac{d^2R}{dq^2}<0.$$

$$\frac{dR}{dq}=12[qe^{-(q/4)}(-1/4)+e^{-(q/4)}]=12e^{-(q/4)}\left(\frac{4-q}{4}\right)\Bigg]$$

8. State the conditions for a maximum profit. Find the profit maximising out-put level if $p=200-10x$ and

$$AC=10+\frac{x}{25}.$$

9. Suppose the total cost function is given by $C=a+bx+cx^2$, where x is the quantity of output produced. Show that the slope of the average cost curve is $\frac{1}{x}(MC-AC)$, where $MC=$ Marginal cost and $AC=$Average cost.

10. A firm produces an output of x tons of a certain product at a total variable cost given by $C=x^3-4x^2+7x$. Find the output at which the average cost is the least and the corresponding value of the average cost.

11. A company notices that higher sales of a particular item which it produces are achieved by lowering the price charged. As a result the total revenue from the sales at first rises as the number of units sold increases. reaches a maximum and then falls off. This pattern of the total revenue is described by the relation :

$y=40{,}00{,}000-(x-2000)^2$ where y is the total revenue and x the number of units sold.

(*i*) Find the number of units that maximizes total revenue.

(*ii*) What is the amount of maximum revenue ?

(*iii*) What would be the total revenue if 2500 units are sold ?

[**Ans.** (*i*) 2000, (*ii*) Rs. 40.00,000, (*iii*) Rs. 37,50,000]

12. If the cost function is $C(x)=4x+9$ and the revenue function is $R(x)=9p-x^2$, where x is the number of units produced (in thousands) and R and C are measured in millions of rupees, find the following :

(*a*) Marginal revenue.

(*b*) Marginal revenue at $x=5$, $x=6$.

(*c*) Marginal cost.

(*d*) The fixed cost.

(*e*) The variable cost at $x=5$.

(*f*) The break-even point, that is, $R(x)=C(x)$.

(*g*) The profit function.

(*h*) The most profitable output.

(*i*) The maximum profit.

(*j*) The marginal revenue at the most profitable output.

(*k*) The revenue at the most profitable output.

(*l*) The variable cost at the most profitable output.

13. Suppose the cost function is given by $C(x)=x^2+5$ and the price function is $p=12-2x$, where p is the price in rupees and x is the number of units produced (in thousands). Answer the question asked in Problem 12.

14. A company has for x items produced the total cost C and the total revenue R given by equations $R=3x$ and $C=1004\ 0{\cdot}015x^2$. Find how many items be produced to maximise the profit. What is this profit ?

15. A sofa-set manufacturer can manufacture x sofa sets per week at a total cost of Rs. $\left(\frac{1}{2}x^2+3x+100\right)$. How many sets per week should he manufacture for maximum monopoly revenue when the demand law of his product is $x=10\sqrt{25-p}$ set per week. Also find the net revenue with this output.

16. The cost function $C(x)$ for producing x units of a commodity is given by

$$C(x)=\frac{1}{3}x^3-5x^2+75x+10$$

At what level of output the marginal cost $\left(i.e., \frac{dC}{dx}\right)$ attains its minimum ? What is the marginal cost at this level of production ?

[*C.A. Intermediate, November, 1991*]

[**Ans.** 5, 50]

17. If q be the number of workers employed, the average cost of production is given by

$$C=\frac{3}{2(q-4)}+24q$$

show that $q=4{\cdot}25$ will make the expression minimum. In the interest of the management will you then advise to employ four or five workers ? Give reasons for your answer.

[*I.C.W.A., June 1990*]

[**Hint.** $$C=\frac{3}{2(q-4)}+24q$$

$$\Rightarrow \quad \frac{dC}{dq}=\frac{-3}{2(q-4)^2}+24$$

C will be minimum if $\frac{dC}{dq}=0$

i.e., when $\frac{-3}{2(q-4)^2}+24=0 \quad \Rightarrow \quad q=4{\cdot}25$

Since the function C is not defined at $q=4$, therefore, the value of q must be 5.]

18 The following expressions define a firm's total revenue and total cost functions :

$$\text{Total revenue}=18x-x^2+24$$

$$\text{Total cost}=\tfrac{1}{3}x^3-2{\cdot}5x^2+50$$

(*a*) Use calculus methods to find the optimum production level.

(*b*) State the firm's profits at the optimum production level.

(*c*) Using the same axes, sketch the graphs of the total revenue and total cost curves, indicating the output at which profit is maximum.

[**Ans.** (*a*) 6, (*b*) 64]

19. A steel plant is capable of producing x tons per day of a low grade steel and y tons per day of high grade steel, where $y=\frac{40-5x}{10-x}$. If the fixed market price of low grade steel is half of the high grade steel, show that about 5·5 tons of low grade steel are produced per day for maximum total revenue.

[**Hint.** Let p_1 be the price of low grade steel. Then $2p_1$ is the price of the high grade steel, p_1 is constant.

Total revenue function, $R=2p_1\left(\frac{40-5x}{10-x}\right)+xp_1$

Show that $\frac{dR}{dx}=0 \quad \Rightarrow \quad x=10\pm2\sqrt{5}$

Further show that $\frac{d^2R}{dx^2}<0$ for $x=10-2\sqrt{5}$ and

$$\frac{d^2R}{dx^2}>0 \text{ for } x=10+2\sqrt{5} \quad]$$

20. *Maximizing Profit.* A tractor company can manufacture at most 1000 heavy duty tractors per year. Furthermore, from past demand data, the company knows that the number of heavy duty tractors it can sell depends only on the price p of each unit. The company also knows that the cost to produce the units is a function of the number x of units sold. Assume that the price function is $p=29{,}000-3x$ and the cost function $C=2{,}000{,}000+20{,}000x+5x^2$. How many units should be produced to maximize profits ?

21. A manufacturer estimates that he can sell 500 articles per week if his unit price is Rs. 20·00, and that his weekly sales will rise by 50 units with each Rs. 0·50 reduction in price. The cost of producing and selling x articles a week is $C(x)=6200+6{\cdot}10x+0{\cdot}0003x^2$. Find

(*a*) The price function.

(*b*) The level of weekly production for maximum profit.

(*c*) The price per article at the maximum level of production.

22. A trucking company has an average engine overhaul cost of Rs. 1000 and routine maintenance cost (in rupees) of $C=0{\cdot}40x+10^{-5}x^2$, where x is the interval in kilometres between engine overhauls.

(*a*) Show that the total engine maintenance cost Rs. (per km) is given by

$$c=\frac{1000}{x}+0\,04+10^{-5}x$$

(*b*) Find the rate of change of the total maintenance cost with respect to the engine overhaul to interval $\frac{dc}{dx}$

(*c*) Find the value of x at which the derivative in (*b*) is equal to zero.

(*d*) Evaluate and compare c for $x=5{,}000$; $10{,}000$; $20{,}000$ kms.

$$\left[\text{Ans. } (b)\ \frac{dc}{dx}=\frac{-1000}{x^2}+10^{-5},\ (c)\ 10000,\ (d)\ 0{\cdot}29,\ 0{\cdot}24,\ 0{\cdot}29.\right]$$

23. Given :

$$p=20-q$$
$$C=2+8q+q^2$$

Find

(*a*) q which maximizes profit and corresponding values of p(=Price) R(=Total revenue) and M (=Profit).

(*b*) q which maximizes sales (total revenue) and corresponding values of p, R and M.

(*c*) q which maximizes sales subject to the constraint $M \leqslant 8$ and corresponding values of p and R.

24. A monopolist has the following demand and cost functions .

$$p=30-q$$
$$C=160+8q$$

The Government levies a tax at the rate of 2 per unit sold. Find profit maximizing price and quantity after tax levy.

[**Ans.** $p \quad 10, p=20$]

25. A firm has the following functions

$$p=100-0{\cdot}01\,q$$
$$\pi=50q+30{,}000$$

and a tax of 10 per unit is levied. What will be the profit maximizing price and quantity before the tax and after the tax ? Which does the monopolist find it better to increase the price by less than the increase in tax ?

[**Ans.** Before tax $q=2500$, $p=75$, Profit$=32{,}500$.
After tax $q=2000$, $p=80$, Profit$=10{,}000$
A price higher than 80 will reduce profit below 10,000]

26. If the relevant position of the demand function is

$$p=100-0{\cdot}01\,q$$

when q is weekly production and p is weekly price and cost function is

$$c=50q+30{,}000$$

(*a*) Find maximum profit, output, price and total profit.

(*b*) If suppose government decides to levy a tax of Rs. 10 per unit of product sold, what will happen to price, quantity sold and total profit ?

27. (*a*) Given the demand function $p=(10-x)^2$ and the cost function $C=55x-8x^2$, find the maximum profit. What would be the effect of an imposition of a tax of Rs. 9 per unit quantity on price ?

[**Ans.** 54 ; Price increase=15].

(*b*) Given the demand function $Y=20-4x$ and the average cost function $Y_c=2$, determine the profit maximising output of a monopolist firm. What would be the impact of a tax of Rs. t per unit of output on profit ?

28. A monopolist has a total cost of output x given by ax^2+bx+c and the demand price for the output x is given by $\beta-\alpha x$. Find his monopoly output, price and net revenue in equilibrium. How will these change if a tax at Rs. t per unit of output is levied ?

[**Ans.** *Before tax* :

$$\text{Output}=\frac{\beta-b}{2(\alpha+a)},\ \text{Price}=\frac{2a\beta+\alpha\beta+\alpha b}{2(\alpha+a)}$$

$$\text{Net revenue}=\frac{(\beta-b)^2}{4(\alpha+a)}-c$$

After tax :

$$\text{Output}=\frac{\beta-b-t}{2(\alpha+a)},\ \text{Price}=\frac{2a\beta+\alpha\beta+\alpha b+\alpha t}{2(\alpha+a)}$$

$$\text{Net revenue}=\frac{(\beta-b-t)^2}{4(\alpha+a)}-c\ \Big]$$

29. A monopolist firm has the following revenue and cost functions :

$$R=-\alpha Q^2+\beta Q,\ (\alpha,\ \beta>0)$$

$$C=aQ^2+bQ+C,\ (a,\ b,\ c>0)$$

The government plans to levy an excise tax on its product and wishes to maximise tax revenue T from this source. What is the desired tax rate t (rupees per unit of output) ?

30. (*a*) The demand and cost functions of a firm are given by

$$q=10{,}000-100p \text{ and}$$
$$c=59q+30{,}000.$$

where q=quantity demanded

p=price/unit

c=total cost,

Determine the optimum level of q that the firm should sell.

(*b*) Assuming that the above firm has to pay a sales tax at the rate of Rs. 10 per unit, find out the optimum sales.

SOME APPLIED PROBLEMS

Example 35. *Prove that a rectangle with sides x and y and a given perimeter P has its area maximised if it is a square.*

[*Delhi Univ., B.A.* (*Hons.*) *Economics 1991*]

Solution. We have

$$P=2(x+y)$$

or
$$y=\frac{1}{2}\ P-x$$

$$A=xy=x\left(\frac{1}{2}\ P-x\right)=\frac{1}{2}\ Px-x^2$$

First order condition :

$$\frac{dA}{dx}=0$$

$\therefore$
$$\frac{dA}{dx}=\frac{1}{2}\ P-2x$$

$$\frac{dA}{dx}=0 \quad \Rightarrow \quad \frac{1}{2}\ P-2x=0$$

or
$$x=\frac{1}{4}\ P$$

Second order condition :

$$\frac{d^2P}{dx^2}<0$$

$$\frac{d^2P}{dx^2}=-2<0$$

$\therefore$
$$x=\frac{1}{4}\ P$$

and
$$y=\frac{1}{2}\ P-\frac{1}{4}\ P=\frac{1}{4}\ P$$

Hence the rectangle has the maximum area if it is a square.

Example 36. *A box with a square base is to be made from a square piece of cardboard 24 centimetres on a side by cutting out a square from each corner and turning up the sides. Find the dimensions of the box that yield maximum volume ?*

Solution. Let the volume of the box be denoted by V and the dimensions of the side of the small square by x. Since the area of sheet metal is fixed, the sides of the square can be changed and thus are treated as variables. Let y denote the portion left after cutting the x's to make the square, we have

$$y=24-2x$$

Since the height of the box is x and the area of the base of the box is y^2, the volume V is given by $V=V(x)=xy^2$

$$\Rightarrow \quad V(x)=x(24-2x)^2=4x^3-96x^2+576x$$

To find the value of x which maximises V, we differentiate and find the critical values, *i.e.*,

$$V'(x)=12x^2-192x+576$$
$$=12(x^2-16x+48)=12(x-12)(x-4)$$

$x=12$ is not admissible as in that case box cannot be formed.

$$\therefore \quad x=4$$

Using second derivative test, we have

$$V''(x)=24x-192$$
$$\therefore \quad V''(4)=96-192<0$$

Hence the dimension $x=4$ maximises the volume and $4\times16\times16$ are the dimensions of the box.

Example 37. *The rate of working of an engine is given by the expression* $10v+\frac{4000}{v}$, *where* v *is the speed of the engine. Find the speed at which the rate of working is the least.*

Solution. We require to find the value of v for which the expression $10v+\frac{4000}{v}$ is a minimum.

Let $$H=10v+\frac{4000}{v}$$

$$\therefore \quad \frac{dH}{dv}=10-\frac{4000}{v^2}$$

$$\therefore \quad \frac{dH}{dv}=0, \text{ when } v^2=400, \text{ i.e., } v=\pm20$$

$v=-20$ is not admissible as speed cannot be negative.

$$\therefore \quad v=20.$$

Also $$\frac{d^2H}{dv^2}=\frac{8000}{v^3}$$

$$\frac{d^2H}{dv^2}>0 \text{ when } v=20$$

$\therefore$ The rate of working, H, is a minimum when $v=20$.

Example 38. *A firm's annual sales are* s *units of a product which the firm buys from a supplier. If the replenishment cost is Rs.* r *per order holding cost Rs.* h *per unit per year, find the economic order quantity by using calculus.* [*Delhi Univ., B. Com. (Hons.), 1991*]

Solution. Let x be the number of units ordered at any time. Then the holding (storage) cost for x units is hx.

$\therefore$ Number of orders $=\dfrac{s}{x}$

$\therefore$ The total cost $=hx+\left(\dfrac{s}{x}\right).r$

or
$$C=hx+\frac{sr}{x}$$

For C to be minimum,

First order condition :

$$\frac{dC}{dx}=0,\ i.e.,\ h-\frac{sr}{x^2}=0$$

or
$$x=\sqrt{\frac{sr}{h}}$$

Second order condition :

$$\frac{d^2C}{dx^2}>0$$

$$\frac{d^2C}{dx^2}=\frac{2sr}{x^3}>0$$

Hence the economic order quantity is $\sqrt{\dfrac{sr}{h}}$, *i.e.*, when $\sqrt{\dfrac{sr}{h}}$ units are ordered at a time the cost is minimum.

Example 39. *The production manager of a company plans to include 180 square centimetres of actual printed matter in each page of a book under production. Each page should have a 2·5 cm. wide margin along the top and bottom and 2·0 cm. wide margin along the sides. What are the most economical dimensions of each printed page.*

Solution. Let x, y denote the length and breadth of the printed matter in each page. Then

Area of each page, $xy=180$ ...(*)

Due to margin, the dimension of each page will be

$$x+2\times2=x+4 \quad\text{and}\quad y+2\times2{\cdot}5=y+5$$

Let A be the area of each page then

$$A=(x+4)\,(y+5)=xy+5x+4y+20$$

$$=200+5x+4\times\frac{180}{x} \qquad ...(**)$$

Differentiating (**) w.r.t. x, we get

$$\frac{dA}{dx}=5-\frac{720}{x^2}=0$$

$$\Rightarrow \quad x^2=144$$

$$\Rightarrow \quad x=12, \text{ discarding the negative value.}$$

Using the Second Derivative Test,

$$\therefore \quad \frac{d^2A}{dx^2}=\frac{2\times 720}{x^3}>0, \text{ when } x=12$$

$$x=12 \text{ minimises } A.$$

Substituting $x=12$ in (*), we have $y=\frac{180}{12}=15$.

Hence the most economical dimensions are :

Length $=x+4=16$ cms.

Breadth $=y+5=20$ cms.

Example 40. *Your company is planning to build a new factory. The rectangular area required for manufacturing and office is 15,000 square metres. A car parking area to a depth of 50 metres is needed at the front of the building, an access drive width of 15 metres is planned for the side and a delivery/loading bay to a depth of 25 metres at the rear.*

You are required to calculate the smallest total site the company should buy to meet these requirements. Workings must be shown : marks will be awarded for method used.

Solution. Let the length of the rectangular area required for manufacturing and office be x and the width be y, then $xy=15/000$ square metres.

$x+50+25$ is the length of the sides including the car park and the delivery/loading bay at the rear, *i.e.*, $x+75$, and

$y+15$ is the width of the site including the access drive. The area of the whole site is then

$$A=(x+75)(y+15)$$

$$=(x+75)\left(\frac{15,000}{x}+15\right)$$

$$\left[\because \quad xy=15,000 \quad \Rightarrow \quad y=\frac{15,000}{x}\right]$$

$$=15,000+15x+\frac{1,125,000}{x}+1125$$

Now $$\frac{dA}{dx}=0 \quad \Rightarrow \quad 15-\frac{1,125\cdot000}{x^2}=0$$

$$\Rightarrow \qquad x=\sqrt{\frac{1{,}125{\cdot}000}{15}}=273{\cdot}86$$

Also $$\frac{d^2A}{dx^2}=\frac{2\times 1{,}125{,}000}{x^3}>0$$

Hence A is minimum when $x=273{\cdot}86$

$$\therefore \qquad y=\frac{15{,}000}{273{\cdot}86}=54{\cdot}77$$

Hence the smallest total site the company should buy to meet its requirement is

$$(273{\cdot}86+75) \text{ metres by } (54{\cdot}77+15) \text{ metres.}$$

Example 41. *A metal box with a square top and bottom of equal size, is to contain 1000 cc. The material for the top and bottom costs one paisa per square cm and the material for the sides costs half paisa per square cm. Find the least cost of the box.*

Solution. As the base of the box is a square, the dimensions can be taken as x, x, y. Then the volume is $x^2\,y$.

$$x^2y=1000 \quad \Rightarrow \quad y=\frac{1000}{x^2} \qquad \ldots(^*)$$

Let C be the total cost.

The area of the top and bottom $=2x^2$ sq. cm.

Cost for the top and bottom $=1\times 2x^2=2x^2$

Cost for the 4 sides $=\frac{1}{2}\times 4xy=2xy$

Total cost, $C=2x^2+2xy$ $\qquad \ldots(^{**})$

$$=2x^2+2x.\ \frac{1000}{x^2}$$

$$=2x^2+\frac{2000}{x} \qquad \ldots(^{***})$$

$$\frac{dC}{dx}=4x-\frac{2000}{x^2}$$

$$\frac{dC}{dx}=0 \text{ gives, } 4x-\frac{2000}{x^2}=0$$

$$\Rightarrow \qquad 4x^3=2000$$

$$\Rightarrow \qquad x^3=500$$

$$\Rightarrow \qquad x=\sqrt[3]{500}=5\times\sqrt[3]{4}=5\times 1{\cdot}59=7{\cdot}95$$

$$\frac{d^2C}{dx^2}=4+\frac{2\times 2000}{x^3}>0, \text{ when } x=7{\cdot}95$$

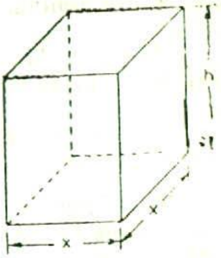
Thus $x=7{\cdot}95$ minimises C.

Substituting $x=7{\cdot}95$ in (***), we get

$$C_{MIN}=2\times(7{\cdot}95)^2+\frac{2000}{7{\cdot}95}=377{\cdot}98=378 \text{ paise}$$

Example 42. *A rectangular block with a square base has the total area of its surface equal to 150 square cms, and the sides of the base are each x cm long. Prove that the volume of the block is* $\frac{1}{2}(75x-x^3)$ *cu. cm., and hence find the maximum volume of the block.*

Solution. In order to obtain the maximum volume of the block, say V, we must first obtain an expression giving V in terms of one variable. As is indicated in the question, we will



first show that $V=\frac{1}{2}(75x-x^3)$ where x is the length of a side of the base.

We have $V=x^2h$ cu. cm., where h is the height of the block.

To obtain h in terms of x, we use the fact that the surface area of the block is equal to 150 sq. cm.

Surface area$=2x^2+4xh=150$

$$\therefore \qquad x^2+2xh=75$$

$$\Rightarrow \qquad h=\frac{75-x^2}{2x}$$

Hence $V=x^2h=x^2\left(\frac{75-x^2}{2x}\right)=\frac{1}{2}(75x-x^3)$ cu. cms.

Having obtained V in terms of one variable, we proceed to find t maximum value in the usual way.

We have $$\frac{dV}{dx}=\frac{75}{2}-\frac{3x^2}{2}$$

$$\therefore \qquad \frac{dV}{dx}=0, \quad \text{when } \frac{75}{2}-\frac{3x^2}{2}=0$$

i.e., when $x^2=25$, or $x=\pm 5$.

In this case, the negative value of x has no meaning and we discard it.

$$\frac{d^2V}{dx^2}=-\frac{6x}{2}=-3x$$

$\therefore$ When $x=5$, $\frac{d^2V}{dx^2}=-15$.

Hence, when $x=5$, $\frac{dV}{dx}=0$ and $\frac{d^2V}{dx^2}$ is negative.

$\therefore$ $x=5$ makes V a maximum.

$\therefore$ Maximum value of $V=\frac{1}{2}\{375-125\}=125$ cu. cm.

Example 43. *A wastepaper basket consists of an open circular cylinder. If the volume of the basket is to be 200 cubic centimetres ; find the radius of its base when the material used is a minimum.*

Solution. The material used in making the basket depends on the surface area of the basket.

Hence we require to find the radius of the base when the surface area is a minimum.

We must first of all obtain an expression giving the surface area (say S sq. cm) in terms of the radius of the base (say r).

The total surface area $S=\pi r^2+2\pi rh$ sq. cm, where 'h' is the height of the cylinder.

To obtain S in terms of r alone, h must be obtained in terms of r. This is done by using the fact that the volume of the basket is equal to 200 cu. cms.

We have, volume $=200=\pi r^2h$

$$\therefore \qquad h=\frac{200}{\pi r^2}$$

$$\text{Hence} \qquad S=\pi r^2+2\pi rh$$

$$\text{or} \qquad S=\pi r^2+2\pi r\,.\,\frac{200}{\pi r^2}=\pi r^2+\frac{400}{r}$$

$$\therefore \qquad \frac{dS}{dr}=2\pi r-\frac{400}{r^2}$$

$$\therefore \qquad \frac{dS}{dr}=0 \text{ when } 2\pi r-\frac{400}{r^2}=0$$

i.e., when $r^3=\frac{200}{\pi}$, or $r=\sqrt[3]{\frac{200}{\pi}}$

$$\text{Also} \qquad \frac{d^2S}{dr^2}=2\pi+\frac{800}{r^3}$$

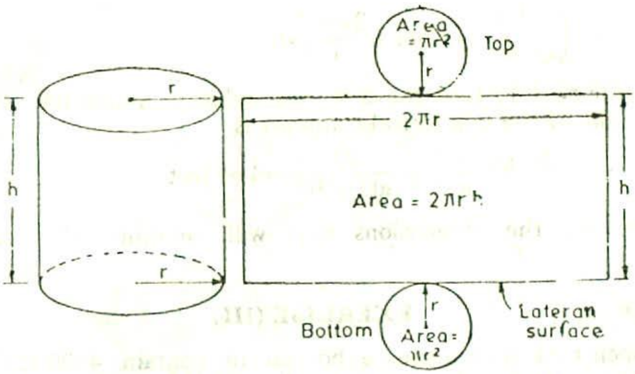
$\therefore$ When $r=\sqrt[3]{\frac{200}{\pi}}$, $\frac{d^2S}{dr^2}$ is positive, *i.e.*, S is a minimum.

Hence, the amount of material used will be a minimum when

$$r=\sqrt[3]{\frac{200}{\pi}}=3{\cdot}99 \text{ cm.}$$

Example 44. *ABC Co. Ltd. wishes to produce a cylindrical container with a capacity of 20 cubic feet. The top and bottom of the container are to be made of a material that costs Rs. 6 per square foot, while the side of the container is made of material costing Rs. 3 per square foot. Find the dimensions that will minimise the total cost of the container.*

Solution. Let h denote the height of the container and r the radius, then the total area of the bottom and the top is $2\pi r^2$ and the area of the lateral surface of the container is $2\pi rh$.



The total cost C of manufacturing the container is

$$C=(\text{Rs. }6)(2\pi r^2)+(\text{Rs. }3)(2\pi rh)$$
$$=12\pi r^3+6\pi rh \qquad ..(*)$$

Since volume of the cylinder is fixed at 20 cubic feet, *i.e.*,

$$V=20=\pi r^2h \quad \Rightarrow \quad h=\frac{20}{\pi r^2} \qquad ...(**)$$

Substituting (**) in (*), we get

$$C=12\pi r^2+6\pi r\cdot\frac{20}{\pi r^2}$$
$$=12\pi r^2+\frac{120}{r}$$

To find the value of r that gives minimum cost, we differentiate C w.r.t. r. Thus

$$\frac{dC}{dr}=C'(r)=24\pi r-\frac{120}{r^2}$$
$$=\frac{24\pi r^3-120}{r^2}$$

The critical values obey $C'(r)=0$

$$\Rightarrow \quad 24\pi r^3-120=0$$

$$\Rightarrow \qquad r^3 = \frac{5}{\pi}$$

$$\Rightarrow \qquad r = \left[\frac{5}{\pi}\right]^{1/2} \simeq 1{\cdot}17$$

Using the Second Derivative test, we have

$$C''(r) = 24\pi + \frac{240}{r^3}$$

and
$$C''\left(\sqrt[3]{\frac{5}{\pi}}\right) = 24\pi + \frac{240\pi}{5} > 0$$

Thus for $r = 1{\cdot}17$ feet, the cost is a relative minimum. The corresponding height of the cylindrical container is

$$h = \frac{20}{\pi r^2} = \frac{20}{\pi[5/\pi]^{2/3}} = 4{\cdot}65 \text{ feet}$$

These are the dimensions that will minimise the cost of the material.

EXERCISE (III)

1. An open tank with a square bottom to contain 4000 C.C. of water is to be constructed. Find the dimensions of the tank so that the surface area may be the least.

   [**Ans.** *Base dimensions 20 cm. Height 10 cm.*]

2. A rectangular box with no top is to be made from a rectangular piece of metal with dimensions 32 cm by 60 cm by cutting equal sized squares from the corners, then turning up the sides. What should be the side of the squares cut off if the box is to have maximum volume ? [**Ans.** 5 cms]

3. A company has scrap pieces of metal sheeting left over at the end of its production line. The company has no other use for the scrap and it can manufacture new boxes on present underutilised machinery. The market is willing to pay Re. 0·50 per cubic centimetre of such boxes, so the company wishes to maximise the volume that can be made by cutting equal squares out of the corners of the scrap pieces that measure 4 cm × 10 cm. The cost of manufacturing and selling the boxes is Rs. 3-00 per box. The production department states that the metal costs Re. 0·10 per square cm. What is the volume of the largest box that can be made from the scrap ? Should the company produce the box ?

   [**Ans.** 16·24 c.c. nearly. The company should produce the box]

4. A box with square top and bottom is to be made to contain 500 cubic cms. Material for top and bottom costs Rs. 4 per square cm and the material for the side costs Rs. 2 per square cm. What is the cost of the least expensive box that can be made ?

[**Hint.** Volume of the box, $x^2 y=500$ ...(*)

Cost for the top and bottom = Rs. $4 \times 2x^2$ = Rs. $8x^2$

Cost for the 4 lateral sides = Rs. $2 \times 4xy$ = Rs. $8xy$

$$C=8x^2+8xy \qquad \text{...(**)}$$

From (*), we get $\quad y=\dfrac{500}{x^2}$

Substituting in (**), we have

$$C=8x^2+8x \cdot \frac{500}{x^2}=8x^2+\frac{4000}{x}\Big]$$

5. A box with a rectangular bottom and no top is to be made from a rectangular piece of material 30 cms. long and 16 cms. wide by cutting equal sized square corners, then turning up the sides. What should be the dimensions of the squares if the box is to have maximum volume ?

[**Hint.** Let x cm. be the side of each square cut off from a corner. Then the dimensions of the box made are :

$$30-2x,\ 16-2x \text{ and } x$$

$$V=(30-2x)(16-2x)x$$

$$=4x^3-92x^2+480x$$

$$\frac{dV}{dx}=0 \quad \Rightarrow \quad 3x^2-46x+120=0$$

$$\Rightarrow \qquad x=12 \quad \text{and} \quad \frac{10}{3}$$

$x=12$ is not admissible.

$$\left(\frac{d^2V}{dx^2}\right)_{x=10/3}=(24x-184)_{x=10/3}<0$$

Hence in order to have maximum volume, the side of the square cut off at a corner should be $\dfrac{10}{3}$ cms.]

6. One side of a rectangular enclosure is formed by a hedge ; the total length of fencing available for the other three sides is 200 yd. Obtain an expression for the area of the enclosure, A sq. yd., in terms of its lengths x yd , and hence deduce the maximum area of the enclosure.

$\left[\right.$ **Ans.** $A=100x-\dfrac{x^2}{2}$, 5000 sq. yd. $\left.\right]$

7. If the volume of a circular cylindrical block is equal to 800 cu, cms., prove that the total surface area is equal to $2\pi x^2+\dfrac{1600}{x}$ sq. cms. where x cms is the radius of the base. Hence obtain the value of x which makes the surface area a minimum.

$\left[\right.$ **Ans.** $x=\sqrt[3]{\dfrac{400}{\pi}}$ - 5·03 cms. $\left.\right]$

8. A closed rectangular box is made of sheet metal of negligible thickness, the length of the box being twice its width. If the box has a capacity of 243 cu. cms., show that its surface area is equal to $4x^2 + \frac{729}{x}$. Hence obtain the dimensions of the box of least surface area.

[**Ans.** 9, 9/2, 6]

9. A rectangular sheet of metal is 8 metre by 3 metre. Equal squares of side x cm. are cut from each of the corners and the whole is folded up to form an open rectangular tray of depth x cms. Find the volume of the tray in terms of x, and its maximum volume.

$\left[\textbf{Ans.}\ \ V = 4x\,(400 - x)(150 - x) \text{ cu. cms., max. volume} = 7\,\frac{11}{27} \text{ cu. m.}\right]$

10. A long strip of metal 60 cms. wide is to be bent to form the base and two sides of a shute of rectangular cross-section. Find the width of the base so that the area of the rectangular cross-section shall be a maximum.

[**Ans.** 30 cms.]

11. An open rectangular box is to be made out of cardboard and to have a volume of 288 c. cms. The length of the box is to be twice the width. If the width is x cms., show that the area of the cardboard required is $2x^2 + \frac{864}{x}$ sq. cms. and find the value of x for this area to be a minimum.

[**Ans.** 6]

12. A rectangular box is to have a volume of 100 c in. and its length is to be twice its breadth. Find an expression for the square of the length of a diagonal of the box in terms of the breadth x in. Find also the minimum possible length of this diagonal. (Find the minimum value of the *square* of the length of the diagonal.)

$\left[\textbf{Ans.}\ \ 5x^2 + \frac{2500}{x^4}\ ;\ \sqrt{75} \text{ in.}\right]$

13. A closed cylindrical can is to have a surface area of 150π sq. cm. Find, in terms of π, the maximum volume of the can.

[**Ans.** 250π c.c.]

14. An open cylindrical can is to have a surface area of 147π sq. cm. Find, in terms of π, the maximum volume of the can.

[**Ans.** 343π c.c.]

15. A skeleton of a box is to be formed from three metres of wire. If the length of the box is to be twice its width, find in cms. its dimensions so that its volume shall be as large as possible. [**Ans** 12, 6, 9 cms.]

16. A closed box is to have a volume of 225 c. cms. and the length of the base is to be $1\frac{1}{2}$ times the width. Find the dimensions for the minimum surface area.

[**Ans.** $7\frac{1}{2}$, 5, 6 cms.]

17. A closed cylindrical can is to have a certain given surface area. Show that the maximum volume is obtained when the height of the can is equal to its diameter.

[**Hint.** $$S=2\pi r^2+2\pi rh \text{ (fixed)}$$

$$\therefore \quad h=\frac{1}{2\pi r}(S-2\pi r^2)$$

$$V=\pi r^2 h=\pi r^2\times\frac{S-2\pi r^2}{2\pi r}=\frac{1}{2}(Sr-2\pi r^3)$$

$$\frac{dV}{dr}=\frac{1}{2}(S-6\pi r^2)$$

$$\frac{dV}{dr}=0 \quad \Rightarrow \quad S=6\pi r^2 \text{ or } 2\pi r^2+2\pi rh=6\pi r^2$$

or $$2\pi rh=4\pi r^2 \quad \text{or} \quad h=2r$$

$$\frac{d^2V}{dr^2}=-6\pi r<0$$

$\therefore$ Volume is maximum when height of the can is equal to its diameter.]

18. An open tank with a square base and vertical sides is to be constructed of sheet metal so as to hold a given quantity of water. Show that the cost of the material will be least when the depth is half of the width.

19. A manager of a printing firm plans to include 200 square centimetres of actual printed matter in each page of a book under production. Each page should have a 2·5 cm. margin along the top and bottom and 2·0 cm. wide margin along the sides. What are the most economical dimensions of each printed page ?

20. A printer plans on having 50 square inches of printed matter per page and is required to allow for margins of 1 inch on each side and 2 inches on the top and bottom. What are the most economical dimensions for each page if the cost per page depends on the area of the page.

21. The total cost C of sampling information is given by $C=a_1 n+\frac{a_2}{n}$, where a_1 is the unit cost of sampling an item, a_2 is the cost of a unit error in estimation and n is the size of the sample. Find the number of items to be sampled that minimises the total sampling cost.

22. There are 60 newly built apartments. At a rental of Rs. 45 per month all apartments will be occupied. But one apartment is likely to remain vacant for each Rs. 1·50 increase in rent. Also an occupied apartment requires Rs. 6·00 more per month than a vacant one for maintenance and service. Find the relationship between the profit and the number of unoccupied apartments. What is the number of vacant apartments for which the profit is maximum ? What is the maximum profit ? [**Ans.** $P=2340+51x-1\cdot5x^2$, 17, Rs. 2773·50]

23. A farmer wishes to enclose 12,000 sq. metres of land in a rectangular plot and then divide it into two plots with a fence parallel to one of the sides. What are the dimensions of the rectangular plot that require the least amount of fence ?

Example 45. *Show that the rate of change of marginal utility of commodity with respect to y is equal to the rate of change of marginal utility of y with respect to x, where utility function is given by*

$$U=3x^2y^2+y^2$$

Solution.

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\,(3x^2y^2+y^2)=3y^2\,\frac{\partial}{\partial x}\,(x^2)+\frac{\partial}{\partial x}\,(y^2)=6y^2x$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\,(3x^2y^2+y^2)=3x^2\,\frac{\partial}{\partial y}\,(y^2)+2y=6x^2y+2y$$

$$\therefore \qquad f_x=6y^2x,\ f_y=6x^2y+2y.$$

$$f_{xy}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}(6y^2x)=6x\,\frac{\partial}{\partial y}\,(y^2)=12xy.$$

$$f_{yx}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\,(6x^2y+2y)$$

$$=\frac{\partial}{\partial x}(6x^2y)+\frac{\partial}{\partial x}\,(2y)=12xy$$

Now f_x is the marginal utility of x.

$\therefore$ f_{xy} will be the rate of change of marginal utility of x *w.r.t.* y.

Similarly f_{yx} will be the rate of change of marginal utility of y *w.r.t.* x.

Hence $\qquad f_{xy}=f_{yx}$

Example 46. *Find the ratio of the marginal utilities for two goods when the utility function is* $U=(x+a)^p.\ (y+b)^q$. *Show that the same result is obtained when the utility function is taken as*

$$U=p\ \log\,(x+a)+q\ \log\,(y+b).$$

Solution. $\qquad U=(x+a)^p\ (y+b)^q$

$$\frac{\partial u}{\partial x}=p(x+a)^{p-1}(y+b)^q \text{ and } \frac{\partial u}{\partial y}=q(x+a)^p(y+b)^{q-1}$$

$$\therefore \qquad \frac{\partial u}{\partial x}:\frac{\partial u}{\partial y}=\frac{p}{x+a}:\frac{q}{y+b}$$

For utility function $U=p\log\,(x+a)+q\log\,(y+b)$,

$$\frac{\partial u}{\partial x}=\frac{p}{x+a} \text{ and } \frac{\partial u}{\partial y}=\frac{q}{y+b}$$

$$\therefore \qquad \frac{\partial u}{\partial x}:\frac{\partial u}{\partial y}=\frac{p}{x+a}:\frac{q}{y+b}$$

Marginal Products

If the output (Q) of a firm is a function of two inputs labour (L) and capital (K), suppose

$$Q=f(L, K)$$

Then, it often becomes necessary to take decisions regarding changes in the inputs with regard to their separate contributions to the enhancement in the rate of output. The partial derivatives, in this case, are known as

$$\text{Marginal productivity (or product) of labour}=\frac{\partial Q}{\partial L}$$

and $$\text{Marginal productivity (or product) of capital}=\frac{\partial Q}{\partial K}.$$

Example 47. *The production function of a firm is given by*

$$Q=4L^{3/4}\ L^{1/4},\ L>0 \text{ and } K>0.$$

Find the marginal productivities of Labour (L) *and Capital* (K). *Is it true that the marginal productivity of labour decreases through positive values as L increases ? Does a similar statement regarding K hold ?*

Also, show that $L\ \dfrac{\partial Q}{\partial L}+K\ \dfrac{\partial Q}{\partial K}=Q$

Solution. The marginal productivity of labour (MPL) is

$$\frac{\partial Q}{\partial L}=4(3/4)\ L^{(3/4)-1}\ K^{1/4}=3K^{1/4}/L^{1/4};$$

and the marginal productivity of capital (MPK) is

$$\frac{\partial Q}{\partial K}=4\ (1/4)\ L^{3/4}\ K^{(1/4)-1}=L^{3/4}/K^{3/4}.$$

Since $L>0$ and $K>0$

$$\frac{\partial Q}{\partial L}>0 \text{ and decreases as } L \text{ increases}$$

and $$\frac{\partial Q}{\partial K}>0 \text{ and decreases as } K \text{ increases.}$$

$$\text{Further } L\ \frac{\partial Q}{\partial L}+K\frac{\partial Q}{\partial K}=L\ \left(\frac{3K^{1/4}}{L^{1/4}}\right)+K\ \left(\frac{L^{3/4}}{K^{3/4}}\right)$$

$$=3L^{3/4}\ K^{1/4}+L^{3/4}\ K^{1/4}$$

$$=4L^{3/4}\ K^{1/4}=Q.$$

Example 48. *Let the production function of a firm be given by*

$$Q=8LK-L^2-K^2.$$

Find the MPL and MPK. Show that

$$L\ \frac{\partial Q}{\partial L}+K\ \frac{\partial Q}{\partial K}=2Q$$

Solution. The marginal productivity of labour (MPL) is

$$\frac{\partial Q}{\partial L}=8K-2L$$

and the marginal productivity of capital (MPK) is

$$\frac{\partial Q}{\partial K}=8L-2K.$$

Therefore

$$L\,\frac{\partial Q}{\partial L}+K\,\frac{\partial Q}{\partial K}=L\,(8K-2L)+K\,(8L-2K)$$
$$=16LK-2L^2-2K^2$$
$$=2(8LK-L^2-K^2)$$
$$=2Q.$$

Example 49. *Given the production function $P=L^2-2KL+2K^2$, where L represents labour and K capital, find marginal physical product of labour $L=2$ and $K=3$.*

Solution. $P=L^2-2KL+2K^2$

$$\frac{\partial P}{\partial L}=2L-2K\;;\;\frac{\partial P}{\partial K}=-2L+4K$$

when $L=2$ and $K=3$, we have

$$\frac{\partial P}{\partial L}=2\times2-2\times3=-2$$

$$\frac{\partial Q}{\partial K}=-2\times2+4\times3=8.$$

Example 50. *Given the production function $P=4KL-2K^2-L^2$, find the maximum P with the constraint $L+K=10$.*

Solution. Since $K+L=10$; $K=10-L$

Now, P can be expressed as a function of L by substituting the value of $K=10-L$

$$P=4\,(10-L)\,L-2\,(10-L)^2-L^2$$
$$=80\,L-7L^2-200$$

For P to be maximum, we have

$$\frac{dP}{dL}=0 \text{ and } \frac{d^2P}{dL^2}<0$$

$$\frac{dP}{dL}=0 \quad\Rightarrow\quad 80-14\,L=0 \quad\text{or}\quad L=40/7$$

and

$$\frac{d^2P}{dL^2}=-14<0$$

Hence maximum P is given by

$$P=80\times\frac{40}{7}-7\left(\frac{40}{7}\right)^2-200=\frac{200}{7}=28{\cdot}57$$

Example 51. *Given the Cobb-Douglas production function :*

$$P=10\,L^{1\cdot25}K^{\cdot5}$$

Find the output levels for

(*a*) *K is fixed at 100 and L rises 5, 10, 15.*

(*b*) *L is fixed at 100 and K rises 5, 10, 15.*

(*c*) *With L is 10 and K is 15.*

Solution. (*a*) With K fixed at 100, the function becomes

$$P=10\ L^{1\cdot25}\ 100^{\cdot5}$$
$$=10\ L^{1\cdot25}\sqrt{100}$$
$$=100\ L^{1\cdot25}$$

with $L=5$, 10 and 15 the production levels will be 747·67, 1778·28 and 2951·98.

(*b*) With L fixed at 100, the function becomes

$$P=10\times100^{1\cdot25}\ K^{\cdot5}$$
$$=316{\cdot}22\ K^{\cdot5}$$

With $K=5$, 10 and 15, the production levels will be 707·11, 999·97 and 1223·86.

(*c*) With $L=10$ and $K=15$

$$P=10\times10^{1\cdot25}\times15^{\cdot5}=688{\cdot}72.$$

Homogeneous Function

If $u=f(x, y)$ be a function of two variables, then this function is said to be a homogeneous function of degree n (or of order n) f the following relationship holds :

$$f(tx, ty)=t^n f(x, y)\ ;\ t>0.$$

Remark. A function is said to be linear homogeneous function if the following relationship holds :

$$f(tx, ty)=t\ f(x, y)$$

Example 52. *Let q be the quantity, p be price and y be income. Show that the demand function shown as*

$$q=f(p, y)=\frac{y}{kp},\ \text{where } k \text{ is a constant, homogeneous of degree zero.}$$

Solution. Here $f(p, y)=\dfrac{y}{kp}$

$$\therefore\ f(tp,\ ty)=\frac{ty}{ktp}=\frac{y}{kp}=t^{\circ}\frac{y}{kp}=t^{\circ}f(p, y)$$

∴ The demand function is homogeneous of degree zero.

Example 53. *Let*

$$Q=10L-0{\cdot}1L^2+15K-0{\cdot}2K^2+2KL$$

be the production function of a commodity with Q standing for output, L for labour and K for capital.

(*a*) *Calculate the marginal products of the two inputs when 10 units each of labour and capital are used.*

(*b*) *Assuming that 10 units of capital are being used, indicate the upper limit for use of labour which a rational producer will never exceed.*

Solution. (*a*) $\frac{\partial Q}{\partial K}=0+0+15\frac{\partial}{\partial K}(K)-0{\cdot}2\frac{\partial}{\partial K}(K^2)+2L\frac{\partial}{\partial K}(K)$

$$=2L-0{\cdot}4K+15$$

Now substituting $L=10$, and $K=10$, we get

Marginal product $=2\times10-0{\cdot}4\times10+15=20-4+15=31$

$$\frac{\partial Q}{\partial L}=10.\frac{\partial}{\partial L}(L)-0{\cdot}1\frac{\partial}{\partial L}(L^2)+0-0+2K\frac{\partial}{\partial L}(L)$$

$$=2K-0{\cdot}2L+10$$

Now substituting $K=10$, and $L=10$, we get

Marginal product for Labour

$$=2\times10-0{\cdot}2\times10+10=28.$$

(*b*) Now, the upper limit for use of labour which a rational producer will never exceed, where 10 units of capital are being used, can be obtained by using the following condition :

$$\left(\frac{\partial Q}{\partial L}\right)_{K=10}\geqslant0$$

$$\Rightarrow\quad\Big[\,2K-0{\cdot}2L+10\,\Big]_{K=10}\geqslant0$$

$$\Rightarrow\quad2\times10-0{\cdot}2\times L+10\geqslant0$$

$$\Rightarrow\quad\frac{30}{0{\cdot}2}\geqslant L$$

$$\Rightarrow\quad L\leqslant150$$

Hence, the upper limit for the use of labour input will be 150 units.

Example 54. *Show that the production function*

$$x=f(l,k)=2\sqrt{lk}$$

(*where x, l and k are the units of output, labour and capital respectively*) *gives constant return to scale and diminishing returns to inputs.*

[*Delhi Univ., B. Com.* (*Hons.*), *1992*]

Solution. The given production function

$$x=f(l,k)=2\sqrt{lk}$$

is homogeneous function of degree one. Replace l by λl and k by λk in x, we have

$$x=f(\lambda l,\lambda k)=2\sqrt{\lambda l.\,\lambda k}=2\lambda\sqrt{lk}$$

Hence the given function is a homogeneous function of degree one. So, the function gives constant returns to scale.

Now $$MP_l = \frac{\partial x}{\partial l} = \sqrt{\frac{k}{l}}$$

and $$\frac{\partial}{\partial l}\left(MP_l\right) = -\frac{1}{2} \cdot \frac{\sqrt{k}}{l^{3/2}} < 0$$

Hence the function gives the diminishing return to labour.

$$MP_k = \frac{\partial x}{\partial k} = \sqrt{\frac{l}{k}}$$

$$\frac{\partial}{\partial k}\left(MP_k\right) = -\frac{1}{2} \cdot \frac{\sqrt{l}}{k^{3/2}} < 0$$

Hence the function gives the diminishing return to capital.

∴ The function gives the diminishing returns to inputs.

Euler's Theorem

Euler has shown that if $Z = f(x_1, x_2)$ is a homogeneous function of degree n, then

$$x_1\frac{\partial Z}{\partial x_1} + x_2\frac{\partial Z}{\partial x_2} = nZ$$

Example 55. *The Cobb-Douglas production function for the economy as a whole is given by*

$$Q = a\, L^{\alpha} \ K^{\beta}$$

where a, α, β are constants such that $\alpha + \beta = 1$.

Show that

(*a*) *Q is linear homogeneous function of L and K.*

(*b*) *Prove that* $L\dfrac{\partial Q}{\partial L} + K\dfrac{\partial Q}{\partial K} = Q.$

Solution. (*a*) Let $Q = f(L, K) = a\, L^{\alpha} K^{\beta}$, $\alpha + \beta = 1$.

Then $$\begin{aligned} f(t\, L,\ t\ K) &= a\, (t\, L)^{\alpha}\, (t\, K)^{\beta} \\ &= t^{\alpha+\beta}\ (a\, L^{\alpha}\, K^{\beta}) \\ &= t^{1} f(L, K). \end{aligned}$$

Hence $Q = f(L, K)$ is a linear homogeneous function of L and K.

(*b*) $\dfrac{\partial Q}{\partial L} = a\alpha\, L^{\alpha-\beta}\, K^{\beta}$ and $\dfrac{\partial Q}{\partial K} = a\beta\, L^{\alpha} \ K^{\beta-1}$.

Hence L $$\frac{\partial Q}{\partial L} + Q\frac{\partial Q}{\partial K} = a\alpha\, L^{\alpha}\, K^{\beta} + a\beta\, L^{\alpha}\, K^{\beta}$$

$$= a\mathrm{L}^{\alpha}\, K^{\beta}\, (\alpha + \beta) = Q.$$

since $\alpha+\beta=1$. [Here, we have verified Euler's Theorem for the Cobb-Douglas (linear homogeneous) production function.]

Example 56. *Verify Euler's Theorem for*

$$u=ax^3+bx^2y+cxy^2+dy^3.$$

Solution. Here the given function is *homogeneous* and of the *third* degree in x and y. It is required to prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$$

Now $$u=ax^3+bx^2y+cxy^2+dy^3$$

$\therefore$ $$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(ax^3+bx^2y+cxy^2+dy^3)$$

$$=\frac{\partial}{\partial x}(ax^3)+\frac{\partial}{\partial x}(bx^2y)+\frac{\partial}{\partial x}(cxy^2)+\frac{\partial}{\partial x}(dy^3)$$

$$=3ax^2+2byx+cy^2 \qquad ...(1)$$

(Here y is constant)

and $$\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}(ax^3+bx^2y+cxy^2+dy^2)$$

$$=\frac{\partial}{\partial y}(ax^3)+\frac{\partial}{\partial y}(bx^2y)+\frac{\partial}{\partial y}(cxy^2)+\frac{\partial}{\partial y}(dy^3)$$

$$=bx^2+2cxy+3dy^2 \qquad ...(2)$$

(Here x is constant)

Multiplying (1) by x and (2) by y, we have

$$x\frac{\partial u}{\partial x}=3ax^3+2bx^2y+cxy^2$$

$$y\frac{\partial u}{\partial y}=bx^2y+2cxy^2+3dy^3$$

Adding, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=3ax^3+3bx^2y+3cxy^2+3dy^3$$

$$=3(ax^3+bx^2y+cxy^2+dy^3)=3u.$$

Example 57. *Verify Euler's theorem for* $u=x^n\log\frac{y}{x}$.

Solution. Here the given function is of the nth degree, the degree of y/x being zero. It is required to prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Since $$u=x^n\log\frac{y}{x}.$$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(x^n \log \frac{y}{x}\right)=x^n \frac{\partial}{\partial x}\left(\log \frac{y}{x}\right)+\log \frac{y}{x}.\ \frac{\partial}{\partial x}(x^n)$$

$$=x^n \frac{1}{(y/x)}\cdot\frac{\partial}{\partial x}\left(\frac{y}{x}\right)+\log \frac{y}{x}.\ nx^{n-1}.$$

$$=x^n \frac{x}{y}\left(\frac{-y}{x^2}\right)+nx^{n-1}.\ \log\frac{y}{x} \qquad \text{(Here } y \text{ is constant)}$$

$$=-x^{n-1}+nx^{n-1}.\ \log\frac{y}{x} \qquad ...(1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(x^n \log \frac{y}{x}\right)=x^n \frac{\partial}{\partial y}\left(\log \frac{y}{x}\right)$$

$$=x^n \frac{1}{(y/x)}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=x^n.\frac{x}{y}\cdot\frac{1}{x}=\frac{x^n}{y} \qquad ...(2)$$

Multiply (1) by x and (2) by y, we get

$$x\frac{\partial u}{\partial x}=-x^n+nx^n \log\frac{y}{x}\cdot$$

$$y\frac{\partial u}{\partial y}=x^n.$$

Adding, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=nx^n \log\frac{y}{x}=nu.$$

Example 58. *Define the degree of homogeneity and state Euler's theorem.*

If the supply function $x=f(p_1, p_2,..., p_m)$ is homogeneous of degree n, show that the sum of the partial price elasticities of supply equals n. (x denotes the quantity supplied of a particular commodity and $p_1, p_2,...,$ p_m are the prices of the different commodities.

[Delhi Univ. B. Com. (Hons), 1991]

Solution. If $u=f(x, y)$ be a function of two variables, then this function is said to be a homogeneous function of degree n if the following relationship holds :

$$f(tx, ty)=t^n f(x, y)\ ;\ t>0.$$

If $Z=f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial Z}{\partial x}+y\frac{\partial Z}{\partial y}=nZ.$$

The partial elasticity of supply x w.r.t. pl.

$$=\frac{pl}{x}\cdot\frac{\partial x}{\partial pl}\ (l=1, 2,..., m)$$

∴ Sum of partial elasticities of supply

$$=\sum_{l=1}^{n}\frac{pl}{x}\cdot\frac{\partial x}{\partial pl}$$

$$=\frac{1}{x} \cdot \sum_{i=1}^{m} pi \cdot \frac{\partial x}{\partial pi}$$

$$=\frac{1}{x}\left(p_1 \cdot \frac{\partial x_1}{\partial p_1}+p_2 \cdot \frac{\partial x_2}{\partial p_2}+\ldots+p_m \frac{\partial x}{\partial p_m}\right)$$

$$=\frac{1}{x} \cdot nx \qquad \text{[By Euler's Theorem]}$$

$$=n.$$

Example 59. *A production function is given by*

$$q=4L^{2/3}\, C^{1/3}$$

where L = labour, C = capital

(*a*) *Find the behaviour of the marginal product of each factor.*

(*b*) *What is the nature of returns to scale?*

(*c*) *What is the reward of labour and capital if each factor is paid a price equal to its marginal product?*

Solution. (*a*) We have $q=4L^{2/3}C^{1/3}$

$$\frac{\partial q}{\partial L}=\frac{\partial}{\partial L}(4L^{2/3}C^{1/3})=4C^{1/3}\ \frac{\partial}{\partial L}(L^{2/3})=4C^{1/3}\ \frac{2}{3}\, L^{(2/3)-1}$$

$$=\frac{8}{3}\ C^{1/3}\, L^{-1/3}$$

$\therefore$ Marginal product of labour $=\frac{8}{3}\ C^{1/3}\, L^{-1/3}$.

Rate of change of Marginal product of labour

$$=\frac{\partial}{\partial L}\left(\frac{\partial q}{\partial L}\right)=\frac{\partial}{\partial L}\left(\frac{8}{3}\ C^{1/3}\, L^{-1/3}\right)=\frac{8}{3}\left(-\frac{1}{3}\right) L^{-4/3}\, C^{1/3}$$

$$=-\frac{8}{9}\ L^{-4/3}\, C^{1/3}$$

hich shows that as L increases, Marginal product of labour decreases.

Again $\frac{\partial q}{\partial C}=\frac{\partial}{\partial C}(4L^{2/3}\, C^{1/3})=\frac{4}{3}\ C^{-2/3}\, L^{2/3}$

$\therefore$ Marginal product of capital $=\frac{4}{3}\ C^{-2/3}\, L^{2/3}$.

Rate of change of Marginal product of capital

$$=\frac{\partial}{\partial C}\left(\frac{\partial q}{\partial C}\right)=\frac{\partial}{\partial C}\left(\frac{4}{3}\ C^{-2/3}\, L^{2/3}\right)=\frac{-8}{9}\ C^{-5/3}\, L^{2/3}$$

which again shows that as C increases Marginal product of capital decreases

(*b*) Let $q=f(L, C)=4L^{2/3}\, C^{1/3}$

$$f(tL, tC)=4\,(tL)^{2/3}\,(tC)^{1/3}=4t^{2/3}\,L^{2/3}\,t^{1/3}\,C^{1/3}$$
$$=t\,[4L^{2/3}\,C^{1/3}]=tf\,(L, C)$$

$\therefore$ The production function is homogeneous of degree one, which shows that the reward to all the factors is exactly equal to total product.

[**Remark.** *If the production function is homogeneous of degree greater than one, we shall have a case of increasing returns and if it is of degree less than or equal to one, this shows we have a case of diminishing returns to scale.*]

(*c*) If each factor is paid a reward equal to its marginal product, then we have

$$L\,\frac{\partial q}{\partial L}+C\,\frac{\partial q}{\partial C}=L\,\left(\frac{8}{3}\,C^{1/3}\,L^{-1/3}\right)+C\,\left(\frac{4}{3}\,C^{-2/3}\,L^{2/3}\right)$$
$$=\frac{8}{3}\,C^{1/3}\,L^{2/3}+\frac{4}{3}\,C^{1/3}\,L^{2/3}=4C^{1/3}\,L^{2/3}=q.$$

Example 60. *The production function is $x=Aa^{\alpha}\;b^{\beta}$, where $\alpha+\beta<1$. Show that there are decreasing returns to scale and deduce that the total product is greater than* a *times the marginal product of Labour plus* b *time the marginal product of capital. What economic interpretation can you give for this ?*

Solution. Here $x=A\;a^{\alpha}\;b^{\beta}$ and $\alpha+\beta<1$

MP_a=Marginal product of labour

$$\frac{\partial x}{\partial a}=A\;\alpha\,a^{\alpha-1}\;b^{\beta}$$

$$\therefore \qquad a\,\frac{\partial x}{\partial a}=A\;\alpha\,a^{\alpha}\;b^{\beta}=\alpha x$$

MP_b=Marginal product of capital

$$\frac{\partial x}{\partial b}=A\,a^{\alpha}\,\beta\,b^{\beta-1}$$

$$\therefore \qquad b\,\frac{\partial x}{\partial b}=\beta x.$$

Thus
$$a\,\frac{\partial x}{\partial a}+b\,\frac{\partial x}{\partial b}=(\alpha+\beta)x<x$$

$$\Rightarrow \qquad x>a\,\frac{\partial x}{\partial a}+b\,\frac{\partial x}{\partial b}$$

Hence there are decreasing returns to scale.

Example 61. *Find the first and second order total differentials of the function*

$$Z=f(x, y)=7y\log\,(1+x)$$

[*Delhi Univ., B.Com. (Hons), 1992*]

Solution. We have

$$Z=7y\log(1+x)$$

$$\therefore \quad dZ=\frac{\partial Z}{dx}\,.\,dx+\frac{\partial Z}{\partial y}\,.\,dy$$

$$\frac{\partial Z}{\partial x}=\frac{7y}{1+x} \text{ and } \frac{\partial Z}{\partial y}=7\log(1+x)$$

$$\therefore \quad dZ=\frac{7y}{1+x}\,.\,dx+7\log(1+x)\,.\,dy$$

$$=7\left[\frac{y\,dx}{1+x}+\log(1+x)\,dy\right]$$

$$d^2Z=\frac{\partial}{\partial x}(dZ)\,.\,dx+\frac{\partial}{\partial y}(dZ)\,.\,dy$$

$$=\left\{-\frac{7y\,dx}{(1+x)^2}+\frac{7dy}{1+x}\right\}.\,dx+\left\{\frac{7dx}{1+x}\right\}.\,dy$$

$$=\frac{14\,dx\,dy}{1+x}-\frac{7y\,d^2x}{(1+x)^2}.$$

Example 62. *Given a linear homogeneous production function* $Z=AL^{\alpha}K^{\beta}P^{\gamma}$, L, *K, P, stand for factor quantities and A is a constant, show that*

(*i*) *the sum of the production elasticities with respect to the factors is unity.*

(*ii*) *the sum of marginal products of factors each multiplied by its respective quantity equals the total output.*

(*iii*) *In* (*i*) *and* (*ii*) *above, consider how these results change if the given production function is not linear homogeneous but homogeneous of degree n.*

Solution. (*i*) $Z=AL^{\beta}K^{\beta}P^{\gamma}$

$$\frac{\partial Z}{\partial L}=\alpha AL^{\alpha-1}K^{\beta}P^{\gamma}$$

$$\therefore \quad \frac{L}{Z}\cdot\frac{\partial Z}{\partial L}=\alpha\,.\,\frac{AL^{\alpha}K^{\beta}P^{\gamma}}{AL^{\alpha}K^{\beta}P^{\gamma}}=\alpha.$$

Similarly

$$\frac{K}{Z}\cdot\frac{\partial Z}{\partial K}=\beta \text{ and } \frac{P}{Z}\cdot\frac{\partial Z}{\partial P}=\gamma.$$

Adding the above production elasticities with respect to the factors, we get

$$\frac{L}{Z}\cdot\frac{\partial Z}{\partial L}+\frac{K}{Z}\cdot\frac{\partial Z}{\partial K}+\frac{P}{Z}\cdot\frac{\partial Z}{\partial P}=\alpha+\beta+\gamma=1.$$

($\because$ For linear homogeneous production function $\alpha+\beta+\gamma=1$)

(*ii*) $$L\frac{\partial Z}{\partial L}+K\frac{\partial Z}{\partial K}+P\frac{\partial Z}{\partial P}=(\alpha+\beta+\gamma)\,Z=Z$$

Hence sum of marginal products of factors multiplied by L, K and P is equal to total output Z.

(*iii*) In each case, the production elasticity will be multiplied by n and also Z will be multiplied by n.

Marginal Demand Functions and Partial Elasticities of Demand

Let the demand functions for two related commodities x_1 and x_2 with the respective prices p_1 and p_2 be

$$x_1=f(p_1, p_2) \quad \text{and} \quad x_2=g(p_1, p_2)$$

Then the partial derivatives of x_1 and x_2 are known as the (partial) *marginal demand function of* x_1 and x_2, respectively.

In particular,

the (*Partial*) *marginal demand of* x_1 w.r.t. p_1 is $\frac{\partial x_1}{\partial p_1}$,

the (*Partial*) *marginal demand of* x_1 w.r.t. p_2 is $\frac{\partial x_1}{\partial p_2}$

the (*Partial*) *marginal demand of* x_2 w.r.t. p_1 is $\frac{\partial x_2}{\partial p_1}$

and the (*Partial*) *marginal demand of* x_2 w.r.t. p_2 is $\frac{\partial x_2}{\partial p_2}$

For the usual demand functions

If p_2 is constant, x_1 increases (decreases) as p_1 decreases (increases) and if p_1 is constant, x_2 increases (decreases) as p_2 decreases (increases) : and hence, $\frac{\partial x_1}{\partial p_1}$ and $\frac{\partial x_2}{\partial p_2}$ are negative for all economically relevant values (positive or zero) of p_1 and p_2. Further,

if $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are both negative for a given (p_1, p_2), then a decrease in either price corresponds to an increase in both demands ; and the commodities X_1 and X_2 are said to be *complementary*.

On the other hand,

if $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are both positive for given (p_1, p_2), then a decrease in either price corresponds to an increase in one demand and a decrease in the other ; and the commodities x_1 and x_2 are said to be *competitive*.

If $\frac{\partial x_1}{\partial p_2}\cdot\frac{\partial x_2}{\partial p_1}<0$, then x_1 and x_2 are neither complementary nor competitive.

The *partial elasticity of demand* is the ratio of the proportional change in quantity demanded of one commodity (say, x_1) to the proportional changes in price of one commodity (p_1, or p_2), with the price of the other commodity (p_2 or p_1) held constant. Thus,

the *partial elasticity of demand* x_1 w.r.t. *price* p_1, with p_2=constant is

$$\eta_{11/11}=\frac{-p_1}{x_1}\cdot\frac{\partial x_1}{\partial p_1}=\frac{\frac{\partial}{\partial x_1}(\log x_1)}{\frac{\partial}{\partial x_1}(\log p_1)};$$

the *partial elasticity of demand* x_1 w.r.t. *price* p_2 with p_1=constant is

$$\eta_{12/12}=\frac{-p_2}{x_1}\cdot\frac{\partial x_1}{\partial p_2}=\frac{\frac{\partial}{\partial x_1}(\log x_1)}{\frac{\partial}{\partial x_1}(\log p_2)};$$

the *partial elasticity of demand* x_2 w.r.t. *price* p_1, with p_2=constant is

$$\eta_{21/21}=\frac{-p_1}{x_2}\cdot\frac{\partial x_2}{\partial p_1}=\frac{\frac{\partial}{\partial x_1}(\log x_2)}{\frac{\partial}{\partial x_2}(\log p_1)};$$

and the *partial elasticity of demand* x_2 w.r.t. *price* p_2, with p_1=constant is

$$\eta_{22/22}=\frac{-p}{x_2}\cdot\frac{\partial x_2}{\partial p_2}=\frac{\frac{\partial}{\partial x_2}(\log x_2)}{\frac{\partial}{\partial x_2}(\log p_2)}$$

Example 63. *For the following pair of demand functions for two commodities X_1 and X_2, determine the four partial marginal demands, the nature of relationship (Complementary, Competitive or neither) between X_1 and X_2 and the four partial elasticities of demand*

$$x_1=\frac{4}{p_1^2p_2} \quad \textit{and} \quad x_2=\frac{16}{p_1p_2^2}.$$

Solution. Partial marginal demands :

$$\frac{\partial x_1}{\partial p_1}=\frac{-8}{p_1^3p_2}<0,$$

$$\frac{\partial x_2}{\partial p_2}=\frac{-32}{p_1p_2^3}<0,\quad \frac{\partial x_1}{\partial p_2}=\frac{-4}{p_1^2p_2^2}<0$$

and $$\frac{\partial x_2}{\partial p_1}=\frac{-16}{p_1^2p_2^2}<0$$

Hence X_1 and X_2 are complementary commodities.

Partial elasticities of demands :

$$|\,\eta_{11}\,|=(-2),\ |\,\eta_{12}\,|=(-1),\ |\,\eta_{21}\,|=(-1),\ |\,\eta_{22}\,|=(-2).$$

Example 64. *The following are the demand functions for two commodities X_1 and X_2*

$$x_1=p_1^{-1\cdot7}\,p_2^{0\cdot8}$$
$$x_2=p_1^{0\cdot5}\,p_2^{-0\cdot2}$$

Determine whether the two commodities are complements or substitutes in some sense. [*Delhi Univ., B.A.* (*Hons.*) *Economics 1991*]

Solution. We have

$$x_1=p_1^{-1\cdot7}\,p_2^{0\cdot8} \text{ and } x_2=p_1^{0\cdot5}\,p_2^{-0\cdot2}$$

$$\therefore \qquad \frac{\partial x_1}{\partial p_1}=-1\cdot7\,p_1^{-1\cdot7}\,p_2^{0\cdot8}$$

$$\frac{\partial x_1}{\partial p_2}=0\cdot8\,p_1^{-1\cdot7}\,p_2^{-0\cdot2}$$

$$\frac{\partial x_2}{\partial p_1}=0\cdot5\,p_1^{-0\cdot5}\,p_2^{-0\cdot2}$$

$$\frac{\partial x_2}{\partial p_2}=-0\cdot2\,p_1^{0\cdot5}\,p_2^{-1\cdot1}$$

Since $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are both positive, the two commodities are substitutes in some sense.

Example 65. *The demand functions of two commodities X_1 and X_2 are $x_1=p_1^{-1\cdot4}\,p_2^{0\cdot6}$ and $x_2=p_1^{0\cdot5}\,p_2^{-1\cdot2}$ respectively, where x_1 and x_2 are the quantities demanded of X_1 and X_2 respectively and p_1 and p_2 are their respective prices. Find the four partial elasticities of demand and determine whether the commodities are competitive or complementary.*

[*Delhi Univ. B. Com.* (*Hons.*), *1991*]

Solution.

The partial elasticity of demand of x_1 w.r.t. prices $p_1=\frac{-p_1}{x_1}\cdot\frac{\partial x_1}{\partial p_1}$

$$=\frac{-p_1}{p_1^{-1\cdot4}\,p_2^{\,0\cdot6}}\times(-1\cdot4)\,p_1^{-2\cdot4}\,p_2^{0\cdot6}=1\cdot4$$

The partial elasticity of demand x_1 w.r.t. price p_2

$$=\frac{-p_2}{x_1}\cdot\frac{\partial x_1}{\partial p_2}=\frac{-p_2}{p_1^{-1\cdot4}\,p_2^{0\cdot6}}\times(0\cdot6)\;p_1^{-1\cdot4}\,p_2^{-0\cdot4}=-0\cdot6$$

The partial elasticity of demand x_2 w.r.t. price p_1

$$=\frac{-p_1}{x_2}\cdot\frac{\partial x_2}{\partial p_1}=\frac{-p_1}{p_1^{0\cdot5}\,p_2^{-1\cdot2}}\times(0\cdot5)\,p_1^{-0\cdot5}\,p_2^{-1\cdot2}=-0\cdot5$$

The partial elasticity of demand x_2 w.r.t. price p_2

$$=\frac{-p_2}{x_2}\cdot\frac{\partial x_2}{\partial p_2}=\frac{-p_2}{p_1^{0\cdot5}\,p_2^{-1\cdot2}}\times(-1\cdot2)\,p_1^{0\cdot5}\,p_2^{-2\cdot2}=1\cdot2$$

Since both $\frac{\partial x_1}{\partial p_2}$ and $\frac{\partial x_2}{\partial p_1}$ are negative, the commodities X_1 and X_2 are said to be complementary.

