14

Circular Functions and Trigonometry

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- measurement of angles, trigonometric ratios, trigonometric functions and their signs.
- trigonometric functions of standard angles ; use of printed tables, *t*-ratios of allied angles, sum and difference of angles, t-ratios of multiple angles.
- changing the sum or difference to products and vice versa for solving problems
- trigonometric identities, properties of a triangle and solution of a triangle.

14·0. INTRODUCTION

The word *Trigonometry* is derived from two Greek words—'trigono' (meaning a triangle) and 'metron' (meaning a measure), and hence the literal meaning *the measurement of a triangle*. Thus trigonometry is that branch of mathematics which deals with the measurement of the sides and the angles of a triangle and the investigation of various relations which exist amongst them.

There are several methods of measuring angles. One of these methods, used mostly in trigonometry is the radian measure which is also called the circular measure. The concept of angle is also somewhat modified in trigonometry. Here it is formed by a moving line from its initial position to the terminal position, clockwise or anti-clockwise. Now the circular functions deal with the relations of these rotating lines and the angles formed by them. Since these are measured in circular or the radian measure, they are called the circular functions. Except this there is no difference between trigonometry and circular functions, and they are so closely interwoven that the difference is not of much relevance.

In trigonometry we have a good deal of combination of algebra and geometry. There are algebraic symbols, formulae and equations which make the subject more interesting and useful for practical applications. To state a few, it is useful in measuring height of the mountains, the summits of which cannot be reached, the distance of inaccessible objects, the width of rivers without undertaking the trouble of actually crossing them, measurement of the size of the earth, etc. It is rather indispensable for industrial engineering, surveying and astronomy. In business the phenomenon of business cycles can be explained by some of the circular functions.

Since trigonometric functions are widely used in mathematics, our understanding of the subject will be incomplete without the knowledge of trigonometry. There are two main branches of trigonometry., *viz*., plane trigonometry and spherical trigonometry, however, we will confine our study only to plane trigonometry.

ANGLES

An angle in trigonometry is defined as the amount of rotation made by a straight line from one position to another about a point. In other words it can be traced by the revolution of a straight line about a point from its initial position to the terminal position as indicated in Fig. 1.

In the adjoining figure, the initial side OX moves anti-clockwise to the terminal side OP from the common point O, called the vertex, to trace a positive angle.

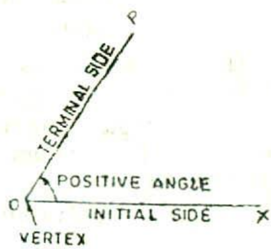
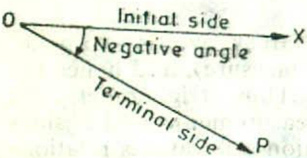


Fig. 1.



A similar movement clockwise will trace a negative angle.

QUADRANTS

Let the two perpendicular lines $X'OX$ and $Y'OY$ divide the plane into four parts, each one of them being called a *quadrant*. Conventionally the region XOY is called the *First quadrant*, the region YOX' is called the *Second quadrant*, the region $X'OY'$ is called the *Third quadrant* and the region $Y'OX$ is called the *Fourth quadrant*.

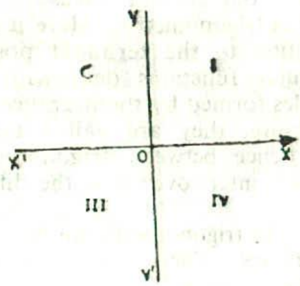


Fig. 3.

14·1. MEASUREMENT OF ANGLES

In geometry, an angle is generally measured in terms of a right angle. This, however, is too large a unit for practical applications in trigonometry, so in trigonometry there are three systems for the measurement of angles.

I. Sexagesimal System (or the English System). In this system a right angle is divided and sub-divided into small parts as shown below :

1 rt. angle $=90$ degrees (written as $90°$)

1 degree $=60$ minutes (written as $60'$)

1 miunte $=60$ seconds (written as $60''$)

II. The Centesimal System (or the French System). In this system, a right angle is divided and sub-divided as shown below :

1 rt. angle $=100$ grades (written as 100^g)

1 grade $=100$ minutes (written as $100'$)

1 minute $=100$ seconds (written as $100''$)

The minutes and seconds used in the centesimal system are distinct from those used in the sexagesimal system.

A right angle being the connecting link between the two systems, an angle in the first system can be converted into the units of the second system and *vice versa*.

$\because$ one rt. angle contains $90°$ and 100^g

$$90°=100^g \Rightarrow 1°=\frac{10^g}{9}$$

Thus to change degrees into grades, multiply by $\frac{10}{9}$, *e.g.*

$$63^\circ = 63 \times \frac{10}{9} = 70^g$$

Similarly to change grades into degrees multiply by $\frac{9}{10}$, *e.g.*

$$40^g = 40 \times \frac{9}{10} = 36^\circ.$$

Remark : A conversion process from one system to the other can be simplified if the given measurement is first converted into a decimal fraction. This, in the case of sexagesimal system, is done by dividing each successive part by 60, 60 and 90 from right to left and in case of it being in centesimal system by 100, 100, 100 in the same manner as illustrated in the following example.

Example 1. (*a*) *Express 17° 15′ 27″ as a decimal fraction of a right angle and then reduce it into centesimal measure.*

(*b*) *Reduce* $36^g\ 32'\ 50''$ *to the sexagesimal measure.*

Solution. (*a*) $27'' = \frac{27'}{60} = 0{\cdot}45'$

$$15'27'' = 15{\cdot}45' = \frac{15{\cdot}45^\circ}{60} = 0{\cdot}2575^\circ$$

$$17^\circ15'27'' = 17{\cdot}2575^\circ = \frac{17{\cdot}2575}{90} \text{ rt. } \angle = \frac{1{\cdot}72575}{9} \text{ rt. } \angle$$

$$= 0{\cdot}19175 \text{ rt. angle}$$

$$\underline{\quad 100 \quad}$$

$$19{\cdot}175 \text{ grades}$$

$$\underline{\quad 100 \quad}$$

$$17{\cdot}5 \text{ minutes}$$

$$\underline{\quad 100 \quad}$$

$$50 \text{ seconds}$$

Thus $17^\circ15'27'' = 19^g17'50''$

(*b*) $$50'' = \frac{50}{100} = 0{\cdot}5'$$

$$32{\cdot}5' = \frac{32{\cdot}5}{100} = 0{\cdot}325^g$$

$$36{\cdot}325^g = \frac{36{\cdot}325}{100} \text{rt.} \angle = 0{\cdot}36325 \text{ rt. } \angle$$

90

32·6925 degrees
60

41·55
60

33·00 seconds

Thus $36^g 32' 50'' = 32°41'33''$

III. The Circular System (or the Circular Measure)

In all the higher branches of mathematics, this system is commonly used for measurement of angles. In this system, the unit of measurement is a radian.

The radian is defined as an *angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.*

Let us draw a circle around the centre O with any radius. From any point A on the circumference cut off an arc AB=radius of the circle. Join OA and OB. Then $\angle AOB = 1$ radian, circular measure of an angle is the number of radians it contains).

Since, the angles at the centre of a circle are proportional to the arcs subtended through them, we have

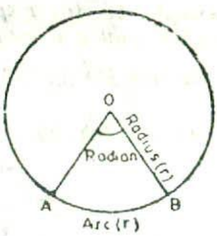


Fig. 4.

$$\frac{\angle AOB}{\text{Total angle at } O} = \frac{\text{arc } AB}{\text{Circumference}} = \frac{r}{2\pi r} = \frac{1}{2\pi} \left[\text{where } \pi = 3 \cdot 1415 \right]$$

$$\Rightarrow \quad \frac{1 \text{ radian}}{4 \text{ rt. } \angle s} = \frac{1}{2\pi}$$

$$\Rightarrow \quad 1 \text{ radian} = \frac{4 \text{ rt. } \angle s}{2\pi} = \frac{2}{\pi} \text{ rt. } \angle s$$

$$\Rightarrow \quad \pi \text{ radians} = 2 \text{ rt. } \angle s = 180° = 200^g$$

which is the relation between three systems for the measurement of angles. Through this, given any one system we can derive in any other system.

Example 2. *Find the circular measure of*

(*i*) *60°*, (*ii*) *112°30′*, (*iii*) *135°*

Solution. We know $180° = \pi$ radians

(*i*) $60° = \frac{\pi}{180} \times 60 = \frac{\pi}{3}$ radians

(*ii*) $112°30' = \frac{225°}{2}$

$$\therefore \qquad \frac{225^\circ}{2}=\frac{\pi}{180}\times\frac{225}{2}\text{ radians}=\frac{5\pi}{8}\text{ radians}$$

(*iii*) $$135^\circ=\frac{\pi}{180}\times 135\text{ radians}=\frac{3\pi}{4}\text{ radians}$$

Example 3. *Find the circular measure of*

(*i*) *40° 27′ 30″*, (*ii*) *65ᵍ 6′ 7″*.

Solution. (*i*) $$30''=\frac{30'}{60}=\frac{1'}{2}$$

$$27'\ 30''=27\ \frac{1'}{2}=\frac{55^\circ}{2\times 60}=\frac{55^\circ}{120}=\frac{11^\circ}{24}$$

$$40^\circ27'30''=40\ \frac{11^\circ}{24}=40{\cdot}4583^\circ$$

Since $$180^\circ=\pi\text{ radians}.\ 1^\circ=\frac{\pi}{180}\text{ radians}$$

$$\therefore \qquad 40{\cdot}4583^\circ=\frac{\pi}{180}\times 40{\cdot}4583\text{ radians}$$

$$=\frac{22}{7}\times\frac{1}{180}\times 40{\cdot}4583=0{\cdot}7064\text{ radians}$$

(*ii*) $$7''=\frac{7'}{100}=0{\cdot}07'$$

$$6'7''=6{\cdot}07'=\frac{6{\cdot}07^g}{100}=0{\cdot}0607^g$$

$$65^g6'7''=65{\cdot}0607^g=65{\cdot}0607\times\frac{\pi}{200}\qquad[\because\ 200^g=\pi\text{ radians}]$$

$$=\frac{0{\cdot}650607}{2}\ \pi\text{ radians}=0{\cdot}3253035\ \pi\text{ radians.}$$

Example 4. *The angles of a triangle are in A.P. and the ratio of the number of degrees in the smallest angle to the number of radians in the greatest angle is 60 : π. Find the angles in degrees.*

Solution. Let the angles be $(a-d)^\circ$, a° and $(a+d)^\circ$.

The sum of the three angles being 180°,

$$(a-d)+a+(a+d)=180^\circ \qquad \Rightarrow \quad a=60^\circ$$

so that the angles are $(60-d)^\circ$, 60°, $(60+d)^\circ$

Now $$1^\circ=\frac{\pi}{180}\text{ radians}$$

$$\therefore \qquad \text{Greatest angle}=(60+d)^\circ=\frac{\pi}{180}\ (60+d)\text{ radians.}$$

Therefore, from the given data

$$\frac{(60-d)}{\frac{\pi}{180}(60+d)}=\frac{60}{\pi}$$

$$\Rightarrow \quad 180(60-d)=60(60+d)$$

$$\Rightarrow \quad 4d=120^\circ, \text{ so that } d=30^\circ$$

Hence the angles are 30°, 60°, 90°

Example 5. (*a*) *If D, G, C are respectively the number of degrees, grades and radians in an angle, show that*

$$(a)\ \frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}, \quad (b)\ G-D=\frac{20C}{\pi}.$$

Solution. (*a*) $D \text{ degrees}=\frac{D}{90} \text{ rt. } \angle s$

$$G \text{ grades}=\frac{G}{100} \text{ rt. } \angle s$$

$$C \text{ radians}=\frac{C\times 2}{\pi} \text{ rt. } \angle s$$

Equating all the three, we get

$$\frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}$$

(*b*) Let $\frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}=k$

$$\therefore \quad D=90k,\ G=100k \text{ and } C=\tfrac{1}{2}\pi k$$

$$\therefore \quad \text{L.H.S.}=G-D=100k-90k$$

$$=10k=\frac{20\times\frac{1}{2}\pi k}{\pi}$$

$$=\frac{20C}{\pi}=\text{R.H.S.}$$

Theorem. *The circular measure of an angle subtended at the centre of a circle by an arc is*

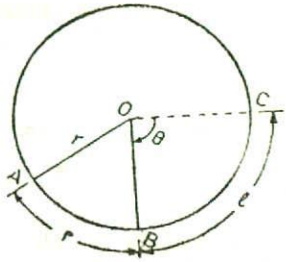
$$\frac{\textit{Subtended arc}}{\textit{Radius}}.$$

Draw a circle with centre O and radius (r) Let $\angle BOC=\theta$ radians and arc $BC=l$.

We have to prove that $\theta=\frac{l}{r}$.

Now cut off arc $BA=r$. Join OA.

Since angles at the centre of a circle are proportional to the arcs subtended by them,



ig. 5.

$$\frac{\angle AOB}{\angle BOC}=\frac{\text{arc } AB}{\text{arc } BC}$$

$\Rightarrow$ $$\frac{1 \text{ radian}}{\theta \text{ radians}}=\frac{r}{l}$$

$\Rightarrow$ $$\theta=\frac{l}{r} \text{ radians.}$$

Remark. $l=r\theta$ and $r=\frac{l}{\theta}$

Thus, if any two of these quantities are given, the third can be determined. But in applying this formula, θ must invariably be expressed in radians and l and r should be expressed in the same units.

Example 6. *Find the angle subtended by an arc 15 cm. long at the centre of a circle whose radius is 60 cm.*

Solution. $\theta=\frac{l}{r}=\frac{15}{60}$ radians $=\frac{\pi}{4}$ radians.

Example 7. *Find the length of an arc which subtends an angle 120° at the centre of a circle whose radius is 6 cm.*

Solution. Here $r=6$, $\theta=120°=120\times\frac{\pi}{180}$ radians

Substituting these values in the formula, we have

$$l=r\theta=6\times120\times\frac{\pi}{180}=4\pi$$

$$=4\times3{\cdot}142=12{\cdot}568 \text{ cm.}$$

Example 8. *A circular wire of 6 cm. radius is cut and bent so as to lie along the circumference of a loop whose radius is 0·96 metre. Find in radians (and also in grades) the angle which is subtended at the centre of the loop.*

Solution. Here l=Length of the arc

=Circumference of the circular wire with 6 cm. radius

$$=2\times\pi\times6 \text{ cm.}=12\pi \text{ cm.}$$

r=radius of the loop=0·96 metre=96 cm.

$$\therefore \quad \theta=\frac{l}{r}=\frac{12\pi}{96}=\frac{\pi}{8} \text{ radians}$$

Also π radians=200 grades

$$\therefore \quad \frac{\pi}{8}\text{radians}=\frac{200}{\pi}\times\frac{\pi}{8} \text{ grades}=25^g$$

Example 9. *The large hand of a big clock is 3 feet long. How many inches does its extremity move in 10 minutes' time ?*

Solution. In 60 minutes, the minute hand turns through 360°.

$$\therefore \quad \text{In 10 minutes it turns through}=\frac{360\times10}{60}=60^\circ$$

$$\therefore \quad \theta=60\times\frac{\pi}{180} \text{ radians}$$

r=radius of the circle=36 inches

$$\therefore \quad l=\text{length of the arc}=r\theta$$

$$=36\times60\times\frac{22}{7}\times\frac{1}{180}$$

$$=\frac{264}{7}=37·7 \text{ inches.}$$

Example 10. *A horse is tied to a post by a rope 8·1 metres long. If the horse moves, always keeping the rope tight, find what distance will it have covered when the rope has traced an angle of 70° ?*

Solution. The distance covered by the horse is the length of the arc of the circle with radius 8·1 metres, the arc subtending an angle of 70° at the centre.

r=radius of the circle=8·1 metres

θ=angle at the centre in radians

$$=70\times\frac{\pi}{180} \text{ radians}$$

l=required length of the arc

$$=r\theta=8·1\times70\times\frac{\pi}{180} \text{ metres}$$

$$=8·1\times70\times\frac{22}{7}\times\frac{1}{180} \text{ metres}=9·9 \text{ metres.}$$

Example 11. *The moon's distance from the earth is 350,000 kilometres and its diameter subtends an angle of 31′ at the eye of the observer. Find the diameter of the moon.*

Solution. Let AB be the diameter of the moon which subtends an angle of 31 at O, the eye of the observer.

Since $\angle AOB$ is very small, therefore, the diameter AB is nearly equal to arc AB of the circle whose centre is O and radius is OA, *i.e.* the distance of the moon from the earth. Let AB be l kilometres.

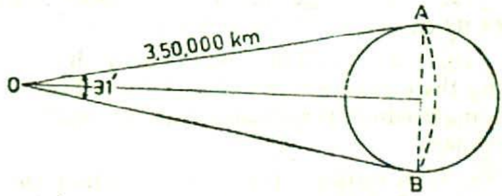


Fig. 6.

Hence $\theta = 31' = \frac{31}{60} \times \frac{\pi}{180}$ radians and $r = 350{,}000$ kilometres

$$\therefore \quad \frac{31 \times \pi}{60 \times 180} = \frac{l}{350{,}000} \qquad \left(\because \ \theta = \frac{l}{r} \right)$$

$$\Rightarrow \quad l = \frac{31 \times 350{,}000 \times 22}{60 \times 180 \times 7} = \frac{85250}{27} \text{ kilometres}$$

$$\Rightarrow \quad = 3157\frac{11}{27} \text{ kilometres.}$$

EXERCISE (I)

1. Give the different methods of measuring angles and give their inter-relations

2. (*a*) Express in terms of a rt. angle, and then reduce to the centesimal system,

(*i*) 20° 44′ 42″, (*ii*) 79°5′ 15″, (*iii*) 135° 24′ 29·34″.

(*b*) Express in terms of a rt. angle and then convert into the sexagesimal system,

(*i*) 35^g 2′ 50″, (*ii*) 26^g 97′ 5″

3. Express in radians the following :

(*a*) (*i*) 225°, (*ii*) 375°, (*iii*) 225^g, (*b*) (*i*) 47°48′ 45″, (*ii*) 56^g 45′ 75″

4 Express both in degrees and radians, the angles of a triangle whose angles are to each other as 1 : 2 : 3.

5. The angles of a triangle are in A.P. and the ratio of the number of grades in the smallest angle to the number of degrees in the greatest angle is 10 : 21. Find the angles in degrees.

6. The angles of a triangle are in A.P. and the number of grades in the least is to the number of radians in the greatest as 40 : π. Find the angles in degrees.

7. (*a*) Find the length of an arc of a circle of radius 7·62 cm. which subtends an angle of 30° at the centre.

(*b*) An arc of a circle of length 38·10 cm. subtends at the centre of the circle an angle of 72°. Find the radius of the circle.

8 A horse is tethered to a stake by a rope 2·23 metres long. If the horse moves along the circumference of a circle always keeping the rope

tight, find how far along the arc it will have gone when the rope has traced an angle of 70°.

9. The large hand of a clock is 60·96 cm long. How many cm does its extremity move in 20 minutes ?

10. (*a*) A circular wire of radius 3 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 48 cm. Find in grades the angle which is subtended at the centre of the hoop. Find also the angle in radians.

(*b*) A railway train is travelling on a curve of 750 metres radius at the rate of 30 km per hour ; through what angle has it turned in 10 seconds ?

ANSWERS

2. (*a*) (*i*) 0·2305 rt. ∠, $23^g5'$, (*ii*) 0·87875 rt. ∠, $0·87^g87'50''$, 1·504535 rt. ∠, $150^g45'\,35''$ (*b*) (*i*) 0·35025 rt. ∠, 31°31′21″, 0·26975 rt. ∠, 24°16′24″. **3.** (*a*) (*i*) $\frac{5\pi}{4}$, (*ii*) $\frac{25}{12}\pi$, (*iii*) $\frac{9}{8}\pi$.

(*b*) (*i*) 0·265625π, (*ii*) 0·282287 π.

4. 30°, 60° and 90° $\left(\text{Let } \frac{A}{1}=\frac{B}{2}=\frac{C}{3}=k\right)$. **5.** 36°, 60°, 84°

6 20°, 60°, 100° **7.** (*a*) 3·99 cm., (*b*) 30·30 cm. **8.** 2·7255 metres.

9. 127·7257 cm. **10.** (*a*) 25^g, $\frac{\pi}{8}$ [**Hint.** Length of arc = circumference of circle of radius 3 cm. = $2\pi \times 3$ cm.], (*b*) 22·9°.

14·2. TRIGONOMETRIC FUNCTIONS

Trigonometric functions are relations between any two of the three sides of a triangle. For the sake of simplicity a right-angled triangle is taken as a starting point to explain these relations. Among other things, the sides of a right-angled triangle are easy to define and grasp. These very ideas will then be extended to all other angles. It may be noticed that an angle in trigonometry is defined in a highly generalised manner which suits the subject because it helps in extending the concepts of trigonometric relations to other angles with no significant changes.

A Right Angle. Let XOY be any angle θ. Take any point P on OY and draw PM perpendicular to OX. A right angled △ OMP is formed. If θ is taken as the angle of reference, MP, the side opposite to θ is called the *perpendicular* and OP, the side opposite to the right angle is called the *hypotenuse* and OM the third side is called the *base*.

