The inventor of this theorem was Isaac Newton who in 1676 gave it through a letter to H. Oldenberg. Before, he could devise a general theorem, the Hindus and Arabs new the expansion up to the 2nd and the 3rd power only to which Vieta added the expansion up to the 4th power. But, all this was through simple multiplication and without the discovery of the law of expansion, the credit for which goes to Newton.

Now, we take a few expansions on the existing lines to enable the reader to grasp the manner in which the terms appear :

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y$$

Some special features of the above expansion are :

(*i*) The total number of terms are one more than the power, *i.e.*, $5+1=6$ in the above case.

(*ii*) The coefficients of the terms are symmetrical, *viz.*, 1, 5, 10, 10, 5, 1. These can also be expressed by combinatorial expressions of the form ${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$, ${}^5C_5$ respectively.

For these coefficients there is a table in the form of a triangle called Pascal's Triangle which can be constructed intuitively beginning with power 1 to n in a ladder like fashion without any difficulty.

Pascal's Triangle

[Showing coefficients of terms $(x+y)^n$]

Size of n	*Binomial coefficients*	*Sum*
1	*1 1*	*2*
2	*1 2 1*	*4*
3	*1 3 3 1*	*8*
4	*1 4 6 4 1*	*16*
5	*1 5 10 10 5 1*	*32*
6	*1 6 15 20 15 6 1*	*64*
7	*1 7 21 35 35 21 7 1*	*128*
8	*1 8 28 56 70 56 28 8 1*	*256*
9	*1 9 36 64 126 126 84 36 9 1*	*512*
10	*1 10 45 120 210 252 210 120 45 10 1*	*1024*

The above triangle is based on the principle that

$${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r,$$ proved in the earlier chapter.

This solves the problem of the coefficients when the indices of the expansion are positive integers. One can note the symmetry of the successive coefficients. This can be observed from the fact that the coefficients equidistant from the middle term are the same.

(*iii*) The coefficients of the first and the last terms are the same since

$${}^nC_0 = {}^nC_n = 1$$

(*iv*) The indices of the terms rise from 0 to 5 in the case of second element of the binomial and come down from 5 to 0 in the first element. Since

$$({}^5C_0)x^5y^0+({}^5C_1)x^{5-1}y^1+({}^5C_2)x^{5-2}y^2+({}^5C_3)x^{5-3}y^3+({}^5C_4)x^{5-4}y^4+({}^5C_5)x^{5-5}y^5$$

Naturally, as $y^0=1$ and ${}^5C_0=1$, the first term remains as x^5 and through a similar logic as ${}^5C_5=1$ and $x^0=1$ the last term remains as y^5.

Further, it should be noted that the sum of the indices of elements in the two terms is equal to the index of the expansion as elaborated below :

Terms	*Sum of indices*
x^5y^0	5
x^4y^1	5
x^3y^2	5

and so on.

10·1. BINOMIAL THEOREM

Statement. *If* $(x+a)$ *is a binomial expression, the expansion of* $(x+a)^n$ *is given by*

$$(x+a)^n={}^nC_0x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+{}^nC_3x^{n-3}a^3+\ldots+{}^nC_rx^{n-r}a^r+\ldots+{}^nC_na^n$$

Proof. The theorem can be proved by the method of induction.

Step I. By actual multiplication, we have

$$(x+a)^2=x^2+2xa+a^2={}^2C_0x^2+{}^2C_1xa+{}^2C_2a^2$$

$$(x+a)^3=x^3+3x^2a+3xa^2+a^3$$

$$={}^3C_0x^3+{}^3C_1x^2a+{}^3C_2xa^2+{}^3C_3a^3$$

Thus the theorem is true when n has the values 2 and 3.

Step II. To prove this theorem by the principle of mathematical induction we shall now assume that the theorem is true for some particular value m of n, and we shall show that it is true for $m+1$ of n also. So we assume

$$(x+a)^m={}^mC_0x^m+{}^mC_1x^{m-1}a+{}^mC_2x^{m-2}a^2+\ldots+{}^mC_rx^{m-r}a^r+\ldots+{}^mC_ma^m$$

Multiplying both sides by $(x+a)$, we have

$$(x+a)^m(x+a)=[x^m+{}^mC_1x^{m-1}a+{}^mC_2x^{m-2}a^2+\ldots+{}^mC_rx^{m-r}a^r+\ldots+{}^mC_ma^m](x+a)$$

$$=x[x^m+{}^mC_1x^{m-1}a+{}^mC_2x^{m-2}a^2+\ldots+{}^mC_rx^{m-r}a^r+\ldots+{}^mC_ma^m]$$

$$+a[x^m+{}^mC_1x^{m-1}a+{}^mC_2x^{m-2}a^2+\ldots+{}^mC_rx^{m-r}a^r+\ldots+{}^mC_m a^m]$$

$$=x^{m+1}+{}^mC_1x^ma+{}^mC_2x^{m-1}a^2+\ldots+{}^mC_rx^{m-r+1}a^r+\ldots+{}^mC_mxa^m$$

$$+x^ma+{}^mC_1x^{m-1}a^2+{}^mC_2x^{m-2}a^3+\ldots+{}^mC_rx^{m-r}a^{r+1}+\ldots+{}^mC_ma^{m+1}$$

$$\therefore\quad (x+a)^{m+1}=x^{m+1}+(1+{}^mC_1)x^ma+({}^mC_1+{}^mC_2)x^{m-1}a^2+\ldots$$

$$+({}^mC_{r-1}+{}^mC_r)x^{m-r+1}a^r+\ldots+{}^mC_ma^{m+1}$$

Now

$${}^mC_{r-1}+{}^mC_r=\frac{m\,!}{(r-1)\,!\,(m-r+1)\,!}+\frac{m\,!}{r\,!(m-r)\,!}$$

$$=\frac{m!\,(r+m-r+1)}{r!\,(m-r+1)!}=\frac{m!\,(m+1)}{r!\,(m-r+1)!}$$

$$=\frac{(m+1)!}{r!\,(m-r+1)!}={}^{m+1}C_r$$

and $\quad {}^mC_m=1={}^{m+1}C_{m+1}$

$$\therefore\ (x+a)^{m+1}={}^{m+1}C_0x^{m+1}+{}^{m+1}C_1x^ma+{}^{m+1}C_2a^{m-1}a^2+{}^{m+1}C_3x^{m-2}a^3+\ldots+{}^{m+1}C_{m+1}a^{m+1}$$

Thus the expansion of $(x+a)^{m+1}$ is exactly of the same form as that of $(x+a)^m$, *i.e.*, the theorem is true for next higher value $(m+1)$ of m.

Step III. But we have seen that the theorem is true for the value $n=3$, therefore it should be true for the value $3+1=4$.

Hence the theorem is true for all positive integral values of n.

Remarks : 1. The expansion of the binomial $(x-a)^n$ is given below :

$$(x-a)^n=x^n+{}^nC_1(-a)x^{n-1}+{}^nC_2(-a)^2x^{n-2}+{}^nC_3(-a)^3x^{n-3}+\ldots+{}^nC_n(-a)^n$$

$$=x^n-{}^nC_1ax^{n-1}+{}^nC_2a^2x^{n-2}-{}^nC_3a^3x^{n-3}+\ldots+(-1)^n\,{}^nC_na^n$$

Thus, the terms in the expansion of $(x-a)^n$ are numerically same as $(x+a)^n$ with the difference that the terms are alternatively positive and negative and the last term is positive or negative depending on whether n is even or odd respectively.

2. We are now aware that in the expansion of $(x+a)^n$, the coefficient of the second term is ${}^nC_1$ and of the third term it is ${}^nC_2$ and so on. Thus, the suffix of each term is one less than the number, therefore ${}^nC_r$ is the coefficient of the $(r+1)$th term.

In the general term we have ${}^nC_r$, by giving numerical values to r, we can find out the coefficient and by assigning appropriate indices to 'x' and 'a', the whole term can be obtained as follows :

$$t_{r+1}={}^nC_rx^{n-r}a^r=\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}\,x^{n-r}a^r$$

It should be noted that the index of 'a' is the same as the suffix of C and that the sum of the indices of x and a is n.

3. The simplest form of binomial expansion in the general form is given below :

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_rx^r+\ldots+{}^nC_nx^n$$

$$=1+nx+\frac{n(n-1)}{1.2}\,x^2+\ldots+x^n$$

e.g., $\quad (1+x)^5=1+5x+10x^2+10x^3+5x^4+x^5$

The general term in the above expansion indicated by t_{r+1} is as follows :

$$t_{r+1}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\,x^r$$

4. $\quad (x+y)^n=\left[x\left(1+\frac{y}{x}\right)\right]^n=x^n(1+z)^n$, where $z=\frac{y}{x}$.

5. The binomial expansion of $(1-x)^n$ is as follows :

$$(1-x)^n=1-nx+\frac{n(n-1)}{1.2}x^2\ \frac{n(n-1)(x-2)}{1.2.3}x^3+\ldots+(-1)^n x^n$$

e.g., $(1-x)^5=1-5x+10x^2-10x^3+5x^4-x^5$

The general term in the above expansion indicated by t_{r+1} is

$$t_{r+1}=(-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}x^r$$

Example 1 *Expand* $\left(x-\frac{1}{x}\right)^5$.

