13

Convergence and Divergence of Series

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- the meaning of convergence and divergence of series
- different methods for finding the convergence and divergence of series.

13·0. INTRODUCTION

The concepts of convergence and divergence are associated with infinite series, their limits and evaluation. Therefore, it becomes necessary to say something about them

13·1. INFINITE SERIES

We know that an expression whose successive terms are formed by some definite law is a sequence. When these individual terms are summed up with plus or minus signs they form a series. Now, if the series terminates at some assigned term it is called a finite series and if the number of terms are unlimited it will be an infinite series. We can define the series formally as follows :

Let $\{u_n\}=u_1, u_2, u_3, \ldots, u_n, \ldots$ be the sequence of real numbers, then the expression in the form

$$u_1+u_2+u_3+\ldots+u_n+\ldots$$

is called an infinite series and is denoted symbolically as

$$\sum_{n=1}^{\infty} u_n \text{ or } \Sigma u_n, \text{ where } u_1, u_2, u_3, \ldots, u_n, \ldots$$

are called the first term, second term, third term,..., nth term,... of the infinite series. The nth term of the infinite series is also sometimes known as the general term.

13·2. CONCEPT OF LIMIT

The concept of x approaching a implies that the difference between x and a (*i.e.*, $x-a$ or $a-x$) is decreasing steadily and is capable of being made as small as we like, say smaller than any positive number ε. We express this as

$$x \to a \text{ (} x \text{ tends towards } a\text{)}$$

which implies that the numerical difference between x and a can be less than any positive number ε, expressed as

$$|x-a| < \varepsilon$$

In case x is infinitely large we express it as

$$x \to \infty$$

which implies that we can take any value of x larger than any large positive number say m (when $x > m$).

For example, consider the function $f(x)=\dfrac{1}{x}$

The tabulated values of $f(x)$ foı larger and larger values of x are as follows :

x	5	10,000	1,000,000	20,000,000
$f(x)$	0·2	0·0001	0·000001	0·00000005

Here, $f(x)$ becomes smaller and smaller as x becomes larger and larger. By making x sufficiently large, we can make $f(x)$ less than any given number.

Thus $$f(x)<0{\cdot}00000005 \text{ if } x>20{,}000{,}000$$

This explains that $1/x$ approaches 0 as x becomes larger and larger *i.e.*, as $x \to \infty$. Therefore

$$\lim_{x\to\infty} f(x)=0$$

i.e., limit of function $1/x$ approaches 0 as x approaches infinity (∞).

Theorems on Limits :

If $\lim_{x\to a} f(x)=l$, and $\lim_{x\to a} g(x)=m$, then

I. $\lim_{x\to a} \{f(x)+g(x)\}=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)=l+m$

II. $\lim_{x\to a} \{f(x)-g(x)\}=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)=l-m$

III. $\lim_{x\to a} \{f(x)\,.\,g(x)\}=\lim_{x\to a} f(x)\,.\,\lim_{x\to a} g(x)=l\,.\,m$

IV. $\lim_{x\to a} \{f(x)/g(x)\}=\lim_{x\to a} f(x)/\lim_{x\to a} g(x)=l/m$, where $m\neq 0$

V. $\lim_{x\to a} x^n=a^n$

VI. $\lim_{x\to a} k=k$, where k is a constant.

13·3. EVALUATION OF LIMITS

Before giving the tests for convergence and divergence of a series, let us study the methods of finding the limits in various cases: The most common type of expressions in n, where limits are required to be found out for testing series for convergence are of the type $\frac{P(n)}{Q(n)}$, $P(n)$ and $Q(n)$ are polynomials in n of a suitable degree.

Case I. $\lim_{n\to\infty} \frac{P(n)}{Q(n)}=0$, if the degree or the power of $P(n)$ is smaller than that of $Q(n)$.

For example, let the degree of $P(n)$ be r and that of $Q(n)$ be s, where $s>r$. Dividing the numerator and the denominator by n^r and taking n large, the limit 0 can be obtained.

Case II. $$\lim_{n\to\infty} \frac{P(n)}{Q(n)}=\pm\infty,$$

if the degree of $Q(n)$ is smaller than that of $P(n)$.

Case III. $$\lim_{n\to\infty} \frac{P(n)}{Q(n)}=\frac{a_0}{b_0}$$

if degrees of $P(n)$ and $Q(n)$ are equal, where a_0 and b_0 are the coefficients of highest power of n in $P(n)$ and $Q(n)$ respectively.

Remark. The method of finding limits requires that the given expression, both in the numerator and the denominator, be divided by n (raised to the highest power of the expression) so that we have nearly all terms of the expression in the form $\frac{1}{n}$, $\frac{1}{n^2}$, etc., which we know approach zero when n becomes indefinitely large. Those terms of the expression which are not of this form, both in the numerator and denominator will determine the limit of the expression as illustrated in the following examples :

Example 1. *Find the value of limits in the following cases :*

(*a*) $\lim\limits_{n\to\infty} \dfrac{7n^3-8n^2+10n-7}{8n^3-9n^2+5}$, (*b*) $\lim\limits_{n\to\infty} \dfrac{n^4-7n^2+9}{3n^2+5}$

(*c*) $\lim\limits_{n\to\infty} \dfrac{3n^3+7n^2-11n+19}{17n^4+18n^3-20n+45}$, (*d*) $\lim\limits_{n\to\infty} \dfrac{2n}{(2n-1)(3n+5)}$ *and*

(*e*) $\lim\limits_{\to\infty} \dfrac{n^{\frac{1}{3}}(n^2+1)^{\frac{1}{3}}}{\sqrt{(2n^2+3n+1)}}$

Solution. (*a*) Denoting the expression by u_n, and dividing both the numerator and the denominator by n^3, we have

$$u_n = \frac{7-\frac{8}{n}+\frac{10}{n^2}-\frac{7}{n^3}}{8-\frac{9}{n}+\frac{5}{n^3}}$$

$$\therefore \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{7-\frac{8}{n}+\frac{10}{n^2}-\frac{7}{n^3}}{8-\frac{9}{n}+\frac{5}{n^3}} = \frac{7}{8}$$

$$\left(\because \ \frac{1}{n}\to 0 \text{ and } \frac{1}{n^2}\to 0 \text{ as } n\to\infty\right)$$

(*b*) Now dividing the numerator and denominator by n^4, we have

$$u_n = \frac{1-\frac{7}{n^2}+\frac{9}{n^4}}{\frac{3}{n^2}+\frac{5}{n^4}}$$

$$\therefore \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1-\frac{7}{n^2}+\frac{9}{n^4}}{\frac{3}{n^2}+\frac{5}{n^4}} = \frac{1}{0} = \infty$$

(*c*) Here dividing the numerator and the denominator by n^4, we have

$$u_n = \frac{\frac{3}{n} + \frac{7}{n^2} - \frac{11}{n^3} + \frac{19}{n^4}}{17 + \frac{18}{n} - \frac{20}{n^3} + \frac{45}{n^4}}$$

$$\therefore \quad \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{\frac{3}{n} + \frac{7}{n^2} - \frac{11}{n^3} + \frac{19}{n^4}}{17 + \frac{18}{n} - \frac{20}{n^3} + \frac{45}{n^4}} = 0$$

(*d*)
$$u_n = \frac{\frac{2}{n}}{\left(2 - \frac{1}{n}\right)\left(3 + \frac{5}{n}\right)}$$

$$\therefore \quad \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{\frac{2}{n}}{\left(2 - \frac{1}{n}\right)\left(3 + \frac{5}{n}\right)} = \frac{0}{6} = 0.$$

(*e*) Dividing the numerator and the denominator by n, we get

$$u_n = \frac{\left(1 + \frac{1}{n^2}\right)^{\frac{1}{3}}}{\sqrt{\left\{2 + \frac{3}{n} + \frac{1}{n^2}\right\}}} = \frac{1}{\sqrt{2}}$$

Since $\frac{1}{n^2}$, $\frac{3}{n}$ and $\frac{1}{n^2}$ tend to zero as n tends to infinity.

13·4. CONVERGENCE OF AN INFINITE SERIES

An infinite series consisting of n terms is a function of n (natural numbers). If n increases infinitely the sum may either tend to a certain finite limit or it may become infinitely large. Now, an infinite series will be convergent when the sum of the first n terms of the series cannot numerically exceed some finite quantity, however large n may be. This has been explained below :

A sequence $\{u_n\}$ is said to converge to a number l if the absolute difference between u_n and l, *i.e.*, $\{ | u_n - l | \}$ can be made as small as we please by taking n sufficiently large. Here l is the limit of the sequence $\{u_n\}$. To define it in a formal way, $\{u_n\}$ converges to a number l if for a given positive number ε, there corresponds a positive integer m such that

$$| u_n - l | < \varepsilon \; \forall n > m$$

Then l is said to be the limit of the sequence, and is written as

$$\lim_{n\to\infty} u_n = l$$

For example, let us take a sequence $\left\{\frac{1}{n}\right\}$ which we shall show converges to the limit 0. Take $\varepsilon > 0$ to be any small number. Then

$$|u_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \text{ if } n > \frac{1}{\varepsilon}.$$

Choosing the positive integer $m > \frac{1}{\varepsilon}$, we see that

$$|u_n - 0| = \frac{1}{n} < \varepsilon \;\forall\; n > m$$

$$\left|\frac{1}{n} - 0\right| < \varepsilon, \text{ when } \frac{1}{n} < \varepsilon.$$

Hence the sequence u_n tends to zero Yet the series formed by such sequence is not convergent which we will discuss later.