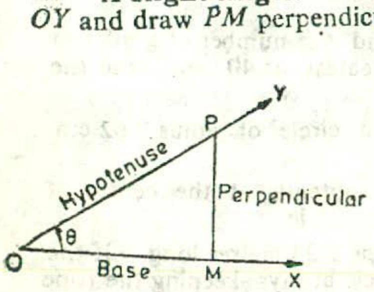


Fig. 7.

General Angles. Let a straight line OA, starting from the position OX and rotating round O trace out an angle XOA. Let θ be the measure of the angle XOA. This angle can be of any magnitude. From any point P in the final position of the revolving line OA, draw PM perpendicular to OX or XO produced if necessary as shown below :

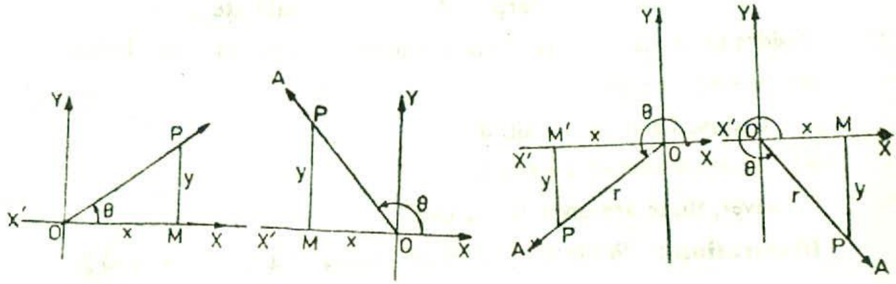


Fig. 8.

14·3. TRIGONOMETRIC RATIOS

Now, the three sides OM, OP and MP can be arranged, two at a time in six $({}^3P_2)$ different ways and hence six ratios can be formed with them. These six ratios are called the trigonometric functions or *t*-ratios or circular functions and are defined as follows :

1. The ratio of the perpendicular to the hypotenuse is called the *sine of the angle* θ and is written as *sin* θ.

$$\sin\theta=\frac{\text{Perp.}}{\text{Hyp.}}=\frac{MP}{OP}=\frac{y}{r}=\frac{\text{Ordinate}}{r}.$$

2. The ratio of the base to the hypotenuse is called the *cosine of the angle* θ and is written as *cos* θ.

$$\cos\theta=\frac{\text{Base}}{\text{Hyp.}}=\frac{OM}{OP}=\frac{x}{r}=\frac{\text{Abscissa}}{r}.$$

3. The ratio of the perpendicular to the base is called the *tangent of the angle* θ and is written as *tan* θ.

$$\tan\theta=\frac{\text{Perp.}}{\text{Base}}=\frac{MP}{OM}=\frac{y}{x}=\frac{\text{Ordinate}}{\text{Abscissa}}.$$

The following three ratios are *reciprocals* of the above ratios.

4. The ratio of the hypotenuse to the perpendicular is called the *cosecant of the angle* θ and is written as *cosec* θ.

$$\text{cosec}\,\theta=\frac{\text{Hyp.}}{\text{Perp.}}=\frac{OP}{MP}=\frac{r}{y}=\frac{r}{\text{Ordinate}}$$

5. The ratio of the hypotenuse to the base is called the *secant of the angle* θ and is written as *sec* θ.

$$\sec\theta=\frac{\text{Hyp.}}{\text{Base}}=\frac{OP}{OM}=\frac{r}{x}=\frac{r}{\text{Abscissa}}$$

6. The ratio of the base to the perpendicular is called the *cotangent of the angle* θ and is written as *cot* θ.

$$\cot\theta=\frac{\text{Base}}{\text{Perp.}}=\frac{OM}{MP}=\frac{x}{y}=\frac{\text{Abscissa}}{\text{Ordinate}}.$$

Besides these six *t*-ratios, there are three more ratios given below :

(*i*) versed sine or $1-\cos\theta$

(*ii*) coversed sine or $1-\sin\theta$

(*iii*) inversed sine or $1+\cos\theta$.

However, these are only rarely used.

Illustration : Write the values of *t*-ratios of angles θ, α, and β.

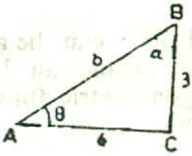


Fig. 9.

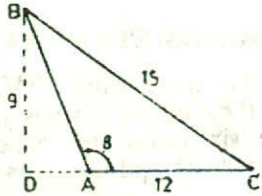


Fig. 10

Solution. It has been presented in tabular form as follows :

t-ratios	θ	α	β
$\sin\delta=\left(\frac{\text{Ordinate}}{r}\right)$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{9}{15}$
$\cos\delta=\left(\frac{\text{Abscissa}}{r}\right)$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{12}{15}$
$\tan\delta=\left(\frac{\text{Ordinate}}{\text{Abscissa}}\right)$	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{9}{12}$
$\text{cosec}\,\delta=\left(\frac{r}{\text{Ordinate}}\right)$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{15}{9}$
$\sec\delta=\left(\frac{r}{\text{Abscissa}}\right)$	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{15}{12}$
$\cot\delta=\left(\frac{\text{Abscissa}}{\text{Ordinate}}\right)$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{12}{9}$

14·4. RELATIONS BETWEEN TRIGONOMETRIC FUNCTIONS

I. Reciprocal Relations. The following relations are obvious from the definitions of *t*-ratios.

(*i*) $\sin\theta \times \operatorname{cosec}\theta = 1$

$\therefore$ $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$.

(*ii*) $\cos\theta \times \sec\theta = 1$

$\therefore$ $\cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$.

(*iii*) $\tan\theta \times \cot\theta = 1$

$\therefore$ $\tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$.

In words : (*i*) cosec θ is reciprocal of sin θ and *vice versa.*

(*ii*) sec θ is reciprocal of cos θ and *vice versa.*

(*iii*) cot θ is reciprocal of tan θ and *vice versa.*

II. Quotient Relations :

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

From Fig. 8 on Page 485, we have

$$\sin\theta = \frac{y}{r},\quad \cos\theta = \frac{x}{r}$$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta \text{ and } \frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{y} = \cot\theta.$$

III. Square Relations :

(*i*) $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \quad \sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$$

From this relation, we can obtain

$$\sin^2\theta = 1 - \cos^2\theta \text{ and } \cos^2\theta = 1 - \sin^2\theta.$$

(*ii*) $\sec^2\theta - \tan^2\theta = 1$

$$\because \quad \sec^2\theta - \tan^2\theta = \frac{r^2}{x^2} - \frac{y^2}{x^2} = \frac{r^2 - y^2}{x^2} = \frac{x^2}{x^2} = 1$$

It follows from above that

$$\sec^2\theta = 1 + \tan^2\theta \text{ and } \tan^2\theta = \sec^2\theta - 1$$

(*iii*) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$$\because \quad \operatorname{cosec}^2\theta - \cot^2\theta = \frac{r^2}{y^2} - \frac{x^2}{y^2} = \frac{r^2 - x^2}{y^2} = \frac{y^2}{y^2} = 1$$

We can obtain from the above relation :

$$\operatorname{cosec}^2\theta = 1+\cot^2\theta \text{ and } \cot^2\theta = \operatorname{cosec}^2\theta - 1$$

The above three inter-relations are very important and are called *Identities.* If any one of the trigonometric function is given, the remaining can easily be found by using the above relations (I), (II) or (III).

Remark. It should be noted that $\sin\theta$ does not mean $\sin\times\theta$, *i.e.,* sin is not a multiplier. The $\sin\theta$ is correctly read as "sin of angle θ". Similar is the case with other trigonometric ratios. Further

$(\sin\theta)^2$ is written as $\sin^2\theta$ (read : sine square θ)

$(\sin\theta)^3$ is written as $\sin^3\theta$ (read : sine cube θ)

Similarly, for other *t*-ratios of θ.

But $(\sin\theta)^{-1}$ is not written as $\sin^{-1}\theta$.

Example 12. *Prove that*

$$2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0.$$

Solution. $\text{L.H.S.} = 2[(\sin^2\theta)^3+(\cos^2\theta)^3]-3(\sin^4\theta+\cos^4\theta)+1$

$$= 2(\sin^2\theta+\cos^2\theta)(\sin^4\theta+\cos^4\theta-\sin^2\theta\cos^2\theta) - 3(\sin^4\theta+\cos^4\theta)+1$$

$$= 2.1.(\sin^4\theta+\cos^4\theta-\sin^2\theta\cos^2\theta)-3(\sin^4\theta+\cos^4\theta)+1$$

$$= -(\sin^4\theta+\cos^4\theta)-2\sin^2\theta\cos^2\theta+1$$

$$= -[(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta]-2\sin^2\theta\cos^2\theta+1$$

$$= -1+2\sin^2\theta\cos^2\theta-2\sin^2\theta\cos^2\theta+1$$

$$= 0 = \text{R.H.S.}$$

Example 13. *Prove that*

$$\frac{\sin\theta}{1-\cos\theta}=\frac{1+\cos\theta}{\sin\theta},\ \theta\neq 0.$$

Solution. $\text{L.H.S} = \dfrac{\sin\theta}{1-\cos\theta}\cdot\dfrac{1+\cos\theta}{1+\cos\theta}$

$$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}=\frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta}=\text{R.H.S.}$$

Example 14. *Prove that*

$$\sqrt{\frac{1-\sin A}{1+\sin A}}=\sec A-\tan A.$$

Solution. $\text{L.H.S} = \sqrt{\dfrac{1-\sin A}{1+\sin A}}$

$$= \sqrt{\frac{1-\sin A}{1+\sin A}\times\frac{1-\sin A}{1-\sin A}}$$

$$=\frac{1-\sin A}{\sqrt{1-\sin^2 A}}=\frac{1-\sin A}{\cos A}=\frac{1}{\cos A}-\frac{\sin A}{\cos A}$$

$$=\sec A-\tan A=\text{R.H.S.}$$

Example 15. *Prove that*

$$(\text{cosec}\ \theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1.$$

Solution. Evidently we have to simplify the L.H.S. This simplification is best done by expressing all t-ratios in terms of the sine and cosine by the use of the formula :

$$\text{cosec}\ \theta=\frac{1}{\sin\theta},\ \sec\theta=\frac{1}{\cos\theta},\ \tan\theta=\frac{\sin\theta}{\cos\theta},\ \cot\theta=\frac{\cos\theta}{\sin\theta}$$

$$\text{L.H.S.}=\left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$=\frac{1-\sin^2\theta}{\sin\theta}\times\frac{1-\cos^2\theta}{\cos\theta}\times\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

But $\quad 1-\sin^2\theta=\cos^2\theta,\ 1-\cos^2\theta=\sin^2\theta,\ \sin^2\theta+\cos^2\theta=1$

$$\therefore \quad \text{L.H.S.}=\frac{\cos^2\theta}{\sin\theta}\times\frac{\sin^2\theta}{\cos\theta}\times\frac{1}{\sin\theta\cos\theta}=1=\text{R.H.S.}$$

Example 16. *Prove that*

$$\sin A(1+\tan A)+\cos A(1+\cot A)=\sec A+\text{cosec}\ A.$$

Solution. $\text{L.H.S.}=\sin A(1+\tan A)+\cos A(1+\cot A)$

$$=\sin A\left(1+\frac{\sin A}{\cos A}\right)+\cos A\left(1+\frac{\cos A}{\sin A}\right)$$

$$=\sin A\left(\frac{\sin A+\cos A}{\cos A}\right)+\cos A\left(\frac{\sin A+\cos A}{\sin A}\right)$$

$$=(\sin A+\cos A)\left(\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)$$

$$=(\sin A+\cos A)\times\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}$$

$$=\frac{\sin A+\cos A}{\cos A\sin A}=\frac{\sin A}{\cos A\sin A}+\frac{\cos A}{\cos A\sin A}$$

$$=\frac{1}{\cos A}+\frac{1}{\sin A}=\sec A+\text{cosec}\ A=\text{R.H.S.}$$

Example 17. *Find the value in terms of p and q of*

$$\frac{(p\cos\theta+q\sin\theta)}{(p\cos\theta-q\sin\theta)}$$

when $\cot\theta=p/q$. *Here use of any figure is not allowea.*

Solution.

$$\frac{p\cos\theta+q\sin\theta}{p\cos\theta-q\sin\theta}=\frac{p\frac{\cos\theta}{\sin\theta}+q}{p\frac{\cos\theta}{\sin\theta}-q}=\frac{p\cot\theta+q}{p\cot\theta-q}=\frac{p.\frac{p}{q}+q}{p.\frac{p}{q}-q}$$

$$=\frac{p^2+q^2}{p^2-q^2}.$$

Example 18. (a) *If* $\tan A+\sin A=m$ ...(1)

and $\tan A-\sin A=n$ ...(2)

prove that $m^2-n^2=4\sqrt{mn}.$

Solution. Adding (1) and (2), we get

$$2\tan A=m+n$$

$$\Rightarrow \quad \tan A=\frac{m+n}{2}$$

$$\Rightarrow \quad \cot A=\frac{2}{m+n} \qquad ...(3)$$

Subtracting (2) from (1), we get

$$2\sin A=m-n$$

$$\Rightarrow \quad \sin A=\frac{m-n}{2},\ i.e.,\ \operatorname{cosec} A=\frac{2}{m-n} \qquad ...(4)$$

Also $\operatorname{cosec}^2 A-\cot^2 A=1$

$$\therefore \quad \frac{4}{(m-n)^2}-\frac{4}{(m+n)^2}=1 \Rightarrow \frac{4\{(m+n)^2-(m-n)^2\}}{(m-n)^2(m+n)^2}=1$$

i.e., $$\frac{4\times 4mn}{\{(m-n)(m+n)\}^2}=1 \Rightarrow (m^2-n^2)^2=16\,mn$$

Hence $(m^2-n^2)=4\sqrt{mn}.$

(b) *If* $m=\operatorname{cosec} A-\sin A$ *and* $n=\sec A-\cos A$ *then prove that*

$$\tan A=\left(\frac{n}{m}\right)^{1/3}$$ [*C.A., May, 1991*]

Solution. We have

$$\frac{n}{m}=\frac{\sec A-\cos A}{\operatorname{cosec} A-\sin A}$$

$$=\frac{\frac{1}{\cos A}-\cos A}{\frac{1}{\sin A}-\sin A}=\frac{\frac{1-\cos^2 A}{\cos A}}{\frac{1-\sin^2 A}{\sin A}}$$

$$=\frac{\sin A}{\cos A}\cdot\frac{\sin^2 A}{\cos^2 A}=\frac{\sin^3 A}{\cos^3 A}=\tan^3 A$$

$$\therefore \qquad \tan A = \left(\frac{n}{m}\right)^{1/3}$$

Example 19. *Show that*

$$(1+\cot\theta-\operatorname{cosec}\theta)\,(1+\tan\theta+\sec\theta)=2.$$

Solution. We have

$$\text{L.H.S.}=\left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)\left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)$$

$$=\frac{\{(\sin\theta+\cos\theta)-1\}\,\{(\sin\theta+\cos\theta)+1\}}{\sin\theta\cos\theta}$$

$$=\frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta}=\frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$$

$$=\frac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta+\cos^2\theta=1]$$

$$=\frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}=2=\text{R.H.S.}$$

Example 20. *Prove that*

$$\frac{\cos\theta}{1-\tan\theta}+\frac{\sin\theta}{1-\cot\theta}=\sin\theta+\cos\theta.$$

Solution.

$$\text{L.H.S.}=\frac{\cos\theta}{1-\dfrac{\sin\theta}{\cos\theta}}+\frac{\sin\theta}{1-\dfrac{\cos\theta}{\sin\theta}}$$

$$=\frac{\cos\theta}{\dfrac{\cos\theta-\sin\theta}{\cos\theta}}+\frac{\sin\theta}{\dfrac{\sin\theta-\cos\theta}{\sin\theta}}$$

$$=\frac{\cos^2\theta}{\cos\theta-\sin\theta}-\frac{\sin^2\theta}{\cos\theta-\sin\theta}$$

$$=\frac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}=\frac{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}{(\cos\theta-\sin\theta)}$$

$$=\cos\theta+\sin\theta=\text{R.H.S.}$$

Example 21. *Show that*

$$\frac{\tan\theta}{\sec\theta-1}+\frac{\tan\theta}{\sec\theta+1}=2\operatorname{cosec}\theta.$$

Solution.

$$\text{L.H.S.}=\frac{\tan\theta}{\sec\theta-1}+\frac{\tan\theta}{\sec\theta-1}$$

$$=\tan\theta\left(\frac{1}{\sec\theta-1}+\frac{1}{\sec\theta-1}\right)$$

$$=\tan\theta\left[\frac{\sec\theta+1+\sec\theta-1}{(\sec^2\theta-1)}\right]$$

$[\because\ \sec^2\theta=1+\tan^2\theta]$

$$=\tan\theta.\frac{2\sec\theta}{\tan^2\theta}=2\frac{\sec\theta}{\tan\theta}=2.\frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}$$

$$=2\operatorname{cosec}\theta=\text{R.H.S.}$$

Example 22. *Prove that*

$$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\frac{1+\sin\theta}{\cos\theta}$$

Solution. $\text{L.H.S.}=\dfrac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$

$[\because\ 1=\sec^2\theta-\tan^2\theta]$

$$=\frac{(\tan\theta+\sec\theta)-(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}{\tan\theta-\sec\theta+1}$$

$$=\frac{(\sec\theta+\tan\theta)(1-\sec\theta+\tan\theta)}{\tan\theta-\sec\theta+1}$$

$$=\sec\theta+\tan\theta=\frac{1+\sin\theta}{\cos\theta}=\text{R.H.S.}$$

Example 23. *Prove that*

$$(\sin A+\operatorname{cosec} A)^2+(\cos A+\sec A)^2=\tan^2 A+\cot^2 A+7.$$

Solution. We have

$$\text{L.H.S.}=(\sin A+\operatorname{cosec} A)^2+(\cos A+\sec A)^2$$

$$=\sin^2 A+\operatorname{cosec}^2 A+2\sin A\operatorname{cosec} A+\cos^2 A+\sec^2 A+2\cos A\sec A$$

$$=(\sin^2 A+\cos^2 A)+\operatorname{cosec}^2 A+\sec^2 A+2\sin A\operatorname{cosec} A+2\cos A\sec A$$

$$=1+(1+\cot^2 A)+(1+\tan^2 A)+2+2$$

$$=\tan^2 A+\cot^2 A+7=\text{R.H.S.}$$

Example 24. *Prove that*

$$\frac{1-\sin x}{1+\sec x}-\frac{1+\sin x}{1-\sec x}=2\cos x(\cot x+\operatorname{cosec}^2 x).$$

Solution. $\text{L.H.S.}=\dfrac{(1-\sin x)(1-\sec x)-(1+\sin x)(1+\sec x)}{(1+\sec x)(1-\sec x)}$

$$=\frac{1-\sec x-\sin x+\sin x \sec x-1-\sec x-\sin x-\sin x \sec x}{1-\sec^2 x}$$

$$=\frac{-2\sec x-2\sin x}{1-\sec^2 x}=\frac{2\sec x+2\sin x}{\sec^2 x-1}=\frac{2\sec x+2\sin x}{\tan^2 x}$$

$$=\frac{2\sec x}{\tan^2 x}+\frac{2\sin x}{\tan^2 x}=\frac{2\cos^2 x}{\cos x\,.\,\sin^2 x}+\frac{2\sin x\,.\,\cos^2 x}{\sin^2 x}$$

$$=2\cos x\left[\frac{1}{\sin^2 x}+\frac{\cos x}{\sin x}\right]=2\cos x\,(\text{cosec}^2\, x+\cot x)=\text{R.H.S.}$$

14·5. SIGNS OF TRIGONOMETRIC FUNCTIONS

The radius vector is always positive in whichever quadrant it lies, therefore, the sign of a trigonometric ratio of an angle will always depend on the sign of the coordinates.

Now, in the first quadrant all t-functions are positive because both the coordinates are positive. In the second quadrant, x-coordinate is negative, therefore, all t-functions which involve x-coordinate are negative, *i.e.*, cos θ and tan θ and their reciprocals while sin θ and cosec θ will be positive. In the third quadrant both the co-ordinates are negative, therefore, t-ratios which involve both these such as tan θ and cos θ are positive but all others are negative. In the fourth quadrant x-coordinate is positive but y-coordinate is negative, therefore, cos θ and sec θ which do not involve y-coordinate are positive and the rest are negative. The following figure is a good aid to memory.

A crude aid to remember the signs is the four worded phrase "*All Silver Tea Cups*" which may be taken to indicate that all, *i.e.*, S (sine), T (tangent) and C (cosine) with their reciprocals are positive in the first quadrant. Only S (sine) and its reciprocal is positive in the second quadrant, only T (tangent) and its reciprocal is positive in the third quadrant and only C (cosine) and its reciprocal is positive in the fourth quadrant.

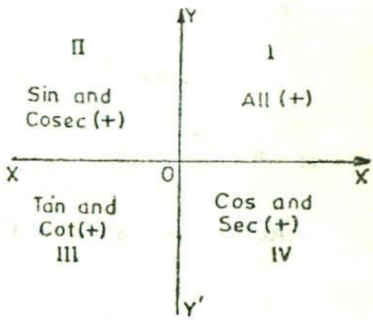


Fig. 11.

Now the radius vector r is always greater than the coordinates. Therefore, sin and cos, where r is in the denominator, can never be greater than unity whereas cosec and sec, where r is in the numerator, can never be less than unity. But the functions tan and cot can have any numerical value depending on the size of their coordinates.

