The corresponding arithmetic series are

(*i*) $1+5+9+13+17+21+25+\ldots$

(*ii*) $4+\frac{7}{2}+3+\frac{5}{2}+2+\frac{3}{2}+1+\frac{1}{2}+0-\frac{1}{2}+\ldots$

Thus if the first term and common difference are known, the A.P. is completely known,

The arithmetic progression

$$\mathbf{a}, (\mathbf{a}+\mathbf{d}), (\mathbf{a}+2\mathbf{d}), (\mathbf{a}+3\mathbf{d}), \ldots$$

whose first term is a and the common difference is **d**, is designated as *the standard form of an arithmetic progression.*

The corresponding arithmetic series

$$\mathbf{a}+(\mathbf{a}+\mathbf{d})+(\mathbf{a}+2\mathbf{d})+(\mathbf{a}+3\mathbf{d})+\ldots$$

is designated as *the standard form of an arithmetic series.* The abbreviatio 'A.P.' for arithmetic progrssion is commonly used.

Definition. *If for a sequence* $\{u_n\}$, $u_{n+1}-u_n$ *remains constant for all natural numbers n, then the sequence is called the A.P. and the numerical difference between two consecutive, terms* n_n *and* u_{n+1}, *is called the common difference of the A.P.*

The nth term of an A.P. Let '*a*' be the first term and '*d*' be the common difference.

Then

$$\text{first term } (u_1)=a$$
$$\text{second term } (u_2)=a+d$$
$$\text{third term } (u_3)=(a+d)+d=a+2d$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\text{seventh term } (u_7)=(a+5d)+d=a+6d$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

*n*th term $(u_n)=a+(n-1)d$, which is also the term indicated by *l*.

This *n*th term is called the *general term* of the standard A.P., as by giving *n*, the values 1, 2, 3, 4, ...the successive terms of the A.P. can be obtained.

Example. 1. *Which term of series*

$$12+9+6+\ldots$$

is equal to (*i*) -30, (*ii*) -100 ?

Solution. (*i*) The series is an A.P. with first term 12 and common difference -3.

$$\therefore \qquad u_n=a+(n-1)d=12+(n-1)(-3)=15-3n$$

Suppose the *n*th term is -30 then

$$15-3n=-30 \Rightarrow n=15$$

-30 is the 15th term.

(*ii*) Now suppose that $u_n=-100$

$$15-3n=-100 \Rightarrow n=\frac{115}{3}$$

which is impossible, because *n* must be a whole number. Hence there exists no term in the series which is equal to -100.

Example. 2. *If a, b, c are the pth, qth and rth terms of an A.P., show that* $a(q-r)+b(r-p)+c(p-q)=0.$

Solution. Let the A.P. be $A,\ A+D,\ A+2D, \ldots\ldots$

Then
$$a=A+(p-1)\,D \quad \ldots(1)$$
$$b=A+(q-1)\,D \quad \ldots(2)$$
$$c=A+(r-1)\,D \quad \ldots(3)$$

Subtracting (2) from (1), we get
$$a-b=(p-q)\,D \quad \ldots(4)$$
and subtracting (3) from (2), we get
$$b-c=(q-r)D \quad \ldots(5)$$

Dividing (4) and (5), we get
$$\frac{a-b}{b-c}=\frac{p-q}{q-r}$$
$$\Rightarrow \quad (a-b)(q-r)=(b-c)(p-q)$$
$$\Rightarrow \quad (a-b)(q-r)-(b-c)(p-q)=0$$
$$\Rightarrow \quad a(q-r)-b(q-r)-b(p-q)+c(p-q)=0$$
$$\Rightarrow \quad a(q-r)-b(q-r+p-q)+c(p-q)=0$$
$$\Rightarrow \quad a(q-r)+b(r-p)+c(p-q)=0$$

Alternative Solution

$$\text{L.H.S.}=a(q-r)+b(r-p)+c(p-q)$$
$$=[A+(p-1)D](q-r)+[A+(q-1)D](r-p)+[A+(r-1)D](p-q)$$
$$=A[(q-r)+(r-p)+(p-q)]+D[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$$
$$=A\times 0+D\times 0=0=\text{R.H.S.}$$

Remark. We have taken A instead of a because a is already present in the question.

Example. 3. *The pth term of an A.P. is q and the qth term is p. Show that the rth term is* $p+q-r$ *and the* $(p+q)$*th term is zero.*

Solution. Let a be the first term and d the common difference of the given series. We are given
$$q=a+(p-1)d \quad \ldots(1)$$
$$p=a+(q-1)d \quad \ldots(2)$$

Subtracting (2) from (1), we get
$$q-p=pd-qd=(p-q)d$$
$$\Rightarrow \quad d(p-q)=-(p-q)$$
$$\Rightarrow \quad d=-1$$

Substituting the value of d in (1), we get

$$a+(p-1)(-1)=q \Rightarrow a-p+1=q$$

$$a=p+q-1$$

The rth term $=a+(r-1)d=(p+q-1)+(r-1)(-1)=p+q-r$

Also $(p+q)$th term is

$$a+(p+q-1)d=(p+q-1)+(p+q-1)(-1)=0$$

12·2. SUM OF A SERIES IN A.P.

The sum of a series in A.P. is an important quantity which yields many other related results. We denote the sum of n terms by S_n and the first and the last terms of the sequence by a and l respectively.

The formula used for finding out the sum of a series in A.P. is

$$S=\frac{n(a+l)}{2} \quad \ldots(1)$$

By substituting $\{a+(n-1)d\}$ for l, the above formula can also be written as

$$S=\frac{n}{2}\{2a+(n-1)d\} \quad \ldots(2)$$

This formula is used when the last term is not known. The proofs of (1) and (2) are as follows :

Proof. Let l denote the nth term and S_n, the sum of the first n terms of the A.P., $a, a+d, a+2d, \ldots, a+(n-1)d$, then

$$S_n=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l \quad \ldots(1)$$

Writing the series in the reverse order, we get

$$S_n=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a \quad \ldots(2)$$

Adding (1) and (2), we get

$$2S_n=(a+l)+(a+l)+(a+l)+\ldots+(a+l)+(a+l)+(a+l)=n(a+l)$$

$$\Rightarrow \qquad S_n=\frac{n}{2}(a+l)$$

Alternative Proof. We prove the same result by means of the Principle of Mathematical Induction. Let $P(n)$ denote the formula for the sum of n terms so that

$$P(n):\ a+(a+d)+(a+2d)+\ldots+\{a+(n-1)d\}$$

$$=\frac{n}{2}\Big[2a+(n-1)d\Big] \quad \ldots(1)$$

(a) By putting $n=1$ in (1), we get

L.H.S. of $(1)=a$

R.H.S. of $(1)=\frac{1}{2}[2a+(1-1)d]$

L.H.S.$=$R.H.S., *i.e.*, $P(1)$ is true.

(*b*) We prove the second part of the formula. We now show the trulth of $P(m)$ for some value m of n, namely

$$a+(a+d)+\ldots+\{a+(m-1)d\}$$
$$=\frac{m}{2}\Big[2a+(m-1)d\Big] \qquad \ldots(2)$$

implies the truth of $P(m+1)$, namely

$$a+(a+d)+\ldots+\{a+(m-1)d\}+(a+md)$$
$$=\frac{m+1}{2}\Big[2a+\{(m+1)-1\}d\Big] \qquad \ldots(3)$$

Now L.H.S. of (3) $=a+(a+d)+(a+2d)+\ldots+\{a+(m-1)d\}+(am+d)$

$$=\frac{m}{2}\Big[2a+(m-1)d\Big]+(a+md) \qquad \Big[\text{from (2)}\Big]$$
$$=ma+\frac{1}{2}(m^2-a)d+a+md$$
$$=(m+1)a+\frac{m}{2}\Big[(m-1)+2\Big]d$$
$$=(m+1)a+\frac{m}{2}(m+1)d=\frac{m+1}{2}\Big[2a+md\Big]$$
$$=\text{R.H.S. of (3)}$$

From the steps (1) and (2), we conclude that

$$P(n)=\frac{n}{2}\Big[2a+(n-1)d\Big]$$

is true for all positive integral values of n.

Example. 4. *Find the sum of the series :*

(*i*) $2+3\frac{1}{2}+5+6\frac{1}{2}+\ldots$ *to 25 term*

(*ii*) $72+70+68+\ldots+40.$

Solution. (*i*) Here the first term a is 2, the common difference d is $3\frac{1}{2}-2=1\frac{1}{2}=\frac{3}{2}$. The last term is not known while the number of terms or n is 25. We apply the formula

$$S_n=\frac{n}{2}\Big[2a+(n-1)d\Big]$$
$$\therefore \qquad S_{25}=\frac{25}{2}\Big[2\times2+(25-1)\frac{3}{2}\Big]=\frac{25}{2}\Big[4+36\Big]=500$$

(*ii*) The given series is an A.P. with $a=72$ and $d=-2$. Let n be the number of terms. Then $u_n=40$

$\therefore \qquad a+(n-1)d=40$

$\Rightarrow \qquad 72+(n-1)(-2)=40$, *i.e.*, $2n=34$ or $n=17$.

Now $\qquad S_n=\frac{n}{2}(a+l)$

$$S_{17}=\frac{17}{2}(72+40)=952$$

Example 5. (*a*) *The first and the last terms of an A.P. are respectively -4 and 146, and the sum of A.P. $=7171$. Find the number of terms of the A.P. and also its common difference.* [*I.C.W.A., December 1990*]

Solution : We have $a=-4$, $l=146$ and $S_n=7171$

Let n be the number of terms of the A.P.

$\therefore \qquad S_n=\frac{n}{2}(a+l) \Rightarrow 7171=\frac{n}{2}(-4+146)$

or $\qquad n=101$

Also $146=(-4)+(101-1)d \Rightarrow 100d=146+4=150$

or $\qquad d=1.5$

Hence $n=101$ and $d=1{\cdot}5$

(*b*) *The sum of a series in A.P. is 72, the first term 17 and the common difference, -2, find the number of terms and explain the double answer.*

Solution. Let n denote the number of terms.

Using $S_n=\frac{n}{2}\left[2a+(n-1)d\right]$, we have $72=\frac{n}{2}\left[34-(n-1)2\right]$

$\Rightarrow \qquad n^2-18n+72=0$, *i.e.*, $(n-6\ (n-12)=0$

$\therefore \qquad n=6$ or 12.

The double answer shows that there are two sets of numbers whose separate sums are 72. The series to 6 terms is 17, 15, 13, 11, 9, 7 and to 12 terms is 17, 15, 13, 11, 9, 7, 5, 3, 1, -1, -3, 5, ; the sum of the last 6 terms in the latter series is zero and so the sum of the 6 earlier terms of the series is the same as that of 12 terms.

Example 6. *Find the sum to n terms of the series :*

$$\left(1-\frac{1}{n}\right)+\left(1-\frac{2}{n}\right)+\left(1-\frac{3}{n}\right)+\ldots$$

Solution. Here

$$u_2-u_1=\left(1-\frac{2}{n}\right)-\left(1-\frac{1}{n}\right)=-\frac{1}{n}$$

nd $\qquad u_3-u_2=\left(1-\frac{3}{n}\right)-\left(1-\frac{2}{n}\right)=-\frac{1}{n}$

$\therefore$ The series is in A.P. with $a=\left(1-\frac{1}{n}\right)$, $d=-\frac{1}{n}$

Hence
$$S_n=\frac{n}{2}\left\{2a+(n-1)d\right\}$$
$$=\frac{n}{2}\left\{2\left(1-\frac{1}{n}\right)+(n-1)\left(-\frac{1}{n}\right)\right\}$$
$$=\frac{n}{2}\left\{2-\frac{2}{n}-1+\frac{1}{n}\right\}$$
$$=\frac{n}{2}\left(1-\frac{1}{n}\right)=\frac{n}{2}\times\frac{n-1}{n}=\frac{n-1}{2}$$

Example 7. *Find the sum of the series :*

$1+3-5+7+9-11+13+15-17+\ldots$ to $3n$ terms

Solution. The series can be easily split into three A.P.'s in each of which, number of terms are n and d is 6.

$\therefore$ The required sum

$=(1+7+13+\ldots$ to n terms$)+(3+9+15+\ldots$ to n terms$)$
$-(5+11+17+\ldots$ to n terms$)$

$$=\frac{n}{2}\left[2+(n-1)6\right]+\frac{n}{2}\left[6+(n-1)6\right]-\frac{n}{2}\left[10+(n-1)6\right]$$
$$=\frac{n}{2}\left[2+6n-6+6+6n-6-10-6n+6\right]$$
$$=\frac{n}{2}(6n-8)=3n^2-4n$$

Example 8. *S_n denotes the sum of the first n terms of a series, If $S_n=(2n^2+3n)$, show that the series is in A.P.*

Solution.
$$S_n=2n^2+3n$$
$\therefore$
$$S_1=2.1^2+3.1=5$$
$$S_2=2.2^2+3.2=14$$
$$S_3=2.3^2+3.3=27$$
$$S_4=2.4^2+3.4=44$$

Now 1st term $=S_1=5$, 2nd term $=S_2-S_1=14-5=9$,
3rd term $=S_3-S_2=27-14=13$; 4th term $=S_4-S_3=44-27=17$.

$\therefore$ The series is 5, 9, 13, 17, ... which is in A.P. with common difference 4.

Example 9. *Find the 20th term of the arithmetic progression 15, 13, 11,... Calculate the number of terms required to make the sum equal to zero.*

Solution. 20th term $=u_{20}=15+(20-1)(-2)=-23$.

In the second case the sum is equal to zero and we have to find n.

$\therefore \qquad 0=\frac{n}{2}\left\{2\times 15+(n-1)\times(-2)\right\}=n(16-n)$

$\Rightarrow \qquad$ either $n=0$ or $n=16$

Hence the only admissible value is $n=16$.

