11

Mathematical Induction, Sequences and Series

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- *mathematical induction and its applications*
- *sequences, series, sigma notations, summation of series using sigma notation.*

11·0. INTRODUCTION

Many important mathematical formulae which cannot be easily derived by direct proof are sometimes established by an indirect method, known as the method of mathematical induction. The proof of a theorem by mathematical induction can be divided into 3 steps. The first step consist of actual verification for particular values of n, usually $n=1, 2, 3$; the second by assuming the theorem to be true for some positive integral value m of n and from there to show that the theorem is true for $(m+1)$. The third step lies in simple reasoning that if the theorem is true for the value m of n, it is also true for the value $(m+1)$, which is the next higher integer. But, by the first step if the theorem is true for the value 3, it should be true for the value $3+1$, *i.e.*, 4 and so on.

11·1. PRINCIPLE OF MATHEMATICAL INDUCTION

Let the formula or the proposition be denoted by $P(n)$. The principle of mathematical induction states that if

(*i*) the formula or the given proposition $P(n)$ involving n is true for $n=1, 2$ and

(*ii*) if the truth of $P(m)$ for $n=m$ implies it is true for $P(m+1)$, then $P(n)$ is true for all positive integral values of n.

Example 1. *Prove by mathematical induction that*

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}, \text{ where } n \text{ is a positive integer.}$$

Solution. Here $P(n)$ is

$$1+2+3+\ldots+n=\frac{n(n+1)}{2} \qquad \ldots(1)$$

Step I. The formula is true for $n=1$, since L.H.S. $=1$

and $$\text{R.H.S.}=\frac{1.(1+1)}{2}=1$$

$\therefore$ L.H.S. $=$ R.H.S. and $P(1)$ is true.

It it also true for $n=2$, because the L.H.S. $=1+2=3$ and the

$$\text{R.H.S.}=\frac{2(2+1)}{2}=3.$$

Step II. We will now show that the truth of $P(n)$ to be true for some positive integral value of n, say $n=m \Rightarrow$ the truth of $P(n)$ for $n=m+1$, *i.e.*, we have to show that the truth of $P(m)$, namely

$$1+2+3+\ldots+m=\frac{m(m+1)}{2} \qquad \ldots(2)$$

implies the truth of $P(m+1)$, namely

$$1+2+3+\ldots+m+(m+1)=\frac{(m+1)[(m+1)+1]}{2} \qquad \ldots(3)$$

$$\text{L.H.S. of (3)}=[1+2+3+\ldots+m]+(m+1)$$

$$=\frac{m(m+1)}{2}+(m+1), \text{ [due to assumption (2)]}$$

$$=(m+1)\left[\frac{m}{2}+1\right]$$

$$=\frac{(m+1)(m+2)}{2}$$

$$=\frac{(m+1)[(m+1)+1]}{2}=\text{R.H.S. of (3)}$$

By the step I, $P(n)$ is true for $n=1, 2$. Therefore, by step II, $P(n)$ is true for any particular value say $n=2$, and hence for $n=3$, and for $n=4$ and so on, *i.e.*, for every natural number n.

Example 2. *Prove by the method of induction :*

$$P(n) : 2+7+12+...+(5n-3)=\tfrac{1}{2}n\,(5n-1) \qquad ...(1)$$

Solution. Step I. We verify the result for $n=1$.

Put $n=1$ in (1). L.H.S.$=2$; R.H.S.$=\tfrac{1}{2}.1.(5-1)=2$

$\therefore$ L.H.S.=R.H.S. and $P(1)$ is true.

Step II. We now show that the assumption of the truth of $P(m)$, namely

$$2+7+12+...+(5m-3)=\tfrac{1}{2}m(5m-1) \qquad ...(2)$$

implies the truth of $P(m+1)$, namely

$$2+7+12+...+(5m-3)+(5m+2)=\tfrac{1}{2}(m+1)\,[5(m+1)-1] \qquad ...(3)$$

L.H.S. of (3)$=[2+7+12+...+(5m-3)]+(5m+2)$

$=\tfrac{1}{2}\,[m(5m-1)]+(5m+2)$, [by assumption (2)]

$$=\frac{5m^2-m+10m+4}{2}$$

$$=\frac{1}{2}\left[5m^2+5m+4m+4\right]$$

$$=\frac{1}{2}\left[5m(m+1)+4(m+1)\right]$$

$$=\frac{1}{2}\,(m+1)\,(5m+4)$$

$$=\frac{1}{2}(m+1)\,[5(m+1)-1]$$

=R.H.S. of (3).

We have thus proved that if the result is true for $n=m$, it remains true for next integral value of n namely $m+1$. Since the result is verified for $n=1$ in step I, it follows that it is true (by step II), for $n=1+1=2$ and hence for $n=2+1=3$ and so on for all positive integral values of n.

Example 3. *Prove by the method of induction :*

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Solution. Let $P(n)$ be $\dfrac{1}{3.5}+\dfrac{1}{5.7}+\dfrac{1}{7.9}+...+\dfrac{1}{(2n+1)(2n+3)}$

$$=\frac{n}{3(2n+3)} \qquad ...(1)$$

Step I. We verify the result (1) for $n=1$.

Put $n=1$ in (1). L.H.S.$=\dfrac{1}{3.5}=\dfrac{1}{15}$

R.H.S.$=\dfrac{1}{3(2+3)}=\dfrac{1}{15}$

$\therefore$ L.H.S.=R.H.S., *i,e.*, P (1) is verified to be true.