Example 25. *If* θ *is in the fourth quadrant and* cos $\theta=\frac{5}{13}$, *find the value of* $\frac{13\sin\theta+5\sec\theta}{5\tan\theta+6\,\text{cosec}\,\theta}$.

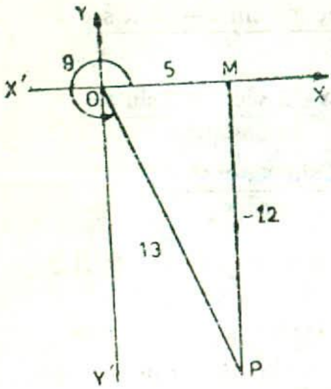


Fig 12.

Solution.

$$\cos\theta=\frac{OM}{OP}=\frac{5}{13}$$

$\therefore\quad OM=5,\ OP=13$ and
$MP^2=OP^2-OM^2$
$=13^2-5^2=144$

$\therefore\quad MP=-12$

($\because$ MP is $-$ive in the 4th quadrant)

$\therefore\ \sin\theta=-\frac{12}{13},\ \sec\theta=\frac{13}{5},$

$\tan\theta=-\frac{12}{5}$ and $\operatorname{cosec}\theta=-\frac{13}{12}.$

$\therefore$ Given expression

$$=\frac{13\sin\theta+5\sec\theta}{5\tan\theta+6\operatorname{cosec}\theta}$$

$$=\frac{13\left(-\frac{12}{13}\right)+5\left(\frac{13}{5}\right)}{5\left(\frac{12}{5}\right)+6\left(-\frac{13}{12}\right)}=\frac{-12+13}{-12-\frac{13}{2}}=-\frac{2}{37}.$$

Example 26. *If $\sin\theta.\sec\theta=-1$ and θ lies in the second quadrant, find $\sin\theta$ and $\sec\theta$.*

Solution. We are given that

$\sin\theta.\sec\theta=-1$

i.e., $\frac{\sin\theta}{\cos\theta}=-1$ or $\tan\theta=-1.$

$\therefore\qquad \tan\theta=\frac{MP}{OP}=-1$

Since θ lies in the second quadrant, MP is $+$ive and OM is $-$ive.

$\therefore$ If $MP=1$, then $OM=-1$.

Also $\quad OP=\sqrt{MP^2+OM^2}$

$=\sqrt{1+1}=\sqrt{2}$

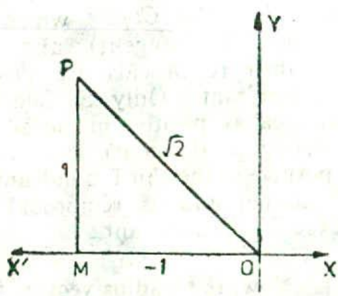


Fig. 13.

$$\therefore\qquad \sin\theta=\frac{MP}{OP}=\frac{1}{\sqrt{2}}\text{ and }\sec\theta=\frac{OP}{MP}=\frac{\sqrt{2}}{-1}=-\sqrt{2}$$

Example 27. *If $\sin\theta=\frac{21}{29}$, prove that*

(*i*) *$\sec\theta+\tan\theta=\frac{5}{2}$, if θ lies between 0 and $\frac{\pi}{2}$.*

What will be the value of the expression when θ lies

(ii) between $\frac{\pi}{2}$ and π, and (iii) between π and $\frac{3\pi}{2}$.

Solution. (*i*)

$$\sin\theta=\frac{MP}{OP}=\frac{21}{29}$$

$\therefore$ $MP=21$ and $OP=29$

Also $OM^2=OP^2-MP^2$

$$=29^2-21^2=400$$

$\Rightarrow$ $OM=20$

($\because$ OM is +ive in the first quadrant)

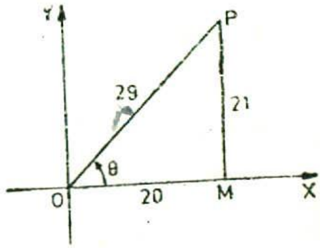


Fig. 14.

$\therefore$ $\sec\theta=\frac{OP}{OM}=\frac{29}{20}$

and $\tan\theta=\frac{MP}{OM}=\frac{21}{20}$

Hence $\sec\theta+\tan\theta=\frac{29}{20}+\frac{21}{20}=\frac{5}{2}$

(*ii*) When θ lies between $\frac{\pi}{2}$ and π, *i.e.*, in the second quadrant, OM is negative. (Draw the actual figure)

$\therefore$ $\sec\theta=\frac{29}{-20}=-\frac{29}{20}$ and $\tan\theta=\frac{21}{-20}=-\frac{21}{20}$

$\therefore$ $\sec\theta+\tan\theta=-\frac{29}{20}-\frac{21}{20}=-\frac{5}{2}$.

(*iii*) The question is impossible since in the third quadrant $\sin\theta$ is negative but it is given to be positive.

Example 28. *If* $\cot\theta=\frac{12}{5}$, $\left(\pi<\theta<\frac{3\pi}{2}\right)$, *find the value of* $\sec\theta$ *and* $\sin\theta$.

Solution. $\text{cosec}^2\,\theta=1+\cot^2\theta=1+\frac{144}{25}=\frac{169}{25}$

$\therefore$ $\text{cosec}\,\theta=-\frac{13}{5}$

(Since cosec θ is negative, θ being in the third quadrant)

$\therefore$ $\sin\theta=-\frac{5}{13}$

Again $\sec\theta=\frac{1}{\cos\theta}=\frac{\sin\theta}{\cos\theta}\,\frac{1}{\sin\theta}=\tan\theta\,.\,\text{cosec}\,\theta$

$$=\frac{5}{12}\cdot\left(-\frac{13}{5}\right)=-\frac{13}{12}.$$

Example 29. *If $\sec\theta=a+\frac{1}{4a}$, prove that*

$$\sec\theta+\tan\theta=2a \text{ or } \frac{1}{2a}.$$

Solution. We know

$$\tan^2\theta=\sec^2\theta-1=\left(a+\frac{1}{4a}\right)^2-1$$

$$=\left(a+\frac{1}{4a}\right)^2-4\,.\,a\,.\frac{1}{4a}$$

$$=\left(a-\frac{1}{4a}\right)^2 \qquad [\because (x+y)^2-4xy=(x-y)^2]$$

$\therefore$ $$\tan\theta=\left(a-\frac{1}{4a}\right) \text{ or } -\left(a-\frac{1}{4a}\right)$$

Taking the first value of $\tan\theta$, we have

$$\sec\theta+\tan\theta=\left(a+\frac{1}{4a}\right)+\left(a-\frac{1}{4a}\right)=2a$$

and taking the second value of $\tan\theta$, we have

$$\sec\theta+\tan\theta=\left(a+\frac{1}{4a}\right)-\left(a-\frac{1}{4a}\right)=\frac{1}{2a}$$

$\therefore$ $$\sec\theta+\tan\theta=2a \text{ or } \frac{1}{2a}.$$

Example 30. *If $\sec\theta+\tan\theta=4$, find*

(*a*) *$\sec\theta$ and $\tan\theta$,* (*b*) *$\sin\theta$ and $\cos\theta$.*

Solution. (*a*) We are given

$$\sec\theta+\tan\theta=4 \qquad ...(1)$$

Also $(\sec^2\theta-\tan^2\theta)=1$

$\Rightarrow$ $(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)=1$

$\Rightarrow$ $$(\sec\theta-\tan\theta)=\frac{1}{4} \qquad ...(2)$$

Adding (1) and (2), we get

$$2\sec\theta=4+\frac{1}{4}=\frac{17}{4},$$

$\Rightarrow$ $$\sec\theta=\frac{17}{8}$$

Substituting (1) and (2), we get

$$2\tan\theta=4-\frac{1}{4}=\frac{15}{4}$$

$\Rightarrow$ $$\tan\theta=\frac{15}{8}$$

(*b*) $$\cos\theta=\frac{1}{\sec\theta}=\frac{8}{17}$$

and $$\sin\theta=\frac{\sin\theta}{\cos\theta}\times\cos\theta=\tan\theta\times\cos\theta$$

$$=\frac{15}{8}\times\frac{8}{17}=\frac{15}{17}.$$

Example 31. *If* $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$,

prove that $$\cos\theta+\sin\theta=\sqrt{2}\cos\theta.$$

Solution. $$\cos\theta-\sin\theta=\sqrt{2}\sin\theta$$

$$\Rightarrow \quad \cos\theta=\sqrt{2}\sin\theta+\sin\theta=(\sqrt{2}+1)\sin\theta$$

$$\Rightarrow \quad \sin\theta=\frac{1}{\sqrt{2}+1}\cos\theta$$

$$=\frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1}\cdot\cos\theta=\frac{\sqrt{2}-1}{(\sqrt{2})^2-(1)^2}\cdot\cos\theta$$

$$\Rightarrow \quad \sin\theta=\sqrt{2}\cos\theta-\cos\theta$$

$$\Rightarrow \quad \cos\theta+\sin\theta=\sqrt{2}\cos\theta.$$

EXERCISE (II)

Prove the following identities :

1. (*a*) $(1-\sin^2\theta)\sec^2\theta=1$, (*b*) $(\sec^2\theta-1)\cot^2\theta=1$

   (*c*) $\tan\theta(1-\cot^2\theta)+\cot\theta(1-\tan^2\theta)=0.$

2. (*a*) $\dfrac{\cot A+\tan B}{\tan A+\cot B}=\dfrac{\cot A}{\tan B}$, (*b*) $\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cot\theta-1}{\cot\theta+1}$

   (*b*) $\dfrac{\cos A+\cos B}{\sin A-\sin B}+\dfrac{\sin A+\sin B}{\cos A-\cos B}=0.$

3. $\sin^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta\cos^2\phi=1.$

4. (*a*) $\sin^3\theta+\cos^3\theta=(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)$

   (*b*) $\sin^6\theta+\sin^4\theta\cos^2\theta-\sin^2\theta\cos^4\theta-\cos^6\theta=\sin^2\theta-\cos^2\theta.$

5. (*a*) $\sqrt{\dfrac{1+\sin A}{1-\sin A}}=\sec A+\tan A$

   (*b*) $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}=\operatorname{cosec}\theta+\cot\theta$

6. (*a*) $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\dfrac{1-\tan\theta}{1-\cot\theta}\right)^2=\left(\dfrac{1+\tan\theta}{1+\cot\theta}\right)^2$

(b) $\frac{\cos^2 A}{1+\tan^2 A}+\frac{\cot^2 A}{(1+\cot^2 A)^2}+\sin^2 A=1.$

7. (a) $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}=\sec A \operatorname{cosec} A+1$

(b) $\frac{\operatorname{cosec} A}{\operatorname{cosec} A-1}+\frac{\operatorname{cosec} A}{\operatorname{cosec} A+1}=2 \sec^2 A$

(c) $\frac{\sec \theta+1}{\tan \theta}+\frac{\tan \theta}{\sec \theta+1}=2 \operatorname{cosec} \theta$

8. (a) $(\sin \theta+\sec \theta)^2+(\cos \theta+\operatorname{cosec} \theta)^2=(1+\sec \theta \operatorname{cosec} \theta)^2$

(b) $(\sin A-\cos A)(1+\cot A+\tan A)=\frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A}$

(c) $(1+\sin \theta-\cos \theta)^2+2(\sin \theta+\cos \theta)^2+(1-\sin \theta+\cos \theta)^2=6.$

9. $\frac{1}{\operatorname{cosec} A+\cot A}-\frac{1}{\sin A}=\frac{1}{\sin A}-\frac{1}{\operatorname{cosec} A-\cot A}$

10. (a) $\frac{\cot \theta+\operatorname{cosec} \theta-1}{\cot \theta-\operatorname{cosec} \theta+1}=\frac{1+\cos \theta}{\sin \theta}$

(b) $\left(\frac{\sec x+\tan x-1}{\tan x-\sec x+1}\right)^2=\frac{\operatorname{cosec} x+1}{\operatorname{cosec} x-1}$

(c) $\cot^2\theta\left(\frac{\sec \theta-1}{1+\sin \theta}\right)+\sec^2\theta\left(\frac{\sin \theta-1}{1+\sec \theta}\right)=0.$

11. (a) Prove that the equation

$$\sin \theta=\left(a+\frac{1}{a}\right)$$

is possible only for imaginary values of a.

12. (a) If $\sin \theta=\frac{8}{17}$, find $\tan \theta+\sec \theta$

(b) If $\sin \theta=\frac{3}{5}$ and $\frac{\pi}{2}<\theta<\pi$, find the value of $\frac{\sec \theta-\tan \theta}{\operatorname{cosec} \theta+\cot \theta}$

13. (a) If $\cos A=\frac{21}{29}$ and A lies in the fourth quadrant, find other t-ratios.

(b) If θ is an angle lying between π and $\frac{3\pi}{2}$ and $\tan \theta=\frac{3}{4}$, find all the other t-ratios of θ.

(c) The sine of an angle is to its cosine as 8 : 15. Find their actual values.

(*d*) If $\sec\theta=\sqrt{2}$ and $\frac{3\pi}{2}<\theta<2\pi$, find the value of

$$\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta}.$$

(*e*) If $\tan\theta=\frac{4}{5}$, find the value of $\frac{2\sin\theta+3\cos\theta}{4\cos\theta+3\sin\theta}$.

14. If $5\sin^2\theta-1=0$, find the other *t*-ratios.

15. (*a*) If $\cos\theta\,.\operatorname{cosec}\theta=-1$ and θ lies in the fourth quadrant, find $\cos\theta$ and $\operatorname{cosec}\theta$.

(*b*) Eliminate θ and ϕ from $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$.

(*c*) State giving reason whether the following equation is possible :

$$2\sin^2\theta-3\cos\theta-6=0$$

ANSWERS

12. (*a*) $\frac{5}{3}$, (*b*) $-\frac{3}{2}$ 13. (*a*) $\sin A=-\frac{20}{29}$, $\tan A=-\frac{20}{21}$,

(*b*) $\operatorname{cosec} A=\frac{-29}{20}$, $\sec A=\frac{29}{21}$, $\cot A=\frac{-21}{20}$.

$\operatorname{cosec}\theta=-\frac{5}{3}$, (*c*) $\sin\theta=-\frac{8}{17}$, $\cos\theta=-\frac{15}{17}$, (*d*) -1.

14. $\cos\theta=\pm\frac{2}{\sqrt{5}}$, $\tan\theta=\pm\frac{1}{2}$, $\cot\theta=\pm2$, $\sec\theta=\pm\frac{\sqrt{5}}{2}$,

$\operatorname{cosec}\theta=\pm\sqrt{5}$. 15. (*a*) $\frac{1}{\sqrt{2}}$, $-\sqrt{2}$, (*b*) $x^2+y^2+z^2=r^2$ (*c*) No.

14·6. TRIGONOMETRIC FUNCTIONS OF STANDARD ANGLES

(*i*) **Angle of 0°.** Let OX be the initial position of the rotating line. Take any point P on this line in this position. Then OP makes an angle of 0° with x-axis. If $OP=r$ and co-ordinates of P be $(r, 0)$, then y definition

$$\text{co } 0°=\frac{x}{r}=\frac{r}{r}=1$$

$$\sin 0°=\frac{y}{r}=\frac{0}{r}=0$$

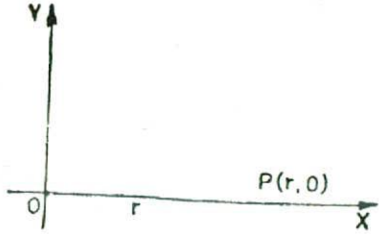


Fig. 15.

$$\therefore \qquad \tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{r} = 0$$

Similarly

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

$$\operatorname{cosec} 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty$$

and
$$\cot 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} = \infty.$$

(ii) Angle of 30° or $\pi/6$. Rotate the straight line through a positive angle XOP of 30°, starting from the initial position OX. Make $\angle QOX = 30°$ in magnitude. Let $P(x, y)$ be any point in this final position of the rotating line. Draw $PM \perp OX$ and extend it to meet OQ in Q. Then evidently $\triangle$s MOP and MOQ are congruent. Therefore $\angle P = \angle Q = 60°$.

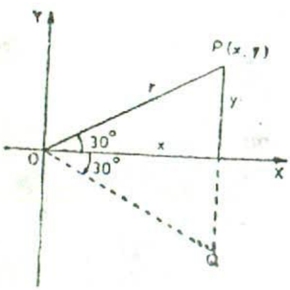


Fig. 16.

Hence $\triangle POQ$ is equilateral.

$$\therefore \qquad MP = \frac{1}{2} \quad PQ = \frac{1}{2} \quad OP = \frac{r}{2}$$

From $\triangle OMP$,

$$OM^2 = OP^2 - MP^2$$

$$r^2 = 1 - \frac{r^2}{4} = \frac{3r^2}{4}$$

$$\therefore \qquad OM = \frac{\sqrt{3r}}{2} \qquad [\because 30° \text{ lies in the first quadrant}]$$

Hence
$$\sin 30^\circ = \frac{MP}{OP} = \frac{r}{2r} = \frac{1}{2},$$

$$\cos 30^\circ = \frac{OM}{OP} = \frac{\sqrt{3}r}{2r} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{MP}{OM} = \frac{r}{2} \cdot \frac{2}{\sqrt{3}r} = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{OM}{MP} = \sqrt{3}$$

$$\sec 30° = \frac{OP}{OM} = \frac{2}{\sqrt{3}}$$

and $$\operatorname{cosec} 30° = \frac{OP}{MP} = 2$$

(*iii*) **Angle of 45° or $\frac{\pi}{4}$**. As before rotate the straight line through a positive angle of 45° with OX.

Take $P(x, y)$ any point in the final position. Draw $PM \perp OX$.

Here $OM = MP = x$

$\therefore \quad OP = \sqrt{OM^2 + MP^2}$

$= x\sqrt{2}$

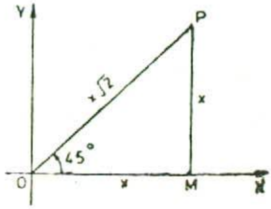


Fig. 17.

From $\triangle OMP$

$$\sin 45° = \frac{MP}{OP} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\cos 45° = \frac{OM}{OP} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{x}{x} = 1.$$

Similarly $\sec 45° = \sqrt{2} = \operatorname{cosec} 45°$ and $\cot 45° = 1$.

(*iv*) **Angle of 60° or $\frac{\pi}{3}$**. Rotate the straight line through a positive angle of 60° starting from the initial position OX. Let $P(x, y)$ be any point on the final position of the straight line. Draw $PM \perp OX$. By geometry, $OP = 2\ OM$

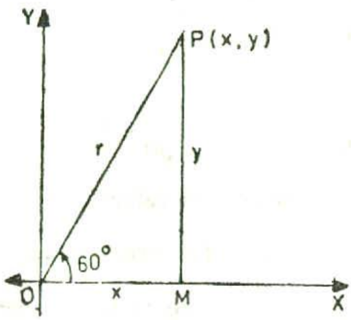


Fig. 18.

$\therefore \quad r = 2x$ and $MP = y = x\sqrt{3}$

From $\triangle OMP$,

$$\sin 60° = \frac{y}{r} = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{x}{r} = \frac{x}{2x} = \frac{1}{2}$$

$$\tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Similarly $\sec 60° = 2$, $\operatorname{cosec} 60° = \frac{2}{\sqrt{3}}$

and $$\cot 60° = \frac{1}{\sqrt{3}}$$

(*v*) **Angle of 90° or $\frac{\pi}{2}$** . When the rotating straight line makes an angle of 90° with *OX*, it lies along *OY*.

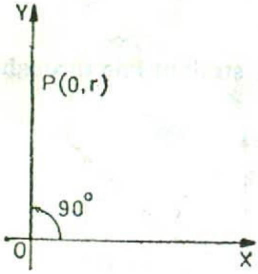


Fig. 19.

Let *OP* be this position. Here in this position, the coordinates of *P* are (0, *r*)

∴ As before,

$$\sin 90° = \frac{y}{r} = \frac{r}{r} = 1$$

$$\cos 90° = \frac{x}{r} = \frac{0}{r} = 0$$

$$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0} = \infty$$

Similarly $\operatorname{cosec} 90° = \frac{1}{\sin 90°} = \frac{1}{1} = 1$

$$\sec 90° = \frac{1}{\cos 90°} = \frac{1}{0} = \infty, \ \cot 90° = \frac{1}{\tan 90°} = \frac{1}{\infty} = 0.$$

TABULAR PRESENTATION

The values of some standard angles dealt above have been presented in a systematised tabular form as an aid to memory. The first four rows are simple steps for assigning suitable values to six *t*-ratios in the remaining six rows. The method of writing these has been explained below :

(*i*) The standard angles from 0° to 90° stated in order.

(*ii*) The numbers 0 to 4 written in the ascending order.

(*iii*) Each number has been divided by 4.

(*iv*) Square root has been taken of the fractions in the previous row.

(*v*) The simplified results of the previous row give the values of sin θ.

(*vi*) Values of sin θ given in the row (*v*) written in the reverse order.

(*vii*) Values of sin θ in row (*v*) divided by cos θ in row (*vi*) give the values of tan θ.

(*viii*) Reciprocals of the values of tan θ given in the previous row.

(*ix*) Reciprocals of the values of cos θ given in the row (*ix*).