Example 10. *Find the sum of all natural numbers between 200 and 400 which are divisible by 7*

Solution. The natural numbers between 200 and 400 which are divisible by 7 are 203, 210, 217,..., 399. They form an arithmetic progression (A.P.) of the type

$$7\times 29,\ 7\times 30,\ 7\times 31,\ \ldots,\ 7\times 57$$

where the first term is 203 and the nth term is 399. The number 'of the terms are

$$399=203+(n-1)\times 7 \qquad \Rightarrow\ n-1=\frac{196}{7}=28$$

$\therefore \qquad n=29$

Now the sum of all the numbers between 200 and 400 which are divisible by 7 can be obtained by applying the formula of A.P., *viz.*,

$$S_n=\frac{n}{2}(a+l)=\left[\frac{29}{2}\left\{7\times 29+7\times 57\right\}\right]$$

$$=7\left[\frac{29}{2}(29+57)\right]=7[29\times 43]=8729$$

Example 11. *Find the sum of natural numbers from 1 to 200 excluding those divisible by 5.*

Solution. The required sum

$$=(1+2+3+\ldots+200)-(5+10+15+\ldots 200)=S_1-S_2$$

The sum of the first bracket is

$$S_1=\frac{n}{2}(a+l)=\frac{200}{2}(1+200)=20{,}100$$

For finding out S_2, the sum of the second bracket, let us first find out the value of n. Since the last term here is 200; we have

$$5+(n-1)5=200 \qquad [\because a+(n-1)d=200]$$

$\Rightarrow \qquad 5+5n-5=200, \quad$ *i.e.*, $\quad 5n=200$

$\therefore \qquad n=40$

Hence, the sum of 40 terms with $a=5$ and $l=200$ is

$$S_2=\frac{40}{2}\left[5+200\right]=4100$$

$\therefore \qquad$ The required sum is $20{,}100-4100=16000$

Example 12. *Show that the sum of all odd numbers between 2 and 1000 which are divisible by 3 is 83,667 and of those not divisible by 3 is 1,66,332.*

Solution. We notice that the first odd number which is divisible by 3 is 3 and the last number < 1000 and divisible by 3 is 999.

We have to find the sum of $3+9+15+21+\ldots+999$. Since 999 is the nth term of this series, we have

$$3+(n-1)6=999$$

$$\Rightarrow \quad (n-1)6=996, \text{ i.e. } \Rightarrow (n-1)=166$$

$$\therefore \quad n=167.$$

$\therefore$ The required sum $=\frac{167}{2}\left[2\times 3+166\times 6\right]=167[3+498]=83,667$

Again by inspection, we observe that the odd numbers between 2 and 1000 not divisible by 3 are

$$5, 7, 11, 13, \ldots, 995, 997$$

For finding their sum we shall arrange them into the arithmetic series as,

$$(5+11+17+\ldots+995)+(7+13+19+\ldots+997)=S_1+S_2 \text{(say)}$$

If 995 is the nth term of the first A.P., then

$$5+(n-1)6=995, \text{ i.e., } (n-1)6=990$$

$$\Rightarrow \quad n=166$$

$$\therefore \quad S_1=\frac{166}{2}\left[2\times 5+165\times 6\right]=83\times 1000=83,000$$

Similarly if 997 is the nth term of second A.P., then

$$997=7+(n-1)6, \text{ i.e., } (n-1)6=990$$

$$\Rightarrow \quad n=166$$

$$\therefore \quad S_2=\frac{166}{2}\left[2\times 7+165\times 6\right]=83[14+990]=83\times 1004=83,332$$

Hence the required sum

$$=S_1+S_2=83,000+83,332=1,66,332.$$

Example 13. *If a, b, c, be the sums of p, q, r terms respectively of an A.P., show that*

$$\frac{a(q-r)}{p}+\frac{b(r-p)}{q}+\frac{c(p-q)}{r}=0$$

Solution. Let A denote the first term and D the common difference of an A.P. Then

$$a=\frac{p}{2}\left[2A+(p-1)D\right]$$

$$b=\frac{q}{2}\left[2A+(q-1)D\right]$$

$$c=\frac{r}{2}\left[2A+(r-1)D\right]$$

The above equations can be written as

$$\frac{a}{p}=A+(p-1)\frac{D}{2} \quad ...(1)$$

$$\frac{b}{q}=A+(q-1)\frac{D}{2} \quad ...(2)$$

$$\frac{c}{r}=A+(r-1)\frac{D}{2} \quad ...(3)$$

Multiplying (1) by $q-r$, (2) by $r-p$, (3) by $p-q$ and adding, we get

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p+)\frac{c}{r}(p-q)=A(q-r+r-p+p-q)$$
$$+\frac{D}{2}\Big[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\Big]$$
$$=0.$$

Example 14. *If S_1, S_2, S_3 be respectively the sum of n, 2n, 3n terms of an A.P., prove that $S_3=3(S_2-S_1)$.*

Solution. Let a be the first term and d the common difference of an A.P., then

$$S_1=\frac{n}{2}\Big\{2a+(n-1)d\Big\} \quad ...(1)$$

Changing n to $2n$ and $3n$, we get

$$S_2=\frac{2n}{2}\Big\{2a+(2n-1)d\Big\} \quad ...(2)$$

and

$$S_3=\frac{3n}{2}\Big\{2a+(3n-1)d\Big\} \quad ...(3)$$

$$\therefore \quad S_2-S_1=\frac{2n}{2}\Big\{2a+(2n-1)d\Big\}-\frac{n}{2}\Big\{2a+(n-1)d\Big\}$$
$$=\frac{n}{2}\Big[\Big\{4a+(4n-2)d\Big\}-\Big\{2a+(n-1)d\Big\}\Big]$$
$$=\frac{n}{2}\Big[2a+(3n-1)d\Big]$$

$$\Rightarrow \quad 3(S_2-S_1)=\frac{n}{2}\Big[2a+(3n-1)d\Big]=S_3$$

Example 15. *If $S_1, S_2, S_3, \ldots, S_p$ are the sums, each to n terms of p arithmetic progressions whose first terms are 1, 2, 3, ... and common difference are 1, 3, 5, 7, ... respectively, then show that*

$$S_1+S_2+S_3+\ldots+S_p=\frac{np(np+1)}{2}$$

Solution. Since S_1 is the sum of n terms of an A.P. $1+2+3+\ldots$

$$S_1=\frac{n}{2}\left[2.1+(n-1).1\right]=\frac{n(n+1)}{2}$$

Again S_2 is the sum of n terms of the A.P. $2+5+8+\ldots$,

$$\therefore \quad S_2=\frac{n}{2}\left[2.2+(n-1).3\right]=\frac{n(3n+1)}{2}$$

and S_3 is the sum of n terms of the A.P. $3+8+13+\ldots$

$$\therefore \quad S_3=\frac{n}{2}\left[2.3+(n-1).5\right]=\frac{n(5n+1)}{2}$$

Lastly S_p is the sum of n terms of the A.P. $p+(3p-1)+(5p-2)+\ldots$

$$S_p=\frac{n}{2}\left[2p+(n-1)(2p-1)\right]$$

$$=\frac{n}{2}\left[(2p-1)n+1\right]$$

$$\therefore \quad S_1+S_2+S_3+\ldots+S_p$$

$$=\frac{n}{2}\left[n+1\right]+\frac{n}{2}\left[3n+1\right]+\frac{n}{2}\left[5n+1\right]$$

$$+\ldots+\frac{n}{2}\left[(2p-1)n+1\right]$$

$$=\frac{n}{2}\left[(n+1)+(3n+1)+(5n+1)+\ldots+\{(2p-1)n+1\}\right]$$

$$=\frac{n}{2}\left[\{n+3n+5n+\ldots+(2p-1)n\}+p\right]$$

$$=\frac{n}{2}\left[n\{1+3+5+\ldots+(2p-1)\}+p\right]$$

$$=\frac{n}{2}\left[n\left\{\frac{p}{2}\left(2+\overline{p-1}.2\right)\right\}+p\right]$$

$$=\frac{n}{2}\left[np(1+p-1)+p\right]$$

$$=\frac{np(np+1)}{2}$$

Example 16. *Let the sum of n terms of two A.P.'s be in ratio $7n-5 : 5n+17$. Show that the 6th terms of two series are equal.*

Solution. Let the two A.P.'s be

$$a,\ a+d,\ a+2d,\ \ldots,\ a+(n-1)d$$
$$a_1,\ a_1+d_1,\ a_1+2d_1,\ \ldots,\ a_1+(n-1)d_1$$

We are given that

$$\frac{S_n}{S_n'}=\frac{\frac{n}{2}\left[\ 2a+(n-1)d\ \right]}{\frac{n}{2}\left[2a_1+(n-1)d_1\right]}=\frac{7n-5}{5n+17}$$

$$\Rightarrow \qquad \frac{2a+(n-1)d}{2a_1+(n-1)d_1}=\frac{7n-5}{5n+17} \qquad ...(1)$$

We want to find the ratio of 6th term, *i.e.*, we want to find $\frac{a+5d}{a_1+5d_1}$ which can be obtained from (1) above, by writting it as $\frac{2a+10d}{2a_1+10d_1}$ *i.e.*, by putting $n=11$ in (1) above. Therefore

$$\frac{a+5d}{a_1+5d_1}=\frac{2a+10d}{2a_1+10d_1}=\frac{2a+(11-1)d}{2a_1+(11-1)d_1}=\frac{7.11-5}{5.11+17}=1$$

$$\Rightarrow \qquad a+5d=a_1+5d_1$$

$\therefore$ 6th terms of the series are equal.

Example 17. *The natural numbers are written as follows :*

```
         1
       2   3
     4   5   6
   7   8   9   10
 ...      ...      ...
 ...      ...      ...
```

Show that the sum of the numbers in the nth row is $\frac{1}{2}n(n^2+1)$.

Solution. Evidently the number of terms in each row is the same as its order, nth row will contain n terms.

The first term of the nth row will be the nth term of the series $1+2+4+7+...$

Let S_n denote the sum and u_n, the nth term of the series

$$S=1+2+4+7+...+u_n$$

Also $$S=\qquad 1+2+4+...+u_{n-1}+u_n$$

Subtracting and transposing, we get

$$u_n=1+(1+2+3+......\text{to } \overline{n-1} \text{ terms})$$

$$=1+\frac{n-1}{2}\left[\ 2+(n-2).1\ \right]=\tfrac{1}{2}(n^2-n+2)$$

This is the first term of the nth row. Thus the terms in the nth row will form a series in A.P. whose first term is $\frac{1}{2}(n^2-n+2)$, the common difference as 1 and the number of terms n.

$\therefore$ Sum of the numbers in the nth row

$$=\frac{n}{2}\left[\ 2\ .\ \tfrac{1}{2}(n^2-n+2)+(n-1).1\ \right]$$

$$=\frac{n}{2}\left(\ n^2+1\ \right)$$

12·3. ARITHMETIC MEAN

When there are three quantities in A.P., the middle one is called the *arithmetic mean* of other two terms. If a, b, c are in A.P., we have

$$b-a=c-b \quad \text{or} \quad 2b=a+c$$

$$\Rightarrow \qquad b=\frac{a+c}{2}$$

In general, when any number of quantities form an A.P., the quantities lying in between the first and the last are called the *Arithmetic means* (briefly written as A.Ms). Thus if the terms $a, A_1, A_2, \ldots, A_n, b$ are in A.P., the quantities $A_1, A_2, \ldots, A_n$, are called the A.Ms. between a and b.

Insertion of Arithmetic Means. Let $A_1, A_2, \ldots, A_n$ be the A.Ms. between a and b. Then $a, A_1, A_2, \ldots, A_n, b$ are in A.P.

b is the $(n+2)$th term of this A.P., let d be the common difference.

$$b=a+(n+2-1)d \quad \Rightarrow \quad d=\frac{b-a}{n+1}$$

Hence
$$A_1=a+d=a+\frac{(b-a)}{n+1}$$

$$A_2=a+2d=a+2\,\frac{(b-a)}{n+1}$$

$$A_3=a+3d=a+3\,\frac{(b-a)}{n+1}$$

$$A_n=a+nd=a+n\frac{(b-a)}{n+1}$$

Example 18. *Find the 14 arithmetic means which can be inserted between 5 and 8 and show that their sum is 14 times the arithmetic mean between 5 and 8.*

Solution. Let $A_1, A_2, A_3, \ldots, A_{14}$ be the 14 A.Ms. between 5 and 8. Then, $5, A_1, A_2, \ldots A_{14}, 8$ form an A.P. whose first term is 5 and whose 16th term is 8. Let d be the common difference of the A.P., then

$$8=5+(16-1)d \quad \Rightarrow \quad d=\tfrac{1}{5}$$

$$A_1=a+d=5+\tfrac{1}{5}=\frac{26}{5}\ ;\ A_2=a+2d=\frac{27}{5}, \ldots$$

Hence the fourteen A.Ms. are $\frac{26}{5}, \frac{27}{5}, \frac{28}{5}, \ldots$

Sum of these means is $\frac{14}{2}\left[2\times\frac{26}{5}+\left(14-1\right)\frac{1}{5}\right]=91$

A.M. between 5 and 8 is $\frac{5+8}{2}=\frac{13}{2}$

and 14 times the A.M. is $14\times\frac{13}{2}=91$

Hence the sum of the 14 A.Ms, = 14 times the A.M. between 5 and 8.

Representation of Terms in A.P. We can conveniently represent the terms in A.P. as follows :

(*i*) 3 terms : $a-d, a, a+d$

(*ii*) 4 terms : $a-3d, a-d, a+d, a+3d$

(*iii*) 5 terms : $a-2d, a-d, a, a+d, a+2d$.

It should be noted that in case of odd number of terms, the middle term is a while in case of even number of terms, the middle terms are $a-d$, $a+d$ and common difference is $2d$. The following examples will illustrate the use of such representation.

Example 19. *Find three numbers in A.P., whose sum is 9 and the product is −165.*

Solution. Let the three numbers in A.P. be $a-d, a, a+d$ ...(1)

Now we are given

$$(a-d)+a+(a+d)=9$$

$$\Rightarrow \quad 3a=9, \text{ i.e., } a=3$$

Also we are given

$$(a-d)\times a\times(a+d)=-165$$

$$\Rightarrow \quad (3-d)\times 3\times(3+d)=-165$$

$$\Rightarrow \quad 9-d^2=-55, \text{ i.e., } d^2=64$$

$$\therefore \quad d=\pm 8$$

Putting $a=3$ and $d=8$ in (1), we get the required numbers as

$$3-8, 3, 3+8, \text{ i.e., } -5, 3, 11$$

If we take $a=3$ and $d=-8$, we get

$$3-(-8), 3, 3+(-8), \text{ i.e., } 11, 3, -5$$

which are the same numbers as before, written in reverse order.