Example 66. *Find the elasticity of substitution* (σ) *for the production function* $Q=f(l, k)=[ak^{-b}+(1-a)\ l^{-b}]^{-1/b}$ *by using the formula*

$$\sigma=\frac{f_e \mid f_k}{k \mid l}.\frac{d(k \mid l)}{d(f_e \mid f_k)}$$

[Delhi Univ. B. Com. (Hons.), 1991]

Solution. We have

$$Q=\left[\ ak^{-b}+(1-a)l^{-b}\right]^{-1/b}$$

or $$Q^{-b}=ak^{-b}+(1-a)l^{-b} \qquad \ldots(1)$$

Differentiating (1) partially w.r.t. k, we get

$$-b.Q^{-b-1}.\frac{\partial Q}{\partial k}=-abk^{-b-1}$$

$$\Rightarrow \qquad \frac{\partial Q}{\partial k}=\frac{ak^{-b-1}}{Q^{-b-1}}=f_k$$

Similarly, $$\frac{\partial Q}{\partial l}=\frac{(1-a)l^{-b-1}}{Q^{-b-1}}=f_l$$

Now $$\frac{\partial^2 Q}{\partial l \partial k}=\frac{\partial}{\partial l}\left(\frac{\partial Q}{\partial k}\right)=\frac{\partial}{\partial l}\left(ak^{-b-1}.Q^{b+1}\right)$$

$$=ak^{-b-1}.(b+1)Q^b\frac{\partial Q}{\partial l}$$

$$=\frac{ak^{-b-1}\ (b+1)Q^b\ (1-a)l^{-b-1}}{Q^{-b-1}}=f_{lk}$$

$$\sigma=\frac{f_l \mid f_k}{k \mid l}\cdot\frac{d(k \mid l)}{d(f_l \mid f_k)}=\frac{\frac{\partial Q}{\partial k}.\frac{\partial Q}{\partial l}}{Q\ .\ \frac{\partial^2 Q}{\partial l \partial k}}$$

$$=\frac{ak^{-b-1}}{Q^{-b-1}}\times\frac{(1-a)l^{-b-1}}{Q^{-b-1}}\times\frac{Q^{-b-1}}{Qak^{-b-1}(b+1)\ Q^b(1-a)l^{-b-1}}$$

$$=\frac{1}{b+1}.$$

Example 67. *The demand* (D) *of passenger automobiles is given by* $D=0{\cdot}90\ I^{1\cdot1}\ p^{-0\cdot7}$, *where* I *is the income and* p *is the price per car. Find the* (i) *income elasticity of demand and* (ii) *price elasticity of demand.*

Solution. The income elasticity of demand is given by

$$\left|\eta_I\right|=\frac{I}{D}\cdot\frac{\partial D}{\partial I}\cdot$$

$$=\frac{I}{0\cdot90\, I^{1\cdot1}p^{-0\cdot7}}\times 0\cdot90\times 1\cdot1 I^{0\cdot1}p^{-0\cdot7}$$

$$=1\cdot1.$$

Price elasticity of demand is given by

$$\left|\eta_p\right|=\frac{p}{D}\cdot\frac{\partial D}{\partial p}$$

$$=-\frac{p}{0\cdot90 I^{1\cdot1}p^{-0\cdot7}}\times 0\cdot90\times(-0\cdot7)I^{1\cdot1}p^{-1\cdot7}$$

$$=0\cdot7.$$

Example 68. *The demand function for a commodity 'X' is given by*

$$x=300-0\cdot5p_x^2+0\cdot02p_0+0\cdot05y \qquad \ldots(*)$$

where x is the quantity demanded of 'X', p_x the price of X, p_0 the price of a related commodity and y is the constant income. Compute

(*i*) *The price elasticity of demand for X,*

(*ii*) *The income elasticity of demand for X, and*

(*iii*) *Cross elasticity of demand for X, w.r.t. p_0*

when $p_x=12$, $p_0=10$ *and* $y=200$.

Solution. (*i*) Price elasticity of demand for X is given by

$$\left|\eta_{p_x}\right|=\frac{p_x}{x}\cdot\frac{\partial x}{\partial p_x}$$

$$=\frac{p_x}{300-0\cdot5p_x^2+0\cdot02p_0+0\cdot05y}\times\{0\cdot5\times(-2p_x)\}$$

[from ...(*)]

$$=\frac{-p_x^2}{300-0\cdot5p_x^2+0\cdot02p_0+0\cdot05y}$$

When $p_x=12$, $p_0=10$ and $y=200$,

$$\left|\eta_{p_x}\right|=\left|\frac{-144}{300-72+0\cdot2+10}\right|=\frac{144}{238\cdot2}=0\cdot60$$

(*ii*) The income elasticity of demand for X is given by

$$\left|\eta_y\right|=\frac{y}{x}\cdot\frac{\partial x}{\partial y}$$

$$=\frac{0\cdot05y}{300-0\cdot5p_x^2+0\cdot02p_0+0\cdot05y} \qquad \text{[from ...(*)]}$$

When $p_x=12$, $p_0=10$ and $y=200$, as in part (*i*)

$$\left|\eta_y\right|=\frac{10}{238\cdot2}=0\cdot04$$

(*iii*) Cross-elasticity of demand for X w.r.t. p_0 is given by

$$\left|\eta_{p_0}\right|=\frac{p_0}{x}\cdot\frac{\partial x}{\partial p_0}$$

(positive sign is taken since, from (*) we see that p_0 and x change in the same direction).

$$\therefore \qquad \left|\eta_{r_0}\right|=\frac{0\cdot 02\,p_0}{300-0\cdot 5p_x^2+0\cdot 02p_0+0\cdot 05y} \qquad \text{[from (*)]}$$

At $p_x=12$, $p_0=10$ and $y=200$, we get

$$\left|\eta_{r_0}\right|=\frac{0\cdot 02\times 10}{238\cdot 2}=0\cdot 0008$$

Maxima and Minima for Function of two Variables

It is beyond the scope of this book to obtain the general conditions. We shall merely state a set of sufficient conditions, which are applicable to a large number of problems.

For a function $Z=f(x, y)$, if at the point (x_1, y_1)

$$(i) \qquad \frac{\partial z}{\partial x}=0=\frac{\partial z}{\partial y}$$

and

$$(ii) \qquad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2>0$$

then Z is maximum or minimum according as

$\frac{\partial^2 z}{\partial x^2}$ is negative or positive at (x_1, y_1).

Example 69. *The joint cost function for two commodities is*

$$C=x_1^2+2x_1x_2+3x_2^2$$

The prices are 8(for x_1) and 12(for x_2) per unit. Find the maximum profit and the total cost.

Solution. Total revenue $=8x_1+12x_2$

Total cost $=x_1^2+2x_1x_2+3x_2^2$

Total profit : $P=TR-TC$

$$=(8x_1+12x_2)-(x_1^2+2x_1x_2+3x_2^2)$$

$$\frac{\partial P}{\partial x_1}=8-2x_1-2x_2$$

and

$$\frac{\partial P}{\partial x_2}=12-2x_1-6x_2$$

$\therefore$ The condition (*i*) gives

$$8-2x_1-2x_2=0 \quad \text{and} \quad 12-2x_1-6x_2=0$$

Solving these simultaneous linear equations in x_1 and x_2, we get

$$x_1=3 \text{ and } x_2=1$$

$\therefore$ P can have a maximum value at (3, 1).

Now $\frac{\partial^2 P}{\partial x_1^2}=-2$, $\frac{\partial^2 P}{\partial x_2^2}=-6$ and $\frac{\partial^2 P}{\partial x_1\partial x_2}=-2$

$$\therefore\ \left(\frac{\partial^2 P}{\partial x_1^2}\right)\left(\frac{\partial^2 P}{\partial x_2^2}\right)-\left(\frac{\partial^2 P}{\partial x_1 \partial x_2}\right)^2=(-2)(-6)-(-2)^2>0$$

$\therefore$ The condition (*ii*) is satisfied at (3, 1).

Also $\dfrac{\partial^2 P}{\partial x_1^2}=-2$ is negative

$\therefore$ P has a maximum at (3, 1) and the maximum profit

$$=(8\times 3+12\times 1)-(9+6+3)=18$$

EXERCISE (IV)

1. Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ for

(*i*) $u=x^2y^2+x^5+y^6$, (*ii*) $u=x^3y+xy^3$, (*iii*) $u=x^2+y^3+3axy$,

(*iv*) $u=\log(x^2+y^2)^{5/3}$, (*v*) $u=1/\sqrt{2x^2+y^2}$.

[**Ans.** (*i*) $2xy^2+5x^4, 2xy^2+6y^5$, (*ii*) $3x^2y+y^3, x^3+3xy^2$,

(*iii*) $2x+3ay, 3y^2+3ax$, (*iv*) $10x/3(x^2+y^2), 10y/3(x^2+y^2)$,

(*v*) $-2x(2x^2+y^2)^{-3/2}, -y(2x^2+y^2)^{-3/2}$.]

2. Find the first order and second order partial derivatives of the following functions :

(*i*) $u=x^2-5xy+y^2$, (*ii*) $u=x^2e^y$.

[**Ans.** (*i*) $f_x=2x-5y, f_{xx}=2, f_{xy}=-5$,

$f_y=-5x+2y, f_{yy}=2, f_{yx}=-5$.]

3. Find the first order and second order partial derivatives of the following functions :

(*i*) $u=x^2-5xy+y^2$, (*ii*) $u=e^{x^{x^2-y^2}}$ (*iii*) $u=x^2e^y$

(*iii*) $u=e^{x^y}$ (*iv*) $u=x^2e^y$.

4. Verify that $\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial y\partial x}$. when $u=x^y+y^x$

5. If $u=\log[x+\sqrt{x^2+y^2}]$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1$.

6. If $u=ax^2+2bxy+by^2$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u.$$

7. A utility function is given by

$$u=2q_1^2q_2+3q_1q_2^3.$$

Show that the rate of change of marginal utility of commodity q_1 w. r. t. q_2 is equal to the rate of change of marginal utility of q_2 w. r. t. q_1.

8. (*a*) If $u=x^3-3x^2y+3xy^2-y^3$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u.$$

(*b*) If $a^2x^2+b^2y^2=c^2u$, show that

$$b^2\frac{\partial^2 u}{\partial x^2}+a^2\ \frac{\partial^2 u}{\partial y^2}=\frac{a^2b^2}{c^2u}.$$

9. Verify that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\ \frac{\partial^2 u}{\partial x\partial y}+y^2\ \frac{\partial^2 u}{\partial y^2}=2u,\ \text{where } u=x^2+2xy-y^2.$$

10. If $u=x^3+y^3-3axy^2$, verify that

$$x^2\ \frac{\partial^2 u}{\partial x^2}+2xy\ \frac{\partial^2 u}{\partial x\partial y}+y^2\ \frac{\partial^2 u}{\partial y^2}=6u.$$

11. If $z=(ax+by)^{-1}$, find the value of $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$.

12. For the production function $z=ax^{\alpha}\ y^{\beta}$, show that

(*i*) $x\ \frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=(\alpha+\beta)z$, and

(*ii*) $x^2\ \frac{\partial^2 z}{\partial x^2}+2xy\ \frac{\partial^2 z}{\partial x\partial y}+y^2\ \frac{\partial^2 z}{\partial y^2}=(\alpha+\beta)(\alpha+\beta-1)z.$

13. Suppose there is a production function of the type

$$z=e^{(x^2+2xy+3y^2)}$$

where z is the product and x and y are different factors of production, find the marginal products of x and y.

14. If $q=3L^2C^2-2L^2C^2$, where L and C are inputs Labour and Capital. Find Average product and Marginal product of labour (L). If input C be fixed, what is the value of input L for which AP be maximum ? Does the maximum of Marginal Product Curve reach at lower level of labour ?

$$\left[\textbf{Hint.}\quad \frac{\partial q}{\partial L}=\frac{\partial}{\partial L}\ (3L^3C^2-2L^2C^3)=\frac{\partial}{\partial L}\ (3L^3C^2)-\frac{\partial}{\partial L}\ (2L^2C^3)\right.$$

$$=3C^2.3L^2-2C^3.2L=9C^2L^2-4C^3L.$$

$\therefore$ Marginal product of labour $(MP)=9C^2L^2-4C^3L$

Also Average product of Labour $(AP)=\frac{q}{L}=\frac{3L^3C^2-2L^2C^3}{L}$

$$=3L^2C^2-2LC^3.$$

Now for maximum value of AP,

$$MP=AP$$

or $\quad 9C^2L^2-4C^3L=3L^2C^2-2LC^3$

$$6C^2L^2-2C^3L=0,\ 6C^2L^2=2C^3L\quad\Rightarrow\quad L=\frac{C}{3}.$$

Again Marginal product of labour (MP) is maximum when slope of MP curve is zero, *i.e.*, when $\frac{\partial}{\partial L}\ (MP)=0$.

$$\frac{\partial}{\partial \mathrm{L}}(9C^2L^2-4C^3L)=0 \quad \Rightarrow \quad 9C^2\frac{\partial}{\partial L}(L^2)-4C^3\frac{\partial}{\partial L}(L)=0$$

$$\Rightarrow \qquad L=\frac{2C}{9}.\ \ ]$$

15. The demand function is $q=3y+2y^2-6x^2-5x^{-4}$, where $x>0$, $y>0$, q is quantity demanded, y is income, x is the price.

Find what is the slope of the demand curve ? Is the commodity normal or inferior ? Is the reaction of demand to price is independent of the level of income ?

[**Hint.** $\frac{\partial q}{\partial x}=\frac{\partial}{\partial x}(3y+2y^2-6x^2-5x^{-4})=-12x+20x^{-5}$

Slope of Demand curve $=\frac{\partial q}{\partial x}=-12x+20x^{-5}$.

If $f_y=\frac{\partial q}{\partial y}>0$, then the commodity will be normal.

$$f_y=\frac{\partial q}{\partial y}=\frac{\partial}{\partial y}(3y+2y^2-6x^2-5x^{-4})=3+4y.$$

Since y is positive, $f_y>0$

$\therefore$ The commodity is normal commodity and is not inferior.

Now the reaction of demand to price is independent of the level of income if $f_{xy}=f_{yx}=0$.

Now $f_y=\frac{\partial q}{\partial y}=3+4y$ and $f_x=\frac{\partial q}{\partial x}=-12x+20x^{-5}$

Now $f_{xy}=\frac{\partial^2 q}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial q}{\partial y}\right)=\frac{\partial}{\partial x}(3+4y)=0$

and $f_{yx}=\frac{\partial^2 q}{\partial y \partial x}=\frac{\partial}{\partial x}\left(\frac{\partial q}{\partial x}\right)=\frac{\partial}{\partial y}(-12x+20x^{-6})=0.$

$\therefore$ The reaction of demand to price is independent of the level of income.]

16. The following are two linear homogeneous production functions where X, L, K represent output, labour and capital respectively. Show that in each case, L times the marginal product of labour plus K times the marginal product of capital equals total product.

(*i*) $X=A\mathrm{L}^{\alpha}K^{1-\alpha}$, (*ii*) $X=a\mathrm{L}+b\mathrm{K}$.

Find what is the sum of the partial elasticities in each case.

17. If 'a' men are employed in planting 'b' acres with timber, the amount of timber cut after 't' years is $x=f(a, b, t)$. What meaning can be attached to

$$\frac{\partial x}{\partial a},\ \frac{\partial x}{\partial b} \text{ ann } \frac{\partial x}{\partial t}\ ?$$

The production of a particular commodity was estimated as :

$X = L^{0.64} K^{0.36}$, where X is the production of that commodity, L is labour and K is capital.

Determine the marginal productivities for $L = 1{\cdot}5$ and $K = 1{\cdot}1$ units.

18. $Q = 1{\cdot}01\ L^{0.75} K^{0.25}$, where Q=output, L=Labour, K=Capital.

Prove that $L\dfrac{\partial Q}{\partial L} + K\dfrac{\partial Q}{\partial K} = Q.$

19. The following is a linear homogeneous production function where X, L, K represent output, labour, and capital respectively :

$$X = \sqrt{aL^2 + 2hLK + bK^2}.$$

Show that L times the marginal product of labour and K time the marginal product of capital equals total product

20. (*a*) For the linear homogeneous production function :

$$X = AL^{\alpha}\ K^{1-\alpha}$$

where X, L and K denote output, Labour and Capital respectively ; show that the average and marginal products of each factor L and K are functions of the relative amounts of L and K used.

(*b*) If the production function is given by $X = AL^{\beta}\ K^{\beta}$, show that there are increasing, decreasing or constant returns to scale as $\alpha + \beta > 1$, < 1 or $= 1$.

(*c*) For the production function $X = AL^{\alpha} K^{\beta}$ where X, L and K represent Output, Labour and Capital respectively, show that—

(*i*) α and β represent the labour share and capital share of the output respectively.

(*ii*) α and β are also the elasticities of output with respect to labour and capital respectively.

$\left[\textbf{Hint.}\ (ii)\ \text{Calculate}\ \dfrac{K}{X}\cdot\dfrac{\partial X}{\partial K},\ \dfrac{L}{X}\cdot\dfrac{\partial X}{\partial L}\ .\right]$

21. The production function is $P = AK^{\alpha} L^{\beta}$ where $\alpha + \beta < 1$. Show that there are decreasing returns to scale. Deduce that total product is greater than total income distributed between K and L, when income is distributed according to each factor's marginal productivity. What will be the economic interpretation of the residual ?

22. Explain what you mean by production function. State the factors which are generally involved in it. State the mathematical form of Cobb-Douglas production function, interpret its constants and describe the method to fit it to the production data.

For the production function

$$x = A\ a^{\beta}\ b^{1-\beta}$$

show that the average products and the marginal products are functions of the ratio of the factors used.

23. A production function is given as $x=A\ a^{\alpha}\ b^{\beta}$ where $\alpha+\beta>1$, and factor quantities are a and b for labour and capital respectively. Show that there is increasing returns to scale and deduce that the total product is greater than a time the marginal product of labour plus b times the marginal product of capital.

24. For the production function

$$Q=A[\alpha L^{-\beta}+(1-\alpha)\ K^{-\beta}]^{-1/\beta},$$

where $A>0$, $0<\alpha<1$ and $\beta\neq 0$ are constants, find the marginal products of labour (L) and capital (K). Further, if

$$\sigma=\frac{\dfrac{\partial Q}{\partial L}\cdot\dfrac{\partial Q}{\partial K}}{Q\cdot\dfrac{\partial^2 Q}{\partial L\partial K}}$$

is the *elasticity of substitution*, show that $\sigma=\dfrac{1}{1+\beta}$ is a constant.

25. If $U=f(x_1, x_2, \ldots\ldots, x_n)$ is the (*total*) *utility* (*index*) *function* in terms of the amounts $x_1, x_2, \ldots, x_n$ consumed of the n respective goods (commodities) $X_1, X_2, \ldots, X_n$, then the *marginal utility of the goods* X_i, is defined to be $\dfrac{\partial U}{\partial x_i}$, at a point $(x_1, x_2, \ldots, x_n)$.

Find :

(*i*) The marginal utilities with respect to two commodities X_1 and X_2, when $x_1=1$ and $x_2=2$ units of the two commodities are consumed, if the utility (index) function of X_1 and X_2 is given by

$$U=(x_1+3)\ (x_2+5).$$

(*ii*) The ratio of the marginal utility of the good X_1 to the marginal utility of the good X_2, if the utility function of the goods X_1 and X_2 is given by

(*a*) $$U=ax_1+bx_2+c\sqrt{x_1x_2}\ ,$$

(*b*) $$U=\log_e\ (ax_1+bx_2+c\sqrt{x_1x_2})\ ;$$

[**Ans.** (*i*) Marginal utilities :

$$\left(\frac{\partial U}{\partial x_1}\right)_{(1,\ 2)}=7\ ;\ \left(\frac{\partial U}{\partial x_2}\right)_{(1,\ 2)}=4$$

(*ii*) In (*a*) as well as in (*b*) :

$$\left(\frac{\partial U}{\partial x_1}\right)\Big/\left(\frac{\partial U}{\partial x_2}\right)=\frac{2a\ \sqrt{x_1x_2}+cx_2}{2b\ \sqrt{x_1x_2}+cx_1}\ ]$$

26. If $X=f(p_x, p_y, M)$ is a homogeneous demand function of degree zero, where p_x and p_y are prices of two commodities x and y, and M is the money income ; then prove that the sum of the partial elasticities is equal to zero.

27. The supply function for

$$x=f(p_1, p_2, p_3, \dots, p_n)$$

(where $p_1, p_2, p_3, \dots, p_n$ are the prices of several goods) is a homogeneous function of degree n. Prove that the sum of partial elasticities of x must total n. [*Delhi Univ., B.A. (Econ. Hons.) 1990 (N.S.)*]

28. A manufacturer finds that his costs are given by the function $Q=a+8b$. Under the assumption that he keeps his cost fixed at the value of $Q=50$ and that his production function is defined by

$$U=32ab-7a^2-16b^2$$

prove that his maximum production is $U=500$.

29. For the linear homogeneous production function :

$$x=\frac{2Hab-Aa^2-Bb^2}{Ca+Db},$$

where H, A, B, C and D are positive constants, and a and b denote Labour and Capital respectively ; show that the average and marginal products of the factors depend only on the ratio of the factors.

30. If x_1 and p_1 are demand and price of tea and x_2 and p_2 are demand and price of coffee, and the demand functions are given by

$$\left.\begin{array}{l} x_1=p_1^{-1\cdot 3}\, p_2^{0\cdot 5} \\ x_2=p_1^{0\cdot 3}\, p_2^{-0\cdot 5} \end{array}\right\} \qquad \dots(*)$$

show that the two commodities are competitive. Also find four partial elasticities of demand.

[**Hint.** $\dfrac{\partial x_1}{\partial p_2}=0\cdot 5_1^{-1\cdot 8}\, p_2^{-0\cdot 5}>0$ and $\dfrac{\partial x_2}{\partial p_1}=0\cdot 3\, p_1^{-0\cdot 7}\, p_2^{-0\cdot 5}>0$

Since both $\dfrac{\partial x_1}{\partial p_2}>0$ and $\dfrac{\partial x_2}{\partial p_1}>0$, the two commodities, *viz.*, tea and coffee, are competitive.

Partial Elasticities :

$$(i) \quad -\frac{p_1}{x_1}\cdot\frac{\partial x_1}{\partial p_1}=-\frac{p_1}{p^{1\cdot 8}\, p_2^{0\cdot 5}}\times\left[\,(-1\cdot 3)\, p_1^{-2\cdot 3}\, p_2^{0\cdot 5}\,\right]=1\cdot 3$$

$$(ii) \quad +\frac{p_2}{x_1}\cdot\frac{\partial x_1}{\partial p_2}=\frac{p_2}{p_1^{-1\cdot 3}\, p_2^{0\cdot 5}}\times\left[p_1^{-1\cdot 3}\,(0\cdot 5)\, p_2^{-0\cdot 5}\right]=0\cdot 5$$

(Note that here positive sign is taken since from (*), x_1 and p_2 move in the same direction).

$$(iii) \quad +\frac{p_1}{x_2}\cdot\frac{\partial x_2}{\partial p_1}=\frac{p_1}{p_1^{0\cdot 3}\, p^{-0\cdot 5}}\times\left[\,0\cdot 3\, p_1^{-0\cdot 7}\, p_2^{-0\cdot 5}\,\right]=0\cdot 3$$

(Here also note the positive sign)

$$(iv) \quad -\frac{p_2}{x_2}\cdot\frac{\partial x_2}{\partial p_2}=-\frac{p_2}{p_1^{0\cdot 3}\, p_2^{0\cdot 5}}\left[\,p_1^{0\cdot 3}\,(-0\cdot 5)\, p_2^{-1\cdot 5}\,\right]=0\cdot 5\ \Big]$$

31 Determine the partial elasticities and nature of commodities for the demand functions,

$$x_1=p_1^{-1\cdot7}\,p_2^{0\cdot8} \qquad x_2=p_1^{0\cdot5}\,p_2^{-0\cdot2}$$

32. When are two goods X_1 and X_2 said to be (*a*) Competitive, (*b*) Complementary in demand ?

Examine the relation between X_1 and X_2 in the case of the following demand functions :

(*a*)
$$x_1=a_1-a_{11}\,p_1+a_{12}\,p_2$$
$$x_2=a_2+a_{21}\,p_1-a_{22}\,p_2$$

(*b*)
$$x_1=\frac{a_1}{p_1+a_{11}}+a_{12}\,p_2$$
$$x_2=\frac{a_2}{p_2+a_{22}}+a_{21}\,p_1$$

(*c*)
$$x_1=p_1^{-a_{11}}\quad e^{(a_{12}p_2+a_1)}$$
$$x_2=p_2^{-a_{22}}\quad e^{(a_{21}p_1+a_2)}$$

33. The demand functions for two commodities X_1 and X_2, in terms of their respective prices p_1 and p_2, are given by

$$x_1=p_1^{-a_1}\quad e^{b_1p_2+c_1} \qquad \text{and} \qquad x_2=p_2^{-a_2}\quad e^{b_2p_1+c_2}$$

where a_1, a_2, b_1, b_2 and c_1, c_2 are constants.

Find the four partial marginal demand functions and show that :
The 'direct' price-elasticities $\left(viz., \dfrac{\partial x_1}{\partial p_1} \text{ and } \dfrac{\partial x_2}{\partial p_2}\right)$ are independent of the prices ; while the 'cross' price-elasticities $\left(viz., \dfrac{\partial x_1}{\partial p_2} \text{ and } \dfrac{\partial x_2}{\partial p_1}\right)$ are determined in sign, by the constants b_1 and b_2.

34. Show that $x_1=a_1\,\dfrac{p_2}{p_1}$ is an example of a demand law for good x_1 in competition with good x_2 and that $x_1=\dfrac{a_1}{p_1\,p_2}$ is a corresponding law where x_1 and x_2 are complementary.

35. The cost functions for two duopolists producing a common good are respectively by

$$c_1=5x_1 \qquad \text{and} \qquad c_2=5x_2^2.$$

The demand function for the good is given by

$$p=100-0\cdot5x$$

where
$$x=x_1+x_2.$$

Assuming the duopolists take independent decisions regarding outputs (*i.e.*, there is no *conjectural variation*), find their equilibrium outputs and profits.

[**Hint.** The profit functions of duopolists are

$$P_1=px_1-c_1 \qquad \text{and} \qquad P_2=px_2-c_2$$

respectively. Equilibrium outputs are determined by the condition that P_1 and P_2 are maximum.]

[**Ans.** Equilibrium outputs : $x_1=3980/43$ and $x_2=210/43$; equilibrium profits : $P_1 \simeq 462{\cdot}8$ and $P_2 \simeq 130$.]

36. A monopolist firm produces chocolates of two types X_1 and X_2. The constant average cost of X_1 and X_2 are respectively, Rs. 2·50 and Rs. 3·00 per kg. For price of p_1 and p_2, the demands for X_1 and X_2 are respectively, given by

$$x_1=5(p_2-p_1) \quad \text{and} \quad x_2=32+5p_1-10p_2.$$

Find the levels at which prices will be fixed for X_1 and X_2 for maximum joint monopoly profit.

Also find the prices of X_1 and X_2 fixed by two independent monopolists.

[**Ans.** For a single monopolist, levels of prices Rs. $p_1 \simeq 4{\cdot}45$ and $p_2 \simeq 4{\cdot}70$; for two independent monopolists, levels of prices is Rs. $p_1 \approx 3{\cdot}2$ and $p_2 \approx 3{\cdot}9$.]

37. A monopolist produces amounts x_1 and x_2 of two goods X_1 and X_2 at a total cost $\pi=x_1^2+2x_1x_2+3x_2^2$. The demands for the two goods in the market are

$$p_1=36-3x_1 \quad \text{and} \quad p_2=40-5x_2$$

where p_1 and p_2 are the prices charged. Determine the quantities and prices which maximise the profit. Find also the value of the maximum profit.

38. The demand for a good x is represented by the demand relation $p=\psi(x)$. The production of the good is shared between two duopolist firms selling at the same price p. The first duopolist produces an output x_1 at a total cost of $\pi_1=F_1(x_1)$ and the second doupolist produces an output x_2 at a total cost ot $\pi_2=F_2(x_2)$. Find the equations which determine the output of the two duopolists. (Assume zero conjectural variation.)

APPLICATIONS OF INTEGRATION

To find the cost function when marginal cost is given :

We know that if the total cost function, say C, is given then the marginal cost function is the first derivative of the total cost function. It follows, therefore, that the total cost function is the integral of the marginal cost function.

If C represent the total cost of producing an output x, then the marginal cost is given by

$$MC=\frac{dC}{dx}$$

$$\therefore \qquad C=\int (MC)dx+k$$

The constant of integration k can be evaluated if the fixed cost (*i.e.*, the cost when $x=0$) is given.

Further the average cost AC can be obtained from the relation :

$$AC=\frac{C}{x}\cdot$$

Example 70. *The marginal cost function of a product is given by*

$$\frac{dC}{dq}=100-10q+0{\cdot}1\ q^2,$$

where q is the output. Obtain the total and the average cost function of the firm under the assumption that its fixed cost is Rs. 500.

Solution. $$\frac{dC}{dq}=100-10q+0{\cdot}1\ q^2=MC$$

Integrating both sides w.r.t. q, we have

$$C=\int(100-10q+0{\cdot}1q^2)\ dq$$

$$=100-10.\frac{q^2}{2}+0{\cdot}1\ .\ \frac{q^3}{3}+k \qquad \ldots(*)$$

Now the fixed cost is 500, *i.e.*, when $q=0$, $C=500$.

$\therefore$ $$k=500.$$

Hence total cost function is

$$C=100q-5q^2+\frac{q^3}{30}+500$$

Average cost function is

$$AC=\frac{C}{q}=100-5q+\frac{q^2}{30}+\frac{500}{q}\cdot$$

Example 71. *The marginal cost function of manufacturing x shoes is $6+10x-6x^2$. The total cost of producing a pair of shoes is Rs. 12. Find the total and average cost function.*

Solution. $$MC=6+10x-6x^2=\frac{d}{dx}(C)$$

$\therefore$ $$C=\int(6+10x-6x^2)dx$$

$$=6x+10\ .\ \frac{x^2}{2}-6\ .\ \frac{x^3}{3}+k,$$

where k is the constant of integration.

Now $$C=12,\text{ when } x=2.$$

$\therefore$ $$12=6(2)+10\times\ \frac{(2)^2}{2}-6\times\ \frac{(2)^3}{3}+k$$

$\Rightarrow$ $$k=12-12-20+16=-4$$

$\therefore$ The total cost function is

$$C=6x+5x^2-2x^3-4$$

Further the average cost function AC is given by

$$AC=\frac{C}{x}=6+5x-2x^2-\frac{4}{x}\cdot$$

Example 72. *The marginal cost function of a firm is given by*

$$MC=3000\ e^{0\cdot 3x}+50,$$

when x is quantity produced. If fixed cost is Rs. 80,000, find the total cost function of the firm. *[Delhi Univ., B. Com. (Hons) 1990]*

Solution. The total cost function of the firm is given by

$$TC=\int (MC)dx+k, \text{ where } k \text{ is constant of integration.}$$

$$\therefore \qquad TC=\int (3000e^{0\cdot 3x}+50)dx+k$$

$$=3000\ .\ \frac{e^{0\cdot 3x}}{0\cdot 3}+50x+k$$

$$=10000e^{0\cdot 3x}+50x+k$$

When $x=0$, $TC=80000$, therefore, we have

$$80000=10000+k$$

$$k=7000$$

$$\therefore \qquad TC=10000e^{0\cdot 3x}+50x+70000.$$

Example 73. *Assume that the marginal cost in lakhs of rupees is given by*

$$MC=4+5x^2+\frac{3}{2}\ e^{-x},$$

where x is the quantity produced. Find the total cost of production when $x=2$, if fixed cost is Rs. 6 lakhs. *[Delhi Univ., B. Com. (Hons.), 1992]*

Solution. We have

$$MC=4+5x^2+\frac{3}{2}\ e^{-x}$$

$$\therefore \qquad TC=\int MC\ dx=\int\left(4+5x^2+\frac{3}{2}\ e^{-x}\right)dx$$

$$=4x+\frac{5x^2}{3}-\frac{3}{2}\ e^{-x}+k,$$

where k is constant of integration.

We are given that when $x=0$, $TC=6$

$$\therefore \qquad 6=-\frac{3}{2}+k \quad \Rightarrow \quad k=\frac{15}{2}$$

$$\therefore \qquad TC=4x+\frac{5x^3}{3}-\frac{3}{2}e^{-x}+\frac{15}{2}$$

$$\therefore \quad TC(\text{at } x=2)=4\times 2+\frac{5\times 8}{3}-\frac{3}{2}e^{-2}+\frac{15}{2}$$

$$=8+\frac{40}{3}-\frac{3}{2}\times 0{\cdot}1353+\frac{15}{2}$$

$$=\text{Rs. } 28\ 63 \text{ lakhs.}$$

[Let $y=e^{-2}$, $\therefore$ $\log y=-2\log e=-2\times 0{\cdot}4343$

$$=1{\cdot}1314$$

$$\therefore \quad y=\text{anti log }(\bar{1}.1314)=0{\cdot}1353]$$

To find the total revenue function and the demand function when the marginal revenue function is given.

If R is the total revenue when the output is x, then the marginal revenue MR is given by

$$MR=\frac{dR}{dx}$$

Hence if the marginal revenue MR is given, then the total revenue R is the indefinite integral of MR with respect to x, *i.e.*,

$$R=\int (MR)\,dx+k$$

The constant of integration k can be evaluated from the fact that the total revenue R is zero when the output x is zero.

Further, since $R=px$, the demand function can be easily obtained as

$$p=\frac{R}{x}\cdot$$

Example 74. *If the marginal revenue function for output is given by* $R_m=\dfrac{6}{(x+2)^2}+5$, *find the total revenue function by integration. Also deduce the demand function.*

Solution. Total revenue function is given by

$$R=\int R_m\,dx=\int\left\{\frac{6}{(x+2)^2}+5\right\}dx$$

$$=\int\frac{6}{(x+2)^2}\,dx+5\int dx$$

$$=-\frac{6}{(x+2)}+5x+k$$

Since total revenue is zero at $x=0$, we get

$$0=-\frac{6}{2}+k \quad \Rightarrow \quad k=3$$

$$\therefore \quad R=3-\frac{6}{x+2}+5x$$

Also we know $R=p\times x$

$$\therefore\ p=\frac{R}{x}=\frac{3-\dfrac{6}{x+2}+5x}{x}$$

$$=\frac{3}{x}-\frac{6}{x(x+2)}+5$$

$$=\frac{3x+6-6}{x(x+2)}+5$$

$$=\frac{3}{x+2}+5$$

Hence $\therefore\ p=\frac{3}{x+2}+5$ is the required demand function.

Example 75. *If the marginal revenue function is*

$$MR=\frac{ab}{(x-b)^2}-c,$$

show that $$p=\frac{a}{b-x}-c$$

is the demand law.

Solution. $$MR=\frac{ab}{(x-b)^2}-c=\frac{d}{dx}(R)$$

Integrating both sides w.r.t. x, we have

$$R=\int\left\{\frac{ab}{(x-b)^2}-c\right\}dx+k,$$

where k is a constant of integration.

$$=ab\int(x-b)^{-2}\,dx-c\int dx+k$$

$$=ab\,\frac{(x-b)^{-2+1}}{-2+1}-cx+k=-\frac{ab}{x-b}-cx+k\quad\ldots(*)$$

Now when $x=0$, total revenue $=0$.

$$\therefore\quad 0=-\frac{ab}{0-b}-0+k=a+k$$

$$\Rightarrow\quad k=-a.$$

Hence the total revenue function is given by

$$TR=\frac{-ab}{x-b}-cx-a=px$$

$$\Rightarrow\quad p=\frac{-ab}{x(x-b)}-c-\frac{a}{x}$$

$$=\frac{-ab-ax+ab}{x(x-b)}-c=\frac{-ax}{x(x-b)}-c$$

$$=\frac{-a}{x-b}-c=\frac{a}{b-x}-c$$

is the required demand law.

To find the consumption function when the marginal propensity to consume (MPC) is given.

If P is the consumption when the disposable income of a person is x, the marginal propensity to consume (MPC) is given by

$$MPC=\frac{dP}{dx}$$

Hence if MPC is given, the consumption P is given by the indefinite integral of MPC with respect to x, *i.e.*,

$$P=\int (MPC)\, dx+k$$

The constant of integration k, can be evaluated if the value of P is known for some x.

Example 76. *If the marginal propensity to save* (MPS) *is* $1{\cdot}5+0{\cdot}2x^{-2}$, *when* x *is the income. Find the consumption function, given that the consumption is* 4·8 *when income is ten.*

Solution. Now "derivative of consumption function w.r.t. output represents marginal propensity to consume".

$$\therefore \qquad MPS=1{\cdot}5+0{\cdot}2x^{-2}=\frac{dP}{dx}$$

$$\therefore \qquad P=\int (1{\cdot}5+0{\cdot}2x^{-2})\, dx=1{\cdot}5x+0{\cdot}2\left(\frac{x^{-2+1}}{-2+1}\right)+k$$

$$=1\ 5x-\frac{0\ 2}{x}+k$$

Now $P=4{\cdot}8$ when $x=10$

$$\therefore \qquad 4{\cdot}8=1{\cdot}5\times 10-\frac{0{\cdot}2}{10}+k$$

$$\Rightarrow \qquad k=-10{\cdot}18$$

Hence the consumption function is

$$P=1{\cdot}5x-\frac{0{\cdot}2}{x}-10{\cdot}18$$

Maximum Profits

Suppose we are required to find the maximum profits of a firm when only the marginal cost and the marginal revenue functions are given. Then our problem is, how to compute maximum profits ? By equating marginal cost to marginal revenue, we can find the output that maximises total profits. To calculate total profits at this output, we have

$$\frac{dP}{dx}=\frac{dR}{dx}-\frac{dC}{dx}$$

where P, R, C, x represents the total profit, total revenue, total cost and output respectively.

Integrating, we have

$$P=\int \frac{dR}{dx}\ dx-\int \frac{dC}{dx}\ .\ dx+k=R-C+k$$

where the constant of integration, k, can be found from the additional information given.

Remark. It may be noted that profit is maximised when marginal revenue equals marginal cost, given the assumption of pure competition Total profit is the integral of marginal revenue minus marginal cost from zero quantity to quantity for which profit is maximised.

Example 77. *The marginal cost of production of a firm is given as*

$$C'(q)=5+0{\cdot}13q$$

Further, the marginal revenue is

$$R'(q)=18$$

Also it is given that $C(0)=\text{Rs. } 120$. *Compute the total profits.*

Solution. Since profit is maximum, where,

marginal cost=marginal revenue

i.e., $C'(q)=R'(q)$

$\Rightarrow$ $5+0{\cdot}13=18$

$\Rightarrow$ $q=\dfrac{13}{0{\cdot}13}=100.$

Also $\dfrac{dP}{dq}=\int R'(q)\ dq-\int C'(q)\ dq$

Now $\int R'(q)=\int 18\ dq=18q+k_1,$

where k_1 is an arbitrary constant.

Put $k_1=0$, as under pure competition, total revenue = output × price.

$\therefore$ $R(q)=\int R'(q)dq=18q.$

Also $\int C'(q)dq=\int(5+0{\cdot}13q)\ dq$

$\Rightarrow$ $C(q)=5q+0{\cdot}13\ \dfrac{q^2}{2}+k_2,$

where k_2 is an arbitrary constant.

From the additional information $C(0)=120$, we have

$$C(0)=5(0)+0{\cdot}13(0)+k_2=120$$

$\Rightarrow$ $k_2=120,\quad \therefore\quad C(q)=5q+0\ 065q^2+120.$

Now $P(q)=R(q)-C(q)$

$$=18q-5q-0{\cdot}065q^2-120$$

$$=13q-0{\cdot}065q^2-120$$

$\therefore$ Total profit, when $q=100$ is

$$P(100)=13-100-0{\cdot}065(100)^2-120$$
$$=1300-650-120=\text{Rs. }530.$$

Example 78. *The ABC Co. Ltd. has approximated the marginal revenue function for one of its products by $MR=20x-2x^2$ The marginal cost function is approximated by $MC=81-16x+x^2$.*

Determine the profit-maximizing output and the total profit at the optimal output.

Solution. Solving for profit-maximizing output, set MR equal to MC, *i.e.*,

$$MR=MC$$
$$\Rightarrow \quad 20x-2x^2=81-16x+x^2$$
$$\Rightarrow \quad -81+36x-3x^2=0$$
$$\Rightarrow \quad x^2-12x+27=0$$
$$\Rightarrow \quad (x-3)(x-9)=0$$
$$\Rightarrow \quad x=3,\ 9.$$

The second derivative of $MR-MC$ is the second derivative of total profit. The sign of $P''(x)$ indicates whether x is a relative maximum or relative minimum.

$$\frac{d(MR-MC)}{dx}=36-6x$$
$$P''(3)=36-6(3)=+18.$$
$$P''(9)=36-6(9)=-18.$$

Therefore, at $x=9$, profit is maximum.

$$\text{Total profit}=\int_0^9(-81+36x-3x^2)\,dx=\Big|(-81x+18x^2-x^3)\Big|_0^9$$
$$=[-81(9)+18(9)^2]=0$$

which indicates no profit. A negative sign would signify a loss.

Example 79. *XYZ Co. Ltd. suffers a loss of Rs. 121·50 if one of its special product does not sell. Marginal revenue is approximated by $MR=30-6x$ and marginal cost by $MC=-24+3x$.*

Determine the total profit function, the break-even points, and the total profit between break-even points.

Solution. Solving for total profit, first determine marginal profit.

$$MP=MR-MC$$
$$=(30-6x)-(-24+3x)$$
$$=54-9x$$

Total profit function

$$=\int MP\,dx$$

$$=\int(54x-9x)\,dx$$

$$=54x-\frac{9x^2}{2}+k$$

Since a loss of Rs. 121·50 occurs when there are no sales, k must equal $-121{\cdot}50$. Consequently, total profit function equals

$$P(x)=-121{\cdot}50+54x-\frac{9}{2}x^2.$$

Solving for break-even points, set $P(x)=0$

$$0=-121{\cdot}50+54x-\frac{9}{2}x^2$$

$$\Rightarrow \quad (x-3)(x-9)=0$$

$$\Rightarrow \quad x=3,\ 9.$$

Integrating the profit function between break-even points will give total profit between break-even points.

$$TP=\int_3^9\left(-121{\cdot}50+54x-\frac{9}{2}x^2\right)dx$$

$$=\left[\left(-121{\cdot}50x+54\,.\,\frac{x^2}{2}-\frac{9}{6}x^3\right)\right|_3^9$$

$$=\left[-121{\cdot}50(9)+54\,\frac{(9)^2}{2}-\frac{3}{9}(9)^3\right]$$

$$-\left[-121{\cdot}50(3)+54\,\frac{(3)^2}{2}-\frac{3}{2}(3)^3\right]$$

$$=\text{Rs. } 4536.$$

Example 80. *The price elasticity of a demand curve $x=f(p)$ is of the form $(a-bp)$ where a and b are given constants. Find the demand curve.*

Solution. We are given

$$\eta_p=-\frac{p}{x}\cdot\frac{dx}{dp}=a-bp.$$

$$\Rightarrow \quad \left(\frac{a-bp}{p}\right)dp+\frac{dx}{x}=0$$

$$\Rightarrow \quad \left(\frac{a}{p}-b\right)dp+\frac{dx}{x}=0$$

Integrating, we get

$$(a\log p-bp)+\log x=\log c$$

$$\Rightarrow \quad \log\left(p^a e^{-bp}\right)+\log x=\log c$$

$$\Rightarrow \quad xp^a e^{-bp}=c$$

$$\Rightarrow \quad x=cp^{-a}e^{bp},$$

where log c is the constant of integration.

Example 81. *Derive the demand function which has the unit price elasticity of demand throughout.* [*Delhi Univ., B. Com.* (*Hons.*) *1991*]

Solution. Since the elasticity of demand is unity throughout, we have

$$-\frac{p}{x}\cdot\frac{dx}{dp}=1$$

or $$\frac{dx}{x}=-\frac{dp}{p}$$

Integrating both sides, we have

$$\int\frac{dx}{x}=-\int\frac{dp}{p}+k,$$

where k is the constant of integration.

or $$\log x=-\log p+k$$

or $$\log x+\log p=k$$

or $$px=e^k=c$$

$\therefore$ $px=c$ is the required demand function.

Consumer's Surplus

Suppose the price p a consumer is willing to pay for a quantity x of a particular commodity is governed by the demand curve

$$p=D(x).$$

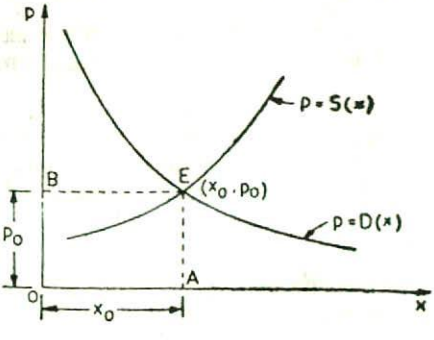
In general, the function $D(x)$ is a decreasing function, since, as the price of a commodity increases, the quantity the consumer is willing to buy declines.

Further, suppose the price p that a producer is willing to charge for a quantity x of a particular commodity is governed by the supply curve

$$p=S(x).$$

In general, the function $S(x)$ is an increasing function since, as the price p of a commodity increases, the more the producer is willing to supply the commodity.

The point of intersection of the demand curve and the supply curve is called the equilibrium point E.



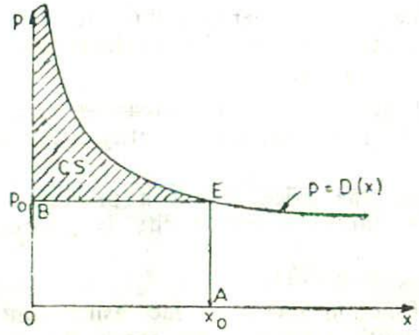
If the coordinates of E are (x_0, p_0) then p_0, the market price, is the price a consumer is willing to pay for and a producer is willing to sell for a quantity x_0, the demand level, of the commodity. The total revenue of the producer at a market price p_0; and a demand level x_0 is $p_0 x_0$ (the price per unit times the number of units) which can be interpreted geometrically as the area of rectangle $OAEB$.

In a free market economy, there are times when some consumers would be willing to pay more for a commodity than the market price p_0 that they actually do pay. The benefit of this to consumers, *i.e.*, the difference between what consumers actually paid and what they were willing to pay, is called *consumer's surplus* (CS). Thus

$$CS = \{\text{Total area under the demand curve } D(x) \text{ from } x=0 \text{ to } x=x_0\}$$

$$-\{\text{the area of the rectangle } OAEB\}$$

$$= \int_0^{x_0} D(x)\, dx - x_0 \times p_0$$



In other words, consumer's surplus is the amount which a consumer is willing to pay for a commodity rather than go without it, minus what he would have to pay actually for it at the market price.

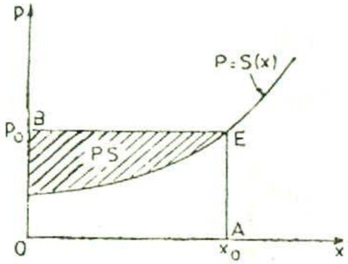
Remarks. 1. Under pure competition, the price p_0 is determined by equating the demand and supply functions, and from this relation the demand x_0 is calculated.

2. Under monopoly, the price p_0 is determined by equating MR and MC functions. From this price value p_0, we obtain the corresponding value of x_0 and then the consumer's surplus is calculated in the usual way.

Producer's Surplus

In a free market economy, there are also times when some producers would be willing to sell at a price below the market price p_0

that the consumer actually pays. The benefit of this to the producer, *i.e.*, the difference between the revenue producers actually receive and what they have been willing to receive, is known as *producer's surplus* (PS).



PS={Area of the rectangle $OAEB$}−{Area below the supply function from 0 to x_0}

$$=x_0 \times p_0-\int_0^{x_0} S(x)\,dx$$

Example 82. *The demand law for a commodity is*

$$p=20-D-D^2$$

Find the consumer's surplus when the demand is 3.

Solution. Here $p=f(D)=20-D-D^2$

Also when the demand $D_0=3$, the price

$$p_0=20-(3)-(3)^2=8$$

$$\therefore \text{ Consumer's surplus}=\int_0^{D_1} f(D)\,dD-p_0 D_0$$

$$=\int_0^3 (20-D-D^2)\,dD-(8\times 3)$$

$$=\left[20\,D-\frac{D^2}{2}-\frac{D^3}{3}\right]_0^3-24$$

$$=\left[20\times 3-\frac{(3)^2}{2}-\frac{(3)^3}{3}\right]-24=\frac{45}{2}$$

Example 83. *If the supply curve is $p=\sqrt{10+x}$ and the quantity sold in market is 6 units, find the producer's surplus.*

Solution. Now $x_0=6 \Rightarrow p_0=\sqrt{10+6}=\pm 4$

$\therefore$ $x_0=6$ and $p_0=4$ ($\because$ $p_0=-4$ is meaningless).

Hence producers' surplus

$$=6\times 4-\int_0^6 \sqrt{10+x}\,dx$$

$$=24-\left|\frac{(10+x)^{3/2}}{3/2}\right|_0^6$$

$$=24-\frac{2}{3}\left[(16)^{3/2}-(10)^{3/2}\right]=2{\cdot}42$$

Example 84. *Determine consumer surplus and producer surplus under pure competition for the demand function $p=36-x^2$ and supply fuuction $p=6+\frac{x^2}{4}$, where p is the price and x is quantity.*

[Delhi Univ. B. Com. (Hons.), 1991]

Solution. Under pure competition, market equilibrium conditions can be obtained by equating the demand and supply.

$$\therefore \quad 36-x^2=6+\frac{x^2}{4} \quad \text{or} \quad \frac{5x^2}{4}=30$$

or $$x^2=\frac{30\times 4}{5}=24 \quad \Rightarrow \quad x=2\sqrt{6}=x_0$$

$$\therefore \quad p_0=36-24=12$$

Consumer's Surplus $$=\int_0^{x_0} D(x)\,dx-p_0x_0$$

$$=\int_0^{2\sqrt{6}} (36-x^2)\,dx-2\sqrt{6}\times 12$$

$$=\left|36x-\frac{x^3}{3}\right|_0^{2\sqrt{6}}-24\sqrt{6}$$

$$=72\sqrt{6}-16\sqrt{6}-24\sqrt{6}=32\sqrt{6}$$

Producer's Surplus $$=p_0x_0-\int_0^{x_0} S(x)\,dx$$

$$=2\sqrt{6}\times 12-\int_0^{2\sqrt{6}}\left(6+\frac{x^2}{4}\right)dx$$

$$=24\sqrt{6}-\left|6x+\frac{x^3}{12}\right|_0^{2\sqrt{6}}$$

$$=24\sqrt{6}-12\sqrt{6}-4\sqrt{6}=8\sqrt{6}$$

Example 85. *Find the consumer surplus and producer surplus under pure competition for demand function* $p=\frac{8}{x+1}-2$ *and supply function* $p=\frac{1}{2}(x+3)$, *where p is price and x is quantity.*

[*Delhi Univ. B. Com.* (*Hons*)., *1992*]

Solution. Under pure competition, market equilibrium conditions can be obtained by equating the demand and supply.

$\therefore$ $$\frac{8}{x+1}-2=\frac{1}{2}(x+3)$$

or $$16-4(x+1)=(x+3)(x+1)$$

or $$16-4x-4=x^2+4x+3$$

or $$x^2+8x-9=0$$

or $$(x+9)(x-1)=0$$

$\therefore$ $$x=1 \text{ or } x=-9$$

$x=-9$ is inadmissible as quantity cannot be negative.

$\therefore$ $$x=1$$

When $x=1$, $$p=\frac{1}{2}(x+3)=\frac{1}{2}(1+3)=2.$$

Consumer surplus $$=\int_0^{x_0} D(x)\,dx-p_0x_0$$

$$=\int_0^1\left(\frac{8}{x+1}-2\right)dx-1\times 2$$

$$=\Big[8\log(x+1)-2x\Big]_0^1-2$$

$$=8 \log 2-2-2=8 \log 2-4.$$

Producer surplus $=p_0x_0-\int_0^{x_0} S(x)\ dx$

$$=1\times 2-\int_0^1 \frac{1}{2}\ (x+3)\ dx$$

$$=2-\left[\frac{1}{2}\left(\frac{x^2}{2}+3x\right)\right]_0^1$$

$$=2-\frac{1}{2}\left(\frac{1}{2}+3\right)$$

$$=2-\frac{7}{4}=\frac{1}{4}$$

Example 86. *The demand and supply function under perfect competition are $y=16-x^2$ and $y=2x^2+4$ respectively. Find the market price, consumer's surplus and producer's surplus.*

Solution. Demand function : $y=16-x^2$ ...(*)

Supply function : $y=2x^2+4$ .. (**)

Subtracting (1) from (2), we have

$$0=12-3x^2$$

$\Rightarrow$ $x=2=x_0$

When $x=2$,

$$y=16-(2)^2=12=y_0$$

Thus when the quantity demanded or supplied is 2 units, the price is 12 units.

Consumers' surplus

$$=\int_0^2 (16-x^2)\ dx-2\times 12$$

$$=\left[16x-\frac{x^3}{3}\right]_0^2-24$$

$$=32-\frac{8}{3}-24=\frac{16}{3}=5{\cdot}33$$

Producers' surplus

$$=2\times 12-\int_0^2 (2x^2+4)\,dx$$

$$=24-\left[\frac{2x^3}{3}+4x\right]_0^2=24-\left[\frac{16}{3}+8\right]=\frac{32}{3}$$

$$=10{\cdot}67$$

Example 87. *Demand and supply functions are $D(x)=(12-2x)^2$ and $S(x)=56+4x$ respectively. Determine CS under monopoly (so as to maximise the profit) and the supply function is identified with the marginal cost function.*

Solution. Total revenue $=TR=x\times D(x)$

$$=(144-48x+4x^2)x$$

$$=144x-48x^2+4x^3$$

$$\therefore \quad MR=144-96x+12x^2$$

Since the supply price is identified with MC, we have

$$MC=56+4x$$

In order to find CS under monopoly, *i.e.*, to maximise profit, we must have

$$MR=MC$$

$$\Rightarrow \quad 144-96x+12x^2=56+4x$$

$$\Rightarrow \quad 12x^2-100x+88=0$$

$$\Rightarrow \quad 3x^2-25x+22=0$$

$$\Rightarrow \quad x=1=x_0 \text{ or } x=\frac{22}{3}=x_0$$

When $x_0=1$, $D(x_0)=p_0=(12-2)^2=100$

$$\therefore \quad CS=\int_0^1 (144-48x+4x^2)\,dx-1\times 100$$

$$=\left[144x-48\cdot\frac{x^2}{2}+4\cdot\frac{x^3}{3}\right]_0^1-100$$

$$=144-24+\frac{4}{3}-100=\frac{64}{3} \text{ units.}$$

Again when $x_0=\frac{22}{3}$; $p_0=\left(12-\frac{44}{3}\right)^2=\frac{64}{9}$

$$\therefore \quad CS = \int_0^{22/3} (144 - 48x + 4x^2)\, dx - \frac{22}{3} \times \frac{64}{9}$$

$$= \left| 144x - 48 \cdot \frac{x^2}{2} + 4 \cdot \frac{x^3}{3} \right|_0^{22/3} - \frac{22}{3} \times \frac{64}{9} = \frac{19360}{81} \text{ units.}$$

Example 88. *When the price of pocket calculators averaged Rs. 400, ABC Co. Ltd. sold 20 every month. When the price dropped to an average of Rs. 100, 120 were sold every month by the same company. When the price was Rs. 400, 200 calculators were available per week for sale. When the price reached Rs. 100, only 50 remained. Determine consumers' and producers' surplus.*

Solution. The demand and supply functions are obtained as follows :

$$D\,(q): \quad \frac{D(q) - 400}{q - 20} = \frac{100 - 400}{120 - 20}$$

$$\Rightarrow \quad D(q) = 460 - 3q$$

$$S\,(q): \quad \frac{S(q) - 400}{q - 200} = \frac{100 \quad 400}{50 - 200}$$

$$\Rightarrow \quad S(q) = 2q$$

At equilibrium, $D(q) = S(q)$

$$\Rightarrow \quad 460 - 3q = 2q$$

$$\Rightarrow \quad q = 92 = q_0$$

With $\quad q_0 = 92\ ;\ p_0 = 184$

$$C.S. = \int_0^{92} (460 - 3q)\, dq - (92 \times 184)$$

$$= \left| 460q - \frac{3q^2}{2} \right|_0^{92} - 16928$$

$$= 460 \times 92 - \frac{3}{2} \times (92)^2 - 16928 = 12696$$

$$P.S. = 92 \times 184 - \int_0^{92} 2q\, dq$$

$$= 16928 - \Big[q^2 \Big]_0^{92} = 8464$$

Example 89. *Let p be the price of rice, q the quantity of rice, and S, the amount of fertiliser used in rice production. Using data for India for 1949—1964 (Tintner and Patel), we find for the per capita demand function for rice* $p=0{\cdot}964-6{\cdot}773q$

and for the supply function

$$q=0{\cdot}063+0{\cdot}036\ S$$

(*i*) *Find the equilibrium in the rice market if* $S=0{\cdot}5$

(*ii*) *Find the consumer's surplus.*

Solution. The demand function for rice is

$$p=0{\cdot}964-6{\cdot}773q \qquad \ldots(*)$$

The supply function is

$$q=0{\cdot}063+0{\cdot}036\ S \qquad \ldots(**)$$

For equilibrium, quantity demanded = quantity supplied.

∴ From the two equations, we have (on eliminating q)

$$p=0{\cdot}964-6{\cdot}773\ (0{\cdot}063+0{\cdot}036\ S)$$

∴ For $S=0{\cdot}5$, $p=0{\cdot}964-6{\cdot}773\ (0{\cdot}063+0{\cdot}036\times{\cdot}5)$

$$=0{\cdot}964-6{\cdot}773\ (0{\cdot}063+0{\cdot}018)=0{\cdot}415=p_0$$

and $q=0{\cdot}063+0{\cdot}036\times0{\cdot}5=0{\cdot}063+0{\cdot}018=0{\cdot}081=q_0$

are the equilibrium price and quantity exchanged.

(*b*) The required consumer's surplus $=\displaystyle\int_0^{0{\cdot}081} p\ dq-p_0\ q_0$

$$=\int_0^{0{\cdot}081}(0{\cdot}964-6{\cdot}773\ q)\ dq-0{\cdot}415\times0{\cdot}081$$

$$=\left[0{\cdot}964\ q-\frac{6{\cdot}773q^2}{2}\right]_0^{0{\cdot}081}-0{\cdot}033615$$

$$=0{\cdot}964\times0{\cdot}081-\frac{6{\cdot}773}{2}\ ({\cdot}081)^2-0{\cdot}033615=0{\cdot}02225$$

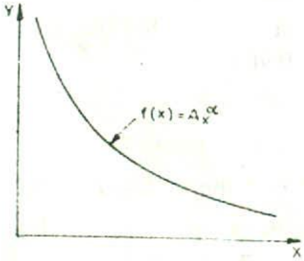
The Learning Curve

In certain industrial operations such as assembling of television sets, cars, home appliances, operating printing presses, workers learn from experience so that the direct labour input per unit of product steadily declines. The rate of reduction in direct labour requirements is described by a curve called *Learning curve*. The general form of the function is usually taken as :

$$f(x)=Ax^{\alpha}$$

where $f(x)$ is the number of hours of direct labour required to produce the xth unit, $-1 \leqslant \alpha < 0$ and $A > 0$. The choice of x^{α}, with $-1 \leqslant \alpha < 0$, guarantees that, as the number of x units produced increases, the direct labour input decreases.

The learning curve can be used as a predictor to determine the number of production hours for future work, once it has been determined for a gross production process. From a given learning curve the total number of labour hours required to produce units numbered 'a' through 'b' is



$$N = \int_a^b f(x)\,dx = \int_a^b Ax^{\alpha}\ dx$$

Example 90. *ABC Co. Ltd. manufactures air-conditioners on an assembly line. From experience it was determined the first 100 air conditioners required 1400 labour hours. For each subsequent 100 air conditioners (1 unit), less labour hours were required according to the learning curve*

$$f(x) = 1400\ x^{-0\cdot3}$$

where $f(x)$ is the rate of labour hours required to assemble the xth unit (each unit being 100 air-conditioners). This curve was determined after 100 units had been manufactured. If the company is in the process of bidding for a large contract involving 20,000 additional air-conditioners or 200 additional units, find the man power required to complete the job.

Solution. The labour hours required to assemble the additional 200 units can be estimated by evaluating

$$N = \int_{100}^{300} f(x)\,dx = \int_{100}^{300} 1400\ x^{-0\cdot3}\,dx$$

$$= \left| \frac{1400\ x^{0\cdot7}}{0\cdot7} \right|_{100}^{300}$$

$$=2000\left[\ (300)^{0{\cdot}7}-(100)^{0{\cdot}7}\ \right]$$

$$=2000\left[\ y-z\ \right],\text{ say} \qquad \ldots(*)$$

Let $y=(300)^{0{\cdot}7}$

$$\log y=0{\cdot}7\log 300=0{\cdot}7\times 2{\cdot}4771=1{\cdot}73397$$
$$y=\text{Antilog }(1{\cdot}73397)=54{\cdot}20$$

Also let $z=(100)^{0{\cdot}7}$

$$\log\ z=0{\cdot}7(\log 100)=0{\cdot}7\times 2=1{\cdot}4$$
$$z=\text{Antilog }1{\cdot}4=25{\cdot}12$$

Substituting the values in (*), we have

$$N=2000(54{\cdot}20-25{\cdot}12)=58{,}160$$

Hence the company can bid estimating the total labour hours needed as 58,1 60.

Example 91. *After producing 35 units, the production manager of a company determines that its production facility is following a learning curve of the form*

$$f(x)=1000\ x^{-0{\cdot}5}$$

where $f(x)$ is the rate of labour hours required to assemble the xth unit. How many total labour hours should they estimate are required to produce an additional 25 units.

Solution.
$$N=\int_{35}^{60} 1000\ x^{-0{\cdot}5}\ dx$$

$$=2000\ \Big|\ x^{1/2}\ \Big|_{35}^{60}=2000\left(\ 60^{1/2}-35^{1/2}\ \right)$$

$$=2000\ (7{\cdot}746-5{\cdot}916)=3660\text{ hours}$$

Rate of Sales

When the rate of sales of a product is a known function of x, say $f(x)$ where x is a time measure, the total sales of this product over a time period T is

$$\int_0^T f(x)\ dx$$

Example 92. *Suppose the rate of sales of a new product is given by*

$$f(x)=200-90\ e^{-x}$$

where x is the number of days the product is on the market. Find the total sales during the first 4 days.

Solution. The total sales $=\int_0^4 f(x)\,dx$

$$=\int_0^4 (200-90\,e^{-x})\,dx=\Big|200x+90\,e^{-x}\Big|_0^4$$

$$=800+90\,e^{-4}-90=710+90\,e^{-4}$$

$$=710+90(0{\cdot}018)=711{\cdot}62 \text{ units.}$$

Example 93. *Assume that in 1990 the annual world use of natural gas was 50 trillion cubic feet The annual consumption of gas is increasing at a rate of 3% compounded continously. How long will it take to use all available gas, if it is known that in 1990 there were 2200 trillion cubic feet of proven reserves ? Assume that no new discoveries are made.*

[*Delhi Univ., B. Com.* (*Hons.*) *1991*]

Solution. We are given that

$$\int_0^t 50.\ e^{0{\cdot}03t}dt=2200$$

$$\Rightarrow \qquad \left|50\ .\ \frac{e^{0{\cdot}03t}}{0{\cdot}03}\right|_0^t=2200$$

$$\Rightarrow \qquad \frac{5000}{3}\left(e^{0{\cdot}03t}-1\right)=2200$$

$$\Rightarrow \qquad e^{0{\cdot}03t}=\frac{2200\times 3}{5000}+1=2{\cdot}32$$

$$0{\cdot}03t \log e=\log 2{\cdot}32$$

$$\Rightarrow \qquad (0{\cdot}03t)(0{\cdot}4343)=(0{\cdot}3655)$$

$$\Rightarrow \qquad t=\frac{0{\cdot}3655}{0{\cdot}03\times 0{\cdot}4343}=28{\cdot}1 \text{ year}$$

Example 94. *A firm has the current sales of Rs. 50,000 per month. The firm wants to embark on a certain advertising campaign that will increase the sales by 2% per month (compounded continuously over the period of the campaign which is 12 months. Find the total increase in sales as a result of the campaign. Use calculus.*

[*Delhi Univ., B. Com.* (*Hons.*) *1990*]

Solution. Total increase in sales is given by

$$=\int_0^{12} 50000\, e^{0\cdot 02t}dt-50000\times 12$$

$$=50000\,.\left|\frac{e^{0\cdot 02t}}{0\cdot 02}\right|_0^{12}-50000\times 12$$

$$=25,00,000(e^{0\cdot 24}-1)-50,000\times 12$$

Let $$y=e^{0\cdot 24}$$

$\therefore\quad \log y=0\cdot 24\log e=0\cdot 24\times 0\cdot 4343=0\cdot 1042$

or $$y=\text{antilog.}(0\cdot 1042)=1\cdot 272$$

$\therefore$ Total increase in sales is given by

$$=25,00,000\ (1\cdot 272-1)-50,000\times 12$$

$$=6,80,000-6,00,000=80,000.$$

Example 95. *A company whose annual sales are currently Rs. 5,00,000 has been experiencing sales increase of 20% per year. Assuming this rate of growth continues, what will the annual sales be in five years.*

Solution. If A is the annual sales in five years, then

$$A=\int_0^5 5,00,000\ e^{0\cdot 2t}\,dt$$

$$=\frac{5,00,000}{0\cdot 2}\Big[e-1\ \Big]$$

$$=5,00,000(8\cdot 59139)=\text{Rs. }4295695$$

Amount of an Annuity

The amount of an annuity is the sum of all payments made, plus all interest accumulated.

If an annuity consists of equal annual payments P in which an interest rate of r% per annum is compounded continuously, the amount A of the annuity after N payment is

$$A=\int_0^N Pe^{rt}\,dt$$

$$=\left|\frac{Pe^{rt}}{r}\right|_0^t=\frac{P(e^{rt}-1)}{r}.$$

Example 96. *XYZ Bank pays 10% per annum compounded continuously. If a person places Rs. 10,000 in a savings account each year, how much will be in the account after 5 years ?*

Solution. Here $P=10{,}000$; $N=5$ and $r=0{\cdot}10$. The amount A after 5 years is

$$A=\int_0^5 10{,}000\, e^{0\cdot 10t}\, dt$$

$$=\frac{10{,}000}{0{\cdot}10}\Big| e^{0\cdot 10t} \Big|_0^5=\frac{10{,}000}{0{\cdot}10}(e^{0\cdot 5}-1)$$

$$=\frac{10{,}000}{0{\cdot}1}\ [0{\cdot}6488]=\text{Rs. } 64880$$

Example 97. *A bank pays interest at the rate of 6% per annum compounded continuously. Find how much should be deposited in the bank each year in order to accumulate Rs. 6,000 in 3 years*

[*Delhi Univ., B. Com.* (*Hons.*) ; *1992*]

Solution. Let Rs. A be deposited each year. Then, we have

$$6000=\int_0^3 A\,.\, e^{0\cdot 06t}\, dt$$

$$=A\,.\left[\frac{e^{0\cdot 08t}}{0{\cdot}06}\right]_0^3=\frac{A}{0{\cdot}06}\ (e^{0\cdot 18}-e^{0})$$

$$=\frac{A}{0{\cdot}06}(e^{0\cdot 18}-1)$$

$$\Rightarrow \qquad 6000\times 0{\cdot}06=A(e^{0\cdot 18}-1)$$

$$\Rightarrow \qquad A=\frac{6000\times 0{\cdot}06}{e^{0\cdot 18}-1}=\frac{360}{e^{0\cdot 18}-1}$$

Let $\qquad y=e^{0\cdot 18}$

$\therefore \quad \log y=0{\cdot}18 \log e=0{\cdot}18\times 0{\cdot}4343=0{\cdot}0782$

$\therefore \qquad y=\text{antilog } (0{\cdot}0782)=1{\cdot}198]$

$\therefore \qquad A=\dfrac{360}{1{\cdot}198-1}=\dfrac{360}{0{\cdot}198}=1818{\cdot}18$

EXERCISES

1. If MC of a firm is given by

$$C(q)=2+5e^{q},$$

find total cost if $C(0)=100$. Also find average cost. What will be the marginal, average and total cost for $q=60$ units ?

2. Let the marginal cost function of a firm be $100-10x+0{\cdot}1\,x^2$, where x is the output. Obtain the total cost function of the firm under the assumption that its fixed cost is Rs. 500.

[Hint. $MC=100-10x+0{\cdot}1x^2$

$$\therefore \quad TC=\int(100-10x+0{\cdot}1x^2)\,dx$$

$$=100x-5x^2+\frac{x^3}{30}+k.$$

Fixed cost is $=500$

$$\therefore \quad TC=100x-5x^2+\frac{x^3}{30}+500]$$

3. The marginal cost of production is found to be

$$MC=2000-40x+3x^2$$

where x is the number of units produced. The fixed cost of production is Rs. 18,000. Find the cost function.

If the manufacturer fixes the price per unit at Rs. 6800,

(*i*) Find the revenue function.

(*ii*) Find the profit function.

(*iii*) Find the sales volume that yields maximum profit ?

(*iv*) What is the profit at this sales volume ?