(*x*) Reciprocals of the values of sin θ given in the row (*x*).

T-ratios of Standard Angles

(*i*)		0°	30°	45°	60°	90°
(*ii*)		0	1	2	3	4
(*iii*)		$\frac{0}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
(*iv*)		$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
(*v*)	**sin θ**	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
(*vi*)	**cos θ**	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
(*vii*)	**tan θ**	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
(*viii*)	**cot θ**	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
(*ix*)	**sec θ**	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
(*x*)	**cosec θ**	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

It may be noted that the *t*-ratios of 0° and 90° are valid only in the limiting sense. For example, $\sin 0° = \underset{\theta \to 0°}{\text{Lt}} \sin \theta$ and $\tan 90° = \underset{\theta \to 90°}{\text{Lt}} \tan \theta$.

Example 32. *Simplify*

$$\cot^2 \frac{\pi}{2} \sec^2 \frac{\pi}{3} \cos \frac{\pi}{2} - 15 \sin^2 \frac{\pi}{2} \cos \frac{\pi}{4} - 4 \cos \frac{\pi}{6} \cos \frac{\pi}{4} \cos \frac{\pi}{2}.$$

Solution. We note that $\cos\frac{\pi}{2}$ is a factor in the first and the third terms of the given expression. But $\cos\frac{\pi}{2}=\cos 90°=0$, therefore each of these terms is 0. Hence there is no need of substituting values in these terms.

Given expression $=0-15\sin^2 90°\cos 45°-0$

$$=-15\times(1)^2\times\frac{1}{\sqrt{2}}=-\frac{15}{\sqrt{2}}.$$

Example 33. *In a triangle ABC, $C>90°$. Find all angles if*

$$\sin(A+B)=\frac{\sqrt{3}}{2} \text{ and } \cos(A-B)=\frac{1}{\sqrt{2}}.$$

Solution. Since $C>90°$, $A+B<90°$, also $A-B<90°$

Now $\sin(A+B)=\frac{\sqrt{3}}{2} \Rightarrow A+B=60°$ ...(1)

and $\cos(A-B)=\frac{1}{\sqrt{2}} \Rightarrow A-B=45°$ ...(2)

Adding (1) and (2), we get $2A=105°$, *i.e.*, $A=52\frac{1}{2}°$

Subtracting (2) from (1), we get $2B=15°$, *i.e.*, $B=7\frac{1}{2}°$

Also $C=180°-(A+B)=180°-60°=120°$

$\therefore$ $A=52\frac{1}{2}°$, $B=7\frac{1}{2}°$, $C=120°$.

14.7. USE OF PRINTED TABLES

The approximate values of *t*-ratios can be found out from the printed tables which are available for both natural and logarithmic expressions. These tables indicate *t*-ratios for acute angles, for other angles we have to calculate on the basis of the values of acute angles. Since the tables have columns for interval of 6′ the mean difference has to be adjusted given for 1 to 5. It must be remembered that in case of natural sine, tangent and secant, the mean difference is added because these values increase as the θ angle increases from 0° to 90° but in case of natural cosine, cotangent and cosecant (all with initial *c*) the mean difference has to be subtracted because their values decrease as the θ angle increases from 0° to 90°.

Example 34. *Find the value of sin 20° 32′.*

Solution. In the table of "natural sines", first run through the first column under degrees till 20° is reached and then look the horizontal row till the minute column under 30′ is reached. We get

$$\sin 20°\,30'=0{\cdot}3502$$

For the second part, we have to add the mean difference for 2′ which is 0·0005 as follows :

$\sin 20°\,30'=0{\cdot}3502$

Diff. for $2'=0{\cdot}0005$

$\therefore$ $\sin 20°\,32'=0{\cdot}3507.$

Example 3 . *Find the value of*

(a) cos 21° 17′, (b) tan 17° 17′.

Solution. (*a*) Table value of cos 21° 12′ is 0·9323. In this case the mean difference for 5′ has to be subtracted as follows :

$$\cos 21^\circ\ 12' = 0{\cdot}9323$$
$$\text{Difference for } 5' = 0{\cdot}0005$$
$$\therefore \quad \cos 21^\circ\ 17' = 0{\cdot}9318$$

(*b*) From the tables, we have

$$\tan 17^\circ\ 18' = 0{\cdot}3115$$
$$\text{Subtract mean difference of } 1' = 0{\cdot}0003$$
$$\therefore \quad \tan 17^\circ\ 17' = 0{\cdot}3112$$

Interpolation. Values have to be interpolated for *t*-ratios of angles higher than 80° where mean differences are not sufficiently accurate. Also interpolation is resorted to where the angles lie between values shown in the table.

Example 36. *Find the value of*

(a) tan 82° 26′, (b) sin 43° 19′ 20″.

Solution. (*a*) The table values for higher and lower values other than the one given are :

$$\tan 82^\circ\ 30' = 7{\cdot}5958$$
$$\tan 82^\circ\ 24' = 7{\cdot}4947$$
$$\text{Difference for } 6' = 0{\cdot}1011$$
$$\text{Difference for } 2' = \frac{0{\cdot}1011 \times 2}{6} = 0{\cdot}0337$$
$$\tan 82^\circ\ 26' = 7{\cdot}4947 + 0{\cdot}0337 = 7{\cdot}5284.$$

(*b*) The table values of

$$\sin 43^\circ\ 20' = 0{\cdot}3960$$
$$\sin 43^\circ\ 19' = 0{\cdot}3958$$
$$\text{Difference for } 1' \text{ or } 60'' = 0{\cdot}0002$$

Therefore difference for $20'' = \dfrac{0{\cdot}0002 \times 20}{60}$

$$\sin 43^\circ\ 19'\ 20'' = 0{\cdot}3958 + \frac{0{\cdot}0002 \times 20}{60} = 0{\cdot}3959$$

Example 37. *Find the θ (angles) if*

(a) sin θ=0·8865, (b) cos θ=0·8719.

Solution. (*a*) The value of sin 62° 24″ in the table is 0·8862, and the difference between 0·8865 and 0·8862 is 0·0003. The mean difference of 0·0003 is under the column of 2′, therefore, the angle is

$$\sin (60^\circ\ 24' + 2') = \sin 60^\circ\ 26'$$

(*b*) The value of cos 29° 18′ in the table is 0·8721. There is now difference of 0·0002 but it being not there in the mean difference, where it is 0·0003; we have to adjust $\frac{2}{3}$rd of this. But, in cos the *t*-ratio decreases with increase in angle, so the angle has to be adjusted for a decrease.

Therefore

$$0{\cdot}8721 = \cos 22^\circ\ 18'$$

$$\therefore \quad 0{\cdot}8719 = \cos\left[29^\circ\ 18' + \left(2' \times \tfrac{2}{3}\right)\right] = \cos\left[29^\circ\ 18' + 1'\ 20''\right]$$

$$= \cos 29^\circ\ 12'\ 20''$$

Logarithms of *t*-ratios are found in the same manner as in the case of natural logarithms. But, the table values have been given with an addition of 10 to avoid expression in negative values.

Example 38. *Find (a) log sin 43° 17′,*

(b) angle whose log cos is 1·2184 or 9·2184.

Solution. (*a*) From the table

$$\log \sin 43^\circ\ 12' = 9{\cdot}8354$$

Add mean difference for 5′ = 7

$$\therefore \quad \log \sin 43^\circ\ 17' = 9{\cdot}8361 \text{ or } 1{\cdot}8361.$$

(*b*) The value of angle whose log cos is 9·2176 = 80° 30′

Subtract angle for mean difference of 8 = 1′

$$\therefore \quad \text{Angle for log cos } 9{\cdot}2184 = 80^\circ\ 29'$$

EXERCISE (III)

1. Find the values of

(*i*) $\dfrac{\tan 45^\circ}{\text{cosec } 30^\circ} + \dfrac{\sec 60^\circ}{\cot 45^\circ} - \dfrac{5}{2} \cdot \dfrac{\sin 90^\circ}{\cos 0^\circ}$,

(*ii*) $\cos^2 0^\circ + \cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 90^\circ$,

(*iii*) $\dfrac{1}{3} \sin^2 60^\circ - \dfrac{1}{2} \sec 60^\circ \tan^2 30^\circ + \dfrac{4}{3} \sin^2 45^\circ \tan^2 60^\circ$.

2. Prove that

(*a*) $4(\sin^4 30^\circ + \cos^4 60^\circ) - 3(\cos^2 45^\circ - \sin^2 90^\circ) - 2 = 0$

(*b*) $4 \cot^2 \dfrac{\pi}{3} + \sec^2 \dfrac{\pi}{6} - \sin^2 \dfrac{\pi}{4} = \dfrac{13}{6}$

(*c*) $\dfrac{(\sin 30^\circ + \cos 60^\circ)(\sin 0^\circ + \cot 45^\circ)(\cot 90^\circ + \tan 60^\circ)}{(\tan 45^\circ + \sec 60^\circ)(\text{cosec } 30^\circ + \tan 0^\circ)} = \dfrac{1}{2\sqrt{3}}$

(*d*) $32 \cot^2 \dfrac{\pi}{4} - 8 \sec^2 \dfrac{\pi}{3} + 8 \cos^3 \dfrac{\pi}{6} - 3\sqrt{3} = 0.$

3. Find the values of θ from the equation

$$\cot^2 \theta - (1 + \sqrt{3}) \cot \theta + \sqrt{3} = 0 \text{ for } 0 < \theta < \frac{\pi}{2}$$

4. (*a*) If $\tan^2 45° - \cos^2 60° = x \sin 45° \tan 60°$, find x

(*b*) Find x from the equation

$$x \sin 30° \cos^2 45° = \frac{\cot^2 30° \sec 60° \tan 45°}{\operatorname{cosec}^2 45° \operatorname{cosec} 30°}$$

5. (*a*) Given $\sin(A-B)=\frac{1}{2}$, and $\cos(A+B)=\frac{1}{2}$, find A and B (A, B being positive acute angles).

(*b*) Given $\tan(A+B)=\sqrt{3}$, and $\tan(A-B)=1$, find A and B (A, B being positive acute angles).

ANSWERS

1. (*i*) 0, (*ii*) $\frac{5}{2}$, (*iii*) $\frac{23}{12}$. 3. 45°, 30°.

4. (*a*) $\frac{\sqrt{3}}{2}$, (*a*) 6. 5. (*a*) $A=45°$, $B=15°$, (*b*) $A=52\frac{1}{2}°$, $B=7\frac{1}{2}°$.

14·8. T-RATIOS OF ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°. The angles $-\theta$, $90°\pm\theta$, $180°\pm\theta$, $360°\pm\theta$ etc., are angles allied to the angle θ, which is being assumed to be expressed in degrees. However if θ is measured in radians (π radians$=180°$), then angles allied to θ are

$$-\theta, \frac{\pi}{2}\pm\theta, \pi\pm\theta, 2\pi\pm\theta, 2n\pi\pm\theta, \text{ etc.}$$

Through the *t*-ratios of allied angles we can find the *t*-ratios of angles of any magnitude. Broadly speaking all angles can be represented by $n.90°\pm\theta$, where n is zero, an even or an odd integer. Thus, if n is zero only $\pm\theta$ angle remains ; if n is even it may be $180°\pm\theta$ or $360°\pm\theta$ and like that ; if n is odd then it may be $90°\pm\theta$ or $270°\pm\theta$ and like that. The detailed break-up of the three groups is presented below. You will find that except the change of signs depending on the quadrant in which the angle falls, the same *t*-ratios are there when n is even and they change to co-ratios if n is odd.

Some important relations between the *t*-ratios of various allied angles are given below without any proof. Students are advised to remember all these results.

I.
$\sin(-\theta) = -\sin\theta$
$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
$\cos(-\theta) = \cos\theta$
$\sec(-\theta) = \sec\theta$
$\tan(-\theta) = -\tan\theta$
$\cot(-\theta) = -\cot\theta$

Since $-\theta$ lies in the fourth quadrant, only cos and sec are +ive, all other *t*-ratios are −ive.

II. The trigonometric ratio of $(n.90°\pm\theta)$, where n is an even integer and θ is acute angle is numerically equal to the *t*-ratio of θ. The algebraic sign is with reference to the quadrant in which $n.90°\pm\theta$ lies.

(*a*)
$$\begin{aligned}\sin(180°-\theta)&=+\sin\theta\\ \cos(180°-\theta)&=-\cos\theta\\ \tan(180°-\theta)&=-\tan\theta\\ \cot(180°-\theta)&=-\cot\theta\\ \sec(180°-\theta)&=-\sec\theta\\ \text{cosec}(180°-\theta)&=+\text{cosec}\,\theta\end{aligned}$$

Since $180°-\theta$ lies in the second quadrant, only sin and cosec are +ive, all other *t* ratios are −ive.

(*b*)
$$\begin{aligned}\sin(180°+\theta)&=-\sin\theta\\ \cos(180°+\theta)&=-\cos\theta\\ \tan(180°+\theta)&=+\tan\theta\\ \cot(180°+\theta)&=+\cot\theta\\ \sec(180°+\theta)&=-\sec\theta\\ \text{cosec}(180°+\theta)&=-\text{cosec}\,\theta\end{aligned}$$

Since $180°+\theta$ lies in the third quadrant, only tan and cot are +ive, all other *t*-ratios are −ive.

(*c*)
$$\begin{aligned}\sin(360°-\theta)&=-\sin\theta\\ \cos(360°-\theta)&=+\cos\theta\\ \tan(360°-\theta)&=-\tan\theta\\ \cot(360°-\theta)&=-\cot\theta\\ \sec(360°-\theta)&=+\sec\theta\\ \text{cosec}(360°-\theta)&=-\text{cosec}\,\theta\end{aligned}$$

Since $360-\theta$ lies in the fourth quadrant, only cos and sec are +ive, all other *t*-ratios are −ive.

(*d*)
$$\begin{aligned}\sin(360°+\theta)&=+\sin\theta\\ \cos(360°+\theta)&=+\cos\theta\\ \tan(360°+\theta)&=+\tan\theta\\ \cot(360°+\theta)&=+\cot\theta\\ \sec(360°+\theta)&=+\sec\theta\\ \text{cosec}(360°+\theta)&=+\text{cosec}\,\theta\end{aligned}$$

Since $360°+\theta$ lies in the first quadrant, all *t*-ratios are +ive.

III. Any trigonometric ratio of $n\,.\,90°\pm\theta$, where n is an odd integer and θ is any acute angle is numerically equal to the corresponding co-ratio of θ and *vice versa*. The algebraic sign, as in the previous case is the one applicable to the quadrant in which $n\,.\,90°\pm\theta$ lies.

(*a*)
$$\begin{aligned}\sin(90°-\theta)&=+\cos\theta\\ \cos(90°-\theta)&=+\sin\theta\\ \tan(90°-\theta)&=+\cot\theta\\ \cot(90°-\theta)&=+\tan\theta\\ \sec(90°-\theta)&=+\text{cosec}\,\theta\\ \text{cosec}(90°-\theta)&=+\sec\theta\end{aligned}$$

Since $90°-\theta$ lies in the first quadrant, all the *t*-ratios are +ive but since $90°=1\times90°=$ odd multiple of 90°, therefore, sin changes to cos, tan changes to cot, sec changes to cosec and *vice versa*.

(*b*)
$$\begin{aligned}\sin(90°+\theta)&=+\cos\theta\\ \cos(90°+\theta)&=-\sin\theta\\ \tan(90°+\theta)&=-\cot\theta\\ \cot(90°+\theta)&=-\tan\theta\\ \sec(90°+\theta)&=-\text{cosec}\,\theta\\ \text{cosec}(90°+\theta)&=+\sec\theta\end{aligned}$$

Since $90°+\theta$ lies in the second quadrant, only sin and cosec are +ive and all the other *t*-ratios are −ive.

(*c*)

$\sin(270°-\theta) = -\cos\theta$
$\cos(270°-\theta) = -\sin\theta$
$\tan(270°-\theta) = +\cot\theta$
$\cot(270°-\theta) = +\tan\theta$
$\sec(270°-\theta) = -\operatorname{cosec}\theta$
$\operatorname{cosec}(270°-\theta) = -\sec\theta$

Since $270°-\theta$ lies in the third quadrant, only tan and cot are +ive and all the other *t*-ratios are −ive.

(*d*)

$\sin(270°+\theta) = -\cos\theta$
$\cos(270°+\theta) = +\sin\theta$
$\tan(270°+\theta) = -\cot\theta$
$\cot(270°+\theta) = -\tan\theta$
$\sec(270°+\theta) = +\operatorname{cosec}\theta$
$\operatorname{cosec}(270°+\theta) = -\sec\theta$

Since $270°+\theta$ lies in the fourth quadrant, only cos and sec are +ive and all the other *t*-ratios are −ive.

The two important rules to bear in mind are :

1. Any *t*-ratio of an angle expressed as 180° or 360° plus or minus an acute angle, *i.e.*, even number$\times\frac{\pi}{2}\pm$ acute angle has numerically the same *t* ratio as that of the acute angle. The proper signs can be ascertained as per the rules stated below depending on where the revolving line term nates.

sin	all
tan	cos

2. Any *t*-ratio of an angle expressed as 90° or 270° plus or minus an acute angle, *i.e.*, odd number$\times\frac{\pi}{2}\pm$ acute angle equals numerically the co-*t*-ratio of the acute angle with the plus or minus sign depending upon the quadrant in which the revolving line terminates.

The same rules can be presented in the form of a table given below :

t-Ratio	$-\theta$	$90°-\theta$	$90°+\theta$	$180°-\theta$	$180°+\theta$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$\sec\theta$	$+\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$
$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$
$\cot\theta$	$-\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$

Example 39. *Find the values of*

(*a*) *sin 315°*, (*b*) *cos (−1760°)*

Solution. (*a*) $\sin 315° = \sin(360°-45°) = \sin(4\times 90°-45°)$

$$= -\sin 45° = -\frac{1}{\sqrt{2}}.$$

(*b*) $\cos(-1760°) = \cos 1760° = \cos(4\times 360°+320°)$

$$= \cos 320° = \cos(360°-40°)$$
$$= \cos 40°$$
$$= 0{\cdot}7660.$$

Illustrations.

1. $\sin 115° = \sin(90°+25°) = \cos 25° = 0{\cdot}9063$
2. $\sin 120° = \sin(180°-60°) = \sin 60° = \frac{\sqrt{3}}{2}$
3. $\cos 134° = \cos(90°+44°) = -\sin 44° = -0{\cdot}6947$
4. $\cos 150° = \cos(180°-30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$
5. $\tan 172° = \tan(180°-8°) = -\tan 8° = -0{\cdot}1405$
6. $\tan 211° = \tan(180°+31°) = \tan 31° = 0{\cdot}6009$
7. $\sin 748° = \sin(2\times 360°+28°) = \sin 28° = 0{\cdot}4695$
8. $\cos 700° = \cos(2\times 360°-20°) = \cos 20° = 0{\cdot}9397$
9. $\cos 246° = \cos(180°+66°) = -\cos 66° = -0{\cdot}4069$
10. $\tan 675° = \tan(2\times 360°-45°) = -\tan 45° = -1$
11. $\sin 315° = \sin(360°-45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$
12. $\tan(-1742°) = -\tan 1742°$
$$= -\tan(5\times 360°-58°)$$
$$= \tan 58° = 1{\cdot}6003$$
13. $\cos(-1760°) = \cos 1760°$
$$= \cos(4\times 360°+320°)$$
$$= \cos 320° = \cos(360°-40°)$$
$$= \cos 40° = 0{\cdot}7660$$
14. Prove that $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$.

Solution. L.H.S. $= \tan(180°+45°)\cot(360°+45°)$
$$+\tan(2\times 360°+45°)\cot(2\times 360°-45°)$$
$$= \tan 45° \cot 45° + \tan 45°(-\cot 45°) = 0 = \text{R.H.S.}$$

Example 40. *Prove that*

$$\cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300° = \tfrac{1}{2}$$

Solution. We have

$$\cos 125° = \cos(180°-55°) = -\cos 55°$$

$\cos 204° = \cos(180° + 24°) = -\cos 24°$

$\cos 300° = \cos(360° - 60°) = \cos(-60°) = \cos 60° = \frac{1}{2}$

$\therefore$ L.H.S. $\cos 24° + \cos 55° - \cos 55° - \cos 24° + \frac{1}{2} = \frac{1}{2}$ = R.H.S.