Example 20. *Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.*

Solution. Let the four numbers in A.P. be

$$a-3d, a-d, a+d, a+3d \qquad \ldots(1)$$

$\because$ Sum of four numbers = 20

$\therefore \quad (a-3d)+(a-d)+(a+d)+(a+3d)=20$

$\Rightarrow \quad 4a=20$, *i.e.*, $a=5$ ...(2)

$\therefore$ Sum of their squares $=120$

$\therefore \quad (a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120$

$\Rightarrow \quad (5-3d)^2+(5-d)^2+(5+d)^2+(5+3d)^2=120$

$\Rightarrow \quad 25-30d+9d^2+25-10d+d^2+25+10d+d^2+25+30d+9d^2=120$

$\Rightarrow \quad 100+20d^2=120$

$\Rightarrow \quad d^2=1 \quad \text{or} \quad d=\pm 1.$

Putting $a=5$ and $d=1$ in (1), we get the required numbers to be

$5-3,\ 5-1,\ 5+1,\ 5+3$, *i.e.*, 2, 4, 6, 8.

If we take $a=5$ and $d=-1$, we get the same numbers in reverse order.

Example 21. *Find the three numbers in A.P., where the sum of the numbers is 24 and the sum of their cubes is 1968.*

Solution. Let the three numbers in A.P. be $a-d,\ a,\ a+d$

Then, as given in the problem,

$(a-d)+a+(a+d)=24$

$\Rightarrow \quad a=8$

Also $\quad (8-d)^3+(8)^3+(8+d)^3=1968$

$\Rightarrow \quad (8)^3-3(8)^2.d+3.(8)d^2-d^3+(8)^3+8^3+3(8)^2.d+3.(8).d^2+d^3=1968$

$\Rightarrow \quad 3.(512)+6.8d^2=1968$

$48d^2=1968-1536=432$

$\Rightarrow \quad d^2=9$

$\therefore \quad d=\pm 3.$

Hence the three numbers are 5, 8, 11.

Example 22. *Divide $12\frac{1}{2}$ into five parts in A.P. such that the first and the last parts are in the ratio 2 : 3.*

Solution. Let the five parts in A.P. be

$a-2d,\ a-d,\ a,\ a+d,\ a+2d$ ... (1)

$\because$ The sum of these parts $=12\frac{1}{2}$

$\therefore \quad (a-2d)+(a-d)+a+(a+d)+(a+2d)=12\frac{1}{2}$

$\Rightarrow \quad 5a=12\frac{1}{2}$, *i.e.*, $a=\frac{5}{2}$

Also first part : fifth part $=2:3$

$\therefore \quad \dfrac{a-2d}{a+2d}=\dfrac{2}{3} \quad \Rightarrow \quad 3a-6d=2a+4d$

$\Rightarrow \quad 10d=a=\dfrac{5}{2}$

$\therefore \quad d=\dfrac{5}{2}\times\dfrac{1}{10}=\dfrac{1}{4}$

Putting $a=\frac{5}{2}$ and $d=\frac{1}{4}$ in (1), we get the parts to be

$$\frac{5}{2}-\frac{1}{2},\ \frac{5}{2}-\frac{1}{4},\ \frac{5}{2},\ \frac{5}{2}+\frac{1}{4},\ \frac{5}{2}+\frac{1}{2}$$

i.e.,
$$2,\ 2\frac{1}{4},\ 2\frac{1}{2},\ 2\frac{3}{4},\ 3$$

Example 23. *If a, b, c are in A.P., then prove that*

(*i*) $$a^3+4b^3+c^3=3b(a^2+c^2)$$

(*ii*) $$a^2+4ac+c^2=2(ab+bc+ca).$$

Solution. (*i*) Let d be the common difference of this A.P. so that

$$b=a+d \text{ and } c=a+2d$$

Now L.H.S. $=a^3+4(a+d)^3+(a+2d)^3$

$=a^3+4(a^3+3a^2d+3ad^2+d^3)+(a^3+6a^2d+12ad^2+8d^3)$

$=a^3+4a^3+12a^2d+12ad^2+4d^3+a^3+6a^2d+12ad^2+8d^3$

$=6a^3+18a^2d+24ad^2+12d^3$

R.H.S. $=3b(a^2+c^2)=3(a+d)[a^2+(a+2d)^2]$

$=3(a+d)[a^2+a^2+4ad+4d^2]$

$=(3a+3d)(2a^2+4ad+4d^2)$

$=6a^3+18a^2d+24ad^2+12d^3$

$\therefore$ L.H.S. = R.H.S.

(*ii*) L.H.S. $=a^2+4ac+c^2=a^2+4a(a+2d)+(a+2d)^2$

$=a^2+4a^2+8ad+a^2+4ad+4d^2$

$=6a^2+12ad+4d^2=2(3a^2+6ad+2d^2)$

R.H.S. $=2(ab+bc+ca)$

$=2[a(a+d)+(a+d)(a+2a)+(a+2d)a]$

$=2[a^2+ad+a^2+3ad+2d^2+a^2+2ad]$

$=2[3a^2+6ad+2d^2].$

$\therefore$ L.H.S. = R.H.S.

Example 24. *If a, b, c are in A.P., show that*

(*i*) $$\frac{a(b+c)}{bc},\ \frac{b(c+a)}{ca},\ \frac{c(a+b)}{ab},$$

(*ii*) $a^2(b+c),\ b^2(c+a),\ c^2(a+b)$ *are also in A.P.*

Solution. (*i*) Since a, b, c are in A.P.,

$$\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc} \text{ are also in A.P.}$$

(by dividing each term by abc)

i.e., $$\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ac} \text{ are in A.P.}$$

$$\therefore \quad \frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab} \text{ are also in A.P.}$$

[by multiplying each term by $ab+bc+ca$]

$$\Rightarrow \frac{ab+bc+ca}{bc}-1, \frac{ab+bc+ca}{ca}-1, \frac{ab+bc+ca}{ab}-1 \text{ are also in A.P.}$$

(by substracting one from each term)

$$\Rightarrow \quad \frac{ab+ca}{bc}, \frac{ab+bc}{ca}, \frac{bc+ca}{ab} \text{ are also in A.P.}$$

$$\Rightarrow \quad \frac{a(a+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab} \text{ are also in A.P.}$$

(*ii*) Multiplying each term by abc, we get

$a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are also in A.P.

Example 25. *If* $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are in A.P., prove that* a^2, b^2, c^2 *are also in A.P.*

Solution. $\because \quad \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\therefore \quad (c+a)(a+b)$, $(b+c)(a+b)$, $(b+c)(c+a)$ are also in A.P.

[by multiplying each term by $(b+c)(c+a)(a+b)$]

$\Rightarrow \quad a^2+(bc+ca+ab)$, $b^2+(bc+ca+ab)$, $c^2(bc+ca+ab)$ are in A.P.

$\Rightarrow \quad a^2$, b^2, c^2 are in A.P.

[by substracting $(bc+ca+ab)$ from each term]

Example 26. *If* a^2, b^2, c^2 *are in A.P., prove that*

(*i*) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are also in A.P.*

(*ii*) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ *are also in A.P.*

Solution. (*i*) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

If $$\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$$

i.e., if $$\frac{b+c-c-a}{(c+a)(b+c)}=\frac{c+a-a-b}{(a+b)(c+a)}$$

i.e., if $(b-a)(a+b)=(c-b)(c+b)$

i.e., if $b^2-a^2=c^2-b^2$

i.e., if a^2, b^2, c^2 are in A.P.

Example 27. *If* $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ *are in A.P., show that* $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are also in A.P.*

Solution. $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

$\therefore$ $\frac{b+c-a}{a}+2, \frac{c+a-b}{b}+2, \frac{a+b-c}{c}+2$ are also in A.P.

$\therefore$ $\frac{b+c+a}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are also in A.P.

$\therefore$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Example 28. *If* $(b-c)^2, (c-a)^2, (a-b)^2$ *are in A.P., show that* $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ *are also in A.P.*

Solution. Put $b-c=x$, $c-a=y$, $a-b=z$

Then $x+y+z=(b-c)+(c-a)+(a-b)=0$

$\Rightarrow$ $x=-(y+z)$ and $z=-(x+y)$

Now $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ will be in A.P.

if $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

i.e., if $\frac{1}{y}-\frac{1}{x}=\frac{1}{z}-\frac{1}{y}$

i.e., if $\frac{x-y}{xy}=\frac{y-z}{yz}$

i.e., if $\frac{x-y}{x}=\frac{y-z}{z}$

i.e., if $\frac{x-y}{-(y+z)}=\frac{y-z}{-(x+y)}$

i.e., if $y^2-x^2=z^2-y^2$

i.e., if x^2, y^2, z^2 are in A.P.

i.e. if $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.

which is true by hypothesis.

Example 29. *If a, b, c are in A.P., show that*

$$(b+c),\ (b+a),\ (a+b) \text{ are also in } A.P.$$

Solution. $b+c,\ c+a,\ a+b$ will be in A.P. if

$$(c+a)-(b+c)=(a+b)-(c+a)$$

i.e., if $\qquad a-b=b-c$

i.e., if $\qquad a, b, c$ are in A.P.

which is true by hypothesis.

Example 30. *A man saved Rs. 16,500 in ten years. In each year after the first he saved Rs. 100 more than he did in the preceding year. How much did he saved in the first year ?*

Solution. Here a=savings in the first year=?

n=number of years=10, d=100, S_n=16,500

Now $$S_n=\frac{n}{2}\,[2a+(n-1)d]$$

$$\Rightarrow \qquad 16{,}500=\frac{10}{2}\,[2a+(10-1)100]$$

$$\Rightarrow \qquad 16{,}500=5(2a+900)$$

$$\Rightarrow \qquad 10a=16{,}500-4{,}500=12{,}000$$

$$\Rightarrow \qquad a=\text{Rs. } 1200.$$

Example 31. *A piece of equipment cost a certain factory Rs. 60,0,000. If it depreciates in value, 15% in the first year, 13½% in the next year, 12% in the third year, and so on, what will be its value at the end of 10 years, all percentages applying to the original cost ?*

Solution. Suppose the cost of an equipment is Rs. 100. Now the percentages of depreciation at the end of 1st, 2nd, 3rd years are 15, 13½, 12,.. which are in A.P., with $a=15$ and $d=-\frac{3}{2}$.

Hence percentage of depreciation in the tenth year

$$=a+(10-1)d=15+9\left(-\frac{3}{2}\right)=\frac{3}{2}$$

Also total value depreciated in 10 years

$$=15+13\,\frac{1}{2}+12+\ldots+\,\frac{3}{2}=\frac{10}{2}\left(15+\frac{3}{2}\right)=\frac{165}{2}$$

Hence the value of equipment at the end of 10 years

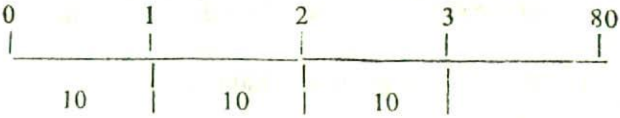
$$=100-\frac{165}{2}=\frac{35}{2}$$

The total cost being Rs. 6,00,000, its value at the end of 10 years

$$=\text{Rs. }\frac{6{,}00{,}000}{100}\times\frac{35}{2}=\text{Rs. } 1{,}05{,}000.$$

Example 32. *80 coins are placed in a straight line on the ground. The distance between any two consecutive coins is 10 metres How far must a person travel to bring them one by one to a basket placed 10 metres behind the first coin ?*

Solution.



Let 1, 2, 3,..., 80 represent the positions of the coins and 0 that of the basket.

The distance covered in bringing the first coin $=10+10=20$

,, ,, ,, ,, ,, 2nd ,, $=20+20=40$

,, ,, ,, ,, ,, 3rd ,, $=30+30=60$

and so on.

$\therefore$ Total distance covered

$$=\frac{n}{2}\{2a+(n-1)d\}$$

$$=\frac{80}{2}\{2\times 20+(80-1)20\}$$

$$=40(40+1580)=64,800 \text{ metres.}$$

Example 33. *A man is employed to count Rs. 10,710. He counts at the rate of Rs. 180 per minute for half an hour. After this he counts at the rate of Rs. 3 less every minute than the preceding minute. Find the time taken by him to count the entire amount.*

Solution. Amount counted in half an hour

$$=\text{Rs. } 180\times 30=\text{Rs. } 5400$$

$\therefore$ Balance to be counted $=$ Rs. 10,710 $-$ Rs. 5,400 $=$ Rs. 5,310

In one minute just after half an hour, the amount counted is Rs. 177.

Let n minutes be taken to count Rs. 5310, then

$$5310=\frac{n}{2}\{2\times 177+(n-1)\times(-3)\}=\frac{n}{2}(357-3n)$$

$$\Rightarrow \quad 3n^2-357n+10620=0$$

$$\Rightarrow \quad n^2-119n+3540=0, \text{ i.e.,}$$

$$\Rightarrow \quad (n-59)(n-60)=0$$

$\therefore$ Either $n=59$ or 60

Now in 59 minutes after half an hour, the amount counted

$$=\frac{59}{2}\{2\times 177-58\times 3\}=5310$$

and nothing is counted in the 60th minute since $a+59d=177-59\times 3=0$

$\therefore$ The required time $=30+59=89$ minutes.

Example 34. *B arranges to pay off a debt of Rs. 9600 in 48 annual instalments which form an arithmetic series. When 40 of these instalments are paid, B becomes insolvent and his creditor finds that Rs. 2400 still remains unpaid. Find the value of each of the first three instalments of B. Ignore interest.*

Solution. We are here given $S_n=9600,\ n=48$

$$\therefore \quad 9600=\frac{48}{2}\{2a+(48-1)d\} \Rightarrow 2a+47d=400 \qquad ...(1)$$

Also $9600-2400=7200$

Again $7200=\frac{40}{2}\{2a+(40-1)d\}=20(2a+39d)$

$$\Rightarrow \quad 2a+39d=360 \qquad ...(2)$$

Solving (1) and (2), we get $a=82\frac{1}{2},\ d=5$.

Hence the first three instalments are Rs. 82·50, Rs. 87·50, and Rs. 92·50.