Step II. We now show that the assumption of the truth of $P(m)$ namely

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2m+1)(2m+3)}=\frac{m}{3(2m+3)} \quad \ldots(2)$$

implies the truth of $P(m+1)$, namely

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2m+1)(2m+3)}+\frac{1}{(2m+3)(2m+5)}$$
$$=\frac{m+1}{3[2(m+1)+3]} \quad \ldots(3)$$

$$\text{L.H.S. of (3)}=\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2m+1)(2m+3)}\right]+\frac{1}{(2m+3)(3m+5)}$$

$$=\frac{m}{3(2m+3)}+\frac{1}{(2m+3)(2m+5)}, \text{ [due to assumption (2)]}$$

$$=\frac{m(2m+5)+3}{3(2m+3)(2m+5)}=\frac{2m^2+5m+3}{3(2m+3)(2m+5)}$$

$$=\frac{(m+1)(2m+3)}{3(2m+3)(3m+5)}=\frac{m+1}{3(2m+5)}=\frac{m+1}{3[2(m+1)+3]}$$

=R.H.S. of (3)

We have thus proved that if the result (1) is true for $n=m$, it remains true for the next integral value of n namely $m+1$. Since the result is verified to be true for $n=1$, in step I, it follows (by step II) that it is true for $n=2$, and hence for $n=3$, and so on for all positive integral values of n.

Example 4. *Prove by the method of induction :*

$$1.3+2.3^2+3.3^3+\ldots+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

Solution. Here $P(n)$: $1.3+2.3^2+3.3^3+\ldots+n.3^n=\dfrac{(2n-1)3^{n+1}+3}{4}$ ...(1)

Step I. The formula is true for $n=1$, since

$$\text{L.H.S.}=1.3=3$$

and $$\text{R.H.S.}=\frac{(2-1)3^{1+1}+3}{4}=3$$

For $n=2$, L.H.S.$=1.3+2.3^2=21$

and $$\text{R.H.S.}=\frac{(4-1)3^3+3}{4}=21$$

Hence the relation is ture for $n=1, 2$.

Step II. We will now show that the truth of $P(m)$, namely

$$1.3+2.3^2+3.3^3+\dots+m.3^m=\frac{(2m-1)\ 3^{m+1}+3}{4} \qquad \dots(2)$$

implies the truth of $P(m+1)$, namely

$$1.3+2.3^2+3.3^3+\dots+m\ 3^m+(m+1).3^{m+1}=\frac{[2(m+1)-1]\ 3^{m+1+1}+3}{4} \qquad \dots(3)$$

$$\text{L.H.S. of (3)}=1.3+2.3^2+3.3^3+\dots+m.3^m+(m+1).3^{m+1}$$

$$=\frac{(2m-1)\ 3^{m+1}+3}{4}+(m+1).3^{m+1}\text{[from (2)]}$$

$$=\frac{1}{4}\left\{2m.3^{m+1}-3^{m+1}+3+4m.3^{m+1}+4.3^{m+1}\right\}$$

$$=\frac{1}{4}\left\{6m.3^{m+1}+3.3^{m+1}+3\right\}$$

$$=\frac{1}{4}\left\{2m.3^{m+2}+3^{m+2}+3\right\}$$

$$=\frac{1}{4}\left\{3^{m+2}\ (2m+1)+3\right\}$$

$$=\frac{1}{4}\left[\{2(m+1)-1\}\ 3^{(m+1)+1}+3\right]=\text{R.H.S. of (3)}$$

By step I. $P(n)$ is true for $n=1, 2$. Therefore, by step II, $P(n)$ is true for $n=2+1$ and so for $n=3+1$ and so on for every natural number n.

Example 5. *By the method of induction, show that* $10^n+3.4^{n+2}+5$ *is divisible by 9.*

Solution. Let $P(n)$: $10^n+3.4^{n+2}+5$ is divisible by 9 ...(1)

Step I. We verify the result (1) for $n=1$.

Put $n=1$ in (1), L.H.S.$=10+3.4^3+5=207=9\times23$ is divisible by 9.

Step II. We now show that the assumption of the truth of $P(m)$ namely $10^m+3.4^{m+2}+5$ is divisible by 9, *i.e.*,

$10^m+3.4^{m+2}+5=9k$, where k is a positive integer ...(2)

implies the truth of $P(m+1)$, namely

$10^{m+1}+3.4^{(m+1)+2}+5$ is divisible by 9 ...(3)

We have

$$10^{m+1}+3.4^{(m+1)+2}+5=10^m.10+3.4^{m+2}.4+5$$
$$=10^m.9+10^m+3.4^{m+2}.3+3.4^{m+2}+5$$
$$=10^m.9+3.4^{m+2}.3+(10^m+3.4^{m+2}+5)$$
$$=10^m.9+9.4^{m+2}+9k \quad \text{[From (2)]}$$
$$=9(10^m+4^{m+2}+k)$$
$$=9p, \text{ where } p=10^m+4^{m+2}+k, \text{ which is divisible by 9.}$$

We have thus proved that if the result (1) is true for $n=m$, it remains true for the next integral value of n, namely $m+1$. Since the result is verified for $n=1$ in step I, it follows by step II that it is true for $n=2$ and hence for $n=3$ and so on for all positive integral values of n.

Example 6. *Prove by induction the inequality :*

$$(1+x)^n > 1+nx, \textit{ for } n=2, 3, 4,\ldots \textit{and } x>-1.$$

Solution. Let $P(n)$ be $(1+x)^n > 1+nx$ ...(1)

Step I. We verify the result (1) for $n=2$.