Example 41. *Prove that*

$$\cos 510° \cos 330° + \sin 390° \cos 120° = -1.$$

Solution. Now $\cos 510° = \cos(360° + 150°)$

$$= \cos 150° = \cos(180° - 30°)$$

$$= -\cos 30° = -\frac{\sqrt{3}}{2}$$

$$\cos 330° = \cos(360° - 30°) = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\sin 390° = \sin(360° + 30°) = \sin 30° = \frac{1}{2}$$

and $\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$

$\therefore$ L.H.S. $= \cos 510° \cos 330° + \sin 390° \cos 120°$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{3}{4} - \frac{1}{4} = -1 = \text{R.H.S.}$$

Example 42. *Simplify* $\dfrac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sec(360° - \theta)\sin(180° + \theta)\cot(90° - \theta)}$

Solution. We know that

$$\cos(90° + \theta) = -\sin\theta,\ \sec(-\theta) = \sec\theta$$

$$\tan(180° - \theta) = -\tan\theta,\ \sec(360° - \theta) = \sec\theta$$

$$\sin(180° + \theta) = -\sin\theta,\ \cot(90° - \theta) = \tan\theta$$

$\therefore$ Given expression $= \dfrac{(-\sin\theta)\sec\theta(-\tan\theta)}{\sec\theta(-\sin\theta)\tan\theta} = -1$

Example 43. *Simplify*

$$\frac{\sin\theta}{\cos(90° - \theta)} + \frac{\tan(-\theta)}{\tan(180° - \theta)} + \frac{\sec(180° - \theta)}{\operatorname{cosec}(90° - \theta)}$$

Solution. We know that

$\tan(-\theta) = -\tan\theta$, $\cos(90° - \theta) = \sin\theta$

$\sec(180° - \theta) = -\sec\theta$, $\tan(180° - \theta) = -\tan\theta$

$\operatorname{cosec}(90° - \theta) = \sec\theta$

$\therefore$ Given expression $= \dfrac{\sin\theta}{\sin\theta} + \dfrac{\sin\theta}{\tan\theta} - \dfrac{\sec\theta}{\sec\theta} = 1.$

Example 44. *Find x from the equation :*

$$\operatorname{cosec}(90° + A) + x\cos A\cot(90° + A) = \sin(90° + A).$$

Solution. We know that

$$\operatorname{cosec}(90° + A) = \sec A = \frac{1}{\cos A}$$

$$\cot(90° + A) = -\tan A = -\frac{\sin A}{\cos A}$$

$$\sin(90° + A) = \cos A$$

Substituting these values in the given equation, we get

$$\frac{1}{\cos A}+x\cos A\left(-\frac{\sin A}{\cos A}\right)=\cos A$$

$$\Rightarrow \quad x\sin A=\frac{1}{\cos A}-\cos A=\frac{1-\cos^2 A}{\cos A}=\frac{\sin^2 A}{\cos A}$$

$$\therefore \quad x=\frac{\sin^2 A}{\cos A}\times\frac{1}{\sin A}=\tan A.$$

EXERCISE (IV)

1. Find the values of the following :
   (a) $\sin 480°$, $\cos(-1125°)$, $\sin(-3060°)$, $\cos 720°$
   (b) $\tan 1170°$, $\cot 570°$, $\sec 3120°$, $\text{cosec}\ 390°$, $\text{cosec}\ 1125°$

2. Prove that
   (a) $\dfrac{\sin 330°\times\tan 495°\times\text{cosec}\ 150°}{\tan 120°}=-\dfrac{1}{\sqrt{3}}$
   (b) $\sin 480°\cos 690°+\cos 780°\sin 1050°=\frac{1}{2}$
   (c) $\cos 570°\sin 510°+\sin(-330°)\cos(-390°)=0.$

3. (a) $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°+\theta)}=-1.$
   (b) $\dfrac{\cos(270°-A)\tan(90°+A)\sin(180°+2A)}{2\sin(270°+A)\cos(180°+A)\sin(360°+A)}=-1.$
   (c) $\dfrac{\cot(450°+A)\sin(180°+2A)\cos(270°-A)}{\cos(180°+A)\sin(360°-A)\cos(630°+A)}=2.$
   (d) $\dfrac{\cos(2\pi+A)\,\text{cosec}(2\pi+A)\tan\left(\frac{\pi}{2}+A\right)}{\sec\left(\frac{\pi}{2}+A\right)\cos A\cot(\pi+A)}=1.$

4. Prove that
   (a) $\dfrac{\text{cosec}(90°+A)+\cot(450°+A)}{\text{cosec}(90°-A)+\tan(180°-A)}+\dfrac{\tan(180°+A)+\sec(180°-A)}{\tan(360°+A)-\sec(-A)}=2$
   (b) $\dfrac{\cos(180°+\theta)\sin(90°+\theta)+\sin(180°-\theta)\sin(180°+\theta)}{\cos(360°-\theta)\tan(90°+\theta)\sec^2(180°+\theta)}=\sin\theta$
   (c) $\dfrac{\sin(270°+\theta)\cos^3(720°-\theta)-\sin(270°-\theta)\sin^3(540°+\theta)}{\sin(90°+\theta)\sin(-\theta)-\cos^2(180°-\theta)}+\dfrac{\cot(270°-\theta)}{\text{cosec}^2(450°+\theta)}=1,$

where θ is taken such that the denominator appearing in any fraction in the expression does not vanish.

5. (*a*) $\left[1+\cot A-\sec\left(A+\frac{\pi}{2}\right)\right]\left[1+\cot A+\sec\left(A+\frac{\pi}{2}\right)\right]$ $=2\cot A$

(*b*) If $\sin\theta=\frac{1}{3}$ and $\frac{\pi}{2}<\theta<\pi$, find the value of

$$2\sqrt{2}\cos\left(\frac{\pi}{2}+\theta\right)\sec(\pi+\theta)-3\sin\left(\frac{\pi}{2}-\theta\right)\tan(\pi-\theta).$$

6. Find x from the equation :

$\operatorname{cosec}(270°-A)+x\cos(180°+A)\tan(180°-A)=\sin(270°+A)$.

7. Find the value of

$$\tan\frac{11\pi}{3}-2\sin\frac{4\pi}{6}-\frac{3}{4}\operatorname{cosec}^2\frac{\pi}{4}+4\cos^2\frac{17\pi}{6}$$

8. $\sin^2\frac{\pi}{18}+\sin^2\frac{\pi}{9}+\sin^2\frac{7\pi}{18}+\sin^2\frac{4\pi}{9}=2.$

9. If A, B, C, D are the angles of a quadrilateral, show that

(*i*) $\sin(A+B)+\sin(C+D)=0$

(*ii*) $\cos(A+B)=\cos(C+D)$.

10. Find the value of $\tan 5°$, $\tan 25°$, $\tan 45°$, $\tan 65°$, $\tan 85°$.

ANSWERS

1. (*a*) $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{2}}$, 0, 1. (*b*) $\pm\infty$, $\sqrt{3}$, -2, 2, $\sqrt{2}$.

6. $\tan A$. 7. $\frac{3-4\sqrt{3}}{2}$, 10. 1.

14·9. T-RATIOS OF SUM AND DIFFERENCE OF ANGLES

So far we have studied the *t*-ratios of single angles. Now, we take up the *t*-ratios of compound angles, *i.e.*, sum or difference of two or more angles such as $A+B$, $A-B$, $A+B+C$, etc.

I. *For any two angles A and B,*

(*i*) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(*ii*) $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

Proof. Let the revolving line start from OX and trace out the $\angle XOY = A$ in the anti-clockwise direction. Let the revolving line further trace out the $\angle YOZ = B$ in the same direction so that $\angle XOZ = A + B$.

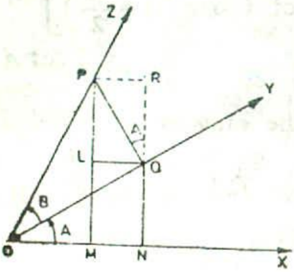


Fig. 20.

From any point P on OZ, draw two perpendiculars PM on OX and PQ on OY and from Q, the foot of perpendicular of PQ, we draw two perpendiculars QN on OX and QL on PM respectively. Further we extend QN to meet PR drawn parallel to LQ.

Then $\angle PQR = 90° - \angle PQL$
$= \angle LQO = \angle QON$
$= A$

(*i*) $\sin(A+B) = \dfrac{PM}{OP} = \dfrac{RN}{OP}$

$= \dfrac{QN + QR}{OP}$

$= \dfrac{QN}{OP} + \dfrac{QR}{OP}$

$= \dfrac{QN}{OQ} \cdot \dfrac{OQ}{OP} + \dfrac{QR}{PQ} \cdot \dfrac{PQ}{OP}$ (Note this step)

$= \sin A \cos B + \cos A \sin B$

(*ii*) $\cos(A+B) = \dfrac{OM}{OP} = \dfrac{ON - NM}{OP} = \dfrac{ON - PR}{OP}$

$= \dfrac{ON}{OP} - \dfrac{PR}{OP} = \dfrac{ON}{OQ} \cdot \dfrac{OQ}{OP} - \dfrac{PR}{PQ} \quad \dfrac{PQ}{OP}$

(Note this step)

$= \cos A \cos B - \sin A \sin B.$

II. *For any two angles A and B,*

(*i*) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(*ii*) $\cos(A-B) = \cos A \cos B + \sin A \sin B.$

Proof. Let the revolving line starting from OX in anti-clockwise direction, trace out the $\angle XOZ = A$ and revolving back in clockwise direction, trace out the $\angle YOZ = B$ so that

$$\angle XOY = A - B$$

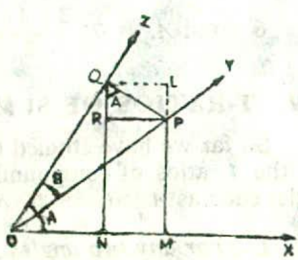


Fig. 21.

From any point P on OY, draw two perpendiculars PQ on OZ and PM on OX respectively. Next we draw another perpendicular QN from Q on OX and extend PM to meet QL drawn parallel to OX.

Then $\angle PQR = 90° - \angle PQL = A$

(*i*) $\sin(A-B) = \dfrac{PM}{OP}$

$$=\frac{ML-PL}{OP}=\frac{QN-QR}{OP}=\frac{QN}{OP}-\frac{QR}{OP}$$

$$=\frac{QN}{OQ}\cdot\frac{OQ}{OP}-\frac{QR}{PQ}\cdot\frac{PQ}{OP} \qquad \text{(Note this step)}$$

$$=\sin A\cos B-\cos A\sin B$$

(*ii*) $$\cos(A-B)=\frac{OM}{OP}=\frac{ON+MN}{OP}=\frac{ON+PR}{OP}$$

$$=\frac{ON}{OP}+\frac{PR}{OP}=\frac{ON}{OQ}\cdot\frac{OQ}{OP}+\frac{PR}{PQ}\cdot\frac{PQ}{OP}$$

$$=\cos A\cos B+\sin A\sin B$$

III. (*i*) $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

(*ii*) $$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Proof.

(*i*) $$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

By dividing both the numerator and denominator by $\cos A\cos B$, we have

$$=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

(*ii*) $$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}.$$

By dividing both the numerator and the denominator by $\cos A\cos B$, we get

$$\tan(A-B)=\frac{\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}}{1+\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

IV. (*i*) $$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A}$$

(*ii*) $$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}$$

Solution. (*i*) $$\cot(A+B)=\frac{1}{\tan(A+B)}=\frac{1-\tan A\tan B}{\tan A+\tan B}$$

$$=\frac{\left[1-\frac{1}{\cot A}\cdot\frac{1}{\cot B}\right.}{\frac{1}{\cot A}+\frac{1}{\cot B}}=\frac{\frac{\cot A\cot B-1}{\cot A\cot B}}{\frac{\cot B+\cot A}{\cot A\cot B}}$$

$$=\frac{\cot A\cot B-1}{\cot B+\cot A}$$

(*ii*) Left as an exercise to the reader.

V. (i) $$sec\,(A+B)=\frac{sec\,A\,sec\,B\,cosec\,A\,cosec\,B}{cosec\,A\,cosec\,B-sec\,A\,sec\,B}$$

(*ii*) $$cosec\,(A+B)=\frac{sec\,A\,sec\,B\,cosec\,A\,cosec\,B}{sec\,A\,cosec\,B+cosec\,A\,sec\,B}$$

Proof. (*i*) $\because \cos(A+B)=\cos A\cos B-\sin A\sin B$

$$\therefore \quad \frac{1}{\sec(A+B)}=\frac{1}{\sec A}\cdot\frac{1}{\sec B}-\frac{1}{\text{cosec } A}\cdot\frac{1}{\text{cosec } B}$$

$$=\frac{\text{cosec } A\text{ cosec } B-\sec A\sec B}{\sec A\sec B\text{ cosec } A\text{ cosec } B}$$

Taking reciprocals of both sides, we get

$$\sec(A+B)=\frac{\sec A\sec B\text{ cosec } A\text{ cosec } B}{\text{cosec } A\text{ cosec } B-\sec A\sec B}$$

(*ii*) Left as an exercise to the reader.

VI. (*i*) $sin\,(A+B)\,sin\,(A-B)=sin^2 A-sin^2 B$ (*or* $cos^2 B-cos^2 A$)

(*ii*) $cos\,(A+B)\,cos\,(A-B)=cos^2 A-sin^2 B$ (*or* $cos^2 B-sin^2 A$)

Proof. (*i*) $\sin(A+B)\sin(A-B)$

$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$

$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$

$=\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$

$=\sin^2 A-\sin^2 A\sin^2 B-\sin^2 B+\sin^2 A\sin^2 B$

$=\sin^2 A-\sin^2 B$ (First form)

$=(1-\cos^2 A)-(1-\cos^2 B)$

$=\cos^2 B-\cos^2 A$ (Second form)

(*ii*) Left as an exercise to the reader.

Example 45. (*a*) *Evaluate* (*i*) *sin 75°*, (*ii*) *tan 75°*, (*iii*) *cos 15°* (*iv*) *tan 15°*.

(*b*) *Prove that tan 75°+cot 75°=4.*

Solution. (*a*) (*i*) $\sin 75°=\sin(45°+30°)$

$=\sin 45°\cos 30°+\cos 45°\sin 30°$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

(*ii*) $$\tan 75°=\tan(45°+30°)=\frac{\tan 45°+\tan 30°}{1-\tan 45°\tan 30°}$$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{3+1+2\sqrt{3}}{2}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}$$

(iii) $\cos 15^\circ=\cos(45^\circ-30^\circ)$

$$=\cos 45^\circ \cos 30^\circ+\sin 45^\circ \sin 30^\circ$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iv) $\tan 15^\circ=\tan(45^\circ-30^\circ)=\dfrac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{3+1-2\sqrt{3}}{2}=2-\sqrt{3}$$

(b) $\tan 75^\circ+\cot 75^\circ=(2+\sqrt{3})+\dfrac{1}{(2+\sqrt{3})}$

$$=(2+\sqrt{3})+\frac{1}{(2+\sqrt{3})}\times\frac{(2-\sqrt{3})}{(2-\sqrt{3})}$$

$$=(2+\sqrt{3})+(2-\sqrt{3})=4.$$

Example 46. (a) *If* $\cos A=\frac{1}{7}$ *and* $\cos B=\frac{13}{14}$, *A and B being positive and acute angles, prove that* $A-B=60^\circ$.

(b) *If* $\sin A=\frac{1}{\sqrt{10}}$ *and* $\sin B=\frac{1}{\sqrt{5}}$, *A and B being positive and acute angles, prove that* $A+B=45^\circ$.

Solution. (a) $\sin^2 A=1-\cos^2 A=1-\frac{1}{49}=\frac{48}{49}$

$\Rightarrow$ $\sin A=\pm\frac{4\sqrt{3}}{7}$

Since A is an acute angle, $\sin A=\frac{4\sqrt{3}}{7}$

Also $\sin^2 B=1-\cos^2 B=1-\frac{169}{196}=\frac{27}{196}$

$\Rightarrow$ $\sin B=\pm\frac{3\sqrt{3}}{14}$

Since B is an acute angle, $\sin B=\frac{3\sqrt{3}}{14}$

Now $\cos(A-B)=\cos A\cos B+\sin A\sin B$

$$=\frac{1}{7}\times\frac{13}{14}+\frac{4\sqrt{3}}{7}\times\frac{3\sqrt{3}}{14}$$

$$=\frac{13}{98}+\frac{36}{98}=\frac{1}{2}=\cos 60^\circ$$

$\therefore \quad A-B=60°.$

(b) Here $\cos A=\sqrt{1-\sin^2 A}=\sqrt{1-\frac{1}{10}}=\frac{3}{\sqrt{10}}$

$\cos B=\sqrt{1-\sin^2 B}=\sqrt{1-\frac{1}{5}}=\frac{2}{\sqrt{5}}$

$\therefore \quad \cos(A+B)=\cos A\cos B-\sin A\sin B$

$$=\frac{3}{\sqrt{10}}\times\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{10}}\times\frac{1}{\sqrt{5}}$$

$$=\frac{5}{\sqrt{50}}=\frac{5}{5\sqrt{2}}=\frac{1}{\sqrt{2}}=\cos 45°$$

$\Rightarrow \quad A+B=45°$

Example 47. (a) *If $A+B+C+D=\pi$, prove that*

$\cos A\cos B+\cos C\cos D=\sin A\sin B+\sin C\sin D$

(b) *In any quadrilateral ABCD, prove that*

$\cos A\cos B+\sin C\sin D=\sin A\sin B+\cos C\cos D$.

Solution. (a) $\because A+B+C+D=\pi$

$\therefore \quad A+B=\pi-(C+D)$

$\Rightarrow \quad \cos(A+B)=\cos\{\pi-(C+D)\}=-\cos(C+D)$

$\Rightarrow \quad \cos(A+B)+\cos(C+D)=0$

$\Rightarrow \cos A\cos B-\sin A\sin B+\cos C\cos D-\sin C\sin D=0$

$\Rightarrow \cos A\cos B+\cos C\cos D=\sin A\sin B+\sin C\sin D.$

(b) Since A, B, C and D are the angles of a quadrilateral, therefore,

$A+B+C+D=360°$

$\Rightarrow \quad A+B=360°-(C+D)$

$\Rightarrow \quad \cos(A+B)=\cos\{360°-(C+D)\}=\cos(C+D)$

$\Rightarrow \quad \cos A\cos B-\sin A\sin B=\cos C\cos D-\sin C\sin D$

$\Rightarrow \quad \cos A\cos B+\sin C\sin D=\sin A\sin B+\cos C\cos D.$

Example 48. *If $A+B=45°$, show that*

(a) *$(1+\tan A)(1+\tan B)=2$*, (b) *$(\cot A-1)(\cot B-1)=2$.*

Solution. (a) L.H.S. $=(1+\tan A)(1+\tan B)$

$=(1+\tan A)[1+\tan(45°-A)]$ $\quad [\because A+B=45°]$

$$=(1+\tan A)\left[1+\frac{\tan 45°-\tan A}{1+\tan 45°\tan A}\right]$$

$$=(1+\tan A)\left[1+\frac{1-\tan A}{1+\tan A}\right]$$

$$=(1+\tan A)\left[\frac{1+\tan A+1-\tan A}{1+\tan A}\right]=2=\text{R.H.S.}$$

(*b*) $\text{L.H.S.}=(\cot A-1)(\cot B-1)=\left(\frac{1}{\tan A}-1\right)\left(\frac{1}{\tan B}-1\right)$

$$=\frac{1-\tan A}{\tan A}\times\left[\frac{1+\tan A}{1-\tan A}-1\right]$$

$$\left(\because \tan B=\frac{1-\tan A}{1+\tan A}\right)$$

$$=\frac{1-\tan A}{\tan A}\times\frac{2\tan A}{1-\tan A}=2=\text{R.H.S.}$$

Example 49. *Prove that*

$$\tan 8A-\tan 5A-\tan 3A=\tan 8A\tan 5A\tan 3A.$$

Solution. $\tan 8A=\tan(5A+3A)=\frac{\tan 5A+\tan 3A}{1-\tan 5A\tan 3A}$

$\Rightarrow$ $\tan 8A(1-\tan 5A\tan 3A)=\tan 5A+\tan 3A$

$\Rightarrow$ $\tan 8A-\tan 5A-\tan 3A=\tan 8A\tan 5A\tan 3A$

Example 50. *Prove that*

$$\tan 75°-\tan 30°-\tan 75°\tan 30°=1$$

Solution. $\tan 45°=\tan(75°-30°)$

$\therefore$ $1=\frac{\tan 75°-\tan 30°}{1+\tan 75°\tan 30°}$

$\Rightarrow$ $1+\tan 75°\tan 30°=\tan 75°-\tan 30°$

$\Rightarrow$ $\tan 75°-\tan 30°-\tan 75°\tan 30°=1.$

Example 51. *Prove that*

(*a*) $$\frac{\cos 13°+\sin 13°}{\cos 13°-\sin 13°}=\tan 58°$$

(*b*) $\tan 69°+\tan 66°+1=\tan 69°\tan 66°$

Solution. (*a*) $\tan 58°=\tan(45°+13°)$

$$=\frac{\tan 45°+\tan 13°}{1-\tan 45°\tan 13°}=\frac{1+\tan 13°}{1-\tan 13°}$$

$$=\frac{1+\frac{\sin 13°}{\cos 13°}}{1-\frac{\sin 13°}{\cos 13°}}=\frac{\cos 13°+\sin 13°}{\cos 13°-\sin 13°}$$

(*b*) The angles involved are 69° and 66°, whose sum is 135° which can be written as 180°−45°

$\therefore$ $\tan 135°=\tan(69°+66°)$

$\Rightarrow$ $-1=\frac{\tan 69°+\tan 66°}{1-\tan 69°\tan 66°}$

$[\because \tan 135°=\tan(180°-45°)=-\tan 45°=-1]$

$$\Rightarrow \qquad \tan 69° + \tan 66° = -1 + \tan 69° \tan 66°$$

$$\Rightarrow \qquad \tan 69° + \tan 66° + 1 = \tan 69° \tan 66°.$$

Example 52. *Prove that*

$$\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)=1.$$

Solution. $\cot\left(\frac{\pi}{4}-\theta\right)=\tan\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\theta\right)\right]$

$$=\tan\left(\frac{\pi}{2}-\frac{\pi}{4}+\theta\right)=\tan\left(\frac{\pi}{4}+\theta\right)$$

$$\therefore \cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)=\cot\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{\pi}{4}+\theta\right)$$

$$=\frac{1}{\tan\left(\frac{\pi}{4}+\theta\right)}\cdot\tan\left(\frac{\pi}{4}+\theta\right)=1.$$

EXERCISE (V)

1. (*a*) If $0<\theta<\frac{\pi}{2}$, $0<\phi<\frac{\pi}{2}$ and $\sin\theta=\frac{3}{5}$ and $\sin\phi=\frac{15}{17}$ find $\cos(\theta+\phi)$ and state in which quadrant $\theta+\phi$ lies.