Example 35. *A man agrees to repay a debt of Rs. 2500 in a number of instalments, each instalment (beginning with the second) exceeding the previous one by Rs. 2. If the first instalment be of Re. 1, find how many instalments will be necessary to wipe out the loan completely?*

Solution. We are given

$$a=1,\ d=2,\ S_n=2500,\ n=?$$

Now $S_n=\frac{n}{2}\{2a+(n-1)d\}$

$$\Rightarrow \quad 2500=\frac{n}{2}\{2\times 1+(n-1)2\}=\frac{n}{2}\{2+2n-2\}$$

$$\Rightarrow \quad 2n^2=5000$$

$$\therefore \quad n=50$$

Example 36. *The rate of monthly salary of a person is increased annually in A.P. It is known that he was drawing Rs. 400 a month during the 11th year of his service, and Rs. 760 during the 29th year. Find his starting salary and the rate of annual increment. What should be his salary at the time of retirement just on the completion of 36 years of service?*

Solution. Let a be his starting salary and d be annual increment.

Now $u_{11}=400 \Rightarrow a+(11-1)d=400 \qquad ...(1)$

and $u_{29}=760 \Rightarrow a+(29-1)d=760 \qquad ...(2)$

Subtracting (1) and (2), we get

$$18d=360 \Rightarrow d=20$$

$\therefore$ From (1), we get $a+200=400 \Rightarrow a=200$

$\therefore$ Starting salary=Rs. 200 and annual increment=Rs. 20

$\therefore$ Salary at the time of retirement$=u_{36}=a+35d=200+35\times 20=900$

Example 37. *Mr. Mohan Lal buys national savings certificates of values exceeding of the last year's purchase by Rs. 100. After 10 years he finds that the total value of the certificates purchased by him is Rs. 5000. Find the value of the certificates purchased by him. (i) in the first year, (ii) in the 8th year.*

Solution. Suppose value of the certificates purchased in the first year=Rs. a

So he has purchased certificates of the value

$$a,\ a+100,\ a+200,\ a+300,\ldots$$

Also $S_{10}=5000,\ d=100,\ n=10$

Now $$S_n=\frac{n}{2}\{2a+(n-1)d\}$$

$\Rightarrow$ $$5000=\frac{10}{2}[2a+(10-1)\ 100]$$

$\Rightarrow$ $$5000=5[2a+900]$$

$\Rightarrow$ $$1000-900=2a$$

$\Rightarrow$ $$a=50.$$

$\therefore$ Value of certificates purchased in first year=Rs. 50

Again value of certificates purchased in the 8th year is

$$u_8=a+(8-1)d=50+7\times 100=\text{Rs. } 750.$$

EXERCISE (I)

1. Find the nth terms of the following and give their 10th term :

   (*i*) 3, 8, 13, 18,......

   (*ii*) $-\frac{15}{8},\ -\frac{7}{8},\ \frac{1}{8},\ \frac{9}{8},\ \ldots$

2. (*a*) Find the sum of the following :

   (*i*) $2+4+6+8+\ldots$to n terms

   (*ii*) $8+13+18+23+\ldots$to 25 terms

   (*iii*) $21+15+9+3+\ldots$to 20 terms.

   (*b*) How many terms are there in each of the following series :

   (*i*) $-3+3+9+\ldots+117$,

   (*ii*) $10+9\frac{1}{2}+9+\ldots+\frac{1}{2}$

3. Find sum of the following series :

(*i*) $7+14+21+...$to 20 terms

(*ii*) $-4-1+2+5+...$to 21 terms.

4. Find the last term and sum of the following series :

(*i*) $1+\frac{4}{3}+\frac{5}{3}+...$to 58 terms

(*ii*) $\frac{a+b}{a+b}, \frac{a+3b}{a+b}, \frac{a+5b}{a+b}, ...$to 15 terms

(*iii*) $(3+4)+(8+9)+(13+14)+...$to 20 terms.

5. Find the nth term and sum to n terms of the following A.P. :

(*i*) $a+b, 2a, 3a-b, ...$

(*ii*) $(x+y)^2, (x^2+y^2), (x-y)^2, ...$

(*iii*) $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+...$

6. How many terms of the series :

(*i*) $5+7+9+...$must be taken so that the sum may be 480 ?

(*ii*) $24+20+16+...$must be taken so that the sum may be 72 ?

7. (*a*) If 10th term of an A.P. is 15 and the 15th term is 10, find the series.

(*b*) Find the 20th term of the A.P. 80, 75, 70,... Calculate the number of terms required to make the sum equal to zero.

(*c*) The 4th term of an A.P. is 64 and the 54th term is -61, show that the 23rd term is $16\frac{1}{2}$.

8. Prove that if unity is added to the sum of any number of terms of the A.P. 3, 5, 7, 9,...the resulting sum is a perfect square.

9. The sum of n terms of an A.P. is $2n^2$. Find the 5th term.

10. The sum of n terms of an A.P. is $2n^2+3n$. Find the nth term and the series.

11. (*a*) If pth term of an A.P. is $\frac{1}{q}$ and qth term is $\frac{1}{p}$, show that the sum of pq terms is $\frac{1}{2}(pq+1)$.

(*b*) The sum of p terms of an A.P. is q and the sum of q terms is p. Find the sum of $(p+q)$ terms.

12. If S_1, S_2, S_3 be the sums of n terms of three arithmetic series, the first term of each being 1 and the respective common differences 1, 2, 3, prove that

$$S_1+S_3=2S_2$$

13. The sum of first 11 terms of an A.P. is 19 and the sum of first 19 terms is 11. Find the sum of the first 30 terms.

14. (*a*) If the 6th term of an A.P. is 121, find the sum of the first 11 terms.

(b) If the 35th term of an A,P. is 30, show that the sum of its first 69 terms is 2070.

15. (a) Find sum of all odd numbers between 200 and 300.

(b) Find the sum of all natural numbers between 500 and 1,000 which are divisible by 13.

(c) Find the sum of all natural numbers from 100 to 300 :

(i) which are exactly divisible by 4,

(ii) excluding those which are divisible by 3,

(iii) which are exactly divisible by 5,

(iv) which are exactly divisible by 4 and 5,

(v) which are not exactly divisible by 4 or 5.

16. Find three numbers in A.P. such that

(i) their sum is 18 and the product is 192,

(ii) their sum is 27 and the sum of their squares is 341.

17. Find four numbers in A.P. such that

(i) their sum is 24 and their product is 945.

(ii) their sum is 20 and the sum of their squares is 120.

(iii) the sum of 2nd and 3rd numbers is 22 and the product of 1st and 4th numbers is 85.

18. Find five numbers in A.P. such that

(i) their sum is 25, and the sum of their squares is 135.

(ii) their sum is 20 and the product of the first and the last is 15.

19. (a) If p, q, r, s are any four consecutive terms of an A.P., show that $p^2-3q^2+3r^2-s^2=0$.

(b) If p, q, r, s, t are in A.P., show that $p+t=q+s=2r$.

20. (a) The sum of n terms of two arithmetic series are in the ratio of $7n-5 : 5n+17$, show that the 6th terms of the two series are equal.

(b) The sum of n terms of two arithmetic progressions are in the ratio $3n+1 : n+4$, find the ratio of the 4th terms.

(c) Divide 20 into 4 parts which are in A.P. and such that the product of the first and fourth is to product of the second and third in the ratio 2:3.

[**Hint.** $20=\frac{4}{2}\left[2a+(4-1)d\right] \Rightarrow 2a+3d=10$ ...(1)

Also we are given

$$\frac{a(a+3d)}{(a+d)(a+2d)}=\frac{2}{3}, \text{ i.e., } a^2+3ad-4d^2=0$$

$$\Rightarrow \quad a=-4d \text{ or } d.$$

Substituting $a=d$ in (1), we get

$$2d+3d=10 \text{ or } 5d=10$$

$$d=2 \text{ and } a=2.$$

∴ The numbers are 2, 4, 6, 8.

Substituting $a=-4d$ in (1), we get

$$-8d+3d=10 \text{ or } -5d=10$$

$$\Rightarrow d=-2 \text{ and } a=8$$

∴ The numbers are 8, 6, 4, 2.

Hence the required parts are 2, 4, 6. 8.]

21. The sequence of natural numbers is written as

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 3 & 4 & \\ 5 & 6 & 7 & 8 & 9 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

Find the sum of the numbers in the rth row.

[**Hint.** Let S_r denote the sum of rth row. Then

$$S_1=1 \qquad =1^3+(1-0)^3$$

$$S_2=2+3+4=9 \qquad =2^3+(2-1)^3$$

$$S_3=5+6+7+8+9=\frac{5}{2}(5+9)=5.7=35=3^3+(3-1)^3$$

$$\vdots \qquad \vdots$$

$$S_r=r^3+(r-1)^3.]$$

22. $S_1, S_2, S_3, \ldots, S_m$ be the sums of n terms of m arithmetic series whose first terms as well as the common differences are $1, 2, 3, \ldots, m$; show that

$$S_1+S_2+S_3+\ldots+S_m=\frac{mn}{4}(m+1)(n+1)$$

23. Find the sum of the series $72+70+68+\ldots+40$. What will be the sum, if all the terms are increased by $12\frac{1}{2}$ per cent ? Express the original sum as a percentage of this sum, giving the result correct to one significant figure only.

24. A man saved Rs. 16,500 in ten years. In each year after the first he saved Rs. 100 more than he did in the preceding year. How much did he save in the first year ?

25. Mr. X takes a loan of Rs. 2000 from Mr. Y and agrees to repay in a number of instalments, each instalment (beginning with the second) exceeding the previous one by Rs. 10. If the first instalment be Rs. 5, find how many instalments will be necessary to wipe out the loan completely ?

26. A class consists of a number of boys whose ages are in arithmetical progressions, the common difference being four months. If the youngest boy of the class be only eight years old and the sum of the ages of all the boys in the class be 168 years, find the number of boys and the age of the oldest boy in the class.

27. A lamp lighter has to light 100 gas lamps. He takes $1\frac{1}{2}$ minutes to go from one lamp post to the next. Each lamp burns 10 cubic feet of gas per hour. How many cubic feet of gas has been burnt by 8·30 P.M. if he lights the first lamp at 6 P.M.

[**Hint.** First term $(a)=150$ min. (*i.e.*, 8.30 − 6 P.M.)

Last term $(l)=a+(n-1)d=150+(100-1)(-1{\cdot}5)=1{\cdot}5$ min.

$$\text{Sum}=\frac{n}{2}(a+l)=50(150+1{\cdot}5)=7575 \text{ min.}$$

$$\text{Total gas burnt}=7575\times\frac{10}{60}=1262{\cdot}5 \text{ cu. ft.}]$$

28. A workman agrees to accept certain wages for the first month, on the understanding that his pay is to be raised one rupee every subsequent month until the maximum (namely Rs. 300 p.m.) is reached. At the end of the month for which he received Rs. 300 for the first time he resigns and finds that his wages during his period of service have averaged Rs. 288 a month. How long has he served ?

[**Hint.** Wages at the nth term$=a+(n-1)\times 1=300$ ...(1)

$$\text{Average wages for } n \text{ months}=\frac{1}{n}\times\frac{n}{2}\left\{2a+(n-1)\times 1\right\}=288 \quad \text{...(2)}$$

Subtracting (1) from (2), $a=276$. $\therefore$ $n=300-276+1=25$ months.]

29. A money-lender lends Rs. 1000 and charges an overall interest of Rs. 140. He recovers the loan and interest by 12 monthly instalments each less by Rs. 10 than the preceding. Find the amount of the first instalment.

30. The monthly salary of a person was Rs. 320 for each of the first three years. He next got annual increments of Rs. 40 per month for each of the following successive 12 years. His salary remained stationary till retirement when he found that his average monthly salary during the service period was Rs. 698. Find the period of his service.

31. Two posts were offered to a man. In the one the starting salary was Rs. 120 per month and the annual increment was Rs. 8, in the other post the salary commenced at Rs. 85 per month but the annual increment was Rs. 12. Then man decided to accept that post which would give him more earnings in the first twenty years of the service. Which post was acceptable to him ? Justify your answer.

[**Hint.** Total earnings in the first job in 20 years

$$=\frac{20}{2}\,[2\times 120+19\times 8]\times 12=47{,}040$$

Total earnings in the second job

$$=\frac{20}{2}[2\times 85+19\times 12]\times 12=47{,}760]$$

32. A person pays Rs. 975 by monthly instalments each less than the former by Rs. 5. The first instalment is of Rs. 100. In what time will the entire amount be paid ?

33. To verify cash balances, the auditor of a certain bank, employs his assistant to count cash in hand of Rs. 4500. At first he counts quickly at the rate of Rs. 150 per minute for 10 minutes only but at the end of that time he begins to count at the rate of Rs. 2 less every minute than he could count in the previous minute. Ascertain how much time he will take to count this sum of Rs. 4,500 ?

34. A man secures an interest-free loan of Rs. 14,500 from a friend and agrees to repay it in ten instalments. He pays Rs. 1000 as first instalment and then increases each instalment by equal amount over the preceding instalment. What will be his last instalment ?

35. The rate of monthly salary of a person increased annually in A.P. It is known that he was drawing Rs. 200 a month during the 11th year of his service, Rs. 380 during the 29th year. Find his initial salary and the rate of annual increment. What should be his salary at the time of retirement just on completion of 35 years of service ?

36. A firm produced 1000 sets of T.V. during its first year. The sum total of the firm's production at the end of 10 years' operation is 14,500 sets.

(*i*) Estimate by how many units, production increased each year.

(*ii*) Forecast based on the estimate of the annual increment in production, the level of output for the 15th year.

37. An enterprise produced 600 units in the 3rd year of existence and 700 units in its 7th year.

(*i*) What was the initial production in the first year ?

(*ii*) What was the production in the fifth year ?

(*iii*) What was the total production in the first five years ?

ANSWERS

1. (*i*) $5n-2$ and 48, (*ii*) $\frac{8n-23}{8}$ and $\frac{57}{8}$

2. (*a*) (*i*) $n(n+1)$, (*ii*) 1700, (*iii*) -720, (*b*) (*i*) 21, (*ii*) 20

3. (*i*) 1470, (*ii*) 546. **4.** (*i*) 20 ; 609, (*ii*) $\frac{a+29b}{a+b}$, $\frac{15(a+15b)}{a+b}$, (*iii*) 197 ; 2040. 5. (*i*) $n(a-b)+2b$, $\frac{1}{2}n(a+3b)+\frac{1}{2}n^2(a-b)$,

(*ii*) $n(x+y)^2-n(n-1)xy$, $n(x+y)^2-n(n-1)xy$, (*iii*) 0, $\frac{1}{2}(n-1)$

6. (*i*) 20, (*ii*) 4 or 9. 7. (*a*) 24, 23, 22..., (*b*) 33

9. 18. **10.** $4n+1$; 5, 9, 13... **11.** (*b*) $-p-q$. **13.** -03.