In short we say that an infinite series of positive terms is said to be convergent if its sum is a finite quantity.

13·5. DIVERGENCE OF AN INFINITF SERIES

An infinite series is divergent when the sum of the first n terms can be made numerically greater than any finite quantity by taking n sufficiently large. It can be defined as follows :

A sequence $\{u_n\}$ is said to diverge to $+\infty$ if u_n remains positive and becomes larger and larger as n becomes large in such a way that u_n can be made to exceed any finite number, however large by taking n sufficiently great.

Formally, $\{u_n\}$ diverges to $+\infty$ if for a given positive number k, howsoever large, there exists a positive number m such that

$$u_n > k \;\forall\; n > m.$$

In this case $\lim_{n\to\infty} u_n = +\infty.$

Similarly, a sequence $\{u_n\}$ is said to diverge to $-\infty$ for each negative number l, howsoever small, there corresponds a positive integer m such that

$$u_n > l \;\forall\; n > m$$

Then $\lim_{n\to\infty} u_n = -\infty$

For example, the sequence $\{u_n\}$ defined by $u_n = n^2$, it diverges to $+\infty$, while defined by $u_n = -(n^2)$, it diverges to $-\infty$.

13.6. OSCILLATORY SERIES

A sequence $\{u_n\}$ is said to be oscillatory when it neither converges nor diverges, *e.g.*,

$$u_n=(-1)^n$$
$$u_n=(-1)^n\, n$$

both are oscillatory.

13.7. SEQUENCE OF PARTIAL SUMS

Let Σu_n be a given infinite series. Let us form a sequence $\{S_n\}$ with the help of the series Σu_n as follows :

$$S_1=u_1$$
$$S_2=u_1+u_2$$
$$\vdots \qquad \vdots$$
$$S_n=u_1+u_2+\ldots+u_n$$

The sequence $\{S_n\}$ is called the *sequence of partial sums* of an infinite series.

The infinite series Σu_n is said to be convergent (*i.e.*, has a finite sum) if the corresponding sequence of partial sums $\{S_n\}$ is convergent.

In this case $l(=\lim\limits_{n\to\infty} S_n)$ is called the sum of the series Σu_n.

If the sequence $\{S_n\}$ is divergent, Σu_n is said to be divergent and if $\{S_n\}$ oscillates, the series Σu_n is said to be oscillatory.

GENERAL RULES

We can ascertain whether a series is convergent or divergent by finding out whether the sum of the first n terms of a given series remain finite or becomes infinite as n is made infinitely large.

(*i*) Consider the following expression of a series :

$$1+r+r^2+r^3+\ldots+r^{n-1}+\ldots$$

In this case if r is equal to 1, the sum of the first n terms is equal to n, therefore, it is divergent as indicated in the series below :

$$1+1+1+1+1\ldots$$

Here the sum of the first n terms can be made sufficiently large by taking n sufficiently large.

If r is equal to -1 then the series is $1-1+1-1+1-1+\ldots$

Here the sum of the series is equal to 0, if the number of terms is even and 1 if the number of terms is odd so the series does not converge nor does it diverge but it oscillates infinitely.

If r is numerically less than 1, then the sum approaches the finite limit $\frac{1}{1-r}$ and the series becomes convergent. For example, the series with $r=0.5$ will be

$$1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}+\ldots$$

$$\Rightarrow \qquad 1+0{\cdot}5+0{\cdot}25+0{\cdot}125+\ldots$$

and the sum will be equal to $\frac{1-r^n}{1-r}$. Therefore, with $r=0{\cdot}5$ the sum

$$S_n=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=2-\frac{1}{2^{n-1}}$$

$$\therefore \qquad \lim_{n\to\infty} S_n=2$$

Thus the series approaches a finite limit 2. It is, therefore, convergent.

If r is numerically greater than 1, the series will be divergent. For example if $r=2$, then the series becomes

$$1+2+2^2+2^3+\ldots+2^{n-1}\ldots \quad \text{or} \quad 1+2+4+8+16+\ldots$$

and the sum of the n terms will be equal to $\frac{r^n-1}{r-1}$.

When $r=2$ and $n=5$, the sum will be

$$S_n=\frac{2^n-1}{2-1}=\frac{2^5-1}{2-1}=\frac{31}{1}=31$$

$$\therefore \qquad \lim_{n\to\infty} S_n=\infty$$

(2) An infinite series of the type

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\left(\frac{1}{2}\right)^{n-1}+$$

has a common ratio $\frac{1}{2}$. It can also be written as

$$1+(\tfrac{1}{2})+(\tfrac{1}{2})^2+\ldots+(\tfrac{1}{2})^{n-1}+\ldots$$

Formally, it is a geometric series of the type

$$a+ar+ar^2+\ldots+ar^{n-1}+\ldots$$

This type of series is convergent when $r<1$ because it approaches a finite limit which in the above case is 2. The sum of the first n terms is

$$S_n=2-(\tfrac{1}{2})^{n-1} \qquad \text{[where } r=\tfrac{1}{2}, \text{ and } a=1]$$

Thus, S_n approaches 2 as $n\to\infty$ and $(\frac{1}{2})^{n-1}\to 0$.

(3) An infinite series of the following type is convergent :

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}+\ldots$$

Here the nth term of the series is

$$\frac{1}{n(n+1)} \quad \text{or} \quad \frac{1}{n}-\frac{1}{n+1}$$

Actually this series can be written in the form :

$$(1-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\ldots$$

The sum of the first n terms is

$$S_n = 1 - \frac{1}{(n+1)}.$$

The series is convergent because as $n \to \infty$, $S_n \to 1$, *i.e.*, its sum does not exceed 1.

13·8. CONVERGENCE OF SERIES

For an infinite series to be convergent its general term should approach zero as n becomes infinitely large. In symbols, for an infinite series

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

$$\Sigma u_n \text{ is convergent } \Rightarrow \lim_{n \to \infty} u_n = 0.$$

Proof. Let us suppose that $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \dots + u_n + \dots$ be a given convergent series. By definition the sequence $\{S_n\}$ of partial sums, where $S_n = u_1 + u_2 + \dots + u_n$ is convergent.

$\therefore \quad \lim_{n \to \infty} S_n = l$ (finite quantity)

Also

$$S_{n-1} = u_1 + u_2 + u_3 + \dots + u_{n-1} = S_n - u_n$$

$$\Rightarrow \quad u_n = S_n - S_{n-1}$$

$$\Rightarrow \quad \lim_{n \to \infty} u_n = \lim_{n \to \infty} (S_n - S_{n-1})$$

$$= \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = l - l = 0.$$

Hence Σu_n convergent $\Rightarrow \lim_{n \to \infty} u_n = 0.$

Remark. It should be noted that this theorem never proves the convergence of a series. It only states that if the above limit is not zero then the series must be divergent. It is still possible for the limit to be zero and the series to be divergent, *e.g.*, in the series

$$\frac{9}{22} + \frac{23}{32} + \frac{17}{42} + \dots + \frac{4n+5}{10n+12} + \dots$$

the nth term is

$$u_n = \frac{4 + \frac{5}{n}}{10 + \frac{12}{n}}$$

$$\therefore \quad \lim_{n \to \infty} u_n = \frac{4}{10}$$

and is not zero, therefore, the series is divergent.

But in the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

and yet the series has been proved to be divergent.

Thus, the condition of limit tending to zero is only a necessary condition but not the sufficient one. Even when the limit tends to zero, the series may not be convergent and for what we have to apply certain other tests, one of which is based on the use of an auxiliary series.

13·9. AUXILIARY SERIES

It is a series whose nature is known to us. The common auxiliary series which are used to test any series are :

I. $1+r+r^2+\ldots+r^{n-1}+\ldots$ where $r<1$.

II. $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$ where $p<1$.

We give the proofs below :

1. *An infinite geometric series*

$$1+r+r^2+r^3+\ldots+r^{n-1}+\ldots$$

(*i*) *is convergent if* $|r|<1$ *and its sum to* ∞ *is* $\frac{1}{1-r}$,

(*ii*) *is divergent if* $r \geqslant 1$,

(*iii*) *is finitely oscillatory if* $r=-1$,

(*iv*) *is infinitely oscillatory if* $r<-1$.