Put $n=2$ in (1), L.H.S. $=(1+x)^2=1+2x+x^2$

R.H.S. $=1+2x$

We obviously have $1+2x+x^2>1+2x$, since $x^2>0$

The result is thus true for $n=2$.

Step II. We now show that the assumption of the truth of $P(m)$, namely

$$(1+x)^m > 1+mx \quad \ldots(2)$$

implies the truth of $P(m+1)$, namely

$$(1+x)^{m+1} > 1+(m+1)x \quad \ldots(3)$$

We have $(1+x)^{m+1}=(1+x)^m(1+x)$

$>(1+mx)(1+x)$, [due to assumption (2) and $x>-1$]

$>1+(m+1)x+mx^2$

$>1+(m+1)x$, since $mx^2>0$

We have thus proved that if the result is true for $n=m$, it remains true for next integral values of n, namely $(m+1)$. Since the result is verified to be true for $n=2$ in step I, it follows by step II that it is true for $n=3$, and hence for $n=4$, and so on for all positive integral values of $n \geqslant 2$.

11 2. SEQUENCES

When individual elements of a set of numbers can be arranged according to some definite rule such that we can find out which of them is the first, second and so on, the set forms a sequence.

Each number of a sequence is called the term so that we have the first term, second term, third term and so on, which form a subset of the set of real numbers.

Definition. *If to every positive integer n, there corresponds a number u_n, such that u_n is related to n by certain law of correspondence, the terms $u_1, u_2, u_3, \ldots, n_n, \ldots\ldots$ are said to form a sequence.* In other words, a set of real numbers in a definite order formed according to some law is called a sequence.

A sequence is sometimes denoted by bracketing its nth term, therefore $\{u_n\}$ or $<u_n>$ means the sequence comprising terms $u_1, u_2, u_3, \ldots u_n$ ... the suffix to 'u' denotes the rank of the term. If

(*i*) $u_n=(-1)^n$, then sequence or $\{u_n\}$ is $1^2, 2^2, 3^2, \ldots, n^2$

(*ii*) $u_n=(-1)^n$, then the sequence or $\{u_n\}$ is $-1, 1, \ldots, (-1)^n \ldots$,

(*iii*) $u_n=4n+3$, then $\{u_n\}$ is $7, 11, 15, 19, \ldots, (4n+3), \ldots$,

(*iv*) $u_n=(-1)^{n-1}\,4^n$, then $\{n_n\}$ is $4, -16, 64, -256, \ldots, (-1)^{n-1}4^n, \ldots$

(*v*) $u_n=\dfrac{1}{4n-5}$, then $\{u_n\}$ is $-1, \dfrac{1}{3}, \dfrac{1}{7}, \dfrac{1}{11}, \ldots, \left\{\dfrac{1}{4n-5}\right\}, \ldots$,

(*vi*) $u_n=\dfrac{n^2}{n+1}$, then $\left\{\dfrac{n^2}{n+1}\right\}$ is $\dfrac{1}{2}, \dfrac{4}{3}, \dfrac{9}{4}, \ldots, \dfrac{n^2}{n+1}, \ldots$

11·3. SERIES

When the terms of a sequence are connected with plus or minus signs, a series is formed. In other wards, a series is an expression consisting of the sum of the terms in *a* sequence. Thus if u_n is the nth term of a sequence then

$$u_1+u_2+u_3+\ldots+u_n \text{ is a series of } n \text{ terms.}$$

11·4. DISCOVERY OF SEQUENCE

It is always convenient for mathematical operations to discover the sequence or the rule of the formation of a series in terms of a sequence related with an order set of real numbers which is not very obvious at times. By the method of trial and error we have to determine, ensuring that at least first few terms are observing the rule so as to generalise ultimately. Let us take a few illustrative examples

(*i*) 2, 6, 12, 20, 30, ... can be written in the form :

$1+(1)^2, 2+(2)^2, 3+(2)^2, 4+(4)^2$, ...and can be formally expressed as

$$u_n=n+n^2, \{n+(n)^2\} \text{ or } \{n+n^2\} \text{ or } \{n(1+n)\}$$

(*ii*) 1, 6, 15, 28, ...which in the form of sequence can be written as :

$$2\times1^2-1, 2\times2^2-2, 2\times3^2-3, 2\times4^2-4, \ldots$$

where $u_n=2n^2-n$ and the sequence can be expressed as

$$\{2n^2-n\}$$

(*iii*) 2, 5, 12, 31,... which in the form of a sequence can be written as :

$$(1+3^{1-1}), (2+3^{2-1}), (3+3^{3-1}), (4+3^{4-1}),...$$

where $u_n=n+3^{n-1}$ and the sequence can be expressed as :

$$\{n+3^{n-1}\}$$

(*iv*) 1, 4, 9, 16, 25, 36, ..., where u_n is the square of n, we can say $u_n=n^2$, and the sequence is $\{n^2\}$.

(*v*) 5, 9, 13, 17, 21, 25,..., where u_n is obtained by adding 4 to the previous term *i.e.*, (u_{n-1}) and $u_1=5$, we can say $u_n=4n+1$, and the sequence is $\{4n+1\}$.

(*vi*) 3, -9, 27, -81,..., where u_n is obtained by multiplying the preceding term (u_{n-1}) by -3 and $u_1=3$, we can say

$$u_n=(-1)^{n-1}3^n, \text{ and the sequence is } \{(-1)^{n-1}3^n\}$$

(*vii*) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, ...$, where the numerator of u_n is 1 and the denominator is obtained by adding 3 to the preceding denominator and $u_1=1/2$. We can say, $u_n=\frac{1}{3n-1}$, and the sequence is $\left\{\frac{1}{3n-1}\right\}$

The two very common types of squences are discussed under the names of arithmetic and geometric progressions (Chapter XII).