14. (*a*) 1331, (*b*) 2070. **15.** (*b*) 28405, (*c*) (*i*) 10200, (*ii*) 30000, (*iii*) 8200, (*iv*) 2200, (*v*) 16200. **16.** (*i*) 4, 6, 8, (*ii*) 2, 9, 16.

17. (*i*) 3, 5, 7, 9, (*ii*) 2, 4, 6, 8, (*iii*) 5, 9, 13, 17. **18.** (*i*) 3, 4, 5, 6, 7, (*ii*) 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5. **20.** (*b*) 2 : 1. **21.** $r^3+(r-1)^3$.

23. $\frac{952}{1071}\times 100=89\%$ **24.** Rs. 1200 **25.** 20

26. Number of boys $=16$, Age of the oldest boy $=13$ years

27. 1262·5 cu. ft. **29** Rs. 150 **30.** 40 years **32.** 15 months

33. 34 minute **34.** Rs. 1900 35. Rs. 100, 10, Rs. 440

36. (*i*) 100, (*ii*) 2400 37. (*i*) 550, (*ii*) 650, (*iii*) 3,000

12·4. GEOMETRIC PROGRESSION

A geometric progression is a sequence whose terms increase or decrease by a constant ratio called the common ratio. A series in geometric progression thus is a multiplicative series whose common ratio can be found by dividing any term by its preceding term. Thus

(*i*) the sequence 1, 2, 4, 8, 16, 32,... is an infinite geometric progression, the first term is 1 and the common ratio is 2. Similarly,

(*ii*) the sequence 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ...is a geometric progression, the first term is 1 and the common ratio is $\frac{1}{2}$.

(*iii*) the sequence 5, -10, 20, -40, 80,...is a geometric progression, the first term is 5 and the common ratio is -2.

(*iv*) the sequence 27, -9, 3, -1, $\frac{1}{3}$,...is a geometric progression, the first term is 27 and the common ratio is $-\frac{1}{3}$.

The corresponding geometric series are :

(*i*) $1+2+4+8+16+32+\dots$

(*ii*) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$

(*iii*) $5-10+20-40+80-\dots$

(*iv*) $27-9+3-1+\frac{1}{3}-\dots$

The geometric progression is, therefore, in the form :

$$a, ar, ar^2, ar^3, \ldots$$

whose first term is a and the common ratio is r, and is designated as the standard form of a geometric progression.

The corresponding geometric series is

$$a+ar+ar^2+ar^3+\ldots$$

and is designated as the standard form of geometric series. The abbreviation commonly used for 'geometric progression' is G.P.

Definition. *If for a sequence, $\frac{u_{n+1}}{u_n}$ remains constant for all natural numbers n, then the sequence is called G.P. and the constant ratio of two consecutive terms u_n and u_{n+1} is called the common ratio of the G.P.*

We can also state the series as

$$S_n=a+ar+ar^2+ar^3+\ldots+ar^{n-3}+ar^{n-2}+ar^{n-1}$$

The sequence has the property that the ratios of successive terms are equal. Thus if $u_1, u_2, \ldots, u_n$ are the terms of a G.P., then

$$\frac{u_2}{u_1}=\frac{u_3}{u_2}=\frac{u_4}{u_3}\ldots=r, \text{ known as the common ratio.}$$

Multiplication of these terms yields

$$\frac{u_n}{u_1}=r^{n-1} \quad \text{or} \quad u_n=u_1r^{n-1}$$

Thus the nth term $u_n=ar^{n-1}$

Now we can illustrate the nth term of the Geometric Progression, $a, ar, ar^2, \ldots$ where a is the first term and r is the common ratio as follows :

1st term $u_1=a=ar^{1-1}$

2nd term $u_2=ar=ar^{2-1}$

3rd term $u_3=ar^2=ar^{3-1}$

It may be noted that the index of r is one less than the suffix of u which denotes the rank of the term in the sequence.

$\therefore$ nth term $u_n=ar^{n-1}$

Alternative Method

We can also prove the formula for the nth term of G.P. by mathematical induction.

Let $P(n)$ be the nth term, $u_n=ar^{n-1}$ ...(1)

Step I. Put $n=1$ in (1), we have $u_1=ar^0=a$

$\therefore$ $P(1)$ is true, since $u_1=a$.

Step II. We now show that the assumption of the truth of $P(m)$, namely

$$u_m=ar^{m-1}$$

implies the truth of $P(m+1)$, namely

$$u_{m+1}=ar^{m+1-1}=ar^m \quad \text{...(2)}$$

L.H.S. of (2), $u_{m+1}=u_m \cdot r$ (by definition)

$=ar^{m-1}\, r$ (by assumption)

$=ar^m$ (by index law)

$=$R.H.S. of (2)

From steps (1) and (2), we conclude that $P(n)$ is true for all positive integral values of n.

Example 38. *If the third term of a G.P. is the square of the first and the fifth term is 64, find the series.*

Solution. Let a be the first term and r the common ratio. Then

$$u_n=ar^{n-1}$$

$\therefore$ $u_3=ar^{3-1}=ar^2$ and $u_5=ar^{5-1}=ar^4$

But $u_3=(u_1)^2$ (given)

$\Rightarrow$ $ar^2=a^2$, *i.e.*, $r^2=a$ ...(1)

Also $u_5=64 \Rightarrow ar^4=64$ ...(2)

Substituting the value of r^2 from (1) in (2), we get

$$a.a^2=64 \Rightarrow a=4$$

Putting this value of a in (1), we have

$$r^2=4 \Rightarrow r=2 \text{ or } -2$$

Taking $a=4$ and $r=2$, the series is

$$4+8+16+32+\dots$$

Taking $a=4$ and $r=-2$, the series is

$$4-8+16-32+\dots$$

Example 39. *Find which term of the series*

$$0{\cdot}004+0{\cdot}02+0{\cdot}1+\dots \text{ is } 12{\cdot}5\,?$$

Solution. Here

$$\frac{u_2}{u_1}=\frac{0{\cdot}02}{0{\cdot}004}=\frac{20}{4}=5 \quad \text{and} \quad \frac{u_3}{u_2}=\frac{0{\cdot}1}{0{\cdot}02}=\frac{10}{2}=5.$$

$\therefore$ The series is a G.P. with $a=0{\cdot}004$ and $r=5$.

Now suppose 12·5 is the nth term.

But $u_n=ar^{n-1}=0{\cdot}004\times 5^{n-1}$

$\Rightarrow$ $0{\cdot}004\times 5^{n-1}=12{\cdot}5$

$\Rightarrow$ $5^{n-1}=\dfrac{12{\cdot}5}{0{\cdot}004}=\dfrac{12500}{4}=3125=5^5$

$\therefore$ $n-1=5$ or $n=6$.

Hence 12·5 is the 6th term.

Example 40. *Three numbers whose sum is 15 are in A.P. If 1, 4 and 19 are added to them respectively, the results are in G.P., find the numbers.*

Solution. Let the three numbers in A.P. be $a-d$, a, $a+d$ so that

$$(a-d)+a+(a+d)=15$$

$$\Rightarrow \qquad a=5.$$

Also we are given

$(a-d+1)$, $(a+4)$, $(a+d+19)$ are in G.P.

$$\therefore \qquad (a-d+1)(a-d+19)=(a+4)^2 \qquad [\because a=5]$$

$$\Rightarrow \qquad (6-d)(24+d)=81$$

$$\Rightarrow \qquad d^2+18d-63=0, \textit{ i.e., } (d-3)(d+21)=0$$

$$\therefore \qquad d=3 \text{ or } -21.$$

Hence the numbers are 2,5,8 or 26,5,-16.

Example 41. *If a, b, c are the pth, qth and rth terms of a G.P., prove that* $a \quad . b^{r-p}. c^{p-q}=1$.

Solution. Let A be the first term and R, the common ratio of the G.P.

Then
$$a=A.\ R^{p-1} \qquad ...(1)$$
$$b=A.\ R^{q-1} \qquad ...(2)$$
$$c=A.\ R^{r-1} \qquad ...(3)$$

Raising (1) to power $q-r$, (2) to power $r-p$ and (3) to power $p-q$ and multiplying them together, we get

$$a^{q-r}.\ a^{r-q}.\ c^{p-q}=A^{q-r}.\ A^{r-p}.\ A^{p-q}.R^{(p-1)(q-r)}\times R^{(q-1)(r-p)}.\ A^{(r-1)(p-q)}$$
$$=A^{\circ}\ .\ R^{\circ}=1$$

Example 42. *If a, b, c are in A.P. and x,y,z in G.P., prove that*

$$x^{b-c}\ y^{c-a}\ z^{a-b}=1.$$

Solution. Since a,b,c are in A.P., $b=a+d$, $c=a+2d$, d being the common difference of A.P.

Also x, y, z are in G.P., means $y=xr$, $z=xr^2$, r being the common ratio of the G.P.

$$x^{b-c}.\ y^{c-a}\ \ z^{a-b}=x^{-d}\ (xr)^{2d}(xr^2)^{-d}=x^{\circ}r^{\circ}=1.$$

Example 43. *If a, b, c are in A.P. and x,y, z in G.P., prove that*

$$x^b y^c z^a=x^c y^a z^b$$

Solution. Let d be the common difference and r the common ratio of the given A.P. and G.P. respectively.

We thus have, $\qquad b=a+d$, $c=a+2d$ and $y=xr$, $z=xr^2$

$$\text{L.H.S.}=x^b.y^c.z^a=x^{a+d}.(xr)^{c+2d}.\ (xr^2)^a=x^{3a+3d}.r^{3a+2d}$$

$$\text{R.H.S}=x^c.y^a.z^b=x^{a+2d}.(xr)^a.(xr^2)^{a+d}=x^{3a+3d}.r^{3a+2d}$$

$\therefore \qquad$ L.H.S. = R.H.S. and hence the result.

12·5. SUM OF A SERIES IN G.P.

The sum of n terms of a series in G.P. can be found out using the following formula :

$$S_n=\frac{a(r^n-1)}{r-1}, \text{ when } r>1$$

$$=\frac{a(1-r^n)}{1-r}, \text{ when } r<1.$$

Proof. Let a be the first term, r the common ratio and n the number of terms. If S_n denotes the sum to n terms, then

$$S_n=a+ar+ar^2+\dots+ar^{n-2}+ar^{n-1} \quad \dots(1)$$

Multiplying both sides by r, we get

$$r.\ S_n=ar+ar^2+\dots+ar^{n-1}+ar^n \quad \dots(2)$$

By substracting (2) from (1), we have

$$S_n-r\ .\ S_n=a-ar^n,\ i.e.,\ (1-r)S_n=a(1-r)^n$$

$$S_n=\frac{a(1-r^n)}{1-r},\ r\neq 1 \quad \dots(3)$$

If l denotes the last term, *i.e.*, the nth term ($l=ar^{n-1}$), the above formula becomes

$$S_n=\frac{a-lr}{1-r} \quad \dots(4)$$

Changing the signs of the numerator and denominator, we can also write

$$S_n=\frac{a(r^n-1)}{r-1} \quad \dots(5)$$

It will be found convenient to remember both the forms (3) and (5) for S_n. The form (3) may be used when $r<1$ and the form (5) may be used when $r>1$.

If $r=1$, the G.P. reduces to $a, a, a, \dots$

$\therefore \quad S_n=na.$

Alternative Method. We now prove the formula for the sum of n terms of the G.P. by the method of mathematical induction.

Let $P(n)$ be : $a+ar+\dots+ar^{n-1}=\dfrac{a(r^n-1)}{r-1},\ r\neq 1 \quad \dots(1)$

Step I. Put $n=1$ in (1), we have

$$\text{L.H.S.}=a \text{ and R.H.S.}=\frac{a(r^1-1)}{r-1}=a$$

Thus, the proposition is true when $n=1$, *i.e.*, $P(1)$ is true.

Step II. Now we assume the proposition to be true for $n=m$

$$i.e., \quad a+ar+ar^2+\dots+ar^{m-1}=\frac{a(r^m-1)}{r-1} \quad \dots(2)$$

We now show that it implies that the formula is valid for $n=m+1$, *i.e.*, the proposition $P(m+1)$ is

$$a+ar+ar^2+\ldots+ar^{m-1}+ar^m=\frac{a(r^{m+1}-1)}{r-1} \qquad \ldots(3)$$

L.H.S. of (3) $=[a+ar+\ldots+ar^{m-1}]+ar^m=\dfrac{a(r^m-1)}{r-1}+ar^m$ [Using (2)]

$$=\frac{a}{r-1}[r^m-1+r^m(r-1)]$$

$$=\frac{a}{r-1}[r^m-1+r^{m+1}-r^m]$$

$$=\frac{a(r^{m+1}-1)}{r-1}=\text{R.H.S. of (3)}$$

From steps (I) and (II), by mathematical induction we prove that $P(n)$ is true for all positive integral values of n.

Example 44. *Find the sum of the series :*
$1+3+9+27+\ldots+$ to 10 terms.

Solution. Using the formula

$$S_n=\frac{a(r^n-1)}{r-1}, \text{ we have}$$

$$S_{10}=\frac{3^{10}-1}{3-1}=\frac{59{,}049-1}{2}=29{,}524.$$

Example 45. *Sum up the series $4+2+1+\frac{1}{2}+\frac{1}{4}+$ to 10 terms.*

Solution. Using the formula given in the previous example, we have

$$u_{10}=\frac{4[1-(\frac{1}{2})^{10}]}{1-\frac{1}{2}}=\frac{4\left[1-\frac{1}{1024}\right]}{\frac{1}{2}}=8\times\frac{1023}{1024}=8 \text{ (approx.)}$$

Example 46. *Find sum of the series :*
$243+324+432+\ldots$ to n terms.