[Hint. $C(x)=\int(2000-40x+3x^2)\,dx=2000x-20x^2+x^3+k$

$C(0)=18{,}000 \Rightarrow C(x)=x^3-20x^2+2000x+18{,}000]$

4. A company determines that the marginal cost of producing x units of a particular commodity during a one-day operation is $MC=16x-1591$, where the production cost is in rupees. The selling price of commodity is fixed at Rs. 9 per unit and the fixed cost is Rs. 1800 per day.

(*a*) Find the cost function.

(*b*) Find the revenue function.

(*c*) Find the profit function.

(*d*) What is the maximum profit that can be obtained in a one-day operation ?

[Hint. (*a*) $C(x)=\int(MC)\,dx=\int(16x-1591)\,dx=8x^2-1591x+k$

$C(0)=1800 \Rightarrow C(x)=8x^2-1591x+1800$

(*b*) $R(x)=9x$

(*c*) $P(x)=R(x)-C(x)=-8x^2+1600x-1800$

(*d*) $P'(x)=-16x+1600=0 \Rightarrow x=100$

$\therefore$ The maximum profit that can be obtained in one day is

$$P(100)=-8(100)^2+1{,}60{,}000-1{,}800=\text{Rs. }78{,}200.]$$

5. If the marginal cost function is given by $\pi_m = \dfrac{3}{\sqrt{3q+4}}$ and fixed cost is 2, find the average cost for 4 units of output. **[Ans.** 8/7**]**

6. Find the total revenue functions and the demand functions corresponding to the following marginal revenue functions.

(*i*) $MR = 9 - 4x^2$, (*ii*) $MR = 7 - 4x - x^2$;

(*iii*) $MR = \dfrac{6}{(q+2)^2} - 5$.

$\Big[$**Ans.** (*i*) $R = 7x - 2x^2 - \dfrac{x^3}{3}$, $AR = 7 - 2x - \dfrac{x^2}{3}$, (*ii*) $p = \dfrac{3}{q+2} - 5.\Big]$

7. The marginal revenue function of a commodity for output q is given by $\dfrac{dR}{dq} = \dfrac{1}{2}\, q^{-\frac{1}{2}}$, where R stands for total revenue. What is the demand function ? **[Ans.** $p = q^{-1/2}$**]**

8. If the marginal revenue of output q is given by the equation $\dfrac{dR}{dq} = \alpha - \beta q$, where R is total revenue. Find the total revenue function and hence deduce the demand function.

$\Big[$**Ans.** $R = \alpha q - \frac{1}{2}\beta q^2$ and $p = \alpha - \dfrac{\beta}{2}\, q \Big]$

9 If the marginal revenue function is $MR = \dfrac{ab}{(x+b)^2} - c$, show that $p = \dfrac{a}{x+b} - c$ is the demand law.

$\Big[$**Hint.** $MR = \left\{ \dfrac{ab}{(x+b)^2} - c \right\} = \dfrac{dR}{dx}$

$$\Rightarrow \qquad R = \int \left\{ \frac{ab}{(x+b)^2} - c \right\} dx = -\frac{ab}{x+b} - cx + k$$

where k is the constant of integration. Now $R = 0$ when $x = 0$.

$$\therefore \qquad \frac{-ab}{b} + k = 0 \qquad \therefore \qquad k = a.$$

$$\therefore \qquad R = -\frac{ab}{x+b} + a - cx = \frac{ax}{x+b} - cx$$

$$\therefore \qquad p = \frac{R}{x} = \frac{a}{x+b} - c. \Big]$$

10. If the marginal revenue and the marginal cost for an output x of a commodity are given as

$$MR = 5 - 4x + 3x^2 \text{ and } MC = 3 + 2x$$

and if the fixed cost is zero find the profit function and the profit when the output is $x=4$.

[**Ans.** Profit function$=2x-3x^2+x^3$; 24]

11. Additional earnings obtained by purchasing a new machine is approximated by $R(x)=50x-x^2$. The annual maintenance costs for the machine are $C(x)=4x^2$. How many years should the machine be maintained, assuming no salvage value ? What are the total net earnings for that period ? Costs are in Rs. 100 units and x is in years.

[**Ans.** 5, Rs. 125]

12. If the marginal cost function is $MC=x^2-16x+20$ and marginal revenue function is $MR=20-2x$, determine the profit-maximizing output and the corresponding total profit. Cost is in units of Rs. 1000 and x is in units of output.

13. The marginal propensity to consume out of income for the economy as a whole is given as $\frac{4}{5}$. It is known that when income is zero, consumption equals Rs. 12 billion. Find the function relating aggregate consumption to national income. Find aggregate saving as function of income.

[**Ans.** $C=\frac{4}{5}Y+12$, $S=\frac{1}{5}Y-12$.]

14. In an economy, the marginal propensity to consume of domestically produced goods is given by

$\frac{dC}{dY}=0{\cdot}6$ and marginal propensity to import is $\frac{dM}{dY}=0{\cdot}2$,

where C, M and Y stand for consumption, imports and income respectively. What will be the equation for aggregate expenditure of the economy ? Also give economic interpretation of the constant of integration.

[**Ans.** $E=K+0{\cdot}8\,Y$, where E is aggregate expenditure of the economy and K represents autonomous expenditure.]

15. Determine the consumer's and the producer's surplus, given the demand function $D(x)=25-5x+(x^2/4)$ and supply function $S(x)=5x+(x^2/4)$. Assume a monoply situation.

[**Ans.** 13·02, 18·25.]

16. Under pure competition for a commodity, the demand and supply laws are :

$$p_d=\frac{8}{x+1}-2 \text{ and } p_s=\frac{1}{2}(x+3) \text{ respectively.}$$

Determine the consumer's surplus and the producer's surplus.

$$\left[\textbf{Ans}\quad \text{C.S}=\int_0^1\left(\frac{8}{x+1}-2\right)dx-2\times 1=8\log 2-2-2\quad\right]$$

17. Find the consumer's surplus (at equilibrium price) if the demand function is $D=\frac{25}{4}-\frac{p}{8}$ and supply function is $p=5+D$.

18. Find consumer's surplus and producer's surplus defined by the demand curve $D(x)=20-5x$ and supply curve $S(x)=4x+8$.

Sketch also the appropriate graphs.

$$\left[\textbf{Hint. } CS=\int_0^{4/3}(20-5x)\,dx-\frac{40}{3}\times\frac{4}{3},\ PS=\frac{160}{9}-\int_0^{4/3}(4x+8)\,dx\right]$$

19. The quantity sold and the corresponding price under monopoly are determined by the demand law $p=16-x^2$ and by the $CM=6+x$ in such a way as to maximise the profit. Determine corresponding $C.S.$

In the above question, if demand law is : $p=45-x^2$ and

$$MC=6+\frac{x^2}{4}, \text{ determine } C.S.$$

20. Assume that the demand and average cost curves of steel are :

$$p=2{\cdot}34-1{\cdot}34x$$

$$\text{and}\quad AC=\frac{1}{x}-0{\cdot}83+0{\cdot}85\,x,$$

x is the quantity of steel demanded or produced.

Show that consumer's surplus under monopoly and perfect competition is 0·351 and 0·129 respectively.

Show also that C.S. would have been equal to 2·043 if steel were a free good.

21. Find the consumer's surplus if the demand curve is

$$D(x)=50-0{\cdot}025x^2$$

and it is known that the market quantity is 20 units.

$$\left[\textbf{Hint. } CS=\int_0^{20}(50-0{\cdot}025\,x^2)\,dx-40\times 20\right]$$

22. A business organisation made an analysis of production which shows that with the present equipment and workers, the production is 10,000 units per day. It is estimated that the rate of change of production P with respect to the change in the number of additional workers x is

$$\frac{dP}{dx}=200-3x^{1/2}$$

What is the production (expressed in units per day) with 25 additional workers ?

[**Hint.** x denotes the change in the number of workers. When there is no change in their number, $x=0$. When 25 additional workers are taken, $x=25$.

$$\frac{dP}{dx}=20-3x^{1/2}.$$

Integrating both sides with respect to x, we get

$$\int dP=\int (200-3x^{1/2})\, dx$$

$$P=200x-\frac{3x^{3/2}}{3/2}+k$$

$$P=200x-2x^{3/2}+k \qquad \ldots(*)$$

Using the condition that when $x=0$, $P=10{,}000$, (*) becomes

$$10{,}000=200\times 0-0+k$$

$$k=10{,}000$$

Hence $\quad P=200x-2x^{3/2}+10{,}000$

When $x=25$, $\quad P=200\times 25-2(25)^{3/2}+10{,}000=14{,}750.$]

23. The production manager of an electronics company obtained the following function :

$$f(x)=1356{\cdot}4x^{-0{\cdot}3218}$$

where $f(x)$ is the rate of labour hours required to assemble the x^{th} unit of a product. The function is based on the experience for assembling the first 50 units of the product. The company was asked to bid on a new order of 100 additional units. Find the total labour hours required for assembling the 100 units. **[Ans.** 31,460]

24. The purchase price of a car is Rs. 75,000. The rate of cost for the repair of the car is given by the function :

$$C=600\,(1-e^{-0{\cdot}5t})$$

where t represents the years of use since purchase and C denotes the cost. Find the cumulative repair cost at the end of 5 years. Also find approximately the time in years at which the cumulative repair cost equals the original cost of the car.

25. If Rs. 500 is deposited each year in a saving account paying 5·5% per annum compounded continuously, how much is in the account after 4 years ?

$$\left[\textbf{Hint}\quad A=\int_0^4 500\, e^{0{\cdot}05t}\, dt=9090(e^{0{\cdot}22}-1)=2236.\right]$$

26. What is the present value of Rs. 1200 per year at 7% for five years ? How does this compare with Rs. 100 per month ? (Assume continuous discounting). **[Ans.** Rs. 5062·49, same]

27. A small data-processing company is planning to acquire additional components for its main computer. Estimated maintenance costs for each unit are $C(x)=3x^2$. Anticipated savings from each added unit

are approximated by $S(x) = 2x^2 + 16$. $C(x)$ is in Rs. 1000 units ; $S(x)$ is in units of Rs. 10,000 ; and x is the number of units added. How many units should be added and what are the resulting earnings ?

28. The anticipated additional sales from a newspaper advertisement campaign are approximated by $R(x) = 16088\ e^{0.04x}$, where $R(x)$ is extra daily sales in rupees and x is in days. Research has found that 10 days is the maximum period of return for an advertisement. If the advertisement cost is Rs. 11·89, what is the expected additional income at the end of the first day ? At the end of the fifth day ? At the end of the tenth day ?

9. Pareto's hypothesis concerning income distribution is given by the equation $y = A\ x^{-(b+1)}$, A and b being positive constants, where y represents the number of persons with an income of Rs. x and is a continuous frequency distribution of persons according to their levels of income.

Find (*i*) the number of income recipients between income levels p and q and (*ii*) their average income

$$\left[\textbf{Ans.}\quad \frac{A}{b}\left[\frac{1}{p^b} - \frac{1}{q^b}\right],\ \frac{b}{1-b} \cdot \frac{q^{-(b-1)} - p^{-(b-1)}}{p^{-b} - q^{-b}} \right]$$

30. Suppose a law of income distribution states that

$$y(x) = \int_x^\infty a\ t^{-u}\ dt$$

where x is income level, u and a are constants and y is a cumulative frequency of income recipients. Find the number of people falling into the income bracket (x_1, x_2).

$$\left[\textbf{Ans.}\quad \frac{u}{1-b}\left[\ x_1^{-b+1} - x_2^{-b+1}\ \right]\right]$$

31. If the investment flow is given by $L_t = 5t^{1/4}$ and the capital stock at $t = 0$ is K_0, find the time path of capital K and also find the capital formation in the t th period.

$$[\textbf{Ans.}\quad 4t^{3/4} + k_0,\ 4\ \{t^{3/4} - (t-1)^{5/4}\}]$$

32. Obtain the demand function for a commodity for which elasticity of demand is constant 'α' throughout.

[Hint. $-\dfrac{p}{x} \cdot \dfrac{dx}{dp} = \alpha \quad \Rightarrow \quad -\dfrac{dx}{x} = \alpha . \dfrac{dp}{p}$

$\Rightarrow \quad -\displaystyle\int \frac{dx}{x} = \alpha \int \frac{dp}{p}$

$\Rightarrow \quad -\log x = \alpha \log p + \log k = \log p^\alpha + \log k$

Hence $\quad xp^\alpha = c.]$

APPLICATIONS TO MATRICES

Example 98. *Mr X is a sole trader, manufacturing tables and chairs. Each table requires 5 hours of labour and 6 units of material. A chair requires 3 labour hours and 3 units of material. If Mr X plans to produce 10 tables and 15 chairs in the next week, how many hours will he need to work and how much material will he require ?*

Solution. The labour requirement is $(10\times5)+(15\times3)=95$ hours

The material requirement is $(10\times6)+(15\times3)=105$ units.

The matrix solution would be :

$$\begin{array}{cc} \textit{Tables} & \textit{Chairs} \\ (10 & 15)_{1\times2} \end{array} \times \begin{array}{c} \begin{array}{cc} \textit{Labour} & \textit{Materials} \end{array} \\ \begin{pmatrix} 5 & 6 \\ 3 & 3 \end{pmatrix}_{2\times2} \end{array} \begin{array}{cc} \textit{Labour} & \textit{Materials} \\ =(95 & 105)_{1\times2} \end{array}$$

It may be noted that

$$\begin{pmatrix} 5 & 6 \\ 3 & 3 \end{pmatrix}_{2\times2} \times \left(\frac{10}{15}\right)_{1\times1} = \left(\frac{140}{75}\right)_{2\times1}$$

is incorrect as labour hours are being added to units of material.

Example 99. *A firm produces different pump units, each of which requires some components shown below in a tabular form :*

Pump	*Housing*	*Impeller*	*Bolts*	*Couplings*	*Inlets*	*Armoured Hose*
Type A	*1*	*1*	*5*	*4*	*2*	*8 m.*
Type B	*1*	*1*	*7*	*3*	*2*	*4 m.*
Type C	*1*	*1*	*3*	*5*	*2*	*3 m.*

The firm receives an order for 8 Type-A pump units, 4 Type-B units and 2 Type-C units. Using the notion of Matrix multiplication, obtain the matrix whose elements may represent the quantities of each item required to make up the order.

Solution. The specifications of the different pump units with their components can be represented by the following matrix,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 5 & 7 & 3 \\ 4 & 3 & 5 \\ 2 & 2 & 2 \\ 8 & 4 & 3 \end{bmatrix}_{6\times3}$$

where each column represents the type of the pump and each row represents the different components required. The firm has received order for 8 type *A*, 4 type *B*, 2 type *C* units. This can be represented by the matrix,

$$\begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}_{3\times1}$$

Therefore the matrix multiplication of these two matrices gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 5 & 7 & 3 \\ 4 & 3 & 5 \\ 2 & 2 & 2 \\ 8 & 4 & 3 \end{bmatrix}_{6\times 3} \times \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1\times 8+1\times 4+1\times 3 \\ 1\times 8+1\times 4+1\times 3 \\ 5\times 8+7\times 4+3\times 3 \\ 4\times 8+3\times 4+5\times 3 \\ 2\times 8+2\times 4+2\times 3 \\ 8\times 8+4\times 4+3\times 3 \end{bmatrix}_{6\times 1}$$

The first element of matrix $(=14)$ gives the number of components for housing, the second $(=14)$ gives that of impeller and so on.

Example 100. *The following matrix gives the number of units of three products (P, Q and R) that can be processed per hour on three machines (A, B and C) :*

$$\begin{array}{c} \\ P \\ Q \\ R \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 10 & 12 & 15 \\ 13 & 11 & 20 \\ 16 & 18 & 14 \end{bmatrix} \end{array}$$

Determine by using matrix algebra, how many units of each product can be produced, if the hours available on machines A, B and C are 54, 46 and 48 respectively. [*Delhi Univ, B. Com.* (*Hons.*), *1992*]

Solution.

$$\text{Units of products} = \begin{array}{c} \\ P \\ Q \\ R \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 10 & 12 & 15 \\ 13 & 11 & 20 \\ 16 & 18 & 14 \end{bmatrix} \end{array} \begin{bmatrix} 54 \\ 46 \\ 48 \end{bmatrix}\begin{array}{c} A \\ B \\ C \end{array}$$

$$= \begin{bmatrix} 540+552+720 \\ 702+506+960 \\ 864+828+672 \end{bmatrix}$$

$$= \begin{bmatrix} 1812 \\ 2168 \\ 2364 \end{bmatrix}\begin{array}{c} P \\ Q \\ R \end{array}$$

$\therefore$ 1812, 2168 and 2364 units of product P, Q and R are produced respectively.

Example 101. *The following matrix gives the proportionate mix of constituents used for three fertilisers :*

		Constituent			
		A	*B*	*C*	*D*
	1	*0·5*	*0*	*0·5*	*0*
Fertiliser	*2*	*0·2*	*0·3*	*0*	*0·5*
	3	*0·2*	*0·2*	*0·1*	*0·5*

(i) If sales are 1000 tins (of one kilogram) per week, 20% being fertiliser 1, 30% being fertiliser 2, and 50% fertiliser 3; how much of each constituent is used ?

(ii) If the cost of each constituent is 50 paise, 60 paise, 75 paise and 100 paise per 100 grams, respectively, how much does a one kilogram tin of each fertiliser cost ?

(iii) What is the total cost per week ?

Express the calculations and answers in matrix form.

Solution. (*i*) The sales of fertilisers per week can be expressed as the following matrix :

$$1000\,(0{\cdot}2 \quad 0{\cdot}3 \quad 0{\cdot}5)=(200 \quad 300 \quad 500)$$

Thus

$$(200 \quad 300 \quad 500)\begin{pmatrix} 0{\cdot}5 & 0 & 0{\cdot}5 & 0 \\ 0{\cdot}2 & 0{\cdot}3 & 0 & 0{\cdot}5 \\ 0{\cdot}2 & 0{\cdot}2 & 0{\cdot}1 & 0{\cdot}5 \end{pmatrix}$$
$$=(260 \quad 190 \quad 150 \quad 400)$$

Requirements of constituents are :

A : 260, B : 190, C : 150, D : 400

(*ii*) Costs of each constituent are 50 p, 60 p, 75 p, and 100 p per 100 grams, *i.e.*, 500 p, 600 p, 750 p and 1000 p per 1,000 grams (one kilogram) of each constituent, respectively.

Thus

$$\begin{pmatrix} 0{\cdot}5 & 0 & 0{\cdot}5 & 0 \\ 0{\cdot}2 & 0{\cdot}3 & 0 & 0{\cdot}5 \\ 0{\cdot}2 & 0{\cdot}2 & 0{\cdot}1 & 0{\cdot}5 \end{pmatrix}\times\begin{vmatrix} 500 \\ 600 \\ 750 \\ 1000 \end{vmatrix}=\begin{pmatrix} 625 \\ 780 \\ 795 \end{pmatrix}$$

Costs per 1 kg tin of fertilizer are :

1 : Rs. 6·25, 2 : Rs. 7·80, 3 : Rs. 7·95.

(*iii*) The total cost of fertiliser if 1,000 one-kilogram tins are needed per week may be calculated by either :

$$(200 \quad 300 \quad 500)\begin{pmatrix} 625 \\ 780 \\ 795 \end{pmatrix}=(7{,}56{,}500)$$

$$\text{or by} \quad (260 \quad 190 \quad 150 \quad 400)\left|\begin{matrix}500\\600\\750\\1000\end{matrix}\right| = (7{,}56{,}500)$$

Hence, total cost per week is Rs. 7,565.

Example 102. *The total cost of manufacturing three types of motor car is given by the following table :*

	Labour (hrs)	*Materials (units)*	*Sub-contracted work (units)*
Car A	*40*	*100*	*50*
Car B	*80*	*150*	*80*
Car C	*100*	*250*	*100*

Labour costs Rs. 20 per hour, units of material cost Rs. 5 each and units of sub-contracted work cost Rs. 10 per unit. Find the total cost of manufacturing 3,000 ; 2,000 and 1,000 vehicles of type A, B and C respectively.

(Express the cost as a triple product of a three element row matrix, a 3×3 matrix and a three element column matrix and perform the multiplication according to the same rules you used for 2×2 matrices.)

Solution. Let matrix P represent labour hours, material used and sub-contracted work for three types of cars A, B, C respectively.

$$\therefore \qquad \mathbf{P} = \begin{bmatrix} 40 & 100 & 50 \\ 80 & 150 & 80 \\ 100 & 250 & 100 \end{bmatrix}$$

Further let matrix **Q** represent labour cost per unit, material cost and cost of sub-contracted work

$$\mathbf{Q} = \begin{bmatrix} 20 \\ 5 \\ 10 \end{bmatrix}$$

The cost of each car A, B, C is now given by the column matrix

$$\mathbf{PQ} = \begin{pmatrix} 1800 \\ 3150 \\ 4250 \end{pmatrix}$$

Let the number of cars A, B, C to be manufactured in that order be represented by the row matrix

$$\mathbf{R} = [3000 \quad 2000 \quad 1000]$$

Hence the total cost of manufacturing three cars A, B and C is given by the matrix

$$\mathbf{PQR}=\begin{bmatrix}1800\\3150\\4250\end{bmatrix}\times[3000 \quad 2000 \quad 1000]$$

$$=[1{,}59{,}50{,}000]$$

Example 103. *A manufacturer produces three products : P, Q and R which he sells in two markets. Annual sales volumes are indicated as follows :*

Markets		*Products*	
	P	*Q*	*R*
I	*10,000*	*2,000*	*18,000*
II	*6,000*	*20,000*	*8,000*

If unit sale prices of P, Q and R are Rs. 2·50, 1·25 and 1·50 respectively, find the total revenue in each market with the help of Matrix Algebra.

If the unit costs of the above 3 commodities are Rs. 1·80, 1·20 and 0·80 respectively, find his gross profits.

Solution. Total revenue in each market is obtained from the matrix product :

$$[2{\cdot}50 \quad 1{\cdot}25 \quad 1{\cdot}50]\times\begin{bmatrix}10000 & 6000\\2000 & 20000\\18000 & 8000\end{bmatrix}=[54500 \quad 52000]$$

$$\text{Total cost}=[1{\cdot}80 \quad 1{\cdot}20 \quad 0{\cdot}80]\times\begin{bmatrix}10000 & 6000\\2000 & 20000\\18000 & 8000\end{bmatrix}$$

$$=[34800 \quad 41200]$$

Profits from market $A=54500-34800=19700$

Profits from market $B=52000-41200=10800$

Example 104. *In a certain city there are 25 colleges and 100 schools. Each school and college has 5 peons, 2 clerks and 1 cashier. Each college in addition has 1 accountant and 1 head-clerk. The monthly salary of each of them is as follows :*

Peon—Rs. 300 ; Clerk—Rs. 500 ; Cashier—Rs. 600 ; Accountant—Rs. 700 ; and Head-clerk—Rs. 800.

Using matrix notation, find

(*a*) *the total number of posts of each kind in schools and colleges taken together.*

(*b*) *the total monthly salary bill of each school and college separately, and* (*c*) *the total monthly salary bill of all the schools and colleges taken together.*

Solution. (*a*) Consider the row matrix of order 1×2

$$\mathbf{A}=[25 \quad 100]$$

This represents the number of colleges and schools in that order.

Let
$$\mathbf{B}=\begin{bmatrix} 5 & 2 & 1 & 1 & 1 \\ 5 & 2 & 1 & 0 & 0 \end{bmatrix}$$

where columns represent number of peons, clerks, cashier, accountant, head-clerk while rows represents colleges and schools in that order. Then

$$\mathbf{AB}=\underset{1\times2}{[25 \quad 100]}\times\underset{2\times5}{\begin{bmatrix} 5 & 2 & 1 & 1 & 1 \\ 5 & 2 & 1 & 0 & 0 \end{bmatrix}}$$

$$=\underset{1\times5}{[625 \quad 250 \quad 125 \quad 25 \quad 100]}$$

where first element represents total number of peons, second represents total number of clerks, third represents total number of cashiers, fourth represents total number of accountants and fifth represents total number of head-clerks.

(*b*) Let the column matrix

$$\mathbf{C}=\begin{bmatrix} 300 \\ 500 \\ 600 \\ 700 \\ 800 \end{bmatrix}$$

represent monthly salary of peon, clerk, cashier, accountant and head-clerk in that order. Then

$$\mathbf{BC}=\underset{2\times5}{\begin{bmatrix} 5 & 2 & 1 & 1 & 1 \\ 5 & 2 & 1 & 0 & 0 \end{bmatrix}}\times\underset{5\times1}{\begin{bmatrix} 300 \\ 500 \\ 600 \\ 700 \\ 800 \end{bmatrix}}$$

$$=\underset{2\times1}{\begin{bmatrix} 1500+1000+600+700+800 \\ 1500+1000+600+0 \quad +0 \end{bmatrix}}=\underset{2\times1}{\begin{bmatrix} 4600 \\ 3100 \end{bmatrix}}$$

Thus, total monthly salary bill of each college is Rs. 4600 and of each school is Rs. 3100.

(*c*) The total monthly salary bill of all schools and colleges taken together is

$$\mathbf{A(BC)}=[25 \quad 100]_{1\times 2} \times \begin{bmatrix} 4600 \\ 3100 \end{bmatrix}_{2\times 1}$$

$$=[1,15,000+3,10,000]_{1\times 1}$$

$$=[4,25,000].$$

Example 105. *The allocation of service department costs to production departments and other service departments is one area where matrix algebra may be used.*

Consider the following data :

	Service departments		*Production department*	
	Maintenance	*Electricity*	*Matching*	*Assembly*
Manhours of maintenance time	—	*3,000*	*16,000*	*1,000*
Units of electricity consumed	*20,000*	—	*1,30,000*	*50,000*
Department costs before any allocation of service departments	*Rs. 50,000*	*Rs. 4,000*	*Rs. 1,40,000*	*Rs. 2,06,000*

You are required to :

(*i*) *Calculate the total costs to be allocated to the production departments using matrix algebra (Formulate the problem and show all workings) :*

(*ii*) *Show the allocation to the production departments, using matrix methods.*

Solution. (*i*) Let X be the total cost of the maintenance department (*i.e.*, including an allocation of electricity costs).

Let Y be the total cost of electricity (*i.e.*, including an allocation of maintenance costs).

Proportion of maintenance time consumed by electricity department is

$$\frac{3000}{3000+16000+1000}=\frac{3000}{20000}=0{\cdot}15$$

i.e., 15% of the maintenance deptt. costs should be allocated to the electricity department.

$$\therefore \qquad Y=4000+0{\cdot}15\,X \qquad \ldots(*)$$

Likewise, the proportion of total electricity consumption used by the maintenance department is

$$\frac{20000}{20000+130000+50000}=\frac{20000}{200000}=0.1$$

so that 10% of the electricity cost should be allocated to the maintenance department.

$$\therefore \qquad X=50000+0.1\,Y \qquad \ldots(^{**})$$

From (*) and (**), we get

$$-0.15X+Y=4000$$

$$X-0.1Y=50000$$

$$\text{i.e.,}\quad \begin{pmatrix} -0.15 & 1 \\ 1 & -0.1 \end{pmatrix}\times\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 4000 \\ 50000 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} -0.15 & 1 \\ 1 & -0.1 \end{pmatrix}^{-1}\times\begin{pmatrix} 4000 \\ 50,000 \end{pmatrix}$$

$$=\frac{1}{0.985}\begin{pmatrix} 0.1 & 1 \\ 1 & 0.15 \end{pmatrix}\times\begin{pmatrix} 4000 \\ 50000 \end{pmatrix}$$

$$=\begin{pmatrix} 51,168 \\ 11,675 \end{pmatrix}$$

Hence $X=$ Rs. 51,168 and $Y=$ Rs. 11,675.

(*ii*) The proportions of maintenance and electricity consumed by the production departments are :

	Maintenance	*Electricity*
Machine	$\frac{16,000}{20,000}=0.8$	$\frac{1,30,000}{2,00,000}=0.65$
Assembly	$\frac{1,000}{20,000}=0.05$	$\frac{50,000}{2,00,000}=0.25$

Accordingly the allocations of maintenance costs to the production department is

$$\begin{pmatrix} 0.8 & 0.65 \\ 0.05 & 0.25 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$=\begin{pmatrix} 0.8 & 0.65 \\ 0.05 & 0.25 \end{pmatrix}\begin{pmatrix} 51,168 \\ 11,675 \end{pmatrix}=\begin{pmatrix} 48,523 \\ 5,477 \end{pmatrix}$$

i.e., Rs. 48,523 to machining and Rs. 5,477 to assembly, a total of Rs. 54,000.

Example 106. *A, B and C has Rs. 480, Rs. 760 and Rs. 710 respectively. They utilised the amounts to purchase three types of shares of prices x, y and z respectively. A purchases 2 shares of price x, 5 of price y and 3 of price z. B purchases 4 shares of price x, 3 of price y and 6 of price z, C purchases 1 share of price x, 4 of price y and 10 of price z. Find x, y and z.*

Solution. We obtain the following set of simultaneous linear equations :

$$\begin{aligned} 2x+5y+3z&=480\\ 4x+3y+6z&=760\\ x+4y+10z&=710 \end{aligned}$$

The above system of equations in the matrix notation is

$$\begin{bmatrix} 2 & 5 & 3 \\ 4 & 3 & 6 \\ 1 & 4 & 10 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 480 \\ 760 \\ 710 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 3 & 6 \\ 1 & 4 & 10 \end{bmatrix}^{-1} \times \begin{bmatrix} 480 \\ 760 \\ 710 \end{bmatrix} \qquad \dots(*)$$

Now $\quad \mathbf{A}^{-1}=\dfrac{Adj\ \mathbf{A}}{|\mathbf{A}|}$; where $|\mathbf{A}|=\begin{vmatrix} 2 & 5 & 3 \\ 4 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}=-119$

and $\quad Adj\ \mathbf{A}=\begin{bmatrix} +6 & -38 & +21 \\ -34 & +17 & 0 \\ +13 & -3 & -14 \end{bmatrix}$ (Try yourself

From (*), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{119} \begin{bmatrix} +6 & -38 & +21 \\ -34 & +17 & 0 \\ +13 & -3 & -14 \end{bmatrix} \times \begin{bmatrix} 480 \\ 760 \\ 710 \end{bmatrix}$$

$$= -\frac{1}{119} \begin{bmatrix} 6\times480-38\times760+21\times710 \\ -34\times480+17\times760+0\ \times710 \\ 13\times480-3\ \times760-14\times710 \end{bmatrix}$$

$$= -\frac{1}{119} \begin{bmatrix} -11090 \\ -3400 \\ -5980 \end{bmatrix} = \begin{bmatrix} 11090/119 \\ 3400/119 \\ 5980/119 \end{bmatrix}$$

Hence $x=\frac{11090}{119},\ y=\frac{3400}{119},\ z=\frac{5980}{119}$.

Example 107. *To control a certain crop disease it is necessary to use 8 units of chemical A, 14 units of chemical B and 13 units of chemical C. One barrel of spray P contains one unit of A, 2 units of B and 3 units of C. One barrel of spray Q contains 2 units of A, 3 units of B and 2 units of C. One barrel of spray R contains one unit of A, 2 units of B and 2 units of C. How many barrels of each type of spray should be used to control the disease ?*

Solution. To grasp the situation easily, let us tabulate the data as follows :

			Spray		*Requirement in chemicals*
		P	*Q*	*R*	
	A	1	2	1	8
Chemical	*B*	2	3	2	14
	C	3	2	2	13
Quantity in each spray		*x*	*y*	*z*	

Let x barrels of spray P, y barrels of spray Q and z barrels of spray R be used to control the disease. Then

$$x+2y+z=8$$
$$2x+3y+2z=14$$
$$3x+2y+2z=13$$

Writing the equations in the matrix form, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix}$$

Now $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} +2 & -2 & +1 \\ +2 & -1 & 0 \\ -5 & +4 & -1 \end{bmatrix}$ (Try yourself)

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +2 & -2 & +1 \\ +2 & -1 & 0 \\ -5 & +4 & -1 \end{bmatrix} \times \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x=1,\ y=2$ and $z=3$

Hence 1 barrel of the spray P, 2 barrels spray Q and 3 barrels of ray R should be used to control the disease.

Example 108. *The XYZ Bakery Ltd. produces three basic pastry mixes A, B and C. In the past the mix of ingredients has been as shown in the following matrix :*

		Flour	*Fat*	*Sugar*
Type	*A*	*5*	*1*	*1*
	B	*6·5*	*2·5*	*0·5*
	C	*4·5*	*3*	*2*

(All quantities in kilogram weight)

Due to changes in consumer tastes it has been decided to change the mixes using the following amendment matrix :

		Flour	*Fat*	*Sugar*
Type	*A*	*0*	*+1*	*0*
	B	*−0·5*	*+0·5*	*−0·5*
	C	*+0·5*	*0*	*0*

Using matrix algebra you are required to calculate :

(i) the matrix for the new mix :

(ii) the production requirements to meet an order for 50 units of type A, 30 units of type B and 20 units of type C of the new mix ;

(iii) the amount of each type that must be made to totally use up 3700 kgs. of flour, 1700 kgs of fat and 800 kgs of sugar that are at present in the stores.

Solution. (*i*) The new mix is given by the addition of the original mix matrix and the amendment matrix.

$$\begin{pmatrix} 5 & 1 & 1 \\ 6{\cdot}5 & 2{\cdot}5 & 0{\cdot}5 \\ 4{\cdot}5 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 0 & +1 & 0 \\ -0{\cdot}5 & +0{\cdot}5 & +0{\cdot}5 \\ +0{\cdot}5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 1 \\ 5 & 3 & 2 \end{pmatrix}$$

Therefore, the answer to part (*i*) is

	Flour	*Fat*	*Sugar*
Type *A*	5	2	1
Type *B*	6	3	1
Type *C*	5	3	2

(*ii*) To determine the production requirements it is necessary to multiply the order vector by the new mix matrix,

$$(50 \quad 30 \quad 20) \begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 1 \\ 5 & 3 & 2 \end{pmatrix} = (530 \quad 250 \quad 120)$$

$\therefore$ 530 kgs of flour, 250 kgs of fat, 120 kgs of sugar

(*iii*)
$$5X_1+6X_2+5X_3=3700$$
$$2X_1+3X_2+3X_3=1700$$
$$X_1+X_2+2X_3=800$$

$$\begin{pmatrix} 5 & 6 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}=\begin{pmatrix} 3700 \\ 1700 \\ 800 \end{pmatrix}$$

$$\Rightarrow \quad AX=B$$

$$\Rightarrow \quad X=A^{-1}B$$

$$\Rightarrow \quad \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}=\begin{pmatrix} 5 & 6 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}^{-1}\times\begin{pmatrix} 3700 \\ 1700 \\ 800 \end{pmatrix}$$

On simplification, we get

$$X_1=400,\ X_2=200 \text{ and } X_3=100.$$

Example 109. *A mixture is to be made of three foods A, B, C. The three foods A, B, C contain nutrients P, Q, R as shown in the tabular column. How to form a mixture which will have 8 gms of P, 5 gms of Q, and 7 gms of R,*

		gms per kg of	
Food	*Nutrient P*	*Nutrient Q*	*Nutrient R*
A	*1*	*2*	*5*
B	*3*	*1*	*0*
C	*4*	*2*	*2*

Solution. Let x kgs of food A, y kgs of food B, and z kgs of food C be chosen to make up the mixture.

Then we have the equations,

$$x+3y+4z=8$$
$$2x+y+2z=5$$
$$5x+2z=7$$

Expressing these equations as a single matrix equation, we have

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 0 & 2 \end{pmatrix}\times\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 5 \\ 7 \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} 1 & 3 & 4 \\ 0 & -5 & -6 \\ 0 & -15 & -18 \end{pmatrix}\times\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ -11 \\ -13 \end{pmatrix} \quad \begin{matrix} \text{Apply} \\ R_2+(-2)R_1 \\ R_3+(-5)R_1 \end{matrix}$$

$$\text{or} \quad \begin{pmatrix} 1 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ 0 \end{pmatrix} \begin{array}{l} \text{Apply} \\ (-1)\, R_2 \\ R_3+(-3)R_2 \end{array}$$

Therefore, we have

$$x+3y+4z=8 \qquad \ldots(*)$$

$$5y+6z=11 \qquad \ldots(**)$$

Let $z=a$. From (**), $5y+6a=11$, *i.e.*, $y=\frac{11-6a}{5}$

Substituting in (*). $x+3\frac{(11-6a)}{5}+4a=8$

$$\Rightarrow \quad 5x+3(11-6a)+20a=40$$

$$\Rightarrow \quad 5x=7-2a \quad \text{or} \quad x=\frac{7-2a}{5}$$

$\therefore$ The solution is $x=\frac{7-2a}{5}$, $y=\frac{11-6a}{5}$, $z=a$.

As 'a' changes, we can get any number of solutions and thus there are any number of mixtures. Since x, y, z take non-negative values $z \geqslant 0$, *i.e.*, $a \geqslant 0$.

Considering the value of x, we have

$$\frac{7-2a}{5} \geqslant 0, \text{ i.e., } 7-2a \geqslant 0, \text{ i.e., } 7 \geqslant 2a, \text{ i.e., } a \leqslant \frac{7}{2} \qquad \ldots(\text{I})$$

Considering the value of y,

$$\frac{11-6a}{5} \geqslant 0, \text{ i.e., } 11-6a \geqslant 0, \text{ i.e., } 11 \geqslant 6a, \text{ i.e., } a \leqslant \frac{11}{6} \qquad \ldots(\text{II})$$

The restriction (II) covers the restriction (I)

Therefore, we have $0 \leqslant a \leqslant \frac{11}{6}$.

$\therefore$ When $a=1$, $x=1$, $y=1$ and $z=1$.

Example 110. *ABC company has two service departments, S_1 and S_2, and four production departments, P_1, P_2 P_3 and P_4.*

Overhead is allocated to the production departments for inclusion in the stock valuation. The analysis of benefits received by each department during the last quarter and the overhead expense incurred by each department were :

Service Department	*Percentages to be allocated to departemnts*					
	S_1	S_2	P_1	P_2	P_3	P_4
S_1	*0*	*20*	*30*	*25*	*15*	*10*
S_2	*30*	*0*	*10*	*35*	*20*	*5*
Direct overhead Expense Rs '000	*20*	*40*	*25*	*30*	*20*	*10*

You are required to :

(*i*) *express the total overhead of the service departments in the form of simultaneous equations :*

(*ii*) *express these equations in a matrix form ;*

(*iii*) *determine the total overhead to be allocated from each of S_1 and S_2 to the production departments.*

Solution. (*i*) Let

S_1 = total overhead of service department S_1

S_2 = total overhead of service department S_2

Then

$$S_1=20{,}000+0{\cdot}3\ S_2$$

$$S_2=40{,}000+0{\cdot}2\ S_1$$

Written as simultaneous equations, this becomes

$$\begin{aligned} S_1-0{\cdot}3\ S_2&=20{,}000\\ -0{\cdot}2\ S_1+\quad S_2&=40{,}000 \end{aligned}$$

(*ii*) In matrix form, the equations are written as

$$\overset{E}{\begin{pmatrix} 20{,}000 \\ 40{,}000 \end{pmatrix}}=\overset{A}{\begin{pmatrix} 1 & -0{\cdot}3 \\ -0{\cdot}2 & 1 \end{pmatrix}}\times\overset{S}{\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}}$$

$$\Rightarrow\quad \overset{S}{\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}}=\overset{A^{-1}}{\begin{pmatrix} 1 & -0{\cdot}3 \\ -0{\cdot}2 & 1 \end{pmatrix}^{-1}}\times\overset{E}{\begin{pmatrix} 20{,}000 \\ 40{,}000 \end{pmatrix}}$$

(*iii*) By the normal rules for finding the inverse of a 2×2 matrix, this equals

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}=\frac{1}{0{\cdot}94}\begin{pmatrix} 1 & 0{\cdot}3 \\ 0{\cdot}2 & 1 \end{pmatrix}\times\begin{pmatrix} 20{,}000 \\ 40{,}000 \end{pmatrix}=\begin{pmatrix} 34{,}043 \\ 46{,}808 \end{pmatrix}$$

The allocation of overhead from S_1 and S_2 becomes :

$$(S_1)\times(0{\cdot}3\ \ 0{\cdot}25\ \ 0{\cdot}15\ \ 0{\cdot}1)=\begin{pmatrix}P_1\\ P_2\\ P_3\\ P_4\end{pmatrix}$$

$$\Rightarrow\qquad (34{,}043)\times(0{\cdot}3\ \ 0{\cdot}25\ \ 0{\cdot}15\ \ 0{\cdot}1)=\begin{pmatrix}10{,}213\\ 8{,}511\\ 5{,}106\\ 3{,}404\end{pmatrix}$$

and

$$(S_2)\times(0{\cdot}1\ \ 0{\cdot}35\ \ 0{\cdot}2\ \ 0{\cdot}05)=\begin{pmatrix}P_1\\ P_2\\ P_3\\ P_4\end{pmatrix}$$

$$\Rightarrow\qquad (46{,}803)\ (0{\cdot}1\ \ 0{\cdot}35\ \ 0{\cdot}2\ \ 0{\cdot}25)=\begin{pmatrix}4{,}681\\ 16{,}383\\ 9{,}362\\ 2{,}340\end{pmatrix}$$

The final allocation becomes :

Department	*Total*	P_1	P_2	P_3	P_4
	Rs.	Rs.	Rs.	Rs.	Rs.
S_1	27,234	10,213	8,511	5,106	3,404
S_2	32,766	4,681	16,383	9,362	2,340
Total	60,000	14,894	24,894	14,468	5,744

LEONTIEF INPUT-OUTPUT MODEL

The Leontief input-output model in economics (named after Wassily Leontief, a recipient of the Noble prize in Economics in 1973) may be characterised as a description of an economy in which input equals output, or in other words, consumption equals production, *i.e.*, the model assumes that whatever is produced is always consumed.

Input-output models are of two types : closed, in which the entire production is consumed by those participating in the production ; and open in which some of the production is consumed by those who produce it and the rest of the production is consumed by external bodies. In the closed model we seek the income of each participant in the system. In the open model, we seek the amount of production needed to achieve a forecasted demand when the amount of production needed to achieve a current demand is known.

Consider an economy consisting of n industries where each industry produces only one type of product (output). There is an inter-dependence of industries in the sense that one must use other products to operate. Also the production of the finished product must meet the final demand as well as the demand of the other industries.

Our problem is to determine the production of each of the industries if the final demand changes, assuming that the structure of the economy does not change. The data is tabulated in the following input-output transaction table :

		To (user)					*Final demand*	*Tatal output*
		1	*2*	*3*	...	*n*		
From (Producer)	1	x_{11}	x_{12}	x_{13}	...	x_{1n}	d_1	X_1
	2	x_{21}	x_{22}	x_{23}	...	x_{2n}	d_2	X_2
	3	x_{31}	x_{32}	x_{33}	...	x_{3n}	d_3	X_3
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	n	x_{n1}	x_{n2}	x_{n3}	...	x_{nn}	d_n	X_n

where x_{ij} is the output of industry i sold to industry j, *i.e.*, it represents the rupee value of the product of industry i used by industry j.

Now $$X_i = x_{i1}+x_{i2}+x_{i3}+\ldots+x_{in}+d_i$$

represents the rupee value of the total output of industry i.

$$\frac{x_{ij}}{X_j}=\frac{\text{Rupee value of the product of industry } i \text{ used by industry } j.}{\text{Rupee value of the total product of industry } j.}$$

$=$ Rupee value of the out-put of industry i that industry j must purchase to produce one rupee worth of its own product.

$=a_{ij}$ (say)

In other words,

$x_{ij}=a_{ij}\ X_j$ amounts to saying that sales of industry i to industry j are a constant proportion a_{ij} of the output of industry j.

$=$ Rupee value of the product of industry i used by industry j.

Now we introduce the matrix of input coefficients.

$$A=\begin{bmatrix} a_{11} & a_{12}\dots a_{1n} \\ a_{21} & a_{22}\dots a_{2n} \\ \vdots & \vdots \\ a_{n1} & a_{n2}\dots a_{nn} \end{bmatrix}, \text{ where } a_{11}=\frac{X_{11}}{X_1}$$

$$a_{12}=\frac{X_{12}}{X_2}, \text{ etc.}$$

Replacing each x_{ij} by $a_{ij}\ X_j$ in the table, we get the set of simultaneous linear equations :

$$\begin{aligned} a_{11}\ X_1+a_{12}\ X_2+\dots+a_{1n}\ X_n+d_1&=X_1 \\ a_{21}\ X_1+a_{22}\ X_2+\dots+a_{2n}\ X_n+d_2&=X_2 \\ \vdots \qquad\qquad\qquad & \vdots \\ a_{n1}\ X_1+a_{n2}\ X_2+\dots+a_{nn}\ X_n+d_n&=X_n \end{aligned}$$

which may be written in the matrix form as

$$X=AX+D$$

$$\Rightarrow \qquad X-AX=D$$

$$\begin{aligned} \therefore \quad (1-a_{11})\ X_1-a_{12}\ X_2-a_{13}\ X_3-\dots-a_{1n}\ X_n&=d_1 \\ -a_{21}\ X_1-(1-a_{22})\ X_2-a_{23}\ X_3-\dots-a_{2n}\ X_n&=d_2 \\ \vdots \qquad\qquad\qquad & \vdots \\ -a_{n1}\ X_1-a_{n2}\ X_2-a_{n3}X_3-\dots-(1-a_{nn})\ X_n&=d_n \end{aligned}$$

In the matrix notation this may be written as :

$$\begin{pmatrix} 1-a_{11} & -a_{12}\dots & -a_{1n} \\ -a_{21} & 1-a_{22}\dots & -a_{2n} \\ \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2}\dots & 1-a_{nn} \end{pmatrix}\times\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}=\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$$\Rightarrow \qquad (I-A)\ X=D$$

$$\Rightarrow \qquad X=(I-A)^{-1}D$$

where I is the matrix of input coefficient, while X and D are the vectors of output and final demand of each industry.

---

*Leontief while developing the input-output analysis made the assumption of direct proportionality between the output and the individual inputs of the industry.

Example 111. *Given the following Transaction matrix, find the input-output coefficient :*

Purchasing sector Producing sector	Agriculture	Industry	Final demand
Agriculture	300	600	100
Industry	400	1200	400
Consumer	300	200	

Find also total output as well as total input.

Solution. Total output for Agriculture is

$300+ 600+100=1000$ and

for industry $400+1200+400=2000$

Similarly total input for Agriculture is

$300+ 400+300=1000$ and

for industry $600+1200+200=2000$

The above transaction can be put in the following way :

Purchasing sector output Producing sector input	Agriculture	Industry	Final demand	Total output
Agriculture	300	600	100	1000
Industry	400	1200	400	2000
Consumer	300	200	0	500
Total input	1000	2000	500	3500

Now to find out input-output coefficients :

A coefficient is obtained by industry's input by total output. It is an indication of the number of any industry's output needed to produce one unit of another industry's output.

Therefore, coefficient of input-output can be obtained as follows :

$$\frac{300}{1000}=0{\cdot}30\ ;\ \frac{600}{2000}=0{\cdot}30$$

$$\frac{400}{1000}=0{\cdot}40\ ;\ \frac{1200}{2000}=0{\cdot}60$$

$$\frac{300}{1000}=0{\cdot}30\ ;\ \frac{200}{2000}=0{\cdot}10$$

which can be represented as follows :

Purchasing sector output / *Producing sector input*	*Agriculture*	*Industry*
Agriculture	0·30	0·30
Industry	0·40	0·60
Consumer	0·30	0·10

Example 112. *Suppose the interrelationship between the production of two industries R and S in a given year is*

	R	*S*	*Current Consumer Demand*	*Total output*
R	*14*	*6*	*8*	*28*
S	*7*	*18*	*11*	*36*

If the forecast demand in two years is

$$D_2=\begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

What should be total output X be ?

Solution. Step I. To obtain the input-output matrix, we determine how much of each of the two products R and S is required to produce one unit of R. For example, to obtain 28 units of R requires the use of 14 units of R and 7 units of S (the entries in column one). Forming the ratios, we find that to produce 1 unit of R requires

$14/28=\frac{1}{2}$ of R, $7/28=\frac{1}{4}$ of S. If we want, say X_1 units of R, we will require $\frac{1}{2}X_1$ units of R, $\frac{1}{4}X_1$ units S.

Continuing in this way, we can construct the input-output matrix as follows :

$$A=\begin{array}{c} \\ R \\ S \end{array}\begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

It may be noted that column 1 represents the amounts R, S required for one unit of R, column 2 represents the amounts of R, S required for one units of S. For example, the entry in row 1, column 2 represents the amount of S needed to produce one unit of S.

As a result of placing the entries this way, if

$$\mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

represents the total output required to obtain a given demand, the product **AX** represents the amounts of R and S required for internal consumption. Here the total output is

$$\mathbf{X}=\begin{bmatrix} 28 \\ 36 \end{bmatrix}$$

The correctness of the values in **A** may be verified by noting that

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}\times\begin{bmatrix} 28 \\ 36 \end{bmatrix}=\begin{bmatrix} 20 \\ 25 \end{bmatrix}$$

where $\begin{bmatrix} 20 \\ 25 \end{bmatrix}$ represents the internal needs of R and S.

If the demand vector is

$$\mathbf{D}_0=\begin{bmatrix} 8 \\ 11 \end{bmatrix},$$

then for production to equal consumption, we must have

Internal needs + Consumer demand = Total output .. (*)

In terms of the input-output matrix **A**, the total output **X**, and the demand vector $\mathbf{D}_0$, (*) becomes

$$\mathbf{AX}+\mathbf{D}_0=\mathbf{X}.$$

Again, the correctness of this result may be verified since for the demand vector $\mathbf{D}_0$, we know the output is

$$\mathbf{X}=\begin{bmatrix} 28 \\ 36 \end{bmatrix}$$

To find the total output X, required to achieve a future demand

$$\mathbf{D}_2=\begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

we need to solve for X in

$$\mathbf{AX}+\mathbf{D}_2=\mathbf{X}$$

Simplifying, we have

$$(\mathbf{I}-\mathbf{A})\ \mathbf{X}=\mathbf{D}_2.$$

Solving for $\mathbf{X}$, we have

$$\mathbf{X}=(\mathbf{I}-\mathbf{A})^{-1}\ .\ \mathbf{D}_2.$$

$$=\begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}^{-1}\times\begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$=\frac{24}{5}\begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}\times\begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$=\frac{24}{5}\begin{bmatrix} 15 \\ 20 \end{bmatrix}=\begin{bmatrix} 72 \\ 96 \end{bmatrix}$$

Hence the total output of R and S for the forecast $\mathbf{D}_2$ is

$$X_1=72, \qquad X_2=96.$$

Example 113. *Given the following transaction matrix, find the gross output to meet the final demdnd of 200 units of Agriculture and 800 units of Industry.*

Producing Sector	*Purchasing Sector*		*Final Demand*
	Agriculture	*Industry*	
Agriculture	*300*	*600*	*100*
Industry	*400*	*1200*	*400*

Solution.

Producing sector	*Purchasing sector*		*Final Demand*	*Total Output*
	Agriculture	*Industry*		
Agriculture	300	600	100	1000
Industry	400	1200	400	2000

The input-output coefficients can be obtained as follows :

$$a_{11}=\frac{300}{1000}=\frac{3}{10},\quad a_{12}=\frac{600}{2000}=\frac{3}{10}$$

$$a_{21}=\frac{400}{1000}=\frac{2}{5},\quad a_{22}=\frac{1200}{2000}=\frac{3}{5}$$

∴ The technology matrix is

$$A=\begin{pmatrix} \frac{3}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

$$(I-A)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} \frac{3}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}=\begin{pmatrix} \frac{7}{10} & -\frac{3}{10} \\ -\frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

$$|I-A|=\frac{7}{10}\times\frac{2}{5}-\left(-\frac{2}{5}\right)\times\left(-\frac{3}{10}\right)=\frac{8}{50}=\frac{4}{25}$$

$$(I-A)^{-1}=\frac{25}{4}\begin{pmatrix} \frac{2}{5} & \frac{3}{10} \\ \frac{2}{5} & \frac{7}{10} \end{pmatrix}$$

Now $$X=(I-A)^{-1}\,D$$

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}=(I-A)^{-1}\,D=\frac{25}{4}\begin{pmatrix} \frac{2}{5} & \frac{3}{10} \\ \frac{2}{5} & \frac{7}{10} \end{pmatrix}\begin{pmatrix} 100 \\ 400 \end{pmatrix}$$

$$=\frac{25}{4}\begin{pmatrix} 160 \\ 320 \end{pmatrix}=\begin{pmatrix} 1000 \\ 2000 \end{pmatrix}$$

which verifies the given data.

The new demand vector is $D=\begin{pmatrix} 200 \\ 800 \end{pmatrix}$

Then

$$X=(I-A)^{-1}\ D=\frac{25}{4}\begin{pmatrix} \frac{2}{5} & \frac{3}{10} \\ \frac{2}{5} & \frac{7}{10} \end{pmatrix}\times\begin{pmatrix} 200 \\ 800 \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}=\frac{25}{4}\begin{pmatrix} 320 \\ 640 \end{pmatrix}=\begin{pmatrix} 2000 \\ 4000 \end{pmatrix}$$

Hence the Agriculture and Industry sector must produce 2000 and 4000 units to meet the final demand.

EXERCISES

1. The prices of 3 commodities *A*, *B* and *C* in a shop are Rs. 5, Rs. 6 and Rs. 10 respectively. Customer *X* buys 8 units of *A*, 7 units of *B* and 6 units of *C*. Customer *Y* buys 6 units of *A*, 7 units of *B* and 8

units of C. Show in matrix notation, the prices of the commodities, quantities bought and the amount spent.

2. Two types of food, 1 and 2 have a vitamin content in units per kg given by the following table :

	Vitamin A	*Vitamin B*
Food 1	3	7
Food 2	2	9

Express the vitamin content of 5 kg of food 1 and 6 kg of food 2 as a matrix product and evaluate it. If food 1 costs 30 paise per kg and food 2 costs 35 paise per kg, express the cost of 5 kg, 6 kg of foods 1, 2 respectively as a matrix product and evaluate it.

[**Hint.** $(5 \quad 6)\begin{pmatrix} 3 & 7 \\ 2 & 9 \end{pmatrix}=(27 \quad 89)$, *i.e.*, 27 units of vitamin A and 89 units of vitamin B.

$(5 \quad 6)\begin{pmatrix} 30 \\ 35 \end{pmatrix}=(360)$, *i.e.*, the cost is Rs. 3·60.]

3. A motor corporation has two types of factories each producing buses and trucks. The weekly production figures at each type of factory are as follows :

	Factory A	*Factory B*
Buses	20	30
Trucks	40	10

The corporation has 5 factories A and 7 factories B. Buses and trucks sell at Rs. 50,000 and Rs. 40,000 respectively. Express in matrix form and hence evaluate :

(*i*) The total weekly production of buses and trucks.

(*ii*) The total market value of vehicles produced each week.

[**Ans.** (*i*) $\begin{pmatrix} 20 & 30 \\ 40 & 10 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \end{pmatrix}=\begin{pmatrix} 310 \\ 270 \end{pmatrix}$, *i.e.*, 310 buses, 270 trucks

(*ii*) $(50000 \quad 40000). \begin{pmatrix} 20 & 30 \\ 40 & 10 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

$=(50000 \quad 40000) \cdot \begin{pmatrix} 310 \\ 270 \end{pmatrix}=(2,63,00,000)$,

i.e., the total weekly value = Rs. 2,63,00,000]

4. In a certain coal mine, the amounts of Grade 1 and Grade 2 coal (in tonnes) obtained per shift from each of two teams, A and B are given by the following table :

	Grade 1	*Grade 2*
Team A	4,000	2,000
Team B	1,000	3,000

Team A has worked 5 shifts per week and team B has worked 4 shifts per week. Grade 1 coal sells at Rs. 9 per tonne and Grade 2 coal sells at Rs. 8 per tonne. Find :

(*i*) the total amount of coal mined each week,

(*ii*) the market value of the coal mined each shift,

(*iii*) the market value of the coal mined each week.

[**Ans.** (*i*) (24,000 ; 22,000) tons of Grade 1 and Grade 2 respectively.

$$(ii)\ \begin{pmatrix} 52{,}000 \\ 33{,}000 \end{pmatrix},\ (iii)\ (5\ \ 4)\begin{pmatrix} 4{,}000 & 2{,}000 \\ 1{,}000 & 3{,}000 \end{pmatrix}\begin{pmatrix} 9 \\ 8 \end{pmatrix}\Bigg]$$

5. A builder develops a site by building 9 houses and 6 bungalows. On the average one house requires 16,000 units of materials and 2,000 hours of labour ; one bungalow requires 50,000 units of materials and 4,800 hours of labour. Labour costs Rs. 5 per hour and each unit of material costs, on the average Rs. 10. Express in matrix form and hence evaluate :

(*i*) The total materials and labour used in completing the site.

(*ii*) The cost of building a house and a bungalow.

(*iii*) The total cost of developing the site.

$$\Bigg[\textbf{Ans.}\ (i)\quad (9\ \ 6)\begin{pmatrix} 16{,}000 & 2{,}000 \\ 50{,}000 & 4{,}800 \end{pmatrix}$$

$$(ii)\quad \begin{pmatrix} 16{,}000 & 2{,}000 \\ 50{,}000 & 4{,}800 \end{pmatrix}\begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$(iii)\quad (9\ \ 6)\begin{pmatrix} 16{,}000 & 2{,}000 \\ 50{,}000 & 4{,}800 \end{pmatrix}\begin{pmatrix} 10 \\ 5 \end{pmatrix}\Bigg]$$

6. Two television companies, TV_1 and TV_2 both televise documentary programmes and variety programmes. TV_1 has two transmitting stations and TV_2 has three transmitting stations. All stations transmit different programmes. On an average the TV_1 stations broadcast 1 hour of documentary and 3 hours of variety programmes each day, whereas each TV_2 station broadcasts 2 hours of documentary and $1\frac{1}{2}$ hours of variety programmes each day. The transmission of documentary and variety programmes costs approximately Rs. 50 and Rs. 200 per hour respectively. Express in matrix form and hence evaluate :

(*i*) The daily cost of transmission from each TV_1 and each TV_2 station.

(*ii*) The total number of hours daily which are devoted to documentary and to variety programmes by both corporations.

(*iii*) The total daily cost of transmission incurred by both corporations.

[**Ans.** (*i*) $\begin{pmatrix} 1 & 3 \\ 2 & 1\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 50 \\ 200 \end{pmatrix} = \begin{pmatrix} 650 \\ 400 \end{pmatrix}$

i.e., Rs. 650, Rs. 400 per day respectively for each TV_1, TV_2 station.

(*ii*) $(2\ \ 3) \begin{pmatrix} 1 & 3 \\ 2 & 1\frac{1}{2} \end{pmatrix} = (8\ \ 10\frac{1}{2})$

i.e., 8 hours documentary and $10\frac{1}{2}$ hours variety.

(*iii*) Rs. $(2\ \ 3) \begin{pmatrix} 1 & 3 \\ 2 & 1\frac{1}{2} \end{pmatrix} \begin{pmatrix} 50 \\ 200 \end{pmatrix} =$ Rs. $(2\ 3) \begin{pmatrix} 650 \\ 400 \end{pmatrix} =$ Rs. 2,500]

7. A firm produces five qualities of its product which needs the following materials :

Quality	*Materials needed*			
	M_1	M_2	M_3	M_4
A_1	6	6	10	8
A_2	3	4	12	6
A_3	4	5	15	8
A_4	2	2	12	5
A_5	3	2	10	

If the firm has to produce, respectively, 3, 22, 20, 12 and 7 units of the five qualities find the amounts of different materials required by writing their requirements as a row vector.

[**Ans.** (169, 194, 658, 324)]

8. A publishing house has two branches. In each branch, there are three offices. In each office, there are 6 peons, 8 clerks and 10 typists. In one office of a branch, 12 salesmen are also working. In each office of other branch 4 head-clerks are also working. Using matrix notation find (*i*) the total number of posts of each kind in all the offices taken together in each branch, (*ii*) the total number of posts of each kind in all the offices taken together from both branches.

9. $$A = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix} \end{array}, \quad B = \begin{pmatrix} 4 & 6 & 8 \\ 10 & 12 & 14 \\ 16 & 18 & 20 \end{pmatrix}, C = \begin{pmatrix} 6 & 10 & 14 \\ 18 & 32 & 26 \\ 30 & 34 & 38 \end{pmatrix}$$

Matrix A shows the stock of 3 types of items I, II, III in three shops A_1, A_2, A_3. Matrix B shows the number of items delivered to three shops at the beginning of a week. Matrix C shows the number of items sold during that week. Using matrix algebra, find

(*i*) the number of items immediately after the delivery,

(*ii*) the number of items at the end of the week.

10. The following matrix gives the vitamin content of food items, in conveniently chosen units

$$\begin{array}{l} \text{Vitamin :} \\ \text{Food } I \\ \text{Food } II \\ \text{Food } III \end{array} \begin{pmatrix} A & B & C & D \\ {\cdot}5 & {\cdot}5 & 0 & 0 \\ {\cdot}3 & 0 & {\cdot}2 & {\cdot}1 \\ {\cdot}1 & {\cdot}1 & {\cdot}2 & {\cdot}5 \end{pmatrix}$$

If we eat 5 units of food I, 10 units of food II, and 8 units of food III, how much of each types of vitamin we have consumed ? If we pay only for the vitamin content of each food, paying 10 paise, 20 paise, 25 paise, 50 paise respectively for units of the four vitamins, how much does a unit of each type of food costs ? Compute the total cost of the food eaten.

$$[\textbf{Ans.}\ (6{\cdot}3\ \ 3{\cdot}3\ \ 3{\cdot}6\ \ 5{\cdot}0)\ ;\ \begin{bmatrix} 15 \\ 13 \\ 33 \end{bmatrix}\ ;\ \text{Rs. } 4{\cdot}69]$$

11. A manufacturing unit produces three types of products A, B, C. The following matrix shows the sale of products in two different cities.

$$\begin{pmatrix} A & B & C \\ 1200 & 900 & 600 \\ 900 & 600 & 300 \end{pmatrix}$$

If cost price of each product A, B, C is Rs. 1000, Rs. 2000, Rs. 3000 respectively and selling price Rs. 1500, Rs. 3000, Rs. 4000 respectively, find the total profits using matrix algebra only.

12. The production of a book involves several steps : first it must be set in type, then it must be printed and finally it must be supplied with covers and bound. Suppose that type setter charges Rs. 6 per hour, paper costs $\frac{1}{4}$ paisa per sheet, that the printer charges 11 paise for each minute that his press runs, that the cover costs 28 paise, and the binder charges 15 paise to bind each book. Suppose now that a publishers wishes to print a book that requires 300 hours of work by the typesetter, 220 sheets of paper per book and five minutes of press time per book.

(*i*) Using matrix multiplication, find the cost of publishing one copy of a book.

(*ii*) Using matrix addition and multiplication find the cost of printing a first edition run of 5000 copies.

(*iii*) Assuming that the type plates from the first edition are used again, find the cost of printing a second edition of 5000 copies.

[**Ans.** (*i*) Rs. 1801·53, (*ii*) Rs. 9450, (*iii*) Rs. 7650]

13. One unit of commodity A is produced by combining 1 unit of land, 2 units of labour and 5 units of capital. One unit of B is produced by 2 units of land, 3 units of labour and 1 unit of capital. One unit of commodity C results if we use 3 units of land, 1 unit of labour and 2 units of capital. Assume that the prices are $P_a=27$, $P_b=16$ and $P_c=19$. Find the rent R, wage W and rate of interest I. (Use matrix method).

14. To control a certain crop disease it is necessary to use 7 units of chemical A, 10 units of chemical B, and 6 units of chemical C. One barrel of spray P contains 1, 4, 2 units of the chemicals, one barrel of spray Q contains 3, 2, 2 units and one barrel of spray R contains 4, 3, 2 units of these chemicals respectively. How much of each type of spray be used to control the disease ?

[**Ans.** $1\frac{1}{2}$ barrels of spray P, $\frac{1}{2}$ barrel of spray Q and one barrel of spray R]

15. A certain company gets the automobile chassis and then builds 3 types of bodies, *viz.*, luxury coaches, ordinary passenger bus and lorries. For a luxury coach 5 supervisors and 20 skilled labourers, for a passenger bus 3 and 12, for a lorry 2 and 11 of these categories, are required for a day's work. If 50 supervisors and 260 skilled labourers are available how many coaches, buses and lorries could be built ?

16. A firm manufactures 3 products P, Q, R using 20 machines of type L, 12 machines of type M and 15 machines of type N. If the machinery time requirements are given in the following table, find the production quantity of each product during a 40-hour week.

	Machines		
Product	L	M	N
P	3 hr.	2 hr.	4 hr.
Q	2 hr.	1 hr.	2 hr.
R	4 hr.	3 hr.	1 hr.

[**Ans.** 16 units of P, 232 units of Q and 72 units of product R.]

17. In a market survey three commodities A, B and C were considered. In finding out the index number some fixed weights were assigned to three varieties in each of the commodities. The table below provides the information regarding the consumption of three commodities according to three varieties and also the total weight received by the commodity :

Commodity	*Variety*			*Total Weight*
	I	*II*	*III*	
A	1	2	3	11
B	2	4	5	21
C	3	5	6	27

Find the weights assigned to the three varieties by using Matrix-inverse method, given that the weights assigned to a commodity are equal to the sum of the weights of the various varieties multiplied by the corresponding consumption. [**Ans.** 2, 3, 1]

18. The monthly expenditure in an office for three months is given below according to the type of staff employed :

Month	*No. of Employees*			*Total monthly salary*
	Clerks	*Typists*	*Peons*	*(Rs.)*
April	4	2	3	4,900
May	3	3	2	4,500
June	4	3	4	5,800

Assuming that the salary in all the three months of different categories of staff did not vary, calculate the salary for each type of staff per mensum using the matrix method. [**Ans.** 700, 600, 300]

19. The following table shows the fixed cost (F) and the variable cost (V) of producing 1 unit of X and 1 unit of Y

		Product		
		X	*Y*	
Cost	*F*	5	8	(Rs. '000)
	V	4	12	

When x units of X and y units of Y are produced, the total fixed cost is Rs. 6,40,000 and total variable cost is Rs. 8,20,000. Express this information as a matrix equation and hence find the quantities of X and Y produced. [**Ans.** $x=40$, $y=55$]

20. A salesman has the following record of sales during three months for three items A, B and C which have different rates of commission.

Months	*Sales of Units*			*Total Commission drawn* (in Rs.)
	A	*B*	*C*	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rates of commission on items *A*, *B* and *C*.
[**Ans.** Rs. 2, 4 and 11]

21. (*a*) We consider buying three kinds of food. Food I has one unit of vitamin *A*, three units of vitamin *B* and four units of vitamin *C*. Food II has two, three and five units respectively. Food III has three units each of vitamin *A* and vitamin *C* and none of vitamin *B*. We need to have 11 units of vitamin *A*, 9 of *B* and 20 of *C*. Find all possible amounts of the three foods that will provide precisely these amounts of the vitamins.

(*b*) One unit of food I contains 100 units of vitamins, 60 units of minerals and 80 calories. One unit of food II contains 150 units of vitamins, 60 units of minerals and 180 calories. One unit of food III contains 90 units of vitamins, 40 units of minerals and 100 calories. Diet requirement for a patient is 1100 units of vitamins, 500 units of minerals and 1200 calories. Find out *either* by matrix method *or* by determinants method how many units of each food be mixed to form the diet which would meet the requirements exactly.

22. An automobile manufacturer uses three different types of trucks T_1, T_2 and T_3, to transport the number of station wagons, full size and intermediate size cars as shown in the following matrix :

$$\begin{array}{ll|ccc|} & & \textit{Station Wagons} & \textit{Full-size Cars} & \textit{Intermediate-size Cars} \\ & T_1 & 2 & 6 & 9 \\ \textbf{Trucks} & T_2 & 3 & 7 & 12 \\ & T_3 & 6 & 6 & 8 \end{array}$$

Using the inverse of the matrix, determine the number of trucks of each type required to supply 58 station wagons, 75 full-size, and 62 intermediate-size cars to a dealer in city *A*.

If a dealer in city *B* orders 46 station wagons, 60 full-size and 64 intermediate-size cars, how many trucks of each type does the factory need to make this delivery.

[**Ans.** $A^{-1}=\frac{1}{40}\begin{bmatrix} -16 & 5 & 9 \\ 48 & -38 & 3 \\ -24 & 24 & -4 \end{bmatrix}$,

City A : Station wagons 2 ; fullsize cars 3 ; Intermediate cars 4

City B : ,, ,, 5 ; ,, 3 ; ,, ,, 2]

23. For the following input-output table, calculate the technolog matrix and also write the balance equation for the two sectors :

Sector	*A*	*B*	*Final demand*
A	50	150	200
B	100	75	100

24. Suppose the interrelationships between the production of two industries P and Q in a given year is

	P	*Q*	*Current Consumer Demand*	*Total output*
P	30	40	60	130
Q	20	10	40	70

If the forecast demand in two years is

$$D_2=\begin{bmatrix} 80 \\ 40 \end{bmatrix}$$

what should the total output X be ?

