15

Coordinate Geometry

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- coordinates, distance between two points, section formula, area of triangle, collinearity of three points, locus of a point
- straight line and different forms of straight lines and their applications in solving problems
- circle, tangent and normal and solution of problems
- ellipse, parabola, tangent and normal to parabola and problems based on these concepts.

15·0. NTRODUCTION

The credit for bringing out this new branch of geometry goes to the French mathematician Renatus Cartesius (1596—1650) popularly known as Rene Descartes and it is after his name that it is sometimes called as Cartesian Geometry.

Coordinate Geometry is that branch of geometry in which two real numbers, called coordinates, are used to indicate the position of a point in a plane. The main contribution of coordinate geometry is that it has enabled the integration of algebra and geometry. This is evident from the fact that algebraic methods are employed to represent and prove the fundamental properties of geometrical theorems. Equations are also employed to represent the various geometric figures. It is because of these features that the coordinate geometry is considered to be a more powerful tool of analysis than the Euclidian Geometry. It is on this consideration that sometimes it is described as Analytical Geometry.

Before we come to the basic concept of coordinates it is necessary to say a word about the directed line.

15·1. DIRECTED LINE

A directed line is a straight line with number units positive, zero and negative. The point of origin is the number 0. The arrow indicates its direction. On the side of the arrow are the positive numbers and on the other side are the negative numbers. It is like a real number scale illustrated below :

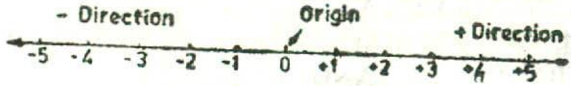


Fig. 1.

A directed line can be horizontal normally indicated by $X'OX$ axis and vertical indicated normally by $Y'OY$ axis. The point where the two intersect each other is called the point of origin. The two lines together are called rectangular axes and are at right angles to each other. If these axes are not at right angles they are said to be oblique axes and the angle between the positive axes XOY is denoted by ω (omega).

Now, remember the three expressions of the coordinates.

(*i*) The coordinates of the origin are (0, 0).

(*ii*) The coordinates of any point on x-axis is $(x, 0)$ say (5, 0) if the point is $+5$ units on x-axis from the origin on the right hand side towards the direction of the arrow.

(*iii*) The coordinates of any point on y-axis is $(0, y)$ say $(0, -5)$ if the point is -5 units on the y-axis from the origin downwards on the vertical axis.

15·2. QUADRANTS

The two directed lines, when they intersect at right angles at the point of origin, divide their plane into four parts or regions namely XOY, $X'OY$, $X'OY'$ and XOY'. These parts are respectively indicated as first (I), second (II), third (III) and fourth (IV) quadrants. The position of the coordinates in a particular quadrant would depend on the positive and negative values of the coordinates shown in Fig. 2.

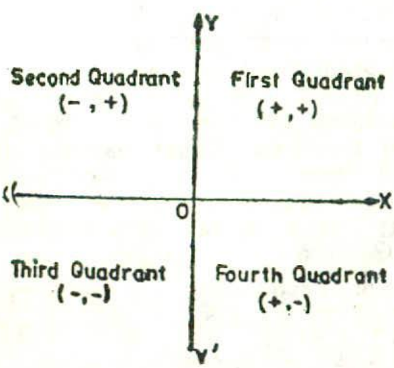


Fig. 2.

15·3. COORDINATES

In a two-dimensional figure a point in plane has two coordinates. The exact position of the point can be located by the unit size of these coordinates. As a matter of convention, the first coordinate is read on the $X'OX$ axis and the second coordinate on the $Y'OY$ axis. Various methods of expressing these pairs of coordinates are :

(*i*) *Varying alphabets*	(x, y)	(a, b)	(h, k)
(*ii*) *Varying subscripts*	(x_1, y_1)	(x_2, y_2)	(x_3, y_3)
(*iii*) *Varying dashes*	(x, y)	(x', y')	(x'', y'')

The diagrammatic presentation of the two coordinates is as follows :

It should be noticed that the horizontal distance of the point from the $Y'OY$ is called the x-coordinate or the abscissa and the vertical distance of the point from the $X'OX$ is called the y-coordinate or the ordinate.

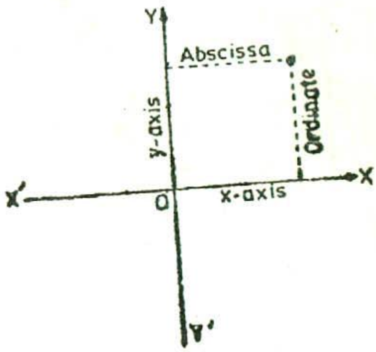


Fig. 3.

Example 1. *Plot the points with the following coordinates :*

$P(-5, -5)$, $R(3, 2)$,

$Q(-4, 6)$, $S(0, -5)$.

Solution. In the following graph, we plot the points. It should be remembered that we read the first coordinate on $X'OX$ axis right or left depending on whether unit numbers are positive or negative respectively. Similarly we read the second coordinate units on the $Y'OY$ axis upwards or downwards depending on whether the units are positive or negative respectively.

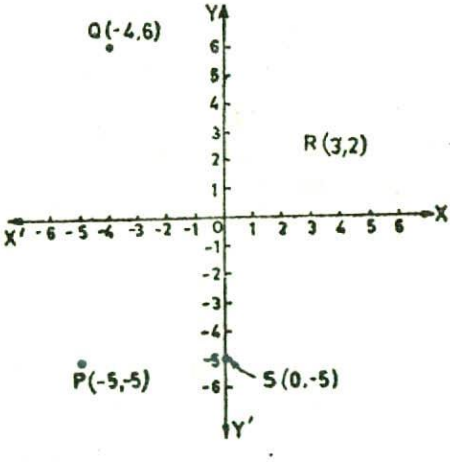


Fig. 4.

15·4. COORDINATES OF MID-POINTS

We can find out the coordinates of a mid-point from the coordinates of the any two points using the following formula :

$$x_m=\frac{x_1+x_2}{2},\ y_m=\frac{y_1+y_2}{2}$$

For example, the coordinates of the mid-point of the join of points

$$(-2, 5), (6, 3) \text{ are } \left(\frac{-2+6}{2}, \frac{5+3}{2}\right), \textit{i.e.}, (2, 4)$$

This is helpful first in finding out the middle point from a join of any two points and secondly in verifying whether two straight lines bisect each other.

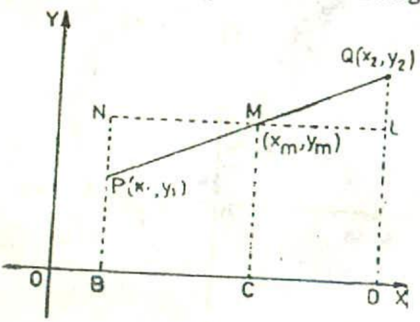


Fig. 5.

In the diagram, the dotted vertical lines are drawn perpendicular to x-axis and the dotted horizontal lines are parallel to the x-axis. The $\triangle NMP$ and $\triangle QML$ are the congruent triangles. It follows, therefore, that

$$NM=ML$$

$$\Rightarrow \quad BC=CD$$

$$\Rightarrow \quad OC-OB=OD-OC$$

$$\Rightarrow \quad (x_m-x_1)=(x_2-x_m)$$

$$\Rightarrow \quad x_m=\frac{x_1+x_2}{2} \qquad \ldots(1)$$

Also from the same congruent triangles, we get

$$\Rightarrow \quad NP=QL$$

$$\Rightarrow \quad NB-PB=QD-LD$$

$$\Rightarrow \quad MC-PB=QD-MC$$

$$\Rightarrow \quad y_m-y_1=y_2-y_m$$

$$\Rightarrow \quad y_m=\frac{y_1+y_2}{2} \qquad \ldots(2)$$

From (1) and (2), we conclude that the coordinates of the mid-point (x_m, y_m) are $\left(\frac{x_1+y_2}{2}, \frac{y_1+y_2}{2}\right)$

Now, the coordinates of the mid-point of the join of the two points $(-1, 5)$ and $(7, 3)$ will be

$$\left(\frac{-1+7}{2}, \frac{2+3}{2}\right), \text{ i.e., } (3, 4).$$

15·5. DISTANCE BETWEEN TWO POINTS

The distance, say d, between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(\text{diff. of abscissae})^2+(\text{diff. of ordinates})^2}$$

Since we take the square of the two differences, we may designate any of the points as (x_1, y_1) and the other (x_2, y_2).

In order to prove the above formula, let us take two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as shown in the following diagram.

The vertical dotted lines PB and QC are perpendiculars from P and Q on the x-axis, and PR is the perpendicular from P on QC. Then

$$PR=BC=OC-OB=x_2-x_1$$

and $QR=QC-RC=y_2-y_1$

From the right angled triangle PRQ, right angled at R, we have by the Pythagoras theorem

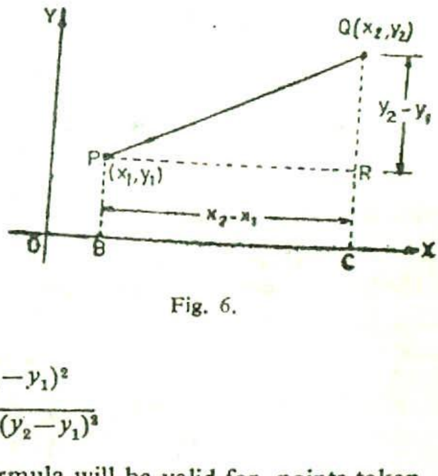


Fig. 6.

$$PQ^2=PR^2+QR^2$$

$$\Rightarrow \quad d^2=(x_2-x_1)^2+(y_2-y_1)^2$$

$$\Rightarrow \quad d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

It may be noted that the above formula will be valid for points taken in other three quadrants as well.

Also the distance of a point $P(x_1, y_1)$ from the origin is

$$d=\sqrt{(x_1-0)^2+(y_1-0)^2}=\sqrt{x_1^2+y_1^2}$$

Thus, the distance between two points say $(4, -1)$ and $(7, 3)$ is

$$d=\sqrt{(7-4)^2+(3+1)^2}=\sqrt{3^2+4^2}=\sqrt{25}=5 \text{ units.}$$

Example 2. (*a*) *Show that the points* $(6, 6)$, $(2, 3)$ *and* $(4, 7)$ *are the vertices of a right-angled triangle.*

(*b*) *Prove that the points (4, 3), (7, −1) and (9, 3) are the vertices of an isosceles triangle.*

Solution. (*a*) Let A, B, C be the points (6, 6), (2, 3) and (4, 7) respectively, then

$$AB^2=[(6-2)^2+(6-3)^2]=16+9=25$$
$$BC^2=[(2-4)^2+(3-7)^2]=4+16=20$$
$$CA^2=[(4-6)^2+(7-6)^2]=4+1\ =5$$
$$\therefore \quad AB^2=BC^2+CA^2$$
$$\Rightarrow \quad \angle ABC=1 \text{ right angle}$$

Hence the points $A(6, 6)$, $B(2, 3)$ and $C(4, 7)$ are the vertices of a right angled triangle.

(*b*) We know that the property of an isosceles triangle is that two of its sides are equal.