Solution. $\left(x-\frac{1}{x}\right)^5={}^5C_0x^5+{}^5C_1x^4\left(-\frac{1}{x}\right)^1+{}^5C_2x^3\left(-\frac{1}{x}\right)^2$

$$+{}^5C_3x^2\left(-\frac{1}{x}\right)^3+{}^5C_4x\left(-\frac{1}{x}\right)^4+{}^5C_5\left(-\frac{1}{x}\right)^5$$

$$=x^5-5x^3+10x-10\,.\frac{1}{x}+5.\frac{1}{x^3}-\frac{1}{x^5}$$

Example 2. *Write down the expansion of* $\left(3x-\frac{1}{2}y\right)^4$ *by the binomial theorem. By giving suitable value to x and y, obtain the value of* $(29{\cdot}5)^4$ *correct to four significant figures.*

Solution. $\left(3x-\frac{1}{2}y\right)^4=(3x)^4+{}^4C_1(3x)^3\left(-\frac{y}{2}\right)$

$$+{}^4C_2(3x)^2\left(-\frac{y}{2}\right)^2+{}^4C_3(3x)\left(-\frac{y}{2}\right)^3+\left(-\frac{y}{2}\right)^4$$

$$=(3x)^4-4(3x)^3\frac{y}{2}+6(3x)^2\frac{y^2}{4}-4(3x)\frac{y^3}{8}+\frac{y^4}{16} \qquad \ldots(1)$$

Now $(29{\cdot}5)^4=(30-0{\cdot}5)^4$ or $\left(3.10-\frac{1}{2}.1\right)^4$

$\therefore$ Substituting $x=10$ and $y=1$ in the above expansion (1), we get

$$(29{\cdot}5)^4=(30)^4-4(30)^3(0{\cdot}5)+6(30)^2(0{\cdot}5)^2+\ldots$$
$$=8{,}10{,}000-54{,}000+1350=7{,}57{,}350$$

(to five significant figures)

Example 3. *Expand and simplify*

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$$

Solution. $(\sqrt{2}+1)^6=(\sqrt{2})^6+{}^6C_1(\sqrt{2})^5.1+{}^6C_2(\sqrt{2})^4.1^2+{}^6C_3(\sqrt{2})^3.1^3$
$+{}^6C_4(\sqrt{2})^2.1^4+{}^6C_5(\sqrt{2}).1^5+{}^6C_6.1^6$

$(\sqrt{2}-1)^6=(\sqrt{2})^6-{}^6C_1(\sqrt{2})^5.1+{}^6C_2(\sqrt{2})^4.1^2-{}^6C_3(\sqrt{2})^3.1^3$
$+{}^6C_4(\sqrt{2})^2.1^4-{}^6C_5(\sqrt{2}).1^5+{}^6C_6.1^6$

$\therefore\ (\sqrt{2}+1)^6+(\sqrt{2}-1)^6=2\{(\sqrt{2})^6+{}^6C_2(\sqrt{2})^4+{}^6C_4(\sqrt{2})^2+{}^6C_6\}$

(terms with odd indices have cancelled out)

$$=2\left\{2^3+\frac{6\times5}{1.2}.2^2+\frac{6\times5\times4\times3}{4.3.2.1}.2+1\right\}$$
$$=2(8+60+30+1)=198$$

Example 4. *Evaluate* $\{x+\sqrt{x^2+1}\}^6+\{x-\sqrt{x^2+1}\}^6$

Solution. Let $\{x+\sqrt{x^2+1}\}^6=(x+y)^6$, where $y=\sqrt{x^2+1}$

$$={}^6C_0x^6+{}^6C_1x^5y+{}^6C_2x^4y^2+{}^6C_3x^3y^3+{}^6C_4x^2y^4+{}^6C_5xy^5+{}^6C_6y^6 \quad ...(1)$$

Also $\{x-\sqrt{x^2+1}\}^6=(x-y)^6$

$$={}^6C_0x^6-{}^6C_1x^5y+{}^6C_2x^4y^2-{}^6C_3x^3y^3+{}^6C_4x^2y^4-{}^6C_5xy^5+{}^6C_6y^6 \quad ...(2)$$

Adding (1) and (2), we get

$$\{x+\sqrt{x^2+1}\}^6+\{x-\sqrt{x^2+1}\}^6$$
$$=2\{{}^6C_0x^6+{}^6C_2x^4y^2+{}^6C_4x^2y^4+{}^6C_6y^6\}$$
$$=2\{{}^6C_0x^6+{}^6C_2x^4(x^2+1)+{}^6C_4x^2(x^2+1)^2+{}^6C_6(x^2+1)^3\}$$
$$=2\left\{x^6+\frac{6.5}{1.2}x^4(x^2+1)+\frac{6.5}{1.2}.x^2(x^4+2x^2+1)\right.$$
$$\left.+(x^6+3x^4+3x^2+1)\right\}$$
$$=2\{32x^6+48x^4+18x^2+1\}$$

Example 5. *Write down the 7th term in the expansion of* $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

Solution. In the expansion of $(y+a)^n$, the general term is

$$t_{r+1}={}^nC_ry^{n-r}a^r$$

Putting $r=6$, $y=\frac{4x}{5}$, $a=\left(-\frac{5}{2x}\right)$ and $n=9$, we get

$$t_{6+1}={}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6$$
$$=\frac{9!}{6!\,3!}\cdot\frac{(4x)^3}{5^3}\cdot\frac{(-5)^6}{(2x)^6}$$
$$=\frac{9\times8\times7}{3\times2}\cdot\frac{64x^3}{5\times5\times5}\cdot\frac{5\times5\times5\times5\times5\times5}{64x^6}$$
$$=\frac{84\times125}{x^3}=\frac{10500}{x^3}$$

Example 6. *Using the binomial theorem, calculate $(1{\cdot}1)^{10}$ correct to 6 decimal places.* [*C.A. Intermediate, November 1981*]

Solution. We have

$$(1{\cdot}1)^{10}=(1+0{\cdot}1)^{10}$$
$$=1+{}^{10}C_1(0{\cdot}1)+{}^{10}C_2(0{\cdot}1)^2+{}^{10}C_3(0{\cdot}1)^3+{}^{10}C_4(0{\cdot}1)^4$$
$$+{}^{10}C_5(0{\cdot}1)^5+{}^{10}C_6(0{\cdot}1)^6+{}^{10}C_7(0{\cdot}1)^7+{}^{10}C_8(0{\cdot}1)^8$$
$$+{}^{10}C_9(0{\cdot}1)^9+{}^{10}C_{10}(0{\cdot}1)^{10}$$

$$= 1 + 10(0{\cdot}1) + 45(0{\cdot}01) + 120(0{\cdot}001) + 210(0{\cdot}0001) + 252(0{\cdot}00001) + 210(0{\cdot}000001) + 120(0{\cdot}0000001) + 45(0{\cdot}00000001) + \ldots\ldots$$

$$= 1 + 1{\cdot}0 + 0{\cdot}45 + 0{\cdot}120 + 0{\cdot}0210 + 0{\cdot}00252 + 0{\cdot}000210 + 0{\cdot}0000120 + 0{\cdot}00000045$$

$$= 2{\cdot}593742$$

EXERCISE (I)

1. Expand

(*i*) $(3x-y)^4$, (*ii*) $\left(\frac{x}{3}+\frac{2}{y}\right)^4$

(*iii*) $\left(\frac{3}{4}x-\frac{4}{3x}\right)^5$, (*iv*) $(1-x+x^2)^4$.

(*v*) $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$. [*I.C.W.A., December 1990*]

2. Write down and simplify

(*i*) the 11th term in the expansion of $(y+4x)^{30}$,

(*ii*) the 5th term in the expansion of $\left(\frac{3x}{4}+\frac{4}{3x}\right)^{12}$

(*iii*) write down the 5th term of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^8$, and show that it does not contain x.

3. Write down

(*a*) 3rd term from the end in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^8$

(*b*) nth term in the expansion of $\left(x-\frac{1}{x^2}\right)^{3n}$

4. Expand $\left(y+\frac{1}{10y}\right)^8$

Simplify each term as far as possible. Use your expansion to evaluate $(1{\cdot}1)^8$ correct to four places of decimal.

5. Expand in ascending powers of x, up to and including the term in x^3,

(*i*) $\left(1+\frac{x}{3}\right)^9$, (*ii*) $(2-x)^6$.

6. Expand $[1+(x+x^2)]^{10}$ as a series in ascending powers of x up to and including the term in x^3. Find the value of $(1{\cdot}0101)^{10}$ correct to three places of decimal.

ANSWERS

1. (*i*) $81x^4-108x^3y+54x^2y^2-12xy^3+y^4$

(ii) $\frac{x^4}{81}+\frac{8x^3}{27y}+\frac{8x^2}{3y^2}+\frac{32x}{3y^3}+\frac{16}{y^4}$

(iii) $\frac{243}{1024}x^5-\frac{135}{64}x^3+\frac{15}{2}x-\frac{40}{3x}+\frac{320}{27x^3}-\frac{1024}{243x^5}$

(iv) $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$ (v) 82.

2. (i) $4^{10}\,{}^{30}C_{10}.y^{30}.x^{10}$ (ii) $\frac{40095}{256}x^4$, (iii) 1120. 3. (a) $\frac{4375}{x^4}$

(b) $(-1)^{n-1}\frac{(3n)!\,x^3}{(n-1)!(2n+1)!}$

4. 2·1436. 5. (i) $1+3x+4x^2$, (ii) $64-192x+240x^2$.

6. $1+10x+55x^2+210x^3$; 1·106.

10·2. POSITIONS OF TERMS

We have already explained that in the expansion of $(1+x)^n$, the coefficients of terms equidistant from the beginning and the end are equal. We also know that the coefficient of $(r+1)$th term from the beginning is ${}^nC_r$. The $(r+1)$th term from the end has $\{(n+1)-(r+1)\}$ or $n-r$ terms before it, therefore, from the beginning it is $(n-r+1)$th term and its coefficient is ${}^nC_{n-r}$, which has been shown to be equal to ${}^nC_r$.