Proof. Let us consider the partial sum

$$S_n = 1+r+r^2+\ldots+r^{n-1}$$

Case. (*i*) When $|r|<1$

$$S_n = 1+r+r^2+\ldots+r^{n-1} = \frac{1-r^n}{1-r}$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1-r^n}{1-r}\right) = \frac{1}{1-r} - \lim_{n\to\infty}\left(\frac{r^n}{1-r}\right)$$

$$= \frac{1}{1-r} \quad (\because \; r^n \to 0 \text{ as } n\to\infty, \text{ when } |r|<1)$$

$=$ finite quantity.

$\Rightarrow$ sequence $\{S_n\}$ is convergent and consequently Σu_r is also convergent. The sum of the series is

$$\lim_{n\to\infty} S_n = \frac{1}{1-r}$$

Case (*ii*) Let $r>1$

$$S_n=1+r+r^2+\ldots+r^{n-1}$$
$$=\frac{r^n-1}{r-1}$$

$$\therefore \lim_{n\to\infty} u_n=\lim_{n\to\infty}\left(\frac{r^n}{r-1}\right)-\frac{1}{r-1}$$

$=\infty$ ($\because r^n\to\infty$ as $n\to\infty$ when $r>1$)

$\Rightarrow$ sequence $\{S_n\}$ is divergent hence the series is also divergent.

Let $r=1$

$$S_n=1+r+r^2+\ldots+r^{n-1}$$
$$=1+1+1+\ldots+1$$
$$=n$$

$$\therefore \lim_{n\to\infty} S_n=\infty$$

$\Rightarrow$ sequence $\{S_n\}$ is divergent and hence the series is divergent.

Case (*iii*) Let $r=-1$

$$S_n=1-1+1-\ldots+(-1)^{n-1}$$

$=1$ or 0 according as n is odd or even

$$\therefore \quad \lim_{n\to\infty} S_n=0 \text{ or } 1.$$

$\Rightarrow$ sequence $\{S_n\}$ oscillates between two points and as such Σu_n is an oscillatory series.

II. *The series*

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots+\frac{1}{n^p}+\ldots$$

is convergent if $p>1$ and divergent if $p\leqslant 1$.

Proof. Case I. When $p>1$

The terms of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\ldots$$

can be grouped as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\ldots$$

Now since $3>2 \Rightarrow 3^p>2^p$

$$\Rightarrow \quad \frac{1}{3^p}<\frac{1}{2^p}$$

$$\Rightarrow \quad \frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}=\frac{1}{2^{p-1}}$$

Using similar arguments, we have

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p}=\frac{1}{2^{2(p-1)}}$$

Also we have

$$\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}<\frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p}=\frac{8}{8^p}=\frac{1}{2^{3(p-1)}}$$

and so on.

$$\therefore \quad \frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5p}+\frac{1}{6p}+\frac{1}{7p}\right)+\ldots$$

$$<\frac{1}{1^p}+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+\ldots$$

The series on the R.H.S. is a geometric series with common ratio $\frac{1}{2^{p-1}}$ which is less than 1 $\left(\text{since } p>1 \Rightarrow \frac{1}{2^{p-1}}<1\right)$ converges (by geometric series test) and consequently the given series being less than a convergent series, is also convergent.

Case II. When $p=1$.

In this case the series is

$$\sum\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$$

can be grouped as follows :

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots\right)+\ldots$$

Now
$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

and so on.

$$\therefore \quad 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots$$

$$>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots \qquad \ldots(1)$$

The series on the R.H.S. is a geometric series with common ratio 1 diverges (by geometric series test) and consequently the given series being greater than a divergent series, is also divergent.

Case III. When $p<1$, we have

$$\frac{1}{n^p} > \frac{1}{n} \ \forall n$$

$$\Rightarrow \qquad \sum \frac{1}{n^p} \geqslant \sum \frac{1}{n}$$

But the series $\sum \frac{1}{n}$ is divergent.

$\therefore$ $\sum \frac{1}{n^p}$ being greater than a divergent series is also divergent.

Hence the auxiliary series $\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ is convergent if $p>1$ and divergent if $p \leqslant 1$

The above two auxiliary series form the basis of comparison test.

13·10. COMPARISON TEST

The convergence or divergence of a series can be determined by comparing it with some suitable auxiliary series (known also as standard series or helping series) whose convergence or divergence is known beforehand. Through this test we can know about the convergence or the divergence without knowing the sum of the series.

In the following comparison methods, Σv_n serves as an auxiliary series.

Test I. *(i)* If Σu_n, Σv_n are two positive term series and there is a positive constant k such that

$u_n < kv_n \ \forall \ n$ and Σv_n is convergent then Σu_n is also convergent.

(ii) If Σu_n, Σv_n are two positive term series and there is a positive constant k such that

$a_n > kv_n \ \forall \ n$ and Σv_n is divergent then Σu_n is also divergent.

Test II If Σu_n, Σv_n are two positive term series and there exists two positive constants A and B such that

$$A < \frac{u_n}{v_n} < B \ \forall \ n$$

then the two series Σu_n and Σv_n converge or diverge together.

Test III. (*Comparison by limits*)

(a) If Σu_n and Σu_n are two series of positive terms and

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l \ (\textit{finite and non-zero})$$

then the two series Σu_n and Σv_n are either both convergent or both divergent.

(b) *If* $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n}=0$ *and* Σv_n *is convergent, then* Σu_n *is also convergent.*

(c) *If* $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n}=\infty$ *and* Σv_n *is divergent, then* Σu_n *is also divergent.*

It should be noted that this test is very useful in practice and as such is worth remembering.

Test IV. If Σu_n and Σv_n be two series of positive terms, then Σu_n is convergent if (*i*) Σv_n is convergent and (*ii*) from and after some particular term

$$\frac{u_{n+1}}{u_n}<\frac{v_{n+1}}{v_n}.$$

Σu_n is divergent if (*i*) Σv_n is divergent, and (*ii*) from and after some particular term

$$\frac{u_{n+1}}{u_n}>\frac{v_{n+1}}{v_n}$$

Since the removal of a finite number of terms does not alter the nature of the series, we shall assume that the inequality is satisfied from the every first term.

How to find an Auxiliary Series. If the given series is indicated by its nth term u_n, we have to take a suitable auxiliary series v_n. First we have to simplify the given series Σu_n and then take the highest power of n, dropping the coefficient, as an auxiliary series, indicated by its nth term v_n.

Example 2. *Prove that the following series is divergent :*

$$\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\ldots+\frac{n+1}{n^2}+\ldots$$

Solution. Here $\quad u_n=\dfrac{n+1}{n^2}=\dfrac{1+\dfrac{1}{n}}{n}$

Let us take the auxiliary series Σv_n, where

$$v_n=\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n+1}{n^2}\div\frac{1}{n}=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1.$$

$\Rightarrow$ By limit comparison test both Σu_n and Σv_n converge or diverge together. But the auxiliary series $\Sigma\dfrac{1}{n}$ is divergent, hence the given series Σu_n is also divergent.

Example 3. *Discuss the convergence or divergence of the series :*

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots$$

Solution. Here $$u_n=\frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$=\frac{1}{n^{\frac{1}{2}}\left[1+\left(1+\frac{1}{n}\right)^{\frac{1}{2}}\right]}$$

Let us take the auxiliary series

$$\sum v_n=\sum\frac{1}{\sqrt{n}}$$

so that $$v_n=\frac{1}{\sqrt{n}}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left[1+\left(1+\frac{1}{n}\right)^{\frac{1}{2}}\right]}=\frac{1}{2}$$

which is a finite quantity other than zero.

$\Rightarrow$ By limit comparison test, Σu_n and Σv_n both converge or diverge together.

But the auxiliary series $\Sigma v_n=\sum\frac{1}{\sqrt{n}}$ is divergent as $p=\frac{1}{2}<1$.

Hence the given series Σu_n is also divergent.

Example 4. *Examine the convergence of the series :*

$$\frac{1\cdot 2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\ldots$$

Solution. Here $$u_n=\frac{(2n-1)\,2n}{(2n+1)^2\,(2n+2)^2}$$

degree of numerator is 2 and that of the denominator is 4, *i.e.*, degree of denominator is in excess by 2. This suggests that the given series may be compared with the auxiliary series Σv_n, where

$$v_n=\frac{1}{n^2}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n}=\frac{(2n-1)(2n)n^2}{(2n+1)^2(2n+2)^2}$$

$$=\frac{\left(2-\frac{1}{n}\right).2.1}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n}=\frac{2.2}{2^2\,.\,2^2}=\frac{1}{4}\neq 0$$

$\Rightarrow$ By limit comparison test Σu_n and Σv_n are either both convergent or both divergent. But $\Sigma v_n=\Sigma\frac{1}{n^2}$ is convergent, hence the given series is also convergent.