Using the distance formula, we have

$$AB=\sqrt{(4-7)^2+(3+1)^2}=\sqrt{9+16}=5$$
$$BC=\sqrt{(7-9)^2+(-1-3)^2}=\sqrt{4+16}=2\sqrt{5}$$
$$AC=\sqrt{(9-4)^2+(3-3)^2}=\sqrt{25}=5$$

Since two of the sides, *i.e.*, AB and AC are equal, the triangle is an isosceles triangle.

Example 3. *Prove that the quadrilateral with vertices (2, −1), (3, 4), (−2, 3) and (−3, −2) is a rhombus.*

(*b*) *Show that the points (4, −5), (8, 1), (14, −3) and (10, −9) are the vertices of a square.*

Solution. (*a*) Let $A(2, -1)$, $B(3, 4)$, $C(-2, 3)$ and $D(-3, -2)$ be the four vertices of the quadrilateral

$$AB=\sqrt{(2-3)^2+(-1-4)^2}=\sqrt{26}$$
$$BC=\sqrt{[3-(-2)]^2+(4-3)^2}=\sqrt{26}$$
$$CD=\sqrt{[(-2)-(-3)]^2+[3-(-2)]^2}=\sqrt{26}$$
$$DA=\sqrt{[(-3)-(2)]^2+[(-2)-(-1)]^2}=\sqrt{26}$$
$$AC=\sqrt{(2+2)^2+(-1-3)^2}=\sqrt{32},\ BD=\sqrt{(3+3)^2+(4+2)^2}=\sqrt{72}$$
$$\Rightarrow \quad AB=BC=CD=DA,\ AC\neq BD$$
$$\Rightarrow \quad ABCD \text{ is a rhombus.}$$

(*b*) Left as an exercise for the student.

Example 4. *Prove that (−2. −1), (1, 0), (4, 3) and (1, 2) are the vertices of a parallelogram.*

Solution. Let $A(-2, -1)$, $B(1, 0)$, $C(4, 3)$ and $D(1, 2)$ be the vertices of a quadrilateral.

Then the mid-point of $AC=\left(\frac{-2+4}{2}, \frac{-1+3}{2}\right)=(1, 1)$ ...(1)

and the mid-point of $BD=\left(\frac{1+1}{2}, \frac{0+2}{2}\right)=(1, 1)$ ...(2)

From (1) and (2), we conclude that AC and BD bisect each other at the same point (1, 1) and hence the quadrilateral $ABCD$ is a parallelogram.

Example 5. *Find the coordinates of the circumcentre of a triangle whose coordinates are* $(3, -2)$, $(4, 3)$ *and* $(-6, 5)$. *Hence find the circumradius.*

Solution. Let $A(3, -2)$, $B(4, 3)$ and $C(-6, 5)$ be the vertices of the triangle and $P(x, y)$ be the circumcentre.

$\therefore$ By definition $PA=PB=PC \Rightarrow PA^2=PB^2=PC^2$

Now by the distance formula :

$$PA^2=(x-3)^2+(y+2)^2=x^2+y^2-6x+4y+13$$
$$PB^2=(x-4)^2+(y-3)^2=x^2+y^2-8x-6y+25$$
$$PC^2=(x+6)^2+(y-5)^2=x^2+y^2+12x-10y+61$$

Now $PA^2=PB^2$

$\Rightarrow$ $x^2+y^2-6x+4y+13=x^2+y^2-8x-6y+25$

$\Rightarrow$ $2x+10y=12$

$\Rightarrow$ $x+5y=6$ ...(1)

and $PB^2=PC^2$

$\Rightarrow$ $x^2+y^2-8x-6y+25=x^2+y^2+12x-10y+61$

$\Rightarrow$ $-20x+4y=36$

$\Rightarrow$ $-5x+y=9$ ...(2)

Solving (1) and (2), we get

$$x=-\frac{3}{2}, \ y=\frac{3}{2}$$

$\therefore$ The circumcentre P is $\left(-\frac{3}{2}, \frac{3}{2}\right)$

Now the circumradius of $\triangle ABC$ is PA or PB or PC. Therefore

$$PA=\sqrt{\left(-\frac{3}{2}-3\right)^2+\left(\frac{3}{2}+2\right)^2}$$

$$=\sqrt{\frac{81}{4}+\frac{49}{4}}=\sqrt{\frac{130}{4}}$$

or $$PA=\frac{\sqrt{130}}{2}$$

Example 6. *Determine the coordinates of the vertices of the triangle ABC if the middle points of its sides BC, CA, AB have coordinates (3, 2), (−1, −2) and (5, −4) respectively.* [*C.A., November, 1991*]

Solution. Let the coordinates of the point A, B and C of the triangle ABC be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. Therefore, we have

$$\frac{x_2+x_3}{2}=3 \quad \text{and} \quad \frac{y_2+y_3}{2}=2$$

or $$x_2+x_3=6 \qquad \ldots(1)$$

and $$y_2+y_3=4 \qquad \ldots(2)$$

$$\frac{x_3+x_1}{2}=-1 \quad \text{and} \quad \frac{y_3+y_1}{2}=-2$$

or $$x_3+x_1=-2 \qquad \ldots(3)$$

and $$y_3+y_1=-4 \qquad \ldots(4)$$

$$\frac{x_1+x_2}{2}=5 \quad \text{and} \quad \frac{y_1+y_2}{2}=-4$$

or $$x_1+x_2=10 \qquad \ldots(5)$$

and $$y_1+y_2=-8 \qquad \ldots(6)$$

Adding (1) and (3), (5) and (2), (4) and (6), we have

$$x_1+x_2+x_3=7$$

and $$y_1+y_2+y_3=-4$$

$$\therefore \quad x_1=1,\ x_2=9,\ x_3=-3 \quad \text{and} \quad y_1=-8,\ y_2=0,\ y_3=4$$

Hence (1, −8), (9, 0) and (−3, 4) are the vertices of the triangle.

15·6. SECTION FORMULA

The coordinates of a point $R(x, y)$ dividing a line in the ratio of $m : n$ connecting the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the diagram below is given by

$$x=\frac{mx_2+nx_1}{m+n} \quad \text{and} \quad y=\frac{my_2+ny_1}{m+n}$$

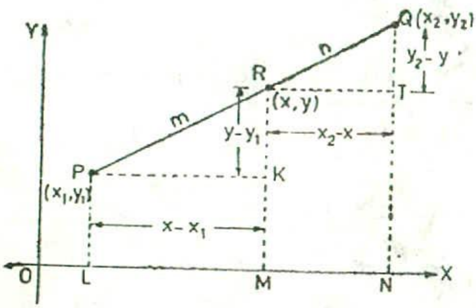


Fig. 7.

Draw PL, RM and QN perpendiculars on $X'OX$ and draw PK and RT perpendiculars on RM and QN respectively.

We are given

$$\frac{PR}{RQ}=\frac{m}{n} \qquad \ldots(*)$$

Suppose PQ makes angle θ with the x-axis. From the figure :

In $\triangle PRK$,

$$\frac{PK}{PR}=\cos\theta$$

$\Rightarrow$ $$PR\cos\theta=x-x_1 \quad \ldots(1)$$

and in $\triangle RQT$,

$$\frac{RT}{RQ}=\cos\theta$$

$\Rightarrow$ $$RQ\cos\theta=x_2-x \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{PR}{RQ}=\frac{x-x_1}{x_2-x}=\frac{m}{n} \quad \text{[Using (*)]}$$

$\Rightarrow$ $nx-nx_1=mx_2-mx$, *i.e.*, $x(m+n)=mx_2+nx_1$

$\therefore$ $$x=\frac{mx_2+nx_1}{m+n}$$

Similarly

$$\frac{PR\sin\theta}{RQ\sin\theta}=\frac{y-y_1}{y_2-y}=\frac{m}{n}$$

$\Rightarrow$ $ny-ny_1=my_2-my$

$\Rightarrow$ $y(m+n)=my_2+ny_1$

$\Rightarrow$ $$y=\frac{my_2+ny_1}{m+n}$$

It may be noted that m, which corresponds to the segment PR multiplies the coordinate of Q, while n, which refers to RQ, multiplies the coordinate of P. If $m=n$, we find that the coordinates of the mid-point are

$$\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}$$

15·7. EXTERNAL DIVISION

In the above it was assumed that the point R divides PQ *internally* in the ratio $m:n$. This means that the point R lies between P and Q. If PQ is divided *externally* by R, then R lies outside PQ. The student should repeat the above method, and using the same diagram with R and Q interchanged, it can be proved that

$$x=\frac{mx_2-nx_1}{m-n},\ y=\frac{my_2-ny_1}{m-n}$$

Note that if the given ratio is 4 : 3, then R lies on PQ produced, whilst if the ratio is 3 : 4, then R is on QP produced and to the left of P. The formula, however, takes care of this.

15·8. COORDINATES OF A CENTROID

The centroid of a triangle is the point of intersection of the three medians of a triangle. Each median bisects the side opposite to the vertex into two equal parts. In order to prove that the medians of a triangle intersect at a point, called centroid, we have to show that its coordinates are

$$x=\frac{x_1+x_2+x_3}{3},\ y=\frac{y_1+y_2+y_3}{3}$$

To prove this, let us take a triangle with its vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ as shown in the following diagram :

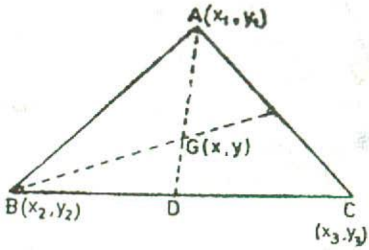


Fig. 8.

In the diagram median AD bisects the base BC at D with coordinates

$$D=\left(\frac{x_2+x_3}{2},\ \frac{y_2+y_3}{2}\right)$$

We take a point G where two medians intersect which divides AD internally, say, in the ratio $2:1$, *i.e.*, $m_1 : m_2=2:1$.

Hence by the section formula, the coordinates of G are

$$x=\frac{2\times\frac{x_2+x_3}{2}+1\times x_1}{2+1}=\frac{x_1+x_2+x_3}{3}$$

and

$$y=\frac{2\times\frac{y_2+y_3}{2}+1\times y_1}{2+1}=\frac{y_1+y_2+y_3}{3}$$

Example 7. *Find the coordinates of the point which divides the points $P(8, 9)$ and $Q(-7, 4)$ internally in the ratio $2:3$ and externally in the ratio $4:3$.*

Solution. Now the values given are $x_1=8$, $y_1=9$, $x_2=-7$ and $y_2=4$. We have to substitute these values into the formula :

(*i*) For internal division where $m_1=2$ and $m_2=3$, we have

$$x=\frac{m_1x_2+m_2x_1}{m_1+m_2}=\frac{2\times(-7)+3\times 8}{2+3}=2$$

$$y=\frac{m_1y_2+m_2y_1}{m_1+m_2}=\frac{2\times 4+3\times 9}{2+3}=7$$

(*ii*) For external division where $m_1=4$ and $m_2=3$, we have

$$x=\frac{m_1x_2-m_2x_1}{m_1-m_2}=\frac{\{4\times(-7)\}-\{3\times8\}}{4-3}=-52$$

$$y=\frac{m_1y_2-m_2y_1}{m_1-m_2}=\frac{4\times4-3\times9}{4-3}=-11$$

Below we will discuss a converse problem in which we have to find the ratio $m_1 : m_2$ when the values of all the coordinates are given.