Therefore, when n is even the greatest coefficient is ${}^nC_{\frac{n}{2}}$ and when n is odd it is ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$, these two coefficients are however equal.

Example 7. (a) *Find the middle term in the expansion of*

$$\left(\frac{a}{x}-bx\right)^{12}$$

(b) *Find the two middle terms in the expansion of*

$$\left(3x-\frac{2x^2}{3}\right)^7$$

Solution. (a) The total number of terms in the expansion is $12+1=13$. Since the number is odd, there is only one middle term, *i.e.*, the 7th terms.

$$t_7={}^{12}C_6\left(\frac{a}{x}\right)^{12-6}(-bx)^6={}^{12}C_6a^6b^6$$

(b) The total number of terms in the expansion are 8 (even) and so there will be two middle terms, *viz.*, the 4th and 5th.

$$\therefore \quad t_4={}^7C_3(3x)^{7-3}\left(-\frac{2x^2}{3}\right)^3=-840.x^{10}$$

$$t_5={}^7C_4(3x)^{7-4}\left(-\frac{2x^2}{3}\right)^4=\frac{560}{3}.x^{11}$$

Example 8. *Show that the middle term in the expansion of* $(1+x)^{2n}$ *is*

$$\frac{1.3.5\ldots(2n-1)}{n\,!}.2^n x^n$$

Solution. In $(1+x)^{2n}$, there are $(2n+1)$ terms and so the middle term is $(n+1)$th term. Now

$$t_{n+1}={}^{2n}C_n x^n=\frac{(2n)\,!}{n\,!\,n\,!}.x^n$$

$$=\frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\ldots5.4.3.2.1}{n\,!\,n\,!}.x^n$$

$$=\frac{[2n(2n-2)(2n-4)\ldots4.2][(2n-1)(2n-3)\ldots5.3.1]}{n\,!\,n\,!}x$$

$$=\frac{2^n[n(n-1)(n-2)\ldots2.1][1.3.5\ldots(2n-1)]}{n\,!\,n\,!}.x^n$$

$$=\frac{2^n n\,!\,[1.3.5\ldots(2n-1)]}{n\,!\,n\,!}..x^n$$

$$=\frac{[1.3.5\ldots(2n-1)]}{n\,!}.2^n x^n$$

Example 9. *Prove that the middle term of*

$$\left(x+\frac{1}{2x}\right)^{2n} \text{ is } \frac{1.3.5\ldots(2n-1)}{n\,!}$$

Solution. In $\left(x+\frac{1}{2x}\right)^{2n}$, there are $(2n+1)$ terms and so the middle term is $(n+1)$th term. Now

$$t_{n+1}={}^{2n}C_n\, x^n\left(\frac{1}{2x}\right)^n={}^{2n}C_n.\frac{1}{2^n}=\frac{2n\,!}{n\,!\,n\,!}\cdot\frac{1}{2^n}$$

$$=\frac{[1.3.5\ldots(2n-1)]}{n\,!}.$$

[See above example]

10·3. BINOMIAL COEFFICIENTS

The coefficients of the expansion of the binomial are the prefixes of each term, the elements are with variable powers of the binomials. The coefficients are ${}^nC_0$, ${}^nC_1\ldots$, ${}^nC_n$ indicated briefly as $C_0, C_1,\ldots, C_n$. The properties of these coefficients are given below :

I. In the expansion of $(1+x)^n$, *where n is a positive integer, coefficients of terms equidistant from the beginning and end are equal.*

Proof. $(r+1)$th term from the beginning $=t_{r+1}$ is ${}^nC_r x^r$

$\Rightarrow$ Coefficient of $(r+1)$th term from the beginning is ${}^nC_r$

Now total number of terms in the expansion are $n+1$.

Since $(r+1)$th term from the end has $(n+1)-(r+1)$, *i.e.*, $n-r$ terms before it, at the beginning it is $(n-r+1)$th term.

$\therefore$ $(r+1)$th term from the end or t_{n-r+1} is ${}^nC_{n-r}x^{n-r}$

$\Rightarrow$ Coefficient of $(r+1)$th term from the end is ${}^nC_{n-r}={}^nC_r$

$(\because\ {}^nC_r={}^nC_{n-r})$

II. Sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n.

Proof. We know that

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$$

Putting $x=1$, we get

$$(1+1)^n=C_0+C_1+C_2+\ldots+C_n=2^n$$

III. In the expansion of $(1+x)^n$, the sum of even coefficients is equal to the sum of the odd coefficients and each is equal to 2^{n-1}

Proof. We know have

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$$

Putting $x=-1$, we get

$$C_0-C_1+C_2-C_3+C_4-\ldots=0$$

$$\Rightarrow \qquad C_0+C_2+C_4+\ldots=C_1+C_3+C_5+\ldots$$

$\Rightarrow$ Sum of odd coefficients = Sum of even coefficients

Also, the sum of each $=\dfrac{2^n}{2}=2^{n-1}$.

IV. The sum of squares of the coefficients in the expansion of $(1+x)^n$

$$\frac{(2n)\,!}{(n\,!)^2}$$

Proof. We know that

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_{n-1}x^{n-1}+C_nx^n \qquad \ldots(1)$$

Also $$(x+1)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_{n-1}x+C_n \qquad \ldots(2)$$

Multiplying (1) and (2), we get

$$(1+x)^{2n}=[C_0+C_1x+C_2x^2+\ldots+C_{n-1}x^{n-1}+C_nx^n] \times[C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_{n-1}x+C_n]$$

Equating the coefficients of x^n on both sides, we get

$${}^{2n}C_n=C_0^2+C_1^2+C_2^2+\ldots+C^2_{n-1}+C_n^2$$

$\therefore$ Sum of the squares of binomial coefficients is

$${}^{2n}C_n=\frac{(2n)\,!}{n\,!\,(2n-n)\,!}=\frac{2n\,!}{(n\,!)^2}$$

Remark. Equating coefficients of x^{n+1} on both sides, we get

$${}^{2n}C_{n+1}=C_0C_1+C_1C_2+C_2C_3+\ldots+C_{n-1}C_n$$

$$\Rightarrow \quad C_0C_1+C_1C_2+C_2C_3+\ldots+C_{n-1}C_n=\frac{(2n)\,!}{(n+1)\,!\,(n-1)\,!}$$

Example 10. *If* $C_0, C_1, C_2, \ldots, C_n$ *denote the coefficients of the expansion of* $(1+x)^n$, *prove that*

(a) $C_1+2C_2+3C_3+\ldots+n.C_n=n.2^{n-1}$

(b) $C_0+2C_1+3C_2+\ldots+(n+1).C_n=(n+2).2^{n-1}$

(c) $C_0+3C_1+5C_2+\ldots+(2n+1).C_n=(n+1)2^n$.

Solution. (a) L.H.S. $=C_1+2C_2+3C_3+\ldots+n.C_n$

$$=n+2.\ \frac{n(n-1)}{2}+3\ .\ \frac{n(n-1)(n-2)}{3.2.1}+\ldots+n$$

$$=n\left[\ 1+(n-1)+\frac{(n-1)(n-2)}{2}+\ldots+1\ \right]$$

$$=n[{}^{n-1}C_0+{}^{n-1}C_1+{}^{n-1}C_2+\ldots+{}^{n-1}C_{n-1}]$$

$$=n[(1+1)^{n-1}]=n.2^{n-1}=\text{R.H.S.}$$

(b) L.H.S. $=C_0+2C_1+3C_2+4C_3+\ldots+(n+1).C_n$

$$=(C_0+C_1+C_2+\ldots+C_n)+(C_1+2.C_2+\ldots+n.C_n)$$

$$=2^n+n.2^{n-1}=2.2^{n-1}+n.2^{n-1}$$

$$=(2+n).2^{n-1}=\text{R.H.S.}$$

(c) L.H.S. $=C_0+3C_1+5C_2+\ldots+(2n+1).C_n$

$$=(C_0+C_1+C_2+\ldots+C_n)+2(C_1+2C_2+\ldots+n.C_n)$$

$$=2^n+2(n.2^{n-1})=2^n(1+n)=\text{R.H.S.}$$

Example 11. *If* $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, *prove that*

(a) $\dfrac{C_1}{C_0}+\dfrac{2C_2}{C_1}+\dfrac{3C_3}{C_2}+\ldots+\dfrac{nC_n}{C_{n-1}}=\dfrac{n(n+1)}{2}$

(b) $C_0+\dfrac{C_1}{2}+\dfrac{C_2}{3}+\ldots+\dfrac{C_n}{n+1}=\dfrac{2^{n+1}-1}{n+1}$

Solution.