(*b*) If $\sin\alpha=-\frac{5}{13}$, $\pi<\alpha<\frac{3\pi}{2}$

and $\cos\beta=-\frac{7}{25}$, $\pi<\beta<\frac{3\pi}{2}$,

find $\sin(\alpha-\beta)$ and $\cos(\alpha+\beta)$.

2. Prove that

(*i*) $\frac{\sin(A+B)}{\cos A\cos B}=\tan A+\tan B$

(*ii*) $\frac{\sin(A+B)}{\sin(A-B)}=\frac{\tan A+\tan B}{\tan A-\tan B}$.

3. $\sin\left(\theta-\frac{\pi}{6}\right)+\cos\left(\theta-\frac{\pi}{3}\right)=\sqrt{3}\sin\theta.$

4. $\tan\left(\frac{\pi}{4}+\theta\right)=\frac{1+\tan\theta}{1-\tan\theta}$, $\tan\left(\frac{\pi}{4}-\theta\right)=\frac{1-\tan\theta}{1+\tan\theta}$.

5. (*a*) $\sin(B-C)\cos A+\sin(C-A)\cos B+\sin(A-B)\cos C=0$

(*b*) $\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}=0.$

6. $\sin 105°+\cos 105°=\cos 45°$

[**Hint.** $\sin 105°=\sin(60°+45°)$.]

7. $\frac{\tan(\alpha+\beta)}{\cot(\alpha-\beta)}=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha-\sin^2\beta}=\frac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\tan^2\beta}$

8. $\sin(n+1)A\sin(n-1)A+\cos(n+1)A\cos(n-1)A=\cos 2A$

9. (a) $\cot\theta-\cot 2\theta=\operatorname{cosec} 2\theta$

(b) $\tan 2\theta-\tan\theta=\tan\theta\sec 2\theta$

10. $\dfrac{1}{\tan 3A+\tan A}-\dfrac{1}{\cot 3A+\cot A}=\cot 4A$

11. Prove that

$$\frac{\cos\theta}{1-\tan\theta}+\frac{\sin\theta}{1-\cot\theta}=\sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right)$$

12. (a) $\tan 13A-\tan 9A-\tan 4A=\tan 13A\tan 9A\tan 4A$

(b) $\tan 23°+\tan 22°+\tan 23°\tan 22°=1$

(c) $\tan 70°=\tan 20°+2\tan 50°$

13. (a) $\dfrac{\cos 29°+\sin 29°}{\cos 29°-\sin 29°}=\tan 74°$

(b) $\dfrac{\cos 15°-\sin 15°}{\cos 15°+\sin 15°}=\dfrac{1}{\sqrt{3}}$

14. (a) $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{\pi}{4}-\theta\right)=1$

(b) $\operatorname{cosec} 2\theta=\dfrac{\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)}{\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)}$

15. If $\tan\beta=\dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, prove that

$$\tan(\alpha-\beta)=(1-n)\tan\alpha.$$

14·10. T-RATIOS OF MULTIPLE ANGLES

The angles 2θ, 3θ, 4θ, ... are called *multiple angles* of θ and the angles $\frac{\theta}{2}$, $\frac{\theta}{3}$, $\frac{\theta}{4}$, ... are called *submultiple angles of* θ.

I. Formula for the t-ratios of 2θ :

(a) (i) $\cos 2\theta=\cos^2\theta-\sin^2\theta$

(ii) $\cos 2\theta=2\cos^2\theta-1$

(iii) $\cos 2\theta=1-2\sin^2\theta$

(iv) $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(b) $1+\cos 2\theta=2\cos^2\theta$; $1-\cos 2\theta=2\sin^2\theta$

(c) (i) $\sin 2\theta=2\sin\theta\cos\theta$

(ii) $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$

(d) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

Proof. We use the results of the addition theorem for cosines, sines and tangents.

(a) (i) $\cos 2\theta=\cos(\theta+\theta)=\cos\theta\cos\theta-\sin\theta\sin\theta=\cos^2\theta-\sin^2\theta$

(ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta=\cos^2\theta-(1-\cos^2\theta)=2\cos^2\theta-1$

(iii) $\cos 2\theta=\cos^2\theta-\sin^2\theta=(1-\sin^2\theta)-\sin^2\theta=1-2\sin^2\theta$

(iv) $$\cos 2\theta=\cos^2\theta-\sin^2\theta=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$ [$\because$ $\cos^2\theta+\sin^2\theta=1$]

$$=\frac{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$$ [dividing numerator and denominator by $\cos^2\theta$]

$$=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

(b) $\cos 2\theta=2\cos^2\theta-1 \Rightarrow 1+\cos 2\theta=2\cos^2\theta$

Also $\cos 2\theta=1-2\sin^2\theta \Rightarrow 1-\cos 2\theta=2\sin^2\theta$

(c) (i) $\sin 2\theta=\sin(\theta+\theta)=\sin\theta\cos\theta+\cos\theta\sin\theta=2\sin\theta\cos\theta$

(ii) $$\sin 2\theta=2\sin\theta\cos\theta=\frac{2\sin\theta\cos\theta}{\cos^2\theta+\sin^2\theta}$$

$$=\frac{\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$$ [Dividing numerator and denominator by $\cos^2\theta$]

$$=\frac{\dfrac{2\sin\theta}{\cos\theta}}{\dfrac{\cos^2\theta}{\cos^2\theta}+\dfrac{\sin^2\theta}{\cos^2\theta}}=\frac{2\tan\theta}{1+\tan^2\theta}$$

(d) $$\tan 2\theta=\tan(\theta+\theta)=\frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}=\frac{2\tan\theta}{1-\tan^2\theta}$$

II. Formula for the t-ratios of 3θ :

(a) $\sin 3\theta=3\sin\theta-4\sin^3\theta$

(b) $\cos 3\theta=4\cos^3\theta-3\cos\theta$

(c) $$\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

Proof. We use the results of the addition theorem and its extension.

(a) $\sin 3\theta=\sin(2\theta+\theta)$

$=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta$

$=2\sin\theta\cos\theta\cos\theta+(1-2\sin^2\theta)\sin\theta$

$=2\sin\theta\cos^2\theta+\sin\theta-2\sin^3\theta$

$=2\sin\theta(1-\sin^2\theta)+\sin\theta-2\sin^3\theta$

$$=2\sin\theta-2\sin^3\theta+\sin\theta-2\sin^3\theta$$
$$=3\sin\theta-4\sin^3\theta$$

(b) $$\cos 3\theta=\cos(2\theta+\theta)$$
$$=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta$$
$$=(2\cos^2\theta-1)\cos\theta-2\sin\theta\cos\theta\,.\,\sin\theta$$
$$=2\cos^3\theta-\cos\theta-2\sin^2\theta\cos\theta$$
$$=2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta$$
$$=2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta$$
$$=4\cos^3\theta-3\cos\theta$$

(c) $$\tan 3\theta=\tan(2\theta+\theta)=\frac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$$
$$=\frac{\frac{2\tan\theta}{1-\tan^2\theta}+\tan\theta}{1-\frac{2\tan\theta}{1-\tan^2\theta}\,.\,\tan\theta}$$
$$=\frac{2\tan\theta+\tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta)-2\tan^2\theta}$$
$$=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

III. Formula for the t-ratios of $\frac{\theta}{2}$.

Changing θ to $\frac{\theta}{2}$ *in (1), we get*

(a) (i) $$\cos\theta=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$$

(ii) $$\cos\theta=2\cos^2\frac{\theta}{2}-1$$

(iii) $$\cos\theta=1-2\sin^2\frac{\theta}{2}$$

(iv) $$\cos\theta=\frac{1-t^2}{1+t^2},\ \text{where } t=\tan\frac{\theta}{2}$$

(b) (i) $$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

(ii) $$\sin\theta=\frac{2t}{1+t^2},\ \text{where } t=\tan\frac{\theta}{2}$$

(c) $$\tan\theta=\frac{2t}{1-t^2},\ \text{where } t=\tan\frac{\theta}{2}$$

(*d*) $$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

(*e*) $$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

(*f*) (*i*) $$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}$$

(*ii*) $$\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

Proof. (*a*) $$\cos\theta=2\cos^2\frac{\theta}{2}-1$$

$$\Rightarrow\quad \cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2}$$

$$\Rightarrow\quad \cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

(*e*) $$\cos\theta=1-2\sin^2\frac{\theta}{2}\quad\Rightarrow\quad\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}$$

$$\therefore\quad \sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

(*f*) (*i*) $$\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\frac{2\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

$$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\frac{\sin\theta}{1+\cos\theta}$$

(*ii*) $$\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

Example 53. *Prove that* $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta$

Solution. $$\text{L.H.S.}=\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$$

$$=\frac{\sin\theta+2\sin\theta\cos\theta}{1+\cos 2\theta+\cos\theta}=\frac{\sin\theta\,(1+2\cos\theta)}{2\cos^2\theta+\cos\theta}$$

$$=\frac{\sin\theta\,(1+2\cos\theta)}{\cos\theta\,(2\cos\theta+1)}=\tan\theta=\text{R.H.S.}$$

Example 54. *Prove that*

$$\tan A+\cot A=2\,\text{cosec}\,2A$$

Solution. L.H.S. $=\tan A+\cot A$

$$=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{\sin^2 A+\cos^2 A}{\sin A\cos A}$$

$$=\frac{1}{\sin A\cot A}=\frac{2}{2\sin A\cos A}=\frac{2}{\sin 2A}$$

$$=2\,\text{cosec}\,2A=\text{R.H.S.}$$

Example 55. *Prove that*

$$\text{cosec}\,A-2\cot 2A\cos A=2\sin A$$

Solution. L.H.S. $=\text{cosec}\,A-2\cot 2A\cos A$

$$=\frac{1}{\sin A}-\frac{2\cos 2A}{\sin 2A}\cdot\cos A$$

$$=\frac{1}{\sin A}-\frac{2\cos 2A\cos A}{2\sin A\cos A}$$

$$=\frac{1}{\sin A}-\frac{\cos 2A}{\sin A}=\frac{1-\cos 2A}{\sin A}$$

$$=\frac{2\sin^2 A}{\sin A}=2\sin A=\text{R.H.S.}$$

Example 56. *Prove that*

$$2\cos\theta=\sqrt{2+\sqrt{2+2\cos 4\theta}}$$

Solution. R.H.S. $=\sqrt{2+\sqrt{2(1+\cos 4\theta)}}$

$$=\sqrt{2+\sqrt{2\times 2\cos^2 2\theta}}=\sqrt{2+\sqrt{4\cos^2 2\theta}}$$

$$=\sqrt{2+2\cos 2\theta}=\sqrt{2(1+\cos 2\theta)}$$

$$=\sqrt{2\times 2\cos^2\theta}=2\cos\theta=\text{L.H.S.}$$

Example 57. *Prove that* $\dfrac{1+\sin A+\cos A}{1+\sin A-\cos A}=\cot\dfrac{A}{2}$

Solution. L.H.S. $=\dfrac{(1+\cos A)+\sin A}{(1-\cos A)+\sin A}$

$$=\frac{2\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{2\cos\frac{A}{2}\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)}{2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$$

$$=\cot\frac{A}{2}=\text{R.H.S.}$$

Example 58. *Prove that* $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

Solution. $\text{L.H.S.}=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$

$$=\sqrt{\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$$

$$=\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$\left[\text{Dividing num. and denom. by }\cos\frac{\theta}{2}\right]$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\text{R.H.S.}$$

Example 59. (*a*) *Prove that* $\cot\alpha-\tan\alpha=2\cot 2\alpha$

Hence deduce that

$$\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha=\cot\alpha$$

Solution. We have

$$2\cot 2\alpha=\frac{2}{\tan 2\alpha}=\frac{2}{\frac{2\tan\alpha}{1-\tan^2\alpha}}=\frac{1-\tan^2\alpha}{\tan\alpha}$$

$$=\frac{1}{\tan\alpha}-\tan\alpha=\cot\alpha-\tan\alpha$$

$\Rightarrow \quad 2\cot 2\alpha=\cot\alpha-\tan\alpha \quad \text{...(1)}$

$\therefore \quad 8\cot 8\alpha=4[2\cot 8\alpha]=4[\cot 4\alpha-\tan 4\alpha]$

$=4\cot 4\alpha-4\tan 4\alpha$

$=2(\cot 2\alpha-\tan 2\alpha)-4\tan 4\alpha \quad \text{...[using (1)]}$

$=2\cot 2\alpha-2\tan 2\alpha-4\tan 4\alpha$

$=\cot\alpha-\tan\alpha-2\tan 2\alpha-4\tan 4\alpha$

$\Rightarrow \quad \tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha=\cot\alpha.$

Example 59. (*b*) *If* $\tan x=\frac{1-\cos y}{\sin y}$, *then prove that one solution is* $y=2x$. *Use this result to prove :*

$$\tan 7\frac{1°}{2}=\sqrt{6}-\sqrt{3}+\sqrt{2}-2. \qquad \text{[C.A., November 1991]}$$

Solution. We have

$$\tan x=\frac{1-\cos y}{\sin y}=\frac{2\sin^2(y/2)}{2\sin(y/2)\cos(y/2)}=\tan(y/2)$$

$$\therefore \quad x=y/2 \quad \text{or} \quad y=2x.$$

$$\cos 15°=\cos(45°-30°)=\cos 45°\cos 30°+\sin 45°\sin 30°$$

$$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\sin 15°=\sin(45°-30°)=\sin 45°\cos 30°-\cos 45°\sin 30°$$

$$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$\therefore \quad \tan 7\frac{1°}{2}=\frac{1-\cos 15°}{\sin 15^2}=\frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{2\sqrt{6}-3-\sqrt{3}+2\sqrt{2}-\sqrt{3}-1}{2}=\sqrt{6}-\sqrt{3}+\sqrt{2}-2.$$

Example 60. *From the formula for the circular functions of* 2θ *and* 3θ, *deduce the values of cosines and sines of*

(*a*) *18°, 72°, 36°, 54°.* (*b*) $22\frac{1}{2}°$

Solution. (*a*) (*i*) Let $\theta=18°$, then $5\theta=90°$ or $2\theta+3\theta=90°$

$$\Rightarrow \quad 2\theta=90°-3\theta$$

$$\Rightarrow \quad \sin 2\theta=\sin(90°-3\theta)=\cos 3\theta$$

$$\Rightarrow \quad 2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$$

Dividing throughout by $\cos\theta$, (which being $\cos 18°$, is not zero), we get

$$2\sin\theta=4\cos^2\theta-3=4(1-\sin^2\theta)-3=1-4\sin^2\theta$$

$$\therefore \quad 4\sin^2\theta+2\sin\theta-1=0, \text{ a quadratic equation in } \sin\theta$$

$$\Rightarrow \quad \sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

As sin 18° is +ive,

$$\sin 18° = \frac{\sqrt{5}-1}{4}$$

$$\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

(*ii*) $\sin 72° = \sin(90°-18°) = \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$

and $\cos 72° = \cos(90°-18°) = \sin 18° = \frac{\sqrt{5}-1}{4}$

(*iii*) $\cos 36° = \cos 2.18° = 1-2\sin^2 18°$

[$\because \cos 2A = 1-2\sin^2 A$]

$$= 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1-\frac{2(6-2\sqrt{5})}{16} = \frac{4+4\sqrt{5}}{16}$$

$$= \frac{\sqrt{5}+1}{4}$$

Again $\sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2}$

$$= \sqrt{1-\frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

(*iv*) $\sin 54° = \sin(90°-36°) = \cos 36° = \frac{\sqrt{5}+1}{4}$

$$\cos 54° = \cos(90°-36°) = \sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

(*b*) We know that

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

Put $A=45°$, then

$$\sin 22\tfrac{1}{2}° = \pm\sqrt{\frac{1-\cos 45°}{2}} = \pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$

$$= \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}$$

Since $22\tfrac{1}{2}°$ lies in the first quadrant, $\sin 22\tfrac{1}{2}°$ is +ive

$\therefore$ $\sin 22\tfrac{1}{2}° = \frac{\sqrt{2-\sqrt{2}}}{2}$

(*b*) $\cos 22\tfrac{1}{2}° = \pm\sqrt{\frac{1+\cos 45°}{2}} = \pm\sqrt{\left(1+\frac{1}{\sqrt{2}}\right)/2}$

$$= \pm\sqrt{\frac{2+\sqrt{2}}{4}} = \pm\frac{\sqrt{2+\sqrt{2}}}{2}$$

Now $\cos 22\frac{1}{2}°$ being +ive, we have

$$\cos 22\tfrac{1}{2}° = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Example 61. *Prove that*

$$\sin 36° \sin 72° \sin 108° \sin 144° = \frac{5}{16}$$

Solution. $\sin 108° = \sin (180° - 72°) = \sin 72°$

and $\sin 144° = \sin (180° - 36°) = \sin 36°$

$$\text{L.H.S.} = \sin^2 36° \,.\, \sin^2 72° = \sin^2 36° \,.\, \cos^2 18°$$

$$[\because \sin 72° = \sin (90° - 18°) = \cos 18°]$$

$$= [1 - \cos^2 36°]\,[1 - \sin^2 18°]$$

$$= \left[1 - \left(\frac{\sqrt{5}+1}{4}\right)^2\right]\left[1 - \left(\frac{\sqrt{5}-1}{4}\right)^2\right]$$

$$= \frac{10-2\sqrt{5}}{16} \times \frac{10+2\sqrt{5}}{16} = \frac{100-20}{16 \times 16} = \frac{5}{16} = \text{R.H.S.}$$

EXERCISE (VI)

1. (*a*) If $\sin A = \frac{1}{7}$, find $\cos 2A$

(*b*) If $\tan \theta = 5$, find $\tan 2\theta$

(*c*) If $\sin \theta = \frac{2}{3}$ and θ lies between $\frac{\pi}{2}$ and π, find the value of $\cos 2\theta + \tan 2\theta$

2. (*a*) $\dfrac{\sin 2\theta}{1 - \cos 2\theta} = \cot \theta$, (*b*) $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A$

(*c*) $\dfrac{\sin 2\theta}{\sin \theta} - \dfrac{\cos 2\theta}{\cos \theta} = \sec \theta$, (*d*) $\cot A - \cot 2A = \operatorname{cosec} 2A$

3. (*a*) $\tan \alpha + \cot \alpha = 2 \operatorname{cosec} 2\alpha$, (*b*) $\cot A - \tan A = 2 \cot 2A$

4. (*a*) $\dfrac{1 + \sin 2\theta - \cos 2\theta}{1 + \sin 2\theta + \cos 2\theta} = \tan \theta$

(*b*) $\dfrac{1 + \sin A + \cos A}{1 + \sin A - \cos A} = \cot \dfrac{A}{2}$

5. (*a*) $\dfrac{1 + \cos A}{\sin A} = \cot \dfrac{A}{2}$, (*b*) $\left(\cos \dfrac{A}{2} + \sin \dfrac{A}{2}\right)^2 = 1 + \sin A$

6. $\dfrac{\cos A + \sin A}{\cos A - \sin A} - \dfrac{\cos A - \sin A}{\cos A + \sin A} = 2 \tan 2A$

7. $\tan\left(\dfrac{\pi}{4} + \theta\right) - \tan\left(\dfrac{\pi}{4} - \theta\right) = 2 \tan 2\theta$

8. $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A$

9. $\sec 2A+\tan 2A=\tan\left(\frac{\pi}{4}+A\right)$

10. If $\cos\theta=\frac{1}{2}\left(x+\frac{1}{x}\right)$, show that

(*i*) $\cos 2\theta=\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)$, (*ii*) $\cos 3\theta=\frac{1}{2}\left(x^3+\frac{1}{x^3}\right)$.

11. $\cos^2 A+\cos^2(120°+A)+\cos^2(120°-A)=\frac{3}{2}$

12. $\tan\theta+\tan(\theta+60°)+\tan(\theta-60°)=3\tan 3\theta$

13. $\frac{\cos A}{1+\sin A}=\tan\left(45°-\frac{A}{2}\right)$

14. If $2\tan\alpha=3\tan\beta$, show that $\tan(\alpha-\beta)=\frac{\sin 2\beta}{5-\cos 2\beta}$

ANSWER

1. (*a*) $\frac{47}{49}$, (*b*) $\frac{-15}{12}$ (*c*) $\frac{1}{9}-4\sqrt{5}$.

14·11. TRANSFORMATION OF PRODUCTS AND SUMS

In the last article we have proved that

$$\sin A\cos B+\cos A\sin B=\sin(A+B)$$

and

$$\sin A\cos B-\cos A\sin B=\sin(A-B)$$

By addition, we have

I $$2\sin A\cos B=\sin(A+B)+\sin(A-B)$$

By subtraction, we have

II $$2\cos A\sin B=\sin(A+B)-\sin(A-B)$$

These formulae enable us to express the product of sine and cosine as the sum and difference of two sines.