Example 8. *Find the ratio in which the point (11, −3) divides the join of points (3, 4) and (7, 11).*

Solution. Let the point $P(11, -3)$ divide the join of points $A(3, 4)$ and $B(7, 11)$ in the ratio $\lambda : 1$

∴ By the section formula, we have

$$11=\frac{7\lambda+3}{\lambda+1}$$

$$\Rightarrow \quad 11\lambda+11=7\lambda+3$$

$$\Rightarrow \quad \lambda=-2$$

∴ P divides AB externally in the ratio $2 : 1$. It should be noted that in finding the ratio of division, the knowledge of one coordinate of the point of division is enough. The application of the section formula to the other coordinate, if known, will give the same ratio.

Example 9. *Find the ratios in which the axes divide the line joining the points (2, 5) and (1, 9). Also find the coordinates of the points in which the coordinate axes intersect this line.*

Solution. Let the x-axis intersect the join of $A(2, 5)$ and $B(1, 9)$ at the point P and the y-axis intersect AB at Q.

Since P lies on the x-axis, its y-coordinate is 0. Also Q lies on the y-axis, its x-coordinate is 0.

(*i*) The point Q divides AB in the ratio $\frac{AQ}{QB}=\frac{\lambda}{1}$, say.

Let the coordinates of Q be $(0, b)$. We have by the section formula

$$0=\frac{\lambda+2}{\lambda+1} \text{ and } b=\frac{9\lambda+5}{\lambda+1}$$

From the former, we have $\lambda=-2$.

Substituting this value in the latter equation, we get

$$b=\frac{9\times(-2)+5}{(-2)+1}=13$$

Since the ratio of division is negative, Q divides AB externally in the ratio $2 : 1$ and the coordinates of the point of division are $(0, 13)$.

(*ii*) The point P divides AB in the ratio $\frac{AP}{PB}=\frac{k}{1}$, (say).

Let the coordinates of P be $(a, 0)$. We have by the section formula

$$a=\frac{k+2}{k+1} \text{ and } 0=\frac{5+9k}{k+1}$$

From the latter equation, we have $k=-\frac{5}{9}$

Substituting this value in the former equation, we get

$$a=\frac{-\frac{5}{9}+2}{-\frac{5}{9}+1}=\frac{13}{4}$$

Since the ratio of division is negative, the division is external, *i.e.*, P divides AB externally in the ratio of $5:9$ and the coordinates of the point of division are $\left(\frac{13}{4}, 0\right)$.

Example 10. *If $(-3, 4)$ is the centroid of the triangle whose vertices are $(6, 2)$, $(x, 3)$, $(0, y)$, find x and y.*

Solution. Using the centroid formula, we have

$$-3=\frac{6+x+0}{3} \Rightarrow x=-15$$

and

$$4=\frac{2+3+y}{3} \Rightarrow y=7$$

15·9. AREA OF A TRIANGLE

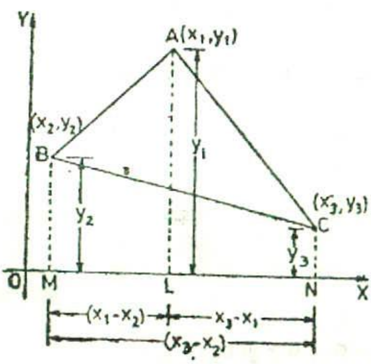


Fig. 9.

We can find out the area of a triangle with the vertices given.

For this let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the coordinates of the vertices of the triangle ABC. From A, B, C draw perpendiculars AL, BM and CN on the x-axis.

As is clear from the figure, area of $\triangle ABC$

$=$Area of trapezium $ABML$

$+$Area of trapezium $ALNC-$Area of trapezium $BMNC$

Since the area of the trapezium

$=\frac{1}{2}$ [Sum of the parallel sides]

[Perpendicular distance between the parallel sides]

Hence the area of the $\triangle ABC$ can be given as

$$\triangle ABC=\tfrac{1}{2}\,(BM+AL)ML+\tfrac{1}{2}(AL+CN)LN-\tfrac{1}{2}(BM+CN)MN$$
$$=\tfrac{1}{2}(y_2+y_1)(x_1-x_2)+\tfrac{1}{2}(y_1+y_3)(x_3-x_1)-\tfrac{1}{2}(y_3+y_2)(x_3-x_2)$$
$$=\tfrac{1}{2}[x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3] \quad \ldots(1)$$

The above on simplification can take the following form :

$$=\tfrac{1}{2}[(x_1y_2-x_1y_3)+(x_2y_3-x_2y_1)+(x_3y_1-x_3y_2)] \quad \ldots(2)$$
$$=\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)] \quad \ldots(3)$$

Remarks 1. The sign of the area of the triangle is positive or negative as the arrangement of vertices are counter-clockwise or clockwise as shown below .

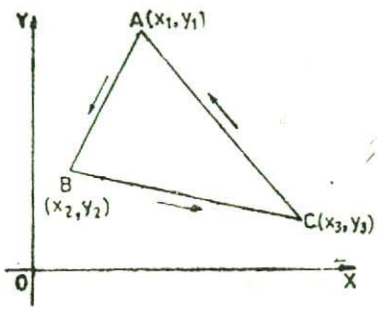


Fig. 10(*i*)

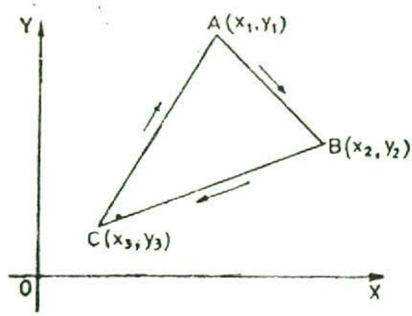


Fig. 10(*ii*)

The proper area formula is therefore given by

$$\text{Area of } \Delta=\pm\tfrac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3)$$

2. *Aid to memory.* The formula for area of $\triangle ABC$ can be conveniently remembered by using the memory chart in the following form :

$$\text{Area of a } \Delta=\tfrac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix}$$

(where dotted arrow is preceded by a minus sign and the other one by a plus sign).

Example 11. *Find the area of the triangle whose vertices are* (2, 3), (5, 7), (−3, 4).

Solution. Using the above aid to memory, *viz.*,

$$\tfrac{1}{2}\begin{bmatrix} 2 & 5 & -3 & 2 \\ 3 & 7 & 4 & 3 \end{bmatrix} \quad \text{we get}$$

$$\text{Area of } \triangle = \tfrac{1}{2}[2.7-3.5+5.4-7.(-3)+(-3).3-4.2]$$
$$=\tfrac{1}{2}(14-15+20+21-9-8)$$
$$=11{\cdot}5 \text{ sq. units}$$

It should be noted that the vertices are taken anti-clockwise and therefore, the result is positive. If we take in the reverse order placing $(-3, 4)$ as (x_1, y_1), the result will be negative.

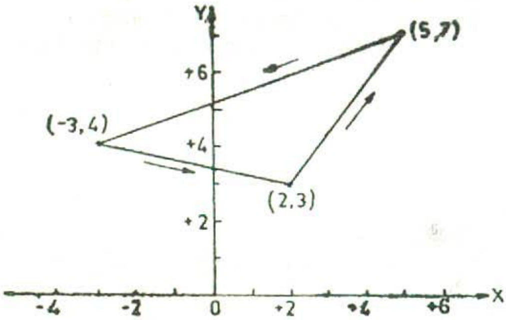


Fig. 11.

Example 12. *The vertices of a triangle ABC are A(5, 2), B(−9, −3) and C(−3, −5), D, E, F are respectively the mid-points of BC, CA and AB. Prove that*

$$\triangle ABC = 4\ \triangle DEF$$

Solution. Area of $\triangle ABC$ is

$$\triangle ABC = \tfrac{1}{2}[5\{-3-(-5)\}+(-9)\{-5-2\}+(-3)\{2-(-3)\}]$$
$$=\tfrac{1}{2}[10+63-15]=29 \text{ sq. units} \qquad \ldots(1)$$

Also the coordinates of D, E and F are

$$D=\left[\frac{(-9)+(-3)}{2},\ \frac{(-3)+(-5)}{2}\right]=(-6, -4)$$

$$E=\left[\frac{-3+5}{2},\ \frac{-5+2}{2}\right]=\left(1, -\frac{3}{2}\right)$$

$$F=\left[\frac{-9+5}{2},\ \frac{-3+2}{2}\right]=(-2, -\tfrac{1}{2})$$

$$\therefore\quad \triangle DEF=\tfrac{1}{2}[(-6)\{-\tfrac{3}{2}-(-\tfrac{1}{2})\}+1\{-\tfrac{1}{2}-(-4)\}+(-2)\{-4-(-\tfrac{3}{2})\}]$$

$$=\frac{1}{2}\left[6+\frac{7}{2}+5\right]=\frac{29}{4} \text{ sq. units} \qquad \ldots(2)$$

From (1) and (2), we conclude that

$$\triangle ABC = 4\triangle DEF$$

Example 13. *Prove that the triangle formed by the points $A(8, -10)$, $B(7, -3)$, $C(0, -4)$ is a right angled triangle.*

Solution. We know that in a right angled triangle

$$AC^2 = AB^2 + BC^2$$

Also distance between two points P and Q is

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$\therefore$
$$AB^2 = (7-8)^2 + (-3+10)^2 = 50$$
$$BC^2 = (0-7)^2 + (-4+3)^2 = 50$$
$$AC^2 = (8-0)^2 + (-10+4)^2 = 100$$

or
$$AC^2 = AB^2 + BC^2$$

and hence the triangle is a right angled triangle.

15·10. COLLINEARITY OF THREE POINTS

In case the three points of a triangle are in a straight line, they are called collinear. The area of such a triangle is equal to zero as indicated below :

$$\text{Area of } \triangle = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) = 0$$

Example 14. *Show that the following points are collinear :*

$P(3, -2)$, $Q(-1, 1)$, $R(-5, 4)$.

Solution.

Memory Chart :

$$\triangle = \frac{1}{2}\begin{bmatrix} 3 & -1 & -5 & 3 \\ -2 & 1 & 4 & -2 \end{bmatrix}$$

$$\text{Area of } \triangle PQR = \tfrac{1}{2}[(3)(1) - (-2)(-1) + (-1)(4) - (1)(-5) + (-5)(-2) - (4\,(3)]$$

$$= \tfrac{1}{2}[3 - 2 - 4 + 5 + 10 - 12] = \tfrac{1}{2}[0] = 0$$

Hence the points P, Q and R are collinear.