Example 5. *Examine the convergence of the series :*

$$\sum\left\{\sqrt{(n^3+1)}-\sqrt{n^3}\right\}$$

Solution. Let u_n denote the nth term of the series, then

$$u_n=\sqrt{(n^3+1)}-\sqrt{n^3}$$

$$=n^{3/2}\left[\left(1+\frac{1}{n^3}\right)^{\frac{1}{2}}-1\right], \text{ (apply Binomial Theorem)}$$

$$=n^{3/2}\left[\left\{1+\frac{1}{2n^3}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}\left(\frac{1}{n^3}\right)^2+\ldots\right\}-1\right]$$

$$=n^{3/2}\left[\left(1+\frac{1}{2n^3}-\frac{1}{8n^6}+\ldots\right)-1\right]$$

$$=\frac{1}{2n^{3/2}}-\frac{1}{8n^{9/2}}+\ldots$$

Comparing the series with the auxiliary series Σv_n, where

$$v_n=\frac{1}{n^{3/2}},$$

then,
$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{8n^3}+\ldots\right]=\frac{1}{2},$$

which is non-zero and finite.

$\Rightarrow$ By limit comparison test, the series Σu_n and Σv_n are either both convergent or both divergent. But the auxiliary series

$$\Sigma v_n=\sum\frac{1}{n^{3/2}}$$

is convergent ($p>1$), hence the given series is also convergent.

Example 6. *Show that the series whose nth term is*

$$\sqrt{(n^4+1)}-\sqrt{(n^4-1)}$$

is convergent.

Solution. Here
$$u_n=(\sqrt{n^4+1}-\sqrt{n^4-1})$$

$$=(\sqrt{n^4+1}-\sqrt{n^4-1})\times\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$=\frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$=\frac{2}{n^2\left[1+\frac{1}{n^4}\right]^{\frac{1}{2}}+n^2\left[1-\frac{1}{n^4}\right]^{\frac{1}{2}}}$$

$$=\frac{1}{n^2}\left[\frac{2}{\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}}+\left(1-\frac{1}{n^4}\right)^{\frac{1}{2}}}\right]$$

Let us take the auxiliary series Σv_n, where $v_n=\dfrac{1}{n^2}$ so that

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2}{\left[1+\left(\frac{1}{n^4}\right)\left(^{\frac{1}{2}}+\ 1-\frac{1}{n^4}\right)^{\frac{1}{2}}\right]}=\frac{2}{1+1}=1\neq 0$$

$\Rightarrow$ By limit comparison test both Σu_n and Σv_n converge or diverge together.

But $\quad \Sigma v_n=\sum\dfrac{1}{n^2}$ is convergent. $\quad (\because \ p>1)$

Hence the given series Σu_n is also convergent.

Example 7. *Examine for convergence the series :*

$$\frac{2^3}{1^k+3^k}+\frac{3^3}{2^k+4^k}+\frac{4^3}{3^k+5^k}+\ldots$$

Solution. Here

$$u_n=\frac{(n+1)^3}{n^k+(n+2)^k}=\frac{n^3\left[1+\frac{1}{n}\right]^3}{n^k\left[1+\left(1+\frac{2}{n}\right)^k\right]}$$

Degree of numerator is 3 and that of denominator is k. The degree of denominator is in excess by $k-3$. Compare the series with the auxiliary series Σv_n, where

$$v_n=\frac{1}{n^{k-3}}$$

It may be seen that $\displaystyle\lim_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{2}\neq 0$

$\Rightarrow$ By limit comparison test Σu_n and Σv_n are either both convergent or both divergent.

But Σv_n is convergent if $k-3>1$, *i.e.*, $k>4$ and divergent if $k-3\leqslant 1$, *i.e.*, $k\leqslant 4$.

Hence the given series is convergent if $k>4$ and divergent if $k\leqslant 4$.

Example 8. *Discuss the convergence of the series :*

$$\sum_{n=1}^{\infty}\frac{1}{(a+n)^p(b+n)^q}, \text{ (}p\text{ and }q\text{ being positive)}$$

Solution. Here

$$u_n=\frac{1}{(a+n)^p(b+n)^q}=\frac{1}{n^{p+q}\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q}$$

Taking auxiliary series

$$\Sigma v_n=\sum\frac{1}{n^{p+q}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q}=1\neq 0$$

which is finite for any given value of p and q and is non-zero.

$\therefore$ By limit comparison test, Σu_n and Σv_n either both converge or both diverge.

But $\Sigma v_n=\sum\frac{1}{n^{p+q}}$ is convergent if $p+q>1$ and divergent if $p+q\leq 1$.

Hence the given series is convergent if $(p+q)>1$ and divergent if $(p+q)\leq 1$.

Example 9. *Examine the convergence of the series :*

$$1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots+\frac{2^n-2}{2^n+1}x^{n-1}+\ldots$$

Solution. Here $\quad u_n=\frac{2^n-2}{2^n+1}x^{n-1}$

Let us take the auxiliary series $\Sigma v_n=\Sigma x^{n-1}$, then

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2^n-2}{2^n+1}=\lim_{n\to\infty}\frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}}=1$$

$\Rightarrow$ By limit comparison test both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n=\Sigma x^{n-1}$ (geometric series) converges if $|x|<1$ and diverges if $x\geq 1$, hence the given series converges if $x<1$ and diverges if $x\geq 1$.

Example 10. *Discuss the convergence or divergence of the series :*

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ldots$$

Solution. Since we know that the convergence or divergence of a series remains unaltered by the omission of a finite number of terms, neglecting the first term, we get

$$u_n=\frac{n^n}{(n+1)^{n+1}}=\frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}$$

$$=\frac{1}{n\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$$

Let us take the auxiliary series $\Sigma v_n=\sum\frac{1}{n}$

$$\therefore \qquad \frac{u_n}{v_n}=\frac{1}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}$$

$$=\frac{1}{e}\neq 0 \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\right]$$

$\Rightarrow$ By limit comparison test, Σu_n and Σv_n both converge or diverge together.

But the auxiliary series $\sum v_n=\sum\frac{1}{n}$ is divergent. Hence the given series Σu_n is also divergent.

EXERCISE (I)

1. Prove that the series :

(i) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$ is divergent.

(ii) $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\ldots$ is convergent.

(iii) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\frac{7}{4.5.6}+\ldots$ is convergent.

[**Hint.** $u_n=\frac{2n-1}{n(n+1)(n+2)}$, $v_n=\frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left\{\frac{1}{n^2}\cdot\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}\cdot\frac{n^2}{1}\right\}=2\Bigg]$$

(iv) $\frac{1}{\sqrt{3.4}}+\frac{1}{\sqrt{4.5}}+\frac{1}{\sqrt{5.6}}+\frac{1}{\sqrt{6.7}}+\ldots$ is divergent.

(v) $\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\sqrt{\frac{4}{5^3}}+\ldots$ is divergent.

(vi) $1+\frac{1}{4^{\frac{2}{3}}}+\frac{1}{9^{\frac{2}{3}}}+\frac{1}{14^{\frac{2}{3}}}+\ldots$ is divergent.

(vii) $1+\frac{1}{2.2^{\frac{1}{10}}}+\frac{1}{3.3^{\frac{1}{10}}}+\frac{1}{4.4^{\frac{1}{10}}}+\ldots$ is convergent.

(viii) $1+\frac{2^{\frac{1}{10}}}{2}+\frac{3^{\frac{1}{10}}}{3}+\frac{4^{\frac{1}{10}}}{4}+\ldots$ is divergent.

2. Examine the convergence of the series :

(i) $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+\ldots$

(ii) $\frac{1}{a.1^2+b}+\frac{2}{a.2^2+b}+\frac{3}{a.3^2+b}+\ldots$

(iii) $\frac{14}{1^3}+\frac{24}{2^3}+\frac{34}{3^3}+\ldots+\frac{4+10n}{n^3}+\ldots$

3. Examine for convergence of the series :

(i) $\sum \frac{n(n+1)}{(n+2)(n+3)(n+4)^2}$ (ii) $\sum_{k=1}^{\infty} \frac{k+1}{k(2k-1)}$

(iii) $\sum_{k=1}^{\infty} \frac{k+3}{k^3-k+1}$, (iv) $\sum (3n-1)^{-1}$, (v) $\sum \frac{\sqrt{n}}{n^2+1}$

(vi) $\sum \sqrt{\frac{n}{n^4+2}}$, (vii) $\sum \frac{n+\sqrt{n}}{n^2-2}$

(viii) $\sum \left(\frac{n^2+2}{15+2n^3}\right)^{\frac{1}{3}}$ (ix) $\sum_{n=1}^{\infty} \frac{(2n-1)}{n^2(n+1)^2}$. [*C.A., Nov., 1991*]

4. Test for convergence of the following series :

(i) $(\sqrt{2}-1)+(\sqrt{5}-2)+(\sqrt{10}-3)+\ldots+(\sqrt{n^2+1}-n)+\ldots$

(ii) $\sum_{1}^{\infty} \left\{\sqrt{(n+1)}-\sqrt{(n-1)}\right\}$

(iii) $\sum_{1}^{\infty} \left\{\sqrt{n^4+1}-n^2\right\}$, (iv) $\sum_{1}^{\infty} \left\{\sqrt{n^3+1}-\sqrt{n^3}\right\}$

(v) $\sum_{1}^{\infty} \left\{\sqrt[3]{(n^3+1)}-n\right\}$, (vi) $\sum_{1}^{\infty} \frac{\sqrt{(n+1)}-\sqrt{(n-1)}}{n}$

5. Examine the following series for convergence :

(i) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$

(ii) $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\ldots$

(iii) $\frac{1^p}{2^q}+\frac{2^p}{3^q}+\frac{3^p}{4^q}+\frac{4^p}{5^q}+\ldots$

(*iv*) $\frac{1^p}{2^{p+q}}+\frac{2^p}{3^{p+q}}+\frac{3^p}{4^{p+q}}+.$

6. Test for convergence the series whose general term is

(*i*) $\frac{1}{(2n+3)^p}$, (*ii*) $\frac{1}{(an+b)^{1+p}}$, (*iii*) $\frac{\sqrt{n+1}-\sqrt{n}}{n^p}$

ANSWERS

2. (*i*) Divergent, (*ii*) Divergent, (*iii*) convergent.