Solution $\dfrac{324}{243}=\dfrac{4}{3}, \quad \dfrac{432}{324}=\dfrac{4}{3}$

$\therefore$ The series is a G.P. with $a=243$, $r=\dfrac{4}{3}>1$.

$$\therefore \quad S_n=\frac{a(r^n-1)}{r-1}=\frac{243\left\{\left(\frac{4}{3}\right)^n-1\right\}}{\frac{4}{3}-1}=\frac{243\left(\frac{4^n}{3^n}-1\right)}{\frac{1}{3}}$$

$$=243\times3\left(\frac{4^n-3^n}{3^n}\right)=\frac{3^6(4^n-3^n)}{3^n}=3^{6-n}(4^n-3^n).$$

Example 47. *Prove that the sum to n terms of the series :*

$11+103+1005+\ldots$ *is* $\frac{10}{9}(10^n-1)+n^2$

Solution. Let $S_n=11+103+1005+\ldots$

$$=(10+1)+(10^2+3)+(10^3+5)+\ldots$$

$$=(10+10^2+10^3+\ldots+10^n)+[1+3+5+\ldots+(2n-1)]$$

Now the series in the first bracket is in G.P. and in the second bracket in A.P.

$$\therefore \quad S_n=\frac{10(10^n-1)}{10-1}+\frac{n}{2}\{2+(n-1)2\}$$

$$=\frac{10}{9}(10^n-1)+\frac{n}{2}\{2+2n-2\}=\frac{10}{9}(10^n-1)+n^2$$

Example 48. *Find the sum to n terms of the series :*

(*a*) $x(x+y)+x^2(x^2+y^2)+x^3(x^3+y^3)+\ldots$

(*b*) $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots$

Solution. (*a*) The series can be split into two parts, each of which is a G.P. to n terms.

$$S_n=(x^2+xy)+(x^4+x^2y^2)+(x^6+x^3y^3)+\ldots \text{to } n \text{ terms}$$

$$=(x^2+x^4+x^6+\ldots\text{to } n \text{ terms})+(xy+x^2y^2+x^3y^3+\ldots\text{to } n \text{ terms})$$

$$=\frac{x^2\{1-(x^2)^n\}}{1-x^2}+\frac{xy\{1-(xy)^n\}}{1-xy}$$

$$=\frac{x^2(1-x^{2n})}{1-x^2}+\frac{xy(1-x^ny^n)}{1-xy}$$

(*b*) Let $S_n=(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots$ to n terms.

Multiplying both sides by $(x-y)$, we get

$$(x-y)S_n=(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots\text{to } n \text{ terms}$$

$$=(x^2+x^3+x^4+\ldots+\text{to } n \text{ terms})-(y^2+y^3+y^4+\ldots\text{to } n \text{ terms})$$

$$=\frac{x^2(1-x^n)}{1-x}-\frac{y^2(1-y^n)}{1-y}$$

$$S_n=\frac{1}{x-y}\left\{\frac{x^2(1-x^n)}{1-x}-\frac{y^2(1-y^n)}{1-y}\right\}$$

Example 49. *Sum to n terms the series :*

(*a*) $7+77+777+\ldots$; (*b*) $\cdot 7+\cdot 77+\cdot 777+\ldots$

Solution. We have

$$S_n=7+77+777+\ldots\text{to } n \text{ terms}$$
$$=7(1+11+111+\ldots\text{to } n \text{ terms})$$
$$=\frac{7}{9}(9+99+999+\ldots\text{to } n \text{ terms})$$
$$=\frac{7}{9}[(10-1)+(10^2-1)+(10^3-1)+\ldots\text{to } n \text{ terms}]$$
$$=\frac{7}{9}[(10+10^2+10^3+\ldots\text{to } n \text{ terms})$$
$$-(1+1+1+\ldots\text{to } n \text{ terms})]$$
$$=\frac{7}{9}\left[10\frac{(10^n-1)}{10-1}-n\right]=\frac{7(10^{n+1}-10)}{81}-\frac{7}{9}n$$

(*b*) $S_n=\cdot7+\cdot77+\cdot777+\ldots\text{to } n \text{ terms}$

$$=7[\cdot1+\cdot11+\cdot111+\ldots\text{to } n \text{ terms}]$$
$$=\frac{7}{9}[\cdot9+\cdot99+\cdot999+\ldots\text{to } n \text{ terms}]$$
$$=\frac{7}{9}[(1-\cdot1)+(1-\cdot01)+(1-\cdot001)+\ldots\text{to } n \text{ terms}]$$
$$=\frac{7}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots\text{to } n \text{ terms}\right]$$
$$=\frac{7}{9}\left[n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots\text{to } n \text{ terms}\right)\right]$$
$$=\frac{7}{9}\left\{n-\frac{1}{10}\quad\frac{1-\left(\frac{1}{10}\right)^n}{1-\left(\frac{1}{10}\right)}\right\}$$
$$=\frac{7}{9}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]$$

Example 50. *Find the least value of n for which the sum $1+3+3^2+$ $+\ldots$to n terms is greater than 7000.*

Solution. Sum to n terms$=1+3+3^2+\ldots+3^{n-1}=\frac{3^n-1}{3-1}=\frac{3^n-1}{2}$

This sum to n terms will be greater than 7000 if

$$\frac{3^n-1}{2}>7000$$

i.e., if $\quad 3^n-1>14000$, *i.e.*, if $3^n>14001$

i.e., if $\quad n\log 3>\log 14001$

i.e., if $\quad n>\frac{\log 14001}{\log 3}=\frac{4\cdot1461}{0\cdot4771}=8\cdot69$ (approx.)

Hence the least value of n is 9.

Example 51. *If S be the sum, P the product and R the sum of the reciprocals of n terms in G.P., prove that*

$$P^2=\left(\frac{S}{R}\right)^n$$

[*Delhi Uni. B.A. (Hons.) Eco., 1991*]

Solution. Let the n terms in G.P. be $a, ar, ar^2, \ldots, ar^{n-1}$

Now $$S=\frac{a(1-r^n)}{1-r}$$

$$P=a\,.\,ar\,.\,ar^2\ldots ar^{n-1}=a^n.r^{1+2+3+\cdots+(n-1)}=a^n\,.\,r^{\frac{(n-1)n}{2}}$$

$\therefore$ $$P^2=a^{2n}\,.\,r^{n(n-1)}$$

$$R=\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\ldots+\frac{1}{ar^{n-1}}=\frac{\frac{1}{a}\left\{\left(\frac{1}{r}\right)^n-1\right\}}{\frac{1}{r}-1}$$

$$=\frac{r}{a}\cdot\frac{(1-r^n)}{(1-r)r^n}=\frac{1-r^n}{ar^{n-1}(1-r)}$$

$\therefore$ $$\frac{S}{R}=\frac{a(1-r^n)}{(1-r)}\;\frac{ar^{n-1}(1-r)}{(1-r^n)}=a^2r^{n-1}$$

$\Rightarrow$ $$\left(\frac{S}{R}\right)^n=(a^2r^{n-1})^n=a^{2n}r^{n(n-1)}=P^2$$

$\Rightarrow$ $$P^2=\left(\frac{S}{R}\right)^n$$

Sum to infinite terms of a G.P. The sum to infinite terms of a G.P. is given by

$$S_\infty=\frac{a}{1-r} \text{ where } |r|<1.$$

Example 52. *Find sum of the following series :*

(*a*) $8+4\sqrt{2}+4+\ldots to\ \infty$

(*b*) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+\ldots to\ \infty$

Solution. (*a*) The series is a G.P. with $a=8$ and $r=\frac{1}{\sqrt{2}}$

$\therefore$ $$S_\infty=\left[\frac{a}{1-r}\right]=\frac{8}{1-\frac{1}{\sqrt{2}}}=\frac{8\sqrt{2}}{\sqrt{2}-1}=\frac{8\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$=\frac{8(2+\sqrt{2})}{2-1}=8(2+\sqrt{2})$$

(b) $S_\infty=\left(\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\ldots \text{to } \infty\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots\text{to } \infty\right)$

$=\left(\text{An infinite G.P. with } a=\frac{1}{2} \text{ and } r=\frac{1}{4}\right)$

$+\left(\text{An infinite G.P. with } a=\frac{1}{9} \text{ and } r=\frac{1}{9}\right)$

$$=\frac{\frac{1}{2}}{1-\frac{1}{4}}+\frac{\frac{1}{9}}{1-\frac{1}{9}}=\frac{2}{3}+\frac{1}{8}=\frac{19}{24}.$$

Example 53. *The sum of an infinite series in G.P. is 57 and the sum of their cubes is 9747, find the series.*

Solution. Let the series be $a+ar+ar^2+\ldots\infty$

Then $$\frac{a}{1-r}=57 \qquad \ldots(1)$$

Also, the series whose terms are the cubes of its terms is

$$a^3+a^3r^3+a^3r^6+\ldots\infty$$

$$\therefore \qquad \frac{a^3}{1-r^3}=9747 \qquad \ldots(2)$$

Dividing the cube of (1) by (2), we get

$$\frac{a^3}{(1-r)^3}\times\frac{1-r^3}{a^3}=\frac{57\times57\times57}{9747}$$

$$\Rightarrow \quad \frac{(1-r)(1+r+r^2)}{(1-r)^3}=19, \text{ i.e., } \frac{1+r+r^2}{1-2r+r^2}=19$$

$$\Rightarrow \quad (1+r+r^2)=19-38r+19r^2$$

$$\Rightarrow \quad 18r^2-39r+18=0$$

$$\Rightarrow \quad (3r-2)(6r-9)=0$$

$$\Rightarrow \quad r=\frac{2}{3} \text{ or } \frac{3}{2}$$

We reject $r=\frac{3}{2}$, because the sum of an infinite G.P. exists only when r is numerically less than 1

$$\therefore \quad r=\frac{2}{3}$$

$\therefore$ From (1), we get $\frac{a}{1-\frac{2}{3}}=57 \Rightarrow a=19$

Hence, the series in G.P. is $19, \frac{38}{3}, \frac{76}{9}, \ldots$

Example 54. *In an infinite G.P. each term is equal to three times the sum of all the terms that follow it and the sum of the first two terms is 15. Find the sum of the series to infinity.*

Solution. Let a be the first term and r the common ratio of the series. Then the series is

$$a+ar+ar^2+ar^3+\ldots\infty$$

Since nth term $=3\times$ Sum of the terms that follows the nth term

$$ar^{n-1}=3(ar^n+ar^{n+1}+ar^{n+2}+\ldots\infty)=3\cdot\frac{ar^n}{1-r}$$

$$\Rightarrow \qquad 1-r=3r$$

$$\Rightarrow \qquad r=\frac{1}{4}$$

Also we are given $a+ar=15$

$$\Rightarrow \qquad a=\frac{15}{1+r}=\frac{15}{1+\frac{1}{4}}=12$$

Hence $$S_\infty=\frac{a}{1-r}=\frac{12}{1-\frac{1}{4}}=16$$

Example 55. *If* $x=1+a+a^2+\ldots\infty$,
$y=1+b+b^2+\ldots\infty$, *prove that*

$$1+ab+a^2b^2+\ldots\infty=\frac{xy}{x+y-1}.$$

when $|a|$ *and* $|b|$ *are less than 1.*

Solution. $x=1+a+a^2+\ldots\infty=\dfrac{1}{1-a}$

because the series is an infinite G.P. with first term 1 and common ratio a and $|a|<1$.

$$y=1+b+b^2+\ldots\infty=\frac{1}{1-b}$$

$$\text{L.H.S.}=1+ab+a^2b^2+\ldots\infty=\frac{1}{1-ab}$$

$$\text{R.H.S.}=\frac{xy}{x+y-1}=\frac{\frac{1}{1-a}\cdot\frac{1}{1-b}}{\frac{1}{1-a}+\frac{1}{1-b}-1}$$

$$=\frac{1}{(1-b)+(1-a)-(1-a)(1-b)}=\frac{1}{1-ab}$$

$$\therefore \qquad \text{L.H.S.}=\text{R.H.S.}$$

Representation of terms in G.P. The following are some convenient ways of representing terms of G.P. by symbols :

(*ii*) Three numbers in G.P. : $\frac{a}{r}$, a, ar,

(*ii*) four numbers in G.P. : $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3, and

(*iii*) five numbers in G.P. : $\frac{a}{r^2}$, $\frac{a}{r}$, a, ar, ar^2 etc.

The usefulness of assuming the terms in the above form will be illustrated in the following examples.

Example 56. *The sum of 3 numbers in G.P. is 35 and their product is 1,000. Find the numbers.*

Solution. Let $\frac{a}{r}$, a, ar be the three numbers in G.P.

The product of these numbers $=\frac{a}{r} \cdot a \cdot ar=1000$

$\Rightarrow \quad a^3=1000$, *i.e.*, $a=10$

The sum of the three numbers is

$$\frac{a}{r}+a+ar=35$$

$$\Rightarrow \quad a\left(\frac{1}{r}+1+r\right)=35$$

$$\Rightarrow \quad 10(1+r+r^2)=35r \qquad [\because \; a=10]$$

$$\Rightarrow \quad 2+2r+2r^2=7r$$

$$\Rightarrow \quad 2r^2-5r+2=0$$

$$\Rightarrow \quad (2r-1)(r-2)=0$$

$$\therefore \quad r=\frac{1}{2} \text{ or } 2$$

Thus, we have $a=10$ and $r=\frac{1}{2}$ or 2

For $r=\frac{1}{2}$, the numbers will be 10×2, 10, $10\times\frac{1}{2}$, *i.e.*, 20, 10, 5 and for $r=2$, the numbers are $\frac{10}{2}$, 10, 10×2, *i.e.*, 5, 10, 20 which are the same but in the reverse order.

Hence the three numbers are 5, 10, 20.

Example 57. *Find three numbers in G.P. such that their sum is 21, and the sum of their squares is 189.*

Solution. Let the three numbers in G.P. be $\frac{a}{r}$, a, ar.