11·5. SIGMA NOTATION

We now introduce a simple summation notation which considerably simplifies the formulae and makes handling of complicated expressions simpler. The letter "Σ" of the greek alphabet (pronounced as 'sigma') is used to denote the sum of a given series. The letter Σ is placed before the rth term say u_r. We thus write Σu_r to denote the sum of r terms of the series u_r. If we want to sum up u_r for values of r corresponding to $r=1, 2, 3, .., n$; we denote the sum by

$$\sum_{r=1}^{r=n} u_r \text{ or simply by } \sum_{r}^{n} u_r$$

Thus $$u_1+u_2+u_3+...+u_r+...+u_n = \sum_{r=1}^{n} u_r$$

[read as 'sigma u_r from $r=1$ to $r=n$']

Some Properties of Sigma Notation :

I. $$\sum_{r=1}^{n} au_r=a\sum_{r=1}^{n} u_r, \text{ where } a \text{ is constant.}$$

Proof. $$\sum_{r=1}^{n} au_r=au_1+au_2+au_3+...+au_n$$

$$=a(u_1+u_2+u_3+...+u_n)=a\sum_{r=1}^{n} u_r$$

II. $$\sum_{r=1}^{n}(u_r+v_r)=\sum_{r=1}^{n}u_r+\sum_{r=1}^{n}v_r$$

Proof. $$\sum_{r=1}^{n}(u_r+v_r)=(u_1+v_1)+(u_2+v_2)+(u_3+v_3)+\ldots+(u_n+v_n)$$

$$=(u_1+u_2+u_3+\ldots+u_n)+(v_1+v_2+v_3+\ldots+v_n)$$

$$=\sum_{r=1}^{n}u_r+\sum_{r=1}^{n}v_r$$

III. $$\sum_{r=1}^{n}(ar^3+br^2+cr+d)=a\sum_{r=1}^{n}r^3+b\sum_{r=1}^{n}r^2+c\sum_{r=1}^{n}r+nd$$

Proof. $$\sum_{r=1}^{n}(ar^3+br^2+cr+d)=a\,.\,1^3+b\,.\,1^2+c\,.\,1+d$$

$$+a\,.\,2^3+b\,.\,3^2+c\,.\,2+d$$
$$+a\,.\,3^3+b\,.\,3^2+c\,.\,3+d$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$+a\,.\,n^3+b\,.\,n^2+c\,.\,n+d$$

Adding these columnwise, we get

$$\sum_{r=1}^{n}(ar^3+br^3+cr+d)=a(1^3+2^3+3^3+\ldots+n^3)$$
$$+b(1^2+2^2+3^2+\ldots+n^2)+c(1+2+3+\ldots+n)$$
$$+d+d+d+\ldots+n \text{ times}$$
$$=a\sum_{r=1}^{n}r^3+b\sum_{r=1}^{n}r^2+c\sum_{r=1}^{n}r+nd$$

11 6. SUM OF NATURAL NUMBERS

I. Sum of the first n natural numbers. The snm of the first n natural numbers is

$$S_1\equiv 1+2+3+\ldots+r+\ldots+n=\sum_{1}^{n} r=\frac{n(n+1)}{2}$$

Proof We can prove this identity by means of difference of squares of the consecutive integers. We have the identity :

$$(x+1)^2-x^2\equiv 2x+1$$

By putting $x=1, 2, 3, \ldots, n$, in the above identity, we get

$$2^2-1^2=2.1+1$$
$$3^2-2^2=2.2+1$$
$$4^2-3^2=2.3+1$$
$$\vdots \qquad \vdots$$
$$n^2-(n-1)^2=2.(n-1)+1$$
$$(n+1)^2-n^2=2.n+1$$

Adding these n equalities columnwise, we get

$$(n+1)^2-1^2=2(1+2+3+\ldots+n)+(1+1+1\ldots n \text{ times})$$

$$\therefore \quad n^2+2n=2S_1+n \;\Rightarrow\; 2S_1=n^2+n=n(n+1)$$

Hence $$S_1=\frac{n}{2}(n+1)$$

II. Sum of the squares of the first n natural numbers. The sum of the squares of the first n natural numbers is

$$S_2\equiv 1^2+2^2+3^2+\ldots+r^2+\ldots+n^2=\sum_1^n r^2=\frac{1}{6}n(n+1)(2n+1)$$

Proof. We can prove this by making use of the following identity

$$(x+1)^3-x^3\equiv 3x^2+3x+1$$

By putting $x=1, 2, 3, \ldots, n$, in the above identity, we get

$$2^3-1^3=3.1^2+3.1+1$$
$$3^3-2^3=3.2^2+3.2+1$$
$$4^3-3^3=3.3^2+3.3+1$$
$$\vdots$$
$$n^3-(n-1)^3=3.(n-1)^2+3(n-1)+1$$
$$(n+1)^3-n^3=3n^2+3n+1$$

Adding these n equalities columnwise, we get

$$(n+1)^3-1^3=3(1^2+2^2+3^2+\ldots+n^2)+3(1+2+3+\ldots+n)$$
$$+(1+1+1+\ldots n \text{ times})$$

$$\therefore \quad (n+1)^3-1=3S_2+3S_1+n$$

$$\Rightarrow \quad n^3+3n^2+3n=3S_2+3.\frac{n(n+1)}{2}+n$$

$$\Rightarrow \quad 3S_2=n^3+3n^2+3n-\frac{3}{2}n(n+1)-n$$

$$=\frac{n}{2}[2n^2+6n+6-3(n+1)-2]$$

$$=\frac{n}{2}[2n^2+3n+1]=\frac{n}{2}(n+1)(2n+1)$$

$$\therefore \quad S_2=\frac{n(n+1)(2n+1)}{6}$$

Alternative Method. We can prove the above result by the method of mathematical induction also.