3. (*i*) Divergent, (*ii*) divergent, (*iii*) convergent, (*iv*) divergent, (*v*) convergent, (*vi*) convergent, (*vii*) divergent, (*viii*) divergent, (*ix*) convergent.

4. (*i*) Divergent, (*ii*) divergent, (*iii*) convergent, (*iv*) convergent (*v*) convergent, (*vi*) convergent.

5. (*i*) Convergent if $p>2$ and divergent if $p\leqslant 2$,
(*ii*) convergent if $p>1$, divergent if $p\leqslant 1$,
(*iii*) convergent if $q-p>1$, divergent if $q-p\leqslant 1$,
(*iv*) convergent if $q>1$, divergent if $q\leqslant 1$.

6. (*i*) Convergent if $p>1$ and divergent if $p\leqslant 1$. (*ii*) divergent, if $p\leqslant 0$, (*iii*) convergent if $p>\frac{3}{2}$ and divergent if $p\leqslant\frac{3}{2}$.

13·11. D' ALEMBERT'S RATIO TEST

If we could find S_n, *i.e.*, the sum of n terms of a series, the question of its convergency could be settled. In most cases this is impossible or very difficult, for which there is a very useful test which requires us only to know the nth term. We will state this theorem without proof.

Let $u_1+u_2+u_3+\ldots+u_n+\ldots$denote a series of positive terms. Find the ratio $\frac{u_n}{u_{n+1}}$ and find the limit of this ratio as n tends to infinity. If this limit is greater than 1, the series converges ; if less than 1 the series diverges, if equal to 1 ; the series may converge or diverge and this test gives no information.

In symbols, if Σu_n is a positive term series, then

(*i*) *it is convergent if* $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}>1$,

(*ii*) *is divergent if* $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}<1$,

(*iii*) *the test fails if* $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=1$

This test is known after the name of a French mathematician Jean e-Round D' Alembert as D' Alembert's test or the generalised ratio test.

Example 11. *Test for convergence of the series* :

(*a*) $\frac{1}{1}+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\ldots$

(b) $\frac{1}{10}+\frac{2}{11}+\frac{2^2}{12}+\frac{2^3}{13}+\ldots$

(c) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots$

Solution. (a) We note that the nth term of the series is

$$u_n=\frac{2n-1}{2^{n-1}} \text{ so that}$$

$$\frac{u_n}{u_{n+1}}=\frac{2n-1}{2^{n-1}}\cdot\frac{2^n}{2n+1}=2\cdot\frac{2n-1}{2n+1}=2\left[\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right]$$

Then $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim\limits_{n\to\infty} 2\left[\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right]=2>1.$

$\therefore$ The series is convergent.

(b) Here $u_n=\frac{2^{n-1}}{10+(n-1)}$ so that

$$\frac{u_n}{u_{n+1}}=\frac{2^{n-1}}{10+(n-1)}\cdot\frac{10+n}{2^n}$$

$$=\frac{1}{2}\cdot\frac{10+n}{9+n}=\frac{1}{2}\cdot\frac{\frac{10}{n}+1}{\frac{9}{n}+1}$$

$\therefore \lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim\limits_{n\to\infty}\left[\frac{1}{2}\cdot\frac{\frac{10}{n}+1}{\frac{9}{n}+1}\right]=\frac{1}{2}<1.$

Hence the series is divergent.

(c) Here $u_n=\frac{n}{n+1}$

$$\therefore \quad \frac{u_n}{u_{n+1}}=\frac{n}{n+1}\cdot\frac{n+2}{n+1}=\frac{1\cdot\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=1$$

Thus the test does not give any information. We note, however, that since

$$u_n=\frac{1}{1+\frac{1}{n}}$$

$$\therefore \qquad \lim_{n\to\infty} u_n=1,$$

and since this limit is not zero, the series is divergent.

Example 12. *Test the convergence or divergence of the series :*

$$\frac{1}{3}+\frac{2\,!}{9}+\frac{3\,!}{27}+\frac{4\,!}{81}+\ldots$$

Solution. The nth term of the series is

$$u_n=\frac{n\,!}{3^n}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}}=\frac{n\,!}{3^n}\cdot\frac{3^{n+1}}{(n+1)\,!}=\frac{n\,!\,3^n.3}{3^n(n+1)\,n\,!}=\frac{3}{n+1}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{3}{n+1}=\lim_{n\to\infty}\frac{\frac{3}{n}}{1+\frac{1}{n}}=0<1.$$

Hence the series is divergent.

Example 13. *Prove that the series* $\sum_{n=0}^{\infty}\frac{n^3+a}{2^n+a}$ *is convergent by using d' Alemoert's Ratio Test.*

Solution. Let u_n be nth term, then

$$u_n=\frac{n^3+a}{2^n+a} \text{ so that}$$

$$\frac{u_n}{u_{n+1}}=\frac{n^3+a}{(n+1)^3+a}\cdot\frac{2^{n+1}+a}{2^n+a}=\frac{1+\frac{a}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}}\cdot\frac{2+\frac{a}{2^n}}{1+\frac{a}{2^n}}$$

$$\text{Now} \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1+0}{1+0}\cdot\frac{2+0}{1+0}=2,$$

which is greater than 1.

Hence by ratio test, the series is convergent.

Example 14. *Test for convergence of the series whose nth term is*

(*i*) $\frac{n^2+1}{5^n}$, (*ii*) $\frac{n^n}{n\,!}$, (*iii*) $\frac{1.3.5\ldots(2n-1)}{n^4}$.

Solution. (*i*) Let u_n denote the nth term of the given series, then

$$\frac{u_n}{u_{n+1}}=\frac{n^2+1}{5^n}\cdot\frac{5^{n+1}}{(n+1)^2+1}=\frac{5(n^2+1)}{n^2+2n+2}$$

Thus $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=5>1$, consequently Σu_n converges.

(*ii*) $\frac{u_n}{u_{n+1}}=\frac{n^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}}=\frac{n^n}{(n+1)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}$

Obviously $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1$

Consequently Σu_n diverges.

(*iii*) $$\frac{u_n}{u_{n+1}}=\frac{1.3.5\ldots(2n-1)}{n^4}\cdot\frac{(n+1)^4}{1.3.5\ldots(2n-1)(2n+1)}$$
$$=\frac{\left(1+\frac{1}{n}\right)^4}{n\left(2+\frac{1}{n}\right)}=\left(\frac{1}{n}\right)\cdot\frac{\left(1+\frac{1}{n}\right)^4}{\left(2+\frac{1}{n}\right)}$$

Obviously $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=0$, consequently Σu_n diverges.

Example 15. *Prove that the series*

$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots$$

is convergent for $0\leqslant x<1$ and divergent for $x\geqslant 1$.

Solution. The nth term of the series is

$$u_n=\frac{x^n}{n}$$

Thus $$\frac{u_n}{u_{n+1}}=\frac{x^n}{n}\times\frac{n+1}{x^{n+1}}=\left(1+\frac{1}{n}\right)\cdot\frac{1}{x}$$

Then $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\cdot\frac{1}{x}\right]=\frac{1}{x}$

So if $\frac{1}{x}>1$, *i.e.*, if $x<1$, the series is convergent,

and if $\frac{1}{x}<1$, *i.e.*, if $x>1$, the series is divergent.

If $x=1$, the test gives no information, but in that case, we note that the given series reduces to the auxiliary series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ which we know is divergent.

Hence the given series is convergent if $x<1$ and is divergent if $x\geqslant 1$.

Example 16. *Discuss the convergence of the series:*

$$\frac{1}{3}+\frac{x}{36}+\frac{x^2}{243}+\ldots+\frac{x^n}{3^n.n^2}+\ldots$$

Solution. Here $u_n=\frac{x^n}{3^n.n^2}$

Thus $\frac{u_n}{u_{n+1}}=\frac{x^n}{3^n.n^2}\cdot\frac{3^{n+1}(n+1)^2}{x^{n+1}}=\left(1+\frac{1}{n}\right)^2\times\frac{3}{x}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^2 \cdot \frac{3}{x}\right] = \frac{3}{x}$$

⇒ By ratio test, the series Σu_n is convergent if $\frac{3}{x}>1$, *i.e.*, if $x<3$ and diverges if $\frac{3}{x}<1$, *i.e.*, $x>3$.