(a) $$\text{L.H.S.}=\frac{n}{1}+\frac{\dfrac{2.n(n-1)}{2\,!}}{n}+\frac{\dfrac{3.n(n-1)(n-2)}{3\,!}}{\dfrac{n(n-1)}{2\,!}}+\ldots+\frac{n.1}{n}$$

$$=n+(n-1)+(n-2)+\ \ldots+1=\frac{n(n+1)}{2}=\text{R.H.S.}$$

(b) $$\text{L.H.S.}=C_0+\ \frac{C_1}{2}\ +\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}$$

$$=1+\ \frac{n}{2}+\frac{\dfrac{n(n-1)}{1.2}}{3}+\ldots+\frac{1}{n+1}$$

$$=1+\frac{n}{2}+\frac{n(n-1)}{1.2.3}+\ldots+\frac{1}{n+1}$$

$$=\frac{1}{n+1}\left[(n+1)+\frac{(n+1)n}{1.2}+\frac{(n+1).n.(n-1)}{1.2.3}+\ldots\ 1\right]$$

$$=\frac{1}{n+1}\left[{}^{n+1}C_1+{}^{n+1}C_2+{}^{n+1}C_3+\ldots+{}^{n+1}C_{n+1}\right]$$

$$=\frac{1}{n+1}\left[(1+1)^{n+1}-1\right]=\frac{2^{n+1}-1}{n+1}=\text{R.H.S.}$$

Example 12. *If a_1, a_2, a_3, a_4 are the coefficients of the second, third, fourth and fifth terms respectively in the binomial expansion $(1+x)^n$, prove that*

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$$

Solution. Since $(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+{}^nC_4x^4+\ldots+{}^nC_nx^n$,

$a_1={}^nC_1,\ a_2={}^nC_2,\ a_3={}^nC_3,\ a_4={}^nC_4$

$$\text{L.H.S.}=\frac{a_1}{a_1+a_2}+\frac{a^2}{a_3+a_4}=\frac{{}^nC_1}{{}^nC_1+{}^nC_2}+\frac{{}^nC_3}{{}^nC_3+{}^nC_4}$$

$$=\frac{{}^nC_1}{{}^{n+1}C_2}+\frac{{}^nC_3}{{}^{n+1}C_4}\qquad\text{[Using the formula } {}^nC_r+{}^nC_{r-1}={}^{n+1}C_r]$$

$$=\frac{n}{\frac{(n+1)n}{2.1}}+\frac{\frac{n(n-1)(n-2)}{3.2.1}}{\frac{(n+1)n(n-1)(n-2)}{4.3.2.1}}$$

$$=\frac{2}{n+1}+\frac{4}{n+1}=\frac{6}{n+1}$$

and $\quad \text{R H.S.}=\dfrac{2a_2}{a_2+a_3}=\dfrac{2.{}^nC_2}{{}^nC_2+{}^nC_3}=\dfrac{2.{}^nC_2}{{}^{n+1}C_3}$

$$=\frac{2.\frac{n(n-1)}{2.1}}{\frac{(n+1)n(n-1)}{3.2.1}}=\frac{3.2.1}{n+1}=\frac{6}{n+1}$$

$\therefore \qquad$ L.H.S. = R.H.S.

Example 13. *Find the coefficient of x^4 in the expansion of*

$$\left(x^4+\frac{1}{x^3}\right)^{15}$$

Solution. t_{r+1} in the expansion of $\left(x^4+\dfrac{1}{x^3}\right)^{15}$

$$={}^{15}C_r(x^4)^{15-r}\left(\frac{1}{x^3}\right)^r$$

$$={}^{15}C_r x^{60-4r}.x^{-3r}$$

$$={}^{15}C_r x^{60-7r}$$

This term will contain x^4 if $60-7r=4$, *i.e.*, $r=8$

$\therefore$ The required coefficient of $x^4={}^{15}C_8=\dfrac{15\,!}{8\,!\,7\,!}$

Example 14. *Find whether there is any term containing x^9 in the expansion of*

$$\left(2x^2-\frac{1}{x}\right)^{20}$$

Solution. t_{r+1} in the expansion of $\left(2x^2-\dfrac{1}{x}\right)^{20}$

$$={}^{20}C_r\,(2x^2)^{20-r}\left(-\frac{1}{x}\right)^r$$

$$=(-1)^r\,{}^{20}C_r(2)^{20-r}x^{40-3r}$$

The term will contain x^9 if $40-3r=9$, *i.e.*, $r=\frac{31}{3}$ which is not possible because r must be a positive integer. Hence there is no term containing x^9 in the given expansion.

Example 15. *Find the term independent of x in the expansion of*

$$\left(\frac{3}{5}x^2-\frac{1}{2x}\right)^9$$

Solution. t_{r+1} in the expansion of $\left(\dfrac{3}{5}x^2-\dfrac{1}{2x}\right)^9$

$$={}^9C_r\left(\frac{3}{5}x^2\right)^{9-r}\left(-\frac{1}{2x}\right)^r$$

$$={}^9C_r\left(\frac{3}{5}\right)^{9-r}(x^2)^{9-r}(-1)^r(2x)^{-r}$$

$$={}^9C_r\left(\frac{3}{5}\right)^{9-r}x^{18-2r}\,(-1)^r\,2^{-r}x^{-r}$$

$$={}^9C_r\left(\frac{3}{5}\right)^{9-r}(-1)^r\,2^{-r}x^{18-3r}$$

This term will be independent of x if

$18-3r=0$, *i.e.*, if $r=6$

$\therefore$ The required term independent of x

$$=(-1)^6.{}^9C_6\left(\frac{3}{5}\right)^{9-6}2^{-6}$$

$$=\frac{9\,!}{6\,!\,3\,!}\cdot\frac{3^3}{5^3}\cdot\frac{1}{2^6}=\frac{2268}{8000}$$

Example 16. *Find the term independent of x (or the constant term or the absolute term) in the expansion of*

$$\left(x-\frac{1}{x^2}\right)^{3n}$$

Solution. $t_{r+1}={}^{3n}C_r x^{3n-r}\left(-\frac{1}{x^2}\right)^r$

$$={}^{3n}C_r\, x^{3n-r}\frac{(-1)^r}{x^{2r}}=(-1)^r.{}^{3n}C_r\, x^{3n-3r}$$

This term will be independent of x, if $3n-3r=0$, *i.e.*, $r=n$.

$\therefore$ Required term $=(-1)^n.{}^{3n}C_n=(-1)^n.\frac{(3n)\,!}{n\,!\,(2n)\,!}$

Example 17. *Find the term independent of x in the expansion of*

$$\left(2x-\frac{3}{x^2}\right)^{15}$$ [*C.A. Entrance, June, 1984*]

Solution. Let us assume that $(r+1)$th term be independent of x in the expansion of $\left(2x-\frac{3}{x^2}\right)^{15}$

$$\therefore \quad t_{r+1}={}^{15}C_r(2x)^{15-r}\left(-\frac{3}{x^2}\right)^r$$
$$={}^{15}C_r.2^{15-r}.x^{15-r}.(-3)^r.x^{-2r}$$
$$={}^{15}C_r.2^{15-r}.(-3)^r.x^{15-3r}$$

Since this term is independent of x, we must have

$$15-3r=0$$

or $$r=5$$

Hence the 6th term is independent of x and its value is given by

$$t_6={}^{15}C_5.2^{10}.(-3)^5=-{}^{15}C_5.2^{10}.3^5$$

Example 18. (*a*) *If the 21st and 22nd terms in the expansion of $(1+x)^{44}$ are equal, find the value of x.*

(*b*) *In the expansion of $(1+x)^{11}$, the fifth term is 24 times the third term. Find the value of x.*

Solution. (*a*) $(r+1)$th term in the expansion of $(1+x)^{44}$

$$t_{r+1}={}^{44}C_r\, x^r$$

$\therefore$ $t_{21}={}^{44}C_{20}x^{20}$ and $t_{22}={}^{44}C_{21}x^{21}$

Now $t_{21}=t_{22}$ (given)

$\Rightarrow$ ${}^{44}C_{20}x^{20}={}^{44}C_{21}x^{21}$

$\Rightarrow$ $$x=\frac{{}^{44}C_{20}}{{}^{44}C_{21}}=\frac{44\,!}{24\,!\,20\,!}\times\frac{23\,!\,21\,!}{44\,!}$$

$$=\frac{23\,!\times 21\times 20\,!}{24\times 23\,!\times 20\,!}=\frac{21}{24}=\frac{7}{8}$$

(b) $t_{r+1}={}^{11}C_r x^r$

$\therefore$ $t_5={}^{21}C_4x^1$ and $t_3={}^{11}C_2x^2$

We are given $t_5=24t_3$

$\Rightarrow$ $${}^{11}C_4x^4=24.{}^{11}C_2x^2$$

$\Rightarrow$ $$x^2=24\times\frac{{}^{11}C_2}{{}^{11}C_4}=24\times\frac{11\,!}{2\,!\,9\,!}\times\frac{4\,!\,7\,!}{11\,!}$$

$$=24\times\frac{4\times 3\times 2\,!\times 7\,!}{2\,!\times 9\times 8\times 7\,!}=24\times\frac{12}{72}=4$$

$\therefore$ $x=\pm 2$

Example 19. (a) *If the coefficient of x in the expansion of* $\left(x^2+\frac{k}{x}\right)^5$ *is 270, find k.*

(b) *If the absolute term in the expansion of* $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ *is 405, find the value of k.*

Solution. (a) t_{r+1} in the expansion of $\left(x^2+\frac{k}{x}\right)^5$

$$={}^5C_r(x^2)^{5-r}\left(\frac{k}{x}\right)^r={}^5C_r\,x^{10-2r}\,k^r\,x^{-r}$$

$$={}^5C_rk^rx^{10-3r}$$

This term will contain x if $10-3r=1$, *i.e.* if $r=3$.

Now coefficient of x is given to be 270.