Again $$\cos A\cos B-\sin A\sin B=\cos(A+B)$$

and

$$\cos A\cos B+\sin A\sin B=\cos(A-B)$$

By addition, we have

III $$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$

By subtraction, we have

IV $$2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

These formulae enable us to express

(*i*) The product of two cosines as the sum of two cosines.

(*ii*) the product of two sines as the difference of two cosines.

For practical purposes, the following verbal statements of the results are more useful.

$$2\sin A\cos B=\sin(\text{sum})+\sin(\text{difference}),$$

$$2\cos A\sin B=\sin(\text{sum})-\sin(\text{difference}),$$

$$2 \cos A \cos B = \cos\text{(sum)} + \cos\text{(difference)}$$
$$2 \sin A \sin B = \cos\text{(difference)} - \cos\text{(sum)}$$

Let $A+B=C$ and $A-B=D$, then

$$A=\frac{C+D}{2} \text{ and } B=\frac{C-D}{2}$$

By substituting for A and B in the formula I, II, III and IV, we obtain

V. $$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

VI. $$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

VII. $$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

VIII. $$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

In practice, it is more convenient to quote the formula we have just obtained verbally as follows :

Sum of two sines = 2 sin (half sum) cos (half difference)
Difference of two sines = 2 cos (half sum) sin (half difference)
Sum of two cosines = 2 sin (half sum) cos (half difference)
Difference of two cosines = −2 sin (half sum) sin (half difference).

Illustrations. 1. $2\sin 9A\cos 6A=\sin(9A+6A)+\sin(9A-6A)$
$$=\sin 15A+\sin 3A$$

2. $2\cos 4\theta\sin 7\theta=\sin(4\theta+7\theta)-\sin(4\theta-7\theta)$
$$=\sin 11\theta-\sin(-3\theta)=\sin 11\theta+\sin 3\theta.$$

3. $$\cos\frac{4A}{2}\cos\frac{7A}{2}=\frac{1}{2}\left\{\cos\left(\frac{4A}{2}+\frac{7A}{2}\right)+\cos\left(\frac{4A}{2}-\frac{7A}{2}\right)\right\}$$
$$=\frac{1}{2}\left\{\cos\frac{11A}{2}+\cos\left(-\frac{3A}{2}\right)\right\}$$
$$=\frac{1}{2}\left\{\cos\frac{11A}{2}+\cos\frac{3A}{2}\right\}$$

4. $$\sin 75^\circ\sin 15^\circ=\frac{1}{2}\left\{\cos(75^\circ-15^\circ)-\cos(75^\circ+15^\circ)\right\}$$
$$=\frac{1}{2}\left\{\cos 60^\circ-\cos 90^\circ\right\}$$

5. $$\sin 16\theta+\sin 8\theta=2\sin\frac{16\theta+8\theta}{2}\cos\frac{16\theta-8\theta}{2}$$
$$=2\sin 12\theta\cos 4\theta$$

6. $$\sin 9A-\sin 7A=2\cos\frac{9A+7A}{2}\sin\frac{9A-7A}{2}$$
$$=2\cos 8A\sin A$$

7. $\cos 2A+\cos 9A=2\cos\frac{11A}{2}\cos\left(-\frac{7A}{2}\right)$

$$=2\cos\frac{11A}{2}\cos\frac{7A}{2}$$

8. $\cos 80°-\cos 20°=2\sin 50°\sin(-30°)=-2\sin 50°\sin 30°$

9. $\frac{\sin 7\theta+\sin 3\theta}{\cos 7\theta+\cos 3\theta}=\frac{2\sin\frac{7\theta+3\theta}{2}\cos\frac{7\theta-3\theta}{2}}{2\cos\frac{7\theta+3\theta}{2}\cos\frac{7\theta-3\theta}{2}}=\tan 5\theta$

10. $\frac{\cos A-\cos 3A}{\sin 3A-\sin A}=\frac{2\sin\frac{A+3A}{2}\sin\frac{3A-A}{2}}{2\cos\frac{3A+A}{2}\sin\frac{3A-A}{2}}=\tan 2A$

Example 62. *Prove that*

$$\cos 20°\cos 40°\cos 60°\cos 80°=\frac{1}{16}$$

Solution. L.H.S. $=\frac{1}{2}(\cos 20°\cos 40°)\cos 80°$

$=\frac{1}{4}\{\cos(20°+40°)+\cos(20°-40°)\}\cos 80°$

$=\frac{1}{4}\{\cos 60°+\cos 20°\}\cos 80°$ $\quad[\because \cos(-20°)=\cos 20°]$

$=\frac{1}{4}\{\frac{1}{2}+\cos 20°\}\cos 80°$

$=\frac{1}{8}\cos 80°+\frac{1}{4}\cos 20°\cos 80°$

$=\frac{1}{8}\cos 80°+\frac{1}{4}\cdot\frac{1}{2}\{\cos(20°+80°)+\cos(20°-80°)\}$

$=\frac{1}{8}\cos 80°+\frac{1}{8}(\cos 100°+\cos 60°)$

$=\frac{1}{8}\cos 80°+\frac{1}{8}(\cos 100°+\frac{1}{2})$

$=\frac{1}{8}\cos 80°+\frac{1}{8}\cos 100°+\frac{1}{16}$

$=\frac{1}{8}\cos 80°-\frac{1}{8}\cos 80°+\frac{1}{16}$

$[\because \cos 100°=\cos(180°-80°)=-\cos 80°]$

$=\frac{1}{16}=$ R.H.S.

Example 63. *Prove that*

$$\sin A\sin(60°-A)\sin(60°+A)=\frac{1}{4}\sin 3A$$

Solution. L.H.S. $=\frac{1}{2}\sin A\,[2\sin(60°-A)\sin(60°+A)]$

$=\frac{1}{2}\sin A\,[\cos 2A-\cos 120°]$

$=\frac{1}{2}\sin A\,[\cos 2A-(-\frac{1}{2})]=\frac{1}{2}\sin A\cos 2A+\frac{1}{4}\sin A$

$=\frac{1}{4}(2\sin A\cos 2A)+\frac{1}{4}\sin A$

$=\frac{1}{4}[\sin(A+2A)+\sin(A-2A)]+\frac{1}{4}\sin A$

$=\frac{1}{4}\sin 3A-\frac{1}{4}\sin A+\frac{1}{4}\sin A$

$=\frac{1}{4}\sin 3A=$ R.H.S.

Example 64. *Prove that*

$$4\cos\alpha\cos\left(\alpha+\frac{\pi}{3}\right)\cos\left(\alpha+\frac{2\pi}{3}\right)=-\cos 3\alpha$$

Solution. L.H.S. $=4\cos\alpha\cos\left(\alpha+\frac{\pi}{3}\right)\cos\left(\alpha+\frac{2\pi}{3}\right)$

$$=4\cos\alpha\left[\cos\left(\alpha+\frac{\pi}{3}\right)\cos\left(\alpha+\frac{2\pi}{3}\right)\right]$$

$$=4\cos\alpha\cdot\frac{1}{2}\left[\cos(2\alpha+\pi)+\cos\left(-\frac{\pi}{3}\right)\right]$$

$$=2\cos\alpha\left[-\cos 2\alpha+\frac{1}{2}\right]=-2\cos 2\alpha\cos\alpha+\cos\alpha$$

$$\left[\because \cos\left(-\frac{\pi}{3}\right)=\cos\frac{\pi}{3}=\frac{1}{2}\right]$$

$$=-2\cdot\frac{1}{2}[\cos 3\alpha+\cos\alpha]+\cos\alpha$$

$=-\cos 3\alpha-\cos\alpha+\cos\alpha=-\cos 3\alpha=$ R.H.S.

Example 65. *Prove that*

$$(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\frac{\alpha-\beta}{2}$$

Solution. L.H.S. $=\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)^2$

$$+\left(2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)^2$$

$$=4\cos^2\frac{\alpha-\beta}{2}\left[\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right]$$

$=4\cos^2\frac{\alpha-\beta}{2}=$ R.H.S.

Example 66. *Prove that*

$$\frac{\cos A+\cos 3A+\cos 5A+\cos 7A}{\sin A+\sin 3A+\sin 5A+\sin+7A}=\cot 4A$$

Solution. L.H.S. $=\frac{(\cos A+\cos 7A)+(\cos 3A+\cos 5A)}{(\sin A+\sin 7A)+(\sin 3A+\sin 5A)}$

(Note this step)

$$=\frac{2\cos 4A\cos 3A+2\cos 4A\cos A}{2\sin 4A\cos 3A+2\sin 4A\cos A}$$

$=\frac{2\cos 4A(\cos 3A+\cos A)}{2\sin 4A(\cos 3A+\cos A)}=\cot 4A=$ **R.H.S.**

Example 67. *Prove that*

$$\frac{\sin 8\theta \cos \theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos \theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$$

Solution. $$\text{L.H.S.} = \frac{\frac{1}{2}\left[2 \sin 8\theta \cos \theta - 2 \sin 6\theta \cos 3\theta\right]}{\frac{1}{2}\left[2 \cos 2\theta \cos\theta - 2 \sin 3\theta \sin 4\theta\right]}$$

$$= \frac{(\sin 9\theta + \sin 7\theta) - (\sin 9\theta + \sin 3\theta)}{(\cos 3\theta + \cos \theta) - (\cos \theta - \cos 7\theta)} = \frac{\sin 7\theta - \sin 3\theta}{\cos 7\theta + \cos 3\theta}$$

$$= \frac{2 \cos \frac{7\theta + 3\theta}{2} . \sin \frac{7\theta - 3\theta}{2}}{2 \cos \frac{7\theta + 3\theta}{2} . \cos \frac{7\theta - 3\theta}{2}} = \frac{2 \cos 5\theta . \sin 2\theta}{2 \cos 5\theta . \cos 2\theta}$$

$$= \tan 2\theta = \text{R.H.S.}$$

Example 68. *Show that*

$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = 2 \cot^n \frac{A-B}{2} \text{ or } 0$$

according as n is even or odd.

Solution. $$\frac{\cos A + \cos B}{\sin A - \sin B} = \frac{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}} = \cot \frac{A-B}{2}$$

$\therefore$ $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n = \cot^n \frac{A-B}{2}$, whenever n is even or odd ...(1)

Also $$\frac{\sin A + \sin B}{\cos A - \cos B} = \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}}$$

$$= \frac{\cos \frac{A-B}{2}}{-\sin \frac{A-B}{2}} = -\cot \frac{A-B}{2}$$

$$\left[\because \sin \frac{B-A}{2} = -\sin \frac{A-B}{2}\right]$$

$\therefore$ $$\left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = (-1)^n \cot^n \frac{A-B}{2}$$

$$= \begin{cases} \cot^n \frac{A-B}{2}, \text{ when } n \text{ is even} \\ -\cot^n \frac{A-B}{2}, \text{ when } n \text{ is odd} \end{cases} \quad ...(2)$$

$$\therefore \quad \left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n=(1)+(2)$$

$$=2 \cot^n \frac{A-B}{2} \text{ or } 0,$$

according as n is even or odd.

Example 69. *Find the value of*

$$\cos 20°+\cos 100°+\cos 140°.$$

Solution. Given expression $=\cos 20°+(\cos 100°+\cos 140°)$

$$=\cos 20°+2 \cos 120° \cos 20°$$
$$=\cos 20°+2(-\tfrac{1}{2}) \cos 20°$$
$$=\cos 20°-\cos 20°=0.$$

Example 70. *Prove that*

(a) $\sin \theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)=0.$

(b) $\cos A+\cos (A+120°)+\cos (A-120°)=0.$

(c) $\cos^2 x+\cos^2 (60°-x)+\cos^2 (60°+x)=\frac{3}{2}.$ [*C.A., May 1991*]

Solution. (a) L.H.S. $=\sin \theta+\sin (\theta+120°)+\sin (\theta+240°)$

$$=\sin \theta+2 \sin \frac{\theta+120°+\theta+240°}{2} \cos \frac{\theta+120°-\theta-240°}{2}$$

$$=\sin \theta+2 \sin (\theta+180°) \cos (-60°)$$
$$=\sin \theta+2(-\sin \theta) \cos 60°=\sin \theta-2 \sin \theta\times\tfrac{1}{2}=0=\text{R.H.S.}$$

(b) L.H.S. $=\cos A+\{\cos (A+120°)+\cos (A-120°)\}$

$$=\cos A+2\left\{\cos \frac{A+120°+A-120°}{2} \cos \frac{A+120°-A+120°}{2}\right\}$$

$$=\cos A+2 \cos A \cos 120°=\cos A+2 \cos A\times(-\tfrac{1}{2})$$
$$=0=\text{R.H.S.}$$

(c) L.H.S. $=\cos^2 x+\cos^2 (60°-x)+\cos^2 (60°+x)$

$$=\frac{1+\cos 2x}{2}+\frac{1+\cos (120°-2x)}{2}+\frac{1+\cos (120°+2x)}{2}$$

$$=\frac{3}{2}+\cos 2x+\cos (120°-2x)+\cos (120°+2x)$$

$$=\frac{3}{2}+\cos 2x+[2 \,.\, \cos 120° \cos 2x]$$

$$=\frac{3}{2}+\cos 2x+2\,.\left(-\frac{1}{2}\right).\cos 2x \qquad \left[\because \cos 120°=-\frac{1}{2}\right]$$

$$=\frac{3}{2}=\text{R.H.S.}$$

Example 71. *Prove that*

$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$$

Solution.

$$\text{R.H.S.}=2\sin\frac{\alpha+\beta}{2}\left[2\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}\right]$$
$$=-2\sin\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha+\beta+2\gamma}{2}-\cos\frac{\alpha-\beta}{2}\right]$$
$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta+2\gamma}{2}$$
$$=\sin\alpha+\sin\beta-\{\sin(\alpha+\beta+\gamma)+\sin(-\gamma)\}$$
$$=\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)=\text{L.H.S.}$$

Example 72. *Prove that*

$$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$$

Solution. L.H.S. $=\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\alpha+\beta+2\gamma}{2}\cos\left(-\frac{\alpha+\beta}{2}\right)$$
$$=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\alpha+\beta+2\gamma}{2}\cos\frac{\alpha+\beta}{2}$$
$$=2\cos\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta+2\gamma}{2}\right]$$
$$=2\cos\frac{\alpha+\beta}{2}\left[2\cos\frac{\alpha+\gamma}{2}\cos\left(-\frac{\beta+\gamma}{2}\right)\right]$$
$$=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}=\text{R.H.S.}$$

EXERCISE (VII)

1. Express the following products as sums :

(*i*) $2\sin\theta\cos 5\theta$

(*ii*) $2\sin(2x+y)\cos(x-2y)$.

2. Express the following sum or difference as products :

(*i*) $\sin 4\theta+\sin 9\theta$, (*ii*) $\sin(x+h)-\sin x$

(*iii*) $\cos 2\theta-\cos 4\theta$, (*iv*) $\sin\frac{\pi}{4}-\sin\frac{\pi}{5}$.

3. Prove that

(*a*) $\sin 10^\circ\sin 30^\circ\sin 50^\circ\sin 70^\circ=\frac{1}{16}$

(b) $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

(c) $\cos 12° \cos 24° \cos 48° \cos 96° = -\frac{1}{16}$ [*C.A., Nov., 1991*]

4. (a) $\sin 38° + \sin 22° = \sin 82°$

(b) $\sin 50° - \sin 70° + \sin 10° = 0$

(c) $\sin 52° + \cos 68° + \cos 172° = 0$

(d) Prove that

$\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200° = -\frac{3}{4}$

5. (a) $\frac{\sin (3A+B) - \sin (A+B)}{\cos (3A+B) + \cos (A+B)} = \tan A$

(b) $\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

(c) $\frac{\sin (A-C) + 2 \sin A + \sin (A+C)}{\sin (B-C) + 2 \sin B + \sin (B+C)} = \frac{\sin A}{\sin B}$.

6. (a) $\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 4 \cos \theta \cos 2\theta \sin 4\theta$.

(b) $\sin (\beta - \gamma) \cos (\alpha - \delta) + \sin (\gamma - \alpha) \cos (\beta - \delta)$
$+ \sin (\alpha - \beta) \cos (\gamma - \delta) = 0.$

7. (a) $\sin \theta + \sin \left(\theta + \frac{2\pi}{3} \right) + \sin \left(\theta + \frac{4\pi}{3} \right) = 0$

(b) $\cos \theta + \cos (\theta + 120°) + \cos (\theta - 120°) = 0$

8. (a) $4 \cos \alpha \cos \beta \cos \gamma = \cos (\alpha + \beta + \gamma) + \cos (\beta + \gamma - \alpha)$
$+ \cos (\gamma + \alpha - \beta) + \cos (\alpha + \beta - \gamma)$

(b) $\cos \alpha + \cos \beta + \cos \gamma + \cos (\alpha + \beta + \gamma)$

$$= 4 \cos \frac{\alpha+\beta}{2} \cos \frac{\beta+\gamma}{2} \cos \frac{\gamma+\alpha}{2}$$

9. (a) $\frac{\sin A \sin 2A + \sin 3A \sin 6A + \sin 4A \sin 13A}{\sin A \cos 2A + \sin 3A \cos 6A + \sin 4A \cos 13A} = \tan 9A$

(b) $\frac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A}$

$= \cot 6A \cot 5A.$

14·12. TRIGONOMETRIC IDENTITIES

When two or more angles are connected by some relation, we can find a relation existing among their circular functions. The method of discovering such a relation is best illustrated by examples.

The student is advised to note carefully the various steps.

Example 73. *If $A+B+C=\pi$, show that*

$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

Solution. L.H.S.$=\sin 2A+\sin 2B+\sin 2C$

$$=2\sin(A+B)\cos(A-B)+2\sin C\cos C$$
$$=2\sin(\pi-C)\cos(A-B)$$
$$+2\sin C\cos[\pi-(A+B)] \quad \text{(Note this step)}$$
$$=2\sin C\cos(A-B)-2\sin C\cos(A+B)$$
$$=2\sin C[\cos(A-B)-\cos(A+B)]$$
$$=2\sin C[2\sin A\sin B]$$
$$=4\sin A\sin B\sin C=\text{R.H.S.}$$

Example 74. *If $A+B+C=180°$, prove that*

$$\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$$

Solution. L.H.S.$=\cos^2 A+\cos^2 B+\cos^2 C$

$$=\tfrac{1}{2}(1+\cos 2A)+\tfrac{1}{2}(1+\cos 2B)+\cos^2 C$$
$$=1+\tfrac{1}{2}(\cos 2A+\cos 2B)+\cos^2 C$$
$$=1+\cos(A+B)\cos(A-B)+\cos^2 C$$
$$=1-\cos C\cos(A-B)+\cos^2 C$$
$$[\because \cos(A+B)=\cos(180°-C)=-\cos C]$$
$$=1-\cos C[\cos(A-B)-\cos C]$$
$$=1-\cos C[\cos(A-B)+\cos(A+B)]$$
$$[\because \cos C=\cos(180°-\overline{A+B})=-\cos(A+B)]$$
$$=1-\cos C[2\cos A\cos B]$$
$$=1-2\cos A\cos B\cos C=\text{R.H.S.}$$

Example 75. *If $A+B+C=\pi$, show that*

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Solution. L.H.S.$=(\sin A+\sin B)+\sin C$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$
$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$
$$\left[\because \sin\frac{A+B}{2}=\sin\frac{180°-C}{2}=\sin\left(90°-\frac{C}{2}\right)=\cos\frac{C}{2}\right]$$
$$=2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\sin\frac{C}{2}\right]$$
$$=2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right]$$
$$=2\cos\frac{C}{2}\;2\cos\frac{A}{2}\cos\frac{B}{2}$$
$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\text{R.H.S.}$$

Example 76. *If $A+B+C=\pi$, prove that*

$$\cos A+\cos B+\cos C=1+4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

Solution. L.H.S.$=(\cos A+\cos B)+\cos C$

$$=2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}+1-2 \sin^2 \frac{C}{2}$$

$$=2 \sin \frac{C}{2} \cos \frac{A-B}{2}-2 \sin^2 \frac{C}{2}+1$$

$$\left[\because \cos \frac{A+B}{2}=\cos \frac{180°-C}{2}=\cos\left(90°-\frac{C}{2}\right)=\sin \frac{C}{2}\right]$$

$$=2 \sin \frac{C}{2}\left\{\cos \frac{A-B}{2} \cos -\frac{A+B}{2}\right\}+1$$

$$\left[\because \sin \frac{C}{2}=\sin\left\{90°-\frac{A+B}{2}\right\}=\cos \frac{A+B}{2}\right]$$

$$=2 \sin \frac{C}{2} . 2 \sin \frac{A}{2} \sin \frac{B}{2}+1$$

$$=4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}+1=\text{R.H.S.}$$

Example 77. *If $A+B+C=\pi$, prove that*

$$\cos^2 \frac{A}{2}+\cos^2 \frac{B}{2}+\cos^2 \frac{C}{2}=2\left(1+\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right)$$