Here $P(n)$ is

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6} \qquad \ldots(1)$$

Step I. For $n=1$, L.H.S. $=1^2=1$

and R.H.S. $=\frac{1}{6}(1+1)(2+1)=1$

$\therefore$ L.H.R. $=$ R.H.S.

For $n=2$, L.H.S. $=1^2+2^2=1+4=5$

and R.H.S. $=\frac{2}{6}(2+1)(4+1)=5$

$\therefore$ L.H.S. $=$ R.H.S.

Thus the result is true for $n=1, 2$.

Step II. Let us assume that the result is true for some particular value, say m of n. We now show that the truth of $P(m)$, namely

$$1^2+2^2+3^2+\ldots+m^2=\frac{m(m+1)(2m+1)}{6} \qquad \ldots(2)$$

implies the truth of $P(m+1)$, namely

$$1^2+2^2+3^2+\ldots+m^2+(m+1)^2=\frac{(m+1)\{(m+1)+1\}\{2(m+1)+1\}}{6} \qquad \ldots(3)$$

Then

$$\begin{aligned}
\text{L.H.S. of } (3) &= (1^2+2^2+3^2+\ldots+m^2)+(m+1)^2 \\
&= \frac{m(m+1)(2m+1)}{6}+(m+1)^2 \quad \text{[From (2)]} \\
&= \frac{(m+1)}{6}[m(2m+1)+6(m+1)] \\
&= \frac{(m+1)}{6}[2m^2+7m+6] \\
&= \frac{(m+1)}{6}[2m^2+4m+3m+6] \\
&= \frac{(m+1)}{6}[2m(m+2)+3(m+2)] \\
&= \frac{(m+1)}{6}[(m+2)(2m+3)] \\
&= \frac{1}{6}(m+1)[\{(m+1)+1\}\{2(m+1)+1\}] \\
&= \text{R.H.S. of (3)}
\end{aligned}$$

From the two steps I and II, it follows from the principle of mathematical induction that the result $P(n)$ is true for every natural number n.

III. Sum of the cubes of the first n natural numbers. The sum of the cubes of the first n natural numbers is

$$S_3 = 1^3 + 2^3 + 3^3 + \ldots + r^3 + \ldots + n^3 = \sum_1^n r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Proof. As discussed in the previous cases we will now use, here, the identity based on the difference of the forrth powers of the consecutive number $x+1$ and x. We have

$$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$$

Putting $x = 1, 2, 3, \ldots\ldots (n-1), n$; successively in the identity, we get

$$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$$
$$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$$
$$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$$
$$\vdots \qquad \vdots$$
$$n^4 - (n-1)^4 = 4.(n-1)^3 + 6.(n-1)^2 + 4.(n-1) + 1$$
$$(n+1)^4 - n^4 = 4.n^3 + 6.n^2 + 4.n + 1$$

Adding these n equalities columnwise, we get

$$\begin{aligned}(n+1)^4 - 1^4 &= 4(1^3 + 2^3 + 3^3 + \ldots + n^3)\\ &\quad + 6(1^2 + 2^2 + 3^2 + \ldots + n^2)\\ &\quad + 4(1 + 2 + 3 + \ldots + n)\\ &\quad + (1 + 1 + 1 + \ldots n \text{ times})\\ &= 4S_3 + 6S_2 + 4S_1 + n\end{aligned}$$

$$\Rightarrow \quad 4S_3 = n^4 + 4n^3 + 6n^2 + 4n - 6S_2 - 4S_1 - n$$

$$= n^4 + 4n^3 + 6n^2 + 4n - 6 \cdot \frac{n(n+1)(2n+1)}{6} - 4 \cdot \frac{n(n+1)}{6} - n$$

$$= n^4 + 4n^3 + 6n^2 + 4n - 2n^3 - 3n^2 - n - 2n^2 - 2n - n$$

$$= n^4 + 2n^3 + n^2$$

$$= n^2(n^2 + 2n + 1) = n^2(n+1)^2$$

$$\Rightarrow \quad S_3 = \sum_{r=1}^n r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 7. *By the method of induction or otherwise prove that*

$$1^3+2^3+3^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}$$

Solution : Step I.

For $n=1$, L.H.S. $=1^3=1$

$$\text{R.H.S.}=\frac{1^2(1+1)^2}{4}=1$$

L.H.S. = R.H.S. and the proposition is true for $n=1$.

Step II. Let us assume that the result is true for some particular value, say m of n. We now show that the truth of $P(m)$, namely

$$1^3+2^3+3^3+\ldots+m^3=\frac{m^2(m+1)^2}{4} \qquad \ldots(1)$$

implies the truth of $P(m+1)$, namely

$$1^3+2^3+3^3+\ldots+m^3+(m+1)^3=\frac{(m+1)^2[(m+1)+1]^2}{4} \qquad \ldots(2)$$

$$\text{L.H.S. of (2)}=(1^3+2^3+3^3+\ldots+m^3)+(m+1)^3$$

$$=\frac{m^2(m+1)^2}{4}+(m+1)^3 \qquad \text{[by assumption (1)]}$$

$$=\frac{(m+1)^2}{4}[m^2+4(m+1)]$$

$$=\frac{(m+1)^2(m+2)^2}{4}$$

$$=\frac{(m+1)^2[(m+1)+1]^2}{4}$$

= R.H.S. of (2)

From step I and II, we conclude by the principle of mathematical induction, that the result is true for every natural number n or all integral values of n.