$\therefore$ $${}^5C_3k^3=270,\ i.e.,\ k^3=\frac{270}{{}^5C_3}=\frac{270}{10}=27$$

$\Rightarrow$ $$k=3$$

(b) t_{r+1} term in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$

$$={}^{10}C_r\,(\sqrt{x})^{10-r}\left(-\frac{k}{x^2}\right)^r$$

$$={}^{10}C_r\,(-1)^r\,k^r\,x^{5-r/2}\,x^{-2r}$$

$$=(-1)^r\,{}^{10}C_r\,k^r.x^{\frac{10-5r}{2}}$$

This term will be independent of x, *i.e.*, the power of x will be zero if

$$\frac{10-5r}{2}=0,\ i.e.,\ \text{if } r=2$$

Thus the third term is independent of x, and its value is

$$t_3=(-1)^2\,{}^{10}C_2k^2$$

and we are given

$$(-1)^2\cdot{}^{10}C_2k^2=405, \text{ i.e., } k^2=\frac{405}{{}^{10}C_2}=\frac{405}{45}=9$$

$\therefore$ $$k=\pm 3$$

Example 20. *If* x^r occurs ^{r} *in the expansion of* $\left(x+\frac{1}{x^2}\right)^{2n}$, *prove that its coefficient is*

$$\frac{(2n)!}{\{\frac{1}{3}(2n-r)\}!\,\{\frac{1}{3}(4n+r)\}!}$$

Solution. In the expansion of $\left(x+\frac{1}{x^2}\right)^{2n}$,

$$t_{p+1}={}^{2n}C_p x^{2n-p}\left(\frac{1}{x^2}\right)^p={}^{2n}C_p x^{2n-3p} \qquad \ldots(1)$$

In (1), the power of x will be r if

$$2n-3p=r \Rightarrow p=\tfrac{1}{3}(2n-r)$$

Substituting this value of p in the coefficient of (1), we have

$$\text{Required coefficient}={}^{2n}C_{\frac{1}{3}(2n-r)}$$

$$=\frac{(2n)!}{\frac{1}{3}\{(2n-r)\}!\,2n-\frac{1}{3}\{(2n-r)\}!}$$

$$=\frac{(2n)!}{\{\frac{1}{3}(2n-r)\}!\,\{\frac{1}{3}(4n+r)\}!}$$

Example 21. *In the expansion of* $(1+x)^{10}$, *the coefficient of* $(2r+1)$*th term is equal to the coefficient of* $(4r+5)$*th term. Find* r.

Solution. $(2r+1)$ term $=t_{2r+1}={}^{10}C_{2r}\,x^{2r}$

$(4r+5)$th term $=t_{4r+5}=t_{(4r+4)+1}={}^{10}C_{4r+4}\,x^{4r+4}$

Coeff. of $t_{2r+1}={}^{10}C_{2r}$

Coeff. of $t_{4r+5}={}^{10}C_{4r+4}$

Now we are given ${}^{10}C_{2r}={}^{10}C_{4r+4}$

$\therefore$ $$2r+(4r+4)=10^*$$

Hence $$r=1$$

Example 22. *Find the eoefficient of* x^{32} *in the expansion of* $\left(x^4-\frac{1}{x^3}\right)^{15}$.

[*C.A. Entrance December 1983*]

* $\because$ If ${}^nC_r={}^nC_{n-r}$, then $r+n-r=n$

Solution. Let us assume that x^{32} occurs in the $(r+1)$th term of the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$.

$$\therefore \quad t_{r+1}={}^{15}C_r \cdot (x^4)^{15-r} \cdot \left(-\frac{1}{x^3}\right)^r$$

$$={}^{15}C_r \cdot x^{60-4r} \cdot (-1)^r \cdot x^{-3r}={}^{15}C_r \cdot (-1)^r \cdot x^{60-7r}$$

Since x^{32} is in the $(r+1)$th term, we must have

$$60-7r=32$$
$$7r=28$$
$$r=4$$

$$\therefore \quad t_5={}^{15}C_4\,(-1)^4 \cdot x^{32}={}^{15}C_4 \cdot x^{32}=1365 \cdot x^{32}$$

$\therefore$ Coefficient of x^{32} is 1365.

Example 23. *Find the coefficient of x^{11} in the expansion of $(1-2x+3x^2)(1+x)^{11}$.* [*C.A. Intermediate November 1982*]

Solution. Using binomial theorem, we have

$$(1-2x+3x^2)\,(1+x)^{11}$$
$$=(1-2x+3x^2)\,(1+{}^{11}C_1x+{}^{11}C_2x^2+{}^{11}C_3x^3+\ldots\ldots+$$
$$ {}^{11}C_9x^9+{}^{11}C_9x^9+{}^{11}C_{10}x^{10}+{}^{11}C_{11}x^{11})$$

$\therefore$ Coefficient of x^{11} in the expansion of

$$(1-2x+3x^2)\,(1+x)^{11}$$
$$=1\times{}^{11}C_{11}-2\times{}^{11}C_{10}+3\times{}^{11}C_9$$
$$=1-2\times{}^{11}C_1+3\times{}^{11}C_2 \qquad [\because \ {}^nC_r={}^nC_{n-r}]$$
$$=1-22+165=144.$$

Example 24. *Find the coefficient of x^4 in the expansion of $(1+x-2x^2)^6$. If the complete expansion of the expression is given by*

$$1+a_1x+a_2x^2+\ldots+a_{12}x^{12},$$

prove that

$$a_2+a_4+a_6+\ldots+a_{12}=31$$

Solution. (*i*) $(1+x-2x^2)^6=[1+x(1-2x)]^6$

$$=1+{}^6C_1\,x(1-2x)+{}^6C_2x^2(1-2x)^2+{}^6C_3x^3(1-2x)^3$$
$$+{}^6C_4x^4(1-2x)^4+\ldots$$
$$=1+{}^6C_1\,x(1-2x)+{}^6C_2x^2(1-4x+4x^2)$$
$$+{}^6C_3x^3(1-6x+\ldots)+{}^6C_4x^4(1-\ldots)+\ldots$$

Coefficient of $x^4=4.\,{}^6C_2-6.\,{}^6C_3+{}^6C_4$

$$=60-120+15=-45$$

(*ii*) Now $(1+x-2x^2)^6=1+a_1x+a_2x^2+\ldots+a_{12}x^{12}$

Putting $x=1$, we have

$$1+a_1+a_2+\ldots+a_{12}=0 \quad \ldots(1)$$

Putting $x=-1$, we have

$$1-a_1+a_2-\ldots+a_{12}=64 \quad \ldots(2)$$

Adding (1) and (2), we get

$$2(1+a_2+a_4+\ldots+a_{12})=64 \quad \Rightarrow \quad a_2+a_4+\ldots+a_{12}=31$$

Example 25. *If the 2nd, 3rd and 4th terms in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, find x, a and n.*

Solution. $(x+a)^n={}^nC_0x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+{}^nC_3x^{n-3}a^3+\ldots$

$$=x^n+nx^{n-1}a+\frac{n(n-1)}{2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{6}x^{n-3}a^3+\ldots$$

Equating 2nd, 3rd and 4th terms, we get

$$nx^{n-1}a=240 \quad \ldots(1)$$

$$\frac{n(n-1)}{2}x^{n-2}a^2=720 \quad \ldots(2)$$

$$\frac{n(n-1)(n-2)}{6}x^{n-3}a^3=1080 \quad \ldots(3)$$

Multiplying (1) and (3), we get

$$\frac{n^2(n-1)(n-2)}{6}x^{2n-4}a^4=240\times 1080 \quad \ldots(4)$$

Squaring (2), we get

$$\frac{n^2(n-1)^2}{4}x^{2n-4}a^4=720\times 720 \quad \ldots(5)$$

Dividing (4) by (5), we get

$$\frac{n^2(n-1)(n-2)\,x^{2n-4}a^4}{6}\times\frac{4}{n^2(n-1)^2x^{2n-4}a^4}=\frac{240\times 1080}{720\times 720}$$

$$\Rightarrow \quad \frac{4(n-2)}{6(n-1)}=\frac{1}{2}$$

$$\Rightarrow \quad 8n-16=6n-6$$

$$\Rightarrow \quad n=5$$

Substituting the value of n in (1) and (2), we get

$$5x^4a=240 \quad \ldots(6)$$

$$10x^3a^2=720 \quad \ldots(7)$$

Divide square of (6) by (7), we get

$$\frac{25x^8a^2}{10x^3a^2}=\frac{240\times 240}{720}$$

$\Rightarrow \qquad x^5=32$, *i.e.*, $x=2$

Substituting values of x and n in (1), we get

$$5\times 2^{5-1}\times a=240 \Rightarrow a=\frac{240}{5\times 16}=3$$

Hence $\qquad x=2,\ a=3$ and $n=5$

EXERCISE (II)

1. Evaluate the following :

$$^{11}C_0+{}^{11}C_1+{}^{11}C_2+\dots+{}^{11}C_{11}$$

2. If $(1+x)^n=C_0+C_1x+C_2x^2+\dots+C_nx^n$, prove the following :

(*a*) $C_0C_2+C_1C_3+C_2C_4+\dots+C_{n-2}C_n=\dfrac{(2n)!}{(n+2)!\,(n-2)!}$

(*b*) $C_0-2C_1+3C_2-\dots+(-1)^n(n+1)C_n=0$

(*c*) $2C_0+\dfrac{2^2C_1}{2}+\dfrac{2^3C_2}{3}+\dots\dots+\dfrac{2^{n+1}C_n}{n+1}=\dfrac{3^{n+1}-1}{n+1}$

(*d*) $C_0-\dfrac{C_1}{2}+\dfrac{C_2}{3}-\dots+(-1)^n\dfrac{C_n}{n+1}=\dfrac{1}{n+1}$

3. Show that

$$({}^nC_0+{}^nC_1+{}^nC_2+\dots+{}^nC_n)^2={}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+\dots+{}^{2n}C_{2n}.$$

4. If P be the sum of odd terms and Q the sum of even terms in the expansion of $(x+a)^n$, prove that

(*a*) $P^2-Q^2=(x^2-a^2)^n$,

(*b*) $4PQ=(x+a)^{2n}-(x-a)^{2n}$

5. Find the middle term in the expansion of $\left(\dfrac{a}{x}+bx\right)^{12}$ in ascending powers of x.

6. Find the middle term of

(*i*) $\left(\dfrac{a}{x}-\dfrac{x}{a}\right)^{10}$, (*ii*) $(1+x)^{2n}$.