$\therefore \quad \frac{a}{r}+a+ar=21$

$\Rightarrow \quad \frac{a}{r}(1+r+r^2)=21 \quad \ldots(1)$

and $\quad \frac{a^2}{r^2}+a^2+a^2r^2=189$

$\Rightarrow \quad \frac{a^2}{r^2}(1+r^2+r^4)=189 \quad \ldots(2)$

Dividing (2) by (1), we get

$$\frac{a(r^4+r^2+1)}{r(r^2+r+1)}=9$$

$\Rightarrow \quad \frac{a}{r} \cdot \frac{(r^2+r+1)(r^2-r+1)}{r^2+r+1}=9$

$\therefore \quad \frac{a}{r}(r^2-r+1)=9 \quad \ldots(3)$

Again dividing (1) by (3), we get

$$\frac{r^2+r+1}{r^2-r+1}=\frac{21}{9}=\frac{7}{3}$$

$\Rightarrow \quad 7r^2-7r+7=3r^2+3r+3$

$\Rightarrow \quad 4r^2-10r+4=0$

$\Rightarrow \quad 2r^2-5r+2=0$

$\Rightarrow \quad (2r-1)(r-2)=0$

$\therefore \quad r=\frac{1}{2}$ or 2

When $r=2$, from (1), we get $\frac{a}{2}(1+2+4)=21$

$\therefore \quad a=6$

Hence the three numbers are 3, 6, 12

For $r=\frac{1}{2}$, we get the same numbers but in the reverse order.

Example 58. *If a, b, c are in G.P., prove that* $a(b^2+c^2)=c(a^2+b^2)$.

Solution. Let r be the common ratio of the G.P., we have

$$b=ar,\ c=ar^2$$

$$\text{L.H.S.}=a(b^2+c^2)=a(a^2r^2+a^2r^4)=a^3r^2(1+r^2)$$

$$\text{R.H.S.}=c(a^2+b^2)=ar^2(a^2+a^2r^2)=a^3r^2(1+r^2)$$

$$\text{L.H.S.}=\text{R.H.S.}$$

Example 59. *If a, b, c, d are in G.P., prove that*

(*i*) $\frac{ab-cd}{b^2-c^2}=\frac{a+c}{b}$, (*ii*) $(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$.

Solution. We are given that a, b, c, d are in G.P., therefore, if r is the common ratio of this G.P., then

$$b=ar,\ c=ar^2,\ d=ar^3$$

(*i*)
$$\text{L.H.S.}=\frac{ab-cd}{b^2-c^2}=\frac{a\ .\ ar-ar^2\ .\ ar^3}{a^2r^2-a^2r^4}$$

$$=\frac{a^2r(1-r^4)}{a^2r^2(1-r^2)}=\frac{(1+r^2)}{r}$$

$$\text{R.H.S.}=\frac{a+c}{b}=\frac{a+ar^2}{ar}$$

$$=\frac{a(1+r^2)}{ar}=\frac{1+r^2}{r}$$

$\therefore$ L.H.S. = R.H.S.

(*ii*)
$$\text{L.H.S.}=(ab+bc+cd)^2$$
$$=(a\ .\ ar+ar\ .\ ar^2+ar^2\ .\ ar^3)^2$$
$$=a^4r^2(1+r^2+r^4)^2$$

$$\text{R.H.S.}=(a^2+b^2+c^2)(b^2+c^2+d^2)=(a^2+a^2r^2+a^2r^4)(a^2r^2+a^2r^4+a^2r^6)$$
$$=a^2(1+r^2+r^4)\ .\ a^2r^2(1+r^2+r^4)=a^4r^2(1+r^2+r^4)^2$$

$\therefore$ L.H.S. = R.H.S.

Example 60. *If a, b, c, d are in G.P., prove that $a+b$ $b+c$, $c+d$ are also in G.P.*

Solution. We are given that a, b, c, d form a G.P. Hence if r is the common ratio of the G.P., then

$$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}=r$$

$\Rightarrow$ $b=ar,\ c=br,\ d=cr$

$\Rightarrow$ $b=ar,\ c=ar^2,\ d=ar^3$

Now $a+b,\ b+c,\ c+d$ will be in G.P.

if $(b+c)^2=(a+b)(c+d)$

i.e., if $(ar+ar^2)^2=(a+ar)(ar^2+ar^3)$

i.e., if $a^2r^2(1+r)^2=a^2r^2(1+r)^2$

which is true. Hence the required result.

Example 61. *If a^2+b^2, $ab+bc$ and b^2+c^2 are in G.P., prove that a, b, c are also in G.P.*

Solution. a^2+b^2, $ab+bc$, b^2+c^2 are in G P.

$$\Rightarrow \quad \frac{ab+bc}{a^2+b^2}=\frac{b^2+c^2}{ab+bc}$$

$$\Rightarrow \quad (a^2+b^2)(b^2+c^2)=(ab+bc)^2$$

$$\Rightarrow \quad a^2b^2+a^2c^2+b^4+b^2c^2=a^2b^2+b^2c^2+2ab^2c$$

$\Rightarrow \quad b^4+a^2b^2-2ab^2c=0$

$\Rightarrow \quad (b^2-ac)^2=0$

$\Rightarrow \quad b^2-ac=0$

$\Rightarrow \quad b^2=ac$

$\Rightarrow \quad a, b, c$ are in G.P.

12·6. GEOMETRIC MEAN

When any number of quantities form a G.P., the quantities lying in between the first and the last are called the *Geometric Means* (briefly written as G.Ms. between the first and the last. Thus if $a, G_1, G_2, G_3, \ldots, G_n$, b are in G.P., the quantities $G_1, G_2, G_3, \ldots, G_n$ are the G.Ms. between a and b. In particular, if we take any three quantities of a series in G.P., the middle term is known as the *Geometric Mean* (G.M.) between the other two.

Let G be the geometric mean between a and b, then a, G, b will be three terms of an order set in geometric progression so that

$$\frac{G}{a}=\frac{b}{G}$$

$\Rightarrow \quad G^2=ab$

$\Rightarrow \quad G=\sqrt{ab}$

For example, if 4, 8, 16 are consecutive terms in G.P., then

$$\frac{16}{8}=\frac{8}{4}$$

i.e., $\quad 64=4\times 16$

$\therefore \quad 8=\sqrt{4\times 16}$

Insertion of Geometric Means. Let $G_1, G_2, \ldots, G_n$ be the n geometric means between a and b. Then $a, G_1, G_2, \ldots, G_n, b$ are in G.P.

Let r denote the common ratio of this G.P., including the given terms a and b, there are $(n+2)$ terms in this

$\therefore \quad b=(n+2)$th term of the G.P.$=ar^{(n+2)-1}=ar^{n+1}$

$\Rightarrow \quad r^{n+1}=\frac{b}{a},$

i.e., $\quad r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ or $\sqrt[n+1]{\frac{b}{a}}$

$\therefore \quad G_1=ar=a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$$G_2=ar^2=a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$

$$\vdots \quad \vdots \quad \vdots$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}} \text{ respectively.}$$

Remark. Product of n geometric means is

$$G_1 . G_2 . G_3 . \ldots G_n = a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \left(\frac{b}{a}\right)^{\frac{2}{n+1}} \ldots \left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}(1+2+\ldots+n)}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}} = a^n \left(\frac{b}{a}\right)^{\frac{n}{2}} = (ab)^{n/2}$$

Now, to find out the value of r we can adopt this simple procedure :

Let a be the first term, l be the last term and n the number of means, then there are $n+2$ terms where

$$l = (n+2)\text{th term}$$

which can also be written as :

$l = ar^{n+1}$ [where n stands for the number of means between a and l]

$$r^{n+1} = \frac{l}{a}$$

$$r = \sqrt[n+1]{\frac{l}{a}} = \left(\frac{l}{a}\right)^{1/(n+1)}$$

Example 62. *Insert 5 geometric means between 320 and 5.*

Solution. We have in all 7 terms of which the first term is 320 and the 7th term is 5. Therefore, using the formula $r^{n+1} = \frac{l}{a}$

$$\begin{bmatrix} \text{where} & l = 5 \\ \text{and} & a = 320 \end{bmatrix}$$

We have $\quad r^6 = \frac{1}{64}$, *i.e.*, $= \sqrt[6]{\frac{1}{64}} = \left(\frac{1}{64}\right)^{1/6}$

$\therefore \quad r = \frac{1}{2}$, which is the common ratio.

Therefore, the series is 320, 160, 80, 40, 20, 10 and 5, and the geometric means are 160, 80, 40, 20 and 10.

Example 63. (*a*) *If a, b, x, y, z are positive numbers such that a, x, b are in A.P., a, y, b are in G.P., and $(a+b)z=2ab$, prove that*

(*i*) *x, y, z, are in G.P., and* (*ii*) $x \geqslant y \geqslant z$.

Solution. (*i*) Numbers a, x, b are in A.P.

$$\Rightarrow \qquad x=\frac{a+b}{2} \qquad \ldots(1)$$

Numbers a, y, b are in G.P.

$$\Rightarrow \qquad y^2=ab \qquad \ldots(2)$$

Also
$$z=\frac{2ab}{a+b} \qquad \ldots(3)$$

Now
$$y^2=ab=\frac{2ab}{a+b}\cdot\frac{a+b}{2}=xz$$

$$\Rightarrow \qquad \frac{x}{y}=\frac{y}{z}, \text{ i.e., } x, y, z \text{ are in G.P.}$$

(*ii*)
$$x-y=\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}\left(a+b-2\sqrt{ab}\right)$$
$$=\frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2 \geqslant 0$$

$$\Rightarrow \qquad x \geqslant y, \text{ i.e., } \frac{x}{y} \qquad \ldots(4)$$

But
$$\frac{y}{z}=\frac{x}{y} \geqslant 1 \quad \Rightarrow \quad y \geqslant z \qquad \ldots(5)$$

From (4) and (5), we have

$$x \geqslant y \geqslant z$$

(*b*) *Find four positive integers x, y, z, w such that y, z, w are in Arithmetic Progression ; x, y, z are in Geometric Progression and $z+w=10$, $x+y=3$.* [*C.A., Nov., 1991*]

Solution. y, z, w are in A.P. $\Rightarrow$ $2z=y+w$ ...(1)

x, y, z are in G.P. $\Rightarrow$ $y^2=xz$ ...(2)

Again
$$z+w=10 \qquad \ldots(3)$$
and
$$x+y=3 \qquad \ldots(4)$$

From (1) and (3), we get

$$y=3z-10 \qquad \ldots(5)$$

From (4) and (5), we get

$$x+(3z-10)=3 \quad \Rightarrow \quad x=13-3z \qquad \ldots(6)$$

Substituting the values of x and y [obtained from (5) and (6)] in (2), we get

$$(3z-10)^2=(10-3z)\,.\,z$$

$$\Rightarrow \quad 12z^2-73z+100=0$$
$$\Rightarrow \quad 12z^2-48z-25z+100=0$$
$$\Rightarrow \quad (12z-25)(z-4)=0$$
$$\Rightarrow \quad z=4 \text{ or } z=\frac{25}{12}$$

But z being an integer so $z=\frac{25}{12}$ is rejected.

Thus $z=4$ is an admissible value.

$$\therefore \quad x=13-3z=13-12=1$$
$$y=3z-10=12-10=2$$
$$w=10-z=10-4=6$$

Hence $x=1, y=2, z=4, w=6$.

Example 64. *For three consecutive months, a person deposits some amount of money on the first day of each month in small savings fund. These three successive amounts in the deposit, the total values of which is Rs. 65, form a G.P. If the two extreme amounts be multiplied each by 3 and the mean by 5, the products form an A.P. Find the amounts in the first and second deposits.*

Solution. Let the three successive deposits be Rs. a, Rs. ar and Rs. ar^2.

Thus $$a+ar+ar^2=65 \quad \ldots(1)$$

Also $3a$, $5ar$ and $3ar^2$ form an A.P.

Thus $$3a-5ar=5ar-3ar^2$$
$$\Rightarrow \quad 3ar^2-10ar+3a=0$$
$$\Rightarrow \quad 3r^2-10r+3=0, \text{ i.e., } (r-3)(3r-1)=0$$
$$\therefore \quad r=3, \tfrac{1}{3}$$

When $r=3$, from (1), we get $a+3a+9a=65$

or $$a=5$$

Thus the amounts are Rs. 5, Rs. 15 and Rs. 45.

Again if $r=\frac{1}{3}$ then from (1), we get $a+\frac{a}{3}+\frac{a}{9}=65$

or $$13a=65\times 9, \text{ i.e., } a=45$$

Thus the three successive deposits are Rs. 45, Rs. 15, Rs. 5. Hence the amounts in the first and second deposits are either Rs. 5, Rs. 15 or Rs. 45, Rs. 15.

Example 65. *At 10% per annum compound interest, a sum of money accumulates to Rs. 8750 in 4 years. Find the sum invested initially.*

Solution. Let P be the principal, then

amount of P after 1 year $=P\left(1+\frac{10}{100}\right)=P\times 1{\cdot}1$

,, ,, ,, ,, 2 years $=P\times(1{\cdot}1)^2$

,, ,, ,, ,, 3 years $=P\times(1{\cdot}1)^3$

,, ,, ,, ,, 4 years $=P\times(1{\cdot}1)^4$

$\therefore \quad P\times(1{\cdot}1)^4=8650$

$$P=\frac{8750}{(1{\cdot}1)^4}=\frac{8750}{1{\cdot}4641}=5976{\cdot}37$$

which is the required principal.

Example 66. *If the value of Fiat car depreciated by 25 per cent annually, what will be its estimated value at the end of 8 years if its present value is Rs. 2048 ?*

Solution Present value of car = Rs. 2048

Value of car depreciated = 25% annually

If present value is 100, then value after one year = Rs. 75

,, ,, ,, ,, 1 ,, ,, ,, ,, ,, $=\frac{75}{100}$

,, ,, ,, ,, 2048 ,, ,, ,, ,, ,, $=\frac{75}{100}\times 2048$

= Rs. 1536

$\therefore \quad a=1536$

We also note that values at the end of second, third, fourth, fifth, sixth, seventh and eighth years form a G.P. with common ratio $(r)=\frac{75}{100}=\frac{3}{4}$

$\therefore$ Value at the end of eight years

$$=ar^{8-1}=ar^7=1536\times\left(\frac{3}{4}\right)^7$$

$$=1536\times\frac{3\times3\times3\times3\times3\times3\times3}{4\times4\times4\times4\times4\times4\times4}=\frac{6561}{32}=205{\cdot}03$$

EXERCISE (II)

1. Find

(*i*) The 6th term of 5, 15, 45,...