Example 8. *Find the sum of the series :*

1.4+3.7+5.10+...to n terms.

Solution. The rth term of the series is equal to the product of the rth terms of the two series

1, 3, 5,... and 4, 7, 10...

$$\therefore \quad u_r=[1+(r-1)2]\,.\,[4+(r-1)3]=(2r-1)(3r+1)=6r^2-r-1$$

Hence $$S_n=\sum_{r=1}^{n} u_r=\sum_{r=1}^{n}(6r^2-r-1)=6\sum_{r=1}^{n} r^2-\sum_{r=1}^{n} r-n$$

$$=\frac{6n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}-n$$

$$=\frac{1}{2}\,n(4n^2+5n-1)$$

Example 9. *Find the sum of the series*

$$1^2+3^2+5^2+7^2+\ldots+ n \text{ terms}$$

Solution. $u_r=[1+(r-1)2]^2=(2r-1)^2=4r^2-4r+1$

$\therefore$
$$S_n=\sum_{r=1}^{n} u_r=\sum_{r=1}^{n}(4r^2-4r+1)=4\sum_{r=1}^{n} r^2-4\sum_{r=1}^{n} r+n$$

$$=4\cdot\frac{n(n+1)(2n+1)}{6}-4\,\frac{n(n+1)}{2}+n$$

$$=\frac{n(4n^2-1)}{3}$$

Example 10. *Find the sum of the series* $1+(1+2)+(1+2+3)\ldots$*to n terms.*

Solution. $u_r=1+2+3+\ldots+r=\frac{r(r+1)}{2}=\frac{(r^2+r)}{2}$

$\therefore$
$$S_n=\sum_{r=1}^{n} u_r=\frac{1}{2}\sum_{r=1}^{n} r^2+\frac{1}{2}\sum_{r=1}^{n} r$$

$$=\frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\cdot\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)(n+2)}{6}$$

Example 11. *Sum to n terms the following series :*

$$\frac{1^2}{1}+\frac{1^2+2^2}{2}+\frac{1^2+2^2+3^2}{3}+\ldots$$

Solution. Here $u_r=\frac{1^2+2^2+3^2+\ldots+r^2}{r}$

$$=\frac{r(r+1)(2r+1)}{6r}=\frac{2r^2+3r+1}{6}$$

$$S_n=\sum_{r=1}^{n} u_r=\sum_{r=1}^{n}\left(\frac{2r^2+3r+1}{6}\right)$$

$$=\frac{1}{3}\sum_{r=1}^{n} r^2+\frac{1}{2}\sum_{r=1}^{n} r+\frac{1}{6}\sum_{r=1}^{n} 1$$

$$=\frac{1}{3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\cdot\frac{n(n+1)}{2}+\frac{1}{6}\cdot n$$

$$=\frac{n(4n^2+15n+17)}{26}$$

Example 12. *Sum the series to n terms :*

$$2.4.6+4.6.8+6.8.10+\ldots\ldots$$

Solution. The first factors of the terms form the sequence 2, 4, 6,... where rth term is $2r$. Also, the second factors of the terms form the sequence 4, 6, 8, ... where rth term is $(2r+2)$. Similarly, the third factors of the terms form the sequence 6, 8, 10,...where the rth term is $(2r+4)$.

∴ The rth term of the given series is

$$u_r=2r(2r+2)(2r+4)=8r^3+24r^2+16r$$

∴ $$S_n=\sum_{r=1}^{n} u_r=8\left[\sum_{r=1}^{n} r^3\right]+24\left[\sum_{r=1}^{n} r^2\right]+16\left[\sum_{r=1}^{n} r\right]$$

$$=8\cdot\frac{n^2(n+1)^2}{4}+24\cdot\frac{n(n+1)(2n+1)}{6}+16\cdot\frac{n(n+1)}{2}$$

$$=2n(n+1)[n(n+1)+2(2n+1)+4]$$

$$=2n(n+1)(n+2)(n+3).$$

Example 13. *Sum to n terms the series :*

$$1.3^2+4.4^2+7.5^2+10.6^2+\ldots$$

Solution. Here $u_r=[1+(r-1).3]\times[3+(r-1).1]^2=(3r-2)(r+2)^2$
$$=3r^3+10r^2+4r-8$$

∴ $$S_n=\sum_{r=1}^{n} u_r=\sum_{r=1}^{n}(3r^3+10r^2+4r-8)$$

$$=3\sum_{r=1}^{n} r^3+10\sum_{r=1}^{n} r^2+4\sum_{r=1}^{n} r-8n$$

$$=3\cdot\frac{n^2(n+1)^2}{4}+10\cdot\frac{n(n+1)(2n+1)}{6}+\frac{4n(n+1)}{2}-8n$$

$$=\frac{n(n+1)}{12}[9n^2+49n+44]-8n$$

Example 14. *Find the nth term and sum to n terms of the series :*

$$4+6+9+13+18+\ldots$$

Solution. The law of formation of the series is not obvious. In such a case we proceed as follows :

Let S_n denote the sum of the first n terms and u_n the nth term of a given series. Then

$$S_n=4+6+9+13+18+\ldots+u_n \qquad \ldots(1)$$

Also $$S_n=\quad 4+6+9+13+\ldots+u_{n-1}+u_n \qquad \ldots(2)$$

The same series is written again with each term shifted by one place to the right.