25. The following table gives the input-output coefficients for a two-sector economy consisting of agriculture and manufacturing industry.

Input-output Coefficient

Input *Industry*	*A*	*M*
A	0·10	1·50
M	0·20	0·25

The final demands for the two industries are 300 and 100 units respectively. Find the gross outputs of the two industries.

If the input coefficients for the labour for two industries are respectively 0·5 and 0·6, find the total units of labour required.

26. Consider an oversimplified two sector economy in which there are two industries, each producing a single commodity. The production of Re. one worth of the first industry's product requires material worth of 30 paisa of the first industry and 20 paisa of the second industry. The production of the second industry's product worth Re. one requires 10 paisa and 30 paisa material of the first and second industries respectively. Determine the output levels of each industry necessary to meet the open sector demand of Rs. 12 million and Rs. 5 million worth of goods of the first and second industries respectively.

[**Ans.** 20, 10]

27. In an economy there are two industries A and B and the following table gives the supply and demand position of these in million rupees :

		User		*Final*	*Total*
		A	*B*	*Demand*	*Output*
Producer	*A*	15	10	10	35
	B	20	30	15	65

Determine the total output if the final demand changes to 12 for A and 18 for B. **[Ans.** 42, 78]

28. In an economy of three industries A, B, C the data is given below (in millions of rupees of products).

		User			*Final*	*Total*
		A	*B*	*G*	*demand*	*output*
	A	80	100	100	40	320
Producer	*B*	80	200	60	60	400
	C	80	100	100	20	300

Determine the output if the final demand changes to

(*i*) 10 for A, 40 for B, 20 for C.

(*ii*) 60 for A, 40 for B, 60 for C.

[Ans. (*i*) 179·13, 245·22, 189·13 ; (*ii*) 417·39, 455·65, 417·39]

29. Suppose that the final demands for steel, coal and electricity in an economy consisting only of these three sectors are Rs. 10 crores, Rs. 5 crores and Rs. 6 crores respectively. It is given that a Rupee worth of steel requires 20 paise, 40 paise and 10 paise worth of steel, coal and electricity respectively as inputs, a Rupee worth of coal requires 30 paise, 10 paise and 30 paise worth of steel, coal and electricity respectively as inputs and that a Rupee worth of electricity requires 20 paise worth of steel, coal and electricity each as inputs respectively. How much of steel, coal and electricity should be produced to satisfy both final and intermediate demands ?

[Hint. Matrix of input-output coefficients is

$$A=\begin{bmatrix} 0{\cdot}20 & 0{\cdot}40 & 0{\cdot}10 \\ 0{\cdot}30 & 0{\cdot}10 & 0{\cdot}30 \\ 0{\cdot}20 & 0{\cdot}20 & 0{\cdot}20 \end{bmatrix}$$

30. A pharmaceutical company produces three products X, Y and Z which are partially used in the manufacture of these products. However, none of the products is used in its own manufacture. The quantities of the outputs of each product which are used as inputs in the manufacture of one unit of each of the other products are :

		Input		
		X	Y	Z
	X	0	0·3	0·4
Output	Y	0·2	0	0·3
	Z	0·1	0·5	0

The production targets for each product are Rs. 1,50,000 for X, Rs. 2,00,000 for Y and Rs. 1,00,000 for Z, these being the amounts of the three products which are to reach the final consumer. Use input-output analysis to determine how much of each of the products should be produced.

31. From the following matrix, find out the final output goals of each industry assuming that consumer output targets are Rs. 80 million in steel, Rs. 30 million in coal and Rs. 50 million in railway transport :

	Steel	*Coal*	*Railway transport*
Steel	0·3	0·2	0·2
Coal	0·2	0·1	0·5
Railway transport	0·2	0·4	0·2
Labour	0·3	0·3	0·1

What would be the labour requirements in final output of three industries ?

[**Hint.** $\therefore\ [1-A]=\begin{bmatrix} +0\cdot7 & -0\cdot2 & -0\cdot2 \\ -0\cdot2 & +0\cdot9 & -0\cdot5 \\ -0\cdot2 & -0\cdot4 & +0\cdot8 \end{bmatrix}$

Substituting in $X=[I-A]^{-1}D$, we get

$$X=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}=\begin{bmatrix} 0\cdot7 & -0\cdot2 & -0\cdot2 \\ -0\cdot2 & 0\cdot9 & -0\cdot5 \\ -0\cdot2 & -0\cdot4 & 0\cdot8 \end{bmatrix}^{-1}\times\begin{bmatrix} 80 \\ 30 \\ 50 \end{bmatrix}$$

After inverting the matrix, we get the required result.]

32. D Limited produces three products, x, y and z on three different types of machine installed in three departments A, B and C. The departmental monthly capacity is limited to :

Department	*Machine hours*
A	1,800
B	2,100
C	1,300

The machines are purpose built and each type can perform specialised task only.

The three products are proposed in all three departments but take varying amounts of time in each as follows :

		Departments	
Products	*A*	*B*	*C*
		Hours per unit	
x	2	6	1
y	2	1	3
z	3	2	2

The production controller has been instructed to obtain the fullest possible utilisation of all machines.

Calculate the number of units of products x, y and z to be produced in order to fill the capacity of all three departments for the month.

[**Ans.** $x=200$, $y=100$, $z=400$]

33. The prices of the three commodities X, Y and Z are x, y and z per unit respectively. A purchases 4 units of Z and sells 3 units of X and 5 units of Y. B purchases 3 units of Y and sells 2 units of X and 1 unit of Z. C purchases 1 unit of X and sells 4 units of Y and 6 units of Z. In the process A, B, C earn Rs. 6,000, Rs. 5,000 and Rs. 13,000 respectively. Using matrices, find the prices per unit of the three commodities. (Note that selling the units is positive earnings and buying the units is negative earnings).

[**Hint.** The above data can be written in the form of simultaneous equations as

$$3x+5y-4z=6{,}000$$
$$2x-3y+z=5{,}000$$
$$-x+4y+6z=13{,}000$$

and the equations can be written in the matrix form as

$$\begin{pmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & 4 & 6 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6{,}000 \\ 5{,}000 \\ 13{,}000 \end{pmatrix}$$

$$\Rightarrow \quad AX=B \quad \Rightarrow \quad X=A^{-1}B$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & 4 & 6 \end{pmatrix}^{-1} \times \begin{pmatrix} 6{,}000 \\ 5{,}000 \\ 13{,}000 \end{pmatrix}$$

$$= -\frac{1}{151} \begin{pmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -7 & -19 \end{pmatrix} \begin{pmatrix} 6{,}000 \\ 5.000 \\ 13{,}000 \end{pmatrix} = \begin{pmatrix} 3{,}000 \\ 1{,}000 \\ 2{,}000 \end{pmatrix}$$

Hence $x=3{,}000$; $y=1{,}000$ and $z=2{,}000$]

SECTION B

Linear Programming

> **"LP is only one aspect of what has been called a systems approach to management where all programmes are designed and evaluated in terms of their ultimate effects in the realisation of business objectives."**
>
> *N. Paul Loomba*

INTRODUCTION

The central theme of economic theory and management science is to optimise the use of scarce resources which include machine, man-power, money, warehouse space or raw material. There are several theoretical tools to accomplish this purpose in both the sciences. But such tools are not adequate for treating a complex economic problem with several alternatives each with its own restrictions and limitations. It is for tackling such problems that the use of linear programming has been found to be most useful. The technique was first invented by the Russian Mathematician L. V. Kantorovich and developed later by George B. Dantzig, the Simplex method is particularly associated with his name.

MEANING

Linear programming is a method or technique of determining an optimum programme of inter-dependent activities in view of available resources. In other words, it is a technique of allocating limited resources in an optimum manner so as to satisfy the laws of supply and demand for the firm's products. In general, Linear Programming is a mathematical technique for determining the optimal allocation of resources and obtaining a particular objective (*i.e.*, cost minimization or inversely profit maximization when there are alternative uses of the resources : Land, Labour, Capital, Materials, Machines, etc.

Programming is just another word for "planning" and refers to the process of determining a particular plan of action from amongst several alternatives. The word *linear* stands for indicating that all relationships involved in a particular problem are of degree one.

APPLICATIONS

The use of LP is made in regard to the problems of allocation, assignment, transportation, etc. But the most important of these is that of allocation of scarce resources on which we shall concentrate. Some allocation problems are as follows :

1. Devising of a production schedule that could satisfy future demands (seasonal or otherwise) for the firm's product and at the same time minimise production (including inventory) costs.

2. Choice of investment from a variety of shares and debentures so as to maximise return on investment.

3. Allocation of a limited publicity budget on various heads in order to maximise its effectiveness.

4. Selection of the product-mix to make the best use of machines, man-hours with a view to maximise profits.

5. Selecting the advertising mix that will maximise the benefit subject to the total advertising budget, Linear Programming can be effectively applied.

6. Determine the distribution system to minimise transport costs from several warehouses to various market places.

Three Typical Problems. Three problems have become classical illustrations in linear programming.

A. The Diet Problem

It is the problem of deciding how much of 'n' different foods to include in a diet, given the cost of each food, and the particular combination of nutrient each food contains. The object is to minimise the cost of diet such that it contains a certain minimum amount of each nutrient.

B. Optimal Product Lines Problem

How much of 'n' different products a firm should produce and sell, when each product requires a particular combination of labour, machine time and warehouse space per unit of output and where there are fixed limits on the amounts of labour, machine time and warehouse space available ?

C. Transportation Problem

It is a problem of determining a shipping schedule for a commodity, say, steel or oil, from each of a number of plants (or oil-fields) at different locations to each of a number of markets (or refineries) at different locations in such a way as to minimise the total shipping cost subject to the constraints that (1) the demand at each market (refinery) will be satisfied, and (2) the supply at the plant (oil field) will not be exceeded.

General Linear Programming Problem

Let Z be a linear function defined by

(*i*) $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

where c 's are constants.

(*ii*) Let (a_{ij}) be mn constants and let (b_i) be a set of m constants such that

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \ (\leqslant, =, \geqslant)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ (\leqslant, =, \geqslant)\ b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \ (\leqslant, =, \geqslant)\ b_m$$

and finally let

(*iii*) $x_1 \geqslant 0,\ x_2 \geqslant 0,\ \dots,\ x_n \geqslant 0.$

The problem of determming the values of $x_1, x_2, \dots, x_n$ which makes Z *a* minimum (or maximum) and which satisfies (*ii*) and (*iii*) is called the *General Linear Programming Problem*.

(*a*) **Objective function.** The linear function

$$Z=c_1x_1+c_2x_2+\ldots+c_nx_n$$

which is to be minimized (or maximized) is called the *Objective function* of the general L. P. P.

(*b*) **Constraints.** The inequalities (*ii*) are called the *constraints* of the General L.P.P.

(*c*) **Non-negative restrictions.** The set of inequalities (*iii*) is usually known as the set of *non-negative restrictions* of the General L.P.P.

(*d*) **Solution** Values of unknowns $x_1, x_2, \ldots, x_n$ which satisfy the constraints of a General L.P.P. is called a *solution* to the General L.P.P.

(*e*) **Feasible Solution.** Any solution to a General L.P.P. which satisfies the non-negative restrictions of the problem, called *feasible solution* to the General L.P.P.

(*f*) **Optimum Solution.** Any feasible solution which optimizes (minimizes or maximizes) the objective function of a General L.P.P. is called an *optimum solution* to the general L.P.P.

Example 1. *A manufacturing firm has discontinued production of a certain unprofitable product line, and this has created considerable excess production capacity. Management is considering to devote this excess capacity to produce one or more of three products 1, 2 and 3. The available excess capacity on the machines which might limit output, is summarised in the following table :*

Machine type	*Available excess capacity (in machine hours per week)*
Milling machine	250
Lathe	150
Grinder	50

The number of machine-hours requires for each unit of the respective product is given below :

	Capacity Requirement (in machine-hours per unit)		
Machine Type	*Product 1*	*Product 2*	*Product 3*
Milling machire	8	2	3
Lathe	4	3	0
Grinder	2	0	1

The per unit contribution would be Rs. 20, Rs. 6 and Rs. 8 respectively for products 1, 2 and 3. Formulate the problem mathematically.

Solution. Step 1. Let the number of units of the products 1, 2 and 3 manufactured be designated by x_1, x_2 and x_3 respectively.

Step 2. Since it is not possible to manufacture any negative quantities, it is quite obvious that in the present situation feasible alternatives are sets of values of x_1, x_2, x_3 satisfying $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$.

Step 3. The objective here is to maximize the profit, which is given by the linear function :

$$(\text{maximize})\ Z = 20x_1 + 6x_2 + 8x_3$$

Step 4. Next we express in words the influencing factors or constraints (or restrictions) which occur generally because of the constraints on availability (resources) or requirements (demands). Here in order to produce x_1 units of product 1, x_2 units of product 2 and x_3 units of product 3, the total time needed on Milling machine, Lathe, and Grinder are given by

$$8x_1 + 2x_2 + 3x_3,\ 4x_1 + 3x_2 \text{ and } 2x_1 + x_3$$

Since the manufacturer does not have more than 250 hours available on Milling machine, 150 hours available on the Lathe and 50 hours available on the Grinder, we must have

$$\begin{aligned} 8x_1 + 2x_2 + 3x_3 &\leqslant 250 \\ 4x_1 + 3x_2 \quad &\leqslant 150 \\ 2x_1 \quad + x_3 &\leqslant 50 \end{aligned}$$

Hence the manufacturing firm problem can be put in the following mathematical form :

Determine three real numbers x_1, x_2 and x_3 such that

$$\begin{aligned} 8x_1 + 2x_2 + 3x &\leqslant 250 \\ 4x_1 + 3x_2 \quad &\leqslant 150 \\ 2x_1 \quad + x_3 &\leqslant 50 \\ x_1,\ x_2,\ x_3 &\geqslant 0 \end{aligned}$$

and for which the expression (objective function)

$$Z = 20\,x_1 + 6x_2 + 8x_3$$

may be maximum.

Example 2. Production Scheduling Problem. *A company is manufacturing two products A and B. The manufacturing times required to make them, the profit and capacity available at each work centre are given by the following table :*

Work Centre / Product	Matching	Fabrication	Assembly	Profit per unit (in Rs.)
A	1 hour	5 hours	3 hours	80
B	2 hours	4 hours	1 hour	100
Total Capacity	720 hours	1800 hours	900 hours	

Formulate the L.P. model.

Solution. Step I. The key decision to be made is to determine the number of units of product A and B to be produced by the company.

Step II. Let x_1 be the number of units of product A and x_2, the number of units of Product B which the company decides to produce.

Step III The total profit that the manufacturer gets after selling the two products A and B is given by

$$Z=80\ x_1+100\ x_2$$

Step IV. Now, in order to produce these two products A and B, the total number of hours required at matching centre is given by

$$x_1+2x_2$$

The total number of hours required at fabrication centre is

$$5x_1+4x_2$$

and the total number of hours required at assembly centre is given by

$$3x_1+x_2$$

Since the matching centre is not available for more than 720 hours, fabrication centre is available only for 1800 hours and assembly centre is available only for 900 hours, we have

$$x_1+2x_2\leqslant 720$$

$$5x_1+4x_2\leqslant 1800$$

$$3x_1+x_2\leqslant 900$$

Step V. Also, since it is not possible for the manufacturer to produce negative number of the products, it is obvious that we must also have

$$x_1\geqslant 0 \text{ and } x_2\geqslant 0$$

Step VI. The above allocation problem of the manufacturer can be mathematically expressed as follows :

Find two real numbers, x_1 and x_2 such that

$$x_1+2x_2 \leqslant 720 \quad \ldots(1)$$

$$5x_1+4x_2 \leqslant 1800 \quad \ldots(2)$$

$$3x_1+x_2 \leqslant 900 \quad \ldots(3)$$

$$x_1, x_2 \geqslant 0$$

and for which the expression (objective function)

$$Z=80x_1+100x_2$$

may be maximum (greatest)

Example 3. *A company produces three products P, Q and R from three raw materials A, B and C. One unit of product P requires 2 units of A and 3 units of B. One unit of product Q requires 2 units of B and 5 units of C and one unit of product R requires 3 units of A, 2 units of B and 4 units of C. The company has 8 units of material A, 10 units of material B and 15 units of material C available to it. Profits per unit of products P, Q and R are Rs. 3, Rs. 5 and Rs. 4 respectively.*

Formulate the problem mathematically.

Decision variables	*Product*	*Type of raw material* A	B	C	*Profit per unit* (Rs.)
x_1	*P*	2	3	—	3
x_2	*Q*	—	2	5	5
x_3	*R*	3	2	4	4
Units of material available :		8 maximum	10 maximum	15 maximum	

x_1=number of units of Product *P*

x_2=number of units of Product *Q*

x_3=number of units of Product *R*

The given problem is formulated as the *LPP* as follows :

$$\text{Maximize } Z=3x_1+5x_2+4x_3$$

Subject to the constraints :

$$\begin{aligned} 2x_1 \quad\; +3x_3 &\leqslant 8 \\ 3x_1+2x_2+2x_3 &\leqslant 10 \\ 5x_2+4x_3 &\leqslant 15 \\ x_1, x_2, x_3 &\geqslant 0. \end{aligned}$$

Example 4. *A diet conscious housewife wishes to ensure certain minimum intake of vitamins A, B and C for the family. The minimum* *daily (quantity) needs of the vitamins A, B, C for the family are respectively 30, 20 and 16 units. For the supply of these minimum vitamin requirements, the housewife relies on two fresh foods. The first one provides 7, 5, 2 units of the three vitamins per gram respectively and the second one provides 2, 4, 8 units of the same three vitamins per gram of the foodstuff respectively. The first foodstuff costs Rs. 3 per gram and the second Rs. 2 per gram. The problem is how many grams of each foodstuff should the housewife buy everyday to keep her food bill as low as possible ?*

Formulate the underlying L.P. problem.

Solution. Step 1. By designating the number of units of foods X and Y by x_1 and x_2 respectively, the data of the given problem can be summarized as below :

Decision variables	*Food*	*Content of vitamins type*			*Cost per unit*
		A	*B*	*C*	(Rs)
x_1	*P*	7	5	2	3
x_2	*Q*	2	4	8	2
Minimum vitamins required		30	20	16	

x_1= number of units of food P

x_2= number of units of food Q

Step 2. Here the objective is to minimize the cost and, therefore, the objective function is

$$Z=3x_1+2x_2$$

As the minimum required amounts of vitamins A, B and C are 30, 20 and 16 respectively, the constraints of the problem are :

$$7x_1+2x_2 \geqslant 30 \; ; \; 5x_1+4x_2 \geqslant 20 \; ; \; 2x_1+8x_2 \geqslant 16$$

Thus the given LP problem is :

Minimize :

$$Z=3x_1+2x_2$$

Subject to the constraints :

$$7x_1+2x_2 \geqslant 30$$
$$5x_1+4x_2 \geqslant 20$$
$$2x_1+8x_2 \geqslant 16$$
$$x_1, x_2 \geqslant 0$$

Example 5. *A city hospital has the following minimal daily requirements for nurses :*

Period	*Clock Time (24 hour day)*	*Minimal Number of Nurses Required*
1	*6 A.M. — 10 A.M.*	*2*
2	*10 A.M. — 2 P.M.*	*7*
3	*2 P.M. — 6 P.M.*	*15*
4	*6 P.M. — 10 P.M.*	*8*
5	*10 P.M. — 2 A.M.*	*20*
6	*2 A.M. — 6 A.M.*	*6*

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimal number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this as a Linear Programming Problem by setting up appropriate constraints and objective function. Do not solve.

Solution. Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number of nurses commencing duty at 6 A.M., 10 A.M.,..., 10 P.M., 2 A.M. respectively.

(*i*) *Requirement Constraints.* Between 10 A.M. and 2 P.M.. the nurses who start work at 6 A.M. (x_1) as well as those who start work at 10 A.M. (x_2) will be available. Since the requirement of nurses during this interval is 7,

$$x_1+x_2\geqslant 7$$

Similarly

$$x_2+x_3\geqslant 15$$

$$x_3+x_4\geqslant 8$$

$$x_4+x_5\geqslant 20$$

$$x_5+x_6\geqslant 6$$

$$x_6+x_1\geqslant 2$$

$$x_1, x_2, x_3, x_4, x_5, x_6\geqslant 0$$

Objective Function : To minimise

$$Z=x_1+x_2+x_3+x_4+x_5+x_6$$

EXERCISES

1. A small manufacturing firm produces two types of gadgets, *A* and *B*, which are first processed in the foundry, then sent to the machine shop for finishing. The number of man-hours of labour required in each shop for the production of each unit of *A* and of *B*, and the number of man-hours the firm has available per week are as follows :

	Foundry	*Machine Shop*
Gadget A	10	5
Gadget B	6	4
Firm's capacity per week	1000	600

The profit on the sale of *A* is Rs. 30 per unit as compared with Rs. 20 per unit of *B*.

The problem is to determine the weekly production of gadgets *A* and *B*, so that total profit is maximized.

[**Hint.** Determine two unknown variables x_1 and x_2, such that

(*i*) $10x_1+6x_2 \leqslant 1000$ (Foundry constraint)

(*ii*) $5x_1+4x_2 \leqslant 600$ (Machine shop constraint)

(*iii*) $x_1, x_2 \geqslant 0$ (non-negativity constraint)

and for which the expression (objective function)

$$Z=30\ x_1+20\ x_2$$

may be a maximum (greatest).]

2. The *ABC* Electric Appliance Company produces two products : refrigerators and ranges. Production takes place in two separate departments. Refrigerators are produced in Department *I* and ranges are produced in department *II*. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators in Department *I* and 35 ranges in Department *II*, because of the limited available facilities in these two departments. The company regularly employs a total of 60 workers in the two departments. A refrigerator requires 2 man-weeks of labour, while a range requires one man-week of labour. A refrigerator contributes a profit of Rs. 60 and a range. a profit of Rs. 40.

The problem is to determine the weekly production of refrigerators and ranges, so that total contribution is maximised.

Formulate the above problem as a linear programming problem

[**Ans** Maximize $Z=60\ x_1+40\ x_2$, subject to the constraints $2\ x_1+x_2 \leqslant 60$; $x_1 \leqslant 25$; $x_2 \leqslant 35$; $x_1,\ x_2 \geqslant 0$, where x_1 and x_2 be the number of units of refrigerators and ranges respectively.]

3. Three products are processed through three different operations. The time (in minutes) required per unit of each product, the daily capacity of the operations (in minutes per day) and the profit per unit sold for each product (in rupees) are as follows :

Operation	*Time per unit (minutes)*			*Operation Capacity (minutes day)*
	Product I	*Product II*	*Product III*	
1	3	4	3	43
2	5	0	4	46
3	3	6	2	42
Profit unit (Rs)	2	2	3	

The zero times indicate the product does not require the given operation. It is assumed that all units produced are sold. Moreover, the given profits per unit are net values that result after all pertinent expenses are deducted. The problem is to determine the optimum daily production for three products that maximizes the profit.

Formulate the above production planning problem in a linear programming format.

[**Hint.** Find the real numbers x_1, x_2, x_2 so as to maximize

$$Z=2x_1+2x_2+3x_3$$

subject to the constraints

$$\begin{aligned}3x_1+4x_2+3x_3&\leqslant 43\\5x_1\qquad+4x_3&\leqslant 46\\3x_1+6x_2+2x_3&\leqslant 42\end{aligned}$$

with restrictions

$$x_1,\ x_2,\ x_3\geqslant 0]$$

4. Vitamins *A* and *B* are found in food F_1 and F_2. One unit of food F_1 contains 20 units of vitamin *A* and 30 units of vitamin *B*. One unit of food F_2 contains 60 units of vitamin *A* and 40 units of vitamin *B*. 1 unit of each of foods F_1 and F_2 cost Rs 3 and Rs. 4 respectively. The minimum daily requirement (for a person) of vitamins *A* and *B* is 80 units and 100 units respectively. Assuming that anything in excess of daily minimum requirements of vitamins *A* and *B* is not harmful, find out the optimum mixture of foods F_1 and F_2 at the minimum cost which meets the daily minimum requirements of vitamins *A* and *B*

Formulate the above problem as a linear programming problem.

[**Hint.** Find two real numbers *x* and *y*, such that

$$\begin{aligned}20x+60y&\geqslant 80\\30x+40y&\geqslant 100\\x,\ y&\geqslant 0\end{aligned}$$

and for which the expression (objective function)

$$z=3x+4y$$

may be a minimum (least)]

5. A feed mixing company purchases and mixes one or more of the three types of grain, each containing different amounts of four nutritional elements ; the data is given below :

Item	*One unit weight of*			*Minimum total requirement over plannihg horizon*
	Grain I	*Grain 2*	*Grain 3*	
Nutritional ingredient *A*	2	4	6	$\geqslant$ 125
Nutritional ingredient *B*	0	2	5	$\geqslant$ 24
Nutritional ingredient *C*	5	1	3	$\geqslant$ 80
Cost per unit weight (Rs.)	25	15	18	Minimize

The production manager specifies that any feed mix for his livestock meet at least minimal nutritional requirements, and he seeks the least costly among all such mixes. Suppose his planning horizon is a two-week period, *i.e.*, he purchases enough to fill his needs for two weeks.

Formulate the above problem as a linear programming problem.

[**Ans.** Find three real numbers x_1, x_2, x_3 so as to minimise

$$Z=25x_1+15x_2+18x_3$$

subject to the constraints :

$$2x_1+4x_2+6x_3 \geqslant 125$$
$$2x_2+5x_3 \geqslant 24$$
$$5x_1+x_2+3x_3 \geqslant 80$$

and $$x_1, x_2, x_3 \geqslant 0]$$

6. The *XYZ* Company Ltd. manufactures two products *A* and *B*. These products are processed on the same machine. It takes 20 minutes to process one unit of product *A* and 15 minutes for each unit of product *B* and machine operates for a maximum of 80 hours in a week. Product *A* requires 3 kg and product *B*, 2 kg of the raw material per unit, the supply of which is 1200 kg per week. Market constraint on product *B* is known to be 1500 units every week.

If the product *A* costs Rs. 10 per unit and can be sold at a price of Rs. 15, product *B* costs Rs. 15 per unit and can be sold in the market at a unit price of Rs. 22 ; the problem is to find out the number of units of *A* and *B* that should be produced per week in order to maximize the profit potentially.

Formulate this problem in the standard linear programming format. Do not solve it.

7. A firm manufactures 3 products *A*, *B* and *C*. The profits are Rs. 6, Rs. 4 and Rs. 8 respectively. The firm has 2 machines and below is the required processing time (in minutes) for each machine on each product :

Machine	*Products*		
	A	*B*	*C*
X	8	6	10
Y	4	4	8

Machine *X* and *Y* have 4,000 and 5,000 machine minutes respectively. The firm must manufacture 200 *A*'s, 400 *B*'s and 100 *C*'s but no more than 300 *A*'s.

Set up a L.P. problem to maximise profit. Do not solve it.

[**Hint.** Find the real numbers x_1, x_2 and x_3 so as to maximize

$$Z=6x_1+4x_2+8x_3$$

subject to the constraints

$$8x_1+6x_2+10x_3\leqslant 4{,}000$$
$$4x_1+4x_2+8x_3\leqslant 5{,}000$$

with restrictions

$$200\leqslant x_1\leqslant 300$$
$$x_2\geqslant 400$$
$$x_3\geqslant 100.]$$

8. The manager of a company, which supplies office furniture, has asked you to prepare a profit maximizing schedule for their production of desks. This particular company sells a basic line of four desks. (Type *A*, Type *B*, Type *C* and Type *D*) to local distributors at the prices given below. Costs of producing each type are also given :

Desk Type	*Selling Price (In Rupees)*	*Production cost (In Rupees)*
A	28	21
B	35	30
C	52	39
D	72	54

For short-run scheduling, labour must be considered a fixed quantity and desks production is a two-step process, requiring labour for carpentry and finishing operations. Labour is not transferable between operations.

6,000 hours and 4,000 hours can be used in carpentry and finishing respectively. The labour hours required for each desk are given below :

Desk Type	*Hours of Carpentry*	*Hours of Finishing*
A	4	1
B	9	1
C	7	3
D	10	40

Formulate this as a Linear Programming problem.

9. A media specialist has to decide on the allocation of advertisement in three media vehicles. Let x_i be the number of messages carried in the i-th media, $i=1, 2, 3$. The unit costs of a message in the 3 media are Rs. 1000, Rs. 750 and Rs. 500. The total budget available is Rs. 20,000 for the campaign period of a year. The first medium is a monthly magazine and it is desired to advertise not more than one insertion in one issue. At least six messages should appear in the second medium. The number of messages in the third medium should strictly lie between 4 and

8. The expected effective audience for unit message in the media vehicles is shown below :

Vehicle	*Expected effective audience*
1	80,000
2	60,000
3	45,000

Build the linear programming model to maximise the total effective audience.

[**Ans.** maximize $Z=80.000x_1+60,000x_2+45,000x_3$ subjects to

$$1,000x_1+750x_2+500x_3 \leqslant 20,000 \quad \text{(budget)}$$
$$x_1 \leqslant 12$$
$$x_2 \leqslant 6$$
$$4 \leqslant x_3 \leqslant 8$$
$$x_1, x_2, x_3 \geqslant 0].$$

10. The manager of ABC Oil Co. wishes to find the optimal mix of two possible blending processes. For process 1, an input of 1 unit of crude oil A and three units of crude oil B produces an output of 5 units of gasoline X and two units of gasoline Y.. For process 2, an input of 4 units of crude oil A and 2 units of crude oil B produces an output of 3 units of gasoline X and 8 units of gasoline Y. Let x_1 and x_2 be the number of units the company decides to use of process 1 and process 2, respectively. The maximum amount of crude oil A available is 100 units and that of crude oil B is 150 units. Sales commitments require that at least 200 units of gasoline X and 75 units of gasoline Y are produced. The unit profits of process 1 and process 2 are p_1 and p_2 respectively. Formulate the blending problem as a linear programming model.

Ans. Maximise $Z=p_1\, x_1+p_2\, x_2$ subject to

$$\left.\begin{array}{r} x_1+4x_2 \leqslant 100 \\ 3x_1+2x_2 \leqslant 150 \end{array}\right\} \text{Availability}$$
$$\left.\begin{array}{r} 5x_1+3x_2 \geqslant 200 \\ 2x_1+8x_2 \geqslant 75 \end{array}\right\} \text{Demand}$$
$$x_1, x_2 \geqslant 0]$$

GRAPHIC METHOD

Summary Procedure for the Graphic Method

Step 1. Formulate the appropriate *LPP*.

Step 2. Construct the graph for the problem as follows :

'Treat each inequality as though it were an equality and for each equation arbitrarily select two sets of points. Plot each set of points and connect them with appropriate line'.

Step 3. Identify the feasible region, *i.e.*, that space which satisfies all the constraints simultaneously. For less than or equal to' and 'less than' constraints this is generally the region below these lines. For 'greater than or equal to' or greater than' constraints, this is generally the region which lies above the lines.

Step 4. By choosing a convenient profit (cost) figure, draw an isoprofit (isocost) line so that it falls within the shaded area.

Step 5. Move this isoprofit (isocost) line parallel to itself and farther (closer) from (to) the origin until an optimum solution is determined.

Example 6. *A factory manufactures two articles A and B. To manufacture the article A, a certain machine has to be worked for 1·5 hours and in addition a craftsman has to work for 2 hours. To manufacture the article B, the machine has to be worked for 2·5 hours and in addition the craftsman has to work for 1·5 hours. In a week the factory can avail of 80 hours of machine time and 70 hours of craftsman's time. The profit on each article A is Rs. 5 and that on each article B is Rs. 4. If all the articles produced can be sold away, find how many of each kind should be produced to earn the maximum profit per week.*

Formulate the linear programming problem.

Solution. Step I.

DATA SUMMARY CHART

Decision variables	*Article*	*Hours on*		*Profit per unit*
		Machine	*Craftsman*	
x_1	*A*	1·5	2	Rs. 5·00
x_2	*B*	2·5	1·5	Rs. 4·00
Hours available (per week)		80 maximum	70 maximum	

x_1 = number of units of article *A*

x_2 = number of units of article *B*

Thus the given problem is formulated as a L.P.P. as follows :

Maximize $Z = 5x_1 + 4x_2$ ...(*)

subject to the constraints :

$$1·5\ x_1 + 2·5\ x_2 \leqslant 80 \qquad ...(**)$$

$$2\ x_1 + 1·5\ x_2 \leqslant 70 \qquad ...(***)$$

$$x_1 .\ x_2 \geqslant 0 \qquad ...(****)$$

Step II. *Construct the graph.* Next we construct the graph by drawing horizontal and vertical axes which are represented by the x_1-axis and x_2-axis in the cartesian X_1OX_2 plane. Since any point which satisfies the conditions $x_1 \geqslant 0$ and $x_2 \geqslant 0$ lies in the first quadrant only our search for the desired pair (x_1, x_2) is restricted to the points of the first quadrant only.

Now the inequalities are graphed taking them as equalities, *e.g.*, the first constraint $1 \cdot 5x_1 + 2 \cdot 5x_2 \leqslant 80$ will be graphed as $1 \cdot 5x_1 + 2 \cdot 5x_2 = 80$, and the second constraint $2x_1 + 1 \cdot 5x_2 \leqslant 70$ as $2x_1 + 1 \cdot 5x_2 = 70$ and the third constraint $x_1, x_2 \geqslant 0$, merely restricts the solution to non-negative values.

Further, since the functions to be graphed are linear we need plot only two points per constraint. Thus to graph each constraint, we arbitrarily assign a value to x_1 and determine the corresponding value of x_2. The procedure is then repeated for another pair of values for the same constraint. Thus for the first constraint we have two such points as $P(0, 32)$ and $Q(53 \cdot 3, 0)$, which upon joining represents

$$1 \cdot 5x_1 + 2 \cdot 5x_2 = 80.$$

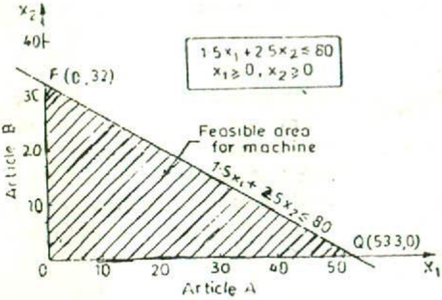


Fig. 1.

If	x_1	0	55·3
then	x_2	32	0

Similarly, by considering the set of points satisfying $x_1 \geqslant 0$, $x_2 \geqslant 0$ and the second constraint $2x_1 + 1 \cdot 5x_2 \leqslant 70$, we obtain the shaded area of Fig. 2 as shown below :

If	x_1	35	0
then	x_2	0	46·7

Step III. *Identify the feasible region.* The feasible region, *i.e.*, *solution space*, is the area of the graph which contains all pairs of values that satisfy all the constraints. In other words, feasible region

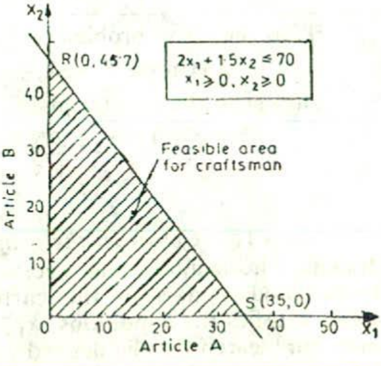


Fig. 2

will be bounded by the two axes, and the two lines $1·5x_1+2·5x_2=80$, $2x_1+1·5x_2=70$, and will be the common area which falls to the left of these constraint equations as both the constraints are of the 'less than equal to' type.

Step IV. *Locate the solution points.* The shaded area *OPTS* represents the set of all feasible solutions. The four corners of the polygon are $O=(0, 0)$, $P=(0, 32)$
$T=(20, 20)$ and
$S=(35, 0)$.

Step V. *Evaluate the objective function.* Dantzig's theorem guarantees that the optimal solution to an L.P.P. occurs at one or more of the corner points, we evaluate the objective function at each of these four points as follows :

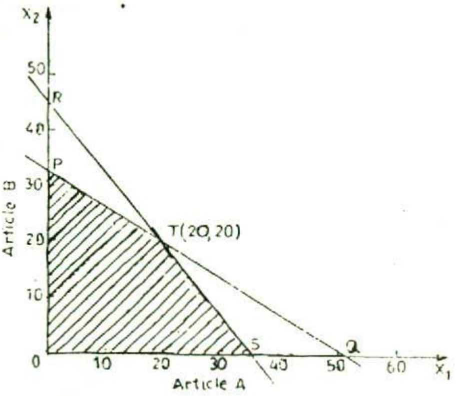


Fig. 3

Corner point (x_1, x_2)	*Objective function* $Z=5x_1+4x_2$	*Value*
$O=(0, 0)$	$5\times0+4\times0$	$Z(O)=0$
$P=(0, 32)$	$5\times0+4\times32$	$Z(P)=128$
$T=(20, 20)$	$5\times20+4\times20$	$Z(T)=180$
$S=(35, 0)$	$5\times35+4\times0$	$Z(S)=175$

Now the optimal solution is that corner point for which the objective function has the largest value. Thus the optimal solution to the present problem occurs at the point $T=(20, 20)$, *i.e.*, $x_1=20$, $x_2=20$ with the objective function value of Rs. 180.

Hence to maximize profit the company should manufacture 20 units of article *A* and 20 units of article *B* per week.

Example 7. *A company produces two articles X and Y. There are two departments through which the articles are processed. viz., assembly and finishing. The potential capacity of the assembly department is 60 hours a week and that of the finishing department is 48 hours a week. Production of one unit of X requires 4 hours in assembly and 2 hours in finishing. Each of the unit Y requires 2 hours in assembly and 4 hours in finishing. If profit is Rs. 8 for each unit of X and Rs. 6 for each unit of Y find out the number of units of X and Y to be produced each week to give maximum profit.*

Solution.

Products	*Time required for producing one unit*		*Total hours available*
	X	*Y*	
Assembly Department	4	2	60
Finishing Department	2	4	48
Profit per unit	Rs. 8	Rs. 6	

Objective function : $Z=8X+6Y$

Subject to constraints : $4X+2Y \leqslant 60$

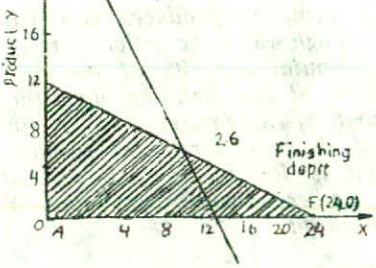
$2X+4Y \leqslant 48$

Non-negativity requirement : $X \geqslant 0$, $Y \geqslant 0$.

Plot the constraints in a graph given below. X is shown on the horizontal axis and Y is shown on the vertical axis. Consider the constraint $4X+2Y \leqslant 60$. When production of X is 0, then $Y=30$. Plot the point (0, 30) in the graph.

Again when production of Y is 0, then $X=15$. Plot the point (15, 0) in the graph. Joining these two points, the resulting straight line BC is such that area ABC of the graph represents the inequality $4X+2Y \leqslant 60$ as long as X and Y are both greater than 0.

Similarly plotting the constraint $2X-4Y \leqslant 48$, *i.e.*, joining $E(0, 12)$ and $F(24, 0)$. The area AEF contains all possible combinations which will satisfy the restriction of the finishing department.



Therefore the best combination of X and Y which must not exceed the available time in either assembly or finishing should fall within the areas ABC and AEF. The area which does not exceed either of the two constraints of the assembly and finishing departments is the shaded area $AEDC$.

Now observing from the graph, the point which yields the greatest profit is the point D (12, 6).

Point	*Total profit (applying objective function)* *Rs. 8X+Rs. 6Y*
$A(0, 0)$	0
$C(15, 0)$	Rs. 8(15)+Rs. 6(0)=Rs. 120
$D(12, 6)$	Rs. 8(12)+Rs. 6(6)=Rs. 132
$E(0, 12)$	Rs. 8(0)+Rs. 6(12)=Rs. 72

This may also be obtained algebraically by solving

$4X+2Y=60$ and $2X+4Y=48$ or $8X+4Y=120$, $2X+4Y=48$

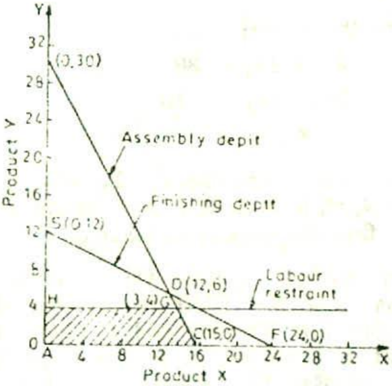
By subtraction $6X=72$ $\Rightarrow$ $X=12$ and $Y=6$

Applying it to the objective function $Z=8X+6Y$, the maximum profit equals to Rs. 8(12)+Rs. 5(6)=Rs. 132. Thus 12 units of X and 6 units of Y give a maximum profit of Rs. 132.

Remark. If there is a third constraint as shortage of labour which restricts the production of Y to a maximum of 4 units per week, then Y is less than or equal to 4 units per week and X and Y are non-negative.

Now plot the constraint in the graph given below and draw a straight line parallel to the horizontal axis. The feasible alternative will be somewhere in the shaded area $AHGC$. The point which yields the greatest profit is found out by testing the four corners of the shaded area. This is the point $G(13, 4)$. Therefore the optimum production per week is 13 units of X and 4 units of Y and the maximum profit

$$\max Z=\text{Rs. } 8(13)+\text{Rs. } 6(4)=\text{Rs. } 128.$$



Example 8. *Solve the following linear programming problem graphically :*

Maximise : $Z=4x+6y$ *subject to constraints* $x+y=5$, $x \geqslant 2$, $y \leqslant 4$, $x, y \geqslant 0$. [*Delhi Univ. B.Com.* (*Hons*) ; *1992*]

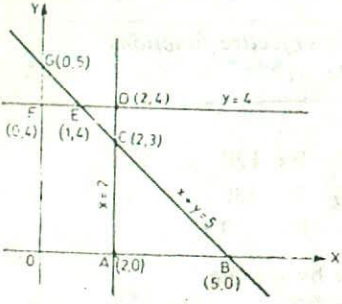


Fig. 4.

Solution. Clearly each point (x, y) satisfying the conditions $x \geqslant 0$, $y \geqslant 0$ must lie in the first quadrant only. Also since $x+y=5$, $x \geqslant 2$ and $y \leqslant 4$, the desired point lies somewhere on the line CB. The co-ordinates of $C=(2, 3)$ and $B=(5, 0)$. The values of the objective function Z at these points are

$$Z(C)=4 \times 2+6 \times 3=26$$

$$Z(B)=4 \times 5+6 \times 0=20$$

Since the maximum value of Z occurs at the point $C(2, 3)$. Thus to maximise Z, $x=2$ and $y=3$.

EXERCISES

1. (*a*) Describe the graphic method of solving a linear programming problem.

(*b*) Solve the following problem by graphic method and for that show

(*i*) Objective function (*ii*) Set of feasible solutions

(*iii*) Optimum solution (*iv*) Extreme points

Maximize $Z=3x_1+4x_2$

subject to the constraints :

$$4x_1+2x_2 \leqslant 80$$
$$2x_1+5x_2 \leqslant 180$$
$$x_1, x_2 \geqslant 0$$

2. It is required to maximise $Z=2x_1+5x_2$ subject to $x_1+x_2 \leqslant 24$, $3x_1+x_2 \leqslant 21$, $x_1+x_2 \leqslant 9$, $x \geqslant 0$, $y \geqslant 0$. Show graphically how to arrive at the solution and find the maximum value of Z.

3. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for at most 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ?

[**Hint.** Maximize $Z=22x_1+18x_2$

s.t. $x_1+x_2 \leqslant 20$

$$360x_1+240x_2 \leqslant 5760$$
$$x_1, x_2 \geqslant 0]$$

[**Ans.** $x_1=8, x_2=12$; max. $Z=$Rs. 392]

4. A manufacturer produces tubes and bulbs. It takes 1 hour of work on machine M and 3 hours of work on machine N to produce one package of bulbs while it takes 3 hours of work on machine M and 1 hour of work on machine N to produce a package of tubes. He earns a profit of Rs. 12·50 per package of bulbs and Rs. 5 per package of tubes. How many packages of each should be produced each day so as to maximize his profit if he operates the machines for at most 12 hours a day.

[**Hint.** Maximize $Z=12{\cdot}50\,x_1+5x_2$

$$\text{s.t.}\quad x_1+3x_2 \leqslant 12$$
$$3x_1+x_2 \leqslant 12$$
$$x_1, x_2 \geqslant 0]$$

[**Ans.** $x_1=3, x_2=3$; max. $Z=$Rs. 52·50]

5. A dealer deals in only two items, cycles and scooters. He has Rs. 50,000 to invest and a space to store at most 60 pieces. One scooter costs him Rs. 2500 and a cycle costs him Rs. 500. He can sell a scooter at a profit of Rs. 500 and a cycle at a profit of Rs. 150. Assuming that he can sell all the items that he buys, how should he invest his money in order that he may maximize his profit ?

[**Hint.** Maximize $Z=500x_1+150x_2$

$$\text{s.t.}\quad x_1+x_2 \leqslant 60$$
$$2500x_1+500x_2 \leqslant 50{,}000$$
$$x_1, x_2 \geqslant 0]$$

[**Ans.** $x_1=10, x_2=50$, Max. $Z=12{,}500$]

6. A firm makes two types of furniture : chairs and tables. The contribution for each product as calculated by the accounting department is Rs. 20 per chair and Rs. 30 per table. Both products are processed on three machines M_1, M_2 and M_3. The time required in hours by each product and total time available in hours per week on each machine are as follows :

Machine	*Chair*	*Table*	*Available Time*
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize contribution ? (Use graphic method only.)

[**Ans.** $x_1=3, x_2=9$, Max. $Z=330$]

7. Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin *A* per gram and 12 units of vitamin *B* and costs 20 paise per gram. The daily minimum requirements of vitamin *A* and vitamin *B* are 100 units and 120 units respectively. Find the minimum cost of product mix using graphic method.

[**Hint.** Minimize $Z=12x_1+20x_2$

subject to the constraints :

$$6x_1+8x_2\geqslant 100$$
$$7x_1+12x_2\geqslant 120$$
$$x_1, x_2\geqslant 0$$

Ans. $x_1=15,\ x_2=\frac{5}{4}$; minimum $Z=205$]

8. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents, it is necessary to buy two products (call them *A* and *B*) in addition. The contents of the various products, per unit, in nutrient constituents (*e.g.*, vitamins, proteins etc.) is given in the following table :

Nutrients	*Nutrient content in product*		*Minimum amount of nutrient*
	A	*B*	
M_1	36	6	108
M_2	3	12	36
M_3	20	10	100

The last column of the above table gives the minimum amounts of nutrient constituents M_1, M_2, M_3 which must be given to the pigs. If the products *A* and *B* cost Rs. 20 and Rs. 40 per unit respectively, how much each of these two products should be bought so that the total cost is minimized ?

[**Hint.** Find real numbers x_1 and x_2 so as to minimize the objective function :

$$Z=20x_1+40x_2$$

subject to the constraints :

$$36x_1+6x_2\geqslant 108$$
$$3x_1+12x_2\geqslant 36$$
$$20x_1+10x_2\geqslant 100$$
$$x_1+x_2\geqslant 0$$

The farm should purchase 4 units of product *A* and 2 units of product *B* in order to maintain a minimum cost of Rs. 160.]

9. A scrap metal dealer has received an order from a customer for at least 2,000 kilograms of scrap metal. The customer requires that at least 1,000 kilograms of the shipment of metal must be high quality copper that can be melted down and used to produce copper tubings. Furthermore,

the customer will not accept delivery of the order if it contains more than 175 kilograms of metal that he deems unfit for commercial use, *i.e.*, metal that contains an excessive amount of impurities and cannot be melted down and refined profitably.

The dealer can purchase scrap metal from two different suppliers in unlimited quantities with the following percentages (by weight) of high quality copper and unfit scrap.

	Supplier A	*Supplier B*
Copper	25%	75%
Unfit scrap	5%	10%

The costs per kilogram of metal purchased from supplier A and supplier B are Re. 1 and Rs. 4 respectively. The problem is to determine the optimum quantities of metal for the dealer to purchase from each of the two suppliers.

[**Hint.** Our problem is to find the real numbers x_1 and x_2 so as to minimize :

$$Z=x_1+4x_2$$

subject to the constraints : $x_1+x_2 \geqslant 2,000$

$$\frac{x_1}{4}+\frac{3x_2}{4} \geqslant 1,000$$

$$\frac{x_1}{20}+\frac{x_2}{10} \leqslant 175$$

$$x_1, x_2 \geqslant 0$$

The dealer should purchase 2,500 kilograms of scrap metal from supplier A and 500 kilograms of scrap metal from supplier B in order to maintain a minimum cost of Rs. 4,500.]

10. A cold drinks company has two bottling plants, located at two different places. Each plant produces three different drinks A, B and C. The capacities of the two plants, in number of bottles per day are as follows :

	Product A	*Product B*	*Product C*
Plant I	3000	1000	2000
Plant II	1000	1000	6000

A market survey indicates that during any particular month there will be a demand of 24,000 bottles of A, 16,000 bottles of B, and 48,000 bottles of C. The operating costs, per day, of running plants I and II are respectively 600 monetary units and 400 monetary units. How many days should the company run each plant during the month so that the production cost is minimised while still meeting the market demand ? (Use graphic method).

[**Hint.** Minimise cost : $Z=600x_1+400x_2$

$$\begin{aligned} \text{s.t.} \quad & 3000x_1+1000x_2 \geqslant 24,000 \\ & 1000x_1+1000x_2 \geqslant 16,000 \\ & 2000x_1+6000x_2 \geqslant 48,000 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0.] \end{aligned}$$

11. The manager of an oil refinery wants to decide on the optimal mix of two possible blending processes 1 and 2 of which the inputs and outputs per production run are as follows :

	Input (Units)		*Output (Units)*	
Process	*Crude A*	*Crude B*	*Gasoline X*	*Gasoline Y*
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of Crudes A and B are 200 units and 150 units respectively. At least 100 units of Gasoline X and 80 units of Y are required. The profit per production run from processes 1 and 2 are Rs. 300 and Rs. 400 respectively. Formulate the above as Linear programming problem and solve it by graphical method.

[**Ans.** Maximize $Z=300x_1+400x_2$

$$\begin{aligned} \text{s.t.} \quad & 5x_1+4x_2 \leqslant 200 \\ & 3x_1+5x_2 \leqslant 150 \\ & 5x_1+4x_2 \geqslant 100 \\ & 8x_1+4x_2 \geqslant 80 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0.] \end{aligned}$$

SIMPLEX METHOD

In most of the linear programming problems, we have more than two variables and, therefore, it cannot be conveniently solved by a graphic method. A procedure known as '*Simplex Method*' can be used to find the optimal solution. The method is in fact an algorithm or a set of instructions which seeks to examine corner point in a methodical manner until the best solution ensuring highest profit or the lowest cost under given constraints is obtained. Fortunately, computer programme is available for dealing with problems involving several variables but to understand its mechanics we shall confine to a few variables only.

Slack and Surplus Variables. The formulation of a linear programming problem for simplex method requires introduction of slack or surplus variable to convert a linear inequality into linear equality.

(*i*) Let the constraint of LP problem be $2x_1+3x_2 \leqslant 10$

Then the non-negative variable S_1 which satisfies

$$2x_1+3x_2+S_1=10$$

is called a *slack* variable.

(*ii*) If the constraint of a LP problem is $4x_1+5x_2 \geqslant 25$

Then the non-negative variable S_2 which satisfies

$$4x_1+5x_2-S_2=25$$

is called a *surplus* variable.

The variable S_1 is called slack variable, because

Slack = Requirement − Production

The variabie S_2 is called surplus variable, because

Surplus = Production − Requirement

These slack or surplus variables introduced in an appropriate manner to linear constraints expressed generally as inequalities get represented in the objective function so that the number of variables in objective function has correspondence with those in the constraints but they do not contribute anything to the objective function and their coefficients in the objective function are only zero.

Illustration.

Problem : Maximise profit $=7x_1+5x_2$

$$\text{Subject to : } 2x_1+1x_2 \leqslant 10$$

$$4x_1+3x_2 \leqslant 24$$

$$x_1 \geqslant 0,\ x_2 \geqslant 0$$

The inequalities expressing constraints are converted into equalities by adding slack variable to each inequality as follows :

$$2x_1+1x_2+S_1=10$$

$$4x_1+3x_2+S_2=24$$

Now, the objective function is being transformed to accommodate slack variables with zero coefficients as follows :

$$\text{Maximise profit} = 7x_1+5x_2+0S_1+0S_2$$

But, since all equations must have equal number of variables that is made possible by incorporating the slack variables of other equations with a zero coefficient as follows :

$$2x_1+1x_2+1S_2+0S_2=10$$

$$4x_1+3x_2+0S_1+1S_2=24$$

A model of simplex tableau to present these is given hereunder :

Simplex Tableau

Coefficients of Programme variables in objective	*Programme variables in objective function*	*Available quantities of variables*	†	†	†	0	0	0	*Objective variable coefficients*
			x_1	x_2	x_3	S_1	S_2	S_3	*Objective variable row*
0	S_1	@	*	*	*	1	0	0	
0	S_2	@	*	*	*	0	1	0	
0	S_3	@	*	*	*	0	0	1	

↑ Structural coefficients, matrix

↑ Identity matrix

Summary Procedure for the Simplex Method (Maximization Case)

The various steps involved in the simplex method may be summed up as follows :

1. Formulate the problem and the objective function.

2. Develop equations from the inequalities by adding slack variables.

3. Develop the initial simplex tableau including the initial (trivial) solution.

4. Obtain the Z_j and $C_j - Z_j$ (index row) for this solution.

5. Choose the highest positive number in the index row.

6. The highest positive number determines the key column.

7. Divide the numbers in quantity column by corresponding numbers in key column.

8. Select the least positive ratio of these quotients.

9. The row containing the least positive ratio is the key row.

10. The key number is at the intersection of the key column and key row.

11. Divide every figure in the key row by the key number.

@ Data on total available capacities.

† Data on coefficients of variables in the objective function.

* Data of coefficients of structural constraints.

12. The quotient of the key row divided by the key number is the main row in the next table. The formula is

$$\text{Main row} = \frac{\text{Key row number}}{\text{Key number}}$$

13. All other numbers for the next table are derived by the ormula

$$\begin{array}{l}\text{Derived} \\ \text{number}\end{array} = \begin{array}{l}\text{Selected} \\ \text{number}\end{array} - \left\{ \frac{\begin{array}{c}\text{Corresponding} \\ \text{number in key} \\ \text{row}\end{array} \times \begin{array}{c}\text{Corresponding} \\ \text{number in key} \\ \text{column}\end{array}}{\text{Key number}} \right\}$$

14. Repeat steps 5 to 13 until no positive numbers exist in the index row. When no positive numbers exist in the index row, an optimum solution has been obtained.

Remarks 1. *Simplification of Calculations*

It is possible to simplify the calculation process by following a few rules :

1. Any variable in the variable column will have a 1 where the row of that variable intersects with the column of that variable, and all other figures in the column of that variable will be zero.

2. If there is a zero in the key column, then the row in which that zero appears will remain unchanged in the subsequent matrix.

3. If there is a zero in the key row, then the column in which that zero appears will remain unchanged in the subsequent matrix.

By observing the above three rules, the number of items for which derived numbers are to be calculated will be greatly reduced. Where a simplex solution has to be worked by hand methods, the saving in time and effort is significant. When computers are used, it is desirable to allow the normal procedure to be followed.

2. *Rules for Ties*

In choosing the key column and key row, whenever there is a tie between two numbers the following rules may be adopted :

1. Select the column farthest to the left, whenever there is a tie between two numbers in the index row.

2. Select the ratio (θ) nearest to the top whenever there is a tie between two ratios in a matrix.

Illustration. *A factory can manufacture 2 products* X_1 *and* X_2. *Each product is manufactured by a two-stage process which involves machines I and II and the time required is as follows* :

Machine	*Product*	
	X_1	X_2
I	*2 hr.*	*1 hr.*
II	*3 hr.*	*2 hr.*

Available hours on machine I is 10 hours and machine II is 16 hours. The contribution for product X_1 Is Rs. 4 per unit and for X_2 is Rs. 3 per unit. What should be the manufacturing policy for the factory ?

Solution. Step I. Formulation of the LP problem.

Maximise (Profit) $Z=4X_1+3X_2$

Subject to : $2X_1+X_2\leqslant 10$ (constraint on Machine I)

$3X_1+2X_2\leqslant 16$ (constraint on Machine II)

$X_1\geqslant 0, X_2\geqslant 0$

Step II. Develop Equations from the Inequalities. The first step in the Simplex Method is to convert the inequalities (or restrictions) into equalities. This is done by adding what are known as slack variables (slack variables in economic terminology represent unused capacity but the contribution associated with them is zero). After adding the slack variables, all the above expressions can be written as

$$2X_1+X_2+S_1=10 \quad \text{...(1)}$$

$$3X_1+2X_2+S_2=16 \quad \text{...(2)}$$

$$4X_1+3X_2+0.S_1+0.S_2=\text{Maximize } Z \quad \text{.. (3)}$$

Here the slack variables S_1 and S_2 represent the idle hours on machines I and II respectively.

Step III. Designing the Initial Programme. Set the basic variables equal to zero in which case the slack variables assume the full value of the resources available and the contribution at this stage is minimum.

A first feasible solution is thus

$$X_1=0,\ X_2=0,\ S_1=10,\ S_2=16$$

The profit contribution resulting from this programme can be determined by substituting the values of the different variables in the objective function. Thus

Profit contribution $=4(0)+3(0)+0(10)+0(16)=0.$

Step IV. Develop Initial Simplex Tableau. We can now set out this whole problem in what is known as a *Simplex Tableau.* The simplex tableau also known as simplex matrix is a table consisting of rows and columns of figures We illustrate below the form of simplex tableau and explain its various parts :

TABLE I. PARTS OF INITIAL SIMPLEX TABLEAU

C column i.e., profit per unit)
Product-mix column)
Constant column (*i.e.*, quantities of product in the mix)
Variable columns

C_j	Product mix	Quantity	Rs. 4	Rs 3	Rs. 0	Rs. 0	← Cj row
			X_1	X_2	S_1	S_2	← Variable row
Rs. 0	S_1	10	2	1	1	0	Rows illustrating constraint equations (Coefficients only)
Rs. 0	S_2	16	3	2	0	1	

Body Matrix consisting of co-efficients of real product variables

Identity Matrix consisting of co-efficients of slack variables.

(*a*) C_j *row or objective row.* On the top row of the tableau known as C_j row or objective row, we insert the coefficients in the objective equation.

(*b*) The identity matrix is formed by the slack variables and consists of a diagonal of 1's and 0's. It may be noted that the identity should never have negative numbers.

(*c*) The *body matrix* consists of all restrictions and equations and includes the coefficient of all variables not in the identity. The numbers in the body can be zero, positive or negative.

(*d*) The *quantity column* represents the list of constants of the equations. Every number in the quantity column (excluding index row) must be zero or positive. This condition is true from the time of setting the matrix until its solution stage.

(*e*) The *product mix column* in the initial programme is a list of the variables in the identity. (It may be noted that the row headed by S_1 and the column headed by S_1 cross in the identity where the 1 occurs. The same is true for S_2 also). It may be noted that the product-mix column shows the variables in the solution. The variables in the first solution are S_1 and S_2 (the slack variables representing unused capacity). In the quantity column, we find the quantities of the variables that are in the solution :

$S_1 = 10$ hours available on Machine I
$S_2 = 16$ hours available on Machine II.

As the variables X_1 and X_2 do not appear in the product-mix, they are equal to zero.

(*f*) C_j *or objective column.* The C_j or objective column at the left end shows the profit per unit for the variables S_1 and S_2. For example, the zero appearing to the left of the S_1 row means that profit per unit of S_1 is zero. Likewise, the zero to the left of S_2 row means that profit per unit of S_2 is zero. The initial simplex tableau will now appear as follows :

C_j	*Product mix*	*Quantity*	Rs. 4 X_1	Rs. 3 X_2	Rs. 0 S_1	Rs. 0 S_2
Rs. 0	S_1	10	2	1	1	0
Rs. 0	S_2	16	3	2	0	1

(g) Z_j *row.* The Z_j is the C_j for a row times the coefficient for that row within the tableau, summed by column. In other words, to arrive at the Z_j value for a particular column, we first multiply each coefficient in that column by the C_j against that coefficient and then add up the products so obtained. The four values of Z_j under the columns of variables X_1, X_2, S_1 and S_2 are likewise computed as follows :

$$Z_j \text{ for column } X_1 = \text{Rs. } 0(2) + \text{Rs. } 0(3) = \text{Rs. } 0$$

$$Z_j \text{ for column } X_2 = \text{Rs. } 0(1) + \text{Rs. } 0(2) = \text{Rs. } 0$$

$$Z_j \text{ for column } S_1 = \text{Rs. } 0(1) + \text{Rs. } 0(0) = \text{Rs. } 0$$

$$Z_j \text{ for column } S_2 = \text{Rs. } 0(0) + \text{Rs. } 0(1) = \text{Rs. } 0$$

The above values of Z_j represent the amounts by which profit would be reduced if 1 unit of any of the variables (X_1, X_2, S_1, S_2) were added to the mix.

(h) $C_j - Z_j$ *(Index) or Net Evaluation row.* $C_j - Z_j$ represents the net profit that will occur from introducing one unit of a variable to the production schedule or solution. For example, if 1 unit of X_1 adds Rs. 4 of profit to the solution and if its introduction causes no loss, then $C_j - Z_j$ for X_1 = Rs. 4. The net profit per unit (*i.e.*, $C_j - Z_j$) of each variable is calculated as shown below :

Variables	*Profit per unit* (C_j)	*Profit lost per unit* (Z_j)	*Net profit per unit* ($C_j - Z_j$)
X_1	4	0	4
X_2	3	0	3
S_1	0	0	0
S_2	0	0	0

TABLE 2. INITIAL SIMPLEX TABLEAU COMPLETED

C_j → ↓	*Product mix*	*Quantity*	4	3	0	0
			X_1	X_2	S_1	S_2
0	S_1	10	2	1		0
0	S_2	16	3	2	0	1
	Z_j	0	0	0	0	0
	$C_j - Z_j$	(Index row)	4	3	0	0

Remark. By examining the numbers in the $(C_j - Z_j)$ row of Table 2, we find that total profit can be increased by Rs. 4 for each unit of X_1 added to the mix or by Rs. 3 for each unit of X_2 added to the mix. Thus a positive number in the $(C_j - Z_j)$ row indicates that profits can be improved by that amount per unit of X_1 added. On the other hand, a negative number in the $(C_j - Z_j)$ row would indicate the amount by which profits would decrease if one unit of the variable heading that column were added to the solution. Hence the optimum solution is reached when no positive numbers are there in $C_j - Z_j$ row.

Step V. Developing Improved Solutions. After the initial simplex tableau is set up, the next step is to determine if the improvement is possible. The computational procedure is as follows :

(*a*) *Choosing the entering variable.* We choose the variable to be added to the first solution which contributes the highest profit per unit. This is done by identifying the column (and hence the variable) which offers the largest positive number in the $(C_j - Z_j)$ row. As will be seen from Table 3, bringing in X_1 will add Rs. 4 per unit to profit. The X_1 column is the *optimum column*, also commonly known as *Pivot Column* or *Key Column*. By definition, the optimum column is that column which has the largest positive value in the $C_j - Z_j$ row, or in other words, the column whose product will contribute the highest profit per unit. Inspection of key or pivot column indicates that the variable X_1 should be added to the product mix replacing one of the variables present in the mix. The variable X_1 is, thus, the *entering* variable.

(*b*) *Choosing the departing variable.* Since we have chosen a variable to enter the solution mix we have to decide which variable is to be replaced. This is done in the following manner.

First, divide each number in the quantity column (also known as constant column), *i.e.*, 10 and 16 by the corresponding numbers in the key column.

Second, select the row with the smallest non-negative ratio as the row to be replaced.

Here the ratios would be :

S_1 row : 10 hours/2 hrs. per unit=5 units of X_1

S_2 row : 16 hours/3 hrs. per unit=$5\frac{1}{3}$ units of X_2

As the S_1 row has the smallest positive ratio, it is called the *replaced row*, or the *pivot row* or *key row*. This row will be replaced in the next solution by 5 units of X_1, *i.e.*, the variable S_1 (unused time) will be replaced by 5 units of X_1 in the next solution.

The number at the intersection of key row and key column is referred to as the *pivot* or *key number* which is 2 in the present case.

TABLE 3. INITIAL SIMPLEX TABLEAU, KEY ROW, KEY NUMBER, KEY COLUMN

Cj	*Product mix*	*Quantity*	Rs. 4 X_1	Rs 3 X_2	Rs. 0 S_1	Rs. 0 S_2	
			↓				
Rs. 0	S_1	10	2	1	1	0	Key number ←Replaced row or Key row
Rs. 0	S_2	16	3	2	0	1	
	Zj	Rs. 0	Rs. 0	Rs. 0	Rs. 0	Rs. 0	
	$Cj-Zj$		4	3	0	0	
				↓			

Optimum Column or Key Column

Step VI. Developing Second Simplex Tableau. Having chosen the optimum solution and the replaced row, a second simplex tableau can be developed, providing an improved solution.

(*a*) *Compute new values for the key row.* For this we have to simply divide each number in the key row by key number. The key row now becomes :

Cj	*Product-mix*	*Quantity*	X_1	X_2	S_1	S_2
4	X_1	5	1	$\frac{1}{2}$	$\frac{1}{2}$	0

It may be noted that in the product mix, S_1 has been replaced by X_1 and the corresponding C_j value also has been replaced (4 for 0).

(*b*) *Compute new values* (*derived numbers*) *for each remaining rows.* To complete the second tableau, we compute new values for the remaining rows. All remaining rows of the variables in the tableau are calculated using the following formula :

$$\begin{pmatrix}\text{New}\\\text{row}\end{pmatrix}=\begin{pmatrix}\text{Elements in}\\\text{the old row}\end{pmatrix}-\left[\begin{pmatrix}\text{Intersection}\\\text{element of old row}\end{pmatrix}\times\begin{pmatrix}\text{Corresponding ele-}\\\text{ments in replacing}\\\text{row}\end{pmatrix}\right]$$

Using this formula, we get the new S_2 row as follows :

Element in old S_2 row	*Intersectional element of S_2 row*	*Corresponding element in key row*	*New S_2 row*
(*1*)	(*2*)	(*3*)	(*1*)−(*2*)×(*3*)
16	3	5	$16-3\times5=1$
3	3	1	$3-3\times1=0$
2	3	$\frac{1}{2}$	$2-3\times\frac{1}{2}=\frac{1}{2}$
0	3	$\frac{1}{2}$	$0-3\times\frac{1}{2}=-\frac{3}{2}$
1	3	0	$1-3\times0=1$

Thus, the new S_2 row will be :

$$(1,\ 0,\ \tfrac{1}{2},\ -\tfrac{3}{2},\ 1)$$

An alternative formula is as follows :

Derived Number = Selected number

$$-\left(\frac{\text{Corresponding number in key row}\times\text{Corresponding number in key column}}{\text{Key number}}\right)$$

The computations will be as under :

$$16-\frac{10\times3}{2}=1\ ;\ 3-\frac{2\times3}{2}=0\ ;\ 2-\frac{1\times3}{2}=\frac{1}{2}\ ;$$

$$0-\frac{1\times3}{2}=-\tfrac{3}{2}\ ;\ 1-\frac{0\times3}{2}=1.$$

(c) *Computing Z_j and C_j-Z_j rows.* Now, we shall compute the Z_j and C_j-Z_j rows (the profit opportunities) according to the methods discussed earlier.

TABLE 4. SECOND SIMPLEX TABLEAU

$C_j\rightarrow$ ↓			*4*	*3*	*0*	*0*	
	Product mix	*Quantity*	X_1	X_1	S_1	S_2	
4	X_1	5	1	$\frac{1}{2}$	$\frac{1}{2}$	0	
0	S_2	1	0	$\frac{1}{2}$	−3/2	1	←Key row
	Z_j	20	4	2	2	0	
		C_j-Z_j	0	1	−2	0	←Index row

↑ Key Column

* Keynumber

The computation of the Z_j row of the second tableau is as follows :

Z_j (*i.e.*, total profit) for quantity column

$$= (4 \times 5) + (0 \times 1) = 20$$

$$Z_j \text{ for } X_1 \text{ column} = (4 \times 1) + (0 \times 0) = 4$$

$$Z_j \text{ for } X_2 \text{ column} = (4 \times \tfrac{1}{2}) + (0 \times \tfrac{1}{2}) = 2$$

$$Z_j \text{ for } S_1 \text{ column} = (4 \times \tfrac{1}{2}) + \{0 \times (-\tfrac{3}{2})\} = 2$$

$$Z_j \text{ for } S_2 \text{ column} = (4 \times 0) + (0 \times 1) = 0$$

The computation of the $(C_j - Z_j)$ row of the second tableau is as follows :

Variables	*Profit per unit* (C_j)	*Profit lost per unit* (Z_j)	*Net profit per unit* $(C_j - Z_j)$
X_1	4	4	0
X_2	3	2	1
S_1	0	2	-2
S_2	0	0	0

Step VII. The presence of a positive number in the X_2 column of the $C_j - Z_j$ row of the second tableau shows that positive improvement is possible. Hence the process used to develop the second solution must now be repeated to obtain a third solution. Accordingly, we find that

(*a*) The variable X_2 will enter the solution by virtue of $C_j - Z_j = 1$ being the largest and only positive number in that row. This means that for every unit of X_2 that we produce, the objective function will increase by Re. 1.