(*ii*) The 8th term of $\frac{1}{2}, -\frac{1}{3}, \frac{2}{9},\ldots$

(*iii*) The 12th term of $2, -2\sqrt{3}, 6,\ldots$

(*iv*) The 7th term of $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \ldots$

(*v*) The 10th term of $\frac{1}{\sqrt{2}}, -1, \sqrt{2}, \ldots$

2. Find the *n*th term of the series :

(*i*) $2\frac{3}{4}, 1\frac{5}{6}, 1\frac{2}{9}, \ldots$

(*ii*) $a^2-b^2, a+b, \frac{a+b}{a-b}, \ldots$

(*iii*) $9, -6, 4, \ldots$

(*iv*) $0{\cdot}004, 0{\cdot}02, 0{\cdot}1, \ldots$

(*v*) $72, -18, \frac{9}{2}, -\frac{9}{8}, \ldots$

(*vi*) $\frac{6}{\sqrt{3}}, 3\sqrt{3}, \frac{27}{2\sqrt{3}}, \frac{27\sqrt{3}}{4}, \ldots$

(*vii*) $\frac{x+y}{x-y}, 1, \frac{x-y}{x+y}, \ldots$

3. (*a*) Find a G.P. whose 3rd and 6th terms are 1 and $-\frac{1}{8}$ respectively. Write down the 10th term also.

(*b*) The third term of a G.P. is $\frac{2}{3}$ and the 6th term is $\frac{2}{81}$, find the 8th term.

(*c*) The product of first and second terms of a G.P. is 256 and that of second and third terms is 16, find the 5th term.

4. (*a*) Which term of the series 1, 2, 4, 8, ... is 256 ?

(*b*) Is $\frac{1}{3125}$ a term of the series 25, 5, 1, ... ?

(*c*) Find *n* if $\frac{1}{2^{17}}$ is the *n*th term of the series 16, 8, 4, ...

5. If the *n*th term of the series 1, 2, 4, 8, ... be the same as the *n*th term of the series 256, 128, 64, ... find out *n*.

6. The 4th term of a G.P. is *x*, the 10th term is *y* and the 16th term is *z*. Show that $xz=y^2$.

7. The *n*th term of a sequence is 2.3^{n-1}, show that it is in G.P. What is the first term and the common ratio ?

8. (*a*) If the mth term of a G.P. be n and the nth term be m, show that the $(m+n)$th term is $n^{m/m-n}/m^{n/m-n}$.

(*b*) In a G.P. if the $(p+q)$th term is m and the $(p-q)$th term is n, prove that the pth term is $\sqrt{mn}$.

9. Sum the following series :

(*i*) $1024+512+256+\ldots$ to 15 terms

(*ii*) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ to 12 terms

(*iii*) $1\frac{1}{2}+2\frac{1}{4}+3\frac{3}{8}+\ldots$ to 8 terms

(*iv*) $\sqrt{2}+\sqrt{6}+\sqrt{18}+\ldots$ to 10 terms

10. Sum to n terms the series :

(*i*) $\frac{1}{\sqrt{3}}+1+\frac{3}{\sqrt{3}}+\ldots$

(*ii*) $2\frac{1}{2}-1+\frac{2}{5}-\ldots$

(*iii*) $0{\cdot}3+0{\cdot}03+0{\cdot}003+\ldots$

(*iv*) $(a+b)+(a^2+2b)+(a^3+3b)+\ldots$

11. (*a*) The sum of the first eight terms of a G.P. is five times the sum of the first four terms. Find the common ratio.

(*b*) The sum of n terms of a G.P. whose first term is one and the common ratio is $\frac{1}{2}$, is $1\frac{127}{128}$, find n.

(*c*) In a G.P., the sum of n terms is 255, the last term is 128 and the common ratio is 2, find n.

(*d*) How many terms of the G.P. 1, 4, 16, ... must be taken to have their sum equal to 341 ?

12. Sum to n terms the series :

(*a*) $5+55+555+\ldots$

(*b*) $8+88+888+\ldots$

(*c*) $0{\cdot}5+0{\cdot}55+0{\cdot}555+\ldots$

13. Sum to n terms the series : (*a*) $0{\cdot}8+0{\cdot}88+0{\cdot}888+\ldots$

(*b*) $1{\cdot}03+(1{\cdot}03)^2+(1{\cdot}03)^3+\ldots$

(*c*) $\frac{a}{1+i}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\ldots+\frac{a}{(1+i)^n}$

14. The ratio of the 4th to the 12th term of a G.P. with positive common ratio is $\frac{1}{256}$. If the difference of the two terms be $61{\cdot}68$, find the sum of the series to 8 terms.

15. Sum the following series to infinity :

(i) $\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\dots$

(ii) $4{\cdot}0+0{\cdot}8+0{\cdot}16+0{\cdot}032+\dots$

(iii) $\sqrt{2}+\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{2}}+\dots$

(iv) $\frac{2}{3}+\frac{5}{9}+\frac{2}{27}+\frac{5}{81}+\frac{2}{243}+\dots$

(v) $(\sqrt{2}+1)+1+(\sqrt{2}-1)+\dots$

(vi) $\left(1+\frac{1}{2^2}\right)+\left(\frac{1}{2}+\frac{1}{2^4}\right)+\left(\frac{1}{2^2}+\frac{1}{2^6}\right)+\dots$

(vii) $\frac{4}{7}-\frac{5}{7^2}+\frac{4}{7^3}-\frac{5}{7^4}+\dots$

16. (a) The sum of infinite terms in a G.P. is 2 and the sum of their squares is $\frac{4}{3}$. Find the series.

(b) Find the infinite G.P. whose first term is $\frac{1}{4}$ and the sum is $\frac{1}{3}$.

(c) The first term of G.P. exceeds the 2nd term by 2 and the sum to infinity is 50, find the series.

17. (a) Find three numbers in G.P. such that

(i) their sum is 130, and their product is 27,000.

(ii) their sum is $\frac{13}{3}$, and the sum of their squares is $\frac{91}{9}$.

(b) Find five numbers in G.P., such that their product is 32 and the product of the last two is 108.

(c) The continued product of three numbers in G.P. is 27, and the sum of their products in pairs is 39. Find the numbers.

18. If a, b, c be in G.P., prove that

(i) $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{a+b+c}{a-b+c}$, (ii) $a(b^2+c^2)=c(a^2+b^2)$, and

(iii) $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3$

19. (a) If a, b, c, d are in G.P., show that

(i) $(a-b)^2, (b-c)^2, (c-d)^2$ are in G.P.

(ii) $(a^2+b^2), (ab+bc), (b^2+c^2)$ are in G.P.

(iii) $(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2$

(b) If $(a-b)$, $(b-c)$, $(c-a)$ are in G.P., then show that

$$(a+b+c)^2=3(ab+bc+ca)$$

20. Three numbers whose sum is 18 are in A.P. ; if 2, 4, 11 are added to them respectively, the resulting numbers are in G.P. Determine the numbers.

21. The sum of four numbers in G.P. is 60 and the A.M. of the first and last is 18. Find the numbers.

22. There are four numbers, the first three are in A.P. and the last three in G.P., the sum of the first and the last is 11, and the sum of the other two is 10, find the numbers.

23. If $\sqrt[x]{a}=\sqrt[y]{b}=\sqrt[z]{c}$ and a, b, c be in G.P., prove that x, y, z are in A.P.

24. The sum of the first three terms of the two series, one an A.P. and the other a G.P. is the same. If the first term of each of these is $\frac{2}{3}$ and the common difference of the A.P. is equal to the common ratio of the G.P., find the sum of each series to 20 terms.

25. If
$$x=a+\frac{a}{r}+\frac{a}{r^2}+\ldots\text{to }\infty$$
$$y=b-\frac{b}{r}+\frac{b}{r^2}-\ldots\text{ to }\infty$$
$$z=c+\frac{c}{r^2}+\frac{c}{r^4}+\ldots\text{ to }\infty$$
show that $\frac{xy}{z}=\frac{ab}{c}$

26. The sum of four numbers in G.P. is 60 and the A.M. between the first and the last is 18. Show that the numbers are 4, 8, 16, 32.

27. The sum of three numbers in G.P. is 70, if the two extreme items are multiplied each by 4 and the mean by 5, the products are in A.P. Show that the numbers are 10, 20, 40.

28. (*a*) If a, b, c are in A.P. and a, x, b and b, y, c are in G.P., show that x^2, b^2, y^2 are in A.P.

(*b*) If a, b, c are three unequal numbers in A.P. and $a, b-a, c-a$ are in G.P., show that $\frac{a}{1}=\frac{b}{3}=\frac{c}{5}$.

(*c*) If a, b, c are in A.P., and $a, b, (c+1)$ are in G.P., show that $c=(a-b)^2$.

(*d*) The numbers $x, 8, y (x\neq y)$ are in G.P. and the numbers $x, y, -8$ are in A.P. Find x and y.

29 The sequence a, b, c is an A.P. whose sum is 18. If a and b are each increased by 4 and c is increased by 36, the new numbers form a G.P. Find a, b, c.

30. If $S_1, S_2, \ldots, S_n$ are the sums of infinite geometric series whose first terms are 1, 2, ..., n and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}$ respectively, show that

$$S_1+S_2+\ldots+S_n=\frac{n(n+3)}{2}.$$

31. If S_n represents the sum of n terms of a G.P. whose first terms and common ratios are a and r respectively, then prove that

$$S_1+S_2+S_3+\ldots+S_n=\frac{na}{1-r}-\frac{ar(1-r^n)}{(1-r)^2}$$

[**Hint.** $S_n=\frac{a(1-r^n)}{1-r}$

$$\therefore \quad S_1+S_2+\ldots+S_n=\frac{a}{1-r}\Big[(1+1+1\ldots n \text{ times}) -(r+r^2+r^3+\ldots+r^n)\Big]\Big]$$

32. A manufacturer reckons that the value of the machine costing him Rs. 18,750 depreciates each year by 20%. Find the estimated value at the end of 5 years.

33. Calculate the population in 1985 if the population in 1975 is 55 crores and is growing at a compound rate of 2% annually.

34. Find the value of the machine after a period of 10 years if at the time of purchase it was worth Rs. 10,600. The machine is depreciated at the rate of 8% for first three years and at the rate of 10% for the rest of the period. The reducing balance method of charging depreciation was followed for the entire period.

35. If the population of a town increases 25 per thousand per year and the present population is 26,24,000, what will be the population in three years' time ? What was it a year ago ?

36. A person proposes to give alms to begger 1 nP. on the first day, 2 nP. on the second day, 4 nP. on the third day, 8 nP. on the fourth day and so on. How much does he need to pay in the month of February, 1972 ?

37. Show that a given sum of money if accumulating at 20 per cent per annum more than doubles in 4 years at compound interest.

ANSWERS

1. (*i*) 1215, (*ii*) $-\frac{64}{2187}$, (*iii*) $-486\sqrt{3}$, (*iv*) $\frac{1}{243\sqrt{3}}$, (*v*) -16.

2. (*i*) $\frac{11}{4}\left(\frac{2}{3}\right)^{n-1}$, (*ii*) $\frac{a+b}{(a-b)^{n-2}}$, (*iii*) $\frac{(-2)^{n-1}}{3^{n-3}}$, (*iv*) $\frac{5^{n-1}}{2}$

(*v*) $(-1)^{n-1}\,\frac{9}{2^{2n-5}}$, (*vi*) $\frac{3^{n-1/2}}{2^{n-2}}$, (*vii*) $\left(\frac{x-y}{x+y}\right)^{n-2}$

3. (*a*) $4, -2, 1, \ldots, -\frac{1}{128}$ (*b*) $\frac{2}{729}$, (*c*) $\frac{1}{8}$

4. (*a*) 9th, (*b*) Yes, (*c*) 22. 5. 5 7. 2, 3

9. (*i*) $2047\frac{15}{16}$, (*ii*) $\frac{1365}{2048}$, (*iii*) $73\frac{227}{256}$, (*iv*) $121(\sqrt{6}+\sqrt{2})$.

10. (*i*) $\frac{1}{6}(3+\sqrt{3})(3^{n/2}-1)$, (*ii*) $\frac{1}{14}\cdot\frac{5^n \pm 2^n}{5^{n-2}}$,

(*iii*) $\frac{1}{3}\left(1-\frac{1}{10^n}\right)$, (*iv*) $\frac{a(1-a^n)}{1-a}+\frac{1}{2}n(n+1)b$

11 (*a*) $\pm\sqrt{2}$ or 1, (*b*) 8, (*c*) 8, (*d*) 5.

12. (*a*) $\frac{50}{81}(10^n-1)-\frac{5}{9}n$,

(*b*) $\frac{80}{81}(10^n-1)-\frac{8}{9}n$, (*c*) $\frac{5}{9}n-\frac{5}{81}\left(1-\frac{1}{10^n}\right)$

13. (*a*) $\frac{8}{9}n-\frac{8}{81}\left(1-\frac{1}{10^n}\right)$, (*b*) $\frac{103}{3}\{(1{\cdot}03)^n-1\}$,

(*c*) $\frac{a}{i}\{1-(1+i)^{-n}\}$. **14.** 7·65.

15. (*i*) $\frac{3}{4}$, (*ii*) 5, (*iii*) $2\sqrt{2}$, (*iv*) $\frac{11}{8}$, (*v*) $\frac{1}{2}(4+3\sqrt{2})$,

(*vi*) $\frac{7}{3}$, (*vii*) $\frac{7}{12}-\frac{5}{48}$. 16. (*a*) $1, \frac{1}{2}, \frac{1}{4}, \ldots$,

(*b*) $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$, (*c*) $10, 8, \frac{32}{5}, \ldots$

17. (*a*) (*i*) 10, 30, 90, (*ii*) $\frac{1}{3}$, 1, 3, (*b*) $\frac{2}{9}, \frac{2}{3}$, 2, 18, (*c*) 1, 3, 9.

20. 3, 6, 9 ; 18, 6, −6. **21.** 4, 8, 16, 32. **22.** 2, 4, 6, 9.

24. $773\frac{1}{3}$, $2\left(1-\frac{2^{10}}{3^{10}}\right)$. **28.** (*d*) $x=16, y=4$

29. −2, 6, 14, or 46, 6, −34. **32.** Rs. 6144. **33.** 6,705 lakhs.

34. Rs. 3646. **35.** 28,25,761 ; 25,60,000. 36. Rs. 26,77,800·96.