Example 15. *Find the area of the triangle with vertices $A(3, 1)$, $B(2k, 3k)$ and $C(k, 2k)$. Show that the three distinct points A, B and C are collinear when $k = -2$.* [*I.C.W.A., December 1990*]

Solution. Area of the $\triangle ABC$

$$= \tfrac{1}{2}[3(3k - 2k) + 2k(2k - 1) + k(1 - 3k)]$$
$$= \tfrac{1}{2}(3k + 4k^2 - 2k + k - 3k^2)$$
$$= \tfrac{1}{2}(k^2 + 2k).$$

The three points A, B and C will be collinear if

$$\tfrac{1}{2}(k^2+2k)=0$$

$$\Rightarrow \qquad k=0 \quad \text{or} \quad k=-2.$$

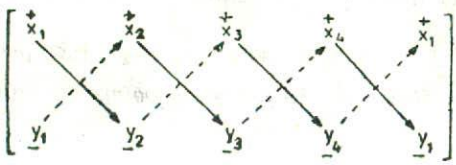
15·11. AREA OF A QUADRILATERAL

In case of a quadrilateral, it is possible to split it into two triangles and then add the area of them, *i.e.*,

Quadrilateral $ABCD = \triangle \text{ABC} + \triangle \text{ADC}$

$$=\tfrac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]$$
$$+\tfrac{1}{2}[(x_1y_3-x_3y_1)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$$
$$=\tfrac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$$

The area of the quadrilateral with 4 sides can be found out by extending the same aid to memory illustrated for a triangle.



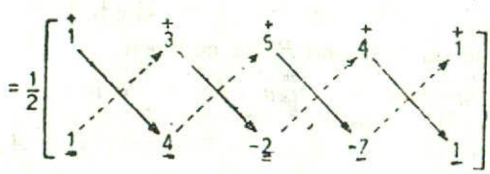
Area of quad. $=\pm\tfrac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)]$

The same as given above because the middle two terms cancel out. The area is indicated by the $\pm$ sign. The sign of area will be positive if the vertices are taken counter clockwise and negative if taken clockwise.

Example 16. *Find the area of a quadrilateral whose vertices are*

$A(1, 1)$; $B(3, 4)$ $C(5, -2)$ and $D(4, -7)$

Solution. Let us make use of the above formula and the memory chart for the sake of convenience.



Area of quad. $=\tfrac{1}{2}[\{(4)-(3)\}+\{(-6)-(20)\}+\{(-35)-(-8)\}+\{(4)-(-7)\}]$

$$=\frac{1}{2}\;[1+(-26)+(-27)+11]=-\frac{41}{2}\text{ sq. units.}$$

EXERCISE (I)

1. Find the distance between the following pair of points .

(*i*) $(0, 0)$; (p, q), (*ii*) $(9, -1)$; $(-2, 10)$, (*iii*) $(\frac{2}{3}, -\frac{1}{2})$; $(-\frac{1}{3}, -\frac{1}{2})$

(*iv*) $(1+\sqrt{2}, 2)$; $(1, 1-\sqrt{2})$, (*v*) $(at_1^2, 2at_1)$; $(at_2^2, 2at_2)$

(*vi*) $(a \cos \alpha, a \sin \alpha)$, $(a \cos \beta, a \sin \beta)$.

2. (*a*) If the point $(a, 3)$ is at a distance of $\sqrt{5}$ units from the point $(2, a)$, find a.

(*b*) What will be the values of x if the distance between $(x, -4)$ and $(-8, 2)$ be 10 ?

(*c*) If the distance between the points $(a, -5)$ and $(2, a)$ is 13, find a.

3. Show that the points

$(1, -1)$, $(-1, 1)$ and $(-\sqrt{3}, -\sqrt{3})$

are the vertices of equilateral triangle.

4. The points $(3, 4)$ and $(-2, 3)$ form with another point (x, y) an equilateral triangle. Find x and y.

5. Prove that the triangle with vertices at the points $(0, 3)$, $(-2, 1)$ and $(-1, 4)$ is right angled.

6. Show that the triangle whose vertices are $(1, 10)$, $(2, 1)$, and $(-7, 0)$ is an isosceles triangle. Find also the altitude of this triangle.

7. Prove that the points

$$\left(\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right), \left(-\frac{\sqrt{3}a}{2}, \frac{a}{2}\right), \left(-\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right)$$

and $\left(\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$

are the vertices of a square.

8. Show that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the vertices of a rectangle.

9. Prove that the following points are the vertices of a parallelogram

(*i*) $(2, 1)$, $(5, 2)$, $(6, 4)$ and $(3, 3)$

(*ii*) $(0, 0)$, $(a, 0)$, $(a+b, c)$, (b, c)

10. Find the coordinates of the circumcentre of a triangle whose coordinates are $(7, -1)$, $(5, 1)$ and $(-3, -7)$. Hence find the circumradius.

11. If $(-3, 2)$; $(1, -2)$ and $(5, 6)$ are the mid-points of the sides of a triangle, find the coordinates of the vertices of the triangle.

12. Prove that the point $(3, 3)$ is equidistant from $(0, -1)$, $(-2, 3)$, $(6, 7)$ and $(8, 3)$. Find this distance and show that the point is the intersection of the diagonals of a rectangle formed by the four points.

13. If $(8, 0)$ is the circumcentre of a triangle whose vertices are $(a, -5)$, $(10, 5)$ and $(3, b)$, find a and b.

14 Find the lengths of the sides and the medians of the triangle whose vertices are $(7, 5)$, $(2, 3)$ and $(6, -7)$

15. Find the coordinates of the point which divides internally the join of the pair of points:

(a) $(6, -5)$ and $(-7, -15)$ in the ratio of $4 : 7$.

(b) $(5, 2)$ and $(7, 9)$ in the ratio of $2 : 7$.

(c) $(a+b, a-b)$ and $(a-b, a+b)$ in the ratio of $a : b$

(d) (p, q) and (q, p) in the ratio of $p-q : p+q$.

16. Find the coordinates of the point which divides externally the join of the pair of points :

(a) $(4, 7)$ and $(1, -2)$ in the ratio of $3 : 2$

(b) $(-3, 2)$ and $(4, -3)$ in the ratio of $5 : 3$.

(c) (p, q) and (q, p) in the ratio of $p-q : p+q$.

17. (a) In what ratio is the line joining the points $A(4, 4)$ and $B(7, 7)$ divided by $P(-1, -1)$?

(b) Determine the ratio in which the join of the points $(-2, 3)$ and $(-4, 6)$ is divided by the point $(2, -3)$.

(c) Find the ratio in which the point $(2, 14)$ divides the line segment joining $(5, 4)$ and $(11, -16)$ externally.

18. (a) In what ratio is the segment joining the pair of points $(2, -4)$ and $(-3, 6)$ is divided by (i) the x-axis and (ii) the y-axis ?

(b) In what ratio is the line segment joining $(2, 3)$ and $(5, -4)$ is divided by (i) x-axis and (ii) y-axis ?

19. Find the length of the medians of a triangle whose vertices are $(1, 2)$, $(2, -1)$ and $(3, 4)$.

20. Show that the join of the points $(4, 3)$ $(2, 1)$ and $(6, -1)$, $(4, 5)$ bisect each other.

21. Find the co-ordinates of the centroid of the triangle whose vertices are

(a) $(3, 2)$, $(-1, -4)$ and $(-5, 6)$

(b) $(a-b, a-c)$; $(b-c, b-a)$; $(c-a, c-b)$.

22. (a) The centroid of a triangle is $(0, 3)$ and if its two vertices are $(-4, 6)$, $(2, -2)$, find the third vertex.

(b) If $(3, 5)$, $(x, 4)$ and $(14, y)$ are the vertices of a triangle and $(5, 6)$ is its centroid, find x and y.

23. G is the centroid of a triangle ABC, where A, B and C are respectively the points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

If $G\equiv(0, 0)$, $A\equiv(2, 3)$, $x_2=3$ and $y_3=-2$, find the values of y_2 and x_3.

24. Find the areas of the triangle whose vertices are

(*i*) (0, 0), (1, 2) and (−1, 2),

(*ii*) (2, −1), (−3, −4) and (0, 2),

(*iii*) $(x, y-z)$, $(-x, z)$ and $(x, y+z)$.

25. Find the area of the quadrilateral whose vertices are

(*i*) (1, 2), (6, 2), (5, 3) and (3, 4)

(*ii*) (−4, 2), (3, −5), (6, −2) and (1, 7)

26. Prove that the following sets of points are collinear

(*i*) (1, −1), (2, 1) and (4, 5)

(*ii*) (1, 4), (3, −2) and (−3, 16)

(*iii*) $(a, b+c)$, $(b, c+a)$ and $(3, a+b)$.

27. Prove that the points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear if

$$\frac{1}{a}+\frac{1}{b}=1.$$

28. (*a*) The points $\left(2, \frac{3}{2}\right)$, $\left(-3, -\frac{7}{2}\right)$, $\left(k, \frac{9}{2}\right)$ are collinear, find k.

(*b*) For what value of y, the points (5, 5), $(10, y)$ and (−5, 1) shall be on the same line ?

29. If the area of the triangle with vertices at $(2, a)$, $(a, 2)$, $(-2, -1)$ is 5, find the possible values of a.

30. (*a*) For what value of x, the area of the triangular region enclosed by the segments joining the points (3, 4), $(x, -1)$ and (4, −6) will be $\frac{15}{2}$ sq. units ?

(*b*) Determine the value of y for which the area of the triangular region enclosed by the segments joining points (−3, −5), (5, 2) and $(-9, y)$ will be 29 sq. units ?

31. (*a*) Find the area of the triangular region bounded by the lines whose equations are $x-y+2=0$, $4x+3y+8=0$, and $9x-2y-17=0$.

(*b*) Show that the area of triangle formed by the lines whose equations are $y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$ is $\frac{1}{2} \cdot \frac{(c_1-c_2)^2}{m_1-m_2}$.

32. A and B are the points (3, 4) and (5, −2). Find a point P such that $PA=PB$ and $\Delta PAB=10$.

33. The vertices of a triangle ABC are $A(3, 0)$, $B(0, 6)$ and $C(6, 9)$. A line DE divides AB and AC in the ratio $1 : 2$ meeting AB in D and AC in E. Prove that $\triangle ABC = 9\triangle ADE$.

34. From two perpendicular roads X and Y, building A is at a distance of 100 yards and 150 yards respectively, building B is at a distance of 150 yards and 100 yards respectively and building C is at a distance of 200 yards and 175 yards respectively. Find the distance between A and B and examine if all the buildings are in a straight line.

ANSWERS

1. (a) (i) $\sqrt{p^2+q^2}$, (ii) $11\sqrt{2}$, (iv) $5+2\sqrt{2}$ 2. (a) 1 or 4. (b) 0 or -16, (c) 7 or -10 3. $\left(\frac{1\pm\sqrt{3}}{2}, \frac{7\mp 5\sqrt{3}}{2}\right)$

4. $x=\frac{1\pm\sqrt{3}}{2}$, $y=\frac{7\pm 5\sqrt{3}}{2}$ 6. $\sqrt{41}$ 10. $(2, -4)$.

11. $(9, 2), (4, 10)$; $(-7, -6)$ 13. $a=6$ or 10 ; $b=\pm 2$

14. Length of the sides are $\sqrt{29}, 2\sqrt{29}$ $\sqrt{145}$; lengths of the medians are $\sqrt{58}$, $\frac{\sqrt{145}}{2}$, $\frac{\sqrt{493}}{2}$ 15. (a) $\left(\frac{14}{11}, -\frac{95}{11}\right)$, (b) $\left(\frac{49}{9}, \frac{32}{9}\right)$

(c) $\left(\frac{a^2+b^2}{a+b} ; \frac{a^2-b^2+2ab}{a+b}\right)$, (d) $\left(\frac{p^2-q^2+2pq}{2p}, \frac{p^2+q^2}{2p}\right)$

16. (a) $(-5, -20)$, (b) $\left(\frac{29}{2}, -\frac{21}{2}\right)$, (c) $\frac{p^2+q^2}{2q}, \frac{-p^2+q^2+2pq}{2q}\Big)$

17. (b) $2 : 3$ externally, (c) $1 : 3$ 18. (a) (i) $2 : 3$ internally, (ii) $2 : 3$ internally, (b) (i) $3 : 4$ internally, (ii) $2 : 5$ externally.