Solution. L.H.S.$=\cos^2 \frac{A}{2}+\cos^2 \frac{B}{2}+\cos^2 \frac{C}{2}$

$$=\frac{1}{2}\left(1+\cos A\right)+\frac{1}{2}\left(1+\cos B\right)+1-\sin^2 \frac{C}{2}$$

$$=2+\frac{1}{2}\left(\cos A+\cos B\right)-\sin^2 \frac{C}{2}$$

$$=2+\cos \frac{A+B}{2} \cos \frac{A-B}{2}-\sin^2 \frac{C}{2}$$

$$=2+\sin \frac{C}{2} \cos \frac{A-B}{2}-\sin^2 \frac{C}{2}$$

$$\left[\because \cos\left(\frac{A+B}{2}\right)=\cos\left(90° \frac{C}{2}\right)=\sin \frac{C}{2}\right]$$

$$=2+\sin \frac{C}{2}\left[\cos \frac{A-B}{2}-\sin \frac{C}{2}\right]$$

$$=2+\sin \frac{C}{2}\left[\cos \frac{A-B}{2}-\cos \frac{A+B}{2}\right]$$

$$\left[\because \sin \frac{C}{2}=\sin\left(90°-\frac{A+B}{2}\right)=\cos \frac{A+B}{2}\right]$$

$$=2+\sin\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$$

$$=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=\text{R.H.S.}$$

Example 78. *If $A+B+C=180°$, prove that*

(*i*) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(*ii*) $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$

(*iii*) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

Solution. (*i*) $A+B+C=180°$

$\Rightarrow$ $A+B=180°-C$

$\therefore$ $\tan(A+B)=\tan(180°-C)$

$\Rightarrow$ $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$

Cross multiplying, we get

$$\tan A+\tan B=-\tan C(1-\tan A\tan B)$$

Transposing, we get tan A + tan B + tan C = tan A tan B tan C

(*ii*) $A+B+C=180° \Rightarrow \dfrac{A+B}{2}=90-\dfrac{C}{2}$

$\therefore$ $\tan\left(\dfrac{A+B}{2}\right)=\tan\left(90°-\dfrac{C}{2}\right)$

$\Leftarrow$ $\tan\left(\dfrac{A}{2}+\dfrac{B}{2}\right)=\cot\dfrac{C}{2}$

$\Leftrightarrow$ $\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\dfrac{1}{\tan\frac{C}{2}}$

Cross multiplying, we get

$$\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

Transposing, we get

$$\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$$

(*iii*) From (*ii*), we have

$$\frac{1}{\cot\frac{B}{2}}\cdot\frac{1}{\cot\frac{C}{2}}+\frac{1}{\cot\frac{C}{2}}\cdot\frac{1}{\cot\frac{A}{2}}+\frac{1}{\cot\frac{A}{2}}\cdot\frac{1}{\cot\frac{B}{2}}=1$$

$$\Rightarrow \quad \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

EXERCISE (VIII)

If $A+B+C=180^\circ$, prove that

1. $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$
2. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$
3. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
4. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
5. $\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
6. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$
7. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$
8. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
9. $\cos\frac{A}{2}\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}+\cos\frac{C}{2}\cos\frac{A-B}{2}$
$=\sin A+\sin B+\sin C$
10. $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

14·13. PROPERTIES OF A TRIANGLE

We have already said in the introduction that the subject of Trigonometry primarily deals with the measurement of sides and angles of a triangle. So far in the chapters we have studied the trigonometric functions of different angles. Now, we develop some important formulae connecting the sides and angles of a triangle. These formulae will be useful in solving the rectilinear figures in general and the triangles in particular.

Notations. In any $\triangle ABC$, A, B, C will represent the angles and a, b, c will denote respectively the *lengths of the sides opposite to those angles.*

I. The Law of Sines. *In any triangle ABC, the sides are proportional to the sines of the opposite angles, i.e.,*

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Proof. In each case, draw $AD \perp BC$ produced if necessary. Then from each figure

$$\frac{AD}{AB} = \sin B \quad \Rightarrow \quad AD = c \sin B \qquad \ldots(1)$$

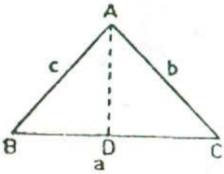


Fig. (*i*)

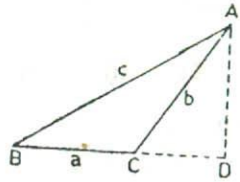


Fig. (*ii*)

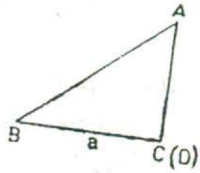


Fig. (*iii*)

In Fig. (*i*), $\frac{AD}{AC} = \sin C \quad \Rightarrow \quad AD = b \sin C$

In Fig. (*ii*), $\frac{AD}{AC} = \sin (180° - C) = \sin C \quad \Rightarrow \quad AD = b \sin C$

In Fig. (*iii*), $AD = AC = b = b \sin C$ $\qquad [\because \sin C = \sin 90° = 1]$

Hence in each case $AD = b \sin C$ $\qquad \ldots(2)$

From (1) and (2), we have

$$c \sin B = b \sin C$$

$$\Rightarrow \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly $\qquad \frac{a}{\sin A} = \frac{b}{\sin B}$ (Considering altitude from C)

Hence $\qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$

II. The Law of Cosines. *The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice their product and the cosine of the included angle, i.e.,*

$$a^2 = b^2 + c^2 - 2\,bc \cos A$$

$$b^2 = c^2 + a^2 - 2\,ca \cos B$$

$$c^2 = a^2 + b^2 - 2\,ab \cos C$$

Proof. Consider any oblique triangle ABC with the altitude AD drawn from the vertex A to the opposite side.

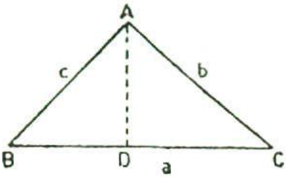


Fig. (*i*)

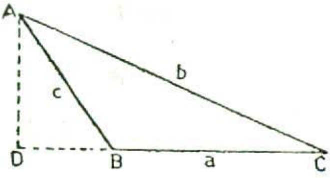


Fig. (*ii*)

In Fig. (*i*), $BD=c\cos B,\ AD=c\sin B$

$CD=BC-BD=a-c\cos B$

In Fig. (*ii*), $BD=-c\cos B,\ AD=c\sin B$

and $CD=CB+BD=a-c\cos B$

In both figures $b^2=AD^2+CD^2$ (Pythagoras theorem)

$=(c\sin B)^2+(a-c\cos B)^2$

$\therefore$ $b^2=c^2\sin^2 B+a^2+c^2\cos^2 B-2ac\cos B$

$=a^2-2ac\cos B+c^2(\sin^2 B+\cos^2 B)$

$=c^2+a^2-2ca\cos B$

The same result may also be written as

$$\cos B=\frac{c^2+a^2-b^2}{2ca}$$

By cyclic changes, we can write two more results of the similar kind, *viz.*,

$$a^2=b^2+c^2-2bc\cos A$$

$$c^2=a^2+b^2-2ab\cos C$$

or $$\cos A=\frac{b^2+c^2-a^2}{2bc} \text{ and } \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

III. The Law of Tangent. *In any* $\triangle ABC$,

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

Proof. We know

$$\frac{\sin B}{b}=\frac{\sin C}{c} \Rightarrow \frac{\sin B}{\sin C}=\frac{b}{c}$$

By componendo and dividendo, we have

$$\frac{\sin B+\sin C}{\sin B-\sin C}=\frac{b+c}{b-c}$$

$$\Rightarrow \quad \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}=\frac{b+c}{b-c}$$

$$\therefore \quad \tan\frac{B+C}{2}=\frac{b+c}{b-c}\tan\frac{B-C}{2}$$

Transposing, we get

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\tan\frac{B+C}{2}=\frac{b-c}{b+c}\tan\left(90°-\frac{A}{2}\right)$$

$$=\frac{b-c}{b+c}\cot\frac{A}{2}$$

IV. The Half angle Formulae. In this article we shall find the trigonometric ratios of half the angles, *viz.*, $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ of a $\triangle ABC$ in terms of the sides and the semi-perimeter of a triangle.

(*a*) *In any* $\triangle ABC$,

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

Proof. Recalling the cosine rule, we have

$$\cos A=\frac{b^2+c^2-a^2}{2bc} \quad \text{...(1)}$$

Also we know that

$$\cos A=1-2\sin^2\frac{A}{2} \quad \text{...(2)}$$

From (1) and (2), we have

$$1-2\sin^2\frac{A}{2}=\frac{b^2+c^2-a^2}{2bc}$$

$$2\sin^2\frac{A}{2}=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-(b^2+c^2-a^2)}{2bc}$$

$$=\frac{a^2-(b^2-2bc+c^2)}{2bc}=\frac{a^2-(b-c)^2}{bc}$$

$$\therefore \quad 2\sin^2\frac{A}{2}=\frac{(a+b-c)(a-b+c)}{2bc} \quad \text{...(3)}$$

Now let the perimeter of the $\triangle ABC$ be denoted by $2s$, where s is often termed as the semi-perimeter of $\triangle ABC$. Thus

$$2s=a+b+c$$

Now $$a+b-c=a+b+c-2c=2s-2c=2(s-c) \quad \ldots(4)$$

$$a-b+c=a+b+c-2b=2s-2b=2(s-b) \quad \ldots(5)$$

Substituting from (4) and (5) in (3), we have

$$2\sin^2\frac{A}{2}=\frac{2(s-c)\times 2(s-b)}{2bc}$$

$$\therefore \quad \sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$$

Taking the positive square root, since $\frac{A}{2}$ is essentially a positive acute angle, we get

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

By cyclic changes, we have

$$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}} \text{ and } \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

(*b*) *In any $\triangle ABC$, prove that*

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

Proof. We know

$$\cos A=2\cos^2\frac{A}{2}-1$$

$$\therefore \quad 2\cos^2\frac{A}{2}-1=\frac{b^2+c^2-a^2}{2bc}$$

Transposing, we get

$$2\cos^2\frac{A}{2}=1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}$$

$$\therefore \quad 2\cos^2\frac{A}{2}=\frac{(b+c)^2-a^2}{2bc}=\frac{(b+c+a)(b+c-a)}{2bc}$$

$$2\cos^2\frac{A}{2}=\frac{2s\times 2\ (s-a)}{2bc} \Rightarrow \cos^2\frac{A}{2}=\frac{s(s-a)}{bc}$$

Taking the positive square root, since $\frac{A}{2}$ is essentially an acute angle, we get

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

By cyclic changes, we have

$$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}} \text{ and } \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$$

(c) *In any $\triangle ABC$, prove that*

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Proof. $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

By cyclic changes, we have

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \text{ and } \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

(d) *In any $\triangle ABC$, prove that*

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Proof. $\sin A=2\sin(A/2)\cos(A/2)$

$$=2\times\sqrt{\frac{(s-b)(s-c)}{bc}}\times\sqrt{\frac{s(s-a)}{bc}}$$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

By cyclic changes, we have

$$\sin B=\frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$$

and $$\sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

Example 79. *Prove that*

$$a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}$$

Solution.

$$\text{R.H.S.}=(b+c)\sin\frac{A}{2}=(k\sin B+k\sin C)\sin\frac{A}{2}$$

$$=k\left[2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\right]\sin\frac{A}{2}$$

$$=k\left[2\sin\left(90°-\frac{A}{2}\right)\cos\left(B+\frac{A}{2}-90°\right)\right]\times\sin\frac{A}{2}$$

$$=k\left[2\sin\frac{A}{2}\cos\frac{A}{2}\right]\cos\left[90°-\left(B+\frac{A}{2}\right)\right]$$

$$=(k\sin A)\sin\left(B+\frac{A}{2}\right)=a\sin\left(B+\frac{A}{2}\right)=\text{R.H.S.}$$

Example 80. *If the cosines of two angles of a triangle are inversely proportional to the opposite sides, show that the triangle is isosceles or right angled.*

Solution. Let $\triangle ABC$ be the given triangle and let

$$\frac{\cos A}{\cos B}=\frac{b}{a}$$

Now $$\frac{\cos A}{\cos B}=\frac{b}{a}=\frac{\sin B}{\sin A} \quad \text{[Law of Sines]}$$

$\Rightarrow$ $$\sin A\cos A=\sin B\cos B$$

$\Rightarrow$ $$2\sin A\cos A=2\sin B\cos B$$

$\Rightarrow$ $$\sin 2A=\sin 2B$$

$\Rightarrow$ $$2A=2B \text{ or } 2A=180°-2B$$

$\Rightarrow$ $$A=B \text{ or } A+B=90°, \text{ i.e., } C=90°$$

Hence the proof.

Example 81. *In any triangle ABC, prove that*

$$\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$$

Solution. From the Law of Sines :

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k, \text{ say}$$

$\therefore$ $a=k\sin A$, $b=k\sin B$ and $c=k\sin C$

Now $$\frac{b-c}{a}=\frac{k\sin B-k\sin C}{k\sin A}=\frac{\sin B-\sin C}{\sin A}$$

$$=\frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} \quad \left[\frac{B+C}{2}=90°-\frac{A}{2}\right]$$

$$=\frac{\sin\frac{B-C}{2}\cos\left(90°-\frac{A}{2}\right)}{\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\sin\frac{B-C}{2}\sin\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}.$$

$$\therefore \qquad \sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$$

Example 82. *Prove that*

$$\frac{a^2 \sin(B-C)}{\sin B+\sin C}+\frac{b^2 \sin(C-A)}{\sin C+\sin A}+\frac{c^2 \sin(A-B)}{\sin A+\sin B}=0$$

Solution. We have

$$\sin(B-C)=\sin B\cos C-\cos B\sin C$$

$$=\frac{b}{k}\cos C-\frac{c}{k}\cos B \qquad \text{[From the Law of Sines]}$$

$$=\frac{1}{k}\left[\frac{b(a^2+b^2-c^2)}{2ab}-\frac{c(a^2+c^2+b^2)}{2ac}\right] \qquad \text{[Law of Cosines]}$$

$$=\frac{1}{k}\quad\frac{a^2+b^2-c^2-a^2-c^2+b^2}{2a}$$

$$=\frac{1}{k}\quad\frac{2(b^2-c^2)}{2a}=\frac{b^2-c^2}{k.a}$$

Similarly $\sin(C-A)=\dfrac{c^2-a^2}{kb}$ and $\sin(A-B)=\dfrac{a^2-b^2}{kc}$

Now

$$\text{L.H.S.}=\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}$$

$$=\frac{a^2(b^2-c^2)}{ka\left(\frac{b}{k}+\frac{c}{k}\right)}+\frac{b^2(c^2-a^2)}{kb\left(\frac{c}{k}+\frac{a}{k}\right)}+\frac{c^2(a^2-b^2)}{kc\left(\frac{a}{k}+\frac{b}{k}\right)}$$

$$=\frac{a^2(b^2-c^2)}{a(b+c)}+\frac{b^2(c^2-a^2)}{b(c+a)}+\frac{c^2(a^2-b^2)}{c(a+b)}$$

$$=a(b-c)+b(c-a)+c(a-b)=0=\text{R.H.S.}$$

14·14. SOLUTIONS FOR A TRIANGLE

We illustrate below the applications of the results derived earlier in the chapter in solving a triangle when some of its elements are known. Usually a triangle can be solved if any three of its elements (excepting the three angles) are known. The general problem then can be divided into four broad cases according to the sets of known elements.

Case I. To solve the triangle, when the three sides a, b, c are known. Either (*i*) use the law of cosines to determine the angles or (*ii*) use the results.

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ and } \tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

to find the angles A and B.

Then $\qquad C=180^\circ-(A+B)$

The latter, however, is adaptable to the use of logarithms.

Example 83. *Solve the triangle if the lengths of its sides are respectively, 52·8, 39 3 and 72·1 cm.*

Solution. Let $a=52{\cdot}8,\ b=39{\cdot}3,\ c=72{\cdot}1$

$\therefore \quad 2s=a+b+c=164{\cdot}2,\ s=82{\cdot}1$

$\Rightarrow \quad s-a=29{\cdot}3,\ s-b=42{\cdot}8\ ,\ s-c=10$

$\therefore \quad \tan\dfrac{A}{2}=\sqrt{\dfrac{42{\cdot}8\times 10}{82{\cdot}1\times 29{\cdot}3}}$

$\Rightarrow \quad \log\tan\dfrac{A}{2}=\dfrac{1}{2}\ [\log 42{\cdot}8+\log 10-\log 82{\cdot}1-\log 29{\cdot}3]$

$$=\tfrac{1}{2}\ [1{\cdot}6314+1-1{\cdot}9143-1{\cdot}4669]=\bar{1}{\cdot}6251$$

Adding 10 to this (for the use of tables), we get

$$\log\tan\frac{A}{2}=9{\cdot}6251$$

$\therefore \quad \dfrac{A}{2}=22^\circ\ 52' \quad \Rightarrow\ A=45^\circ\ 44'$

Similarly, we find

$$\log\tan\frac{B}{2}=\log\sqrt{\frac{10\times 29{\cdot}3}{82{\cdot}1\times 42{\cdot}8}}$$

$$=\tfrac{1}{2}\ [\log 29{\cdot}3+\log 10-\log 82{\cdot}1-\log 42{\cdot}8]=\bar{1}{\cdot}4606$$

$$\log\tan\frac{B}{2}=9{\cdot}4606 \qquad \text{[if 10 is added]}$$

$$\frac{B}{2}=16^\circ\ 6' \quad \Rightarrow \quad B=32^\circ\ 12'$$

Hence $\quad C=180^\circ-(A+B)=102^\circ\ 4'$

Case II. To solve the triangle if one side and two angles are given.

Procedure. Let the side a and angle B and C be given. Then

$$A=180^\circ-(B+C) \qquad \ldots(1)$$

Now using the law of sines

$$b=\frac{\sin B}{\sin A}a \text{ and } c=\frac{\sin C}{\sin A}a$$

which will give values for b and c. Logarithms may be used for calculations.

Case III. To solve the triangle when the two sides and the included angle are given.

Procedure. Let the sides b, c and the included angle A be given

Then $\quad B+C=180^\circ-A \qquad \ldots(1)$

Now using the law of tangents, we have

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\tan\frac{B+C}{2} \quad ...(2)$$

R.H.S. can be computed from the data.
Hence we find (using logarithms, if necessary) $B-C$. ..(3)
From (1) and (3), we can obtain B and C.

The side a can be determined using the result $a=\frac{\sin A}{\sin B}\,b$

Example 84. *Two sides of a triangle are $\sqrt{3}+1$, $\sqrt{3}-1$ and the included angle 50°. Find the other side and angles.*

Solution. Let $b=\sqrt{3}+1$, $c=\sqrt{3}-1$ and $A=60°$

Also $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\tan\frac{B+C}{2}=\frac{2}{2\sqrt{3}}\times\sqrt{3}=1$

$\therefore$ $\frac{B-C}{2}=45°$

$\therefore$ $B-C=90°$

Also $B+C=120°$

$\therefore$ $B=105°$ and $C=15°$

Now $a=\frac{b\sin A}{\sin B}=\frac{(\sqrt{3}+1)\sin 60°}{\sin(60°+45°)}$

$$=\frac{(\sqrt{3}+1)\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)}=\frac{\sqrt{2}(\sqrt{3}+1)\sqrt{3}}{\sqrt{3}+1}=\sqrt{6}$$

EXERCISE (IX)

In any $\triangle ABC$, prove that

1. $(a-b)\sin C+(b-c)\sin A+(c-a)\sin B=0$.

2. (*i*) $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$

   (*ii*) $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0$.

3. (*i*) $a\cos\frac{B-C}{2}=(b+c)\sin\frac{A}{2}$

   (*ii*) $(a-b)\cos\frac{C}{2}=c\sin\frac{A-B}{2}$

4. $(a^2-b^2)\cos^2 C+(b^2-c^2)\cos^2 A+(c^2-a^2)\cos^2 B=0$.

5. $a^2=(b-c)^2 \cos^2 \dfrac{A}{2}+(b+c)^2 \sin \dfrac{A}{2}$

6. $2\left(a \sin^2\dfrac{C}{2}+c \sin^2 \dfrac{A}{2}\right)=c+a-b$

7. $\dfrac{b^2(c^2+a^2-b^2)}{\sin 2B}=\dfrac{c^2(a^2+b^2-c^2)}{\sin 2C}$

8. $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}$

9. $\dfrac{b^2-c^2}{a}\cos A+\dfrac{c^2-a^2}{b}\cos B+\dfrac{a^2-b^2}{c}\cos C=0$

10. $(a^2-b^2)\cos 2C+(b^2-c^2)\cos 2A+(c^2-a^2)\cos 2B=0$

11. $(a+b+c)\sin \dfrac{A}{2}=2a \cos\dfrac{B}{2} \cos\dfrac{C}{2}$

12. $(s-a)\tan \dfrac{A}{2}=(s-b)\tan\dfrac{B}{2}=(s-c)\tan\dfrac{C}{2}$

13. $(a+b+c)\left(\tan\dfrac{B}{2}+\tan\dfrac{C}{2}\right)=2a \cot\dfrac{A}{2}$

14. $\dfrac{\cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}}{\cot A+\cot B+\cot C}=\dfrac{(a+b+c)^2}{a^2+b^2+c^2}$

15. If $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$, then $C=60°$

16. With usual notations, prove that in a $\triangle ABC$,

$$a \sin \frac{A}{2} \sin \frac{B-C}{2}+b \sin\frac{B}{2} \sin \frac{C-A}{2}$$
$$+c \sin\frac{C}{2} \sin \frac{A-B}{2}=0$$