By subtraction, we have

$$0=4+2+3+4+5+\ldots+(u_n-u_{n-1})-u_n$$

$$\Rightarrow \quad u_n=4+(2+3+4+5+\ldots \text{to } \overline{n-1} \text{ terms})$$

$$=4+\frac{n-1}{2}[4+(n-2).1]$$

$$=4+\frac{n-1}{2}(n+2)=\frac{1}{2}(n^2+n+6)$$

$$\therefore \quad S_n=\Sigma u_n=\frac{1}{2}\left[\Sigma n^2+\Sigma n+6n\right]$$

$$=\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}+6n\right]$$

$$=\frac{1}{12}[2n^3+6n^2+40n]=\frac{n}{6}(n^2+3n+20)$$

Example 15. *Find the nth term and sum to n terms of the series .*

$$11+23+59+167+\ldots$$

Solution. Let S_n, be the sum and u_n, the nth term of the given series.

Then $$S_n=11+23+59+167+\ldots+u_n$$

Also $$S_n=\quad 11+23+59+\ldots+u_{n-1}+u_n$$

By subtracting, we have

$$0=11+12+36+108+\ldots+(u_n-u_{n-1})-u_n$$

$$\Rightarrow \quad u_n=11+\{12+36+108+\ldots\ldots \text{to } (n-1) \text{ terms}\}$$

$$=11+12\left(\frac{3^{n-1}-1}{3-1}\right)=11+6(3^{n-1}-1)$$

$$=2\,.\,3^n+5$$

Giving to n the values 1, 2, 3,..., n, we have

$$u_1=2\,.\,3^1+5$$
$$u_2=2\,.\,3^2+5$$
$$u_3=2\,.\,3^3+5$$
$$\vdots \qquad \vdots$$
$$u_n=2\,.\,3^n+5$$

$$\therefore \quad S_n=2(3+3^2+3^3+\ldots+3^n)+5n$$
$$=\frac{2\,.\,3(3^n-1)}{3-1}+5n$$
$$=3^{n+1}+5n-3$$

Example 16. *Sum to n terms the series :*

$$\frac{1}{4.9}+\frac{1}{9.14}+\frac{1}{14.19}+\ldots$$

Solution. $u_n=\frac{1}{[4+(n-1)5][9+(n-1)5]}=\frac{1}{(5n-1)(5n+4)}$

$$=\frac{1}{5}\;\frac{(5n+4)-(5n-1)}{(5n-1)(5n+4}$$
$$=\frac{1}{5}\left[\frac{1}{5n-1}-\frac{1}{5n+4}\right]$$

Giving to n the values 1, 2, 3,..., $n-1$, n ; we have

$$u_1=\frac{1}{5}\left[\frac{1}{4}-\frac{1}{9}\right]$$
$$u_2=\frac{1}{5}\left[\frac{1}{9}-\frac{1}{14}\right]$$
$$\vdots \qquad \vdots$$
$$u_{n-1}=\frac{1}{5}\left[\frac{1}{5n-6}-\frac{1}{5n-1}\right]$$
$$u_n=\frac{1}{5}\left[\frac{1}{5n-1}-\frac{1}{5n+4}\right]$$

Adding columnwise, we get

$$S_n=\frac{1}{5}\left(\frac{1}{4}-\frac{1}{5n+4}\right)=\frac{n}{4(5n+4)}$$

Remark. The sum to infinity of the above series

$$=\frac{1}{5}\left(\frac{1}{4}\right)=\frac{1}{20}.$$

Example 17. *Find the sum of the first n terms of the infinite series* $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ *and then prove that the series converges to the sum one.*

[C.A., May, 1991]

Solution. We have

$$u_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

Giving to n the values 1, 2, 3,..., $n-1$, n ; we have

$$u_1 = 1 - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

....

....

$$u_{n-1} = \frac{1}{n-1} - \frac{1}{n}$$

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding columnwise, we get

$$S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$\therefore$ The sum to infinity of the above series $=1$

Hence the series converges to the sum one.

EXERCISES

1. Use the method of induction to prove the following results :

   (*i*) $1+3+5+\ldots+(2n-1)=n^2$ [*C.A., Nov. 1991*]

   (*ii*) $2+6+10+\ldots+(4n-2)=2n^2$

   (*iii*) $1.2+2.3+3.4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$

   (*iv*) $1.2.3+2.3.4+3.4.5+\ldots+n(n+1)(n+2)$

   $$=\frac{n}{4}(n+1)(n+2)(n+3)$$

   (*v*) $1.2+3.2^2+5.2^3+\ldots+(2n-1)\,2^n=(n-1)\,2^{n+2}-2^{n+1}+6$

   (*vi*) $\frac{1}{3.8}+\frac{1}{8.13}+\frac{1}{13.18}+\ldots+\frac{1}{(5n-2)(5n+3)}=\frac{n}{3(5n+3)}$

   (*vii*) $2^{4n}-1$ is divisible by 15.

   (*viii*) 3^n-2n-1 is divisible by 4.

   (*ix*) $n(n-1)(2n-1)$ is divisible by 6.

   (*x*) $7^{2n}+16n-1$ is divisible by 64.