7. Find the two middle terms in the expansion of

(*i*) $\left(3x-\dfrac{x^3}{6}\right)^9$ (*ii*) $\left(\dfrac{2x}{3}-\dfrac{3y}{2}\right)^9$.

8. Expand $\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^{2n+1}$ giving, in particular, the general term and two middle terms.

9. Find the middle term of $(1-3x+3x^2-x^3)^{2n}$.

10. If k is a real number and if the middle term in the expansion of $\left(\frac{k}{2}+2\right)^8$ is 1120, find k.

11. Find the term independent of x in

(i) $\left(2x+\frac{1}{3x^2}\right)^9$, (ii) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$, (iii) $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)^9$

12. Show that the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5\ldots(2n-1)}{n\,!}.2^n$

13. Show that there is no term independent of x in the expansion of $\left(2x^2-\frac{1}{4x}\right)^{11}$

14. Find the term containing y^3 in the expansion of $\left(2x-y^{1/2}\right)^{10}$

15. Find the coefficient of

(i) x^{18} in the expansion of $(ax^4-bx)^9$

(ii) x^{10} in the expansion of $\left(2x^3-\frac{1}{x}\right)^{20}$

(iii) x^{32} and x^{-17} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ and show that their sum is zero.

(iv) x^{-2} in the expansion of $\left(2x-\frac{1}{x^2\sqrt{3}}\right)^{10}$

16. (a) If the coefficients of x^2 and x^3 in the expansion of $(3+kx)^9$ are equal, find the value of k.

(b) If the coefficient of x^7 and x^8 in the expansion of $\left(3+\frac{x}{2}\right)^n$ are equal, find the value of n.

17. In the expansion of $(1+x)^{20}$, the coefficient of the rth term is to that of the $(r+1)$th term is in the ratio 1 : 2: Find the value of r.

18. Find the coefficient of $x^2y^3z^4$ in the expansion of $(ax-by+cz)^9$.

19. (a) Show that the coefficient of x^5 in the expansion of $(1+3x)^4(1-x)^3$ is 27.

(b) Show that the coefficient of x^5 in $(1+x)^4\,(1+x^2)^5$ is 60.

20. In the expansion of $(1+x)^{2n+1}$, the coefficient of x^r and x^{r+1} are equal, find r.

21. If x^3 occurs in the expansion of $\left(x+\frac{1}{x^3}\right)^{4n}$, show that its coefficient is

$$\frac{(4n)\,!}{\{\frac{1}{4}(4n-3p)\}\,!\,\{\frac{3}{4}(4n+p)\}\,!}$$

22. Find the value of r if the coefficients of $(2r+4)$th term and $(r-2)$th term in the expansion of $(1+x)^{18}$ are equal.

23. Write down the fourth term in the binomial expansion of the function

$$\left(px+\frac{q}{x}\right)^n$$

(*i*) If this term is independent of x, find the value of n.

(*ii*) With this value of n, calculate the values of p and q given that the fourth term is equal to 160, both p and q are positive and $p-q=1$.

24. (*a*) The first three terms in the expansion of $(a+b)^n$ are 1, 14 and 84 respectively. Determine a, b and n.

(*b*) The first three terms in the expansion of the power of the binomial are 625, 3500, 7350 respectively. Find them.

25. If in the expansion of $(1+x)^n$, the fifth term be four times the fourth term and the fourth term be six times the third term. Find the value of n and x.

ANSWERS

1. 2080

3. [**Hint.** $({}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_nx^n)^2$
$=[(1+x)^n]^2=(1+x)^{2n}$
$={}^{2n}C_0+{}^{2n}C_1x+{}^{2n}C_2x^2+\ldots+{}^{2n}C_nx^n$

Put $x=1$ on both sides.]

4. [**Hint.** We have

$$(x+a)^n={}^nC_0x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+{}^nC_3x^{n-3}a^3+\ldots$$

$$\Rightarrow \quad (x+a)^n=({}^nC_0x^n+{}^nC_2x^{n-2}a^2+\ldots)+({}^nC_1x^{n-1}a+{}^nC_3x^{n-3}a^3+\ldots) \quad \ldots(1)$$

Changing a to $-a$, we get

$$(x-a)^n=({}^nC_0x^n+{}^nC_2x^{n-2}a^2+\ldots)-({}^nC_1x^{n-1}a+{}^nC_3x^{n-3}a^3+\ldots) \quad \ldots(2)$$

$$\therefore \quad (x+a)^n=P+Q \quad \ldots(3)$$

and $$(x-a)^n=P-Q \quad \ldots(4)$$

Multiplying (3) and (4), in terms of P and Q

$$(x^2-a^2)^n=P^2-Q^2$$

Squaring and subtracting (3) and (4), we get

$$(x+a)^{2n}-(x-a)^{2n}=4PQ]$$

5. $924\,a^6b^6$. 6. (*i*) -252, (*ii*) $\frac{2n\,!}{(n!)^2}\,x$

7. (*i*) $\frac{189}{8}x^{17}$, $-\frac{21}{16}x^{19}$, (*ii*) $85x^5\,y^4$, $-189x^6\,y^5$

8. ${}^{2n+1}C_2\left(\frac{x}{y}\right)$, ${}^{2n+1}C_{n+1}\left(\frac{y}{x}\right)$

9. [**Hint.** $(1-3x+3x^2-x^3)^{2n}=[(1-x)^3]^{2n}=(1-x)^{6n}$

There are $6n+1$ terms so $(3n+1)$th term is the middle term.]

10. $k=\pm1$ 11. (*i*) 4th term, (*ii*) 6th, $\frac{17}{18}$, (*iii*) 6th term, 2268.

14. 7th term, $1920\,x^4y^3$ 15. (*i*) ${}^9C_3\,a^3\,b^6$, (*ii*) ${}^{20}C_{10}2^{10}$,

(*iii*) ${}^{15}C_4$, $-{}^{15}C_{11}$, (*iv*) $\frac{4480}{3}$

16. (*a*) $k=\frac{9}{7}$, (*b*) $n=55$. 17. $\frac{{}^{20}C_{r-1}}{{}^{20}C_r}=\frac{1}{2} \Rightarrow r=7$.

18. [**Hint.** Grouping the term as

$$\{(ax-by)+cz\}^9$$

The term with z^4 will be ${}^9C_4(ax-by)^5(cz)^4$.

The term containing x^2y^3 will be ${}^5C_3\,(ax)^2\,(-by)^3$

$$=-\frac{5\times4\times3}{3\times2\times1}\,a^2x^2b^3y^3,$$

Hence the coefficient of $x^2y^3z^4$ is

$${}^9C_4(-10a^2b^3)\,c^4=-1260\,a^2b^3c^4]$$

19. (*a*) [**Hint.** $(1+3x)^4(1-x)^3=(1+12x+54x^2+108x^3+81x^4)$
$\times(1-3x-3x^2-x^3)$.

Multiplying the two factors on the R.H.S. and collecting the term in x^5 in the product, we get the coefficient of x^5

$$=(81)(-3)+(108)(3)+54(-1)=27$$

20. [**Hint** ${}^{2n+1}C_r={}^{2n+1}C_{r+1}$

$\Rightarrow \qquad r+r+1=2n+1 \quad \Rightarrow \quad r=n$.

22. $r=6$. 23. (*i*) $n=6$, (*ii*) $p=2$, $q=1$.

24. (*a*) $a=1$, $b=2$, $n=7$, (*b*) $(5+7)^4$. 25. $n=11$, $x=2$

10·4. BINOMIAL THEOREM WITH ANY INDEX

We illustrate these with reference to the simplest form of expressions $(1+x)^{1/n}$ and $(1-x)^{-n}$ from which other binomials can be reduced. The two such expansions are (*i*) with a fractional index and (*ii*) with a negative index, *e.g.*,

$$(1+x)^{1/2}=\sqrt{1+x}=1+\frac{1}{2}\,x-\frac{1}{8}\,x^2+\frac{1}{19}\,x^3-\ldots$$

$$(1-x)^{-2}=\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\ldots$$

In each of these series the number of terms are infinite. The general form of expansion which can be used in the first case is

$$1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots$$

If we put $\frac{1}{2}$ for n, in that case the above expansion will take the following form :

$$1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-1\right)}{1.2.3}x^3+\ldots$$

Now, if the index is negative, the general form of the expansion is

$$1+(-n)x+\frac{(-n)(-n-1)}{1.2}x^2+\frac{(-n)(-n-1)(-n-2)}{1.2.3}x^3+\ldots$$

Example 26. *Expand $(1-x)^{3/2}$ up to four terms.*

Solution. $$(1-x)^{3/2}=1+\frac{3}{2}(-x)+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1.2}(-x)^2$$

$$+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1.2.3}(-x)^3+\ldots$$

$$=1-\frac{3}{2}x+\frac{3}{8}x^2+\frac{1}{16}x^3+\ldots$$

Example 27. *Expand $(2+3x)^{-4}$ up to four terms.*

Solution. $$(2+3x)^{-4}=2^{-4}\left(1+\frac{3x}{2}\right)^{-4}$$

$$=\frac{1}{2^4}\left[1+(-4)\left(\frac{3x}{2}\right)+\frac{(-4)(-5)}{1.2}\left(\frac{3x}{2}\right)^2\right.$$

$$\left.+\frac{(-4)(-5)(-6)}{1.2.3}\left(\frac{3x}{2}\right)^3+\ldots\right]$$

$$=\frac{1}{16}\left(1-6x+\frac{45}{2}x^2-\frac{135}{2}x^3+\ldots\right)$$

Remark. The general term of the expansion can be obtained by the formula :

$$(r+1)\text{th term}=\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r\,!}x^r.$$