If $x=3$, then $u_n = \frac{3^n}{3^n . n^2} = \frac{1}{n^2}$.

But the series $\sum \frac{1}{n^2}$ is convergent, being an auxiliary series $(p>1)$.

Hence Σu_n is convergent if $x<3$ and divergent if $x>3$.

Example 17. *Test the convergence of the series :*

$$\frac{2}{1}x + \frac{3}{8}x^2 + \frac{4}{27}x^3 + \frac{5}{64}x^4 + \ldots + \frac{n+1}{n^3}x^n + \ldots$$

Solution. If Σu_n denotes the given series, then

$$u_n = \frac{n+1}{n^3} . x^n \; ; \; u_{n+1} = \frac{(n+1)+1}{(n+1)^3} . x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n+1}{n^3} . x^n . \frac{(n+1)^3}{n+2} . \frac{1}{x^{n+1}}$$

$$= \frac{(n+1)^4}{n^3(n+2)} \cdot \frac{1}{x} = \frac{n^4\left(1+\frac{1}{n}\right)^4}{n^3 . n\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

Hence by d' Alembert's Ratio test, the series is convergent if $\frac{1}{x}>1$, *i.e.*, $x<1$ and divergent if $\frac{1}{x}<1$, *i.e.*, $x>1$.

If $x=1$, then $u_n = \frac{n+1}{n^3} = \frac{\left(1+\frac{1}{n}\right)}{n^2}$

Taking the auxiliary series as $\Sigma u_n = \sum \frac{1}{n^2}$.

$$\frac{u_n}{v_n} = \left(1+\frac{1}{n}\right)$$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$, a non-zero finite quantity.

$\therefore$ Σu_n and Σv_n are either both convergent or both divergent.

Since $\sum v_n = \sum \frac{1}{n^2}$ is convergent, Σu_n is also convergent.

Hence the series is convergent if $x \leqslant 1$ and divergent if $x > 1$.

Example 18. *Examine the convergence of the series :*

$$\sum_{1}^{\infty} \frac{(n+1)}{(n+2)(n+3)} x^n, \ (x>0)$$

Solution. Here $u_n = \frac{(n+1)}{(n+2)(n+3)} \cdot x^n$

and $$u_{n+1} = \frac{(n+2)}{(n+3)(n+4)} \cdot x^{n+1} \text{ so that}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(n+4)}{(n+2)^2} \cdot \frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{4}{n}\right)}{\left(1+\frac{2}{n}\right)^2} \cdot \frac{1}{x}$$

$$\therefore \qquad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

By ratio test, it follows that the series Σu_n is convergent if

$$\frac{1}{x} > 1, \ i.e., \ x < 1$$

and divergent if $\frac{1}{x} < 1$, *i.e.*, $x > 1$

For $x = 1$, we have

$$u_n = \frac{(n+1)}{(n+2)(n+3)} = \frac{\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}$$

Choosing $v_n = \frac{1}{n}$, we get

$$\frac{u_n}{v_n} = \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)} = 1, \text{ a non-zero finite quantity.}$$

and thus the two series Σu_n and Σv_n converge or diverge together by comparison test.

But $\sum v_n = \sum \frac{1}{n}$ is divergent, therefore, the given series Σu_n is also divergent.

Hence Σu_n is convergent if $x<1$ and divergent if $x \geqslant 1$.

Example 19. *Test for convergence of the series :*

(a) $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots$

(b) $\frac{1}{1^p}+\frac{1}{3^p}x+\frac{1}{5^p}x^2+\ldots+\frac{1}{(2n+1)^p}x^n+\ldots$

Solution. (a) Here $u_n=\frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\therefore \quad \frac{u_n}{u_{n+1}}=\frac{n+2}{n+1}\cdot\sqrt{\left(\frac{n+1}{n}\right)}\cdot\frac{1}{x^2}$$

$$=\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\cdot\left(1+\frac{1}{n}\right)^{\frac{1}{2}}\cdot\frac{1}{x^2}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left[\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\cdot\left(1+\frac{1}{n}\right)^{\frac{1}{2}}\cdot\frac{1}{x^2}\right]=\frac{1}{x^2}$$

Hence the given series is convergent if $\frac{1}{x^2}>1$, *i.e.*, $x^2<1$ and divergent if $\frac{1}{x^2}<1$, *i.e.*, $x^2>1$.

But if $x^2=1$, the nth term of the given series becomes

$$u_n=\frac{1}{(n+1)\sqrt{n}}=\frac{1}{n^{3/2}}\left(1+\frac{1}{n}\right)^{-1}=\frac{1}{n^{3/2}}\left[1-\frac{1}{n}+\ldots\right]$$

Comparing it with the auxiliary series

$$\sum v_n=\sum\frac{1}{n^{3/2}}$$

we find that

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(1-\frac{1}{n}+\ldots\right)=1,$$ which is non-zero and finite.

The both Σu_n and Σv_n converge or diverge together. But the auxiliary series $\sum v_n=\sum\frac{1}{n^{3/2}}$ is convergent, hence the given series is also convergent.

(b) $$\frac{u_n}{u_{n+1}}=\frac{(2n+1)^p}{(2n-1)^p}\cdot\frac{1}{x}=\frac{\left(1+\frac{1}{2n}\right)^p}{\left(1-\frac{1}{2n}\right)^p}\cdot\frac{1}{x}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{2n}\right)^p}{\left(1-\frac{1}{2n}\right)^p}\cdot\frac{1}{x}\right]=\frac{1}{x}$$

Hence the given series is convergent if $\frac{1}{x}>1$, *i.e.*, if $x<1$ and divergent if $\frac{1}{x}<1$, *i.e.*, if $x>1$.

But if $x=1$, the given series becomes

$$\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\ldots$$

$$u_n=\frac{1}{(2n-1)^p}=\frac{1}{n^p\left(2-\frac{1}{n}\right)^p}$$

Choosing the auxiliary series $\Sigma\, v_n=\sum\frac{1}{n^p}$, we have

$$\therefore \qquad \frac{u_n}{v_n}=\frac{n^p}{n^p\left(2-\frac{1}{n}\right)^p}=\frac{1}{\left(2-\frac{1}{n}\right)^p}$$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(2-\frac{1}{n}\right)^p}=\frac{1}{(2-0)^p}=\frac{1}{2^p}$$

which is non zero and finite for any given value of p.

Thus Σu_n and Σv_n either both converge or diverge together.

But $\sum v_n=\sum\frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p\leqslant 1$.

Hence the given series is convergent if $x<1$ and divergent if $x>1$ and when $x=1$, the given series is convergent if $p>1$ and divergent if $p\leqslant 1$.

13·12. CAUCHY'S ROOT TEST

According to this test if Σu_n is a positive term series, then

(*i*) it is covergent if $\lim_{n\to\infty}(u_n)^{\frac{1}{n}}<1$,

(*ii*) is divergent if $\lim_{n\to\infty}(u_n)^{\frac{1}{n}}>1$, and (*iii*) the test is inconclusive if $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = 1.$

Example 20. *Show that the series :*

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots \text{ is convergent.}$$

Solution. Here the nth term of the series is

$$u_n=\frac{1}{n^n}$$

Thus $$(u_n)^{\frac{1}{n}}=\left(\frac{1}{n^n}\right)^{\frac{1}{n}}=\frac{1}{n}$$

$\therefore$ $$\lim_{n\to\infty}\left[(u_n)^{\frac{1}{n}}\right]=\lim_{n\to\infty}\frac{1}{n}=0<1$$

$\Rightarrow$ By Cauchy's root test, Σu_n is convergent.

Example 21. *Test for the convergence the series whose genera term is*

$$\left(1-\frac{1}{n}\right)^{n^2}$$

Solution. Here $u_n=\left(1-\frac{1}{n}\right)^{n^2}$

Thus $$(u_n)^{\frac{1}{n}}=\left(1-\frac{1}{n}\right)^n=\left[\left(1-\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$\therefore \lim_{n\to\infty}(u_n)^{\frac{1}{n}}=\lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{-n}\right]^{-1}=e^{-1}=\frac{1}{e}<1$$

$\Rightarrow$ By Cauchy's root test, Σu_n is convergent.

Example 22. *Test for convergence the series whose nth term is*

$$\frac{n^{n^2}}{(n+1)^{n^2}}$$

Solution. Here $$(u_n)^{\frac{1}{n}}=\frac{n^n}{(1+n)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$\therefore$ $$\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1$$

Hence the series converges by the root test.