19. $\sqrt{\frac{5}{2}}$, 4, $\sqrt{\frac{29}{2}}$ 21. (a) $\left(-1, \frac{4}{3}\right)$, (b) $(0, 0)$ 22. (a) $(2, 5)$

(b) $-2, 9$ 23. $y_2=-1$, $x_3=-5$ 24. (i) 2, (ii) $\frac{21}{2}$ (iii) $2xz$

25. (i) $\frac{11}{2}$, (ii) 56. 28. (a) 5, (b) 7 29. $(0, -3)$ 30. (a) 2,

(b) -3 31 (a) $\frac{35}{2}$ 32. $(7, 2)$. 34. $50\sqrt{2}$, No.

15·12. LOCUS OF A POINT

The locus of a point is the path through which a point passes under certain given conditions. For example, the condition of a line parallel to x-axis is that the distance of all its points is equal. Similarly, if a point moves equal distance from a fixed point, the locus of such a point will be a circle.

An equation of a locus is a relation between x and y which is satisfied by the coordinates of all points on the locus and by the coordinates of no other point. Normally every equation can be represented by a locus and the *vice versa* is also true.

Steps to find out the equation of the locus can be summarised as follows :

In form I :

(*i*) Take the moving point as $P(x, y)$ on the locus.

(*ii*) Form an equation in x and y using the given conditions.

(*iii*) Simplify, if necessary, the relation so obtained.

In form II :

(*i*) Take any point (h, k) or (α, β) on the locus.

(*ii*) Write the relation between h, k using the conditions given.

(*iii*) Simplify the relation, if necessary.

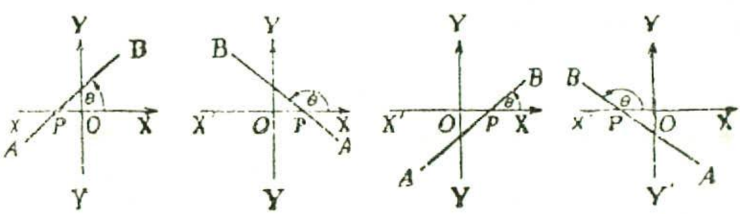
(*iv*) Change (h, k) into (x, y).

15·13. THE STRAIGHT LINE

The study of curves starts with the straight line which is the simplest geometrical entity. Mathematically it is defined as the shortest distance between two distinct points.

15·14. SLOPE OR GRADIENT OF A STRAIGHT LINE

The slope of the line is the tangent of the angle formed by the line above the x-axis towards its positive direction whatever be the position of the line as shown below :



Slope of a line is generally denoted by m. Thus if a line makes an angle θ with the positive direction of the x-axis, its slope is

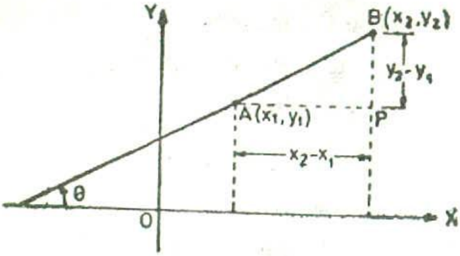
$$m = \tan\theta$$

If θ is acute, slope is positive and if θ is obtuse, the slope is negative.

In terms of the coordinates, the slope of the line joining two points, say $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m=\tan\theta=\frac{y_2-y_1}{x_2-x_1}=\frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

The following diagram will make the explanation more clear.



$$\tan\theta=\frac{BP}{AP}=\frac{y_2-y_1}{x_2-x_1}$$

Fig. 15.

15·15. DIFFERENT FORMS OF EQUATIONS OF THE STRAIGHT LINE

I. Equations of the Coordinate Axes :

(*i*) If $P(x; y)$ be any point on the x-axis, then its ordinate y is always zero for any position of the point P on the x-axis and for no other point.

$\therefore$ $\quad$ $\mathbf{y=0}$,

is the equation of x-axis.

(*ii*) If $P(x, y)$ is any point on the y-axis, then its abscissa x is always zero for any position of the point P on the y-axis and for no other point.

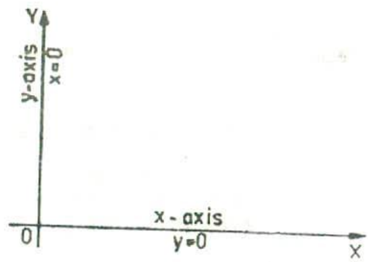


Fig. 16.

$\therefore$ $\quad$ $\mathbf{x=0}$, is the equation of y-axis.

Equations of Lines Parallel to the Coordinate Axes :

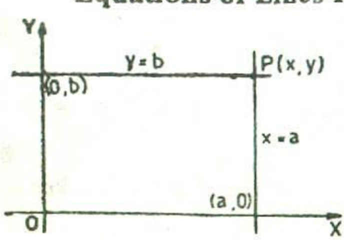


Fig. 17.

(*i*) Let $P(x, y)$ be any point on a line parallel to y-axis at a distance a from it. For any position of the point P lying on this line, and for no other point, the abscissa x is always constant and is equal to a.

$\therefore$ $\mathbf{x=a}$ is the equation of the line parallel to the y-axis and at a distance a from it.

(*ii*) Similarly $\mathbf{y=b}$ is the equation of the line parallel to the x-axis and at a distance b from it.

III. Origin-slope Form. *The equation of a line passing through the origin and having slope m* :

Let a straight line pass through the origin O and have a slope m. Let $P(x, y)$ be any point on the line. From P draw PM perpendicular on the x-axis, then

$$\frac{MP}{OM}=\tan\theta$$

$$\therefore \quad y=x\tan\theta$$

$$\Rightarrow \quad y=mx,$$

which is the required equation of the line.

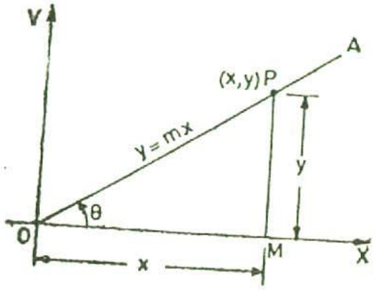


Fig. 18.

IV. A Line Intercepting the Axes. In case the straight line meets the x-axis and the y-axis at points other than the origin, the respective points will be called x-intercept and y-intercept. The diagram shows the two intercepts of a straight line AB which meet the x-axis in A and the y-axis in B, the OA is called the intercept of the line on the x-axis, and OB as the intercept of the line on the y-axis, and the two intercepts OA and OB, taken in this particular order, are called the intercepts of the line on the axes.

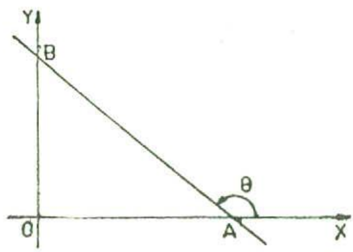


Fig. 19

It may be noticed that at the point of y-intercept, the value of x is equal to zero and at the point of x-intercept, the value of y is equal to zero. Therefore, in order to find out the value of say y-intercept we have to put x equal to zero in the equation. Similarly, to find out the value of x-intercept, we put y equal to zero in the equation.

V. Slope-intercept Form. *The equation of the line with the slope m and an intercept c on y-axis.*

Let a straight line of slope m intersect the y-axis, in K. Let OK, the intercept on the y-axis, be c. Then the coordinates of K are $(0, c)$. Take $P(x, y)$, any variable point on the line

$$\therefore \text{ Slope of } KP=\frac{y-c}{x-0} \qquad \ldots(1)$$

Slope of the line, as given, is m ...(2)

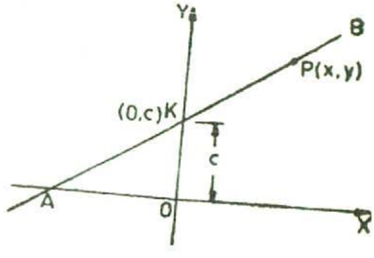


Fig. 20.

Equating (1) and (2), we get the required equation as

$$\frac{y-c}{x}=m \quad \Rightarrow \quad \mathbf{y=mx+c.}$$

VI. Two-intercept Form. *The equation of a line having intercepts a and b on the coordinate axes*

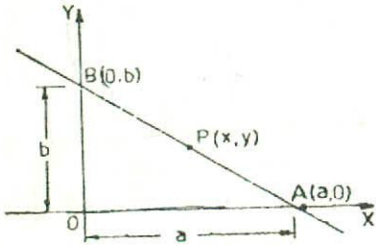


Fig. 21.

Let a straight line intersect the coordinate axes making intercepts of a and b on x-axis and y-axis respectively. Therefore

A is $(a, 0)$ and B is $(0, b)$

Let $P(x, y)$ be any point on the line AB.

$\therefore$ Slope of AB

$$=\frac{b-0}{0-a}=-\frac{b}{a} \qquad \text{...(1)}$$

and slope of $AP=\dfrac{y-0}{x-a}=\dfrac{y}{x-a}$ ...(2)

Now the points A, P, B are on the same line.

$\therefore$ Slope of AP=Slope of AB

$$\Rightarrow \quad \frac{y}{x-a}=-\frac{b}{a}$$

$$\Rightarrow \quad ay=-bx+ab$$

$$\Rightarrow \quad bx+ay=ba$$

$\Rightarrow$ $\mathbf{\dfrac{x}{a}+\dfrac{y}{b}=1}$ is the required equation of the line.

VII Slope-point Form. *The equation of a straight line having a slope m and passing through the point* (x_1, y_1)

Let the straight line passing through a given point $R(x_1, y_1)$ be inclined at an angle θ with the x-axis. The slope of the straight line is, therefore, $m=\tan\theta$. Take any point $P(x, y)$ on the straight

$\therefore$ Slope of $RP=\dfrac{y-y_1}{x-x_1}$

(by def.)

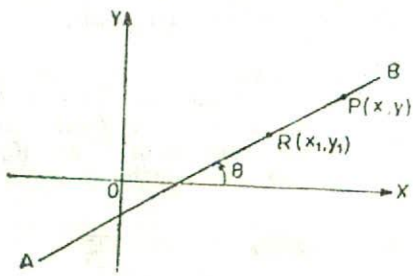


Fig. 22.

Since the points A and P are on the same line, we have

$$m=\tan\theta=\frac{y-y_1}{x-x_1}$$

$\therefore$ The equation of the required line is

$$y-y_1=m(x-x_1)$$

The above equation can also be written as

$$y-y_1=\frac{\sin\theta}{\cos\theta}(x-x_1)$$

$$\Rightarrow \qquad \frac{y-y_1}{\sin\theta}=\frac{x-x_1}{\cos\theta}$$

VIII. Parametric Form. In the Fig. 22, if the point $P(x, y)$ is taken at a distance r from the point R on the line, then

$$\frac{x-x_1}{r}=\cos\theta,\ \frac{y-y_1}{r}=\sin\theta$$

$\therefore$ $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$, is the required equation of the straight line in parametric form.