Example 28. *Find the general term of the expansion of $(1-x)^{-3}$.*

Solution. The $(r+1)$th term$=\frac{(-3)(-4)(-5)\ldots(-3-r+1)}{r\,!}(-x)^r$

$$=(-1)^r\frac{3.4.5\ldots.(r+2)}{r\,!}(-1)^r x^r$$

$$=(-1)^{2r}\frac{3.4.5\ldots(r+2)}{1.2.3\ldots r}x^r$$

$$=\frac{(r+1)(r+2)}{1.2}\,x^r$$

Remark. Students should remember the following expansions :

$$(1-x)^{-1}=1+x+x^2+x^3+\ldots+x^r+\ldots$$
$$(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots+(r+1)x^r+\ldots$$
$$(1-x)^{-3}=1+3x+6x^3+10x^3+\ldots+\frac{(r+1)(r+2)}{1.2}x^r+\ldots$$

Example 29. *Using Binomial theorem, find the value of $\sqrt[3]{126}$ to four decimal places.*

Solution. $\sqrt[3]{126}=(126)^{\frac{1}{3}}=(125+1)^{\frac{1}{3}}$

$$=(125)^{\frac{1}{3}}\left[1+\frac{1}{125}\right]^{\frac{1}{3}}=5[1+0{\cdot}008]^{\frac{1}{3}}$$

$$=5\left[1+\frac{1}{3}\times0{\cdot}008+\frac{\frac{1}{3}(-\frac{2}{3})}{2.1}\times(0{\cdot}008)^2\right]$$

(neglecting the other terms)

$\therefore$ $\quad\sqrt[3]{125}=5[1+0{\cdot}002666-0{\cdot}00000711]=5(1{\cdot}002659]$

$$=5{\cdot}013295=5{\cdot}0133$$

(correct up to four places of decimal)

Example 30. *Find the value of $(630)^{1/4}$ correct to five places of decimal.*

Solution. The number nearest to 630 which is the fourth power of whole number 25 is 625. Thus

$$(630)^{\frac{1}{4}}=(625+5)^{\frac{1}{4}}=(625)^{\frac{1}{4}}\left[1+\frac{5}{625}\right]^{\frac{1}{4}}$$

$$=5(1+0{\cdot}008)^{\frac{1}{4}}$$

$$=5\left[1+\frac{1}{4}(0{\cdot}008)+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2\,!}(0{\cdot}008)^2+\ldots\right]$$

$$=5\left[1+0{\cdot}002-\frac{3}{32}(0{\cdot}000064)+\ldots\right]$$

$$=5[1{\cdot}002-0{\cdot}00006]=5\times1{\cdot}001994$$

$=5{\cdot}009970$, correct to five decimal places.

Example 31. *Extract the fifth root of 244 correct to the three places of decimal.*

Solution. $(244)^{\frac{1}{5}}=(243+1)^{\frac{1}{5}}=\left[243\left(1+\frac{1}{243}\right)\right]^{\frac{1}{5}}$

$$=3\left[1+\frac{1}{3^5}\right]^{\frac{1}{5}}$$

$$=3\left[1+\frac{1}{5}\cdot\frac{1}{3^5}+\frac{\frac{1}{5}\left(-\frac{4}{5}\right)}{2!}\cdot\left(\frac{1}{3^5}\right)^2+\ldots\right]$$

$$=3\left[1+\frac{1}{5}\cdot\frac{1}{3^5}-\frac{2}{25}\cdot\frac{1}{3^{10}}+\ldots\right]$$

$$=3+\frac{1}{5}\cdot\frac{1}{3^4}-\frac{8}{100}\cdot\frac{1}{3^9}+\ldots$$

$$=3+\frac{1}{5}(0{\cdot}01234)-\frac{8}{100}(0{\cdot}00005)$$

$$=3+0{\cdot}00247-0{\cdot}000004=3{\cdot}002466$$

Example 32. *Find the coefficient of x^{10} in the expansion of*

$$\frac{1+2x}{(1-2x)^2}.$$

Solution. $\frac{1+2x}{(1-2x)^2}=(1+2x)(1-2x)^{-2}$

$$=(1+2x)[1+2(2x)+3(2x)^2+\ldots+10(2x)^9+11(2x)^{10}+\ldots]$$

$$=(1+2x)[1+4x+12x^2+\ldots+10.2^9.x^9+11.2^{10}.x^{10}+\ldots]$$

$\therefore$ Required coefficient $=11.2^{10}+2.10.2^9=21504$

Example 33. *Find the coefficient of x^n in the expansion of $(1+x+x^2+x^3+x^4+\ldots\ldots)^{2/3}$ where $|x|<1$.*

[*I.C.W.A. June, 1990*]

Solution. We have

$$(1+x+x^2+x^3+x^4+\ldots\ldots)^{2/3}$$

$$=\left(\frac{1}{1-x}\right)^{2/3}=\left\{(1-x)^{-1}\right\}^{2/3}=(1-x)^{-2/3}=\left\{1+(-x)\right\}^{-2/3}$$

$$=1+\left(-\frac{2}{3}\right)(-x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{2!}(-x)^2$$

$$+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{3!}(-x)^3$$

$$+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)\left(-\frac{2}{3}-3\right)}{4!}(-x)^4+\ldots\ldots$$

$$=1+\frac{\left(\frac{2}{3}\right)}{1\,!}x+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{2\,!}x^2+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)}{3\,!}x^3$$
$$+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)}{4\,!}x^4+\ldots\ldots$$

$\therefore$ Coefficient of x^n in the expansion of

$(1+x+x^2+x^3+x^4+\ldots\ldots)^{2/3}$ is give by

$$\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)\ldots\ldots\text{to } n \text{ factors}}{n\,!}$$

$$=\frac{2.5.8.11\ldots\ldots(3n-1)}{3^n\, n\,!}$$

Example 34. *If x is very small compared to 1, prove that*

$$\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{(1+x)+\sqrt{1+x}}=1-\frac{5x}{6}\ \textit{nearly}$$

Solution. The given expansion

$$=\frac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}}{(1+x)+(1+x)^{\frac{1}{2}}}=\frac{\left(1+\frac{1}{2}x\right)+\left(1-\frac{2}{3}x\right)}{(1+x)+\left(1+\frac{1}{2}x\right)},$$

(neglecting the other terms)

$$=\frac{2-\frac{1}{6}x}{2+\frac{3}{2}x}=\frac{1-\frac{1}{12}x}{1+\frac{3}{4}x}=\left(1-\frac{1}{12}x\right)\left(1+\frac{3}{4}x\right)^{-1}$$

$=\left(1-\frac{1}{12}x\right)\left(1-\frac{3}{4}x\right)$, omitting other terms

$$=1-\frac{1}{12}x-\frac{3}{4}x=1-\frac{5}{6}x$$

Example 35. *Find the first three terms of*

$$1/(1+x)^2\sqrt{1+4x}$$

assuming the validity of the expansion in ascending powers of x.

Solution. Given expansion $=(1+x)^{-2}(1+4x)^{-1/2}$

$$=\left[1+\frac{(-2)x}{1\,!}+\frac{(-2)(-2-1)}{2\,!}x^2+\ldots\right]$$
$$\times\left[1+\frac{\left(-\frac{1}{2}\right)4x}{1\,!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2\,!}(4x^2)+\ldots\right]$$

$= (1-2x+3x^2+...)(1-2x+6x^2+...),$

ignoring higher powers of x

$= 1-4x+13x^2+..............$

EXERCISE (III)

1. Express $(4+3x)^{\frac{1}{2}} - \left(1-\frac{1}{2}x\right)^{-2}$

as a series of ascending powers of x up to and including the term in x^2.

2. Expand $\frac{1}{\sqrt{4-3x^2}}$ up to the term containing x^6 when $x < \frac{\sqrt{3}}{2}$.

3. Find the coefficient of x^8 in the expansion of $\frac{3x^2-2}{x+x^2}$.

4. Find the coefficient of x^7 in the expansion of $(1-x+x^2)^{-3}$.

5. Product in thousand kilograms of a certain firm in first, second, third, etc. weeks is the same as the coefficient of first, second, third, etc., powers of x in the expansion of $(1+x)(1-x)^{-2}$. Find the production in the sixth week.

6. If x is small, show that

$$\frac{(1-2x)^{-\frac{1}{2}} - (1+2x)^{\frac{1}{2}}}{(1-x)^{-\frac{1}{2}} - (1+x)^{\frac{1}{2}}} = 4+2x \text{ (approx.)}$$

7. Evaluate the following :

(*i*) $(1{\cdot}03)^{1/3}$ to 4 places of decimal.

(*ii*) $(998)^{1/8}$ to 5 places of decimal.

8. Evaluate $\frac{1}{\sqrt[3]{128}}$ up to 4 decimal places.

ANSWERS

1. $1-\frac{x}{4}-\frac{57x^2}{64}$. 3. -1; 4. 18, 5. 13, 8. 0·22046.