(*b*) The optimum column or key column is X_2 column.

(*c*) The replaced row is S_2 row ; also known as key row or pivot row. This is found by (*i*) dividing 5 and 1 in the quantity column by their corresponding numbers in the key column, *i.e.*, $\frac{1}{2}$ and $\frac{1}{2}$ respectively, (*ii*) choosing the row with the smaller ratio as the key row.

(*d*) Intersectional element of X_1 row is $\frac{1}{2}$, and the intersectional element of S_2 row is also $\frac{1}{2}$. This will be the pivot number or the key number.

(*e*) The key row is replaced by dividing every number in it by the key number, *i.e.*, $\frac{1}{2}$, the key row now becomes :

C_j	*Product mix*	*Quantity*	X_1	X_2	S_1	S_2
3	X_2	2	0	1	-3	2

(*f*) The new values of the X_1 row (third tableau) are :

$$\begin{array}{ccccccc}
\left(\begin{array}{c}\textit{Element in}\\ \textit{old } X_1 \textit{ row}\end{array}\right) & - & \left(\begin{array}{c}\textit{Intersectional}\\ \textit{element of}\\ X_1 \textit{ row}\end{array}\right. & \times & \left.\begin{array}{c}\textit{Corresponding}\\ \textit{element of}\\ \textit{new } X_2 \textit{ row}\end{array}\right) & = & \textit{New } X_1 \textit{ row}\\
5 & - & (\frac{1}{2} & \times & 2) & = & 4\\
1 & - & (\frac{1}{2} & \times & 0) & = & 1\\
\frac{1}{2} & - & (\frac{1}{2} & \times & 1) & = & 0\\
\frac{1}{2} & - & (\frac{1}{2} & \times & -3) & = & 2\\
0 & - & (\frac{1}{2} & \times & 2) & = & -1
\end{array}$$

(We can compute the new X_1 row through the alternative formula as well.)

TABLE 5. THIRD SIMPLEX TABLEAU

$C_j \to$ ↓			4	3	0	0	
	Product mix	*Quantity*	X_1	X_2	S_1	S_2	
4	X_1	4	1	0	2*	−1	← Key row
3	X_2	2	0	1	−3	2	
	Z_j	22	4	3	−1	2	
	$C_j - Z_j$		0	0	1	−2	← Index row

↑ Key column

Step VIII. Once again, we find that all the values of this row are not zero or negative, therefore, we have to proceed a little further. However, the key row, key column as well as the key number have been indicated in the third simplex tableau.

Step IX. By repeating what has been done earlier we arrive at the final tableau IV.

TABLE 6. FOURTH SIMPLEX TABLEAU

$C_j \to$ ↓			4	3	0	0
	Product mix	*Quantity*	X_1	X_2	S_1	S_2
0	S_1	2	$\frac{1}{2}$	0	1	$-\frac{1}{2}$
3	X_2	8	$\frac{3}{2}$	1	0	$\frac{1}{2}$
	Z_j	24	$\frac{1}{2}$	3	0	$\frac{3}{2}$
	$C_j - Z_j$		$-\frac{1}{2}$	0	0	$-\frac{3}{2}$

As there are no positive values in $C_j - Z_j$ row, no further improvement is possible, and the optimum solution has now been obtained This solution is $X_1=0,\ X_2=8,\ S_1=2$

The Z_j total, *i.e.*, Rs. 24, represents the profit obtained under the optimum solution.

Example 9. *An electronics firm is undecided as to the most profitable mix for its products. The products now manufactured are transistors, resistors and carbon tubes with a profit (per 100 units) of Rs. 10, Rs. 6 and Rs. 4 respectively. To produce a shipment of transistors containing 100 units requires 1 hour of engineering, 10 hours of direct labour and 2 hours of administration service. To produce 100 resistors are required 1 hour, 4 hours and 2 hours of engineering, direct labour and administration time respectively. To produce one shipment of the tubes (100 units) requires 1 hour of engineering, 5 hours of direct labour and 6 hours of administration. There are 100 hours of engineering services available, 600 hours of direct labour and 300 hours of administration. What is the most profitable mix ?*

Solution. For the sake of convenience, we tabulate the data in the following manner :

	Products			*Available hours*
	Transistors	*Resistors*	*Carbon Tubes*	
Engineering	1	1	1	100
Labour	10	4	5	600
Administration	2	2	6	300
Profit per 100 units	Rs. 10	Rs 6	Rs. 4	

Objective Function : Maximise Profit

$$Z = 10x_1 + 6x_2 + 4x_3 \qquad \text{...(1)}$$

Subject to the constraints :

$$x_1 + x_2 + x_3 \leqslant 100 \qquad \text{...(2)}$$
$$10x_1 + 4x_2 + 5x_3 \leqslant 600 \qquad \text{.. (3)}$$
$$2x_1 + 2x_2 + 6x_3 \leqslant 300 \qquad \text{...(4)}$$
$$x_1,\ x_2,\ x_3 \geqslant 0$$

The first step in the Simplex Method is to convert the inequalities (or restrictions) into equalities. This is done by adding a slack variable (unused capacity of the department). After adding the slack variables, all the expressions (1) to (4) can be written as

$$x_1 + x_2 + x_3 + S_1 = 100$$
$$10x_1 + 4x_2 + 5x_3 + S_2 = 600$$
$$2x_1 + 2x_2 + 6x_3 + S_3 = 300$$
$$10x_1 + 6x_2 + 4x_3 + 0 \cdot S_1 + 0 \cdot S + 0 \cdot S_3 = \text{Maximise } Z$$

The simplex method always begins with a zero solution, *i.e.*, it starts at the point of no production whatsoever. This enables the steps in the solution to determine the appropriate quantity of each item to produce, subject to the objective function and the restrictions.

In other words, if x_1. x_2 and x_3 are not produced, then the unused capacity of the three departments as given by S_1, S_2 and S_3 will be 100, 600 and 300 hours respectively. The solution at the first programme is given by the quantity column and the product mix column. The solution at this point is $S_1=100$, $S_2=600$, $S_3=300$.

SIMPLEX TABLEAU I

$C_j \rightarrow$ $\downarrow$	*Product Mix*	*Quantity*	10	6	4	0	0	0	*Ratio Column*
			X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	100	1	1	1	1	0	0	100
0	S_2	600	10*	4	5	0	1	0	← 60
0	S_3	300	2	2	6	0	0	1	150
	Z_j		0	0	0	0	0	0	
	C_j-Z_j		10 ↑	6	4	0	0	0	

SIMPLEX TABLEAU II

0	S_1	40	0	6/10*	5/10	1	−1/10	0	← 67
10	X_1	60	1	4/10	5/10	0	1/10	0	150
0	S_3	180	0	12/10	5	0	−2/10	1	150
	Z_j		10	4	5	0	1	0	
	C_j-Z_j		0	2 ↑	−1	0	−1	0	

SIMPLEX TABLEAU III

6	x_2	400/6	0	1	5/6	10/6	−1/6	0
10	x_1	100/3	1	0	1/6	−2/3	1/6	0
0	S_3	100	0	0	4	—2	0	0
	Z_j		10	6	20/3	10/3	2/3	0
	Z C_j-Z_j		0	0	−20/3	−10/3	−2/3	0

Hence the most profitable mix is $\frac{400}{6}$ resistor and $\frac{100}{3}$ transistors.

The maximum profit is $400+\frac{1000}{3}=$Rs. $733\frac{1}{3}$.

Example 10. *Vitamins A, B and C are found in foods F_1 and F_2. One unit of F_1 contains 1 mg of A, 100 mg of B and 10 mg of C. One unit of F_2 contains 1 mg of A, 10 mg of B and 100 mg of C. The minimum daily requirements of A, B and C are 1 mg, 50 mg and 10 mg respectively. The cost per unit of F_1 and F_2 are Re. 1 and Rs. 1·50 respectively. You are required to (i) formulate the above as a linear programming problem minimising the cost per day, (ii) write the dual of the problem and (ii) solve the dual by using simplex method and read there from the answer to the primal.*

[Delhi Univ., B. Com. (Hons.), 1992]

Solution. (*i*) Let x_1 units of F_1 and x_2 units of F_2 be purchased.

Primal : Minimise (cost per day) : $Z=x_1+1{\cdot}5x_2$ subject to

$$x_1+x_2\geqslant 1,\ 100x_1+10x_2\geqslant 50,\ 10x_1+100x_2\geqslant 10$$
$$x_1\geqslant 0,\ x_2\geqslant 0.$$

(*ii*) *Dual* : Let p, q, and r be the dual variables. Then we have

Minimise $C=p+50q+10r$ subject to

$$p+100q+10r\leqslant 1,\ p+10q+100r\leqslant\ ,\ p,\ q,\ r\geqslant 0.$$

(*iii*) *Solution to Dual* : Introducing slack variables s_1 and s_2, the dual may be written as under

Maximise $C=p+50q+10r+0.s_1+0.s_2$ subject to

$$p+100q+10r+s_1+0{\cdot}s_2=1$$
$$p+10q+100r+0{\cdot}s_1+s_2=\tfrac{3}{2}$$
$$p,\ q,\ r,\ s_1,\ s_2\geqslant 0$$

$C_j\rightarrow$ ↓	*Basic Variables*	*Values of Basic Variables*	*1*	*50*	*10*	*0*	*0*	*Ratio*
			p	q	r	s_1	s_2	
0	$\leftarrow s_1$	1	1	100	10	1	0	$\frac{1}{100}$
0	s_2	$\frac{3}{2}$	1	10	100	0	1	$\frac{3}{20}$
	Z_j	0	0	0	0	0	0	
	C_j-Z_j		1	50 ↑	10	0	0	
50	q	$\frac{1}{100}$	$\frac{1}{100}$	1	$\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{10}$
0	$\leftarrow s_2$	$\frac{7}{5}$	$\frac{9}{10}$	0	99*	$-\frac{1}{10}$	1	$\frac{7}{495}$
	Z_j	$\frac{1}{2}$	$\frac{1}{2}$	50	5	$\frac{1}{2}$	0	
	C_j-Z_j		$\frac{1}{2}$	0	5	$\frac{1}{2}$	0	
50	q	$\frac{17}{1980}$	$\frac{1}{110}$	1	0	$\frac{1}{90}$	$\frac{1}{990}$	$\frac{187}{198}$
10	r	$\frac{7}{495}$	$\frac{1}{110}$	0	1	$\frac{1}{990}$	$\frac{1}{99}$	$\frac{710}{495}$
	Z_j	$\frac{143}{198}$	$\frac{8}{11}$	50	10	$\frac{49}{99}$	$\frac{5}{99}$	
	C_j-Z_j		$\frac{5}{11}$	0	0	$\frac{49}{99}$	$\frac{5}{99}$	
1	p	$\frac{187}{198}$	1	110	0	$\frac{10}{9}$	$\frac{1}{9}$	
10	r	$\frac{11}{1980}$	0	-1	1	$\frac{1}{90}$	$\frac{1}{90}$	
	Z_j	1	1	100	10	0	0	
	C_j-Z_j		0	-50	0	1	0	

Answer to primal : $x_1=1$, $x_2=0$ and total cost $=1$

EXERCISES

1. Why is the simplex method a better technique than the graphical approach for most real cases ?

2. Give outlines of 'Simplex Method' in Linear programming.

3. (*a*) A manufacturer produces two items X_1 and X_2. X_1 needs 2 hours on machine *A* and 2 hours on machine *B*. X_2 needs 3 hours on machine *A* and 1 hour on machine *B*. If machine *A* can run for a maximum of 12 hours per day and *B*. 8 hours per day and profits from X_1 and X_2 are Rs 4 and Rs. 5 per item respectively, find by simplex method how many items per day be produced to have maximum profit. Give the interpretation for the values of 'indicators' corresponding to slack variables in the final iteration.

(*b*) A manufacturer produces bicycles and scooters, each of which must be processed through two machines *A* and *B*. Machine *A* has a maximum of 120 hours available and machine *B* has a maximum or 180 hours available. Manufacturing a bicycle requires 6 hours in machine *A* and 3 hours in machine *B*. Manufacturing a scooter requires 4 hours in machine *A* and 10 hours in machine *B*. If profits are Rs. 45 for a bicycle and Rs. 55 for a scooter, determine the number of bicycles and the number of scooters that should be manufactured in order to maximize the profit.

4. A novelty manufacturer makes two types of emblems, *A* and *B*. He uses three departments : preparation, cutting and packaging. Each department is used for both types of emblems. Processing rates are :

	Type A (min/pc)	*Type B* (min/pc)
Preparation	4	3
Cutting	8	2
Packaging	6	3

The profit per unit is Rs. 2 and Rs. 3 for type *A* and type *B* respectively. If 1.200 minutes are available in each of the departments, determine the optimal production schedule. Use Simplex Method.

5. A firm makes two types of furniture : chairs and tables. Profits are Rs. 20 per chair and Rs. 30 per table. Both products are processed on three machines M_1, M_2, and M_3. The time required for each product in hours and total time available in hours on each machine are as follows :

Machine	*Chair*	*Table*	*Available Time*
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

(*a*) Formulate the above as a linear programming problem to maximise the profit ; (*b*) Write its dual : and (*c*) Solve the primal by simplex method.

[Ans. 3, 9 ; Rs. 330.]

6. A manufacturing company contemplates to produce two additional products, called *A* and *B*, which can be marketed at prevailing prices in any reasonable quantities without difficulty. It is known that product *A* requires 10 and 5 man-hours per unit in the foundry and the machine departments respectively ; and that product *B* requires only 6 and 4. However, the profit margin of *A* is Rs. 30 per unit as compared with Rs. 20 per unit of *B*. In the week immediately ahead, it is estimated that there will be 1000 and 600 man-hours available in the foundry and the machine departments respectively. How much of *A* and *B* should be produced in order to most profitably utilize the excess capacities ?

7. A company makes three products *X*, *Y*, *Z* out of three matarials P_1, P_2 and P_3. The three products use units of three materials according to the following table :

		Materials		
		P_1	P_2	P_3
Products	*X*	1	2	3
	Y	2	1	1
	Z	3	2	1

The unit profit contributions from the three products are :

Product :	*X*	*Y*	*Z*
Profit Contribution (in Rs.) :	3	4	5

and availabilities of the three materials are :

Material :	P_1	P_2	P_3
Amount available (in units) :	10	12	15

The problem is to determine the product mix which will maximize the total profit.

[Hint. **SIMPLEX MATRIX V**

$C_j \rightarrow$	*Product*	*Quantity*	3	4	5	0	0	0
↓	*Mix*		*X*	*Y*	*Z*	S_1	S_2	S_3
0	S_2	1	0	0	4/5	−1/5	1	−3/5
3	*X*	4	1	0	−1/5	−1/5	0	2/5
4	*Y*	3	0	1	8/5	3/5	0	−1/5
	$C_j - Z_j$		0	0	−4/5	−9/5	0	−2/5

The optimal solution of the primal problem is to produce 3 units of product *X*, 4 units of product *Y* and no units of product *Z* which gives a maximum profit of Rs. 24.]

8. A manufacturer of leather belts makes three types of belts *A*, *B* and *C* which are processed on three machines M_1, M_2 and M_3. Belt

A requires 2 hours on machine M_1 and 3 hours on machine M_2. Belt *B* requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 and Belt *C* requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours of time per day available on machine M_2, and 15 hours of time per day available on machine M_3. The profit gained from belt *A* is Rs. 3·00 per unit, from Belt *B* is Rs. 5·00 per unit, from belt *C* is Rs. 4·00 per unit. What should be the dailv production of each type of belts so that the profit is maximum.

[**Hint.** Maximize

$$z=3x_1+5x_2+4x_3$$

Subject to the constraints :

$$\begin{aligned} 2x_1+3x_2 \quad &\leqslant 8 \\ 2x_2+5x_3 &\leqslant 10 \\ 3x_1+2x_2+4x_3 &\leqslant 15 \\ x_1, x_2, x_3 &\geqslant 0. \end{aligned}$$

Using simplex method, we get

$$x_1=\frac{89}{41}, x_2=\frac{50}{41}, x_3=\frac{62}{41} \text{ and max. } Z=\frac{765}{41}\Big]$$

9. Explain the nature and significance of L.P.

A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yields a profit of Rs. 30. An acre of wheat costs Rs. 120 to prepare, requires 10 man days of work and yields a profit of Rs. 40. An acre of soyabeans costs Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 80,000 man-days work, how many acres should be allocated to each crop to maximise the total profit ?

[**Ans.** Corn 250, wheat 625, soyabeans 0, Profit Rs. 32,500.]

10. A small-scale industrialist produces four types of machine components M_1, M_2, M_3 and M_4 made of steel and brass. The amounts of steel and brass required for each component and the number of man-weeks of labour required to manufacture and assemble 1 unit of each component are as follows :

	M_1	M_2	M_3	M_4	*Availability*
Steel	6	5	3	2	100 kg.
Brass	3	4	9	2	75 kg.
Man-weeks	1	2	1	2	20

The industrialist's profit on each unit of M_1, M_2, M_3 and M_4 is respectively Rs. 6, Rs. 4, Rs. 7 and Rs. 5.

How many of each should he produce to optimize his profit, and how much is his profit ? (Note that the values given are the average values per week. If a fractional value appears in the answer, it should be interpreted as an average value.)

[**Ans.** $M_1 : 14$; $M_2 : 0$; $M_3 : 10/3$; $M_4 : 0$;

Profit ; Rs. $13\frac{1}{3}$ per week]

DUALITY IN LINEAR PROGRAMMING

Associated with every linear-programming problem is a related *dual* linear-programming problem. The *originally formulated* problem, in relation to the dual problem, is known as the *primal* linear programming problem. If the objective in the primal problem is *maximization* of some function, then the objective in the dual problem is *minimization* of a related (but different) function. Conversely, a primal minimization problem has a related dual maximization problem. The concept of duality is more effectivity demonstrated in the following illustration :

Primal	*Dual*
Maximize : $Z=3x_1+5x_2+4x_3$	Minimize : $Z^*=8y_1+10y_2+15y$
Subject to :	Subject to :
$2x_1 \quad +3x_2 \leqslant 8$	$2y_1+ \quad 3y_3 \geqslant 3$
$2x_2+5x_3 \leqslant 10$	$3y_1+2y_2+2y_3 \geqslant 5$
$3x_1+2x_2+4x_3 \leqslant 15$	$5y_2+4y_3 \geqslant 4$
$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0$	$y_1 \geqslant 0,\ y_2 \geqslant 0,\ y_3 \geqslant 0$

		PRIMAL				
	Variables	x_1	x_2	x_3	*Relation*	*Constants*
DUAL	y_1	2	3	—	$\leqslant$	8
	y_2	—	2	5	$\leqslant$	10
	y_3	3	2	4	$\leqslant$	15
	Relation	$\geqslant$	$\geqslant$			min Z^*
	Constants	3	5	4		max Z

It will be seen that

1.	**Primal, here, involves maximization.**	Dual involves minimizatio
2.	**In primal, we write objective function as Z.**	**In dual, we write objective function as Z^*.**
3.	**In primal, the variables are x_1, x_2 and x_3.**	**Dual has a new set of variables, *i.e.*, y_1, y_2 and y_3.**

4.	Primal has three variables, *viz*., x_1, x_2 and x_3.	The dual, therefore, has three constraints.
5.	The primal has three constraints.	The dual, therefore, has three variables. *viz*., y_1, y_2 and y_3.
6.	In primal's objective function, 3, 5 and 4 are the coefficients.	In dual. 3, 5 and 4 become constants of constraints on the right hand side.
7.	In primal, the coefficients of constraints, columnwise, are	In dual, each column takes the position row-wise as under :
	$\begin{array}{ccc} 2 & 3 & - \\ - & 2 & 5 \\ 3 & 2 & 4 \end{array}$	$\begin{array}{ccc} 2 & - & 3 \\ 3 & 2 & 2 \\ - & 5 & 4 \end{array}$
8.	In primal, the signs of constraints are less than or equal to	In dual, the signs of the constraints are just the reverse, *i.e*., greater than or equal to.
9.	The non-negativity constraints are as many as the variables in the primal, *i.e*., 3.	The non-negativity constraints are as many as the variables in the dual, *i.e*., 3.
10.	The signs in the non-negativity constraints are greater than or equal to.	The signs in the non-negativity constraints do not change and remain the same.

Conclusion

The foregoing examples make it clear that the transformation of a given primal problem involves the following considerations :

1 If the primal involves maximization, the dual involves minimization, and *vice versa*.

2. A new set of variables appears in the dual.

3. Ignoring the number of non negativity constraints, if there are n variables and m inequalities in the primal, in the dual, there will be m variables and n inequalities.

4. The coefficients in the primal's objective function are put as dual's constraint constants, and *vice versa*.

5. Of the primal's constraint inequalities, the coefficients columnwise (from top to bottom) are positioned in the dual's constraint inequalities row-wise (from left to right), and *vice-versa*.

6. If the primal's constraints involve $\leqslant$ signs, the dual's constraints involve $\geqslant$ signs, and *vice versa*.

7. The signs in the non-negativity constraints are $\geqslant$ both in the primal and the dual.

Example 9. *Food F_1 contains 6 units of vitamin A, 7 units of vitamin B and 8 units of vitamin C. It costs Rs. 10 per unit. Food F_2 contains 7* *units of vitamin A, 6 units of vitamin B and 10 units of vitamin C. It costs Rs. 12 per unit. Food F_3 contains 8 units of vitamin A, 9 units of vitamin B and 6 units of vitamin C. It costs Rs. 15 per unit. The daily minimum requirement of vitamins A, B and C are 100 units, 120 units and 150 units respectively. Write the dual of the above linear programming problem. Solve the dual. From the optimal solution of the dual, find the optimum solution of the primal problem.*

Solution. The given problem, *i.e.*, the primal problem, stated in an appropriate mathematical form is as follows :

Minimize C (Total cost)$=10x_1+12x_2+15x_3$

Subject to the constraints :

$$6x_1+7x_2+8x_3\geqslant 100$$
$$7x_1+6x_2+10x_3\geqslant 120$$
$$8x_1+9x_2+6x_3\geqslant 150$$
$$x_1\geqslant 0,\ x_2\geqslant 0,\ x_3\geqslant 0$$

where.

$x_1=$Number of units of Food F_1

$x_2=$Number of units of Food F_2

$x_3=$Number of units of Food F_3

The dual to the above problem is

Maximise $Z=100y_1+120y_2+150y_3$

Subject to the constraints :

$$6y_1+7y_2+8y_3\leqslant 10$$
$$7y_1+6y_2+9y_3\leqslant 12$$
$$8y_1+10y_2+6y_3\leqslant 15$$
$$y_1\geqslant 0,\ y_2\geqslant 0,\ y_3\geqslant 0$$

Solution of Dual Problem

For solving the Dual problem, we convert the inequalities by adding slack variables S_1, S_2 and S_3.

Maximize $Z=100y_1+120y_2+150y_3+0.S_1+0.S_2+0.S_3$

$$6y_1+7y_2+8y_3+S_1=10$$
$$7y_1+6y_2+9y_3+S_2=12$$
$$8y_1+10y_2+6y_3+S_3=15$$
$$y_1,\ y_2,\ y_3,\ S_1,\ S_2,\ S_3\geqslant 0$$

As usual, if we make an initial decision of no production, this decision summarized in tabular form will be as follows :

SIMPLEX MATRIX I

Cj→ ↓	*Product Mix*	*Quantity*	*100* y_1	*120* y_2	*150* y_3	*0* S_1	*0* S_2	*0* S_3	
0	S_1	10	6	7	8*	1	0	0	←Key Column
0	S_2	12	7	6	9	0	1	0	
0	S_3	15	8	10	6	0	0	1	
	$Cj-Zj$		100	120	150	0	0	0	

↑ Key Row

In simplex matrix I, we find that key column is corresponding to the variable y_3 and key row is corresponding to the variable S_1. We now proceed to Simplex Matrix II.

SIMPLEX MATRIX II

Cj→ ↓			*100*	*120*	*150*	*0*	*0*	*0*
	Product Mix	*Quantity*	y_1	y_2	y_3	S_1	S_2	S_3
150	y_3	5/4	3/4	7/8	1	1/8	0	0
0	S_2	3/4	1/4	−15/8	0	−9/8	1	0
0	S_3	15/2	7/2	19/4	0	−3/4	0	1
		$Cj-Zj$	−25/2	−45/4	−0	150/8	0	0

The optimal solution to the dual problem gives a maximum value or $\frac{750}{4}$ for the objective function.

Interpreting Primal-Dual Optimal Solutions

Once the dual problem has been formulated and solved, there, remains the vital step of correctly interpreting the optimal solution to the primal. The solution values for the primal can be read directly from the optimal solution of the dual. This can be described in the following steps :

Step 1. Locate the slack variables in the dual programme. These correspond to the primal basic variables in the optimal solution.

Step 2. Read the values of the index numbers in the index row corresponding to the columns of the slack variables. This directly gives the optimal values of the basic primal variables.

Step 3. The optimal values of the objective function for the problems are the same.

$$\therefore \quad x_1=\frac{150}{8}\ ;\ x_2=x_3=0 \quad \text{and Min. Cost}=\frac{750}{4}$$

Remark. It may be noted that if the primal problem involves lesser number of variables than the number of restrictions (constraints), the computational procedure can be considerably reduced by converting it into dual and then solving it This offers advantage in number of applications.

EXERCISES

1. Solve the following primal graphically. Write down its dual and solve this also graphically.

Maximize $\qquad Z=x_1+5x_2$

subject to the constraints :

$$5x_1+6x_2\leqslant 30$$
$$3x_1+2x_2\leqslant 12$$
$$x_1,\ x_2\geqslant 0$$

[**Ans.** $x_1=0,\ x_2=5,\ Z_{max.}=25$]

2. Some town, a small township of 15,000 people, requires, on the average, 3,00,000 litres of water daily. The city is supplied from a central waterworks where the water is purified by such conventional methods as filtration and chlorination In addition, two different chemical compounds (*i*) softening chemical and (*ii*) health chemical, are needed for softening the water and for health purposes. The waterworks plans to purchase two popular brands that contain these chemicals. One unit of *ABC* Corporation's products give 8 kilogram of softening chemical and 3 kilogram of health chemical. One unit of *XYZ* chemical company's product contains 4 kilogram and 9 kilogram per unit, respectively.

To maintain the water at a minimum level of softness and to meet a minimum in health protection, experts have decided that 150 and 100 kilogram of the two chemicals that make up each product must be added to water daily. If costs per unit. for *ABC* corporation's and *PQR* chemical company's products are, Rs. 8 and Rs. 10 respectively, what is the optimum quantity of each product that should be used to meet the minimum level of softness and a minimum health standard ? Write also the dual to the above linear programming problem and solve it.

[**Hint.** The relevant data may be tabulated as below :

Chemical	*Brand*		*Daily Requirement*
	ABC	*XYZ*	
(*i*) Softening	8	4	150
(*ii*) Health	3	9	100
Cost/unit of each brand	8	10	

Primal : Minimise (cost), $Z=8x_1+10x_2$

Subject to the constraints :

$$8x_1+4x_2\geqslant 150$$
$$2x_1+9x_2\geqslant 100$$

and $$x_1\geqslant 0,\ x_2\geqslant 0.$$

Minimum cost = Rs. 185.

Dual : Maximise, $Z^*=150y_1+100y_2$

Subject to the constraints :

$$8y_1+2y_2\leqslant 8$$
$$4y_1+9y_2\leqslant 10$$
$$y_1,\ y_2\geqslant 0]$$

3. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 of calories. Two foods A and B are available at a cost of Rs. 4/- and Rs. 3/- per unit respectively. If one unit of A contains 200 units of vitamins, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, 2 units of minerals and 40 calories, find by simplex method what combination of foods be used to have least cost.

[**Hint** **Primal :** Minimize $Z=4x_1+3x_2$

Subject to the constraints :

$$200x_1+100x_2\geqslant 4000$$
$$x_1+2x_2\geqslant 50$$
$$40x_1+40x_2\geqslant 1400$$
$$x_1,\ x_2\geqslant 0.$$

Dual : Maximize $Z^*=4000y_1+50y_2+1400y_3$

Subject to the constraints :

$$200y_1+y_2+40y_3\leqslant 4$$
$$100y_1+2y_2+40y_3\leqslant 3$$
$$y_1,\ y_2,\ y_3\geqslant 0$$

$x_1=5,\ x_2=30,\ Z_{min}=$ Rs. 110]

4. A company makes three products, X, Y, Z out of three materials P_1, P_2 and P_3. The three products use units of three materials according to the following table :

	P_1	P_2	P_3
X	1	2	3
Y	2	1	1
Z	5	2	1

The unit profit contribution of the three products are Rs. 3, Rs. 4 and Rs. 5 respectively. Availabilities of the materials are 10, 12 and 15 units respectively. The problem is to determine the product mix that will maximise the total profit Solve the primal problem, write its dual and give the economic interpretation.

SECTION C

Probability

INTRODUCTION

Two types of phenomena have usually been observed in nature and in everyday life. These are :

(*i*) deterministic, and

(*ii*) probabilistic.

In the first type, the hypotheses are stated exactly and no 'chance elements' are involved subsequently during the analysis of the phenomenon. Consequently, in such a case predictions of complete reliability can be made, *e g*., if we are given that a train is running at a uniform speed of sixty kilometres per hour, then we can predict with cent per cent surety that it will cover one hundred twenty kilometres after two hours, assuming, of course, that it never stopped during these two hours. Most of the phenomena in physical and chemical sciences are of a deterministic nature. However, there exists a number of phenomena where we cannot make predictions with certainty or complete reliability and are known as *unpredictable* or *probabilistic* phenomenon. Such phenomena are frequently observed in business, economics and social sciences or even in our day-to-day life. For example :

(*i*) In toss of a uniform coin we are not sure of getting the head or tail.

(*ii*) A manufacturer cannot ascertain the future demand of his product with certainty.

(*iii*) A sales manager cannot predict with certainty about the sales target next year.

(*iv*) If an electric tube has lasted for one year, nothing can be predicted about its future life.

Probability is also used informally in day-to-day life. We daily come across the sentences like :

(*i*) Possibly, it will rain to-night.

(*ii*) There is a high chance of your getting the job in October.

(*iii*) This year's demand for the product is likely to exceed that of the last year's.

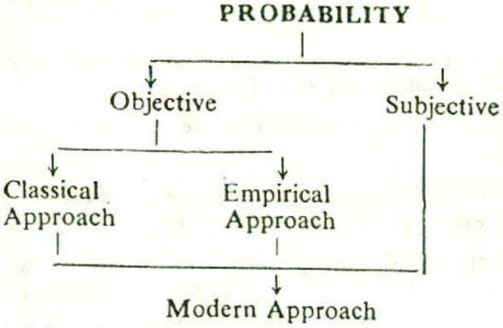
(*iv*) The odds are 2 : 1 in favour of getting the contract applied for.

All the above sentences, with words like 'possibly', 'high chance', 'likely' and 'odds' are expressions indicating a degree of uncertainty about the happening of the event. A numerical measure of uncertainty is provided by a very important branch of statistics called the "Theory of Probability ". Broadly, there are three possible states of expectation— *certainty*', '*impossibility*' and '*uncertainty*'. The probability theory describes 'certainty' by 1, impossibility by 0 and the various grades of uncertainties by coefficients ranging between 0 and 1.

According to Prof Ya-Lin-chou "*Statistics is the science of decision-making with calculated risks in the face of uncertainty.*"

MEASUREMENT OF PROBABILITY

The following is the broad classification of the different schools of thought in probability :



Brief description of these concepts is given below.

OBJECTIVE PROBABILITY

The objective probability is based on certain laws of nature, which are undisputed, or on some experiments conducted for the purpose. This is not based on the impressions of the individuals as is the case with subjective probability. These theories, therefore, are free from personal bias and ensure objectivity. The two approaches to objective probability are (*a*) classical approach, (*b*) empirical approach.

Fundamental Concepts

1. Random Experiment. An operation which can produce any result or outcome is called an experiment. An experiment is called a *random experiment* if, when conducted repeatedly under essentially homogeneous conditions, the result is not unique but may be any one of the various possible outcomes (The word random may be taken as one '*depending on chance*' without any bias). For example :

(*i*) Tossing a fair coin is an experiment. (A coin is a circular metal disc, the two faces of which are somehow distinguishable and are called 'head' and 'tail'.) Whether the coin will throw up head or tail is unpredictable.

(*ii*) Rolling an unbiased die is an experiment. (A die is a solid cube, the six faces of which are marked with 1, 2, 3, 4, 5 and 6 dots or actual figures 1, 2, 3, 4, 5, 6 respectively.) How many dots it will actually throw up is unpredictable and is subject to chance.

(*iii*) Drawing a card from a well-shuffled pack of playing cards is an experiment and as there are 52 cards in the pack and any of these may be drawn in a specific trial, which card it will turn out is unpredictable.

(*iv*) Drawing two balls at random from a box containing, say, 8 white, 9 red and 7 green balls, all well-mixed is an experiment. Which particular ball will be drawn is unpredictable.

(*v*) When a coin is tossed 100 times or 100 coins are tossed together, there are hundred experiments.

(*vi*) Experiments in business world can be in regard to the observation of the number of defective items produced by a machine, or recording the number of customers visiting a sale counter. In an advertising campaign for a new product launched, the number of items sold may be observed.

2. **Elementary Event.** Each one of the possible outcome in a single experiment is called an elementary event.

(*i*) In an experiment of tossing a coin there are 2 possible elementary events, the head and the tail'.

(*ii*) In an experiment which consists of throwing a six-faced die, the possible elementary events are 1, 2, 3, 4, 5 and 6.

(*iii*) In an experiment of drawing a card of a given designation from a pack of cards, there are 4 possible outcomes corresponding to 4 suits with designations of heart, diamond, spade and club.

(*iv*) In a trial of drawing a card from a suit of spade alone, there are 13 elementary events, *viz.*, 1 to 13 cards.

(*v*) In a trial amongst 12 face cards, there are 4 elementary events, *viz.*, king, queen and jack.

3. **Exhaustive Cases or Outcomes.** The total number of possible outcomes of a random experiment is called the *exhaustive cases* for the experiment. Thus in toss of a single coin, we can get head (H) or tail (T). Hence exhaustive number of cases is 2, *viz*, (H, T). If two coins are tossed, the various possibilities are HH, HT, TH, TT, where HT means head on the first coin and tail on second coin and TH means tail on the first coin and head on the second coin and so on. Thus in case of toss of two coins, exhaustive number of cases is 4, *i.e.*, 2^2. Similarly, in a toss of three coins the possible number of outcomes is :

$$(H, T)\times(H, T)\times(H, T)$$

$$=(HH, HT, TH, TT)\times(H, T)$$

$$=(HHH, HTH, THH, TTH, HHT, HTT, THT, TTT)$$

4. **Favourable Cases.** The number of outcomes of a random experiment which entail (or result in) the happening of an event are termed as the cases favourable to the event. For example :

(*i*) In a toss of two coins, the number of cases favourable to the event "*exactly one head*" is 2, HT, TH and for getting '*two heads*' is one, *viz.*, HH.

(*ii*) In drawing a card from a pack of cards, the cases favourable to 'getting a club' are 13 and to 'getting an ace of club' is only 1.

5. **Mutually Exclusive Events or Cases.** Two or more events are said to be mutually exclusive if the happening of any one of them precludes the happening of all others in the same experiment. For example, in tossing of a coin the events 'head' and 'tail' are mutually exclu-

sive because if head comes, we can't get tail and if tail comes we can't get head. Similarly, in the throw of a die, the six faces numbered 1, 2, 3, 4, 5 and 6 are mutually exclusive. Thus events are said to be mutually exclusive if no two or more of them can happen simultaneously.

6. **Equally Likely Cases.** The outcomes are said to be *equally likely* or *equally probable* if none of them is expected to occur in preference to other. Thus, in tossing of a coin (die), all the outcomes, *viz.*, H, T (the faces 1, 2, 3, 4, 5, 6) are equally likely if the coin (die) is unbiased.

Independent Events. Events are said to be independent if the occurrence of one event in no way affects the occurrence of the other. For example :

(*i*) In tossing of a coin, the event of getting 'head' in first throw is independent of getting 'head' in second, third or subsequent throws.

(*ii*) In drawing cards from a pack of cards, the result of the second draw will depend upon the card drawn in the first draw. However, if the card drawn in the first draw is replaced before drawing the second card, then the result of second draw will be independent of the 1st draw.

Similarly, drawing of balls from an urn gives independent events if the draws are made with replacement. If the ball drawn in the earlier draw is not replaced, the resulting draws will not be independent.

Mathematical or Classical or 'a Priori' Probability Definition. *If a random experiment results in N exhaustive, mutually exclusive and equally likely cases (outcomes), out of which m are favourable to the happening of an event A, then the probability of occurrence of A, usually denoted by $P(A)$ is given by :*

$$P(A)=\frac{\text{Number of outcomes favourable to the occurrence of } A}{\text{Exhaustive number of outcomes}}$$

$$=\frac{m}{n}$$

This definition was introduced by James Bernoulli.

Remarks. 1. Probability that event A will not occur, denoted by $P(\overline{A})$ is

$$P(\overline{A})=\frac{\text{Number of outcomes not favourable to occurrence of } A}{\text{Exhaustive number of outcomes}}$$

$$=\frac{N-m}{N}=1-\frac{m}{N}=1-P(A)$$

2 Probability of any event is a number lying between 0 and 1, *i.e.*,

$$0\leqslant P(A)\leqslant 1,$$

for any event A. If $P(A)=0$, then A is called an *impossible* or *null* event. If $P(A)=1$, then A is called a *certain* or *sure* event.

3. The probability of happening of the event A, *i.e*, $P(A)$ is also known as the probability of success and is usually written as p and

the probability of the non-happening, *i.e.*, $P(\overline{A})$ is known as the probability of failure, which is usually denoted by q. Thus,

$$q=1-p \quad \Rightarrow \quad p+q=1$$

4. Limitations. The classical probability has its shortcomings and fails in the following situations :

(*i*) If N, the exhaustive number of outcomes of the random experiment, is infinite.

(*ii*) If the various outcomes of the random experiment are not equally likely. For example, if a person jumps from the running train, then the probability of his survival will not be 50%, since in this case the two mutually exclusive and exhaustive outcomes, *viz.*, survival and death, are not equally likely.

Statistical or Empirical Probability

Definition (Von Mises). *If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.*

Suppose that an event A occurs m times in N repetitions of a random experiment. Then the ratio m/N, gives the *relative frequency* of the event A and it will not vary appreciably from one trial to another. In the limiting case when N becomes sufficiently large, it more or less corresponds to a number which is called the probability of A. Symbolically,

$$P(A)=\lim_{N\to\infty}\frac{m}{N}$$

Remarks. 1. Since in the relative frequency approach, the probability is obtained objectively by repetitive empirical observations, it is also known as *Empirical Probability*.

2. The empirical probability provides validity to the classical theory of probability. If an unbiased coin is tossed at random, then the classical probability gives the probability of a head as ½. Thus, if we toss an unbiased coin 10 times, then classical probability suggests we should have 5 heads. However, in practice, this will not generally be true. In fact in 10 throws of a coin, we may get no head at all or 1 or 2 heads. J.E. Kerrich conducted coin tossing experiment with 10 sets of 1,000 tosses each during his confinement in World War II. The number of heads found by him were :

502, 511. 497, 529, 504, 476, 507, 520, 504, 529

This shows that the probability of getting a head in a toss is nearly ½. Thus, the empirical probability approaches the classical probability as the number of trials becomes indefinitely large.

3. *Limitations.* (*i*) The experimental conditions may not remain essentially homogeneous and identical in a large number of repetitions of the experiment.

(*ii*) The relative frequency m/N, may not attain a unique value no matter however large N may be.

Axiomatic Probability

The modern theory of probability is based on the axiomatic approach introduced by the Russian mathematician A.N. Kolmogorov in 1934. Kolmogorov axiomised the theory of probability and his small book '*Foundations of Probability*' published in 1933, introduces probability as a set function and is considered as a classic. In axiomatic approach, to start with some concepts are laid down and certain *properties* or *postulates* commonly known as *axioms* are defined and from these axioms alone the entire theory is developed by logic of deduction. The axiomatic definitions of probability includes both the classical and empirical definitions of probability and at the same time is free from their drawbacks. Before giving axiomatic definition of probability, we shall explain certain concepts, used therein.

1. **Sample Space.** A set whose elements represent the possible outcomes of an experiment is called a *sample space*, which is a universal set and is denoted by S. Each possible outcome given in the sample space is called a *sample point*. The number of sample points in S may be denoted as $N(S)$. For example :

(*i*) If one item is drawn from a manufactured product, the item selected may either be defective D, or non-defective N. Then the sample space is

$$S = \{D, N\}$$

(*ii*) When a coin and a die are tossed together, there are twelve sample points in the sample space :

$$S = \{(T, 1), (T. 2), (T, 3), (T, 4), (T, 5), (T, 6), (H, 1), (H, 2), (H, 3), (H. 4), (H, 5), (H, 6)\}.$$

(*iii*) If a pair of dice is to be cast once, the 36 possible outcomes of this experiment will be :

Outcome of First Die	*Outcome of Second Die*					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6. 1)	(6. 2)	(6. 3)	(6. 4)	(6. 5)	(6, 6)

The sample space of the experiment.

2. **Event Sets** An 'event set' is a subset of the sample space. Thus on a sample space there can be two or more event sets consisting of a group of elementary events (sample point).

For example, in the experiment of picking two items, one at a time, at random, from a box containing defective and non-defective items, "both items are defective" is one event, "both items are non-defective" is another event.

The event sets are denoted by capital letter A, B, C or $E_1, E_2, \ldots$ etc. The sample points in each set may be denoted by small letters, say a, b, c or a_1, a_2, a_3, or by any other suitable description. The number of sample points in an event set A may be denoted by $n(A)$.

Remarks 1. An event E defined over a sample space S is said to be a *sample*, or *elementary*, or *fundamental* event if it contains exactly one sample point in S. An event E defined over a sample space S is called a *composite*, or *compound event* or simply an *event*, if it contains more than one sample point in S. Thus, when a die is tossed, each of the elements in the sample space $S=\{1, 2, 3, 4, 5, 6\}$ is a simple event ; but the events $E_1=\{1, 3, 5\}$ and $E_2=\{2, 4, 6\}$ are composite.

2. A set of events defined over the same sample space is said to be *mutually exclusive*, or *disjoint*, if no sample point is contained in more than one of these events, *i.e.*, a set of events $\{E_1, E_2, \ldots\}$ is mutually exclusive if no two sets have any sample points in common.

3. Two or more events defined over the same sample space are said to be *collectively exhaustive* if their union is equal to the sample space.

Axiomatic probability (Definition). Given a sample space of a random experiment, the probability of the occurrence of any event A is defined as a probability function $P(A)$ satisfying the following axioms.

Axiom 1. The probability of an event exists, is real and non-negative, *i.e.*,

$$P(A) \geqslant 0$$

Axiom 2. The probability of the entire sample space is 1, *i.e.*,

$$P(S)=1$$

Axiom 3. If $A_1, A_2, A_3, \ldots$ be a finite or infinite sequence of disjoint events of S, then

$$P(A_1+A_2+\ldots)=P(A_1)+P(A_2)+P(A_3)+\ldots$$

Remark. The above axioms are also krown as axioms of *positiveness*, *certainty* and *complete additivity* respectively.

Example 1. (*a*) *Find the probability of getting head in a throw of a coin.*

(*b*) *If two coins are tossed once, what is the probability of getting* (*i*) *both heads*, (*ii*) *at least one head* ?

Solution. (*a*) When a coin is tossed, there are two possible outcomes—head or tail.

$$n=2$$

The outcome 'head' is the favourable case.

$\therefore \quad m=1$

Hence $\quad P(\text{Head})=\frac{1}{2}$

(*b*) When two coins are tossed there are four possible cases, *viz.*,

HH : Head on the first coin and head on the second coin

HT : Head on the first coin and tail on the second.

TH : Tail on the first coin and head on the second.

TT : Tail on the first and tail on the second.

$\therefore \quad n=4$

(*i*) Out of these 4 cases, we need heads on both, *i.e.*, the HH

$\therefore \quad m=1$

Hence $\quad P(\text{both heads})=P(HH)=\frac{1}{4}$

(*ii*) In three cases HH, HT and TH, we get at least one head.

$\therefore \quad P(\text{at least one head})=\frac{3}{4} \text{ or } 1-\frac{1}{4}$

Example 2. *What is the chance that a leap year selected at random will contain 53 Sundays ?*

Solution. In a leap year (which consists of 366 days) there are 52 complete weeks and 2 more days. The following are the possible combinations for these two 'over' days :

(*i*) Sunday and Monday, (*ii*) Monday and Tuesday, (*iii*) Tuesday and Wednesday, (*iv*) Wednesday and Thursday, (*v*) Thursday and Friday, (*vi*) Friday and Saturday, and (*vii*) Saturday and Sunday.

In order that a leap year selected at random should contain 53 Sundays, one of the two 'over' days must be a Sunday. Since out of the above 7 possibilities, 2, *viz.*, (*i*) and (*vii*) are favourable to this event,

the required probitability$=\frac{2}{7}$

Example 3. *Three unbiased coins are tossed. What is the probability of obtaining* (*i*) *all heads,* (*ii*) *two heads,* (*iii*) *one head,* (*iv*) *at least one head,* (*v*) *at least two heads,* (*vi*) *all tails ?*

Solution. There are 2^3 or 8 possible cases, *viz.*, HHH, HHT, HTH, THH, HTT, THT, TTH and TTT (the three letters in each case denoting the results on the 1st, 2nd and 3rd coins respectively). These are mutually exclusive, exhaustive and equally likely cases.

The cases favourable to the events are as follows :

	Event	*Favourable cases*	*Number of favourable cases*
A :	All heads	HHH	1
B :	Two heads	HHT, HTH, THH	3
C :	One head	HTT, THT, TTH	3
D :	At least one head	HTT, THT, TTH, HHT, HTH, THH, HHH	7
E :	At least two heads	HHT, HTH, THH, HHH	4
F :	All tails	TTT	1

Applying the classical definition of probability, we have

(*i*) $P(A)=P(\text{all heads})=1/8$

(*ii*) $P(B)=P(\text{two heads})=3/8$

(*iii*) $P(C)=P(\text{one head})=3/8$

(*iv*) $P(D)=P(\text{at least one head})=7/8$

(*v*) $P(E)=P(\text{at least two heads})=4/8=1/2$

(*vi*) $P(F)=P(\text{all tails})=1/8$

Example 4. *Two unbiased dice are thrown. Find the probability at*

(*a*) *both the dice show the same number,*

(*b*) *the first dice shows 6,*

(*c*) *the total of the numbers on the dice is greater than 8.*

Solution. Each of the six faces of one die can be associated with each of the six faces of the other die, so that the total number of equally likely cases which can arise would be 6×6, *i.e.*, 36. These can be denoted as

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

The expression say (3, 4) means the first die shows 3 and the second die 4. The total of all possible events is

$$N=36$$

(*a*) The favourable cases are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6). Therefore $m=6$

∴ Probability that the two dice show the same number

$$=\frac{6}{36}=\frac{1}{6}$$

(*b*) For out of 36 cases, the first die shows 6 in the following cases : (6, 1), (6, 2), (6, 3), (6, 4), (6, 5) and (6, 6). $m=6$.

$\therefore$ Probability that the first die shows '6' $=\frac{6}{36}=\frac{1}{6}$

(*c*) The cases which give a total of more than 8 are (3, 6), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4) (6, 5) and (6, 6).

$$\therefore \qquad m=10$$

$\therefore$ Probability that the total is greater than $8=\frac{10}{36}=\frac{5}{18}$

Example 5. *A bag contains 5 green and 7 red balls. Two balls are drawn. What is the probability that one is green and the other red ?*

Solution. Total number of balls $=5+7=12$

Now, out of 12 balls, 2 can be drawn in ${}^{12}C_2$ ways.

$\therefore$ Exhaustive number of cases $={}^{12}C_2=\frac{12\times 11}{2}=66$

Out of 5 green balls, 1 green ball can be drawn in ${}^5C_1$ ways and out of 7 red balls, one red ball can be drawn in ${}^7C_1$ ways. Since each of the former cases can be associated with each of the latter cases, the total number of favourable cases is ${}^5C_1\times{}^7C_1=5\times 7=35$.

$\therefore$ Required probability $=\frac{35}{66}$

Example 6. *Five men in a company of 20 are graduates. If 3 men are picked out of the 20 at random, what is the probability that they are all graduates ? What is the probability of at least one graduate ?*

Solution. There are ${}^{20}C_3$ possible ways of selecting groups of 3 men out of 20, and these groups are mutually exclusive, exhaustive and equally likely.

However, a group of 3 men (all graduates) out of 5 can be obtained in ${}^5C_3$ ways. Similarly, a group of no graduate out of remaining 15 can be obtained in ${}^{15}C_0$ ways. Therefore, the number of cases favourable to the event is ${}^5C_3\times{}^{15}C_0$.

Hence, the probability that all are graduates

$$=\frac{{}^5C_3\times{}^{15}C_0}{{}^{20}C_3}=\frac{10\times 1}{1140}=\frac{1}{114}$$

In order to find the probability of at least one graduate, it will be easier to find the probability of the complementary event, *viz.*, that 'none is a graduate'.

Probability that there is no graduate $=\frac{{}^{15}C_3\times{}^5C_0}{{}^{20}C_3}=\frac{455}{1140}=\frac{91}{228}$

Hence, the probability that there is at least one graduate

$$=1-\frac{91}{228}=\frac{137}{228}$$

ADDITION RULE OF PROBABILITY

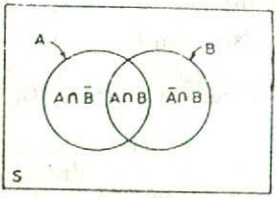
Statement. *The probability of occurrence of at least one of the two events A and B is given by :*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof. Let us suppose that a random experiment results in a sample space S with N sample points (exhaustive number of outcomes). Then by definition :

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A \cup B)}{N},$$

where $n(A \cup B)$ is the number of outcomes (sample points) favourable to the event $(A \cup B)$.



From the above diagram, we get

$$P(A \cup B) = \frac{[n(A) - n(A \cap B)] + n(A \cap B) + [n(B) - n(A \cap B)]}{N}$$

$$= \frac{n(A) + n(B) - n(A \cap B)}{N}$$

$$= \frac{n(A)}{N} + \frac{n(B)}{N} - \frac{n(A \cap A)}{N}$$

$$= P(A) + P(B) - P(A \cap B).$$

Remarks 1.. If the events A and B are mutually exclusive, *i.e.*, if $(A \cap B) = \phi$, then

$$P(A \cap B) = \frac{n(A \cap B)}{N} = \frac{n(\phi)}{N} = 0$$

Thus, the probability of happening of any one of the two mutually disjoint events is equal to the sum of their individual probabilities. Symbolically,

$$P(A \cup B) = P(A) + P(B)$$

2. For three events A, B and C, the probability of occurrence of at least one of them is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

Example 7. *A card is drawn from a well shuffled pack of playing cards. Find the probability that it is either a king or a spade.*

Solution. Let A denote the event of drawing a king and B denote the event of drawing a spade from a pack of cards. Then we have

$$P(A)=\frac{4}{52}=\frac{1}{13} \quad \text{and} \quad P(B)=\frac{13}{52}=\frac{1}{4}$$

There is only one outcome favourable to the event $A\cap B$, *viz.*, king of spade. Hence $P(A\cap B)=\frac{1}{52}$.

$$\therefore \qquad P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\text{or} \qquad P(A\cup B)=\frac{1}{4}+\frac{1}{13}-\frac{1}{52}=\frac{4}{13}.$$

Example 8. *The probability that a student passes an Accountancy test is $\frac{2}{3}$ and the probability that he passes both an Accountancy and Law test is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that he passes in the Law test ?*

Solution. Let us define the following events :

A : The student passes an Accountancy test.

B : The student passes a Law test.

We are given :

$$P(A)=\frac{2}{3}, \quad P(A\cap B)=\frac{14}{45} \text{ and } P(A\cup B)=\frac{4}{5}.$$

$$\text{Now} \qquad P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\Rightarrow \qquad \frac{4}{5}=\frac{2}{3}+P(B)-\frac{14}{45}$$

$$\Rightarrow \qquad P(B)=\frac{4}{5}+\frac{14}{45}-\frac{2}{3}=\frac{4}{9}$$

Example 9. *The probability that a contractor will get a plumbing contract is 2/3, and the probability that he will not get an electric contract is 5/9. If the probability of getting at least one contract is 4/5, what is the probability that he will get both the contracts ?*

Solution. Let A and B denote the events that the contractor will get a 'plumbing' contract and 'electric' contract respectively. Then we are given :

$$P(A)=\frac{2}{3}\ ;\ \ P(\overline{B})=\frac{5}{9}$$

$$\Rightarrow \qquad P(B)=1-P(\overline{B})=\frac{4}{9}$$

and $\qquad P(A\cup B)=$ Prob. that contractor gets at least one contract
$=4/5$

$$\Rightarrow \quad P(A)+P(B)-P(A\cap B)=\frac{4}{5} \quad \text{[By addition rule of probability]}$$

$$\Rightarrow \qquad \frac{2}{3}+\frac{4}{9}-P(A\cap B)=\frac{4}{5}$$

$$\Rightarrow \qquad P(A\cap B)=\frac{2}{3}+\frac{4}{9}-\frac{4}{5}=\frac{14}{45}$$

Hence the probability that the contractor will get both the contracts is 14/45.

Example 10. *A question paper contains 6 questions of equal value divided into two sections of three questions each. If each question poses the same amount of difficulty to Mr. X, an examinee, and he has only 50% chance of solving it correctly, find the answer to any one of the following :*

(*i*) *If Mr. X is required to answer only three questions from any one of the sections, find the probability that he will solve all the three questions.*

(*ii*) *If Mr. X is given the option to answer the three questions by selecting one question out of the two standing at serial number one in the two sections, one question out of the two standing at serial number two in the two sections, and one question out of the two standing at serial number three in the two sections, find the probability that he will solve all the three questions correctly.* [*Delhi Univ., B. Com.* (*Hons*), *1992*]

Solution. (*i*) Mr. X will solve all the three questions correctly, if he is able to solve :

(1) all the questions of the first section and not all the questions of the second section ;

(2) all the questions of the second section and not all the questions of the first section ; or

(3) all the questions of both the sections.

Hence required probability

$$=\left(\frac{1}{8}\right)\left(1-\frac{1}{8}\right)+\left(\frac{1}{8}\right)\left(1-\frac{1}{8}\right)+\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)$$

$$=\frac{7}{64}+\frac{7}{64}+\frac{1}{64}=\frac{15}{64}.$$

(*iii*) Mr. X will solve a question correctly, if he is able to solve at least one of the questions standing at the particular serial number in the two sections, the probability of which is $1-\frac{1}{4}=\frac{3}{4}$.

Hence required probability

$$=\left(\frac{3}{4}\right)^3=\frac{27}{64}.$$

MULTIPLICATION RULE OF PROBABILITY

Statement. *The probability of simultaneous occurrence of two events A and B is given by*

$$\left.\begin{array}{l} P(A\cap B)=P(A)\,.\,P(B\mid A)\,;\; P(A)\neq 0 \\ P(B\cap A)=P(B)\,.\,P(A\mid B)\,;\; P(B)\neq 0 \end{array}\right]$$

or

where $P(B \mid A)$ is the conditional probability of happening of B under the condition that A has already happened and $P(A \mid B)$ is the conditional probability of happening of A under the condition that B has already happened.

Proof. Let A and B be the events associated with the sample space S of a random experiment with exhaustive number of outcomes (sample points) N, *i.e.*, $n(S)=N$. Then by definition :

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)} \qquad \ldots(*)$$

For the conditional event $A \mid B$ (*i.e.*, the happening of A under the condition that B has happened), the favourable outcomes (sample points) must be out of the sample points of B In other words, for the event $A \mid B$, the sample space is $n(B)$ and hence

$$P(A \mid B)=\frac{n(A\cap B)}{n(B)}$$

Similarly, we have

$$P(B \mid A)=\frac{n(B\cap A)}{n(A)} \qquad \ldots(**)$$

Rewriting (*), we get

$$P(A\cap B)=\frac{n(A)}{n(S)}\times\frac{n(A\cap B)}{n(A)}$$

$$=P(A)\,.\,P(B \mid A) \qquad [\text{From } ..(**)]$$

Also

$$P(A\cap B)=\frac{n(B)}{n(S)}\times\frac{n(A\cap B)}{n(B)}$$

$$=P(B)\,.\,P(A \mid B)$$

Remarks. 1. Multiplication Rule for Independent Events. If A and B are *independent* so that the probability of occurrence or non-occurrence of A is not affected by the occurrence or non-occurrence of B, we have

$$P(A \mid B)=P(A) \quad \text{and} \quad P(B \mid A)=P(B)$$

Hence substituting in (**), we get

$$P(A\cap B)=P(A)\ P\ B)$$

Hence *the probability of simultaneous happening of two independent events is equal to the product of their individual probabilities.*

2. The multiplication rule of probability can be extended to more than two events. Thus, for three events A, B and C, we have

$$P(A\cap B\cap C)=P(A)\ P(B \mid A)\ P(C \mid A\cap B)$$

3. *If events A and B are independent then the complementary events $\overline{A}$ and $\overline{B}$ are also independent.*

Proof. We know

$$P(A\cup B)+P(\overline{A\cup B})=1$$

$$\Rightarrow \quad P(A\cup B)+P(\overline{A}\cap\overline{B})=1 \qquad \text{(By De-Morgan's Law)}$$

$$\Rightarrow \quad P(\overline{A}\cap\overline{B})=1-P(A\cup B)$$

$$=1-[P(A)+P(B)-P(A\cap B)$$

$$=1-P(A)-P(B)+P(A)P(B)$$

$$(\because \ A \text{ and } B \text{ are independent events})$$

$$=1-P(A)-P(B)\,[1-P(A)]$$

$$=[1-P(A)]\,[1-P(B)]=P(\overline{A}).P(\overline{B})$$

$\Rightarrow$ $\overline{A}$ and $\overline{B}$ are independent events.

4. P(happening of at least one of the events A, B and C)

$$=1-P(\text{none of the events } A, B, C \text{ happens})$$

or equivalently,

$$P(A\cup B\cup C)=1-P(\overline{A}\cap\overline{B}\cap\overline{C})$$

$$=1-P(\overline{A}).\ P(\overline{B}).\ P(\overline{C})$$

(If A, B and C are independent events).

Example 11. *A bag containts 8 red and 5 white balls. Two successive drawings of 3 balls are made such that (i) balls are replaced before the second trial, (ii) the balls are not replaced before the second trial. Find the probability that the first drawing will give 3 white and the second 3 red balls.*

Solution. Let A denote the event of drawing 3 white balls in the first draw and B denote the event of drawing 3 red balls in the second draw. Then we have to find the probability $P(A\cap B)$.

(*i*) *Draws with replacement.* If the balls drawn in the first draw are replaced back in the bag before the 2nd draw then the event A and B are independent and the required probability is given (by the multiplication rule of probability) by the expression

$$P(A\cap B)=P(A)\ .\ P(B) \qquad \ldots(^*)$$
$$=\frac{^5C_3}{^{13}C_3}\times\frac{^8C_3}{^{13}C_3}$$

(*ii*) *Draws without replacement.* If the balls drawn are not replaced back before the second draw, then the events A and B are not independent and the required probability is given by

$$P(A\cap B)=P(A)\ .\ P(B\mid A) \qquad \ldots(^{**})$$

As discussed in part (*i*),

$$P(A)=\frac{^5C_3}{^{13}C_3}$$

Now, if the 3 white balls which were drawn in the first draw are not replaced back, there are $13-3=10$ balls left in the bag and $P(B\mid A)$ is the conditional probability of drawing 3 red balls from the bag containing 10 balls out of which 2 are white and 8 are red.

Hence $$P(B\mid A)=\frac{^8C_3}{^{13}C_3}$$

Substituting in (**), we get

$$P(A\cap B=\frac{^5C_3}{^{13}C_3}\times\frac{^8C_3}{^{10}C_3}$$

Example 12. *Let A and B be the two possible outcomes of an experiment and suppose*

$P(A)=0\cdot4,\qquad P(A\cup B)=0\cdot7\qquad$ *and* $\qquad P(B)=p$

(*i*) *For what choice of p are A and B mutually exclusive ?*

(*ii*) *For what choice of p are A and B independent ?*

Solution : (*i*) We have

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$\Rightarrow \quad P(A\cap B)=P(A)+P(B)-P(A\cup B)$$
$$=0\cdot4+p-0\cdot7$$
$$=p-0\cdot3$$

If A and B are mutually exclusive, then

$$P(A\cap B)=0 \quad\Rightarrow\quad p-0\cdot3=0 \quad\Rightarrow\quad p=0\cdot3$$

(*ii*) A and B are independent if

$$P(A\cap B)=P(A)\ .\ P(B)$$
$$\Rightarrow \quad p-0\cdot3=0\cdot4\times p$$
$$\Rightarrow \quad 0\cdot6p=0\cdot3$$
$$\Rightarrow \quad p=\frac{0\cdot3}{0\cdot6}=0\cdot5$$

Example 13. *The probability that a management trainee will remain with a company is 0·60. The probability that an employee earns more than Rs. 10,000 per year is 0·50. The probability that an employee is a management trainee who remained with the company or who earns more than Rs. 10,000 per year is 0·70. What is the probability that an employee* *earn more than Rs. 10,000 per year given that he is a management trainee who stayed with the company ?*

Solution. Let us define the events :

A : A management trainee will remain with the company.

B : An employee who earns more than Rs. 10,000/-

Then we are given

$$P(A)=0{\cdot}60 \text{ and } P(B)=0{\cdot}50$$

Also we are given

P(A management trainee remains with the company *or* earns more than Rs. 10,000 per year)$=0{\cdot}70$

$$\Rightarrow \quad P(A\cup B)=0{\cdot}70$$

$$\Rightarrow \quad P(A)+P(B)-P(A\cap B)=0{\cdot}70$$

$$\Rightarrow \quad P(A\cap B)=P(A)+P(B)-0{\cdot}7$$

$$=0{\cdot}6+0{\cdot}5-0{\cdot}7-0{\cdot}4$$

Required probability is

$$P(B \mid A)=\frac{P(A\cap B)}{P(A)}=\frac{0{\cdot}4}{0{\cdot}6}=\frac{2}{3}$$

Example 14. *The odds against student X solving a Business Statistics problem are 8 : 6, and odds in favour of student Y solving the same problem are 14 : 16.*

(*i*) *What is the probability that neither solves the problem, if they both try, independently of each other ?*

(*ii*) *What is the chance that the problem will be solved.*

Solution. Let A denote the event that student X solves the problem and B denote the event that the student Y solves the problem. Then we are given :

$$P(\bar{A})=\frac{8}{14}=\frac{4}{7} \quad \Rightarrow \quad P(A)=1-P(\bar{A})=\frac{6}{14}=\frac{3}{7}$$

$$P(B)=\frac{14}{30}=\frac{7}{15} \quad \Rightarrow \quad P(\bar{B})=1-P(B)=\frac{16}{30}=\frac{8}{15}$$

(*i*) The probability that neither X nor Y solves the problem is given by :

$$\bar{P}(\bar{A}\ \bar{\cap}\ \bar{B})=P(\bar{A})\times P(\bar{B})$$

[Since A and B are independent $\Rightarrow$ $\bar{A}$ and $\bar{B}$ are independent]

$$=\frac{4}{7}\times\frac{8}{15}=\frac{32}{105}$$

(*ii*) The problem will be solved if at least one of the students X and Y solves the problem. Hence the required probability is given by :

$$P(A \cup B) = \text{Probability that at least one of } X \text{ and } Y \text{ solves the problem}$$

$$= 1 - \text{Probability that none solves the problem}$$

$$= 1 - P(\bar{A} \cap \overline{B}) = 1 - \frac{32}{105} = \frac{73}{105}.$$

Example 15. *It is 8 : 5 against a husband who is 55-year-old living till he is 75 and 4 : 3 against his wife who is now 48, living till she is 68. Find the probability that (i) the couple will be alive 20 years hence, and (ii) at least one of them will be alive 20 years hence.*

Solution. Let A denote the event that husband will be alive 20 years hence and B denote the event that wife will be alive 20 years hence. Then we are given that

$$P(A) = \frac{5}{13} \quad \Rightarrow \quad P(\bar{A}) = 1 - \frac{5}{13} = \frac{8}{13}$$

$$P(B) = \frac{3}{7} \quad \Rightarrow \quad P(\overline{B}) = 1 - \frac{3}{7} = \frac{4}{7}$$

(*i*) The event that couple is alive 20 years hence is given by $A \cap B$.

$\therefore$ Required probability $= P(A \cap B)$

$$= P(A) \times P(B)$$

(By multiplication rule of probability, since A and B are independent and consequently $\bar{A}$ and $\overline{B}$ are independent).

$$= \frac{5}{13} \times \frac{3}{7} = \frac{15}{91}$$

(*ii*) The event that at least one of the persons A and B is alive 20 years hence is given by $A \cup B$.

$\therefore$ Required probability $= P(A \cup B)$

$= 1 - P$ (None of A and B is alive 20 years hence)

$$= 1 - P(\overline{A}) \cap B)$$

$$= 1 - P(\overline{A}) \,.\, P(\overline{B})$$

$$= 1 - \frac{8}{13} \times \frac{4}{7} = \frac{59}{91}$$

Example 16. *A candidate is selected for interview for three posts. For the first post there are 3 candidates for the second there are 4 and for the third there are 2. What are the chances of his getting at least one post ?*

Solution. Let A, B and C denote the events that the candidate is selected for the first, second and third post respectively. Since the selection of each candidate is equally likely, we have

$$P(A)=\frac{1}{3} \quad \Rightarrow \quad P(\overline{A})=\frac{2}{3}$$

$$P(B)=\frac{1}{4} \quad \Rightarrow \quad P(\overline{B})=\frac{3}{4}$$

$$P(C)=\frac{1}{2} \quad \Rightarrow \quad P(C)=\frac{1}{2}$$

The probability that the candidate is selected for at least one post is given by

$$P(A\cup B\cup C)=1-P(\overline{A}\cap\overline{B}\cap\overline{C})$$

$$=1-P(\overline{A})\,.\,P(\overline{B})\,.\,P(\overline{C})$$

[Since the events A, B and C are independent]

$$=1-\frac{2}{3}\times\frac{3}{4}\times\frac{1}{2}=\frac{3}{4}$$

Example 17. *A piece of equipment will function only when all the three components A, B and C are working. The probability of A failing during one year is 0·15, that of B failing is 0·05 and that of C failing is 0·10. What is the probability that the equipment will fail before the end of the year ?*

Solution. Let us define the events :

A_1 : Component A fails

A_2 : Component B fails

A_3 : Component C fails

We are given

$$P(A_1)=0{\cdot}15,\ P(A_2)=0{\cdot}05,\ P(A_3)=0{\cdot}10$$

Probability that equipment will fail before the end of the year is given by :

$$P(A_1\cup A_2\cup A_3)=1-P(\overline{A}_1\cap\overline{A}_2\cap\overline{A}_3)$$

$$=1-P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)$$

$$=1-(1-0{\cdot}15)\times(1-0{\cdot}05)\times(1-0{\cdot}10)$$

$$=1-0{\cdot}72675=0{\cdot}27325$$

Example 18. *A bag contains 5 white and 3 black balls and four are successively drawn out and not replaced. What is the probability that they are alternatively of same colours ?*

Solution The required event can materialise in the following mutually exclusive ways :

(*i*) The balls are white, black, white and black in the first, second, third and fourth draw respectively.

(*ii*) The balls are black, white, black and white in the first, second, third and fourth draw respectively.