IX. Two-point Form. *The equation of a straight line passing through two point (x_1, y_1) and (x_2, y_2).*

Let $Q(x_1, y_1)$ and $R(x_2, y_2)$ be the two points on the line.

Take any point $P(x, y)$ on the line. Then by def.,

$$\text{Slope of } QP=\frac{y-y_1}{x-x_1} \qquad \ldots(1)$$

$$\text{Slope of } QR=\frac{y_1-y_2}{x_1-x_2} \qquad \ldots(2)$$

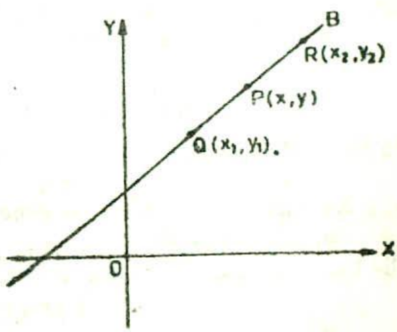


Fig 23.

Since A, P, B are collinear points, from (1) and (2), we have

$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$$

Hence the equation of the required straight line is

$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1).$$

X. Normal or Perpendicular Form. *The equation of a straight line in terms of the perpendicular from the origin and the inclination of the perpendicular with the axis :*

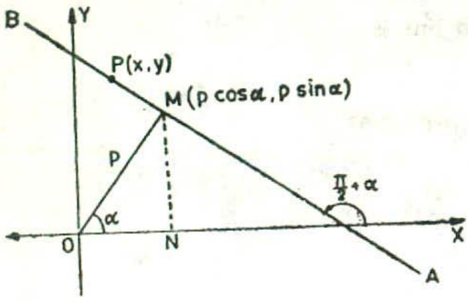


Fig. 24.

Let a straight line be at a perpendicular distance p from the origin and the inclination of the perpendicular OM with the x-axis be α. The coordinates of M, the foot of the perpendicular, in terms of the given constants are ($p \cos \alpha$, $p \sin \alpha$).

The inclination of the line with the x-axis is $\frac{\pi}{2}+\alpha$

$\therefore$ Slope of the given line

$$=\tan\left(\frac{\pi}{2}+\alpha\right)=-\cot \alpha=-\frac{\cos \alpha}{\sin \alpha} \quad \ldots(1)$$

Let $P(x, y)$ be any point on the line, then

$$\text{Slope of } MP=\frac{y-p \sin \alpha}{x-p \cos \alpha} \quad \ldots(2)$$

Since A, P, B are collinear, from (1) and (2), we have

$$\frac{y-p \sin \alpha}{x-p \cos \alpha}=-\frac{\cos a}{\sin a}$$

$\Rightarrow$ $y \sin \alpha-p \sin^2 \alpha=-x \cos \alpha+p \cos^2 \alpha$

$\Rightarrow$ $x \cos \alpha+y \sin \alpha-p (\sin^2 \alpha+\cos^2\alpha)=0$

$\therefore$ $x \cos \alpha+y \sin \alpha=p$, is the required equation.

15·16. GENERAL EQUATION OF A STRAIGHT LINE

An equation of the form $ax+by+c=0$, where a, b, c are constants and x, y are variables, is called the general equation of the straight line.

Proof Let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be the three points on the locus represented by the equation

$$ax+by+c=0 \quad \ldots(1)$$

which is the first degree equation in x and y.

$\because$ (x_1, y_1) lies on (1), we get

$$ax_1+by_1+c=0 \quad \ldots(2)$$

Similarly $ax_2+by_2+c=0$ ...(3)

and $ax_3+by_3+c=0$ ...(4)

Multiplying (2), (3) and (4) by y_2-y_3, y_3-y_1 and y_1-y_2 respectively and adding, we have

$$a[x_1(v_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]+b[y_1(y_2-y_3)+y_2(y_3-y_1)$$
$$+y_3(y_1-y_2)]+c[y_2-y_3+y_3-y_1+y_1-y_2]=0$$

$\Rightarrow$ $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$

$\Rightarrow$ $\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]=0$

$\Rightarrow$ The area of the triangle formed by P, Q, R is zero.

$\Rightarrow$ P, Q, R are collinear.

But they are any three points on the locus of (1).

Hence the locus is a straight line.

Remark. *Slope of the line $ax+by+c=0$.* Writing the equation as $y=-\frac{a}{b}x-\frac{c}{b}$, and comparing it with slope-intercept form, we find that

$$\textbf{Slope of the line} = \mathbf{m} = -\frac{\mathbf{a}}{\mathbf{b}} = -\frac{\textbf{coeff of x}}{\textbf{coeff. of y}}$$

Example. 17. *The line passing through the point $A(2, -1)$, and inclined at 45° to the x-axis, meets the line $\sqrt{2}x+2\sqrt{2}y-6=0$ in the point P. Find the distance AP.*

Solution Equation of the straight line through A (2, −1) and having slope 1 (tan 45°) is given by

$$y+1=1.(x-2)$$

$$\Rightarrow \quad y=x-3 \qquad \ldots(1)$$

To find the coordinates of P, substituting the value of y from (1) in the given equation, we get

$$\sqrt{2}x+2\sqrt{2}(x-3)-6=0$$

$$\Rightarrow \quad 3\sqrt{2}x=6(\sqrt{2}+1)$$

$$\Rightarrow \quad x=\frac{6(\sqrt{2}+1)}{3\sqrt{2}}=\sqrt{2}(\sqrt{2}+1)=2+\sqrt{2}$$

$$\Rightarrow \quad y=2+\sqrt{2}-3=\sqrt{2}-1$$

$\therefore$ Coordinate of P is $[(2+\sqrt{2}), (\sqrt{2}-1)]$

Hence $PA=\sqrt{(2+\sqrt{2}-2)^2+(\sqrt{2}-1+1)^2}=2$ units.

Example 18. *The coordinates of two points A and B are $(-1, 2)$ and $(2, -1)$ respectively. Find the equation and the slope of the line AB.*

Solution. Here $x_1=-1$, $y_1=2$, $x_2=2$ and $y_2=-1$

$\therefore$ The required equation of AB is

$$y-2=\frac{-1-2}{2-(-1)}[x-(-1)]$$

$$\Rightarrow \quad y-2=(x+1)$$

$$\Rightarrow \quad x+y=1$$

$$\text{Slope of } AB=-\frac{\text{Coeff of } x}{\text{Coeff. of } y}=-1.$$

Example 19. *Find the equation of the straight line through (2, 5) and making equal intercepts of opposite sign on the axis.*

Solution. Let a and $-a$ be the intercepts on the x-axis and y-axis respectively. Therefore, the equation of the line is

$$\frac{x}{a}+\frac{y}{-a}=1 \quad ...(1)$$

Since the line (1) passes through (2, 5), we have

$$\frac{2}{a}+\frac{5}{-a}=1$$

or $$a=-3$$

$\therefore$ The equation of the straight line is

$$\frac{x}{-3}+\frac{y}{3}=1$$

or $$x-y+3=0.$$

Example 20. *Find the equation of the straight line passing through the point (4, 5) and the sum of its intercepts on the axes is 18.*

[*I.C.W.A., June 1990*]

Solution. Let the equation of the required line be

$$\frac{x}{a}+\frac{y}{b}=1$$

This line passes through the point (4, 5), therefore, we have

$$\frac{4}{a}+\frac{5}{b}=1 \quad ..(1)$$

and $$a+b=18 \quad ...(2)$$

Putting the value of b from (2) in (1), we have

$$\frac{4}{a}+\frac{5}{18-a}=1$$

or $$a^2-17a+72=0$$

or $$(a-8)(a-9)=0$$

$\therefore$ $$a=8 \text{ or } a=9$$

When $a=8, \quad b=18-8=10$

When $a=9, \quad b=18-9=9$

$\therefore$ The equations are

$$\frac{x}{8}+\frac{y}{10}=1$$

or $$\frac{x}{9}+\frac{y}{9}=1$$

Example 21. *Find the equation of the straight line through $P(-4, 3)$ such that the portion between the axis is divided by P in the ratio 5 : 3.*

Solution. Here $A\equiv(a, 0)$, $B\equiv(0, b)$, then the line AB is

$$\frac{x}{a}+\frac{y}{b}=1 \quad ...(1)$$

Since the point $P(-4, 3)$, divides AB in the ratio $5 : 3$,

$$-4=\frac{3a+0\times 5}{5+3} \quad \text{or} \quad a=-\frac{32}{3}$$

and
$$3=\frac{3\times 0+5b}{5+3} \quad \text{or} \quad b=\frac{24}{5}$$

Substituting the values of a and b in (1), we get the required equation as

$$\frac{3x}{-32}+\frac{5y}{24}=1 \quad \text{or} \quad 9x-20y+96=0$$

Example 22. *Find the ratio in which the join of $(-5, 1)$ and $(1, -3)$ divides the straight line passing through $(3, 4)$ and $(7, 8)$.*

Solution. The equation of the straight line joining $(-5, 1)$ and $(1, -3)$ is

$$y-1=\frac{-3-1}{1+5}(x+5) \Rightarrow 3(y-1)+2(x+5)=0$$

$$\therefore \qquad 2x+3y+7=0 \qquad (1)$$

If the required ratio is $\lambda : 1$ in which (1) divides the join of $A(3, 4)$ and $B(7, 8)$. Then the coordinates of required point are

$$\left(\frac{7\lambda+3}{\lambda+1}, \frac{8\lambda+4}{\lambda+1}\right)$$

These coordinates must satisfy equation (1), therefore, we have

$$\frac{2(7\lambda+3)}{\lambda+1}+\frac{3(8\lambda+4)}{\lambda+1}+7=0$$

$$\Leftrightarrow \qquad (14\lambda+24\lambda+7\lambda)+(6+12+7)=0$$

$$\therefore \qquad \lambda=-\frac{5}{9}$$

Hence the line joining $(-5, 1)$ and $(1, -3)$ divides the join of $(3, 4)$ and $(7, 8)$ externally in the ratio $5 : 9$.

Example 23. *A firm invested Rs. 10 million in a new factory that has a net return of 500,000 per year. An investment of Rs. 20 million would yield a net income of Rs. 2 million per year. What is the linear relationship between investment and annual income ? What would be the annual return on an investment of Rs. 15 million ?*

Solution. Let x coordinate represent the investment and y co ordinate represent the annual income.

Then the required linear relationship between investment and income is the equation of the straight line joining the points [(10,000,000), (5,00,000)] and [(20,000,000), (2,000,000)] and its equation is

$$y-5{,}00{,}000=\frac{2{,}000{,}000-5{,}00{,}000}{20{,}000{,}000-10{,}000{,}000}(x-10{,}000{,}000)$$

$$=\frac{15}{100}(x-10,000,000)$$

$$\Rightarrow \quad 20y-10,000,000=3x-30,000,000$$

$$\Rightarrow \quad 3x-20y-20,000,000=0$$

Again when investment $x=15,000,000$, the annual income y can be found by putting the value of x in the equation obtained, *i.e.*,

$$3(15,000,000)-20y-20,000,000=0$$

$$\therefore \quad y=\text{Rs. } 1,250,000$$

Example 24. *As the number of units manufactured increases from 4,000 to 6,000, the total cost of production increases from Rs. 22,000 to Rs. 30,000. Find the relationship between the cost (y) and the number of units made (x), if the relationship is linear.*

Solution. When $x=4,000$, $y=22,000$

and when $x=6,000$, $y=30,000$

As the relationship between x and y is linear, we have to find the equation of line through (4,000, 22,000) and (6,000, 30,000).