Example 23. *Test the convergence of the series :*

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$$

Solution. Here $u_n=\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}$

$$=\left(\frac{n+1}{n}\right)^{-n}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{n}$$

$\therefore$
$$(u_n)^{\frac{1}{n}}=\left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{-1}$$

$$=\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

Thus $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}}=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\Big/\left[\left(1+\frac{1}{n}\right)^{n}-1\right]$

$=\dfrac{1}{e-1}$, which is less than 1.

Hence by Cauchy's root test, the given series is convergent.

Example 24. *Show that the series* $\sum\frac{\{(n+1)x\}^n}{n^{n+1}}$ *is convergent if* $x<1$ *and divergent if* $x>1$.

Solution. Here $u_n=\dfrac{\{(n+1)x\}^n}{n^{n+1}}=\left[\dfrac{(n+1)x}{n^{1+\frac{1}{n}}}\right]^n$

$$(u_n)^{\frac{1}{n}}=\frac{(n+1)x}{n.n^{\frac{1}{n}}}=\left(1+\frac{1}{n}\right).\frac{x}{n^{\frac{1}{n}}}$$

$\therefore$ $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}}=x$ $\left[\because \lim\limits_{n\to\infty}n^{\frac{1}{n}}=1\right]$

$\Rightarrow$ By Cauchy's root test, the series is convergent if $x<1$ and divergent if $x>1$.

If $x=1$, then $u_n=\dfrac{(n+1)^n}{n^{n+1}}=\dfrac{1}{n}\cdot\left(\dfrac{n+1}{n}\right)^n=\dfrac{1}{n}\left(1+\dfrac{1}{n}\right)^n$

Taking the auxiliary series Σv_n, where $v_n=\dfrac{1}{n}$

$$\frac{u_n}{v_n}=\left(1+\frac{1}{n}\right)^n$$

$\therefore$ $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e\neq0,$

⇒ By limit comparison test, Σu_n and Σv_n are either both convergent or both divergent, but $\Sigma v_n = \sum \frac{1}{n}$ is divergent, therefore, Σu_n is also divergent.

Hence the given series is convergent if $x<1$ and divergent if $x>1$.

13·13. RAABE'S TEST

Let Σu_n be a given positive term series such that

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=l, \text{ then}$$

(i) *Σu_n is convergent if $l>1$*

(ii) *Σu_n is divergent if $l<1$.*

(iii) *if $l=1$, then the test fails, the series may either converge or diverge.*

Example 25. *Determine the convergence of the series :*

$$\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\ldots$$

Solution. Here $u_n=\dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\; x^n$

$$u_{n+1}=\frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots(2n)(2n+2)}\,.\,x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}}=\frac{(2n+2)}{(2n+1)}\cdot\frac{1}{x}=\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\cdot\frac{1}{x} \quad \ldots(1)$$

$$\therefore \quad \lim_{n\to\infty} = \frac{u_n}{u_{n+1}}=\frac{1}{x}$$

⇒ By d'Alembert's Test, the series is convergent or divergent according as $\frac{1}{x}>1$ or $\frac{1}{x}<1$ respectively, *i.e.*, according as $x<1$ or $x>1$ respectively. Since this test is inconclusive when $x=1$, we use Raabe's test. If $x=1$, (1) can be written as

$$\frac{u_n}{u_{n+1}}=\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)^{-1}$$

$$=\left(1+\frac{1}{n}\right)\left(1-\frac{1}{2n}+\frac{1}{2^2n^2}-\ldots\right)=1+\frac{1}{2n}-\frac{1}{4n^2}+\ldots$$

i.e., $$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{1}{2}-\frac{1}{4n}+\ldots$$

$$\therefore \quad \lim_{n\to\infty}\left\{ n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\frac{1}{2}<1$$

⇒ By Raabe's test the series is convergent.

Hence the given series is convergent if $x<1$ and divergent if $x\geqslant 1$.

Example 26. *Examine the convergence or the divergence of the series :*

$$x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\ldots$$

Solution. Here $u_n=\dfrac{2^2.4^2.6^2\ldots(2n-2)^2}{3.4.5.6\ldots.2n}\cdot x^{2n}$

$$u_{n+1}=\frac{2^2.4^2.6^2\ldots(2n-2)^2(2n)^2}{3.4.5.6\ldots.2n(2n+1)(2n+2)}x^{2n+2}$$

Thus $\dfrac{u_n}{u_{n+1}}=\dfrac{(2n+1)(2n+2)}{(2n)^2}\cdot\dfrac{1}{x^2}=\dfrac{4n^2+6n+2}{4n^2}\cdot\dfrac{1}{x^2}$ ...(1)

$\therefore$
$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x^2}$$

$\Rightarrow$ By d'Alembert test, the series is convergent if $\dfrac{1}{x^2}>1$, *i.e.*,

if $x^2<1$ and divergent if $\dfrac{1}{x^2}<1$, *i.e.*, $x^2>1$.

If $x^2=1$, (1) may be rewritten as

$$\frac{u_n}{u_{n+1}}=\frac{4n^2+6n+2}{4n^2}$$

$$\Rightarrow\quad n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left(\frac{4n^2+6n+2}{4n^2}-1\right)=\frac{6n^2+2n}{4n^2}$$

$$\Rightarrow\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\lim_{n\to\infty}\frac{6n^2+2n}{4n^2}=\frac{3}{2}>1$$

$\Rightarrow$ By Raabe's test, the series is convergent.

Hence the series is convergent if $x<1$ and divergent if $x>1$.

EXERCISE (II)

1. Find whether the following series converge or diverge.

(*i*) $\dfrac{1}{2}+\dfrac{2}{2^2}+\dfrac{3}{2^3}+\dfrac{4}{2^4}+\ldots$

(*ii*) $\dfrac{2}{2.3}+\dfrac{2^2}{3.4}+\dfrac{2^3}{4.5}+\ldots+\dfrac{2^n}{(n+1)(n+2)}+\ldots$

(*iii*) $\dfrac{2}{1^2}+\dfrac{2^2}{2^2}+\dfrac{2^3}{3^2}+\ldots+\dfrac{2^n}{n^2}+\ldots$

(*iv*) $\dfrac{2}{1^2+1}+\dfrac{2^2}{2^2+1}+\dfrac{2^3}{3^2+1}+\dfrac{2^4}{4^2+1}+\ldots$

(*v*) $\dfrac{1}{1+2^{-1}}+\dfrac{2}{1+2^{-2}}+\dfrac{3}{1+2^{-3}}+\ldots$

2. Examine the convergence of the series:

(i) $\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$

(ii) $\frac{1}{1!}+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots$

(iii) $\frac{1}{2}+\frac{2!}{8}+\frac{3!}{32}+\frac{4!}{128}+\ldots$

3. Examine the convergence of the following series :

(i) $1+3x+5x^2+7x^3+9x^4+\ldots$

(ii) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\ldots$

4. Examine the convergence of the series .

(i) $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+\ldots$

(ii) $\frac{1}{1.2.3}+\frac{x}{4.5.6}+\frac{x^2}{7.8.9}+\ldots$

(iii) $1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\ldots$

(iv) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots$

5. Test the convergence of the series :

(i) $\frac{x}{a+\sqrt{1}}+\frac{x^2}{a+\sqrt{2}}+\frac{x^3}{a+\sqrt{3}}+\ldots+\frac{x^n}{a+\sqrt{n}}+\ldots$

(ii) $\frac{\sqrt{1}}{\sqrt{1^2+1}}\,x+\frac{\sqrt{2}}{\sqrt{2^2+1}}\,x^2+\frac{\sqrt{3}}{\sqrt{3^2+1}}\,x^3+\ldots+\frac{\sqrt{n}}{\sqrt{n^2+1}}\,x^n+\ldots$

(iii) $\frac{2x}{1^2}+\frac{3x^2}{2^2}+\frac{4x^3}{3^2}+\ldots+\frac{(n+1)x^n}{n^2}+\ldots$

(iv) $\frac{1}{2}+\frac{4}{9}\,x+\frac{9}{28}\,x^2+\ldots+\frac{n^2}{(n^3+1)}\,x^{n-1}+\ldots$

6. Examine the convergence of the following series :

(i) $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\ldots+\frac{x^n}{x+n}+\ldots$

(ii) $x+\frac{3}{5}\,x^2+\frac{8}{10}\,x^3+\frac{15}{17}\,x^4+\ldots+\frac{n^2-1}{n^2+1}\,x^n+\ldots$

(iii) $1+\frac{x^2}{2^p}+\frac{x^4}{4^p}+\frac{x^6}{6^p}+\ldots$

(iv) $\frac{x}{1+x}+\frac{x^2}{2+4x^2}+\frac{x^3}{3+9x^3}+\ldots+\frac{x^n}{n+n^2x^n}+\ldots$

7. Examine the convergence of the following series :

(*i*) $\frac{1.2}{x}+\frac{2.3}{x^2}+\frac{3.4}{x^3}+\ldots$

(*ii*) $2\left(\frac{x}{a}\right)+\frac{5}{8}\left(\frac{x^2}{a^2}\right)+\frac{10}{27}\left(\frac{x^3}{a^3}\right)+\frac{17}{64}\left(\frac{x^4}{a^4}\right)+\ldots$