Hence by addition rule, the required probability 'p' is given by

$$p = P(i) + P(ii) \qquad \ldots(*)$$

Let A, B, C and D denote the event of drawing a white, black, white and black in the first, second, third and fourth draw respectively. Since the balls drawn are not replaced before the next draw, the constitution of the bag in the four draws is respectively :

1st draw	2nd draw	3rd draw	4th draw
$5W$ | $3B$	$4W$ | $3B$	$4W$ | $2B$	$3W$ | $2B$

$$\therefore \quad P(i) = P(A \cap B \cap C \cap D)$$

$$= P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B) \cdot P(D \mid A \cap B \cap C)$$

$$= \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} = \frac{1}{14}$$

Similarly $\quad P(ii) = \frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{1}{14}$

Substituting in (*), the required probability is

$$p = \frac{1}{14} + \frac{1}{14} = \frac{1}{7}$$

Example 19. *A bag contains 5 red and 3 black balls and the second one 4 red and 5 black balls. One of these is selected at random and a draw of two balls is made from it. What is the probability that one of them is red and the other black ?*

Solution. Two balls (one red and one black) can be obtained in the following mutually exclusive ways :

A : when bag I is selected and two balls are drawn from it.

B : when bag II is selected and two balls are drawn from it.

Hence by the *addition rule*, the required probability is given by

$$p = P(A) + P(B)$$

But A is itself a compound event consisting of (*i*) the selection of bag I, with probability $\frac{1}{2}$, and (*ii*) the drawing of two balls, one red and other black from it, with probability $\dfrac{{}^5C_1 \times C_1}{{}^8C_2}$

Hence by the multiplication rule, we have

$P(A)$=(Probability of selection of bag I)×(Probability of drawing one red and one black ball assuming that bag I is selected)

$$=\frac{1}{2}\times\frac{{}^5C_1\times{}^3C_1}{{}^8C_2}=\frac{1}{2}\times\frac{15}{28}=\frac{15}{56}$$

Similarly, $P(B)=\frac{1}{2}\times\frac{{}^4C_1\times{}^5C_1}{{}^9C_2}=\frac{1}{2}\times\frac{20}{36}=\frac{5}{18}$

Hence the required probability is

$$p=\frac{15}{56}+\frac{5}{18}=\frac{275}{504}$$

Example 20. *The odds that a book on Business Mathematics will be favourably reviewed by 3 independent critics are 3 to 2, 4 to 3 and 2 o 3 respectively. What is the probability that, of the three reviews*

(*a*) *all will be favourable.*

(*b*) *majority of the reviews will be favourable,*

(*c*) *exactly one review will be favourable, and*

(*d*) *exactly two reviews will be favourable,*

(*e*) *at least one of the reviews will be favourable.*

Solution. Let A, B and C denote respectively the events that the book is favourably reviewed by first, second and third critic respectively. Then we are given that

$$P(A)=\frac{3}{5},\ P(B)=\frac{4}{7},\text{ and }P(C)=\frac{2}{5}$$

$$\Rightarrow\qquad P(\overline{A})=\frac{2}{5},\ P(\overline{B})=\frac{3}{7}\text{ and }P(\overline{C})=\frac{3}{5}$$

(*i*) P(all the three reviews will be favourable)

$$=P(A\cap B\cap C)$$
$$=P(A).P(B).P(C)$$

[∵ A, B and C are independent]

$$=\frac{3}{5}\times\frac{4}{7}\times\frac{2}{5}=\frac{24}{175}$$

(*ii*) *P*(*majority, i.e.,* at least 2 reviews will be favourable)

$$=P(A\cap B\cap\overline{C})+P(A\cap\overline{B}\cap C)+P(\overline{A}\cap B\cap C)+P(A\cap B\cap C)$$
$$=P(A)P(B)P(\overline{C})+P(A)P(\overline{B})P(C)+P(\overline{A})P(B)P(C)$$
$$+P(A)P(B)P(C)$$

(∵ A, B and C are independent)

$$=\frac{3}{5}\times\frac{4}{7}\times\frac{3}{5}+\frac{3}{5}\times\frac{3}{7}\times\frac{2}{5}+\frac{2}{5}\times\frac{4}{7}\times\frac{2}{5}$$
$$+\frac{3}{5}\times\frac{4}{7}\times\frac{2}{5}=\frac{94}{175}$$

(*iii*) The probability that exactly one review will be favourable is given by

$$P(A\cap\overline{B}\cap\overline{C})+P(\overline{A}\cap B\cap\overline{C})+P(\overline{A}\cap\bar{B}P\cap C)$$

$$=P(A)P(\overline{B})P(\overline{C})+P(\overline{A})P(B)P(\overline{C})+P(\overline{A})P(\overline{B})P(C)$$

$$=\frac{3}{5}\times\frac{3}{7}\times\frac{3}{5}+\frac{2}{5}\times\frac{4}{7}\times\frac{3}{5}+\frac{2}{5}\times\frac{3}{7}\times\frac{2}{5}=\frac{63}{175}$$

(*iv*) Similarly, the probability that exactly two reviews will be favourable is given by

$$P(A\cap B\cap\overline{C})+P(A\cap\bar{B}\cap C)+P(\overline{A}\cap B\cap C)$$

$$=P(A)P(B)P(\overline{C})+P(A)P(\overline{B})P(C)+P(\overline{A})P(B)P(C)$$

$$=\frac{3}{5}\times\frac{4}{7}\times\frac{3}{5}+\frac{3}{5}\times\frac{3}{7}\times\frac{2}{5}+\frac{2}{5}\times\frac{4}{7}\times\frac{2}{5}=\frac{105}{175}$$

(*iv*) The probability that at least one of the reviews will be favourable is given by

$$P(A\cup B\cup C)=1-P(\overline{A}\cap\overline{B}\cap\overline{C})$$

$$=1-P(\overline{A})P(\overline{B})P(\overline{C})$$

$$=1-\frac{2}{5}\times\frac{3}{7}\times\frac{3}{5}=\frac{157}{175}$$

BAYES' RULE

One of the important applications of the conditional probability is in the computation of unknown probabilities, on the basis of the information supplied by the experiment or past records. For example, suppose we have two boxes containing defective and non-defective items. One item is picked at random from either one of the boxes and is found defective, and now we might like to know the probability that it came from Box 1 or Box 2. These probabilities are computed by *Bayes' Rule*, named so after the British Mathematician Thomas Bayes who propounded it in 1763.

Quite often the businessman has the extra information in a parttcular event, either through a personal belief or from the past history of the event Probabilities assigned on the basis of personal experience, before observing the outcomes of the experiment, are called prior probabilities. For example, probabilities assigned to past sales records, to past number of defectives produced by a machine. are examples of *prior probabilities*. When the probabilities are revised with the use of Bayes' rule, they are called *posterlor probabilities*. Bayes' rule is very useful in solving practical business problems in the light of additional information to arrive at valid decisions in the face of uncertainties.

Statement. *If an event B can only occur in conjunction with one of the n mutually exclusive and exhaustive events* $A_1, A_2.A_3, A_n$ *and if B actually happens, then the probability that it was preceded by the parti-*

cular event A_i $(i=1, 2, \ldots, n)$ is given by

$$P(A_i \mid B)=\frac{P(B\cap A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}=\frac{P(A_i)\, P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}$$

Proof. Since the event B can occur in combination with any of the mutually exclusive and exhaustive events $A_1, A_2, \ldots, A_n$, we have

$$B=(B\cap A_1) \cup (B\cap A_2) \cup \ldots \cup (B\cap A_n)$$

where $B\cap A_1$, $B\cap A_2$, ..., $B\cap A_n$ are all disjoint (mutually exclusive) events. Hence, by addition rule of probability, we have

$$\begin{aligned} P(B) &= P(B\cap A_1)+P(B\cap A_2)+\ldots+P(B\cap A_n) \\ &= P(A_1)\, P(B \mid A_1)+P(A_2)\, P(B \mid A_2)+\ldots+P(A_n)\, P(B \mid A_n) \\ &= \sum_{i=1}^{n} P(A_i)\, P(B \mid A_i) \end{aligned}$$

For any particular event A_i, the conditional probability $P(A_i \mid B)$ is given by

$$P(A_i\cap B)=P(B)\, P(A_i \mid B)$$

$$\Rightarrow \quad P(A_i \mid B)=\frac{P(A_i\cap B)}{P(B)}$$

$$=\frac{P(A_i)P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}$$

which is the Bayes' rule for obtaining the conditional probabilities.

Remark. The probabilities $P(A_1)$, $P(A_2)$, ..., $P(A_n)$ which are already given or known before conducting the experiment are termed as *a priori* or *prior* probabilities. The conditional probabilities $P(A_1 \mid B)$, $P(A_2 \mid B)$, ..., $P(A_n \mid B)$ which are computed after conducting the experiment, *viz.*, occurrence of A_i are called *a posteriori* or *posterior* probabilities.

Example 21. *Two sets of candidates are competing for the positions on the Board of Directors of a company. The probabilities that the first and second sets will win are 0·6 and 0·4 respectively. If the first set wins, the probability of introducing a new product is 0·8, and the corresponding probability if the second set wins is 0·3. What is the probability that the product will be introduced?*

Solution. Let A_1, A_2 denote the events that the first and second sets of candidates win respectively. Let B denote the event that '*new product*' is introduced.

We are given

$$P(A_1)=0\cdot6,\ P(A_2)=0\cdot4$$

$P(B \mid A_1)=0\cdot8=$ Probability that 'new product' will be introduced given that first set wins.

$$P(B \mid A_2)=0\cdot3$$

The event B can materialise in the following mutually exclusive ways :

(*i*) First set wins and the new product is introduced, *i.e.*, $A_1 \cap B$ happens.

(*ii*) Second set wins and the new product is introduced, *i.e.*, $A_2 \cap B$ happens. Thus

$$B=(A_1 \cap B) \cup (A_3 \cap B),$$

where $A_1 \cap B$ and $A_2 \cap B$ are disjoint.

Hence using addition rule of probability, we have

$$\begin{aligned} P(B) &= P(A_1 \cap B)+P(A_2 \cap B) \\ &= P(A_1)\ P(B \mid A_1)+P(A_2)\ P(B \mid A_2) \\ &= 0\cdot6\times0\cdot8+0\cdot4\times0\cdot3 \\ &= 0\cdot6. \end{aligned}$$

Example 22. *Suppose that a product is produced in three factories, A, B and C. It is known that factory A produces twice as many items as factory B, and that factories B and C produce the same number of items. Assume that it is known that 2 per cent of the items produced by each of the factories A and C are defective while 4 per cent of those manufactured by factory B are defective. All the items produced in the three factories are stocked, and an item of product is selected at random. What is the probability that this item is defective ?*

Solution. Let the number of items produced by each of factories B and C be n. Then the number of items produced by the factory A is $2n$. Let A_1, A_2 and A_3 denote the events that the item is produced by factory A, B and C respectively and let E be the event of the item being defective. Then we have :

$$P(A_1)=\frac{2n}{2n+n+n}=\frac{2n}{4n}=\frac{1}{2}=0\cdot5$$

$$P(A_2)=\frac{n}{4n}=\frac{1}{4}=0\cdot25$$

$$P(A_3)=\frac{n}{4n}=\frac{1}{4}=0\cdot25$$

$$P(E \mid A_1)=P(E \mid A_3)=0\cdot02 \text{ and } P(E \mid A_2)=0\cdot04 \text{ (Given)}$$

The probability that an item selected at random from the stock is defective is given by

$$P(E)=P[(E \cap A_1) \cup (E \cap A_2) \cup (E \cap A_3)]$$

$$=P(E\cap A_1)+P(E\cap A_2)+P(E\cap A_3)$$

[By addition rule of probability]

$$=P(A_1)\ P(E \mid A_1)+P(A_2)\ P(E \mid A_2)+P(A_3)\ P(E \mid A_3)$$
$$=0{\cdot}5\times0{\cdot}02+0{\cdot}25\times0{\cdot}04+0{\cdot}25\times0{\cdot}02$$
$$=0{\cdot}025.$$

Example 23. *A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooters and Plant II manufactures 30%. At plant I, 80% of scooters are rated standard quality and at plant II 90% of scooters are rated standard quality. A scooter is picked up at random and is found to be of standard quality. What is the chance that it has come from plant I ?*

Solution. Let us define the following events :

A_1 : Scooter is manufactured by plant I

A_2 : Scooter is manufactured by plant II

B : Scooter is rated as standard quality.

Then we are given :

$$P(A_1)=0{\cdot}70,\ P(A_2)=0{\cdot}30,$$
$$P(B \mid A_1)=0{\cdot}80,\ P(B \mid A_2)=0{\cdot}90$$

Using Bayes' rule, required probability is

$$P(A_1 \mid B)=\frac{P(A_1)\ P(B \mid A_1)}{P(A_1)\ P(B \mid A_1)+P(A_2)\ P(B \mid A_2)}$$
$$=\frac{0{\cdot}70\times0{\cdot}80}{0{\cdot}70\times0{\cdot}80+0{\cdot}30\times0{\cdot}90}=\frac{0{\cdot}56}{0{\cdot}83}=\frac{56}{83}.$$

Example 24. *In an automobile factory, certain parts are to be fixed to the chasis in a section before it moves into another section. On a given day, one of the three persons A, B and C carries out this task. A has 45%, B has 35% and C has 20% chance of doing it. The probabilities that A, B and C will take more than the ellotted time are 1/16, 1/10 and 1/20 respectively, If it is found that none of them has taken more time, what is the probability that A has taken more time ?*

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Solution. Let E_1, E_2 E_3 denote the events of carrying out the task by A, B and C respectively. Let H denote the event of taking more time. Then we have

$$P(E_1)=0{\cdot}45,\ P(E_2)=0{\cdot}35,\ P(E_3)=0{\cdot}20$$
$$P(H \mid E_1)=\frac{1}{16},\quad P(H \mid E_2)=\frac{1}{10},\ P(H \mid E_3)=\frac{1}{20}.$$

∴ The required probability

$$=\frac{P(E_1)\ .\ P(H \mid E_1)}{P(E_1)\ .\ P(H \mid E_1)+P(E_2)\ .\ P(H \mid E_2)+P(E_3)\ .\ P(H \mid E_3)}$$

$$=\frac{0\cdot45\times\frac{1}{16}}{0\cdot45\times\frac{1}{16}+0\cdot35\times\frac{1}{10}+0\ 20\times\frac{1}{20}}$$

$$=\frac{5}{13}$$

Example 25. *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5, 4, 2 per cents are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C ?*

Solution. Let us define the events :

A_1 = Bolt is manufactured by machine A.
A_2 = Bolt is manufactured by machine B.
A_3 = Bolt is manufactured by machine C.

The data of the problem give the following probabilities :

$$P(A_1)=0\cdot25,\ P(A_2)=0\cdot35,\ P(A_3)=0\cdot40$$
$$P(B \mid A_1)=0\cdot05,\ P(B \mid A_2)=0\cdot04,\ P(B \mid A_3)=0\cdot02$$
$$P(B\cap A_1)=P(A_1)\ P(B \mid A_1)=0\cdot25\times0\cdot05=0\cdot0125$$
$$P(B\cap A_2)=0\cdot35\times0\cdot04=0\cdot0140$$
$$P(B\cap A_3)=0\cdot40\times0\cdot02=0\cdot0080$$

Hence the probability that a defective bolt chosen at random is manufactured by factory A is given by Bayes' rule as

$$P(A_1 \mid B)=\frac{P(A_1)\ P(B \mid A_1)}{P(A_1)\ P(B \mid A_1)+P(A_2)\ P(B \mid A_2)+P(A_2)\ P(B \mid A_3)}$$

$$=\frac{0\cdot0125}{0\cdot0125+0\ 0140+0\cdot0080}=\frac{0\cdot0125}{0\cdot0345}=\frac{25}{69}.$$

Similarly, we get

$$P(A_2 \mid B)=\frac{0\ 0140}{0\cdot0345}=\frac{28}{69}$$

$$P(A_3 \mid B)=\frac{0\cdot0080}{0\cdot0345}=\frac{16}{69}$$

The above information concerning various probabilities may be summarized in the following table :

Event	*Prior Probability*	*Conditional Probability*	*Joint Probability*	*Posterior Probability*
A_1	0·25	0·05	0·0125	$\frac{25}{69}$
A_2	0 35	0·04	0·0140	$\frac{28}{69}$
A_3	0·40	0 02	0·0080	$\frac{16}{69}$
Total	1·00		0·0345	1·00

Important Remark. $P(A_3)$ is greatest, on the basis of '*a prior*' probabilities alone we are likely to conclude that a defective bolt drawn at random from the product is manufactured by machine *C*. After using the additional information we obtain the '*posterior*' probabilities which give $P(A_2\ \ B)$ as maximum. Thus, we shall now say that it is probable that the defective bolt has been manufactured by machine *B*, a result which is different from the earlier conclusion. However, latter conclusion is a much valid conclusion as it is based on the entire information at our disposal. Thus, Bayes' rule provides a very powerful tool in improving the quality of probability and this helps the management executive in arriving at valid decisions in the face of uncertainty. Thus, the additional information reduces the importance of the prior probabilities. The only requirement for the use of *Bayesian Rule* is that all the hypotheses under consideration must be valid and that none is assigned 'a prior' probability 0 or 1.

EXERCISES

1. (*a*) Define random experiment, trial and event.

(*b*) What do you understand by (*i*) equally likely, (*ii*) mutually exclusive and (*iii*) independent events.

(*c*) Define independent and mutually exclusive events,. Can two events be mutually exclusive and independent simultaneously ? Support your answer with an example.

2. Discuss the different schools of thought on the interpretation of probability How does each school define probability ?

3. Explain the meaning and illustrate by an example how probability can be calculated in the following cases :

(*i*) Mutually exclusive events, (*ii*) Dependent events.

(*iii*) Independent events.

4. Differentiate the following pairs of concepts :

(*i*) Mutually exclusive events and overlapping events.

(*ii*) Simple events and composite events.

(*iii*) Mutually exclusive events and independent events.

5. Define independent and mutually exclusive events. Can the two events be mutually exclusive and independent simultaneously. Support your answer with examples.

6. Explain with examples the rules of Addition and Multiplication in theory of probability.

7. A card is drawn from a pack of cards. Find the probability that it is

(*i*) queen, (*ii*) queen of diamond or heart, (*iii*) not a diamond, (*iv*) a ten, a jack, a queen or a king.

[**Ans.** (*i*) 1/13, (*ii*) 1/25, (*iii*) 3/4, (*iv*) 4/13]

8 (*a*) Given the following data :

x :	0—10	10—20	20—30	30—40	40—50
f :	2	8	13	7	5

What is the probability that an item chosen at random from the data falls between 30 and 40 ? **[Ans.** 1/5]

(*b*) Given the following probabilities concerning the number of accounting personnel that will be needed in a company during the next two years.

No. of Accountants :	<100	100—199	200—299	300—399	400—499	500—599
Probability :	0·10	0·15	0·30	0·30	0·10	0·05

(*i*) What is the probability that the company will need 400 or more additional accountants during the next two years.

(*ii*) What is the probability that the company will need at least 200 but not more than 399 additional Accountants ?

[Ans. (*i*) 0·10+0·05, (*ii*) 0·30+0·30]

9. The following data shows the length of life of wholesale grocers in a particular city :

Length of Life (years)	*Percentage of wholesalers*
0— 5	65
5—10	16
10—15	9
15—25	5
25 and over	5
Total	100

(*i*) During the period studied, what is the probability that an entrant to this business will fail within five years ?

(*ii*) That he will survive at least 25 years ?

[Ans. (*i*) 0·65, (*ii*) 0·95]

10. From 30 tickets marked with the first 30 numerals, one is drawn at random. Find the chance that,

(*i*) it is a multiple of 5 or of 7, (*ii*) it is a multiple of 3 or of 7.

$$\left[\textbf{Ans.}\ (i)\ \frac{1}{3},\ (ii)\ \frac{13}{30}\right]$$

11. A number is chosen from each of the two sets :

1, 2, 3, 4, 5, 6, 7, 8, 9 ; 1, 2, 3, 4, 5, 6, 7, 8, 9.

If p_1 is the probability that the sum of the two numbers be 10 and p_2 the probability that their sum be 8, find p_1+p_2. [**Ans.** 16/18]

12. From a pack of 52 cards, 2 are drawn at random. Find the chance that one is a king and the other a queen.

$$\left[\textbf{Ans.}\quad \frac{^4C_1\times{}^4C_1}{^{52}C_2}\right]$$

13. A bag contains 3 red, 4 white and 5 black balls. Three balls are taken from the bag. Find the probability that

(*i*) all are black,

(*ii*) all are of different colours.

$$\left[\textbf{Ans.}\quad (i)\ \frac{^5C_3}{^{12}C_3},\quad (ii)\ \frac{^3C_1\times{}^4C_1\times{}^5C_1}{^{12}C_3}\right]$$

14. Two cubical dice are tossed. Find the probabilities of the following events :

The sum of numbers

(*i*) Divisible by three, (*ii*) Less than 7,

(*iii*) Not less than 7 (or at least 7 or more than 6).

[**Ans.** (*i*) 1/3, (*ii*) 15/36, (*iii*) 21/36]

15. An urn contains 5 white, 3 black and 6 red balls, 3 balls are drawn at random. Find the probability that

(*i*) two of the balls drawn are white, (*ii*) one of each colour,

(*iii*) none is black, and (*iv*) at least one is white.

$$\left[\textbf{Ans.}\quad (i)\ \frac{^5C_2\times{}^9C_1}{^{14}C_3},\ (ii)\ \frac{5\times3\times6}{^{14}C_3},\ (iii)\ \frac{^{11}C_3}{^{14}C_3},\ (iv)\ 1-\frac{^9C_3}{^{14}C_3}\right]$$

16 There are 3 economists, 4 engineers, 2 statisticians and 1 doctor. A committee of 4 from among them is to be formed. Find the probability that the committee :

(*i*) consists of one of each kind ; (*ii*) has at least one economist ;

(*ii*) has the doctor as a member and three others.

$$\left[\textbf{Ans.}\quad (i)\ \frac{24}{210},\quad (ii)\ 1-\frac{35}{210},\quad (iii)\ \frac{84}{210}\right]$$

17. A bag contains 12 rupee coins, 7 fifty paise coins and 4 twenty-five-paise coins. Find the probability of drawing :

(*i*) a rupee coin ; (*ii*) three rupee coins, and

(*iii*) three coins, one of each type.

18. The Federal Match Company has forty female employees and sixty male employees. If two employees are selected at random, what is the probability that

(*i*) both will be males ? (*ii*) both will be females ?

(*iii*) there will be one of each sex ?

Since the three events are collectively exhaustive and mutually exclusive, what is the sum of the three probabilities ? [**Ans.** One]

19. In a box there are 4 granite stones, 5 sand stones and 6 bricks of identical size and shape. Out of them 3 are chosen at random. Find the chance that

(*i*) They all belong to different varieties.

(*ii*) They all belong to the same variety.

(*iii*) They are all granite stones.

20. If the probability is 0 30 that a Management Accountant's job applicant has a post-graduate degree, 0·70 that he has had some work experience as a Chief Financial Accountant, and 0·20 that he has both. Out of 300 applicants, approximately what number would have either a post graduate degree or some professional work experience ?

[**Ans.** 240]

21. Find the probability of getting 6 at least once in two tosses of a die.

[**Hint.** Using Addition rule, the required probability is

$$P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)=\frac{6}{36}+\frac{6}{36}-\frac{1}{36}=\frac{11}{36}\Big]$$

22. (*a*) A chartered Accountant applies for a job in two firms *X* and *Y*. He estimates that the probability of his being selected in firm *X* is 0·7, and being rejected at *Y* is 0·5 and the probability of at least one of his applications being rejected is 0·6. What is the probability that he will be selected in one of the firms ?

[**Hint.** Let *A* and *B* denote the events of his being selected in firms *X* and *Y* respectively.

$$P(A)=0{\cdot}7,\ P(\bar{B})=0{\cdot}5,\ P(\bar{A} \text{ or } \bar{B})=0{\cdot}6$$

The required probability that he will be selected in one of the firms is obtained by using addition rule as follows :

$$P(A \text{ or } B)=P(A)+\text{P}(B)-P(A \text{ and } B)$$

Also we know

$$P(A \text{ and } B)=1-P(\bar{A} \text{ or } \bar{B})=1-0{\cdot}6=0{\cdot}4$$

Hence $P(A \text{ or } B)=0{\cdot}7+0\ 5-0{\cdot}4=0{\cdot}8$]

23. Two vacancies exist at the junior executive level of a certain company. Twenty people, fourteen men and six women, are eligible and equally qualified. The company has decided to draw two names at random from the list of eligibles. What is the probability that :

(*a*) both positions will be filled by women ?

(*b*) at least one of the position will be filled by women ?

(*c*) neither of the position will be filled by women ?

$$\left[\textbf{Ans.}\ (a)\ \frac{^{6}C_2}{^{20}C_2},\quad (b)\ 1-\frac{^{14}C_2}{^{20}C_2},\ (c)\ \frac{^{14}C_2}{^{20}C_2}\right]$$

24. Sixty per cent of the employees of the ABC Corporation are college graduates. Of these, ten per cent are in sales. Of the employees who did not graduate from college, eighty per cent are in sales.

(*i*) What is the probability that an employee selected at random is in sales ?

(*ii*) What is the probability that an employee selected at random is neither in sales nor a college graduate ?

[**Ans.** (*a*) 0·38, (*b*) 0·08]

25. A small insurance company has written theft insurance for two different businesses. In any one year, the probability that business *A* is burglarized is 0·01. In any one year, the probability that business *B* is burglarized is 0·15. (Assume these are independent events.) Find the probability that :

(*a*) both will be burglarized this year.

(*b*) neither will be burglarized this year.

(*c*) exactly one will be burglarized this year.

26. The probability that a person stopping at a gas station will ask to have his tyres checked is 0·12, the probability that he will ask to have his oil checked is 0·29 and the probability that he will ask to have them both checked is 0·07.

(*i*) What is the probability that a person stopping at this gas station will have either his tyres or his oil checked ?

(*ii*) What is the probability that a person who has his tyres checked will also have his oil checked ?

(*iii*) What is the probability that a person who has his oil checked will also have his tyres checked ?

[**Ans.** (*i*) 0·34, (*ii*) 0·58, (*iii*) 0·24]

27. A card is drawn from a full pack of cards. What is the probability of drawing a "black" king (either spade or club) given that the card drawn was "face" card (jack, queen or king) ?

28. A bag contains 6 white and 9 black balls. Two drawings of 4 balls (in each draw) are made in such a way that

(*i*) the balls are replaced before the second trial.

(*ii*) the balls are not replaced before the second trial.

Find the probability that first drawings will give 4 white and the second 4 black balls in each case.

$$\left[\textbf{Ans.}\ (i)\ \frac{^{6}C_4}{^{15}C_4}\times\frac{^{9}C_4}{^{15}C_4}\quad (ii)\ \frac{^{6}C_4}{^{15}C_4}\times\frac{^{9}C_4}{^{11}C_4}\right]$$

29. If the probability that 'A' project will have an economic life of 20 years is 0·7 and the probability that 'B' project will have an economic life of 20 years is 0·5. What is the probability that both will have an economic life of 20 years ?

[**Ans.** $0{\cdot}7 \times 0{\cdot}5$]

30 A salesman has a 10 per cent chance of making a sale to each customer. The behaviour of successive customers is assumed to be independent. If two customers A and B enter, what is the probability that the salesman will make a sale to A or B ?

[**Ans.** 0·19.]

31. It is known that bolts produced by a certain process are too large 10 per cent of the time and are too small 5 per cent of the time. If a prospective buyer selects a bolt at random from a lot of 500 such bolts, what is the probability that it will be neither too long nor too short ?

32, A problem in Statistics is given to three students A, B and C whose chances of solving it are $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$ respectively. Find the probability that the problem will be solved by at least one of them.

[**Ans.** 3/5]

33. The probabilities that three drivers will be able to drive home safely after drinking are $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{5}$ respectively. If they set out to drive home after a party, what is the probability that all three drivers will have accident ? What is the probability that at least one driver will drive home safely ?

34. (*a*) Find the probability of throwing 6 at least once in six throws, with a single die.

[**Ans.** $1-(5/6)^6$]

(*b*) Suppose two six-faced dice are thrown 10 times. What is the probability of getting a double six in at least one of the throws ?

[**Ans.** $1-(35/36)^{10}$]

35. In the milk section of a self-service market there are 150 quarts, 100 of which are fresh and 50 are a day old.

(*i*) If two quarts are selected, what is the probability that both will be fresh ?

(*ii*) Suppose two quarts are selected after 50 quarts have been removed from the selection. What is the probability that both will be fresh ?

(*iii*) What is the conditional probability that both will be fresh, given that at least one of them is fresh.

36. An urn A contains 2 white and 4 black balls, Another urn B contains 5 white and 7 black balls. A ball is transferred from urn A to the urn B. Then a ball is drawn from the urn B. Find the probability that it will be white.

37. A bag contains 5 white and 3 black balls. Another bag contains 4 white and 5 black balis. From any one of these bags single draw

of two balls is made. Find the probability that one of them would be white and another black ball.

38. An urn contains 10 white and 3 red balls. Another urn contains 3 white and 5 red balls. Two balls are transferred from the first urn and placed in the second, and then one ball is taken from the latter. What is the probability that it is a white ball ?

39. There are two groups of subjects, one of which consists of 5 science subjects and 3 engineering subjects and the other consists of 3 science subjects and 5 engineering subjects. An unbiased die is cast. If number 3 or number 5 turns up, a subject is selected at random from the first group. Otherwise, a subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately.

$$\left[\textbf{Ans.}\ \left[\left(\frac{1}{3}\times\frac{3}{8}\right)+\left(\frac{2}{3}\times\frac{5}{8}\right)\right]\right.$$

40. An urn contains 7 red marbles and 3 white marbles. Three marbles are drawn from the urn, one after the other, without replacement. Find the probability that the first two are red and the third is white.

41. One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first. second and third gun respectively. If $P(E_1)=0{\cdot}5$, $P(E_2)=0{\cdot}6$ and $P(E_3)=0{\cdot}8$ and E_1, E_2, E_3 are independent events, find the probability that (*a*) exactly one hit is registered, (*b*) at least two hits are registered. [**Ans.** (*a*) 0·26 (*b*) 0·70]

42. A certain part can be defective because it has one or more out of three possible defects : insufficient tensile strength, a burr, or a diameter outside tolerance limits. In a lot of 1000 pieces it is known that

120 have a tensile strength defect.
80 have a burr.
60 have an unacceptable diameter.
22 have tensile strength and burr defects.
16 have tensile strength and diameter defects.
20 have burr and diameter defects.
8 have all three defects.

If a piece is drawn at random from the lot, what is the probability that the piece :

(*a*) is not defective ?

(*b*) has at least one defect, and

(*c*) has exactly two defects ?

43. An investment firm purchases 3 stocks for one-week trading purposes. It assesses the probability that the stocks will increase in value over the week as 0·8, 0·7 and 0·6 respectively. What is the chance that (*i*) all three stocks will increase, and (*ii*) at least 2 stocks will increase ?

[Assume that the movements of these stocks are independent.]

Also find the probability that : (*iii*) Exactly one stock will increase in value, (*iv*) Exactly two stocks will increase in value and (*v*) At least one of the stocks will increase in value.

[**Hint.** Let A, B and C denote respectively the events that 1st, 2nd and 3rd stocks increase in value. Then we are given that :

$$P(A)=0\cdot8,\ P(B)=0\cdot7 \text{ and } P(C)=0\cdot6$$

$$\Rightarrow \quad P(\overline{A})=0\cdot2,\ P(\overline{B})=0\cdot3 \text{ and } P(\overline{C})=0\cdot4$$

(*i*) The probability that all the three stocks will increase in value is

$$P(A\cap B\cap C)=P(A)P(B)P(C)$$

[∵ Movements of the stocks are independent]

(*ii*) The event that at least two of the stocks increase in value can materialise in the following mutually exclusive ways :

(*a*) $A\cap B\cap \overline{C}$ happens, (*b*) $A\cap \overline{B}\cap C$ happens,

(*c*) $\overline{A}\cap B\cap C$ happens, and (*d*) $A\cap B\cap C$ happens.

Hence by the addition rule the required probability is given by :

$$P(A\cap B\cap \overline{C})+P(A\cap \overline{B}\cap C)+P(\overline{A}\cap B\cap C)+P(A\cap B\cap C)$$
$$=P(A)P(B)P(\overline{C})+P(A)P(\overline{B})P(C)+P(\overline{A})P(B)P(C)+P(A)P(B)P(C)$$

[∵ A, B, C are independent]

(*iii*) Arguing as in case (*ii*) the probability that exactly one stock will increase in value is given by :

$$=P(A)P(\overline{B})P(\overline{C})+P(\overline{A})P(B)P(\overline{C})+P(\overline{A})P(B)P(\overline{C})$$

(∵ Movements of stocks are independent]

(*iv*) Similarly, the probability that exactly two stocks will increase in value is given by :

$$P(A\cap B\cap \overline{C})+P(A\cap \overline{B}\cap C)+P(\overline{A}\cap B\cap C)$$
$$=P(A)P(B)P(\overline{C})+P(A)P(\overline{B})P(C)+P(\overline{A})P(B)P(C)$$

(*v*) The probability that at least one of the stocks will increase in value is given by :

$$P(A\cup B\cup C)=1-P(\overline{A}\cap \overline{B}\cap \overline{C})=1-P(\overline{A})P(\overline{B})P(\overline{C})$$

44. In a multiple choice examination there are 20 questions. Each question has 4 alternative answers following it and the student must select one correct answer. 4 marks are given for the correct answer and 1 mark is deducted for every wrong answer. A student must secure at least 50% of maximum possible marks to pass the examination. Suppose a student has not studied at all so that he decides to select the answers to the question on a random basis. What is the probability that he will pass in the examination ?

45. A speaks truth 4 out of 5 times. He throws a die and reports that there was a six. What is the chance that actually there was a six ?

$$\left[\textbf{Hint.}\quad P(A\cap B)=\frac{1}{6}\times\frac{4}{5}=\frac{4}{30},\ P(\overline{A}\cap\overline{B})=\frac{5}{30}\right.$$

$$\left.\text{Required probability}=\frac{4/30}{5/30}=\frac{4}{5}\right]$$

46. (*a*) In 1992 there will be three candidates for the position of principal Dr. Singhal, Mr. Mehra and Dr. Chatterji whose chances of getting appointment are in the proportion 4 : 2 : 3 respectively. The probability that Dr. Singhal if selected will abolish co-education in the college is 0·3. The probability of Mr. Mehra and Dr. Chatterji doing the same are respectively 0·5 and 0·8. What is the probability that co-education will be abolished from the college in 1992 ? **[Ans.** 23/45]

(*b*) Suppose that one of three men, a politician, a businessman, and an educator will be appointed as the vice-chancellor of a university. The respective probabilities of their appointments are 0·50, 0·30, 0·20. The probabilities that research activities will be promoted by these people if they are appointed are 0·30, 0·70 and 0·80 respectively. What is the probability that research will be promoted by the new vice-chancellor ?

[Ans. 0·52]

47. Electric light bulbs are manufactured at two plants. The first plant furnished 70% and second 30% of all required production of bulbs. At the first plant, among every 100 bulbs, 83 are on the average standard, whereas only 63 per hundred are standard at the second plant. What is the probability that a bulb chosen at random is manufactured at the second plant, given that the bulb is standard. **[Ans.** 0·245]

48. Suppose that there is a chance for a newly constructed house to collapse whether the design is faulty or not. The chance that the design is faulty is 20%. The chance that the house collapses if the design is faulty is 98% and otherwise it is 25%. It is seen that the house collapsed. What is the probability that it is due to faulty design ?

[**Hint.** We are given

$P(A_1)=0·2$ and $P(A_2)=0·8$; $P(B\mid A_1)=0·98$ and $P(B\mid A_2)=0·25$.

Using Bayes' rule, we have

$$P(A_1\mid B)=\frac{P(A_1)\,.\,P(B\mid A_1)}{P(A_1)\,.\,P(B\mid A_1)+P(A_2)P(B\mid A_2)}$$

$$\left.=\frac{(0·2)\,(0·98)}{(0·2)\,(0·98)+(0·8)\,(0·25)}\right]$$

49. The president of a company must decide which of two actions to take, say whether to rent or buy expensive machinery. His vice-president is likely to make a faulty analysis and thus recommend the wrong decision with probability 0·05. The president hires two consultants, who separately study the problem and make their recommendations.

After watching them at work, the president estimates that one consultant is likely to recommend the wrong decision with probability 0·05, the other with probability 0·10. He decides to take the action recommended by a majority of the three reports he receives. What is the probability that he will make a wrong decision? Does the assumption of independence. you have made seem reasonable for this problem?

[**Ans.** 0·012.]

50. A factory produces a certain type of output by three types of machines. The respective daily production figures are :

Machine *I* : 3,000 units
Machine *II* : 2,500 units
Machine *III* : 4,500 units

Past experience shows that 1· per cent of the output produced by Machine I is defective The corresponding fraction of defectives for the other two machines are respectively 1·2 per cent and 2 per cent. An item is drawn at random from the day's production run and is found to be defective. What is the probability that it comes from the output of (*a*) Machine I, (*b*) Machine II, and (*c*) Machine III ?

[**Ans.** (*a*) 1/5, (*b*) 1/5, (*c*) 3/5]

MATHEMATICAL EXPECTATION

If X is a random variable which can assume any one of the values $x_1, x_2, \ldots\ldots x_n$ with respective probabilities $p_1, p_2, \ldots. p_n$ then the mathematical expectation of X usually called the expected value of X and denoted by $E(X)$ is defined as

$$E(X)=x_1\,p_1+x_2\,p_2+\ldots+x_n\,p_n$$

$$=\sum_{i=1}^{n} x_i\,p_i$$

Properties of Expected Value

(*i*) The expected value of a constant is the constant itself, *i.e.*, $E(k)=k$, for every constant k.

(*ii*) The expected value of the product of a constant and a random variable is equal to the product of the constant with expected value of the random variable, *i e.*,

$$E(kX)=kE(X)$$

(*iii*) The expected value of the sum or difference of two random variables is equal to the sum or difference of the expected values of the individual random variables, *i.e.*,

$$E(X\pm Y)=E(X)\pm E(Y)$$

(*iv*) The expected value of the product of two independent random variables is equal to the product of their individual expected values, *i e.*,

$$E(XY)=E(X)\,.\,E(Y)$$

(v) $E[X-E(X)]=0$

Illustration. A dealer in radio sets estimates from his past experience the probabilities of his selling radio sets in a day. These are given below :

No. of radio sets sold in a day	0	1	2	3	4	5	6
Probability	·02	·10	·21	·32	·20	·09	·06

We observe now that the number of radio sets sold in a day is a random variable which can assume values 0, 1, 2, 3, 4, 5, 6 with the respective probabilities given in the table. We may also note that the dealer has estimated the probability zero of selling seven or more radio sets in a day.

Now

Mean number of radio sets sold in a day

$$=0\times{\cdot}02+1\times{\cdot}10+2\times{\cdot}21+3\times{\cdot}32+4\times{\cdot}20+5\times{\cdot}09+6\times{\cdot}06$$
$$={\cdot}10+{\cdot}42+{\cdot}96+{\cdot}80+{\cdot}45+{\cdot}36=3{\cdot}09$$

Example 26. *A bakery has the following schedule of daily demand for cakes. Find the expected number of cakes demanded per day.*

No. of cakes demanded in hundreds	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Probability	*0·02*	*0·07*	*0·09*	*0·12*	*0·20*	*0·20*	*0·18*	*0·10*	*0·01*	*0·01*

Solution. We observe that number of cakes demanded per day is a random variable (X) which can assume the values 0, 1, 2, ... , 9 with respective probabilities given in the table.

Now

$$\begin{aligned}E(X)&=0\times0{\cdot}02+1\times0{\cdot}07+2\times0{\cdot}09+3\times0{\cdot}12\\&\quad+4\times0{\cdot}20+5\times0{\cdot}20+6\times0{\cdot}18+7\times0{\cdot}10\\&\quad+8\times0{\cdot}01+9\times0{\cdot}01\\&=4{\cdot}36\end{aligned}$$

Example 27. *Anil & Company estimates the net profit on a new product it is launching to be Rs. 3,000,000 during the first year if it is 'successful'; Rs. 1,000,000 if it is 'moderately successful' and a loss of Rs. 1,000,000 if it is 'unsuccessful'. The firm assigns the following probabilities to first year prospects for the product : Successful – 0·15, moderately successful – 0·25. What is the expected value of first year net profit for this product ?*

Solution. Taking loss as negative profit, the probability distribution of net profit (x) on the new product in the first year is

Profit (*in million* Rs.)	3	1	—1
Probability $p(x)$	0·15	0·25	1—0·15—0·25 =0 60

∴ Expected value of first year net profit is

$$E(X)=\Sigma x\ p(x)$$
$$=3\times0{\cdot}15+1\times0{\cdot}25+(-1)\times0{\cdot}60$$
$$=0{\cdot}10 \text{ million Rs.}=\text{Rs. } 1{,}00{,}000$$

Example 28 *A lottery sells 10,000 tickets at Re. 1 per ticket, a prize of Rs. 5,000 will be given to the winner of the first draw. Suppose you have bought a ticket, how much should you expect to win ?*

Solution. Here, the random variable 'win', W, has two possible values : – Re 1 and Rs. 4,999. Their respective probabilities are

$$\frac{9999}{10000} \text{ and } \frac{1}{1000}$$

Thus $E(W)=(-1)\times\frac{9999}{10000}+4999\times\frac{1}{10000}=-\text{Re } 0{\cdot}50$

Hence a minus 50 paisa is the amount we expect to win on the average if we play this game over and over again.

Example 29. *A box contains 6 tickets. Two of the tickets carry a prize of Rs. 5 each, the other four a prize of Re. 1. (a) If one ticket is drawn, what is the expected value of the prize ? (b) If two tickets are drawn what is the expected value of the game ?*

Solution. (*a*) The sample space consists of ${}^6C_1=6$ sample points. Let X be the random variable associated with the experiment and let it denote the amount of prize associated with the sample point. Here X assumes values Rs. 5 and Re. 1 respectively for 2 and 4 sample points.

Also $$p(5)=\frac{2}{6}=\frac{1}{3} \text{ and } p(1)=\frac{4}{6}=\frac{2}{3}$$

∴ $E(X)$ = Expected value of the prize

$$=x_1 . p(x_1)+x_2 . p(x_2)$$
$$=5\times\frac{1}{3}+1\times\frac{2}{3}=\frac{5}{3}+\frac{2}{3}=\frac{7}{3}=\text{Rs. } 2{\cdot}33$$

The expected amount of prize is Rs. 2·33.

(*b*) The sample space consists of ${}^6C_2=15$ sample points. Let X be random variable associated with the experiment and let it denote the amount of prize associated with sample points. Then X assumes following values :

(*i*) Rs. 10 (when both the tickets carry prize Rs. 5 each *i.e.*, Number of sample points ${}^2C_2=1$)

(*ii*) Rs. 6 (when one ticket carries prize Rs. 5 and the other Re. 1 *i.e.*, Number of sample points $={}^2C_1\times{}^4C_1=8$)

(*iii*) Rs. 2 (when both the tickets carry prize Re. 1 each, *i.e.*, No. of sample points $={}^4C_2=6$)

Also $\quad p(10)=\frac{1}{15},\ p(6)=\frac{8}{15},\ p(2)=\frac{6}{15}$

$$\therefore \quad E(X)=x_1.p(x_1)+x_2.p(x_2)+x_3.p(x_3)$$

$$=10\times\frac{1}{15}+6\times\ \frac{8}{15}+2\times\ \frac{6}{15}$$

$$=\frac{2}{3}+\frac{16}{5}+\frac{4}{5}=\frac{10+48+12}{15}=\frac{70}{15}=\frac{14}{3}=4{\cdot}67$$

Hence expected amount of prize is Rs. 4·67.

Example 30. *A player pays Re. 1 to play a game. The game consists of repeatedly tossing a coin and recording the number of times it falls heads. The game ends as soon as the coin falls tails or when it has fallen 3 heads in succession. The player is paid Rs. 2 for each head which appears. Calculate (a) his expectation in each game, (b) the amount won or lost, on the average, in 20 games.*

Solution. According to the rules, the game ends with either of the following outcomes :

T	Tail in 1st throw
HT	Head in 1st throw and Tail in 2nd (*i.e.*, 1 head)
HHT	Head in 1st, Head in 2nd and Tail in 3rd throw, (*i.e.*, 2 heads)
HHH	Head in 1st, 2nd and 3rd throws (*i.e.*, 3 heads)

The probabilities of these events and the amounts received are shown below :

Outcomes	*Probability*	*Amount Received (Rs)* (*Rs. 2 for each head*)
T	1/2	0
HT	1/4	2
HHT	1/8	4
HHH	1/8	6

(*a*) Mathematical expectation $=\frac{1}{2}\times 0+\ \frac{1}{4}\times 2+\ \frac{1}{8}\times 4+\ \frac{1}{8}\times 6$

$$=\frac{7}{4}$$

(*b*) Average loss in one game $=1-\frac{7}{4}=\frac{-3}{4}$

$\therefore$ Loss in 20 games $=-\frac{3}{4}\times 20=-$**Rs.** 15

Example 31. *The manager of a machine shop has a choice of competing for one of the two contracts shown in the table below :*

Event	*Contract A*		*Contract B*	
	Probabilities	*Consequences*	*Probabilities*	*Consequences*
Contract awarded	*0·50*	*+Rs. 60,000*	*0·40*	*+Rs. 80,000*
Contract not awarded	*0·50*	*–Rs. 10,000*	*0·60*	*–Rs. 14,000*

Which contract should be preferred if the expected monetary value is considered as an oppropriate measure.

Solution. *For contract A* : Let X be the random variable which assumes the values 60,000 and $-10{,}000$ with probabilities 0·50 and 0·50 respectively.

Then

Mean of $X=60{,}000\times 0{\cdot}50-10{,}000\times 0{\cdot}50$

$=25{,}000$

Similarly, for contract B, we can define a random variable Y, and we find that

Mean of $Y=80{,}000\times 0{\cdot}40-14{,}000\times 0{\cdot}60$

$=23{,}600$

Thus, if the expected monetary value is considered as an appropriate measure, then contract A should be preferred.

Example 32. *There are four different choices available to a customer who wants to buy a transistor set. The first type costs Rs 800, the second type Rs. 680, the third type Rs. 880 and the fourth type Rs. 760. The probabilities that the customer will buy these types are 1/3, 1/6, 1/4 and 1/4 respectively. The retailer of these sets gets a commission @ 20%, 12%, 25% and 15% for these sets respectively. What is the expected commission of the retailer ?* [*Delhi Univ., B. Com.* (*Hons.*), *1992*]

Solution. We have

Type	*Price (Rs.)*	*Commission*	*Probability*	*Expectation*
(1)	*(2)*	*(3)*	*(4)*	$(2)\times(3)\times(4)=5$
Frist	800	20%	$\frac{1}{3}$	53·33
Second	680	12%	$\frac{1}{5}$	13·60
Third	880	25%	$\frac{1}{4}$	55·00
Fourth	760	15%	$\frac{1}{4}$	28·50
Total				150·43

Hence the retailer's expactation is Rs. 150·43.

EXERCISES

1. (*a*) What do you understand by 'the expectation of a random variable' ? Explain as clearly as you can ?

(*b*) A balanced coin is tossed 4 times. Find probability distribution of the number of heads and its expectation.

(*c*) In a business venture a man can make a profit of Rs. 2,000 with a probability of 0 4 or have a loss of Rs. 1,000 with a probability of 0·6. What is his expected profit ? [**Ans.** Rs. 200]

2. A random variable X has the following probability distribution :

X	:	-1	0	1	2
Probability	:	1/3	1/6	1/6	1/3

Compute the expectation of X. [**Ans.** 1/2]

3. Calculate the expected value of X, the sum of the scores when two dice are rolled. [**Ans.** 7]

4. A box contains 8 items of which 2 are defective. A man selects 3 items at random. Find the expected number of defective items he has drawn. [**Ans.** 3/4]

5. A player tosses two fair coins. He wins Rs. 5 if 2 heads appear, Rs. 2 if 1 head appears and Re. 1 if no head occurs. Find his expected amount of winning. [**Ans.** Rs. 2·5]

6. A player tosses 3 fair coins. He wins Rs. 5 if 3 heads appear, Rs. 3 if 2 heads appear, Re. 1 if 1 head occurs. On the other hand, he losses Rs. 15 if 3 tails occur. Find expected gain of the player.

[**Ans.** Rs 0·25]

7. An urn contains 7 white and 3 red balls. Two balls are drawn together, at random, from this urn. Compute the probability that neither of them is white. Find also the probability of getting one white and one red ball. Hence compute the expected number of white balls drawn.

$$\left[\textbf{Hint.}\quad E(X)=0\times\frac{{}^3C_2}{{}^{10}C_2}+1\times\frac{{}^7C_1\times{}^3C_1}{{}^{10}C_2}+2\times\frac{{}^7C_2}{{}^{10}C_2}=\frac{63}{45}\right]$$

8. The monthly demand for transistors is known to have the following probability distribution :

Demand :	1	2	3	4	5	6
Probability :	0·10	0·15	0·20	0·25	0·18	0·12

Determine the expected demand for transistors. Also obtain the variance. Suppose that the cost (C) of producing (n) transistors is given by the rule, $C=10{,}000+500\ n$. Determine the expected cost.

[**Hint.** $E(C)=E[10{,}000+500\ n]$

$=10{,}000+500\ E(n)$

$=10{,}000+500\ [\Sigma(n\times p)]=10{,}000+500\times 3{\cdot}62]$

9. The probability that there is at least one error in accounts statement prepared by A is 0·2 and for B and C they are 0·25 and 0·4 respectively. A, B and C prepared 10, 16 and 20 statements respectively. Find the expected number of correct statements in all.

[**Hint.** Expected number of correct statements is :

$(1-0{\cdot}2)\times 10+(1-0{\cdot}25)\times 16+(1-0{\cdot}4)\times 20$

$=0{\cdot}8\times 10+0{\cdot}75\times 16+0{\cdot}6\times 20$

$=32]$

10 (*a*) Suppose an insurance company offers a 45 year old man a Rs. 1,000 one year term insurance policy for an annual premium of Rs. 12. Assume that the number of deaths per one thousand is five for persons in this age group. What is the expected gain for the insurance company on a policy of this type.

[**Hint.** Expected gain $=12\times(1-0{\cdot}005)-(1000-12)\times 0{\cdot}005]$

(*b*) The probability that a house of a certain type will be burned by fire in any twelve month period is 0·005. An insurance company offers to sell the owner of such a house Rs. 29,000 one year term fire insurance policy for a premium of Rs. 150. What is the company expected to gain ? [**Ans.** Rs. 5]

11. A firm plans to bid Rs. 300 per tonne for a contract to supply 1,000 tonnes of a metal. It has two competitors A and B and it assumes that the probability that A will bid less than Rs. 300 per tonne is 0·3 and that B will bid less than Rs. 300 per tonne is 0·7. If the lowest bidder gets all the business and the firms bid independently, what is the pxeected value of the contract to the firm ?

[**Hint** $300\times 1000\,[P(\text{both bid less than } 300)$

$+P(A \text{ bids less than } 300 \text{ but } B \text{ bids more } 300)$

$+P(A \text{ bids more than } 300 \text{ but } B \text{ bids less than } 300)$

$=300000(0{\cdot}3\times 0{\cdot}7+0{\cdot}3\times 0{\cdot}3+0{\cdot}7\times 0{\cdot}7)=$Rs. 2,37,000.]

12. A gamester has a disc with a freely revolving needle. The disc is divided into 20 equal sectors by thin lines and the sectors are marked 0, 1, 2 .., 19. The gamester treats 5 or any multiple of 5 as lucky numbers and zero as a special lucky number. He allows a player to whirl the needle on a charge of 10 paise. When the needle stops at the lucky number the gamester pays back the player twice the sum charged and at the special lucky number the gamester pays to the player 5 times of the sum charged. Is the game fair ? What is the expectation of the player ?

[**Hint.**

Event	*Favourable cases*	$p(x)$	*Gain* (x)
Lucky number	5, 10, 15	3/20	$20-10=10\ P$
Special Lucky No.	0	1/20	$50-10=40\ P$
Others	1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19	16/20	$-10\ P$

$$E(X)=\frac{3}{20}\times 10+\frac{1}{20}\times 40-\frac{16}{20}\times 10=-\frac{9}{2}\Big]$$

13. In a college fete a stall is run where on buying a ticket a person is allowed one throw of two dice. If this gives a double six, 10 times the ticket money is refunded, if only one six turns up, double the ticket money is refunded and in other cases nothing is refunded. Will it be profitable to run such a stall ? What is the expectation of a player ? State clearly the assumptions, if any, for your answer.

Some Additional Topics

DE-MOIVRE'S THEOREM

Statement. *For all rational values of n (positive, negative or fraction) cos $n\theta+i$ sin $n\theta$ is the value or one of the values of* $(\cos\theta+i\sin\theta)^n$.

Proof. Case I. *When n is a positive integer.*

By actual multiplication, we have

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$$
$$=(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)$$
$$=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$$

Again $(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3)$

$$=[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)](\cos\theta_3+i\sin\theta_3)$$
$$=\cos(\theta_1+\theta_2+\theta_3)+i\sin(\theta_1+\theta_2+\theta_3),\text{ as before.}$$

Proceeding as above, the product of n factors

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)...(\cos\theta_n+i\sin\theta_n)$$
$$=\cos(\theta_1+\theta_2+...+\theta_.)+i\sin(\theta_1+\theta_2+...+\theta_n) \quad ...(*)$$

Putting $\theta_1=\theta_2=...=\theta_n=\theta$ on both sides of (*), we have

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

Aliter. The proof can be obtained by the method of mathematical induction also.

For $n=1$, the result is obviously true.

For $n=2$, consider

$$(\cos\theta+i\sin\theta)^2=\cos^2\theta+i^2\sin^2\theta+2i\sin\theta\cos\theta$$
$$=(\cos^2\theta-\sin^2\theta)+i(2\sin\theta\cos\theta)$$
$$=\cos 2\theta+i\sin 2\theta$$

Hence the result is also true for $n=2$.

Let the result be true for $n=m$, *i.e.*,

$$(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta \quad ...(*)$$

Now $(\cos\theta+i\sin\theta)^{m+1}=(\cos\theta+i\sin\theta)^m(\cos\theta+i\sin\theta)$

$$=(\cos m\theta+i\sin m\theta)(\cos\theta+i\sin\theta)$$
$$=\cos(m+1)\theta+i\sin(m+1)\theta.$$

Hence the result is true for $n=m+1$ also.

Thus we conclude that if the result is true for $n=2$, then it should be true for $n=2+1$, *i e.*, $n=3$. Therefore, proceeding in this manner we find that the theorem is true for all positive integral values of n.

Case II. *When n is a negative integer.*

Let us suppose $n=-m$, where m is a positive integer.

$$\therefore\quad (\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)^{-m}$$
$$=\frac{1}{(\cos\theta+i\sin\theta)^m}=\frac{1}{(\cos m\theta+i\sin m\theta)}$$
$$=\frac{1}{(\cos m\theta+i\sin m\theta)}\times\frac{(\cos m\theta-i\sin m\theta)}{(\cos m\theta-i\sin m\theta)}$$
$$=\frac{\cos m\theta-i\sin m\theta}{(\cos^2 m\theta+\sin^2 m\theta)}=\cos m\theta-i\sin m\theta$$
$$=\cos(-m\theta)+i\sin(-m\theta)$$
$$=\cos n\theta+i\sin n\theta \qquad [\because\ n=-m]$$

Case III. *When n is a fraction, positive or negative.*

Let $n=\frac{p}{q}$, where q is a positive integer and p an integer positive or negative.

By case I, we have

$$\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right)^q=\cos\left(q\times\frac{\theta}{q}\right)+i\sin\left(q\times\frac{\theta}{q}\right)$$
$$=\cos\theta+i\sin\theta$$

Taking the qth root of both sides, we get

$\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right)$ is one of the values of $(\cos\theta+i\sin\theta)^{1/q}$

Raising both sides to the power p, we get

$\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right)^p$ is one of the values of $(\cos\theta+i\sin\theta)^{p/q}$

$\Rightarrow$ $\left(\cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta\right)$is one of the values of $(\cos\theta+i\sin\theta)^{p/q}$

$\Rightarrow$ $(\cos n\theta+i\sin n\theta)$ is one of the values of $(\cos\theta+i\sin\theta)^n$

Remarks. 1. (*i*) $(\cos\theta+i\sin\theta)^{-n}=\cos(-n\theta)+i\sin(-n\theta)$
$$=\cos n\theta-i\sin n\theta$$
$$[\because\ \sin(-\theta)=-\sin\theta\ ;\ \cos(-\theta)=\cos\theta]$$

(*ii*) $(\cos\theta-i\sin\theta)^n=\{\cos(-\theta)+i\sin(-\theta)\}^n$
$$=\cos(-n\theta)+i\sin(-n\theta)$$
$$=\cos n\theta-i\sin n\theta$$

(*iii*) $(\cos\theta-i\sin\theta)^{-n}=\{\cos(-\theta)+i\sin(-\theta)\}^{-n}$
$$=\cos n\theta+i\sin n\theta$$

2. Students often wrongly apply De-Moivre's theorem in the following way :

$$(\sin\theta+i\cos\theta)^n=\sin n\theta+i\cos n\theta$$

It should be noted that the *real part must be cos θ and imaginary part should be sin θ*, but θ must be the same with cos and sin both.

$\therefore \qquad (\sin \theta + i \cos \theta)^n \neq \sin n\theta + i \cos n\theta$

3. $\qquad (\cos \theta + i \sin \phi)^n \neq \cos n\theta + i \sin n\phi$

4. $\qquad \dfrac{1}{\cos \theta \pm i \sin n\theta} = \cos \theta \mp i \sin \theta$

5. Every complex quantity can be put in the form $r(\cos \theta + i \sin \theta)$, where r and θ are both real.

Let a given complex quantity be $x+iy$.

Also let $\qquad x+iy = r(\cos \theta + i \sin \theta)$

or $\qquad x+iy = r \cos \theta + ir \sin \theta$

Equating the real and imaginary parts on both sides, we get

$$x = r \cos \theta \qquad \ldots(^*)$$

and

$$y = r \sin \theta \qquad \ldots(^{**})$$

Squaring and adding (*) and (**), we have

$$r^2 = x^2 + y^2 \quad \text{or} \quad r = \sqrt{x^2 + y^2}$$

Dividing (**) by (*), we get $\tan \theta = \dfrac{y}{x} \qquad \therefore \quad \theta = \tan^{-1} \dfrac{y}{x}$

Here r is always positive and is called the Modulus of the complex number. θ is called the *amplitude* of the given complex quantity. That value of θ *which* satisfies equations (*) and (**) also lying between π and $-\pi$ is called the *principal value of amplitude*. We shall always take principal value of the amplitude expressing any complex quantity in the form

$$r(\cos \theta + i \sin \theta)$$

Example 1. *Simplify*

$$\frac{(\cos 3\theta + i \sin 3\theta)^5 (\cos \theta - i \sin \theta)^3}{(\cos 5\theta + i \sin 5\theta)^7 (\cos 2\theta - i \sin 2\theta)^5}$$

Solution. Expression

$$= \frac{(\cos 15\theta + i \sin 15\theta)[\cos(-\theta) + i \sin(-\theta)]^8}{(\cos 35\theta + i \sin 35\theta)[\cos(-2\theta) + i \sin(-2\theta)]^5}$$

$$= \frac{(\cos 15\theta + i \sin 15\theta)[\cos(-3\theta) + i \sin(-3\theta)]}{(\cos 35\theta + i \sin 35\theta)[\cos(-10\theta) + i \sin(-10\theta)]}$$

$$= \frac{\cos(15\theta - 3\theta) + i \sin(15\theta - 3\theta)}{\cos(35\theta - 10\theta) - i \sin(35\theta - 10\theta)}$$

$$= \frac{\cos 12\theta + i \sin 12\theta}{\cos 25\theta + i \sin 25\theta}$$

$$= (\cos 12\theta + i \sin 12\theta)(\cos 25\theta + i \sin 25\theta)^{-1}$$

$$= (\cos 12\theta + i \sin 12\theta)[\cos(-25\theta) + i \sin(-25\theta)]$$

$$= \cos(12-25)\theta + i \sin(12-25)\theta$$

$$= \cos 13\theta - i \sin 13\theta$$

Example 2. *Show that*

$$\left[\frac{1+\sin \phi + i \cos \phi}{1+\sin \phi - i \cos \phi}\right]^n = \cos\left(\frac{n\pi}{2} - n\phi\right) - i \sin\left(\frac{n\pi}{2} - n\phi\right)$$

Solution. L.H.S. $=\left[\dfrac{(\sin^2\phi+\cos^2\phi)+(\sin\phi+i\cos\phi)}{1+\sin\phi-i\cos\pi}\right]^n$

$$=\left[\frac{(\sin\phi+i\cos\phi)(\sin\phi-i\cos\phi)+(\sin\phi+i\cos\phi)}{(1+\sin\phi-i\cos\phi)}\right]^n$$

$$=\left[\frac{(\sin\phi+i\cos\phi)(1+\sin\phi-i\cos\phi)}{(1+\sin\phi-i\cos\phi)}\right]^n$$

$$=(\sin\phi+i\cos\phi)^n$$

$$=\left[\cos\left(\frac{\pi}{2}-\phi\right)+i\sin\left(\frac{\pi}{2}-\phi\right)\right]^n$$

$$=\cos\left(\frac{n\pi}{2}-n\phi\right)+i\sin\left(\frac{n\pi}{2}-n\phi\right)=\text{R.H.S}$$

Example 3. *Prove that*

$$[(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)]^n+[(\cos\theta+\cos\phi)-i(\sin\theta+\sin\phi)]^n$$

$$=2^{n+1}\cos^n\frac{(\theta-\phi)}{2}\cos n\frac{(\theta+\phi)}{2}$$

Solution. L.H.S.

$$=\left[2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}+i\,2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right]^n$$

$$+\left[2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}-i\,2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right]^n$$

$$=2^n\cos^n\frac{\theta-\phi}{2}\left[\left\{\cos\frac{\theta+\phi}{2}+i\sin\frac{\theta+\phi}{2}\right\}^n+\left\{\cos\frac{\theta+\phi}{2}-i\sin\frac{\theta+\phi}{2}\right\}^n\right]$$

$$=2^n\cos\frac{\theta-\phi}{2}\left[\cos n\frac{(\theta+\phi)}{2}+i\sin n\frac{(\theta+\phi)}{2}+\cos n\frac{(\theta+\phi)}{2}-i\sin n\frac{(\theta+\phi)}{2}\right]$$

$$=2^n\cos^n\frac{(\theta-\phi)}{2}\,.\,2\cos n\frac{(\theta+\phi)}{2}$$

$$=2^{n+1}\cos^n\frac{(\theta-\phi)}{2}\cos n\frac{(\theta+\phi)}{2}$$

Example 4. *If $x=\cos\theta+i\sin\theta$, $y=\cos\phi+i\sin\phi$ and m and n are integers, prove that*

$$\frac{x^m}{y^n}+\frac{y^n}{x^m}=2\cos(m\theta-n\phi)$$

Solution. $\dfrac{x^m}{y^n}+\dfrac{y^n}{x^m}=\dfrac{\cos m\theta+i\sin m\theta}{\cos n\phi+i\sin n\theta}+\dfrac{\cos n\phi+i\sin n\phi}{\cos m\theta+i\sin m\theta}$

$$=(\cos m\theta+i\sin m\theta)(\cos n\phi+i\sin n\phi)^{-1}$$

$$+(\cos n\phi+i\sin n\phi)(\cos m\theta+i\sin m\theta)^{-1}$$

$$=(\cos m\theta+i\sin m\theta)\,[\cos(-n\phi)+i\sin(-n\phi)]$$

$$+(\cos n\phi+i\sin n\phi)\,[\cos(-m\theta)+i\sin(-m\theta)]$$

$$=\cos(m\theta-n\phi)+i\sin(m\theta-n\phi)+\cos(m\theta-n\phi)$$

$$-\,i\sin(m\theta-n\phi)$$

$$=2\cos(m\theta-n\phi)$$

Example 5. *If* $(a_1+ib_1)(a_2+ib_2)(a_3+ib_3)\dots(a_n+ib_n)=A+iB$, *prove that*

$$(a)\ (a_1^2+b_1^2)(a_2^2+b_2^2)(a_3^2+b_3^2)\dots\dots(a_n^2+b_n^2)=A^2+B^2$$

$$(b)\ tan^{-1}\frac{b_1}{a_1}+tan^{-1}\frac{b_2}{a_2}+taa^{-1}\frac{b_3}{a_3}+\dots\dots+tan^{-1}\frac{b_n}{a_n}=tan^{-1}\frac{A}{B}$$

Solution. (*a*) Let $a_1+ib_1=r_1(\cos\theta_1+i\sin\theta_1)$

$$a_2+ib_2=r_2(\cos\theta_2+i\sin\theta_2)$$

and similar other expressions.

$$\therefore\qquad r_1=\sqrt{a_1^2+b_1^2},\ r_2=\sqrt{a_2^2+b_2^2},\ \dots,\ \text{etc.}$$

and $$\theta_1=\tan^{-1}\frac{b_1}{a_1},\ \theta_2=\tan^{-1}\frac{b_2}{a_2},\ \dots,\ \text{etc.}\qquad\dots(*)$$

Now it is given that

$$(a_1+ib_1)(a_2+ib_2)(a_3+ib_3)\dots(a_n+ib_n)=A+iB$$

or $$[r_1(\cos\theta_1+i\sin\theta_1)][r_2(\cos\theta_2+i\sin\theta_2)][r_3(\cos\theta_2+i\sin\theta_3)]$$

$$\dots[r_n(\cos\theta_n+i\sin\theta_n)]=A+iB$$

or $$r_1r_2r_3\dots r_n[\cos(\theta_1+\theta_2+\dots+\theta_n)+i\sin(\theta_1+\theta_2+\dots+\theta_n)]=A+iB$$

Equating real and imaginary parts on both sides, we get

$$A=r_1r_2r_3\dots\dots r_n\cos(\theta_1+\theta_2+\dots+\theta_n)\qquad\dots(**)$$

$$B=r_1r_2r_3\dots\dots r_n\sin(\theta_1+r_2+\dots+\theta_n)\qquad\dots(***)$$

Squaring and adding (**) and (***), we get

$$A^2+B^2=r_1^2\,r_2^2\,r_3^2\,\dots\,r_n^2$$

$$=(a_1^2+b_1^2)(a_2^2+b_2^2)\dots(a_n^2+b_n^2)$$

(*b*) Dividing (***) by (**), we get

$$\frac{B}{A}=\tan(\theta_1+\theta_2+\dots+\theta_n)$$

or $$\tan^{-1}\frac{B}{A}=\theta_1+\theta_2+\dots+\theta_n$$

$$=\tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\dots+\tan^{-1}\frac{b_n}{a_n}$$

Example 6. *Show that*

$$(1+i)^n+(1-i)^n=2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}.$$

Solution. Let $1+i=r(\cos\theta+i\sin\theta)$

Equating real and imaginary parts of both sides, we have

$$r\cos\theta=1 \quad \text{and} \quad r\sin\theta=1$$

Squaring and adding, we have

$$r^2=1+1=2 \quad \text{or} \quad r=\sqrt{2}$$

Dividing, we have $\tan\theta=1 \quad \Rightarrow \quad \theta=\pi/4$

$$\therefore \quad 1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$(1+i)^n=2^{n/2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]^n$$

$$=2^{n/2}\left[\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right]$$

Similarly $(1-i)^n=2^{n/2}\left[\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right]$

$$\therefore \quad (1+i)^n+(1-i)^n=2^{n/2}\left[2\cos\frac{n\pi}{4}\right]=2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}.$$

Example 7. *Prove that*

$$(a+ib)^{m/n}+(a-ib)^{m/n}=2(a^2+b^2)^{m/2n}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$$

Solution. The cartesian coordinates can be transformed into polar coordinates by means of the following relation :

$$a=r\cos\theta, \qquad b=r\sin\theta,$$

where $\quad r^2=a^2+b^2 \quad \text{and} \quad \theta=\tan^{-1}\dfrac{b}{a}.$

Putting this value in L.H.S., we get

$$(a+ib)^{\frac{m}{n}}+(a-ib)^{\frac{m}{n}}=(r\cos\theta+ir\sin\theta)^{\frac{m}{n}}$$

$$+(r\cos\theta-ir\sin\theta)^{\frac{m}{n}}$$

$$=r^{\frac{m}{n}}\left[(\cos\theta+i\sin\theta)^{\frac{m}{n}}+(\cos\theta-i\sin\theta)^{\frac{m}{n}}\right]$$

$$=r^{\frac{m}{n}}\left[\cos\frac{m}{n}\theta+i\sin\frac{m}{n}\theta+\cos\frac{m}{n}\theta-i\sin\frac{m}{n}\theta\right]$$

[By using De-Moivre's Theorem]

$$=r^{\frac{m}{n}} \,.\, 2 \cos \frac{m}{n}\, \theta = 2\, (a^2+b^2)^{\frac{m}{2n}} \cos \left[\frac{m}{n} \tan^{-1} \frac{b}{a} \right]$$

$$\left[\because \quad r=\sqrt{a^2+b^2},\ \theta=\tan^{-1}\frac{b}{a} \right]$$

Thus $(a+ib)^{\frac{m}{n}}+(a-ib)^{\frac{m}{n}}=2(a^2+b^2)^{\frac{m}{2n}} \cos \left[\frac{m}{n} \tan^{-1} \frac{b}{a} \right]$

THEOREM. Find q roots of $(\cos \theta+i \sin \theta)^{p/q}$, where p and q are integers prime to each other.