$\therefore$ The reauired relationship is

$$y-22,000=\frac{22,000-30,000}{4,000-6,000}(x-4,000)$$

$$\Rightarrow \quad y-22,000=4(x-4,000)$$

$$\Rightarrow \quad 4x-y+6,000=0$$

Example 25. *The total expenses of a mess y, are partly constant and partly proportional to the number of the inmates of the mess x. The total expenses are Rs. 1040 when there are 12 members in the mess, and Rs. 1600 for 20 members.*

(*i*) *Find the linear relationship between y and x,*

(*ii*) *Find the constant expenses and the variable expenses per member, and* (*iii*) *what would be the total expenditure if the mess has 15 members ?*

Solution. (*i*) Corresponding to 12 members, the total expenses are Rs. 1040 and corresponding to 20 members, total expenses are Rs. 1600.

$\therefore$ The equation of the straight line joining the points (12, 1040) and (20, 1600) is given by

$$y-1040=\frac{1600-1040}{20-12}(x-12)$$

$$\Rightarrow \quad y-1040=70(x-12)=70x-840$$

$$\Rightarrow \quad y=70x+200 \qquad (1)$$

which is the required relationship between x and y.

(*ii*) Comparing the equation (1) with slope-intercept form ($y=mx+c$), we find

the constant expenses (c) = Rs. 200

and variable expenses per member (m) = Rs. 70

(*iii*) When the number ot members in the mess is 15, the total expenses $y=70\times15+200=$ Rs. 1250.

Example 26. *Find the equations of the diagonals of the rectangle, whose sides are $x=2$, $x=-1$, $y=6$ and $y=-2$.*

Solution. Since $x=2$ and $x=-1$ are two lines parallel to each other, they form a pair of opposite side say AB and CD. The equations $y=6$ and $y=-2$ give the other two sides BC and DA.

$\therefore$ A is solution of AB and DA, *i.e.*, $x=2$ and $y=-2$, *i.e.*, $A(2, -2)$.

Similarly $B(2, 6)$, $C(-1, 6)$ and $D(-1, -2)$.

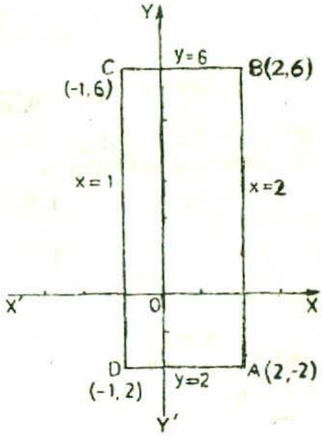


Fig. 25.

Now the equation of CA is given by

$$y-6=\frac{-2-6}{2-(-1)}(x+1)$$

$$\Rightarrow \quad 8x+3y-10=0$$

and the equation of BD is given by

$$y-(-2)=\frac{6-(-2)}{2-(-1)}(x+1)$$

$$\Rightarrow \quad 8x-3y+2=0$$

15·17. INTERSECTING LINES

Let the lines be given by

$$a_1x+b_1y+c_1=0 \qquad ...(1)$$

and

$$a_2x+b_2y+c_2=0 \qquad ...(2)$$

Since the point of intersection lies on both the lines, the coordinates of the point of intersection of the two lines satisfy the equations of both the lines.

Solving equations (1) and (2) by the rule of cross-multiplication, we have

$$\frac{x}{b_1c_2-c_1b_2}=\frac{y}{c_1a_2-a_1c_2}=\frac{1}{a_1b_2-b_1a_2}$$

$\therefore$ The point of intersection is

$$\left(\frac{b_1c_2-c_1b_2}{a_1b_2-b_1a_2}, \frac{c_1a_2-a_1c_2}{a_1b_2-b_1a_2}\right)$$

15·18. CONCURRENT LINES

Let the three lines be

$$a_1x+b_1y+c_1=0 \qquad ...(1)$$

$$a_2x+b_2y+c_2=0 \qquad ...(2)$$
$$a_3x+b_3y+c_3=0 \qquad ...(3)$$

The three straight lines will be concurrent if the point of intersection of any two of the lines lies on the third.

The point of intersection of (2) and (3) is

$$\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}, \frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right)$$

It lies on (1), if $a_1\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}\right)+b_1\left(\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right)+c_1=0$, *i.e.*, if $a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0$, which may be written in the determinant form as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

Example 27. *Prove that the lines $3x-4y+5=0$, $7x-8y+5=0$ and $4x+5y=45$ are concurrent.*

Solution. Consider the first two equations

$$3x-4y=-5 \qquad ...(1)$$
$$7x-8y=-5 \qquad ...(2)$$

Multiplying equations (1) and (2) by 7 and 3 respectively and then subtracting (2) from (1), we get

$$y=5$$

Substituting this value of y in (1), we get

$$3x-4\times5=-5 \quad \text{or} \quad x=5$$

Hence the point of intersection of (1) and (2) is (5, 5) which also satisfy

$$4x+5y=45$$

Hence the three lines are concurrent.

15·19. THE ANGLE BETWEEN TWO STRAIGHT LINES

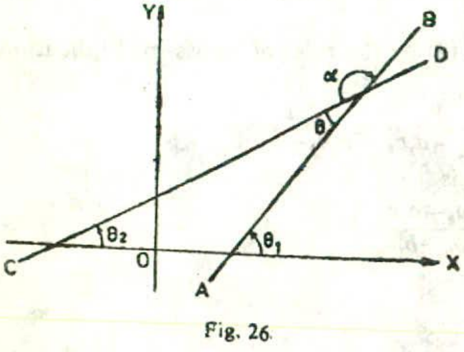


Fig. 26

Let AB and CD be two straight lines with given inclinations to the x-axis as θ_1 and θ_2, form interior and exterior angles θ and α respectively as shown below :

Since $\theta_1=\theta+\theta_2$,

$$\theta=\theta_1-\theta_2 \qquad ...(1)$$

Also $\pi=\theta+\alpha$

$$\therefore \alpha=\pi-\theta \qquad ...(2)$$

(*i*) For the interior angle, we have

$$\tan\theta=\tan(\theta_1-\theta_2)$$
$$=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}=\frac{\text{Difference of the slopes}}{1+\text{Product of the slopes}}$$

If we express slopes by m_1 and m_2, the formula can also be expressed as

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

(*ii*) For the exterior angle, therefore

$$\tan\alpha=\tan(\pi-\theta)=-\tan\theta=-\frac{m_1-m_2}{1+m_1m_2}$$

We can, therefore, say that if the angle formula gives a positive result, it is the tangent of the acute angle between the lines, but if the angle formula gives a negative result, it is the tangent of the obtuse angle between the lines.

Condition of Parallelism. If two lines are parallel, the angle between them is zero degree.

$$\therefore\quad \tan\theta=\tan 0^\circ=0$$
$$\Rightarrow\quad \tan 0^\circ=\frac{m_1-m_2}{1+m_1m_2}=0$$
$$\Rightarrow\quad m_1-m_2=0,\ i.e.,\ m_1=m_2$$

Thus if the lines are parallel, their slopes will be equal.

Condition of Perpendicularism. The two lines are perpendicular, if θ, the angle between them is 90°

$$\therefore\quad \tan\theta=\tan 90^\circ=\infty$$
$$\Rightarrow\quad \frac{m_1-m_2}{1+m_1m_2}=\infty$$
$$\Rightarrow\quad 1+m_1m_2=0$$
$$\Rightarrow\quad m_1m_2=-1$$

Thus if two lines are perpendicular, slope of one line is the negative reciprocal of the other line.

If the lines are given in the form

$$a_1x+b_1y+c_1=0 \quad\text{and}\quad a_2x+b_2y+c_2=0$$

then their slopes are given by $-\frac{a_1}{b_1}$ and $-\frac{a_2}{b_2}$ respectively and if θ is the angle between these lines, then

$$\tan\theta=\frac{\left(-\frac{a_1}{b_1}\right)-\left(-\frac{a_2}{b_2}\right)}{1+\left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\qquad\ldots(1)$$

It immediately follows from (1) that

1. The acute angle θ between the lines is

$$\theta=\tan^{-1}\left(\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right)$$

2. The lines are parallel if

$$a_2b_1-a_1b_2=0,\ i.e.,\ \frac{a_2}{a_1}=\frac{b_2}{b_1}$$

3. The lines are mutually perpendicular if

$$a_1a_2+b_1b_2=0.$$

Remarks. 1. To write down the equation of any line parallel to a given line whose equation is in the general form, change the constant term to a new constant k.

2. To write down the equation of any line perpendicular to the given line whose equation is in general form, we (*i*) interchange the coefficients of x and y, (*ii*) change the sign of one of them, and (*iii*) change the constant term to a new constant k.

Example 28. (*a*) *In a triangle with vertices $A(0, 6)$, $B(-2, -2)$ and $C(4, 2)$, find the equation of the perpendicular bisector of the side BC.* [*C.A., November 1991*]

Solution. Slope of the line $BC=\dfrac{-2-2}{-2-4}=\dfrac{-4}{-6}=\dfrac{2}{3}$

$\therefore$ The slope of the perpendicular bisector of the side $BC=-\dfrac{3}{2}$

Mid-point of $BC=\left(\dfrac{-2+4}{2},\ \dfrac{-2+2}{2}\right)=(1, 0)$

$\therefore$ The equation of the perpendicular bisector of the side BC is

$$y-0=-\frac{3}{2}(x-1)$$

or $$3x+2y-3=0$$

(*b*) *The vertices of a triangle are $(-4, 0)$, $(1, 1)$ and $(3, -1)$. Find the equations of any two of its altitude.* [*C.A., May, 1991*]

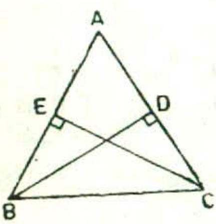


Fig. 27.

Solution. Let $A(-4, 0)$, $B(1, 1)$ and $C(3, -1)$ be the vertices of the triangle ABC.

Slope of the line $AB=\dfrac{0-1}{4-1}=\dfrac{1}{5}$

$\therefore$ Slope of the altitude $CE=-5$

$\therefore$ Equation of the altitude CE is

$$y+1=-5(x-3)$$

or $$5x+y-14=0$$

Slope of the line $AC=\dfrac{-1-0}{3+4}=-\dfrac{1}{7}$

$\therefore$ Slope of the altitude $BD = 7$

$\therefore$ Equation of the altitude BD is

$$y-1=7(x-1)$$

or
$$7x-y+6=0.$$

Example 29. *Find the equation of the straight line passing through the point $(-3, 1)$ and perpendicular to the line $5x-2y+7=0$.*
[I.C.W.A., December, 1990]

Solution. The slope of the line $5x-2y+7=0$ is $\frac{5}{2}$.

$\therefore$ The slope of the line perpendicular to the line $5x-2y+7=0$ is $-\frac{2}{5}$.