8. Test for convergence or divergence of the series :

(*i*) $\sum_{n=1}^{\infty}\frac{3n-4}{n^2-2n}x^n$, (*ii*) $\sum_{n=1}^{\infty}\frac{n^{1/2}}{(n^2+1)^{3/2}}x^n$

(*iii*) $\sum_{n=1}^{\infty}\left[\sqrt{n^2+1}-n\right]x^{2n}$

(*iv*) $\sum_{n=1}^{\infty}\frac{2.5.8\ldots(3n-1)}{1.5.9\ldots(4n-3)}$, (*v*) $\sum_{n=1}^{\infty}\frac{1.3.5\ldots(2n-1)}{3^n.n!}$,

(*vi*) $\sum_{n=1}^{\infty}\frac{x^n}{1+x^n}$ (given $0\leqslant x\leqslant 1$) [*C.A., May, 1991*]

9. Test for convergence the following series :

(*i*) $\sum_{n=1}^{\infty}\left[\frac{n}{(2n+1)}\right]^n$, (*ii*) $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$

(*iii*) $\sum_{n=1}^{\infty}\left[1+\frac{1}{n}\right]^{-n^2}$, (*iv*) $\sum_{n=1}^{\infty}\frac{[(n+1)/n]^{n^2}}{3^n}$;

(*v*) $\sum_{n=1}^{\infty}\left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$ (*vi*) $\sum_{n=1}^{\infty}2^{-n-(-1)^n}$

10. Test for convergence the following series :

(*i*) $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\ldots$

(*ii*) $\sum_{n=1}^{\infty}\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-1)}\cdot\frac{1}{(2n-2)}$

(*iii*) $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}+\ldots$

(*iv*) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots$

ANSWERS

1. (*i*) Convergent, (*ii*) divergent, (*iii*) divergent, (*iv*) divergent, (*v*) divergent.

2. (*i*) Convergent, (*ii*) divergent, (*iii*) divergent.

3. (*i*) Convergent, if $x<1$ and divergent if $x\geqslant 1$,

(*ii*) convergent for all x.

4. (*i*) Convergent for $x\leqslant 1$ and divergent for $x>1$,

(*ii*) same as in (*i*) ; (*iii*) same as in (*i*) ; (*iv*) same as in (*i*),

5. (*i*) Convergent if $x<1$ and divergent if $x\geqslant 1$, (*ii*) same as in (*i*), (*iii*) and (*iv*) same as in (*i*).

6. (*i*) Convergent if $x<1$ and divergent if $x>1$, (*ii*) same as in (*i*), (*iii*) convergent if $x^2<1$, divergent if $x^2>1$. Also convergent if $x^2=1$ and $p>1$ and divergent if $x^2=1$ and $p<1$, (*iv*) convergent.

7. (*i*) Convergent if $x>1$ and divergent if $x\leqslant 1$,

(*ii*) convergent if $x<a$ and divergent if $x\geqslant a$.

8. (*i*) Convergent if $x<1$ and divergent if $x\geqslant 1$,

(*ii*) convergent if $x\leqslant 1$ and divergent if $x>1$, (*iii*) same as in (*ii*).

(*vi*) convergent if $0\leqslant x<1$ and divergent if $x=1$.

9. (*i*) to (*vi*) convergent.

10. (*i*) Divergent, (*ii*) convergent, (*iii*) divergent,

(*iv*) convergent if $x^2<1$ and divergent if $x^2>1$.

13·14. ALTERNATING SERIES

An infinite series in which the terms are alternately positive and negative and in which each term is numerically less than the preceding term is called an *Alternating Series*. Symbolically, a series of the form

$$u_1-u_2+u_3-u_4+\ldots,\ i.e.,\ \sum_{n=1}^{\infty}(-1)^{n-1}u_n$$

where $u_n>0\ \forall n$, is called an alternating series.

13·15. LEIBNITZ TEST

An alternating series is convergent if each term is numerically less than the preceding term and the nth term is infinitely small when n is taken infinitely large. Symbolically, the alternating series

$$u_1-u_2+u_3-u_4+\ldots+(-1)^{n-1}u_n+\ldots$$

is convergent if (i) $u_{n+1} \leq u_n \forall n$, and (ii) $\lim_{n \to \infty} u_n = 0$

and its sum always lies between 0 and u_1.

Example 27. *Test the convergence of the series :*

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$$

Solution. In order to discuss the convergence of the given series, we will apply Leibnitz test for the convergence of an alternating series and we shall verify whether the terms of the series satisfy the two conditions or not.

(i) Obviously $u_{n+1} \leq u_n$, (ii) $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n!} = 0$

$\Rightarrow$ By Leibnitz test, the given series is convergent.

Example 28. *Test the convergence of the series :*

$$\frac{1}{abc} - \frac{1}{(a+1)(b+1)(c+1)} + \frac{1}{(a+2)(b+2)(c+2)} - \frac{1}{(a+3)(b+3)(c+3)} + \ldots$$

a, b, c are positive.

Solution. Let the given series be represented by

$$u_1 - u_2 + u_3 - u_4 + \ldots$$

$$u_n = \frac{1}{(a+n-1)(b+n-1)(c+n-1)}, \quad u_{n+1} = \frac{1}{(a+n)(b+n)(c+n)}$$

Since a, b and c are all positive,

$$\frac{1}{a+n-1} > \frac{1}{a+n}, \quad \frac{1}{b+n-1} > \frac{1}{b+n} \text{ and } \frac{1}{c+n-1} > \frac{1}{c+n}$$

$$\therefore \quad \frac{1}{(a+n-1)(b+n-1)(c+n-1)} > \frac{1}{(a+n)(b+n)(c+n)}$$

$$\Rightarrow \quad u_n > u_{n+1}$$

$$\Rightarrow \quad u_{n+1} < u_n \qquad \ldots(1)$$

Also $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{(a+n-1)(b+n-1)(c+n-1)}$

$$= \lim_{n \to \infty} \frac{1}{n^3\left[1 + \frac{a-1}{n}\right]\left[1 + \frac{b-1}{n}\right]\left[1 + \frac{c-1}{n}\right]}$$

$$= 0 \qquad \ldots(2)$$

From (1) and (2), we conclude that the given series is convergent.

Example 29. *Examine the convergence of the series :*

$$\sum_{n=1}^{\infty} (-1)^n [\sqrt{n+1} - \sqrt{n}]$$

Solution. Here $u_n = \sqrt{(n+1)} - \sqrt{n}$

$$= [\sqrt{(n+1)} - \sqrt{n}] \times \frac{\sqrt{(n+1)} + \sqrt{n}}{\sqrt{(n+1)} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{(n+1)} + \sqrt{n}} = \frac{1}{\sqrt{(n+1)} + \sqrt{n}}$$

Now we shall apply the Leibnitz test for the convergence of an alternating series and we shall verify whether the terms of the series satisfy the two conditions or not.

(*i*) $$u_{n+1} - u_n = \frac{1}{\sqrt{(n+2)} + \sqrt{n+1}} - \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{\sqrt{n+1} + \sqrt{n} - \sqrt{n+2} - \sqrt{n+1}}{(\sqrt{n+2} + \sqrt{n+1})\ (\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{\sqrt{n} - \sqrt{n+2}}{(\sqrt{n+2} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{\sqrt{n} - \sqrt{n+2}}{(\sqrt{n+2} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n})} \times \frac{\sqrt{n} + \sqrt{n+2}}{\sqrt{n} + \sqrt{n+2}}$$

$$= \frac{n-(n+2)}{(\sqrt{n+2} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n})(\sqrt{n} + \sqrt{n+2})}$$

$$< 0$$

$\Rightarrow$ $u_{n+1} - u_n < 0$, *i.e.*, $u_n > u_{n+1}$.

(*ii*) $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\left[\left(1+\frac{1}{n}\right)^{1/2} + 1\right]} = 0.$$

Hence by Leibnitz test, the series is convergent

EXERCISE (III)

1. Test the convergence of the following series :

(*i*) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$

(*ii*) $1-\frac{1}{2\sqrt{2}}+\frac{3}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\ldots$

2. Show that the following series are convergent :

(*i*) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,!}$, (*ii*) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{a+n}$

(*iii*) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+a^2}$

3. Examine the convergence of the following series :

(*i*) $\frac{1}{x}-\frac{1}{x+a}+\frac{1}{x+2a}-\frac{1}{x+3a}+\ldots$ where x and $a>0$

(*ii*) $\frac{1}{xy}-\frac{1}{(x+1)(y+1)}+\frac{1}{(x+2)(y+2)}-\frac{1}{(x+3)(y+3)}+\ldots$
where x and $y>0$.

ANSWERS

1. (*i*) Convergent, (*ii*) convergent,

2. (*i*) Convergent, (*ii*) convergent.