[**Hint.** We know that

$$\cos (2n\pi+\theta)=\cos \theta \text{ and } \sin (2n\pi+\theta)=\sin \theta,$$

where n is any integer.

$$(\cos \theta+i \sin \theta)^{p/q}=\cos \{(2n\pi+\theta)+i \sin (2n\pi+\theta)\}^{p/q}$$

$$=\cos \left\{ \frac{p(2n\pi+\theta)}{q} \right\}+i \sin \left\{ \frac{p(2n\pi+\theta)}{q} \right\}$$

Now giving n the successive values $0, 1, 2, \ldots, (q-1)$, we obtain the q values as

$$\cos \frac{p}{q}\theta+i \sin \frac{p}{q}\theta \text{ when } n=0$$

$$\cos \frac{p(2\pi+\theta)}{q}+i \sin \frac{p(2\pi+\theta)}{q}, \text{ when } n=1$$

$$\cos \frac{p(4\pi+\theta)}{q}+i \sin \frac{p(4\pi+\theta)}{q}, \text{ when } n=2$$

$$\cos \frac{p}{q}\{2(q-1)\pi+\theta\}+i \sin \frac{p}{q}\{2(q-1)\pi+\theta\},$$

when $n=q-1$

When $n=q$, we obtain the values as

$$\cos \frac{p(2q\pi+\theta)}{q}+i \sin \frac{p(2q\pi+\theta)}{q}$$

$$=\cos p \left(2\pi+\frac{\theta}{q} \right)+i \sin p \left(2\pi+\frac{\theta}{q} \right)$$

$$=\cos \left(2p\pi+\frac{p\theta}{q} \right)+i \sin \left(2p\pi+\frac{p\theta}{q} \right)$$

$$=\cos \frac{p\theta}{q}+i \sin \frac{p\theta}{q}$$

which is the same value as obtained by putting $q=0$.

Example 8. *Find all the values of* $(16)^{1/4}$

Solution. $16^{1/4}=2(1)^{1/4}$

$$=2\,(\cos\theta+i\sin\theta)^{1/4}=2\,(\cos 2n\pi+i\sin 2n\pi)^{1/4}$$

$$=2\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right)$$

Giving n the values 0, 1, 2 and 3, we get the required values as

$$2,\ 2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right),\ 2\,(\cos\pi+i\sin\pi)$$

and $$2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$$

i.e., $2,\ 2i,\ -2$ and $-2i$.

Example 9. *Find the all values of* $(8i)^{\frac{1}{3}}$

Solution. Since $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$

$$\therefore\quad 8i=8\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]$$

$$(8i)^{\frac{1}{3}}=8^{\frac{1}{3}}\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]^{\frac{1}{3}}$$

$$=2\left[\cos\left(2n\pi+\frac{\pi}{2}\right)+i\sin\left(2n\pi+\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

$$=2\left[\cos\frac{4n\pi+\pi}{6}+i\sin\frac{4n\pi+\pi}{6}\right]$$

Giving n the values 0, 1 and 2, the required values are

$$2\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right],\ 2\left[\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right]$$

and $$2\left[\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right]$$

i.e., $2\left[\frac{\sqrt{3}}{2}+i\ \frac{1}{2}\right],\ 2\left[\frac{-\sqrt{3}}{2}+i.\ \frac{1}{2}\right]$ and $2[0+i\,(-1)]$

i.e., $2\left(\frac{\sqrt{3}+i}{2}\right),\ 2\left(\frac{-\sqrt{3}+i}{2}\right)$ and $-2i$

i.e., $\sqrt{3}+i,\ -\sqrt{3}+i$ and $-2i$.

Example 10. *Find all the values of* $(-1+\sqrt{3}i)^{7/3}$

Solution. Let $-1+\sqrt{3}i=r\,(\cos\theta+i\sin\theta)$, then

$r\cos\theta=-1$ and $r\sin\theta=\sqrt{3}\Rightarrow r=2$

and $\cos\theta=-\frac{1}{2}$ and $\sin\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{2\pi}{3}$

$$\therefore \quad (-1+\sqrt{3}i)^7=2^7\left[\cos\frac{2\pi}{3}+i\ \sin\frac{2\pi}{3}\right]^7$$

$$=2^7\left[\cos\frac{14\pi}{3}+i\ \sin\frac{14\pi}{3}\right]$$

$$=2^7\left[\cos\left(2n\pi+\frac{14\pi}{3}\right)+i\ \sin\left(2n\pi+\frac{14\pi}{3}\right)\right]$$

or $(-1+\sqrt{3}i)^{7/3}=2^{7/3}\left[\cos\left(\frac{6n\pi+14\pi}{9}\right)+i\ \sin\left(\frac{6n\pi+14\pi}{9}\right)\right]$

Giving n the volues 0, 1 and 2, the required values are

$$2^{7/3}\left[\cos\frac{14\pi}{9}+i\ \sin\frac{14\pi}{9}\right]\ 2^{7/3}\left[\cos\frac{20\pi}{9}+i\ \sin\frac{20\pi}{9}\right]$$

and $2^{7/3}\left[\cos\frac{26\pi}{9}+i\ \sin\frac{26\pi}{9}\right]$

Example 11 *Find the continued product of the four values of*

$$\left(cos\ \frac{\pi}{3}+i\ sin\ \frac{\pi}{3}\right)^{3/4}$$

Solution. $\left(\cos\frac{\pi}{3}+i\ \sin\frac{\pi}{3}\right)^{3/4}=\left[\cos\frac{3\pi}{3}+i\ \sin\frac{3\pi}{3}\right]^{1/4}$

$$=(\cos\pi+i\ \sin)^{1/4}$$

$$=[\cos(2n\pi+\pi)+i\ \sin(2n\pi+\pi)]^{1/4}$$

$$=\cos\left(\frac{2n\pi+\pi}{4}\right)+i\ \sin\left(\frac{2n\pi+\pi}{4}\right)$$

Required continued product

$$=\left(\cos\frac{\pi}{4}+i\ \sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4}+i\ \sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$
$$\times\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

$$=\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)+i\ \sin\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)$$

$=\cos 4\pi+i\sin 4\pi=1+i\times 0=1.$

Example 12. *Expand cos 8θ and sin 8θ.*

Solution. By De-Moivre's Theorem :

$$(\cos\theta+i\sin\theta)^8=\cos 8\theta+i\ \sin\ 8\theta \qquad \ldots(*)$$

Also by Binomial Theorem, we have

$$(\cos\theta+i\sin\theta)^8=\cos^8\theta+{}^8c_1\cos^7\theta\,(i\sin\theta)+{}^8c_2\cos^6\theta\,(i\sin\theta)^2$$
$$+{}^8c_3\cos^5\theta\,(i\sin\theta)^3+{}^8c_4\cos^4\theta\,(i\sin\theta)^4$$
$$+{}^8c_5\cos^3\theta\,(i\sin\theta)^5+{}^8c_6\cos^2\theta\,(i\sin\theta)^r$$
$$+{}^8c_7\cos\theta\,(i\sin\theta)^7+{}^8c_8.(i\sin\theta)^8$$

$$=\cos^8\theta+8i\cos^7\theta\sin\theta-48\cos^6\theta\sin^2\theta-56\,i\cos^5\theta\sin^3\theta$$
$$+70\cos^4\theta\sin^4\theta+56i\cos^3\theta\sin^5\theta-48\cos^2\theta\sin^6\theta$$
$$-8i\cos\theta\sin^7\theta+\sin^8\theta$$

or
$$(\cos\theta+i\sin\theta)^8=(\cos^8\theta-48\cos^6\theta\sin^2\theta+70\cos^4\theta\sin^4\theta$$
$$-48\cos^2\theta\sin^6\theta+\sin^8\theta)$$
$$+i\,(8\cos^7\theta\sin\theta-56\cos^5\theta\sin^3\theta+56\cos^3\theta\sin^5\theta$$
$$-8\cos\theta\sin^7\theta) \qquad \ldots(**)$$

From (*) and (*), we have

$$\cos 8\theta+i\sin 8\theta=(\cos^8\theta-48\cos^6\theta\sin^2\theta+70\cos^4\theta\sin^4\theta$$
$$-48\cos^2 0\sin^6\theta+\sin^8\theta+i\,(8\cos^7\theta\sin\theta-56\cos^5\theta\sin^8\theta$$
$$+56\cos^3\theta\sin^5\theta-8\cos\theta\sin^7\theta$$

Equating imaginary and real parts on both sides, we have

$$\sin 8\theta=8\cos^7\theta\sin\theta-56\cos^5\theta\sin^3\theta+56\cos^3\theta\sin^5\theta$$
$$-8\cos\theta\sin^7\theta$$

and
$$\cos 8\theta=\cos^8\theta-43\cos^6\theta\sin^2\theta+70\cos^4\theta\sin^4 0$$
$$-48\cos^2\theta\sin^6\theta+\sin^8\theta$$

Example 13. *Express :*

(*a*) *$\cos^7\theta$ in a series of cosines of multiples of θ.*

(*b*) *$\sin^{10}\theta$ in a series of cosines of multiples of θ.*

Solution. Let $x=\cos\theta+i\sin\theta$ so that $\frac{1}{x}=\cos\theta-i\sin\theta$

$$x^n=\cos n\theta+i\sin n\theta \text{ and } \frac{1}{x^n}=\cos n\theta-i\sin n\theta$$

$$\therefore \qquad x+\frac{1}{x}=2\cos\theta$$

and
$$x^n+\frac{1}{x^n}=2\cos n\,\theta$$

(*a*) $(2\cos\theta)^7=\left(x+\frac{1}{x}\right)^7$

$$={x^7}+{}^7c_1\,.\,x^6\,.\,\frac{1}{x}+{}^7c_2\,.\,x^5\,.\,\frac{1}{x^2}+{}^7c_3\,.\,x^4\,.\,\frac{1}{x^3}$$
$$+{}^7c_4\,.\,x^3\,.\,\frac{1}{x^4}+{}^7c_5\,.\,x^2\,.\,\frac{1}{x^5}+{}^7C_6\,.\,x\,.\,\frac{1}{x^6}+{}^7c_7\,.\,\frac{1}{x^7}$$
$$=x^7+7x^5+21x^3+35x+35\,.\,\frac{1}{x}+21\,.\,\frac{1}{x^3}+7\,.\,\frac{1}{x^5}+\frac{1}{x^7}$$
$$=\left(x^7+\frac{1}{x^7}\right)+7\left(x^5+\frac{1}{x^5}\right)+21\left(x^3+\frac{1}{x^3}\right)+35\left(x+\frac{1}{x}\right)$$

$$\therefore \quad 2^7\cos^7\theta=2\cos 7\theta+7\,.\,2\cos 5\theta+21\,.\,2\cos 3\theta+35\,.\,2\cos\theta$$

Hence $\cos^7 \theta = (\frac{1}{2})^6 [\cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta]$

(*b*) We have

$$(2\, i \sin \theta)^{10} = \left(x - \frac{1}{x}\right)^{10} = x^{10} + {}^{10}c_1\, x^9 \left(-\frac{1}{x}\right) + {}^{10}c_2\, x^8 \left(-\frac{1}{x}\right)^2$$

$$+ {}^{10}c_3\, x^7 \left(-\frac{1}{x}\right)^3 + {}^{10}c_4\, x^6 \left(-\frac{1}{x}\right)^4$$

$$+ {}^{10}c_5\, x^5 \left(-\frac{1}{x}\right)^5 + {}^{10}c_6\, x^4 \left(-\frac{1}{x}\right)^6$$

$$+ {}^{10}c_7\, x^3 \left(-\frac{1}{x}\right)^7 + {}^{10}c_8\, x^2 \left(-\frac{1}{x}\right)^8$$

$$+ {}^{10}c_9\, x \left(-\frac{1}{x}\right)^9 + {}^{10}c_{10} \left(-\frac{1}{x}\right)^{10}$$

$$= x^{10} - 10x^8 + 45x^6 - 120x^4 + 210x^2 - 252$$

$$+ 210 \frac{1}{x^2} - 120 \frac{1}{x^4} + 45 \frac{1}{x^6} - 10 \frac{1}{x^8} + \frac{1}{x^{10}}.$$

$$= \left(x^{10} + \frac{1}{x^{10}}\right) - 10\left(x^8 + \frac{1}{x^8}\right) - 45\left(x^6 + \frac{1}{x^6}\right)$$

$$-120\left(x^4 + \frac{1}{x^4}\right) + 210\left(x^2 + \frac{1}{x^2}\right) - 252.$$

or $2^{10}\, i^{10} \,.\, \sin^{10} \theta = 2 \,.\, \cos 10\, \theta - 10 \,.\, 2 \cos 8\theta + 45 \,.\, 2 \cos 6\theta$

$- 120\; 2 \,.\, \cos 4\theta + 210 \,.\, 2 \cos 2\theta - 252.$

Hence $\sin^{10} \theta = (-\frac{1}{2})^9 [\cos 10\theta - 10 \cos 8\theta + 45 \cos 6\theta$

$- 120 \cos 4\theta + 210 \cos 2\theta - 126]$

$[\because\; i^{10} = (i^2)^5 = (-1)^5 = -1]$

Example 14. *Prove that*

$$\sin^6 \theta \cos^2 \theta = \frac{1}{2^7} [-\cos 8\theta + 4 \cos 6\theta - 4 \cos 4\theta - 4 \cos 2\theta + 5]$$

Solution. Put

$$x = \cos \theta + i \sin \theta = C + iS$$

$$\frac{1}{x} = \cos \theta - i \sin \theta = C - S$$

$\therefore \quad 2 \cos \theta = x + \frac{1}{x},\; 2iS = x - \frac{1}{x}$

and $\quad x^n = \cos n\theta + i \sin n\theta,\; \frac{1}{x^n} = \cos n\theta - i \sin n\theta.$

$\therefore \quad 2 \cos n\theta = x^n + \frac{1}{x^n},\; 2\, i \sin n\theta = x^n - \frac{1}{x^n}$

$$(2i \sin \theta)^6 (2 \cos \theta)^2$$

$$=\left(x-\frac{1}{x}\right)^6\left(x+\frac{1}{x}\right)^2$$

$$=\left(x-\frac{1}{x}\right)^4\left(x^2-\frac{1}{x^2}\right)^2$$

$$=\left(x^4-4x^3 \,.\, \frac{1}{x}+6x^2 \,.\, \frac{1}{x^2}-4x \,.\, \frac{1}{x^3}+\frac{1}{x^4}\right)\left(x^4-2+\frac{1}{x^4}\right)$$

$$=\left(x^8+\frac{1}{x^8}\right)-4\left(x^6+\frac{1}{x^6}\right)+4\left(x_4+\frac{1}{x^4}\right)+4\left(x^2+\frac{1}{x^2}\right)+10$$

$\therefore \quad -2^8 \sin^6 \theta \cos^2 \theta$

$$=2 (\cos 8\theta-4 \cos 6\theta+4 \cos 4\theta+4 \cos 2\theta-10)$$

Hence $\sin^6 \theta \cos^2 \theta$

$$=\frac{1}{2^7} (-\cos 8\theta+4 \cos 6\theta-4 \cos 4\theta-4 \cos 2\theta+5)$$

SOME IMPORTANT THEOREMS ON MATRICES

Theorem 1. *Matrix multiplication is associative, i.e., if* **A**, **B** *are conformal for the product* **AB**, *and* **C** *are conformal for the product* **BC**, *then,*

$$(\mathbf{AB})\ \mathbf{C}=\mathbf{A}\ (\mathbf{BC})$$

Proof. Let **A, B, C** be the $m \times n$, $n \times p$ and $p \times q$ matrices and

$$\mathbf{A}=[a_{ij}], \quad \mathbf{B}=[b_{ij}], \quad \mathbf{C}=[c_{ij}]$$

Here **A, B** and **C** are conformal for the product **AB** and **BC**.

$$\mathbf{AB}=[a_{ij}]\times[b_{ij}]=[\ \sum_{k=1}^{n} b_{ik}\ bk_j\ ]=[u_{ij}], \text{ say}$$

$[u_{ij}]$ is an $m \times p$ matrix. $i=1, 2, \ldots\ldots, m$

$j=1, 2, \ldots\ldots, p.$

$\therefore$ (**AB**) and **C** are conformal for the product (**AB**) **C**.

$$(\mathbf{AB})\ \mathbf{C}=[u_{il}]\times[c_{lj}]=[\ \sum_{l=1}^{p} u_{il}\ c_{lk}]$$

$$=[\ \sum_{l=1}^{p} (\ \sum_{k=1}^{n} a_{ik}\ b_{kl})\ c_{lj}]$$

$$=[\ \sum_{l=1}^{p} \sum_{k=1}^{n} a_{ik}\ bk_l\ c_{lj}]\ ;\ i=1, 2, \ldots\ldots, m.$$

$j=1, 2, \ldots\ldots, p.$

It is an $m \times q$ matrix.

$$\mathbf{BC}=[b_{ij}]\times[c_{ij}]=[\sum_{s=1}^{p} b_{is}\, c_{sj}]=[v_{ij}], \text{ (say) } i=1, 2, \ldots, n.$$
$$j=1, 2, \ldots, q$$

$[v_{ij}]$ is an $n \times q$ matrix.

Therefore, **A** and (**BC**) are conformal for the product **A**(**BC**).

$$\mathbf{A(BC)}=[a_{ij}]\times[v_{ij}]=[\sum_{t=1}^{p} a_{it}\, v_{tj}]$$

$$=[\sum_{t=1}^{n} a_{it}\,(\sum_{s=1}^{p} b_{ts}\, c_{sj}]=\sum_{s=1}^{p}\sum_{t=1}^{n} a_{it}\, b_{ts}\, c_{sj}$$
$$i=1, 2, \ldots, m,$$
$$j=1, 2, \ldots, q.$$

Here $$\mathbf{(AB)\ C=A\ (BC)}$$

Theorem 2. *Matrix multiplication is distributive with respect to addition of matrices, i.e., if* **A** *and* **B** *are conformal for the product* **AB**, **B** *and* **C** *are conformal for addition, then*

$$\mathbf{A(B+C)=AB+AC}$$

Proof. Let $\mathbf{A}=[a_{ij}]$ be $m \times n$ matrix

and $\mathbf{B}=[b_{ij}]$ and $\mathbf{C}=[c_{ij}]$ be each $n \times p$ matrices, so that ($\mathbf{B}+\mathbf{C}$) is also $n \times p$ matrix.

Thus **A** (**B**+**C**) is of order $m \times p$.

Also **AB**+**AC** is of order $m \times p$.

Therefore, the matrices **A**(**B**+**C**) and **AB**+**AC** are conformable. Further.

(i, j) th element of **A** (**B**+**C**)

= Sum of the product of the corresponding elements of ith row of **A** and jth column of **B**+**C**.

$$=\sum_{k=1}^{n} a_{ik}\,(b_{kj}+c_{kj})$$

$$=\sum_{k=1}^{n} a_{ik}\, b_{jk}+\sum_{k=1}^{n} a_{ik}\, c_{kj}$$

(Since in case of real number, multiplication is distributive w.r.t. addition

$=(i, j)$ th element of **AB**$+(i, j)$ th element of **AC**

$=(i, j)$ th element of (**AB**+**AC**).

Hence $$\mathbf{A\ (B+C)=AB+AC}$$

Theorem 3. *The transpose of the product of two matrices is equal to the product of transposes taken in reverse order, i.e., if the matrices* **A** *and* **B** *are conformable for the product* **AB**, *then the matrices* **B'**, **A'** *are conformable for the product* **B' A'** *and*

$$(\mathbf{AB})' = \mathbf{B'}\ \mathbf{A'}$$

Proof. Let $\mathbf{A} = [a_{ij}]$ be an $m \times p$ matrix and

$\mathbf{B} = [b_{ij}]$ be an $n \times m$ matrix. Then

$\mathbf{A'} = [a_{ji}]$ is an $n \times m$ matrix.

and $\mathbf{B'} = [b_{ji}]$ is a $p \times n$ matrix

$$\mathbf{AB} = [a_{ij}] \times [b_{jk}] = [\sum_{j=1}^{n} a_{ij}\, b_{jk}]$$

It is an $m \times p$ matrix.

$$\therefore \qquad (\mathbf{AB})' = [\sum_{j=1}^{p} b_{jk}\, a_{ij}]\ ; \qquad k = 1, 2, \ldots, p$$
$$i = 1, 2, \ldots, m$$

The elements in the kth row of **B'** are the elements of the kth column of **B**.

They are $b_{1k}, b_{2k}, \ldots\ldots, b_{nk}$. Similarly the elements of the ith column of **A'** are $a_{i1}, b_{i2} \ldots\ldots, a_{in}$.

The scalar product of these two sets of elements $= \sum_{j=1}^{n} b_{jk}\, a_{ij}$

$$\therefore \qquad \mathbf{B'}\ \mathbf{A'} = [\sum_{j=1}^{n} b_{jk}\, a_{ij}]\ ;\ k = 1, 2, \ldots\ldots, p$$
$$i = 1.\ 2, \ldots\ldots, m$$

Hence $\qquad (\mathbf{AB})' = \mathbf{B'A'}$

Theorem 4. *Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.*

Proof. Let **A** be a square matrix and **A'** its transpose.

Then we have

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{A}+\mathbf{A'}) + \tfrac{1}{2}(\mathbf{A}-\mathbf{A'})$$
$$= \mathbf{P} + \mathbf{Q}, \text{ (say)} \qquad \ldots(1)$$

where $\qquad \mathbf{P} = \tfrac{1}{2}(\mathbf{A}+\mathbf{A'})$ and $\mathbf{Q} = \tfrac{1}{2}(\mathbf{A}-\mathbf{A'}) \qquad \ldots(2)$

Now $\qquad \mathbf{P'} = \tfrac{1}{2}(\mathbf{A}+\mathbf{A'})' = \tfrac{1}{2}[\mathbf{A'}+(\mathbf{A'})'] = \tfrac{1}{2}(\mathbf{A'}+\mathbf{A}) = \mathbf{P}$

and $\qquad \mathbf{Q'} = \tfrac{1}{2}(\mathbf{A}-\mathbf{A'})' = \tfrac{1}{2}[\mathbf{A'}-(\mathbf{A'})'] = -\tfrac{1}{2}(\mathbf{A}-\mathbf{A'}) = -\mathbf{Q}$

Thus **P** is a symmetric matrix and **Q** is a skew-symmetric matrix. Hence from (1), we conclude that a square matrix can be expressed as the sum of a symmetrio and skew-symmetric matrix.

To prove the uniqueness of representation (1), let, if possible

$$\mathbf{A} = \mathbf{B} + \mathbf{C} \qquad \ldots(3\ )$$

where **B** is symmetric and **C** is skew-symmetric matrix so that

$$\mathbf{B}'=\mathbf{B} \quad \text{and} \quad \mathbf{C}'=-\mathbf{C}$$

Then $\mathbf{A}'=(\mathbf{B}+\mathbf{C})'=\mathbf{B}'+\mathbf{C}'=\mathbf{B}-\mathbf{C}$ ...(4)

On adding and subtracting (3) and (4), we get respectively

$$\mathbf{B}=\tfrac{1}{2}(\mathbf{A}+\mathbf{A}')=\mathbf{P}$$
$$\mathbf{C}=\tfrac{1}{2}(\mathbf{A}-\mathbf{A}')=\mathbf{Q}$$

This establishes the uniqueness of (1).

Theorem 5. *If* $\mathbf{A}=[a_{ij}]$ *is a square matrix of order n, prove that*

$$\mathbf{A}(adj\ \mathbf{A})=(adj\ \mathbf{A})\ \mathbf{A}=|\ \mathbf{A}\ |\ \mathbf{I}_n$$

Proof. We have

$$\mathbf{A}\ (adj\ \mathbf{A})=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & |A| \end{bmatrix}$$

since from the property of determinants, we have

$$\sum_{j=1}^{n} a_{ij}\ A_{kj}=\begin{cases} |A|, & \text{if } i=k \\ 0, & \text{if } i\neq k \end{cases}$$

Hence $\mathbf{A}\ (adj\ \mathbf{A})=|\ \mathbf{A}\ |\ \mathbf{I}_n$.

Similarly it can be proved that

$$(adj\ \mathbf{A})\ .\ \mathbf{A}=|\ \mathbf{A}\ |\ \mathbf{I}_n.$$

Theorem 6. *The necessary and sufficient condition for the existence of the inverse of square matrix* **A** *is that* **A** *is non-singular.*

Proof. *The necessary condition* : Let **B** be the inverse of **A**.

$\therefore$ $\mathbf{AB}=\mathbf{BA}=\mathbf{I}$

$|\ \mathbf{AB}\ |=|\ \mathbf{A}\ |\times|\ \mathbf{B}\ |=|\ \mathbf{I}\ |=1$

$\therefore$ $|\ \mathbf{A}\ |\neq 0$. Thus **A** is non-singular.

The sufficient condition : If $|\ \mathbf{A}\ |\neq 0$.

$$\mathbf{A}\left(\frac{adj\ \mathbf{A}}{|\ \mathbf{A}\ |}\right)=\mathbf{I}=\left(\frac{adj\ \mathbf{A}}{|\ \mathbf{A}\ |}\right).\ \mathbf{A}$$

$\therefore$ $B=\dfrac{adj\ \mathbf{A}}{|\ \mathbf{A}\ |}$ and it exists.

Theorem 7. *$\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices of the same order, then $\mathbf{AB}$ is also non-singular and*

$$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$$

i.e., the inverse of the product of two non-singular matrices $\mathbf{A}$ and $\mathbf{B}$ is equal to the product of the inverses $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ in the reverse order.

Proof. If $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices of order n, then

$$|\mathbf{A}|\neq 0,\quad |\mathbf{B}|\neq 0.$$

Also $$|\mathbf{AB}|=|\mathbf{A}|\times|\mathbf{B}|\neq 0$$

$\Rightarrow$ $\mathbf{AB}$ is also non-singular and hence has an inverse $(\mathbf{AB})^{-1}$

We have

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})=\mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B}=\mathbf{A}^{-1}\mathbf{I}_n\mathbf{B}=\mathbf{B}^{-1}\mathbf{B}=\mathbf{I}_n$$

and $$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})=\mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1}=\mathbf{A}\,\mathbf{I}_n\mathbf{A}^{-1}=\mathbf{AA}^{-1}=\mathbf{I}_n$$

Hence $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is an inverse of $\mathbf{AB}$. In other words,

$$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$$

EXAMPLES ON DETERMINANTS

Example 15. *Show that*

$$\begin{vmatrix}\beta+\gamma & \alpha & 1\\ \gamma+\alpha & \beta & 1\\ \alpha+\beta & \gamma & 1\end{vmatrix}=0$$

Solution. Operating $C_1\to C_1+C_2$, we have

$$\Delta=\begin{vmatrix}\beta+\gamma+\alpha & \alpha & 1\\ \gamma+\alpha+\beta & \beta & 1\\ \alpha+\beta+\gamma & \gamma & 1\end{vmatrix}=(\alpha+\beta+\gamma)\begin{vmatrix}1 & \alpha & 1\\ 1 & \beta & 1\\ 1 & \gamma & 1\end{vmatrix}=(\alpha+\beta+\gamma)\times 0=0$$

Example 16. *Show that* $$\begin{vmatrix}1 & a & a^2\\ a^2 & 1 & a\\ a & a^2 & 1\end{vmatrix}=(1-a^3)^2$$

Solution. Operating $C_1\to C_1+C_2+C_3$, we get

$$\Delta=\begin{vmatrix}1+a+a^2 & a & a^2\\ 1+a+a^2 & 1 & a\\ 1+a+a^2 & a^2 & a\end{vmatrix}$$

$$=(1+a+a^2)\begin{vmatrix}1 & a & a^2\\ 1 & 1 & a\\ 1 & a^2 & 1\end{vmatrix}$$

$$=(1+a+a^2)\begin{vmatrix}1 & a & a^2\\ 0 & 1-a & a-a^2\\ 0 & a^2-a & 1-a^2\end{vmatrix}$$ Operating and $R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$

$$=(1+a+a^2)\begin{vmatrix}1-a & a(1-a)\\ a(a-1) & 1-a^2\end{vmatrix}$$

$$=(1+a+a^2)(1-a)^2\begin{vmatrix}1 & a\\ -a & 1+a\end{vmatrix}$$

$$=(1-a)\,(1-a)\,(1+a+a^2)\,(1+a+a^2)=(1-a^3)^2$$

Example 17. *Show that*

$$\begin{vmatrix}1 & a & a^2-bc\\ 1 & b & b^2-ca\\ 1 & c & c^2-ab\end{vmatrix}=0$$

Solution. $$\triangle=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}-\begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}-\frac{1}{abc}\begin{vmatrix}a & a^2 & abc\\ b & b^2 & abc\\ c & c^2 & abc\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}-\begin{vmatrix}a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}+\begin{vmatrix}a & 1 & a^2\\ b & 1 & b^2\\ c & 1 & c^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}-\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=0$$

Example 18. *Show that*

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

Solution. Operating $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ we get

$$\triangle = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c-a)\,(b+c+a) & 0 & a^2 \\ 0 & (c+a-b)(c+a+b) & b^2 \\ (c-a-b)(c+a+b) & (c-a-b)(c+a+b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$ Operating $R_3 \to R_3 - (R_1 + R_2)$

Operating $C_1 \to C_1 + \frac{1}{a}\ C_3$ and $C_2 \to C_2 + \frac{1}{b}\ C_3$, we get

$$\triangle = (a+b+c)^2 \begin{vmatrix} b+c & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

$$= (a+b+c)^2 \,.\, 2ab \begin{vmatrix} b+c & \frac{a^2}{b} \\ \frac{b^2}{a} & c+a \end{vmatrix}.$$

(expanding the determinant along its third column)

$$= 2ab\,(a+b+c)^2\,[(b+c)\,(c+a) - ab]$$

$$= 2abc\,(a+b+c)^3$$

Example 19. *Show that*

$$\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^3 (x+a+b+c+d)$$

Solution. Denoting the given determinant by $\triangle$, and operating $C_1 \to C_1 + C_2 + C_3 + C_4$, we get

$$\Delta = \begin{vmatrix} x+a+b+c+d & b & c & d \\ x+a+b+c+d & x+b & c & d \\ x+a+b+c+d & b & x+c & d \\ x+a+b+c+d & b & c & x+d \end{vmatrix},$$

$$= (x+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 1 & x+b & c & d \\ 1 & b & x+c & d \\ 1 & b & c & x+d \end{vmatrix}$$

[$\because$ $(x+a+b+c+d)$ is common in C_1]

$$= (x+a+b+c+d) \begin{vmatrix} 1 & b & c & d \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}, \text{ operating } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1 \text{ and } R_4 \to R_4 - R_1$$

$$= (x+a+b+c+d) \begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}, \text{ expanding the above determinant along } C_1$$

$$= (x+a+b+c+d) \cdot x \begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix}, \text{ expanding the above determinant along } C_1$$

$$= (x+a+b+c+d) \cdot x \cdot x^2$$

$$= x^2 (x+a+b+c+d).$$

Example 20. *Prove that*

$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$$

Solution. Denoting the given determinant by $\triangle$ and operating $C_1 \to C_1 - C_2 + C_3 - C_4$, we get

$$\triangle = \begin{vmatrix} a^3-3a^2+3a-1 & 3a^2 & 3a & 1 \\ 0 & a^2+2a & 2a+1 & 1 \\ 0 & 2a+1 & a+2 & 1 \\ 0 & 3 & 3 & 1 \end{vmatrix},$$

$$= (a^3-3a^2+3a-1) \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix},$$

(expanding the above determinant along C_1)

$$= (a-1)^3 \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix},$$

(operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$)

$$= (a-1)^5 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix},$$ $(a-1)$ is common to R_1 and R_2

$$= (a-1)^5 \begin{vmatrix} a+1 & 1 \\ 2 & 1 \end{vmatrix}$$ expanding the above determinant along C_3

$$= (a-1)^5 \, [(a+1) \,.\, 1 - 1 \,.\, 2] = (a-1)^5 \, (a-1) = (a-1)^6$$

Example 21. *Prove that*

$$\begin{vmatrix} 1+a^2 & ab & ac & ad \\ ab & 1+b^2 & bc & bd \\ ac & bc & 1+c^2 & cd \\ ad & bd & cd & 1+d^2 \end{vmatrix} = (1+a^2+b^2+c^2+d^2)$$

Solution. Multiplying C_1, C_2, C_3 and C_4 by a, b, c and d respectively and dividing by $abcd$, we get

$$\Delta = \frac{1}{abcd} \begin{vmatrix} a(1+a^2) & ab^2 & ac^2 & ad^2 \\ a^2b & b(1+b^2) & bc^2 & bd^2 \\ a^2c & b^2c & c(1+c^2) & cd^2 \\ a^2d & b^2d & c^2d & d(1+d^2) \end{vmatrix}$$

$$= \frac{abcd}{abcd} \begin{vmatrix} 1+a^2 & b^2 & c^2 & d^2 \\ a^2 & 1+b^2 & c^2 & d^2 \\ a^2 & b^2 & 1+c^2 & d^2 \\ a^2 & b^2 & c^2 & 1+d^2 \end{vmatrix}$$

(Taking a, b, c and d common from R_1, R_2, R_3 and R_4 respectively).

Now operating $C_1 \rightarrow C_1 + C_2 + C_3 + C_4$, we get

$$\Delta = (1+a^2+b^2+c^2+d^2) \begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 1 & 1+b^2 & c^2 & a^2 \\ 1 & b^2 & 1+c^2 & d^2 \\ 1 & b^2 & c^2 & 1+d^2 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2+d^2) \begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad \begin{array}{l} \text{Operating,} \\ R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array}$$

$$= 1+a^2+b^2+c^2+d\ ) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (1+a^2+b^2+c^2+d^2)$ [Expanding along the first column]

Example 22. *Prove that*

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Solution. Dividing C_1, C_2, C_3 and C_4 by a, b, c and d respectively, we get

$$\Delta = abcd \begin{vmatrix} (1/a)+1 & 1/b & 1/c & 1/d \\ 1/a & (1/b)+1 & 1/c & 1/d \\ 1/a & 1/b & (1/c)+1 & 1/d \\ 1/a & 1/b & 1/c & (1/d)+1 \end{vmatrix}$$

Operating $C_1 \rightarrow C_1 + C_2 + C_3 + C_4$, we get

$$\Delta = abcd \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

$$\times \begin{vmatrix} 1 & 1/b & 1/c & 1/d \\ 1 & (1/b)+1 & 1/c & 1/d \\ 1 & 1/b & (1/c)+1 & 1/d \\ 1 & 1/b & 1/c & (1/d)+1 \end{vmatrix}$$

$$= abcd \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \begin{vmatrix} 1 & 1/b & 1/c & 1/d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

[Operating $R_1 \rightarrow R_i - R_1$; $i=2, 3, 4$]

$$= abcd \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(Expanding by first column)

PRODUCT OF TWO DETERMINANTS

The product of two determinants of third order is a determinant of third order. More precisely if Δ_1 and Δ_2 are two determinants each of order 3 as given below :

$$\triangle_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \triangle_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

then their product $\triangle_1 \,.\, \triangle_2$ is the determinant $\triangle$ of order 3 given by

$$\triangle = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_2\gamma_3 \end{vmatrix}$$

Now $\triangle$ can be split up into $3 \times 3 \times 3$, *i.e.*, 27 determinants, since there are three constituents in each column. These 27 determinants are :

$$\begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 \end{vmatrix} \quad \begin{vmatrix} a_1\alpha_1 & a_1\beta_2 & a_1\gamma_3 \\ a_2\alpha_1 & a_2\beta_2 & a_2\gamma_3 \\ a_3\alpha_1 & a_3\beta_2 & a_3\gamma_3 \end{vmatrix}$$

$$\begin{vmatrix} b_1\beta_1 & a_1\alpha_2 & c_1\gamma_3 \\ b_2\beta_1 & a_2\alpha_2 & c_2\gamma_3 \\ b_3\beta_1 & a_3\alpha_2 & c_3\gamma_3 \end{vmatrix}$$

The first of these determinants

$$= \alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{vmatrix} = 0, \text{ since the columns in the determinant are identical.}$$

The second of these determinants

$$= \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3 \,.\, \triangle_1$$

The third of these determinants

$$= \beta_1\alpha_2\gamma_3 \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = -\beta_1\alpha_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= -\alpha_2\beta_1\gamma_3 \times \triangle_1$$

In this way we can easily find out that 21 out of these 27 determinants will vanish and the remaining six determinants will be

$$\alpha_1\beta_2\gamma_3\Delta_1-\alpha_1\beta_3\gamma_2\Delta_1+\alpha_2\beta_3\gamma_1\Delta_1-\alpha_2\beta_1\gamma_3\Delta_1$$
$$+\alpha_3\beta_1\gamma_2\Delta_1-\alpha_3\beta_2\gamma_1\Delta_1$$
$$=(\alpha_1\beta_2\gamma_3-\alpha_1\beta_3\gamma_2+\alpha_2\beta_3\gamma_1-\alpha_2\beta_1\gamma_3+\alpha_3\beta_1\gamma_2-\alpha_3\beta_2\gamma_1)\ \Delta_1$$
$$=[\alpha_1\ (\beta_2\gamma_3-\beta_3\gamma_2)-\alpha_2\ (\beta_1\gamma_3-\beta_3\gamma_1)+\alpha_3\ (\beta_1\gamma_2-\beta_2\gamma_1)]\ \Delta_1$$
$$=\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}\times\Delta_1=\Delta_2\times\Delta_1$$
$$\therefore \qquad \Delta=\Delta_2\times\cdot\Delta_1$$

Remark. Formula gives the so-called *row by row rule of the multiplication of two determinants* and may be described in words as follows :

"*The three elements of the first row of the product determinant* $\Delta=\Delta_1\Delta_2$ *are given by the sum of the products of the elements of the first row of* Δ_1 *with the corresponding elements of the first, second and third row of* Δ_2 *respectively. Similarly the elements of the second row of* Δ *are given by the sum of the products of the constituents of 2nd row of* Δ_1 *with the constituents of the first, 2nd and 3rd row of* Δ_2 *respectively ; and so on for the third row of* Δ".

Similarly, we can obtain the value of Δ by *column by column or column by row rule* of multiplication of two determinants Δ_1 and Δ_2. Thus the value of Δ can be obtained by any of the four rules of multiplication, *viz*., row by column rule, row by row rule, column by column rule or column by row rule. In other words, *we can multiply two determinants of the same order by any of the four rules of multiplication.*

Example 22. *Express the product*

$$\begin{vmatrix} a & 0 & 1\\ 1 & b & 0\\ 0 & 1 & c\end{vmatrix}\times\begin{vmatrix} 0 & 1 & a\\ b & 0 & 1\\ 1 & c & 0\end{vmatrix}$$

as a determinant and find the value of it

Solution.

$$\begin{vmatrix} a & 0 & 1\\ 1 & b & 0\\ 0 & 1 & c\end{vmatrix}\times\begin{vmatrix} 0 & 1 & a\\ b & 0 & 1\\ 1 & c & 0\end{vmatrix}$$
$$=\begin{vmatrix} a\times0+0\times1+1\times a & a\times b+0\times0+1\times1 & a\times1+0\times c+1\times0\\ 1\times0+1\times1+0\times a & 1\times b+b\times0+0\times1 & 1\times1+b\times c+0\times0\\ 0\times0+1\times1+c\times a & 0\times b+1\times0+c\times1 & 0\times1+1\times c+c\times0\end{vmatrix}$$

$$=\begin{vmatrix} a & ab+1 & a \\ b & b & 1+bc \\ 1+ca & c & c \end{vmatrix}$$

Also

$$\Delta=\begin{vmatrix} a & 0 & 1 \\ 1 & b & 0 \\ 0 & 1 & c \end{vmatrix}=-\begin{vmatrix} 0 & a & 1 \\ b & 1 & 0 \\ 1 & 0 & c \end{vmatrix}=-\begin{vmatrix} 0 & 1 & a \\ b & 0 & 1 \\ 1 & c & 0 \end{vmatrix}$$

(by interchanging the columns)

$=(abc+1)$ [by expanding]

$\therefore$ Product$=(abc+1)^2$.

Example 23. *Prove that*

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

where capital letters denote the cofactors of the corresponding small letters in the determinant on the right hand side, provided it is not zero.

Solution. Let us write

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta=\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

Now

$$\Delta\Delta'=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\times\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2B_2 & a_3A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_3 & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}$$

(By row-by-row rule of multiplication)

$$=\begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}=\Delta^3$$

Hence if $\Delta\neq 0$; then $\Delta_1=\Delta^2$

Example 24. *Express*

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$$

as a product of two determinants and prove that the value of the determinant is

$$2\,(b-c)\,(c-a)\,(a-b)\,(y-z)\,(z-x)\,(x-y)$$

Solution. Given determinant

$$\Delta=\begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$=\begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}\times\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

(by inspection and trial)

$$=2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\times\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

(Interchanging the first and third column)

$$=2\,(a-b)\,(b-c)\,(c-a)\,(x-y)\,(y-z)\,(z-x)$$

(On simplification)

Example 26. *Solve the following equations by Cramer's rule :*

$$x-2y+3z=5$$
$$4x+3y+4z=7$$
$$x+y-z=-4$$

Solution. We have

Δ = Determinant of coefficients of x, y, z

$$=\begin{vmatrix} 1 & -2 & 3 \\ 4 & 3 & 4 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=1\,.\,(-3-4)+(3-2)+1\,(-8-9)=-20\neq 0$$

Since $\triangle \neq 0$, the unique solution of the system is given by

$$x=\frac{\Delta_1}{\Delta},\ y=\frac{\Delta_2}{\Delta},\ z=\frac{\Delta_3}{\Delta} \qquad \ldots(*)$$

where $\Delta_1=\begin{vmatrix} 5 & -2 & 3 \\ 7 & 3 & 4 \\ -4 & 1 & -1 \end{vmatrix}=40$ (on simplification)

$$\Delta_2=\begin{vmatrix} 1 & 5 & 3 \\ 4 & 7 & 4 \\ 1 & -4 & -1 \end{vmatrix}=-20 \text{ (on simplification)}$$

$$\Delta_3=\begin{vmatrix} 1 & -2 & 5 \\ 4 & 3 & 7 \\ 1 & 1 & -4 \end{vmatrix}=-60 \text{ (on simplification)}$$

Substituting in (*), we get

$$x=-2, \qquad y=1 \qquad \text{and} \qquad z=3.$$

Example 26. *Use determinants to solve the following equations :*

$$ax+by+cz=k$$
$$a^2x+b^2y+c^2z=k^2$$
$$a^3x+b^3y+c^3z=k^3$$

Solution. The determinant of the system

$$\Delta=\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$=abc(a-b)(b-c)(c-a)$$

Let us suppose that a, b, c are three distinct numbers and they are different from zero.

$$\therefore \qquad \triangle \neq 0.$$

$$\therefore \qquad x=\frac{\begin{vmatrix} k & b & c \\ k^2 & b^2 & c^2 \\ k^3 & b^3 & c^3 \end{vmatrix}}{\Delta}=\frac{kbc\begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix}}{\Delta}$$

$$=\frac{kbc(k-b)(b-c)(c-k)}{abc(a-b)(b-c)(c-a)}=\frac{k(k-b)(c-k)}{a(a-b)(c-a)}$$

$$y=-\frac{\begin{vmatrix} a & k & c \\ a^2 & k^2 & c^2 \\ a^3 & k^3 & c^3 \end{vmatrix}}{\Delta}=-\frac{kac\begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix}}{\Delta}$$

$$=\frac{kac(a-k)(k-c)(c-a)}{abc(a-b)(b-c)(c-a)}=\frac{k(a-k\ (k-c)}{a(a-b)(b-c)}$$

Similarly, we shall get

$$z=\frac{k(b-k)(k-a)}{c(b-c)(c-a)}$$

CHARACTERISTIC EQUATION AND ROOTS OF A MATRIX

Let $A=[a_{ij}]$ be a $n \times n$ square matrix. Then matrix $\mathbf{A}-\lambda\mathbf{I}$ is called the characteristic matrix of **A**. The determinant

$$|\mathbf{A}-\lambda\mathbf{I}|=\phi(\lambda), \text{ say}$$

which on expansion gives a polynomial of degree n in λ is called the characteristic polynomial or characteristic determinant of characteristic function of A. The equation

$$\phi(\lambda)=|\mathbf{A}-\lambda\mathbf{I}|=0$$

is known as characteristic equation of **A** and its roots, say, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the characteristic roots or lateral roots of **A**.

Example 27. *Find the characteristic equation and roots of*

$$A=\begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

Solution. The characteristic equation is

$$|\mathbf{A}-\lambda\mathbf{I}|=0$$

$$\Rightarrow \qquad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow \quad (6-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix}+2\begin{vmatrix} -2 & -1 \\ 2 & 3-\lambda \end{vmatrix}+2\begin{vmatrix} -2 & 3-\lambda \\ 2 & -1 \end{vmatrix}=0$$

$$\Rightarrow \quad (6-\lambda)\,[(3-\lambda)^2-1]+2[-6+2\lambda+2]+2[2-6+2\lambda]=0$$

$$\Rightarrow \quad \lambda^3-12\lambda^2+44\lambda-48=0 \text{ (on simplification)}$$

$$\Rightarrow \quad (\lambda-2)\,(\lambda-4)\,(\lambda-6)=0$$

Hence characteristic roots are 2, 4 and 6

Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic equation. Thus if

$$\phi(\lambda)=|\mathbf{A}-\lambda\mathbf{I}|=(-1)^n(\lambda^n+p_1\lambda^{n-1}+p_2\lambda^{n-3}+\ldots+p_n)=0$$

is the characteristic equation of the matrix **A**, then

$$\phi(\mathbf{A})=(-1)^n[\mathbf{A}^n+p_1\mathbf{A}^{n-1}+p_2\mathbf{A}^{n-2}+\ldots+p_n\mathbf{I})=\mathbf{0}\ldots\ldots(*)$$

Remark. Cayley-Hamilton theorem may also be used to obtain the inverse of a non-singular matrix **A**. If **A** is non-singular ($|\mathbf{A}|\neq 0$), then premultiplying (*) by $\mathbf{A}^{-1}$ and transposing, we get

$$\mathbf{A}^{1}=-\frac{1}{p_n}[\mathbf{A}^{n-1}+p_1\mathbf{A}^{n-2}+\ldots+p_{n-1}\mathbf{I}]$$

Example 28. *Verify that the matrix*

$$\mathbf{A}=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

satisfies its characteristic equation. Hence compute $\mathbf{A}^{-1}$.

Solution. The characteristic equation of **A** is

$$|\mathbf{A}-\lambda\mathbf{I}|=0$$

$$\Rightarrow \quad \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}+1\begin{vmatrix} -1 & -1 \\ 1 & 2-\lambda \end{vmatrix}+1\begin{vmatrix} -1 & 2-\lambda \\ 1 & -1 \end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)^2-1]+(-2+\lambda+1)+[1-(2-\lambda)]=0$$

$$\Rightarrow \quad \lambda^3-6\lambda^2+9\lambda-4=0 \qquad \text{(On simplification)}$$

By Cayley-Hamilton theorem, we get

$$\mathbf{A}^3-6\mathbf{A}^2+9\mathbf{A}-4\mathbf{I}=\mathbf{O} \qquad \ldots(*)$$

Verification of (*).

$$\mathbf{A}^2=\mathbf{A}\,.\,\mathbf{A}=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\times\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\mathbf{A}^3=\mathbf{A}^2\,.\,\mathbf{A}=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

We have $A^3-6A^2+9A-4I$

$$=\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}-6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}-4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=O$$

Premultiplying (*) by A^{-1}, we get

$$A^2-6A+9I-4A^{-1}=O$$

$$\Rightarrow \qquad A^{-1}=\tfrac{1}{4}\,[A^2-6A+9I]$$

$$=\tfrac{1}{4}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}-6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}+9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\tfrac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}=\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Example 29. *Obtain the characteristic equation of the matrix*

$$A=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Hence or otherwise calculate its inverse.

[*Delhi Univ., B.A. (Hons.) Eco , 1992*]

Solution. The characteristic equation of **A** is

$$|\,A-\lambda I\,|=O$$

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=0$$

$$\Rightarrow \qquad \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow \quad (1-\lambda)\,.\,\{(2-\lambda)\,(3-\lambda)-0\}-0\,.\,\{0-2\}+2\,.\,\{0-2(2-\lambda\}=0$$

$$\Rightarrow \qquad -\lambda^3+6\lambda^2-7\lambda-2=0$$

By Cayley-Hamilton Theorem, **A** satisfies its characteristic equation. herefore, we have

$$-A^3+6A^2-7A-2I=0.$$

Premultiplying by A^{-1}, we have

$$-A^2+6A-7I-2A^{-1}=0$$

$$\therefore \qquad A^{-1}=\tfrac{1}{2}[-A^2+6A-7I]$$

$$\mathbf{A}^2=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}=\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$\therefore\ -\mathbf{A}^2+6\mathbf{A}-7\mathbf{I}=-\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}+6.\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}-7.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} -5+6-7 & 0+0-0 & -8+12-0 \\ -2+0-0 & -4+12-7 & -5+6-0 \\ -8+12-0 & 0+0-0 & -13+18-7 \end{bmatrix}=\begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$\therefore\ \mathbf{A}^{-1}=\tfrac{1}{2}\begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix}=\begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$$

EXERCISES

1. Determine the characteristic roots of each of the following matrices :

$$(i)\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix},\quad (ii)\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix},\quad (iii)\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

2. Prove that each of the matrices

$$\mathbf{A}=\begin{pmatrix} o & h & g \\ h & o & f \\ g & f & o \end{pmatrix},\quad \mathbf{B}=\begin{pmatrix} o & f & h \\ f & o & g \\ h & g & o \end{pmatrix},\quad \mathbf{C}=\begin{pmatrix} o & g & f \\ g & o & h \\ f & h & o \end{pmatrix}$$

has the same characteristic roots.

3. Prove that the following matrices have the same characteristic equation

$$\mathbf{A}_1=\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},\ \mathbf{A}_2=\begin{pmatrix} b & c & a \\ c & a & b \\ b & c & a \end{pmatrix},\ \mathbf{A}_3=\begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$$

4. Find the characteristic equation of the matrix

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

Verify Cayley-Hamilton theorem. Hence or otherwise compute $\mathbf{A}^{-1}$

[**Hint.** Characteristic equation of **A** is

$$\lambda^3+2\lambda^2-\lambda-20=0$$

By Cayley-Hamilton theorem, we have

$$\mathbf{A}^3+2\mathbf{A}^2-\mathbf{A}-20\mathbf{I}=\mathbf{O}$$

This gives $\mathbf{A}^{-1}=\frac{1}{20}\,[\mathbf{A}^2+2\mathbf{A}-\mathbf{I}]=\frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

5. Show that the matrix

$$\mathbf{A}=\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

satisfy Cayley-Hamilton theorem.

[**Hint.** Characteristic equation of **A** is

$$\lambda^3+\lambda(a^2+b^2+c^2)=0$$

To satisfy Cayley's Theorem, we have to verify that

$$\mathbf{A}^3+(a^2+b^2+c^2)\ \mathbf{A}=\mathbf{O}]$$

6. If

$$\mathbf{A}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

show that for every integer $n \geqslant 3$,

$$\mathbf{A}^n=\mathbf{A}^{n-2}+\mathbf{A}^2-\mathbf{I}$$

Hence determine $\mathbf{A}^{50}$ and $\mathbf{A}^{100}$

[**Hint.** Cayley-Hamilton theorem gives

$$\mathbf{A}^3-\mathbf{A}^2-\mathbf{A}+\mathbf{I}=\mathbf{O} \qquad ...(*)$$

$$\Rightarrow \qquad \mathbf{A}(\mathbf{A}^2-\mathbf{I})=\mathbf{A}^2-\mathbf{I}$$

Premultiplying (*) by $\mathbf{A}^{k-3}$, $k \geqslant 3$, we get

$$\mathbf{A}^{k-2}(\mathbf{A}^2-1)=\mathbf{A}^{k-3}\ (\mathbf{A}^2-1) \qquad ...(**)$$

Putting $k=n, n-1, \ldots, 3$ in succession and multiplying the resulting equations, we shall get

$$\mathbf{A}^{n-2}\ (\mathbf{A}^2-1)=\mathbf{A}^2-\mathbf{I}\ ;\ n \geqslant 3$$

$$\Rightarrow \qquad \mathbf{A}^n-\mathbf{A}^{n-2}=\mathbf{A}^2-\mathbf{I}\ ;\ n \geqslant 3 \qquad ...(***)$$

as desired.

To obtain $\mathbf{A}^{50}$, put $n=50,\ 48,\ \ldots,\ 4$ successively in (***) and add the resulting equations. Similarly for $\mathbf{A}^{100}$]

SUCCESSIVE DIFFERENTIATION

Example 30. *If* $x=\sin t$, $y=\sin pt$; *prove that*

$$(1-x^2)\ y_2-xy_1+p^2y=0$$

Solution. $\dfrac{dx}{dt}=\cos t, \qquad \dfrac{dy}{dt}=p \cos pt$

$$\therefore \qquad \frac{dy}{dx}=\frac{p \cos pt}{\cos t}$$

$$\Rightarrow \qquad \cos t\ .\ y_1=p \cos pt$$

$$\Rightarrow \qquad \cos^2 t\ .\ y_1{}^2=p^2 \cos^2 pt$$

$$\Rightarrow \quad (1-\sin^2 t)\ y_1^2 = p^2(1-\sin^2 pt)$$

$$\Rightarrow \quad (1-x^2)y_1^2 = p^2(1-y^2)$$

Differentiating again, we get

$$(1-x^2)\ 2y_1y_2 + y_1^2\ (-2x) = -p^2\ .\ 2yy_1$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 + p^2y = 0$$

Example 29. *If* $y = Ae^{-kt}\ cos\ (pt++e)$, *show that*

$$\frac{d^2y}{dt^2} + 2k\ \frac{dy}{dt} + n^2y = 0, \text{ where } n^2 = p^2 + k^2.$$

Solution. $\frac{dy}{dt} = -kAe^{-kt} \cos\ (pt+e) - pAe^{-kt} \sin\ (pt+e)$

$$= -ky - pAe^{-kt} \sin\ (pt+e) \qquad \ldots(*)$$

or $$pAe^{-kt}\ \sin\ (pt+e) = -ky - \frac{dy}{dt} \qquad \ldots(**)$$

Differentiating (*), we get

$$\frac{d^2y}{dt^2} = -k\ .\ \frac{dy}{dt} - pA\ [-ke^{-kt} \sin\ (pt+e) + e^{-kt}\ .\ p \cos\ (pt+e)]$$

$$= -k\ .\ \frac{dy}{dt} + k\ .\ pAe^{-kt} \sin\ (pt+e) - p^2y$$

$$= -k\ \frac{dy}{dt} + k\left(-ky - \frac{dy}{dt}\right) - p^2y \qquad \text{(Using **)}$$

$$= -2k\ \frac{dy}{dt} - (p^2+k^2)y$$

$$= -2k\ \frac{dy}{dt} - n^2y.$$ Transposing we get the result.

Example 30. *If* $p^2 = a^2\ cos^2\theta + b^2\ sin^2\theta$, *prove that*

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p}$$

Solution. We have

$$p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta \qquad \ldots(1)$$

$$= a^2\ (1-\sin^2\theta) + b^2 \sin^2\theta$$

$$= a^2 - (a^2-b^2) \sin^2\theta \qquad \ldots(2)$$

Again $$p^2 = a^2 \cos^2\theta + b^2\ (1-\cos^2\theta)$$

$$= (a^2-b^2) \cos^2\theta + b^2$$

$$\therefore \quad (a^2-b^2) \cos^2\theta = p^2 - b^2 \qquad \ldots(3)$$

Differentiating (1), we get

$$2p\,\frac{dp}{d\theta}=-2a^2\sin\theta\cos\theta+2b^2\sin\theta\cos\theta$$

$$\Rightarrow\qquad p\,\frac{dp}{d\theta}=-(a^2-b^2)\sin\theta\cos\theta \qquad ...(4)$$

Differentiating again, we get

$$p\,\frac{d^2p}{d\theta^2}+\left(\frac{dp}{d\theta}\right)^2=-(a^2-b^2)(\cos^2\theta-\sin^2\theta) \qquad ...(5)$$

From (4), we get $\frac{dp}{d\theta}=-\frac{(a^2-b^2)\sin\theta\cos\theta}{p}$

Substituting in (5), we have

$$p\,\frac{d^2p}{d\theta^2}+\frac{(a^2-b^2)^2\sin^2\theta\cos^2\theta}{p^2}$$
$$=-(a^2-b^2)(\cos^2\theta-\sin^2\theta)$$

$$\Rightarrow\qquad p\,\frac{d^2p}{d\theta^2}+\frac{(a^2-b^2)\sin^2\theta\,.\,(a^2-b^2)\cos^2\theta}{p^2}$$
$$=-(a^2-b^2)\cos^2\theta+(a^2-b^2)\sin^2\theta$$

$$\Rightarrow\qquad p\,\frac{d^2p}{d\theta^2}+\frac{(a^2-p^2)(p^2-b^2)}{p^2}=-(p^2-b^2)+(a^2-p^2)$$

[Using (2) and (3)]

$$\Rightarrow\qquad p\,\frac{d^2p}{d\theta^2}+a^2-\frac{a^2b^2}{p^2}-p^2+b^2=a^2+b^2-2p^2$$

$$\Rightarrow\qquad p\,\frac{d^2p}{d\theta^2}+p^2=\frac{a^2b^2}{p^2}.$$

Dividing by p, we get the result.

Example 31. *If* $x^2+2xy+3y^2=1$, *show that*

$$(x+3y)^3\,\frac{d^2y}{dx^2}+2=0.$$

Solution. Differentiating the given relation, we get

$$2x+2\left\{x\,\frac{dy}{dx}+y\right\}+6y\,\frac{dy}{dx}=0$$

$$\Rightarrow\qquad \frac{dy}{dx}=-\frac{x+y}{x+3y}$$

$$\because\qquad \frac{d^2y}{dx^2}=-\frac{\left[(x+3y)\left\{1+\frac{dy}{dx}\right\}-(x+y)\left\{1+3\,\frac{dy}{dx}\right\}\right]}{(x+3y)^2}$$

$$= -\frac{\left[(x+3y)-(x+y)+(x+3y-3x-3y)\dfrac{dy}{dx}\right]}{(x+3y)^2}$$

$$= -\left[2y-2x\frac{dy}{dx}\right] \div (x+3y)^2$$

$$= -2\left[y+\frac{x(x+y)}{x+3y}\right] \div (x+3y)^2$$

$$= -2\,(xy+3y^2+x^2+xy) \div (x+3y)^3$$

$$= -2\,(x^2+2xy+3y^2) \div (x+3y)^3$$

$$= -\frac{2}{(x+3y)^3} \qquad (\because \; x^2+2xy+3y^2=1)$$

$$\Rightarrow \qquad (x+3y)^3\frac{d^2y}{dx^2}+2=0$$

LEIBNITZ'S THEOREM

Statement. If $f(x)$ and $g(x)$ be two functions differentiable up to order n, then

$$(fg)_n = \sum_{r=0}^{n} {}^{n}C_r\, f_{n-r} g_r$$

$$= {}^{n}C_0 f_n g + {}^{n}C_1 f_{n-1}\, g_1 + {}^{n}C_2 f_{n-2}\, g_2 + \ldots + {}^{n}C_r f_{n-r}\, g_r + \ldots + {}^{n}C_n f g_n$$

where the suffixes in f and g denote the order of differentiation *w.r.t.* x.

Proof The theorem can be established using the '*Principal of Mathematical Induction*'.

Step I. By actual differentiation, we have

$$(fg)_1 = f_1 g + f g_1 = {}^{1}C_0\, f_1 g + {}^{1}C_1\, f g_1$$

$$(fg)_2 = (f_2 g + f_1 g_1) + (f_1 g_1 + f g_2)$$

$$= {}^{2}C_0\, f_2 g + {}^{2}C_1\, f_1 g_1 + {}^{2}C_2\, f g_2$$

Thus the theorem is true fot $n=1$ and $n=2$.

Step II. Let us assume that the theorem is true for $n=m$, so that

$$(fg)_m = \sum_{r=0}^{m} {}^{m}C_r\, f_{m-r} g_r$$

Differentiating both sides, we have

$$(fg)_{m+1} = \sum_{r=0}^{m} {}^{m}C_r \left\{ f_{m-r+1}\, g_r + f_{m-r}\, g_{r+1} \right\}$$

$$= {}^{m}C_0 \left\{ f_{m+1}\, g + f_m g_1 \right\} + {}^{m}C_1 \left\{ f_m g_1 + f_{m-1} g_2 \right\}$$

$$+ {}^{m}C_2 \left\{ f_{m-1} g_2 + f_{m-2}\, g_3 \right\} + \ldots + {}^{m}C_m \left\{ f_1 g_m + f g_{m+1} \right\}$$

$$= {}^{m}C_0\, f_{m+1}\, g + ({}^{m}C_0 + {}^{m}C_1)\, f_m g_1 + ({}^{m}C_1 + {}^{m}C_2)\, f_{m-1} g_2 + \ldots + {}^{m}C_m\, f g_{m-1}$$

Step III. We know

$${}^{m}C_0 = {}^{m+1}C_0,\; {}^{m}C_m = {}^{m+1}C_{m+1},\; {}^{m}C_r + {}^{m}C_{r-1} = {}^{m+1}C_r$$

$$\therefore \quad (fg)_{m+1} = {}^{m+1}C_0 f_{m+1} g + {}^{m+1}C_1 f_m g_1 + {}^{m+1}C_2 f_{m-1} g_2 + \ldots + {}^{m+1}C_{m+1} f g_{m+1}$$

$$= \sum_{r=0}^{m+1} {}^{m+1}C_r f_{m-r+1} g_r$$

Step IV. Thus if the theorem is true for $n=m$, it is certainly true for $n=m+1$. It is already verified for $n=1$ and 2, hence the theorem is true for all positive integral values of n.

Remark. We choose g for a function whose nth derivative is known, and f should be such function that vanishes after a few differentiations.

Example 32. *Find the nth derivative of*

$$y = x^3 \sin ax$$

Solution. Here we take $\sin ax$ as f and x^3 as g.

Now $\quad g_1 = 3x^2, \quad g_2 = 3.\ 2x, \quad g_3 = 3.\ 2, \quad g_4 = 0$

Also $\quad f_n = a^n \sin\left(ax + \frac{n\pi}{2}\right)$, etc.

Hence by Leibnitz's theorem ; we have

$$y_n = x^3 a^n \sin\left(ax + \frac{n\pi}{2}\right) + n\ .\ 3x^2\ .\ a^{n-1} \sin\left(ax + \frac{n-1}{2}\pi\right)$$

$$+ \frac{n\ n-1)}{2\,!}\ .\ 3\ .\ 2x\ a^{n-2} \sin\left(ax + \frac{n-2}{2}\pi\right)$$

$$+ \frac{n(n-1)(n-2)}{3\,!}\quad 3\ .\ 2\ .\ 1\ a^{n-3} \sin\left(ax + \frac{n-3}{2}\pi\right)$$

Remark. If one of the factors be a power of x it will be advisable to take that factor as g

Example 33. *Let* $y = x^4\ .\ e^{ax}$; *Find* y_5.

Solution. Here $\quad g = x^4, f = e^{ax}$

so that $\quad g_1 = 4x^3\ ;\ g_2 = 12x^2,\ g^3 = 24x,\ g_4 = 24$

and g_5 etc. all vanish.

Also $\quad f_n = a^n e^{ax}$; etc,

whence $\quad y_5 = a^5\ e^{ax}\ x^4 + 5a^4\ e^{ax}\ .\ 4x^3 + 10\ .\ a^3\ e^{ax}\ .\ 12x^2 + 10a^2\ e^{ax}\ .\ 24x + 5a\ e^{ax}\ .\ 24$

$$= a\ e^{ax}\ \{a^4x^4 + 20a^3x^3 + 120a^2x^2 + 240\ ax + 120\}$$

Example 34. *Differentiate n times the equation :*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Solution. $\frac{d^n}{dx^n}(x^2y_2) = x^2y_{n+2} + n\ .\ 2x\ .\ y_{n+1} + \frac{n(n-1)}{2\,!}\ .\ 2y_n.$

$$\frac{d^n}{dx^n}(xy_1) = xy_{n+1} + ny_n.$$

$$\frac{d^n y}{dx^n} = y_n \; ;$$

therefore, by addition,

$$x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0$$

or
$$x^2 \frac{d^{n+2}y}{dx^{n+2}} + (2n+1)\, x \frac{d^{n+1}y}{dx^{n+1}} + (n^2+1) \frac{d^n y}{dx^n} = 0$$

Example 35. *If* $y = a \cos(\log x) + b \sin(\log x)$, *show that*

$$x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0.$$

Solution. Differentiating, we have

$$y_1 = -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow \quad xy_1 = -a \sin(\log x) + b \cos(\log x)$$

Differentiating again, we get

$$xy_2 + y_1 = -\frac{a \cos(\log x)}{x} - \frac{b \sin(\log x)}{x}$$

$$\Rightarrow \quad x^2 y_2 + xy_1 = -[a \cos(\log x) + b \sin(\log x)] = -y$$

$$\Rightarrow \quad x^2 y_2 + xy_1 + y = 0$$

Differentiating this equation n times using Leibniz's theorem,

$$[x^2 y_{n+2} + {}^nC_1 \cdot 2x \cdot y_{n+1} + {}^nC_2 \cdot 2 \cdot y_n] + [xy_{n+1} + {}^nC_1 \cdot y_n] + y_n = 0$$

$$\Rightarrow \quad x^2 y_{n+2} + 2nxy_{n+1} + n(n-1)\, y_n + xy_{n+1} + ny_n + y_n = 0,$$

$$\Rightarrow \quad x^2 y_{n+2} + (2n+1)\, xy_{n+1} + [n(n-1) + n + 1]\, y_n = 0.$$

Remark. It may be noted that nth derivative of $y = y_n$; nth derivative of $y_1 = y_{n+1}$; nth derivative of $y_2 = y_{n+2}$.