   (*xi*) $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1}$

2. If the matrix P is given by

$$P=\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$

then using the Principle of Mathematical Induction establish that

$$P^n=\begin{pmatrix} 1+2n & 4n \\ -n & 1-2n \end{pmatrix}$$

For all positive integers $n \geqslant 1$. [*C.A., May, 1991*]

3. Sum to n terms the series :

(*i*) $2^2+5^2+8^2+\ldots$, (*ii*) $1^3+3^3+5^3+\ldots$

(*iii*) $1.4+3.7+5.10+\ldots$ (*iv*) $1.3.5+3.5.7+5.7.9+\ldots$

4. Find the nth term and the sum to n terms of the series :

$$2.3^2+5.4^2+8.5^2+11.6^2+\ldots$$

5. Sum to n terms of the series :

(*i*) $1+(1+3)+(1+3+5)+\ldots$

(*ii*) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots$

(*iii*) $1+\left(1+\frac{1}{3}\right)+\left(1+\frac{1}{3}+\frac{1}{3^2}\right)+\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}\right)+\ldots$

6. Find the sum of the following series :

(*i*) $n.1+(n-1).2+(n-2).3+\ldots+2.(n-1)+1.n$

(*ii*) $\left(1-\frac{1}{n}\right)+\left(1-\frac{2}{n}\right)+\left(1-\frac{3}{n}\right)+\ldots$ to n terms.

7. Find the sum of n terms of the series whose nth terms are

(*i*) $3n^2+2n$, (*ii*) $n.2^n$. (*iii*) $5.3^{n+1}+2n$

8. Find the nth terms and sum to n terms of the series :

(*i*) $1+5+12+22+\ldots$

(*ii*) $4+14+30+52+80+\ldots$

(*iii*) $3+6+11+20+37+\ldots$

9. Find the sum to n terms and to infinite terms of the following series :

(*i*) $\frac{1}{4.7}+\frac{1}{7.10}+\frac{1}{10.13}+\ldots$

(*ii*) $\frac{1}{(p+2)(p+4)}+\frac{1}{(p+4)(p+6)}+\frac{1}{(p+6)(p+8)}+\ldots$

10. Prove that : $n^2+2n[1+2+3+\ldots+(n-1)]=n^3$.

11. (*a*) Find the sum of the series :

$$\frac{1^3}{1}+\left(\frac{1^3+2^3}{2}\right)+\left(\frac{1^3+2^3+3^3}{3}\right)+\ldots \text{to } n \text{ terms}$$

(*b*) $\frac{1^2}{1}+\frac{1^2+2^2}{1+2}+\frac{1^2+2^2+3^2}{1+2+3}+\ldots$to n terms

ANSWERS

3. (*i*) $\frac{n}{2}(6n^2+3n-1)$, (*ii*) $n^2(2n^2-1)$, (*iii*) $\frac{1}{2}n(4n^2+5n-1)$,

(*iv*) $n(n+1)(2n^2+6n+1)-3n$

4. $3n^3+11n^2+8n-4$; $\frac{n}{12}(9n^3+62n^2+123n+22)$.

5. (*i*) $\frac{1}{6}n(n+1)(2n+1)$, (*ii*) $\frac{1}{12}n(n+1)^2(n+2)$,

(*iii*) $\frac{3}{2}\left(1-\frac{1}{3^n}\right)$, $\frac{3}{2}\left[n-\frac{1}{2}\left(1-\frac{1}{3^n}\right)\right]$

6. (*i*) $\frac{1}{6}n(n+1)(n+2)$ (*ii*) $\frac{1}{2}(n-1)$

7. (*i*) $\frac{1}{2}n(n+1)(2n+3)$, (*ii*) $(n-1)2^{n+1}+2$, (*iii*) $\frac{5}{2}[3^{n+2}-9]+n(n+1)$

8. (*i*) $\frac{n}{2}(3n-1)$, $\frac{n^2(n+1)}{8}\left[3n+2\right]$ (*ii*) $3n^2+n$, $n(n+1)^2$

(*iii*) 2^n+n, $2^{n+1}+\frac{1}{2}n(n+1)-2$

9. (*i*) $\frac{1}{3}\left(\frac{1}{3n+1}-\frac{1}{3n+4}\right)$, $\frac{n}{4(3n+4)}$, 1/12

(*ii*) $\frac{1}{2}\cdot\frac{2n}{(p+2)(p+2n+2)}=\frac{n}{(p+2)(p+2n+2)}$; $\frac{1}{2(p+2)}$

11. (*a*) $\frac{n(n+1)(n+2)(3n+5)}{48}$, (*b*) $\frac{n(n+2)}{3}$

12

Arithmetic and Geometric Progressions

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- arithmetic progression, its sum, arithmetic mean and its applications in solving problems.
- geometric mean and its applications in solving problems.

12·0. INTRODUCTION

In this chapter we shall discuss two special types of series with sequences increasing or decreasing by an absolute quantity or a certain ratio designated as arithmetic and geometric progressions respectively.

12·1. ARITHMETIC PROGRESSION

An arithmetic progression is a *sequence* whose terms increase or decrease by a constant number called the common difference. A series in arithmetic progression thus becomes an additive series in which the common difference can be found by subtracting each term from its preceding one. Thus

(*i*) the sequence 1, 5, 9, 13, 17, 21, 25,... is an infinite arithmetic progression of seven terms, the first term is 1 and the common difference is 4. Similarly,

(*ii*) the sequence 4, $\frac{7}{2}$, 3, $\frac{5}{2}$, 2, $\frac{3}{2}$, 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, ...is an arithmetic progression whose first term is 4 and the common difference is $-\frac{1}{2}$.