10·5. SUMMATION OF A SERIES

We can find approximate value of a series if it is on lines of the expansion of the binomial. We know $(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+...$

If x is small compared to 1, we find that term which get smaller and smaller and we take 1 as the first approximation of the value of $(1+x)^n$ and $(1+nx)$ as a second approximation etc. The following examples will illustrate.

Example 36. *Find the sum of the infinite series :*

$$1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2.5}{3.6}\cdot\frac{1}{2^2}+\frac{2.5.8}{3.6.9}\cdot\frac{1}{2^3}+\ldots$$

Solution. Comparing the given series with the expansion

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\ldots, \text{ we have}$$

$$nx=\frac{2}{3}\cdot\frac{1}{2} \quad \ldots(1)$$

$$\frac{n(n-1)}{1.2}x^2=\frac{2.5}{3.6}\cdot\frac{1}{2^3} \quad \ldots(2)$$

From (1), we get $x=\frac{1}{2n}$

Substituting the value of x in (2), we have

$$\frac{n(n-1)}{2}\cdot\frac{1}{9n^2}=\frac{2.5}{3.6}\cdot\frac{1}{2^2}$$

$$\Rightarrow \quad \frac{n-1}{n}=\frac{5}{2}, \text{ i.e., } 2n-2=5n$$

$$\Rightarrow \quad n=-\frac{2}{3}$$

and from (1), we get $x=-\frac{1}{2}$.

Hence the given series $=(1+x)^n=\left(1-\frac{1}{2}\right)^{-2/3}$,

$$=\left(\frac{1}{2}\right)^{-2/3}=2^{2/3}$$

Example 37. *Sum the following series :*

$$2+\frac{5}{2\,!\,3}+\frac{5.7}{3\,!\,3^2}+\frac{5.7.9}{4\,!\,3^3}+\ldots.$$

Solution. The given series is

$$\left[1+1+\frac{5}{2\,!}\cdot\frac{1}{3}+\frac{5.7}{3\,!}\cdot\left(\frac{1}{3}\right)^2+\ldots\right]$$

Comparing the series in the bracket with the expansion

$$(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\ldots, \text{ we have}$$

$$nx=1 \quad \ldots(1)$$

$$\frac{n(n-1)}{2}\cdot x^2=\frac{5}{2\,!}\cdot\frac{1}{3} \quad \ldots(2)$$

From (1), we get $x=\frac{1}{n}$

Subs tituting in (2), we get $\frac{n(n-1)}{2}\cdot\frac{1}{n^2}=\frac{5}{6}$

$\Rightarrow \qquad \frac{n-1}{2n}=\frac{5}{6}$, *i.e.*, $6n-6=10n$

$\Rightarrow \qquad n=-\frac{3}{2}$

From (1), we get $x=-\frac{2}{3}$

Hence the given series

$$=\left(1-\frac{2}{3}\right)^{-\frac{3}{2}}=\left(\frac{1}{3}\right)^{-\frac{3}{2}}=(3)^{3/2}=3\sqrt{3}$$

Example 38. *If* $y=\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots$

prove that $\qquad y^2+2y-2=0.$

Solution. The given series may be written as

$$y=-1+\left[1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots\right]$$

Comparing the series in the bracket with

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}.x^2+\ldots, \text{ we have}$$

$$nx=\frac{1}{3} \qquad \ldots(1)$$

and $$\frac{n(n-1)}{1.2}.x^2=\frac{1.3}{3.6} \qquad \ldots(2)$$

Solving for n and x, we get

$$n=-\frac{1}{2} \text{ and } x=-\frac{2}{3}$$

$\therefore \qquad y=(1+x)^n-1=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}-1=\sqrt{3}-1$

$\Rightarrow \qquad y+1=\sqrt{3}$

$\Rightarrow \qquad (y+1)^2=3$

$\Rightarrow \qquad y^2+2y+1=3$

$\Rightarrow \qquad y^2+2y-2=0.$

Example 39. *Sum the series :*

$$1+\frac{1}{3}x+\frac{1.4}{3.6}x^2+\frac{1.4.7}{3.6.9}x^3+\ldots$$

Solution. Let the given series be identical with

$$(1+y)^n=1+ny+\frac{n(n-1)}{2\,!}y^2+\ldots \qquad \ldots(1)$$

Equating the second and third terms in the two series, we get

$$ny=\frac{1}{3}x \qquad \ldots(2)$$

and

$$\frac{n(n-1)}{2\,!}y^2=\frac{1.4}{3.6}x^2 \qquad \ldots(3)$$

From (2), we get $y=\frac{x}{3n}$. Substituting this value of y in (3), we get

$$\frac{n(n-1)}{2}\cdot\frac{x^2}{9n^2}=\frac{4}{18}x^2$$

$$\Rightarrow \quad \frac{n-1}{n}=4$$

$$\Rightarrow \quad n-1=4n$$

$$\Rightarrow \quad n=-\frac{1}{3}$$

$$\therefore \quad y=\frac{3}{3n}=\frac{x}{3.\left(-\frac{1}{3}\right)}=-x$$

Hence the sum of the given series

$$=(1+y)^n=(1-x)^{-1/3}$$

Example 40. *If a and b are values of the second and third terms respectively in the expansion of $(1+x)^n$, prove that*

$$n=\frac{a^2}{a^2-2b} \text{ and } x=\frac{a^2-2b}{a}.$$

Solution. We know that

$$(1+x)^n=1+nx+\frac{n(n-1)}{2\,!}x^2+\ldots$$

We are given $\quad nx=a \qquad \ldots(1)$

and $\quad \frac{n(n-1)}{2\,!}x^2=b \qquad \ldots(2)$

Dividing (2) by the square of (1), we get

$$\frac{n(n-1)}{2\,!}\cdot\frac{x^2}{n^2x^2}=\frac{b}{a^2} \Rightarrow \frac{n-1}{2n}=\frac{b}{a^2}$$

$\Rightarrow \quad a^2n-a^2=2bn \quad$ or $\quad n(a^2-2b)=a^2$

$$\Rightarrow \quad n=\frac{a^2}{a^2-2b}$$

From (1), we get $x=\frac{a}{n}=\frac{a^2-2b}{a}$.

EXERCISE (IV)

1. Sum the following series :

(a) $1+\frac{3}{4}+\frac{3\cdot5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots\infty$

(b) $1+2.\ \frac{1}{3}+3.\frac{1}{3^2}+4.\frac{1}{3^3}+\ldots$

(c) $1+\frac{2}{9}+\frac{2.5}{9.18}+\frac{2.5.8}{9.18.27}+\ldots$

(d) $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2.5}{3.6}\cdot\frac{1}{2^2}+\frac{2.5.8}{3.6.9}\cdot\frac{1}{2^3}+\ldots$

2. Find the sum of the series to infinity :

(a) $1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+\ldots$

(b) $1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1.4}{3.6}\cdot\frac{1}{4^2}+\frac{1.4.7}{3.6.9}\cdot\frac{1}{4^3}+\ldots$

(c) $1+\frac{1}{3^2}+\frac{1.4}{1.2}\cdot\frac{1}{3^4}+\frac{1.4.7}{1.2.3}\cdot\frac{1}{3^6}+\ldots$

(d) $1+\frac{1}{5}+\frac{1.5}{2\,!}\cdot\frac{1}{5^2}+\frac{1.5.9}{3\,!}\cdot\frac{1}{5^3}+\ldots$

(e) $1-\frac{3}{16}+\frac{1.4}{1.2}\left(\frac{3}{16}\right)^2-\frac{1.4.7}{1.2.3}\cdot\left(\frac{3}{16}\right)^3+\ldots$

3. Prove that

$$\sqrt{2}=\frac{7}{5}\left[1+\frac{1}{10^2}+\frac{1.3}{1.2}\cdot\frac{1}{10^4}+\frac{1.3.5}{1.2.3}\cdot\frac{1}{10^6}+\ldots\right].$$

4. Prove that

$$\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots=0\cdot04 \text{ nearly.}$$

5. Sum the following series

$$\frac{5}{3.6}+\frac{5.7}{3.6.9}+\frac{5.7.9}{3.6.9.12}+\ldots$$

[**Hint.** The given series may be written as

$$=\frac{1}{3}\left[\frac{3.5}{3.6}+\frac{3.5.7}{3.6.9}+\ldots\right]$$

$$=\frac{1}{3}\left[\left\{1+\frac{3}{1!}\cdot\left(\frac{1}{3}\right)+\frac{3.5}{2!}\cdot\left(\frac{1}{3}\right)^2+\ldots\right\}-2\right]$$

$$=\frac{1}{3}\left[S-2\right]$$

Here $S=(1-x)^{-n}=\left(1-\frac{2}{3}\right)^{-3/2}=3^{3/2}=3\sqrt{3}$

$\therefore$ $$S_{\infty}=\frac{3\sqrt{3}-2}{3}.\ ]$$

ANSWERS

1. (*a*) $2\sqrt{2}$, (*b*) $\frac{9}{4}$, (*c*) $\left(1+\frac{8}{81}\right)^{\frac{9}{4}}$, (*d*) $4^{1/3}$

2. (*a*) $\frac{2\sqrt{2}}{3\sqrt{3}}$, (*b*) $\left(\frac{4}{3}\right)^{\frac{1}{3}}$, (*c*) $\left(\frac{2}{3}\right)^{-\frac{1}{3}}$
   (*d*) $5^{1/4}$, (*e*) $(4/5)^{2/3}$

3. [**Hint.** The series in the bracket can be proved by

$$\left(1-\frac{1}{50}\right)^{-\frac{1}{2}}=\frac{\sqrt{50}}{7}=\frac{5}{7}\sqrt{2}\ ]$$