Example 36. *If* $y = A\,(x + \sqrt{x^2-1})^n + B(x - \sqrt{x^2-1})^n$: *prove that*

(*a*) $(x^2-1)\, y_2 + xy_1 = n^2 y,$

(*b*) $(x^2-1)\, y_{n+2} + (2n+1) xy_{n+1} = 0$

Solution.
$$y_1 = nA\,(x + \sqrt{x^2-1})^{n-1} \left\{ 1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x \right\}$$

$$+ nB\,(x - \sqrt{x^2-1})^{n-1} \left\{ 1 - \frac{1}{2\sqrt{x^2-1)}} \cdot 2x \right\}$$

$$= nA\,(x + \sqrt{x^2-1})^{n-1} \cdot \frac{(\sqrt{x^2-1} + x)}{\sqrt{x^2-1}}$$

$$+ nB\,(x - \sqrt{x^2-1})^{n-1} \cdot \frac{(\sqrt{x^2-1} - x)}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad (\sqrt{x^2-1})\, y_1 = nA\,(x + \sqrt{x^2-1})^n - nB\,(x - \sqrt{x^2-1})^n$$

Differentiating again, we get

$$(\sqrt{x^2-1})\, y^2 + \frac{1}{2\sqrt{x^2-1}} \,.\, 2x \,.\, y_1$$

$$= n^2A\,(x+\sqrt{x^2-1})^{n-1} \,.\, \left\{1+\frac{1}{2\sqrt{x^2-1}} \,.\, 2x\right\}$$

$$- n^2B\,(x-\sqrt{x^2-1})^{n-1} \,.\, \left\{1-\frac{1}{2\sqrt{x^2-1}} \,.\, 2x\right\}$$

$$= n^2A\,(x+\sqrt{x^2-1})^{n-1}\, \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}$$

$$- n^2B\,(x-\sqrt{x^2-1})^{n-1} \,.\, \frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}$$

⇒ $$(x^2-1)\, y_2 + xy_1 = n^2\, [A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n]$$
$$= n^2y \qquad \ldots(*)$$

Differentiating equation (*) n times by Leibniz's theorem,

$$(x^2-1)\, y_{n+2} + {}^nC_1 \,.\, 2xy_{n+1} + {}^nC_2 \,.\, 2 \,.\, y_n + xy_{n+1} + {}^nC_1 \,.\, 1 \,.\, y_n = n^2y_n$$

⇒ $$(x^2-1)\, y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n = n^2y_n$$

⇒ $$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} = 0$$

Example 37. *If* $y^{1/m}+y^{-1/m}=2x$, *prove that*

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$$

Solution. $y^{1/m}+y^{-1/m}=2x$ gives $(y^{1/m})^2-2x(y^{1/m})+1=0$

∴ $y^{1/m}=\{2x\pm\sqrt{(4x^2-4)}\}/2=x\pm\sqrt{(x^2-1)}$

Thus $y=[x\pm\sqrt{(x^2-1)}]^m$

If $y=(x+\sqrt{x^2-1})^m$, then

$$y_1=m(x+\sqrt{x^2-1})^{m-1} \,.\, \left[1+\frac{x}{\sqrt{x^2-1}}\right]$$

$$= m(x+\sqrt{x^2-1})^m/\sqrt{(x^2-1)} = my/\sqrt{x^2-1}$$

If $y=(x-\sqrt{x^2-1})^m$, then

$$y_1=m(x-\sqrt{x^2-1})^{m-1}\left[1-\frac{x}{\sqrt{x^2-1}}\right]$$

$$= -m(x-\sqrt{x^2-1})^m/\sqrt{x^2-1} = -my/\sqrt{x^2-1}$$

Thus in either case

$$y_1^2=\frac{m^2y^2}{(x^2-1)} \quad \Rightarrow \quad (x^2-1)y_1^2=m^2y^2$$

Differentiating, we get

$$(x^2-1)2y_1y_2+2xy_1^2=2m^2yy_1$$

⇒ $$(x^2-1)y_2+xy_1-m^2y=0$$

Using Leibnitz's theorem for n-times differentiation, we get

$$[(x^2-1)y_{n+2}+{}^nC_1(2x)y_{n+1}+{}^nC_2(2)y_n]+[xy_{n+1}+{}^nC_1y_n]-m^2y_n=0$$

$$\Rightarrow \quad (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+[n(n-1)+n-m^2]y_n=0$$

$$\Rightarrow \quad (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$$

Example 38. *If* $\cos^{-1}(y/b)=\log(x/n)^n$, *prove that*

(a) $x^2y_2+xy_1+n^2y=0$

(b) $x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$

Solution. $\cos^{-1}(y/b)=n\log(x/n)=n[\log x-\log n]$

Differentiating, $\dfrac{-1}{\sqrt{1-(y^2/b^2)}}\cdot\dfrac{1}{b}\cdot y_1=\dfrac{n}{x}$

$$\Rightarrow \quad \frac{y_1}{\sqrt{b^2-y^2}}=\frac{n}{x}$$

$$\Rightarrow \quad x^2y_1{}^2=n^2(b^2-y^2)$$

Differentiating again, we get

$$x^2\,.\,2y_1y_2+2xy_1{}^2=n^2\,.\,(-2yy_1)$$

Dividing by $2y_1$, we get

$$x^2y_2+xy_1+n^2y=0$$

Differentiating n times using Leibnitz theorem, we get

$$[x^2y_{n+2}+{}^nC_1\,.\,2x\,.\,y_{n+1}+{}^nC_2\,.\,2\,.\,y_n]+[xy_{n+1}+{}^nC_1\,.\,1\,.\,y_n]+n^2y_n=0$$

$$\Rightarrow \quad x^2y_{n+2}+2n\;xy_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n+n^2y_n=0$$

$$\Rightarrow \quad x^2y_{2n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$$

Example 39. *If* $y=(x+\sqrt{(x^2+1)})^p$, *prove that*

$$(1+x^2)\,y_2+xy_1-p^2y=0.$$

Hence find the value of y_n *when* $x=0$, *n being an even integer. Also find* $y_n(0)$ *when n is an odd integer.*

Solution. We have $y=(x+\sqrt{x^2+1})^p$ ...(1)

Differentiating, $y_1=p(x+\sqrt{x^2+1})^{p-1}\,.\left(1+\dfrac{2x}{2\sqrt{x^2+1}}\right)$

$$=p(x+\sqrt{x^2+1})^{p-1}\,.\,\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$$

$$=p(x+\sqrt{x^2+1})^p/\sqrt{x^2+1}$$

$$=py/\sqrt{x^2+1}$$

or $\quad y_1\sqrt{x^2+1}=py$

i.e., $\quad (x^2+1)y_1{}^2=p^2y^2.$ ...(2)

Differentiating, we get

$$(x^2+1)\ 2y_1y_2+2x.y_1{}^2=2p^2yy_1$$

Dividing by $2y_1$. we get

$$(x^2+1)y_2+xy_1=p^2y \qquad \ldots(3)$$

which was to be proved.

Differentiating (3) n times by Leibnitz's theorem, we get

$$[(x^2+1)y_{n+2}+{}^nC_1\ .\ 2xy_{n+1}+{}^nC_2\ .\ 2\ .\ y_n]+[xy_{n+1}+{}^nC_1y_n]=p^2x_n$$

Simplifying, we get

$$(x^2+1)\ y_{n+2}+(2n+1)\ xy_{n+1}+(n^2-p^2)y_n=0.$$

Putting $x=0$,

$$y_{n+2}(0)=(p^2-n^2)y_n(0) \qquad \ldots(4)$$

From (1). putting $x=0,\ \ y(0)=1.$

From (2), $y_1=py/\sqrt{x^2+1}$;

$\therefore$ $y_1(0)=py(0)/1=p.$

From (3), putting $x=0$,

$$y_2(0)=p^2\ .\ y(0)=p^2 \quad [\because \quad y(0)=1]$$

In (4), put $n=2, 4, 6, \ldots\ldots$ successively ; then

$$y_4(0)=(p^2-2^2)\ y_2(0)=(p^2-2^2).p^2$$

$$y_6(0)=(p^2-4^2).y_4(0)=(p^2-4^2)\ (p^2-2^2)p^2$$

$$y_8(0)=(p^2-6^2)\ (p^2-4^2)\ (p^2-2^2)\ p^2, \text{ etc.}$$

Hence, when n is an even integer,

$$y_n(0)=[p^2\quad (n-2)^2][p^2-(n-4)^2]\ldots(p^2-2^2)p^2.$$

In (4), put $n=1, 3, 5, \ldots$ successively then

$$y_3\ (0)=(p^2-1^2)y_1\ (0)=(p^2-1^2)\ p$$

$$y_5\ (0)=(p^2-3^2)\ y_3(0)=(p^2-3^2)\ (p^2-1^2)p, \text{ etc.}$$

Hence, when n is an odd integer,

$$y_n(0)=[p^2-(n-2)^2]\ [p^2-(n-4)^2]\ldots(p^2-1^2)p.$$

EXERCISES

1. If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that

$$(1-x^2)\ y_{n+1}-(2n+1)x\ .\ y_n-n^2y_{n-1}=0$$

[**Hint.** $\sqrt{1-x^2}\ .\ y=\sin^{-1}x$

Differentiating w.r.t. x, we have

$$\sqrt{1-x^2}\ .\ y_1+y\ .\ \tfrac{1}{2}(1-x^2)^{-1/2}\ (-2x)=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \qquad y_1\ .\ (1-x^2)-y\ .\ x-1=0 \qquad \ldots(*)$$

We now apply Leibnitz rule for n-times differentiation.

$$[(1-x^2)\ y_{n+1}+{}^nC_1\ (-2x)y_n+{}^nC_2\ (-2)y_{n-1}]-[xy_{n+1}+{}^nC_1y_n]=0$$

⇒ $$(1-x^2)\, y_{n+1}-(2n+1)xy_n-n^2y_{n-1}=0]$$

2. If $y=\sin(m\sin^{-1}x)$, prove that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$$

3. If $y=\cos(m\log x)$, show that

$$x^2y_{n+2}+(2n+1)xy_{n+1}+(m^2+n^2)\,y_n=0.$$

[**Hint.** $y_1=-\sin(m\log x)\, m/x$

∴ $$\begin{aligned}x^2y_1^2&=m^2\sin^2(m\log x)\\&=m^2\{1-\cos^2(m\log x)\}\\&=m^2(1-y^2).\end{aligned}$$

Differentiating, we get

$$x^2\,.\,2y_1y_2+2xy_1^2=m^2(-2yy_1).$$

Divide by $2y_1$ and differentiate n times by Leibnitz's theorem.]

4. If $y=e^{\alpha\sin^{-1}x}$, show that

(*i*) $(1-x^2)y_2-xy_1-\alpha^2y=0.$

(*ii*) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+\alpha^2)y_n=0.$

[**Hint.** $y_1=e^{\alpha\sin^{-1}x}\,.\,\alpha/\sqrt{1-x^2}=\alpha y/\sqrt{1-x^2}.$

∴ $$y_1^2\,(1-x^2)=\alpha^2y^2.$$

Differentiating, we get

$$(1-x^2).2y_1y_2-2xy_1^2=2\alpha^2.yy_1.$$

Divide by $2y_1$ and transpose. Then differentiate n times]

5. If $y=(x^2-1)^n$, prove that

$$(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0.$$

Hence if $P_n=\dfrac{d_n}{dx^n}(x^2-1)^n$, show that

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right]+n(n+1)\,P_n=0.$$

[**Hint.** $y_1=n(x^2-1)^{n-1}\,.\,2x$

Multiplying by x^2-1, we get

$$(x^2-1)y_1=n(x^2-1)^n\,.\,2x=ny\,.\,2x.$$

Differentiate $(n+1)$ times to get the first result.

Now $P_n=D^n(x^2-1)^n=y_n.$

Hence the second result required is

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}\,(y_n)\right]+n(n+1)\,y_n=0.$$

$$\frac{d}{dx}\,[(1-x^2)y_{n+1}]+n(n+1)y_n=0,$$

⇒ $$(1-x^2)y_{n+2}-2xy_{n+1}+n(n+1)\,y_n=0$$

Multiplying by -1, we get

$$(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)\,y_n=0,$$

which is the same as the first result already proved].

6. If $f(x)=\tan x$, prove that

$$f^n(0)-{}^nC_2\,.\,f^{n-2}(0)+{}^nC_4(0)-\ldots=\sin\,(n\pi/2)$$

[Hint. $\qquad f(x)=\sin x/\cos x$

or $\qquad \cos x\,.\,f(x)=\sin x.$

Differentiating n times, we get

$$\cos x\,.\,f^n(x)+{}^nC_1\,.\,(-\sin x)\,.\,f^{n-1}(x)+{}^nC_2\,.\,(-\cos x)\,.\,f^{n-2}(x)+\ldots$$
$$=\sin\,(x+n\pi/2).$$

Putting $n=0$, we get the required result. $f^n(0)$ means the value of $f^n(x)$, when $x=0$]

PARTIAL DIFFERENTIATION

Example 40. *Find the partial derivatives with respect to x and y if*

$$Z=3xy-y^3+(y^2-2x)^{3/2}$$

Solution. $\qquad Z=3xy-y^3+(y^2-2x)^{3/2}$

$$\frac{\partial z}{\partial x}=3y+\frac{3}{2}\,(y^2-2x)^{1/2}\,(-2)=3[y-(y^2-2x)^{1/2}]$$

$$\frac{\partial z}{\partial y}=3x-3y^2+\frac{3}{2}\,(y^2-2x)^{1/2}\,(2y)$$

$$=3[x-y^2+y(y^2-2x)^{1/2}]$$

Example 4 *If $u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$, show that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

Solution. $\qquad u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)$$

$$=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

$$=\frac{-x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\left[\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}\right]$$

$$+y\left[\frac{-x}{y\sqrt{y^2-x^2}}+\frac{x}{x^2+y^2}\right]=0.$$

Example 42 *If* $u=f\left(\frac{y}{x}\right)$, *prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

Solution. Differentiating partially w.r.t. x, y ; we get

$$\frac{\partial u}{\partial x}=f'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right)$$

and

$$\frac{\partial u}{\partial y}=f'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right)$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x.\left(\frac{-y}{x^2}\right)f'\left(\frac{y}{x}\right)+y.\frac{1}{x}.f'\left(\frac{y}{x}\right)=0.$$

Example 43. *If* $f(x, y)=\log(x^2+y^2)+\tan^{-1}\frac{y}{x}$, *then prove that*

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

Solution. $f(x, y)=\log(x^2+y^2)+\tan^{-1}\frac{y}{x}$

$$\frac{\partial f}{\partial x}=\frac{2x}{x^2+y^2}+\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$=\frac{2x-y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{(x^2+y^2).2-(2x-y).2x}{(x^2+y^2)^2}$$

$$=\frac{2y^2-2x^2+2xy}{(x^2+x^2)^2}$$

$$\frac{\partial f}{\partial y}=\frac{2y}{x^2+y^2}+\frac{1}{1+\frac{y^2}{x^2}}.\frac{1}{x}$$

$$=\frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{(x^2+y^2).2-(2y+x).2y}{(x^2+y^2)^2}$$

$$=\frac{2x^2-2y^2\quad 2xy}{(x^2+y^2)^2}$$

$$\therefore \qquad \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

Example 44. *If* $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$, *show that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

Solution. We have

$$\frac{\partial u}{\partial x}=-\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}.\,2x=-x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2}=-(x^2+y^2+z^2)^{-3/2}+3x^2\,(x^2+y^2+z^2)^{-5/2}$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2}=-(x^2+y^2+z^2)^{-3/2}+3y^2\,(x^2+y^2+z^2)^{-5/3}$$

and $$\frac{\partial^2 u}{\partial z^2}=-(x^2+y^2+z^2)^{-3/2}+3z^2(x^2+y^2+z)^{-5/2}$$

Adding, we get $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial z^2}$

$$\begin{aligned}&=-3(x^2+y^2+z^2)^{-3/2}+3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)\\&=-3(x^2+y^2+z^2)^{-3/2}+3(x^2+y^2+z^2)^{-3/2}\\&=0\end{aligned}$$

Example 45. *If* $u=e^{x-at}\ \cos(x-at)$, *show that*

$$\frac{\partial^2 u}{\partial t^2}=a^2\ \frac{\partial^2 u}{\partial x^2}$$

Solution $$\frac{\partial u}{\partial x}=e^{x-at}\cos(x-at)+e^{x-at}\,[-\sin(x-at)]$$

$$=e^{x-at}\,[\cos(x-at)-\sin(x-at)]$$

$$\frac{\partial^2 u}{\partial x^2}=e^{x-at}\,[\cos(x-at)-\sin(x-at)]$$

$$+e^{x-at}\,[-\sin(x-at)-\cos(x-at)]$$

$$=-2e^{x-at}\ \sin(x-at)$$

$$\frac{\partial u}{\partial t}=e^{x-at}(-a)\cos(x-at)+e^{x-at}\,[-\sin(x-at)].(-a)$$

$$=ae^{x-at}\ [\cos\ (x-at)-\sin(x-at)]$$

$$\frac{\partial^2 u}{\partial t^2}=-ae^{x-at}\,(-a)[\cos(x-at)-\sin(x-at)]$$

$$-ae^{x-at}\ [\sin(x-at)+a\cos(x-at)]$$

$$=-2a^2e^{x-at}\ \sin(x-at)$$

$$\therefore \qquad \frac{\partial^2 u}{\partial t^2}=a^2\ \frac{\partial^2 u}{\partial x^2}$$

Example 46. *If* $u=\log r$, *where* $r^2=(x-a)^2+(y-b)^2+(z-c)^2$, *show that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{r^2}$$

Solution. Differentiating partially, we get

$$\frac{\partial u}{\partial x}=\frac{1}{r}\cdot\frac{\partial r}{\partial x}=\frac{1}{r^2}\ (x-a) \qquad \left[\because\ 2r\ \frac{\partial r}{\partial x}=2(x-a)\right]$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{r^2}+(x-a)\left(\frac{-2}{r^3}\right)\frac{\partial r}{\partial x}=\frac{1}{r^2}-\frac{2(x-a)^2}{r^4}$$

$$=\frac{r^2-2(x-a)^2}{r^4}$$

Similarly $\dfrac{\partial^2 u}{\partial y^2}=\dfrac{r^2-2(y-b)^2}{r^4}$, $\dfrac{\partial^2 u}{\partial z^2}=\dfrac{r^2-2(z-c)^2}{r^4}$

Adding, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{3r^2-2[(x-a)^2+(x-b)^2+(x-c)^2]}{r^4}$$

$$=\frac{r^2}{r^4}=\frac{1}{r^2}$$

Example 47. *If* $u=f(r)$, *where* $r^2=x^2+y^2$, *prove that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\ \frac{1}{r}f'(r)$$

Solution. $r=\sqrt{x^2+y^2} \Rightarrow \dfrac{\partial r}{\partial x}=\dfrac{1}{2}\ .(x^2+y^2)^{-1/2}\,.\,2x=\dfrac{x}{r}$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}=f'(r)\,.\,\frac{x}{r}$$

$$\therefore \qquad \frac{\partial^2 u}{\partial x^2}=\frac{r\left[\ f'(r)+xf''(r)\dfrac{\partial r}{\partial x}\right]-xf'(r)\dfrac{\partial r}{\partial x}}{r^2}$$

$$=\frac{1}{r^2}\ [rf'(r)+x^2f''(r)-(x^2/r)\ f'(r)]$$

$$=\frac{1}{r}\ f'(r)+x^2\left[\frac{f''(r)}{r^2}-\frac{f'(r)}{r^3}\right] \qquad \ldots(*)$$

Since r is a symmetric function, by interchanging x, y ; we get

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}\ f'(r)+y^2\left[\frac{f''(r)}{r^2}-\frac{f'(r)}{r^3}\right] \qquad \ldots(**)$$

Adding (*) and (**), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2}{r}f'(r)+(x^2+y^2)\left[\frac{f''(r)}{r^2}-\frac{f'(r)}{r^3}\right]$$

$$=\frac{2}{r}\;f'(r)+r^2\left[\frac{f''(r)}{r^2}-\frac{f'(r)}{r^3}\right]$$

$$=\frac{2}{r}\;f'(r)+f''(r)-\frac{1}{r}\,f''(r)$$

Hence $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}\;f'(r)+f''(r).$$

Homogeneous function. A function $f(x, y)$ is said to be a homogeneous of degree n if on replacing x by kx and y by ky, the function is multiplied by k^n, *i.e.*, if

$$f(kx, ky)=k^n f(x, y)$$

For example $\log x-\log y$ is of zero degree since

$$\log kx-\log ky=\log x-\log y=k^0\,(\log x-\log y)$$

Again $\sqrt{x^2-y^2}\,\sin^{-1}\frac{y}{x}$ is a homogeneous function of degree 1

since $$\sqrt{(kx)^2-(ky)^2}\,\sin^{-1}\frac{ky}{kx}=k^1\,\sqrt{x^2-y^2}\,\sin^{-1}\frac{y}{x}.$$

Another way of defining the homogeneous function $f(x, y)$ of degree n is that it can be expressible as

$$x^n\,f\left(\frac{y}{x}\right)$$

Now $\log x-\log y=x^\circ\,\log\frac{x}{y}$

and $\sqrt{x^2-y^2}\,\sin^{-1}\frac{y}{x}$ is $x^1\left[\sqrt{1-\left(\frac{y}{x}\right)^2}\,\sin^{-1}\frac{y}{x}\right]$

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

Statement. If $z=f(x, y)$ be a homogeneous function of x and y of degree n and possesses continuous partial derivatives, then

$$x\frac{\partial z}{\partial x}+y\,\frac{\partial z}{\partial y}=nz$$

Proof. Step I. Since $z=f(x, y)$ is a homogeneous function of degree n in x and y,

$$z=f(x,\,y)=x^n\phi\,(y/x) \quad \text{...(1)}$$

Step II. Differentiating (1) partially w.r.t. x, we have

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\,[x^n\phi(y/x)]$$

$$=[\phi(y/x)\,.\,nx^{n-1}]+[x^n\phi'\,(x/y)\,(-y/x^2)]$$

or
$$x\frac{\partial z}{\partial x}=nx^n\phi(y/x)-x^{n-1}y\phi'(y/x) \qquad ..(2)$$

Step III. Again, differentiating (1) partially w.r.t. y, we have

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}[x^n\phi(y/x)]$$

$$=x^n\phi'(y/x)(1/x)=x^{n-1}\phi'(y/x)$$

or
$$y\frac{\partial z}{\partial y}=y\,.\,x^{n-1}\phi'(y/x)$$

Step IV. Adding (1) and (2), we have

$$x\frac{\partial z}{\partial y}+y\frac{\partial z}{\partial y}=nx^n\phi(y/x)=nz$$

This proves the theorem

Deduction. *If $z=f(x, y)$ be a homogeneous function of degree n, then*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial z}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Since z is a homogeneous function of degree n, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are homogeneous functions of degree $(n-1)$.

Applying Euler's theorem to the functions $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we have

$$x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=(n-1)\frac{\partial z}{\partial x}$$

and
$$x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=(n-1)\frac{\partial z}{\partial y}$$

i.e.,
$$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y\partial z}=(n-1)\frac{\partial z}{\partial x} \qquad ..(1)$$

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} \qquad ...(2)$$

Multiply (1) by x and (2) by y and add, taking $\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]$$

But by Euler's theorem $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$

$$\therefore \quad x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1z.)$$

Example 48. *If* $f(x, y)=\sqrt{y^2-x^2}\ \sin^{-1}\dfrac{x}{y}+\dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$

show that $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}-f(x, y)=0$

Solution. $f(x, y)=\sqrt{y^2-x^2}\ \sin^{-1}\dfrac{x}{y}+\dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$

$$f(kx, ky)=\sqrt{k^2y^2-k^2x^2}\ \sin^{-1}\frac{kx}{ky}+\frac{k^2x^2-k^2y^2}{\sqrt{(k^2x^2+k^2y^2)}}$$

$$=k\left[\sqrt{y^2-x^2}\ \sin^{-1}\frac{x}{y}+\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right]$$

$f(x, y)$ is a homogeneous function of degree 1.

Hence $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f(x, y)$$

or $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}-f(x, y)=0$$

Example 49. *If* $u=\cos\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$, *prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$$

Solution. $$u=f(x, y, z)=\cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$$

$$f(kx, ky, kz)=\cos\left(\frac{k^2 . xy+k^2 . yz+k^2 . zx}{k^2x^2+k^2y^2+k^2z^2}\right)$$

$$=\cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)=k^\circ f(x, y, z)$$

∴ u is a homogeneous function of zero degree.

Hence $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0\times u=0$$

Example 50. *If* $u=\sin^{-1}\dfrac{x^2+y^2}{x+y}$, *show that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

Solution. Here u is not a homogeneous function but if $z=\sin\ u=\dfrac{x^2+y^2}{x+y}$, then z is a homogeneous function of x, y of degree 1.

∴ By Euler's theorem, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$ ...(*)

Since $z = \sin u$ is a function of x, y

$$\therefore \qquad \frac{\partial z}{\partial x} = \frac{dz}{du} \quad \frac{\partial u}{\partial x} = \cos u \cdot \frac{\partial u}{\partial x}.$$

and
$$\frac{\partial z}{\partial y} = \frac{dz}{\partial u} \quad \frac{\partial u}{\partial y} = \cos u \cdot \frac{\partial u}{\partial y}$$

Substituting these values in (*), we get

$$x \frac{\partial u}{\partial x} \cos u + y \frac{\partial u}{\partial y} \cos u = \sin u$$

or
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$$

Example 51. *If* $z = sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$, *show that*

$$\frac{\partial u}{\partial x} = -\frac{y}{x} \cdot \frac{\partial u}{\partial y}$$

Solution. $z = \sin^{-1} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right) = x^{\circ} \sin^{-1} \left(\frac{1 - \sqrt{y/x}}{1 - \sqrt{y/x}} \right)$

$\therefore$ z is a homogeneous function of degree zero.

Using Euler's theorem, we have

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0 \cdot z = 0$$

$$\Rightarrow \qquad \frac{\partial z}{\partial x} = -\frac{y}{x} \cdot \frac{\partial z}{\partial y}.$$

Example 52. *If* $u = log \left(\frac{x^3 + y^3}{x^2 + y^2} \right)$, *prove that* $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1$

Solution. $u = \log \left(\frac{x^3 + y^3}{x^2 + y^2} \right) \Rightarrow e^u = \frac{x^3 + y^3}{x^2 + y^2}$

$$= x^1 \left[\frac{1 + (y^3/x^3)}{1 + (y^2/x^2)} \right]$$

Here e^u is a homogeneous function of degree one.

$\therefore$ By Euler's theorem, we have

$$x \frac{\partial}{\partial x} (e^u) + y \frac{\partial}{\partial y} (e^u) = 1 \cdot e^u$$

$$\Rightarrow \qquad e^u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = e^u$$

$$\Rightarrow \qquad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1$$

EXERCISES

1. Find the partial derivatives with respect to each variable of

(*i*) $f(x, y, z, \omega)=x^2 e^{2y+3z} \cos 4\omega$

(*ii*) $f(r, \theta, z)=\dfrac{r(r-\cos 2\theta)}{r^2+z^2}$

2. If $z(x+y)=x^2+y^2$, show that

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

3. If $u=\sqrt{x^2+y^2+z^2}$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1$$

4. If $x^2+y^2+z^2=\dfrac{1}{u^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

5. If $u=x^2 \tan^{-1}\dfrac{y}{x}-y^2 \tan^{-1}\dfrac{x}{y}$, prove that

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{x^2-y^2}{x^2+y^2}$$

6. If $u=\tan^{-1}\dfrac{xy}{\sqrt{1+x^2+y^2}}$, show that

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{1}{(1+x^2+y^2)^{3/2}}$$

7. If (*i*) $z=\log\dfrac{x^2+y^2}{xy}$, (*ii*) $z=(x-y)\sqrt{x^2+y^2}$,

verify the relation $\dfrac{\partial^2 z}{\partial x \partial y}=\dfrac{\partial^2 z}{\partial y \partial x}$.

8. If $z=\tan(y+ax)-(y-ax)^{3/2}$, show that

$$\frac{\partial^2 z}{\partial x^2}=a^2 \frac{\partial^2 z}{\partial y^2}.$$

9. If $z=3xy-y^3+(y^2-2x)^{3/2}$, verify that

$$\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x} \text{ and } \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

10. If $u=\log(x^2+y^2+z^2-3xyz)$, show that

(*a*) $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=\dfrac{3}{x+y+z}$

(b) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{(x+y+z)^2}$

11. If $u=\log(x^2+y^2+z^2)$, show that

$$x\frac{\partial^2 u}{\partial y \partial z}=y\frac{\partial^2 u}{\partial z \partial x}=z\frac{\partial^2 u}{\partial x \partial y}$$

12. If $u=\phi(y+ax)+\psi(y-ax)$, show that

$$\frac{\partial^2 u}{\partial x^2}-a^2\frac{\partial^2 u}{\partial y^2}=0$$

[**Hint.**

$$\frac{\partial u}{\partial x}=\phi'(y+ax) \cdot a+\psi'(y-ax)(-a)$$

$$\frac{\partial^2 u}{\partial x^2}=\phi''(y+ax) \cdot a^2+\psi''(y-ax) \cdot a^2$$

$$\frac{\partial u}{\partial y}=\phi'(y+ax) \cdot 1+\psi'(y-ax) \cdot 1$$

$$\frac{\partial^2 u}{\partial y^2}=\phi''(y+ax)+\psi''(y-ax)\ \Big]$$

13. If $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

14. If $z=xyf(y/x)$, prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

15. If $z=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\tan z$$

16. $z=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\sin 2x$$

17. If $u=\sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}\tan u$$

18. If $u=\cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$$

EXAMPLES ON INTEGRATION

Example 53. *Evaluate* $\displaystyle\int\frac{dx}{\sin^2 x\cos^2 x}$

Solution. $\displaystyle\int\frac{dx}{\sin^2 x\cos^2 x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}\,dx$

$$=\int\sec^2 x\,dx+\int\operatorname{cosec}^2 x\,dx=\tan x-\cot x.$$

Example 54. *Evaluate* $\displaystyle\int\frac{\cos x}{\cos(x+a)}\,dx.$

Solution. Let $I=\displaystyle\int\frac{\cos x}{\cos(x+a)}\,dx.$

Put $x+a=t$ $\quad\because\quad dx=dt$

$\therefore$ $\displaystyle I=\int\frac{\cos(t-a)}{\cos t}\,dt=\int\frac{\cos a\cos t+\sin t\sin a}{\cos t}\,dt$

$$=\cos a\int dt+\sin a\int\tan t\,dt$$

$$=t\cos a+\sin a\log\sec t$$

$$=(x+a)\cos a+\sin a\log\sec(x+a).$$

Example 55. *Evaluate* $\displaystyle\int\frac{dx}{\sqrt{1+\sin x}}$

Solution. $\displaystyle\int\frac{dx}{\sqrt{1+\sin x}}=\int\frac{dx}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}}$

$$=\int\frac{dx}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}=\frac{1}{\sqrt{2}}\int\frac{dx}{\cos\frac{\pi}{4}\cos\frac{x}{2}+\sin\frac{\pi}{4}\sin\frac{x}{2}}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}=\frac{1}{\sqrt{2}}\int\sec\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$=\frac{1}{\sqrt{2}}(-2)\log\tan\left(\frac{\pi}{4}+\frac{\pi}{4}-\frac{x}{2}\right)$$

Example 56. *Evaluate* $\displaystyle\int\frac{dx}{\sin(x-a)\sin(x-b)}$

Solution. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$=\frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)} \qquad \text{(Note this step)}$$

$$=\frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\left[\int \cot(x-a)\,dx-\int \cot(x-b)\,dx\right]$$

$$=\text{cosec}\,(a-b)\,[\log\sin(x-a)-\log\sin(x-b)]$$

$$=\text{cosec}\,(a-b)\log\left\{\frac{\sin(x-a)}{\sin(x-b)}\right\}$$

Example 57. *Evaluate* $\int \frac{\sin 2x\, dx}{\sin^4 x+\cos^4 x}$

Solution. Let $I=\int\frac{\sin 2x\,dx}{\sin^4 x+\cos^4 x}=\int\frac{2\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

$$=\int\frac{(2\sin x\cos x)/\cos^4 x}{(\sin^4 x+\cos^4 x)/\cos^4 x}\,dx$$

$$=\int\frac{2\tan x\sec^2 x}{1+\tan^4 x}\,dx$$

Put $\tan^2 x=t \quad \therefore \quad 2\tan x\sec^2 x\,dx=dt$

$\therefore$ $$I=\int\frac{dt}{1+t^2}=\tan^{-1} t=\tan^{-1}(\tan^2 x)$$

Example 58. *Evaluate* $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx.$

Solution. Put $x=\cos\theta$

$$\therefore\ \tan^{-1}\sqrt{\frac{1-x}{1+x}}=\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$=\frac{\theta}{2}=\frac{1}{2}\cos^{-1} x$$

$$\therefore\ \int\tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx=\frac{1}{2}\int(\cos^{-1}x)\,.\,1\,dx$$

$$=\frac{1}{2}\left[x\cos^{-1}x+\int\frac{x}{\sqrt{1-x^2}}\,dx\right]$$

$$=\frac{1}{2}\left[x\cos^{-1}x-\sqrt{1-x^2}\right]$$

Example 59. *Evaluate* $\int \sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right) dx.$

Solution. Let $I=\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx.$

Put $x=a\tan^2\theta \quad \therefore \quad dx=2a\tan\theta\sec^2\theta\,d\theta.$

$$\therefore \quad I=\int \sin^{-1}\left(\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\right).\,2a\tan\theta\sec^2\theta\,d\theta$$

$$=2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right).\tan\theta\,.\sec^2\theta\,d\theta$$

$$=2a\int \theta\,(\tan\theta\sec^2\theta)\,d\theta$$

$$=2a\left[\theta\,.\,\tfrac{1}{2}\tan^2\theta-\int 1\,.\,\tfrac{1}{2}\tan^2\theta\,d\theta\right]$$

$$=a\left[\theta\tan^2\theta-\int(\sec^2\theta-1)\,d\theta\right]$$

$$=a\left[\theta\tan^2\theta-\tan\theta+\theta\right]$$

$$=a\left[\left\{\tan^{-1}\sqrt{\frac{x}{a}}\right\}.\,\frac{x}{a}-\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right]$$

$$=(x+a)\tan^{-1}\left(\sqrt{\frac{x}{a}}\right)-\sqrt{ax}.$$

Example 60. *Evaluate* $\int \frac{dx}{x(x^n+1)}$

Solution Let $I=\int \frac{dx}{x\,(x^n+1)}$

Put $x^n=t \quad \therefore \quad nx^{n-1}\,dx=dt$

$$\therefore \quad I=\frac{1}{n}\int\frac{dt}{t\,(t+1)}=\frac{1}{n}\left[\int\frac{dt}{t}-\int\frac{dt}{t+1}\right]$$

$$=\frac{1}{n}\,[\log\,t-\log\,(t+1)]=\frac{1}{n}\log\left(\frac{t}{t+1}\right)$$

$$=\frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right).$$

Example 61. *Evaluate* $\int \log\,(x+\sqrt{a^2+x^2})\,dx.$

Solution. $\int 1\,.\log\,(x+\sqrt{a^2+x^2})\,dx=x\log\,(x+\sqrt{x^2+a^2})$

$$-\int x\,.\,\frac{1}{x+\sqrt{a^2+x^2}}\,.\left(1+\frac{x}{\sqrt{a^2+x^2}}\right)dx$$

$$= x \log\ (x+\sqrt{a^2+x^2}) - \int \frac{x}{\sqrt{a^2+x^2}}\ dx$$

$$= x \log\ (x+\sqrt{a^2+x^2}) - \tfrac{1}{2} \int 2x(a^2+x^2)^{-\frac{1}{2}}\ dx$$

$$= x \log\ (x+\sqrt{a+x}) - \sqrt{a^2+x^2}$$

Example 62. *Evaluate* $\int \frac{\sin 3x}{\cos x}\ dx.$

Solution. $\sin\ 3x = \sin\ (x+2x) = \sin x \cos 2x \cos x \sin 2x$

$$= \sin x\ (2 \cos^2 x - 1) + \cos x \sin 2x$$

$$\therefore \quad \int \frac{\sin 3x}{\cos x}\ dx = \int \frac{2 \sin x \cos^2 x - \sin x + \cos x \sin 2x}{\cos x}\ dx$$

$$= \int (2 \sin x \cos x - \tan x + \sin 2x)\ dx$$

$$= \int (2 \sin 2x - \tan x)\ dx$$

$$= -\cos 2x + \log \cos x.$$

Example 63. *Evaluate* $\int \cos 2x \log\ (1+\tan x)\ dx$

Solution. Integrating by parts, we get

$$\int \cos 2x \log\ (1+\tan x)\ dx = \frac{\sin 2x}{2} \log\ (1+\tan x) - \int \frac{\sin 2x}{2} \cdot \frac{\sec^2 x}{1+\tan x}\ dx$$

$$= \tfrac{1}{2} \sin 2x \log\ (1+\tan x) - \int \frac{\sin x}{\sin x + \cos x}\ dx.$$

$$= \tfrac{1}{2} \sin 2x \log\ (1+\tan x) - \int \left[A + \frac{B\ (\cos x - \sin x)}{\sin x + \cos x} \right] dx.$$

$$= \tfrac{1}{2} \sin 2x \log\ (1+\tan x) - \int \left[\frac{1}{2} - \frac{1}{2} \quad \frac{\cos x - \sin x}{\sin x + \cos x} \right] dx.$$

$$= \tfrac{1}{2} \sin 2x \log\ (1+\tan x) - \frac{x}{2} + \frac{1}{2} \log\ (\sin x + \cos x).$$

Example 64. *Evaluate* $\int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{(1-x^2)}}\ dx.$

Solution. Here $\frac{d}{dx}\ (\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$

$\therefore$ Substitution is $\sin^{-1} x = t$ so that $\frac{1}{\sqrt{(1-x^2)}}\ dx = dt$

Also when $x=0$, $t=0$ and when $x=1$, $t=\frac{\pi}{2}$ since $\sin\frac{\pi}{2}=1$

$$\therefore \quad \int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{(1-x^2)}}\,dx=\int_0^{\pi/2} t\sin^3 t\,dt=\int_0^{\pi/2}\frac{t}{4}\,(3\sin t-\sin 3t)\,dt.$$

$$=\frac{3}{4}\int_0^{\pi/2} t\sin t\,dt-\frac{1}{4}\int_0^{\pi/2} t\sin 3t\,dt$$

$$=\frac{3}{4}\Big[-t\cos t\Big]_0^{\pi/2}+\frac{3}{4}\int_0^{\pi/2}\cos t\,dt+\frac{1}{4}\Big[\frac{t\cos 3t}{3}\Big]_0^{\pi/2}-\frac{1}{4}\int_0^{\pi/2}\frac{\cos 3t}{3}\,dt$$

$$=0+\frac{3}{4}\Big[\sin t\Big]_0^{\pi/2}+0-\frac{1}{4}\Big[\frac{\sin 3t}{9}\Big]_0^{\pi/2}$$

$$=\frac{3}{4}+\frac{1}{36}=\frac{28}{36}=\frac{7}{9}.$$

Integrals of the type $\int\frac{dx}{X\sqrt{Y}}$, where X and Y are linear or quadratic expressions in x.

The following substitutions will render the above type to the integrable forms :

Case I. *X and Y are both linear.*

The substitution is $Y=t^2$.

Case II. *X is quadratic and Y is linear.*

The substitution of $Y=t^3$.

Case III. *X is linear and Y is quadratic.*

The substitution is $X=\frac{1}{t}$.

Case IV. *X and Y are both quadratic.*

The substitution is $\frac{Y}{X}=t^2$.

Example 65. *Evaluate (i)* $\int\frac{dx}{x\sqrt{(1-x)}}$

(ii) $\int\frac{x^2+1}{(3x+2)\sqrt{(x-1)}}\,dx.$

Solution. (*i*) Put $1-x=t^2$ $\quad\therefore\quad dx=-2t\,dt$ and $x=1-t^2$

$$\therefore \qquad I=\int\frac{-2t\,dt}{(1-t^2)\,t}=-2\int\frac{dt}{1-t^2}=-2\,.\,\frac{1}{2}\log\frac{1+t}{1+t}$$

$$=-\log\frac{1+t}{1-t}=\log\frac{1-t}{1+t}=\log\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}$$

(*ii*) Put $x-1=t^2$ so that $dx=2t\,dt$.

$$\int\frac{x^2+1}{(3x+2)\sqrt{(x-1)}}\,dx=\int\frac{[(t^2+1)^2+1]\,2t}{(3t^2+3+2)\,.\,t}\,dt$$

$$=2\int\frac{t^2+2t^2+2}{3t^2+5}\,dt=2\int\left[\frac{t^2}{3}+\frac{1}{9}+\frac{13}{9(3t^2+5)}\right]dt$$

$$=\frac{2t^3}{9}+\frac{2t}{9}+\frac{26}{9\sqrt{15}}\tan^{-1}\frac{\sqrt{3}\,t}{\sqrt{5}}$$

$$=\frac{2}{9}\,x\sqrt{(x-1)}+\frac{26}{9\sqrt{15}}\tan^{-1}\sqrt{\left(\frac{3x-3}{5}\right)}$$

Example 66 *Evaluate* $\displaystyle\int\frac{x+2}{(x^2+3x+2)\sqrt{(x+1)}}\,dx$

Solution. Put $x+1=t^2$ so that $dx=2t\,dt$

$$\therefore\quad \int\frac{x+2}{(x^2+3x+3)\sqrt{(x+1)}}\,dx=\int\frac{(t^2+1)\,.\,2t\,dt}{[(t^2-1)^2+3(t^2-1)+3]\,t}$$

$$=2\int\frac{(t^2+1)\,dt}{t^4+t^2+1}=2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt.$$

$$=2\int\frac{du}{u^2+3}\qquad\left(\text{ where } t-\frac{1}{t}=u\right)$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{t^2-1}{\sqrt{3}t}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{(3x+3)}}$$

Example 67. *Evaluate* $\displaystyle\int\frac{dx}{(x-3)\sqrt{(x^2-6x+8)}}$

Solution. Put $x-3=\dfrac{1}{t}$ so that $dx=-\dfrac{dt}{t^2}$ and $x=\dfrac{1+3t}{t}$

$$\therefore\qquad I=\int\frac{-dt}{t^2\times\frac{1}{t}\sqrt{\frac{(1+3t)^2}{t^2}-6\frac{(1+3t)}{t}+8}}$$

$$=-\int\frac{dt}{\sqrt{(1+6t+9t^2)-6t-18t^2+8t^2}}$$

$$= - \int \frac{dt}{\sqrt{1-t^2}} = -\sin^{-1} t = -\sin^{-1} \frac{1}{x-3}$$

Example 68. *Evaluate* $\int \frac{dx}{(x^2+1)\sqrt{(1-x^2)}}$

Solution. Put $\frac{1-x^2}{x^2+1} = t^2$ so that $x^2 = \frac{1-t^2}{1+t^2}$

Taking logarithmic differentiation, we get

$$\left(\frac{-2x}{1-x^2} - \frac{2x}{x^2+1}\right) dx = \frac{2}{t}\, dt$$

$$\frac{2x^3+2x+2x-2x^3}{(1-x^2)(x^2+1)} = -\frac{2}{t}\, dt$$

⇒ $$\frac{4x\, dx}{(1-x^2)(x^2+1)} = -\frac{2}{t}\, dt$$

∴ $$\int \frac{dx}{(x+1)\sqrt{(1-x^2)}} = -\int \frac{\sqrt{(1-x^2)}}{2x\,.\,t}\, dt$$

$$= -\int \sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)}\, \frac{1}{2\,.\,\sqrt{1-t^2}}\, \frac{\sqrt{1+t^2}}{t}\, dt$$

$$-= \int \frac{\sqrt{2t}}{\sqrt{1+t^2}\,2\sqrt{1-t^2}} \cdot \frac{\sqrt{1+t^2}}{t}\, dt$$

$$= -\frac{\sqrt{2}}{2}\int \frac{1}{\sqrt{1-t^2}}\, dt = -\sqrt{2}\,\sin^{-1} t$$

$$= -\sqrt{2}\,\sin^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right).$$

Example 69. *Evaluate* $\int_0^{\pi/4} \sqrt{\tan x}\, dx$

Solution. Put $\sqrt{\tan x} = t$ so that $\tan x = t^2$ and $\sec^2 x\, dx = 2t\, dt$

⇒ $$dx = \frac{2t}{1+\tan^2 x}\, dt = \frac{2t\, dt}{1+t^4}$$

Also when $x=0$, $t=\sqrt{\tan 0}=0$

when $$x = \frac{\pi}{4},\quad t = \sqrt{\tan\frac{\pi}{4}} = 1$$

Hence the given integral becomes

$$\int_0^1 \frac{t\,.\,2t\, dt}{1+t^4} = 2\int_0^1 \frac{t^2}{1+t^4}\, dt$$

$$=2\left[\frac{\sqrt{2}}{8}\log\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}+\tfrac{1}{4}\sqrt{2}\tan^{-1}\frac{t^2-1}{t\sqrt{2}}\right]_0^1$$

$$=2\left[\left\{\frac{\sqrt{2}}{8}\log\frac{2-\sqrt{2}}{2+\sqrt{2}}+\frac{\sqrt{2}}{4}\tan^{-1}0\right\}\right.$$
$$\left.-\left\{\frac{\sqrt{2}}{8}\log 1-\frac{\sqrt{2}}{4}\tan^{-1}\infty\right\}\right]$$

$$=2\left[\frac{\sqrt{2}}{8}\log\frac{2-\sqrt{2}}{2+\sqrt{2}}+\frac{\sqrt{2}}{4}\cdot\frac{\pi}{2}\right]$$

$$=\frac{\sqrt{2}}{4}\log\frac{2-\sqrt{2}}{2+\sqrt{2}}+\frac{\sqrt{2}}{4}\pi=\frac{1}{2\sqrt{2}}\log\frac{2-\sqrt{2}}{2+\sqrt{2}}+\frac{1}{2\sqrt{2}}\pi.$$

Example 70. *Evaluate* $\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx$ *and hence find the value of* $\int_0^a\frac{dx}{x+\sqrt{(a^2-x^2)}}$.

Solution. Let $I=\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx$, then

$$I=\int_0^{\pi/2}\frac{\sin\left(\frac{x}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$=\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}\,dx$$

$$2I=\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx+\int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}\,dx$$

$$=\int_0^{\pi/2}dx=\frac{\pi}{2}\qquad\therefore\quad I=\frac{\pi}{4}$$

To evaluate $\int_0^a\frac{dx}{x+\sqrt{(a^2-x^2)}}$, put $x=a\cos t$, then

$$dx=-a\sin t\,dt$$

Also when $x=0$, $t=\frac{\pi}{2}$ and when $x=a$, $t=0$

$$\therefore \quad \int_0^a \frac{dx}{x+\sqrt{(a^2-x^2)}}=-\int_{\pi/2}^{0} \frac{a \sin t\, dt}{a \cos t+a \sin t}=\int_0^{\pi/2} \frac{\sin t\, dt}{\cos t+\sin t}$$

$$=\frac{\pi}{4}$$

Example 71. *Prove that* $\int_0^{\pi/4} \log (1+\tan \theta)\, d\theta=\frac{\pi}{8} \log 2$

and hence find the value $\int_0^1 \frac{\log (1+x)}{1+x^2}\, dx$.

Solution. Let $I=\int_0^{\pi/4} \log (1+\tan \theta)\, d\theta$, then

$$I=\int_0^{\pi/4} \log \left[1+\tan \left(\frac{\pi}{4}-\theta \right)\right] d\theta$$

$$=\int_0^{\pi/4} \log \left[1+\frac{1-\tan \theta}{1+\tan \theta}\right] d\theta=\int_0^{\pi/4} \log \frac{2}{1+\tan \theta}\, d\theta$$

$$=\int_0^{\pi/4} \log 2\, d\theta-I$$

$$\therefore \quad 2I=\int_0^{\pi/4} \log 2\, d\theta=\frac{\pi}{4} \log 2$$

$$\Rightarrow \quad I=\frac{\pi}{8} \log 2$$

For the second integral put $x=\tan \theta$, then $dx=\sec^2 \theta\, d\theta$

Also when $x=0$, $\theta=0$ and when $x=1$, $\theta=\frac{\pi}{4}$

$$\therefore \qquad \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \int_0^1 \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log 2$$

Example 72. *Evaluate* $\int_0^\infty \log\left(x+\frac{1}{x}\right)\cdot\frac{dx}{1+x^2}$.

Solution. Let $I = \int_0^\infty \log\left(x+\frac{1}{x}\right)\cdot\frac{dx}{1+x^2}$

Put $\qquad x = \tan\theta$

$\therefore \qquad dx = \sec^2\theta\,d\theta \qquad \ldots(*)$

Also, when $x=\infty$, (*) gives $\theta=\frac{\pi}{2}$

and when $x=0$, $\theta=0$

$$\therefore \qquad I = \int_0^{\pi/2} \log\left(\tan\theta+\frac{1}{\tan\theta}\right)\cdot\frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1+\tan^2\theta}{\tan\theta}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2}{\sin 2\theta}\right)d\theta$$

$$= \int_0^{\pi/2} (\log 2 - \log\sin 2\theta)\,d\theta$$

$$= \frac{\pi}{2}\log 2 - \int_0^{\pi/2} \log\sin 2\theta\,d\theta$$

$$=\frac{\pi}{2}\log 2-\int_0^{\pi/2}(\log 2)\ d\theta-\int_0^{\pi/2}\log\sin\theta\,d\theta$$

$$-\int_0^{\pi/2}\log\cos\theta\,d\theta$$

$$=\frac{\pi}{2}\log 2-\frac{\pi}{2}\log 2+\frac{\pi}{2}\log 2+\frac{\pi}{2}\log 2$$

$$=\pi\log 2.$$

Example 73. *Evaluate* $\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx.$

Solution. Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx$, then

$$I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}\,dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\tan x}\,dx$$

$$=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\tan x}\,dx-\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx$$

$$=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\tan x}\,dx-I$$

$$2I=\pi\int_0^{\pi}\frac{\tan x\,dx}{\sec x+\tan x}$$

$$=\pi\int_0^{\pi}\frac{(\sec x-\tan x)\tan x}{\sec^2 x-\tan^2 x}\,dx$$

$$=\pi\int_0^{\pi}[\sec x\tan x-(\sec^2 x-1)]\,dx$$

$$=\pi\left[\sec x-\tan x+x\right]_0^{\pi}=\pi(-1+\pi-1)$$

$$\therefore \qquad I=\pi\left(\frac{\pi}{2}-1\right)$$

REDUCTION FORMULAE

Example 74. *Prove that* $\int_0^{\pi/2} sin^n\, x\ dx=\int_0^{\pi/2} cos^n\, x\, dx$

$$=\begin{cases}\dfrac{(n-1)(n-3)(n-5)\ldots3\ .\ 1}{n(n-2)(n-4)\ldots4\ .\ 2}\cdot\dfrac{\pi}{2}, & \textit{when } n \textit{ is even}\\[2ex] \dfrac{(n-1)(n-3)(n-5)\ldots4\ .\ 2}{n(n-2)(n-4)\ldots5\ .\ 3}\ .\ 1, & \textit{when } n \textit{ is odd}\end{cases}$$

Solution. $\int_0^{\pi/2} \sin^n x\ dx=\int_0^{\pi/2} \sin^n\left(\frac{\pi}{2}-x\right)=\int_0^{\pi/2} \cos^n x\, dx$

$$\left[\because \int_0^a f(x)\, dx=\int_0^a f(a-x)\, dx\right]$$

Now $\int \sin^n x\, dx=\int \sin^{n-1} x\ .\ \sin x\, dx$

$$=-\sin^{n-1} x\,(-\cos x)-\int (n-1)\sin^{n-2} x \cos x\,(-\cos x)\, dx$$

(Integrating by parts)

$$=-\sin^{n-1} x \cos x+(n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\, dx$$

$$=-\sin^{n-1} x \cos x+(n-1)\int \sin^{n-2} x\, dx-(n-1)\int \sin^n x\, dx$$

Let $\qquad I=\int_0^{\pi/2} \sin^n x\, dx.$ Then by transposition, we get

$$I_n+(n-1)\, I_n=\left[-\sin^{n-1} x \cos x\right]_0^{\pi/2}+(n-1)\, I_{n-2}$$

$$\therefore \qquad I=\frac{n-1}{n}\ .\ I_{-2}$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\ .\ I_{n-4} \text{ (Changing } n \text{ to } n-2)$$

$$=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot I_{n-6}$$

(Changing n to $n-4$)

and so on.

$$\therefore \qquad I=\frac{(n-1)(n-3)(n-5)\ldots 3\,.\,1}{n(n-2)(n-4)\ldots 4\,.\,2}\,I_0,\ \text{when } n \text{ is even}$$

$$=\frac{(n-1)(n-3)(n-5)\ldots 4\,.\,2}{n(n-2)(n-4)\ldots 5\,.\,3}\,I_1,\ \text{when } n \text{ is odd}$$

Now $$I_0=\int_0^{\pi/2}\sin^0 x\,dx=\int_0^{\pi/2}dx=\Big[\,x\,\Big]_0^{\pi/2}=\frac{\pi}{2}$$

and $$I_1=\int_0^{\pi/2}\sin x\,dx=\Big[-\cos x\Big]_0^{\pi/2}=1.$$

Example 75. *Prove that* $\int_0^{\pi/2}\sin^m x\cos^n dx=\int_0^{\pi/2}\cos^m x\sin^n x\,dx$

$$=\begin{cases}\dfrac{\{1.3.5\ldots(m-1)\}\,.\,\{1.3.5\ldots(n-1)\}}{2.4.6\ldots(m+n)}\cdot\dfrac{\pi}{2} & \textit{when both m and n are even integers.}\\[2ex] \dfrac{2.4.6\ldots\ldots(m-1)}{(n+1)(n+3)\ldots(n+m)}, & \textit{when one of the two indices. say m is an odd integer.}\end{cases}$$

Solution. $\int\sin^m x\cos^n x\,dx=\int\cos^{n-1}x\,(\sin^m x\cos x)\,dx.$

Integrating by parts, we get

$$\int\sin^m x\cos^n x\,dx=\frac{\sin^{m+1}x}{m+1}\cos^{n-1}x-\int(n-1)\cos^{n-2}x\,(-\sin x)\times\left(\frac{\sin^{m+1}}{m+1}\right)dx$$

$$=\frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}\int\sin^{m+2}x\cos^{n-2}x\,dx$$

$$=\frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}\int\sin^m x\cos^{n-2}x(1-\cos^2 x)\,dx.$$

$$=\frac{\sin^m x\cos^{n-1}}{m+1}+\frac{n-1}{m+1}\int\sin^{m-1}x\cos^{n-2}x\,dx$$

$$-\frac{n-1}{m+1}\int\sin^m x\cos^n x\,dx.$$

By transposition, we get

$$\left(1+\frac{n-1}{m+1}\right)\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1}$$

$$+\frac{n-1}{m+1}\int \sin^n x \cos^{n-2} x \, dx.$$

$$\therefore \qquad \int_0^{\pi/2} \sin^n x \cos^n x \, dx = \left[\frac{\sin^{m+1} x \cos^{n-1} x}{m+n}\right]_0^{\pi/2}$$

$$+\frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx.$$

$$=0+\frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} dx$$

or $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx. \qquad \ldots(1)$$

It can be similarly proved that

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \int_0^{\pi/2} \sin^{m-1} x \, (\cos^n x \sin x) \, dx$$

$$=\frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x \cos^n x \, dx \text{ (proceeding as before)} \qquad \ldots(2)$$

Here, with the notation.

$$I_{m,\,n}=\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

We get, from (1) and (2)

$$I_{m,\,n}=\frac{n-1}{m+n}\; I_{m,\,n-2}=\frac{m-1}{m+n}\; I_{m-2,\,n}$$

Thus $I_{m,\,n}=\dfrac{n-1}{m+n}\, I_{m,\,n-2}$

$$=\frac{(m-1)(n-3)}{(m+n)(m+n-2)}-I_{m,\,n-4}$$

$$=\frac{(n-1)(n-3)(n-5)}{(m+n)(m-n-2)(m+n-4)}\, I_{m;\, n-6} \text{ and so on.}$$

$$=\frac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots(m+2)} \cdot I_{m,\,0}$$

where n is an even integers

$$=\begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots 1 \,.\, (m-1)(m-3)\ldots 1}{(m+n)(m+n-2)\ldots 4 \,.\, 2} \cdot \dfrac{\pi}{2} \\ \qquad \text{when } m \text{ and } n \text{ are both even integers} \\ \dfrac{(n-1)(n-3)(n-5)\ldots 3 \,.\, 1(m-1)(m-3)\ 4 \,.\, 2}{(n+m)(n+m-2)\ldots(n-3)(n-1)\ldots 3 \,.\, 1} \\ \qquad \text{when } n \text{ is even and } m \text{ is odd} \end{cases}$$

$$=\begin{cases} \dfrac{(m-1)(m-3)\ldots 4 \,.\, 2}{(m+n)(m+n-2)\ldots(n+1)}, \\ \qquad \text{when } n \text{ is even and } m \text{ is odd} \\ \dfrac{(n-1)(n-3)\ldots 4 \,.\, 2}{(m+n)(m+n-2)\ldots(m+1)}, \\ \qquad \text{when } n \text{ is odd and } m \text{ is even} \end{cases}$$

When both m and n are odd ;

$$I_{m,\,n}=\begin{cases} \dfrac{(n-1)(n-3)\ldots 4 \,.\, 2}{(m+n)(m+n-2) \,.\, (m+1)} \\ \dfrac{(m-1)(m-3)\ldots 4 \,.\, 2}{(m+n)(m+n-2)\ldots(n+1)} \end{cases}$$

Also $\displaystyle\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$=\int_0^{\pi/2} \sin^m\left(\frac{\pi}{2}-x\right)\cos^n\left(\frac{\pi}{2}-x\right)=\int_0^{\pi/2} \sin^n x \cos^n x \, dx$$

i.e., $\qquad I_{m,\,n}=I_{n,\,m}$

Here $\qquad I_{m,\,n}=I_{n,\,m}=\dfrac{\{1.3.5\ldots(m-1)\}\,(1.3.5\ldots(n-1)\}}{2.4.6\ldots(m+n-2)(m+n)} \cdot \dfrac{\pi}{2}$

when both m and n are even integers

$$=\frac{2.4.6\ldots(m-1)}{(n+1)(n+3)\ldots(n+m)},$$

when any of the indices is odd.

Example 76. *If* $I_{m,\,n}=\int \cos^m x \sin nx \, dx$; *show that*

$$I_{m,\,n}=\frac{-\cos^m x \cos nx}{m+n}+\frac{m}{m+n}\, I_{m-1,\,n-1}.$$

Solution. Taking $\cos^m x$ as the first function and $\sin nx$ as the second function, we have on integrating by parts,

$$I_{m, n}=\cos^m x\left(-\frac{\cos nx}{n}\right)$$
$$-\int m\cos^{m-1} x\,(-\sin x)\left(-\frac{\cos nx}{n}\right)dx$$
$$=-\frac{\cos^m x\cos nx}{n}-\frac{m}{n}\int\cos^{m-1} x\sin x\cos nx\,dx \qquad \ldots(^*)$$

Now $\sin(n-1)x=\sin(nx-x)=\sin nx\cos x-\cos nx\sin x$

$\Rightarrow \quad \sin x\cos nx=\sin nx\cos x-\sin(n-1)x$

Substituting this value in (*), we get

$$I_{m, n}=-\frac{\cos^m x\cos nx}{n}$$
$$-\frac{m}{n}\int\cos^{m-1} x\,\{\sin nx\cos x-\sin(n-1)x\}\,dx$$
$$=-\frac{\cos^m x\cos nx}{n}-\frac{m}{n}I_{m, n}$$
$$+\frac{m}{n}\int\cos^{m-1} x\sin(n-1)x\,dx$$

$$\Rightarrow \quad \left(1+\frac{m}{n}\right)I_{m, n}=-\frac{\cos^m x\cos nx}{n}$$
$$+\frac{m}{n}\int\cos^{m-1} x\sin(n-1)x\,dx$$

$$\Rightarrow \quad I_{m, n}=-\frac{\cos^m x\cos nx}{m+n}+\frac{m}{m+n}I_{m-1, n-1}$$

Remark. There is another form of the reduction formula for

$$\int\cos^m x\sin nx\,dx$$

$$\int\cos^m x\sin nx\,dx=\frac{\cos^m x\cos nx}{m-n}+\frac{m}{m-n}\int\cos^{m-1} x\sin(n+1)x\,dx.$$

This is left as an *exercise* for the students.

Example 77. *If* $I_{m, n}=\int\cos^m x\cos nx$, *show that*

$$I_{m, n}=\frac{\cos^m x\sin nx}{m+n}+\frac{m}{m+n}I_{m-1, n-1}.$$

Solution. Taking $\cos^m x$ as the first function and $\cos nx$ as the second function, we have on integrating by parts,

$$I_{m, n}=\cos^m x\,.\,\frac{\sin nx}{n}-\int m\cos^{m-1} x\,(-\sin x)\,.\,\frac{\sin nx}{n}\,dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \, (\sin nx \sin x) \, dx \quad \ldots(*)$$

But $\cos(n-1)\,x = \cos(nx - x) = \cos nx \cos x + \sin nx \sin x$

∴ $\sin nx \sin x = \cos(n-1)\,x - \cos nx \cos x$

∴ (*) gives $I_{m,\,n} = \dfrac{\cos^m x \sin nx}{n}$

$$+ \frac{m}{n} \int \cos^{m-1} x \, \{\cos(n-1)\,x - \cos nx \cos x)\} \, dx$$

$$\Rightarrow \quad I_{m,\,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos(n-1)\,x \, dx$$

$$- \frac{m}{n} \int \cos^m x \cos nx \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \; I_{m-1,\,n-1} - \frac{m}{n} \; I_{m,\,n}$$

$$\Rightarrow \quad \left(1 + \frac{m}{n}\right) I_{m,\,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,\,n-1}$$

$$\Rightarrow \quad I_{m,\,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,\,n-1}$$

which is the required reduction formula.

Remark. There are two other forms of the reduction formulae for

$\int \cos^m x \cos nx \, dx$

(*i*) $$\int \cos^m x \cos nx \, dx = - \frac{\cos^m x \sin nx}{m-n}$$

$$+ \frac{m}{m-n} \int \cos^{m-1} x \cos(n+1)\,x \, dx$$

(*ii*) $$\int \cos^m x \cos nx \, dx = \frac{n \cos^m x \sin nx}{m^2 - n^2} + \frac{m \cos^{m-1} x \sin x \cos nx}{m^2 - n^2}$$

$$+ \frac{m(m-1)}{m^2 - n^2} \int \cos^{m-2} x \cos nx \, dx.$$

Example 78. *Prove that* $\displaystyle\int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}$, *n being a positive integer.*

Solution Let $\displaystyle I_n = \int_0^{\pi/2} \cos^n x \cos nx \, dx$

$$=\left[\frac{\cos^n x \sin nx}{2n}\right]_0^{\pi/2}+\frac{1}{2}\int_0^{\pi/2}\cos^{n-1}x \cos (n-1)\, x \; dx$$

$$=\frac{1}{2}\, I_{n-1} \qquad \ldots(*)$$

Writing $(n-1)$ for n in (*), we have

$$I_{n-1}=\frac{1}{2}\, I_{n-2}$$

$\therefore$ (*) gives $\quad I_n=\frac{1}{2}\cdot\frac{1}{2}\, I_{n-2}$

Proceeding in this way and applying the reduction formula n times, we have

$$I_n=(\tfrac{1}{2}\cdot\tfrac{1}{2}\ldots\ldots\text{to } n \text{ factors})\; I_0$$

$$=\frac{1}{2^n}\int_0^{\pi/2}(\cos x)^0 \cos 0x \; dx=\frac{1}{2^n}\int_0^{\pi/2} 1\,.\,dx$$

$$=\frac{1}{2^n}\Big[\; x \;\Big]_0^{\pi/2}=\frac{\pi}{2^{n+1}}$$

Example 79. *Find a reduction formula for* $\int x^m \sin nx \, dx$.

Solution. Let $I_{m,\,n}=\int x^m \sin nx \; dx$

Taking x^m as the first function and $\sin nx$ as the second function, we have on integrating by parts,

$$I_{m,\,n}=x^m\left(-\frac{\cos nx}{n}\right)-\int m x^{m-1}\left(-\frac{\cos nx}{n}\right)dx$$

$$=-\frac{x^m \cos nx}{n}+\frac{m}{n}\int x^{m-1}\cos nx \; dx \qquad \ldots(*)$$

$$=-\frac{x^m \cos nx}{n}+\frac{m}{n}\Big[x^{m-1}\left(\frac{\sin nx}{n}\right)$$

$$-\int (m-1)\, x^{m-2}\left(\frac{\sin nx}{n}\right)dx\;\Big]$$

$$=-\frac{x^m \cos nx}{n}+\frac{m}{n^2}\; x^{m-1}\sin nx$$

$$-\frac{m(m-1)}{n^2}\int x^{m-2}\sin \; nx \; dx$$

Hence $\displaystyle\int x^m \sin nx \; dx=\frac{-nx^m \cos nx+mx^{m-1}\sin nx}{n^2}$

$$-\frac{m(m-1)}{n^2}\int x^{m-2}\sin \; nx \; dx$$

the required reduction formula.

Remark. A reduction formula for $\int x^m \cos nx\, dx$ is given by

$$\int x^m \cos nx\, dx = \frac{nx^m \sin nx + mx^{m-1} \cos nx}{n^2} - \frac{m(m-1)}{n^2} \int x^{m-2} \cos nx\, dx$$

This is left as an exercise for the reader.

Example 80. *If* $U_n = \int_0^{\pi/2} x^n \sin x\, dx$ *and* $n > 1$, *prove that*

$$U_n + n(n-1)\; U_{n-2} = n \left(\frac{\pi}{2}\right)^{n-1}$$

Solution. We have $U_n = \int_0^{\pi/2} x^n \sin x\, dx$

$$= \Big| -x^n \cos x \Big|_0^{\pi/2} + n \int_0^{\pi/2} x^{n-1} \cos x\, dx$$

$$= n \int_0^{\pi/2} x^{n-1} \cos x\, dx$$

$$= n \left[\Big| x^{n-1} \sin x \Big|_0^{\pi/2} - (n-1) \int_0^{\pi/2} x^{n-2} \sin x\, dx \right]$$

$$= n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, U_{n-2}$$

Hence $\qquad U_n + n(n-1)\, U_{n-2} = n \left(\frac{\pi}{2}\right)^{n-1}$

Example 81. (*a*) *Find a reduction formula for* $\int \tan^n x\, dx$.

(*b*) *If* $I_n = \int_0^{\pi/4} \tan^n x\, dx$, *prove that*

(*i*) $I_n + I_{n-2} = \dfrac{1}{n-1}$

(*ii*) $n(I_{n-1} + I_{n+1}) = 1$.

Solution. (*a*) $\int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx$

$$= \int \tan^{n-2} x\, (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n\;2} x\, dx$$

Hence $$\int \tan^n x\, dx=\frac{\tan^{n-1} x}{n-1}-\int \tan^{n-2} x\, dx,$$

which is the required reduction formula.

Remark. The reduction formula for $\int \cot^n x\, dx$, *viz* ,

$$\int \cot^n x\, dx=-\frac{\cot^{n-1} x}{n-1}-\int \cot^{n-2} x\, dx$$

is left as an exercise for the reader.

(*b*) (*i*) $$I_n=\int_0^{\pi/4} \tan^n x\, dx=\int_0^{\pi/4} \tan^{n-2} x \tan^2 x\, dx$$

$$=\int_0^{\pi/4} \tan^{n-2} x\,(\sec^2 x-1)\, dx$$

$$=\int_0^{\pi/4} \tan^{n-2} x \sec^2 x\, dx-\int_0^{\pi/4} \tan^{n-2} x\, dx$$

$$=\left|\frac{\tan^{n-1} x}{n-1}\right|_0^{\pi/4}-I_{n-2}=\frac{1}{n-1}-I_{n-2}$$

$$\Rightarrow \qquad I_n+I_{n-2}=\frac{1}{n-1}$$

(*ii*) $$n\,(I_{n-1}+I_{n+1})=n\left[\int_0^{\pi/4} \tan^{n-1}\theta\, d\theta+\int_0^{\pi/4} \tan^{n+1}\theta\, d\theta\right]$$

$$=n\int_0^{\pi/4} (\tan^{n-1}\theta+\tan^{n+1}\theta)\, d\theta$$

$$=n\int_0^{\pi/4} \tan^{n-1}\theta\,(1+\tan^2\theta)\, d\theta$$

$$=n\int_0^{\pi/4} \tan^{n-1}\theta \sec^2\theta\, d\theta=n\left|\frac{\tan^n\theta}{n}\right|_0^{\pi/4}=1\cdot$$

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0632	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6095	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

ANTILOGARITHS

x	0	1	2	3	4	5	6	7	8	9	Δm	1	2	3	4	5	6	7	8	9
												ADD								
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	7	1	1	2	3	4	4	5	6	6
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	7	1	2	2	3	4	5	6	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	7	1	2	2	3	4	5	6	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	7	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	8	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	1597	3606	3614	3622	8	1	2	2	3	4	5	6	6	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	8	1	2	2	3	4	5	6	6	7
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	8	1	2	3	4	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	9	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	9	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	3055	4064	9	1	2	3	4	5	5	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	10	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	10	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	10	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	10	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	10	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	11	1	2	3	4	5	7	8	9	10
·67	4677	4688	4699	4710	4731	4732	4742	4753	4764	4775	11	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	11	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	11	1	2	3	4	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	12	1	2	4	5	6	7	8	10	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	12	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	12	1	2	4	5	6	7	8	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	12	1	2	4	5	6	7	8	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	13	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	13	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	13	1	3	4	5	7	8	9	10	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5993	6012	14	1	3	4	6	7	8	10	11	13
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	14	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	14	1	3	4	6	7	8	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	15	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	15	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	15	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	15	2	3	5	6	8	10	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	16	2	3	5	6	8	10	11	13	14
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	16	2	3	5	6	8	10	11	13	14
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	17	2	3	5	7	8	10	12	14	15
·87	7413	7430	7447	7464	7481	7499	7516	7534	7551	7568	17	2	3	5	7	9	10	12	14	15
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	18	2	4	5	7	9	11	13	14	16
·89	7761	7780	7798	7816	7834	7852	7870	7889	7907	7925	18	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	18	2	4	5	7	9	11	13	14	16
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	19	2	4	6	8	10	11	13	15	17
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·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	20	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	21	2	4	6	8	10	13	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	21	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	22	2	4	7	9	11	13	15	18	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	22	2	4	7	9	11	13	15	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	23	2	5	7	9	11	14	16	18	21

NATURAL SINES

Degrees	0 0°·0	6 0°·1	12 0°·2	18 0°·3	24 0°·4	30 0°·5	36 0°·6	42 0°·7	48 0°·8	54' 0°·9	Mean Differences 1	 2	 3	 4	 5
0	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	17	15
5	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences				
											1	2	3	4	5
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0
90	1·000														

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