$\therefore$ The equation of the required line is

$$y-1=-\frac{2}{5}(x+3)$$

or
$$2x+5y+1=0$$

Example 30. *Find the equation of a line which passes through the point $(2, 5)$ and makes an angle of $45°$ with the line $x-3y+6=0$.*
[C.A., May, 1991]

Solution. Let the slope of the required line be m. Therefore, we have

$$\tan 45° = \left|\frac{m_1-\frac{1}{3}}{1+m_1(\frac{1}{3})}\right|$$

$$\therefore \qquad 1=\frac{3m_1-1}{3+m_1} \quad \text{or} \quad -1=\frac{3m_1-1}{3+m_1}$$

$$\Rightarrow \qquad m_1=2 \quad \text{or} \quad m_1=-\frac{1}{2}$$

$\therefore$ The equation of the required line is

$$y-5=2(x-2) \quad \text{or} \quad y-5=-\frac{1}{2}(x-2)$$

$$\Rightarrow \quad 2x-y+1=0 \quad \text{or} \quad x+2y-12=0.$$

Example 31. *Find the equations of two straight lines through the point $(2, -1)$ making an angle of $45°$ with the line $6x+5y-1=0$. Show that these lines are at right angles to one another.*

Solution. The equation of straight line passing through $(2, -1)$ is

$$y+1=m(x-2) \tag{1}$$

Slope of the given line $6x+5y-1=0$ is $-\frac{6}{5}$.

Now the angle between the given straight line and the required straight line is either 45° or 135°.

$$\therefore \tan 45^\circ = \frac{m+\frac{6}{5}}{1-\frac{6}{5}m} \qquad\Big|\qquad \tan 135^\circ = \frac{m+\frac{6}{5}}{1-\frac{6}{5}m}$$

$$\Rightarrow \quad 1=\frac{5m+6}{5-6m} \qquad\Big|\qquad \Rightarrow \quad -1=\frac{5m+6}{5-6m}$$

$$\Rightarrow \quad m=-\frac{1}{11} \qquad\Big|\qquad \Rightarrow \quad m=11$$

$\therefore$ The required straight lines are

$$y+1=-\frac{1}{11}(x-2) \quad \text{and} \quad y+1=11(x-2)$$

or

$$x+11y+9=0 \quad \text{and} \quad 11x-y-23=0$$

Since the slopes of the required lines are $-\frac{1}{11}$ and 11, the product of the slopes is -1, hence the required lines are at right angles to each other.

Example 32. *Find the equation of straight lines each of which makes a positive intercept of 5 units on the y-axis and is inclined at an angle of 45° to the line $2x+3y-7=0$.*

Solution. Let m be the slope of the required lines.

$$\text{Slope of } 2x+3y-7=0 \text{ is } -\frac{2}{3}$$

The angle between the required line and $2x+3y-7=0$ is given to be equal to 45°.

$$\therefore \qquad \tan 45^\circ = \pm\frac{m+\frac{2}{3}}{1-\frac{2}{3}m} \quad \Rightarrow \quad 1=\pm\frac{3m+2}{3-2m}$$

Taking the positive sign, we get

$$3-2m=3m+2, \text{ i.e., } m=\frac{1}{5}$$

Taking the negative sign, we get

$$3-2m=-3m-2, \text{ i.e., } m=-5$$

$\therefore$ The equations of the straight lines are

$$y=\frac{1}{5}x+5 \text{ and } y=-5x+5$$

$$\Rightarrow \qquad x=5y-25 \text{ and } 5x+y=5.$$

Example 33. *Find the orthocentre of the triangle formed by the straight lines $x-y=5$, $2x-y=8$ and $3x-y=9$.*

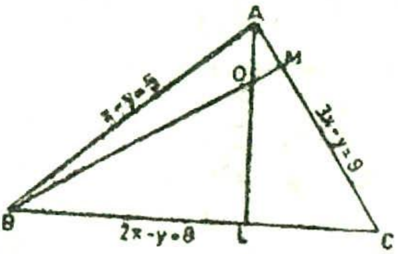


Fig. 28.

Solution. Let ABC be the triangle formed by the given lines.

Let
$$AB \equiv x-y-5=0 \qquad \ldots(1)$$
$$BC = 2x-y-8=0 \qquad \ldots(2)$$
$$CA \equiv 3x-y-9=0 \qquad \ldots(3)$$

Solving equations (1) and (2), (2) and (3), (3) and (1), we get the coordinates of the vertices as $A \equiv (2, -3)$ $B \equiv (3, -2)$ and $C \equiv (1, -6)$.

The orthocentre is the point of intersection of the three altitudes of a triangle. It is sufficient to find the equations of two of the altitudes AL and BM to determine the orthocentre.

Now slope of $BC = 2$.

Since AL is perpendicular to BC, slope of $AL = \frac{1}{2}$

$\therefore$ Equation of AL with slope $-\frac{1}{2}$ and passing through the point $A(2, -3)$ is

$$y+3=-\tfrac{1}{2}(x-2)$$
$$\Rightarrow \quad x+2y+4=0 \qquad \ldots(4)$$

Similarly BM is perpendicular to CA, slope of $BM = -\frac{1}{3}$

$\therefore$ Equation of BM with slope $-\frac{1}{3}$ and passing through the point $B(3, -2)$ is

$$y+2=-\tfrac{1}{3}(x-3)$$
$$\Rightarrow \quad x+3y+3=0 \qquad \ldots(5)$$

Solving equations (4) and (5), we get co-ordinates of the orthocentre as $(-6, 1)$.

Example 34. *Find the equation of the straight line passing through the intersection of $4x-3y-1=0$ and $2x-5y+3=0$ and (i) parallel to $4x+5y=6$, (ii) perpendicular to $2x+3y=12$.*

Solution. The equation of any line passing through the intersection of the given lines is

$$(4x-3y-1)+\lambda(2x-5y+3)=0,$$

where λ is some constant.

This equation can be written as

$$(4+2\lambda)x-(3+5\lambda)y-(1-3\lambda)=0 \qquad \ldots(1)$$

$\therefore$ Its slope is $=\dfrac{4+2\lambda}{3+5\lambda}$

(*i*) Since the line (1) is parallel to $4x+5y=6$,

$$\therefore \quad \frac{4+2\lambda}{3+5\lambda}=-\frac{4}{5}$$

$$\Rightarrow \quad 20+10\lambda=-12-20\lambda$$

$$\Rightarrow \quad \lambda=-\frac{16}{15}$$

$\therefore$ The equation of the required line is

$$\left(4-\frac{32}{15}\right)x-\left(3-\frac{80}{15}\right)y-\left(1+\frac{48}{15}\right)=0$$

$$\Rightarrow \quad 28x+35y-63=0$$

$$\Rightarrow \quad 4x+5y-9=0$$

(*ii*) Since the line (1) is perpendicular to $2x+3y=12$,

$$\therefore \quad \frac{4+2\lambda}{3+5\lambda}=\frac{3}{2} \qquad \left(\because \text{ slope of } 2x+3y=12 \text{ is } -\frac{2}{3}\right)$$

$$\Rightarrow \quad 8+4\lambda=9+15\lambda$$

$$\Rightarrow \quad \lambda=-\frac{1}{11}$$

$\therefore$ Equation of the required line is

$$\left(4-\frac{2}{11}\right)x-\left(3-\frac{5}{11}\right)y-\left(1+\frac{3}{11}\right)=0$$

$$\Rightarrow \quad 42x-28y-14=0$$

$$\Rightarrow \quad 3x-2y-1=0$$

15·20. LENGTH OF PERPENDICULAR FROM A POINT

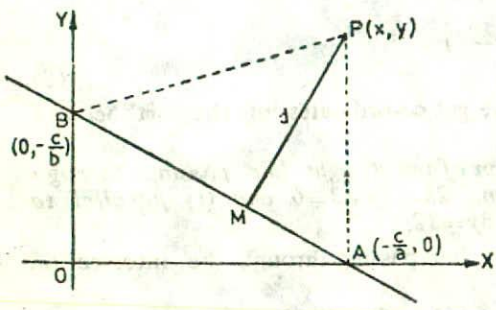


Fig. 29

Let AB be the line

$$ax+by+c=0 \qquad \ldots(1)$$

Let $P(x_1, y_1)$ be the point and $PM \perp AB$. Join AP and PB.

AB meets the x-axis, where putting $y=0$ in (1), we get

$$ax+c=0 \text{ or } x=-\frac{c}{a}$$

$\therefore$ Coordinate of A is

$$\left(-\frac{c}{a}, 0\right)$$

Similarly co-ordinate of B is

$$\left(0, -\frac{c}{b}\right)$$

Now $$\triangle BAP = \pm\frac{1}{2}\ AB.PM$$

$\therefore$ $$d = \pm\frac{2\triangle BAP}{AB} \quad \ldots(2)$$

Now $$\triangle BAP = \frac{1}{2}\left[0-\left(-\frac{c}{a}y_1\right)+\frac{c^2}{ab}-0+0-\left(-\frac{c}{b}x_1\right)\right]$$

$$= \frac{1}{2}\left[\frac{cy_1}{a}+\frac{c^2}{ab}+\frac{cx_1}{b}\right]$$

$$= \frac{c}{2ab}\left[ax_1+by_1+c\right]$$

$$AB = \sqrt{\left(0+\frac{c}{a}\right)^2+\left(-\frac{c}{b}\right)^2} = \sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}} = \frac{c}{ab}\sqrt{a^2+b^2}$$

$\therefore$ From (2), we get $d = \pm\dfrac{2\cdot\dfrac{c}{2ab}(ax_1+by_1+c)}{\dfrac{c}{ab}\sqrt{a^2+b^2}} = \pm\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$

Example 35. *Find the equations of the straight lines through (4, −2) and at a perpendicular distance of 2 units from the origin.*

Solution. The equation of straight line through (4, −2) is

$$y+2 = m(x-4) \quad \ldots(1)$$

or $$mx-y-4m-2=0$$

Since the distance of the origin, *i.e.*, (0, 0) from this line is 2 units,

$$\frac{-(4m+2)}{\sqrt{1+m^2}} = 2$$

$\Rightarrow$ $$-(2m+1) = \sqrt{1+m^2}$$

$\Rightarrow$ $$4m^2+1+4m = 1+m^2$$

$\Rightarrow$ $$m(3m+4) = 0$$

$\Rightarrow$ $$m = 0 \text{ or } m = -\frac{4}{3}$$

Substituting these values of m in (1), we get the required equation of the straight lines as

$$y+2=0$$

and $$y+2 = -\frac{4}{3}(x-4) \text{ or } 4x+3y=10.$$

Example 36. *Find the coordinates of the foot of the perpendicular from (a, 0) on the line* $y=mx+\frac{a}{m}$.

Solution. Slope of the line perpendicular to the given line $=-\frac{1}{m}$

$\therefore$ The equation of the line through $(a, 0)$ and perpendicular to the given line is

$$(y-0)=-\frac{1}{m}(x-a) \text{ or } my+x=a \quad \ldots(1)$$

The equation of the given line may be written as

$$my-m^2x=a \quad \ldots(2)$$

The foot of the perpendicular is the point of intersection of (1) and (2).

Subtracting (1) and (2), we have

$$(1+m^2)x=0, \text{ i.e., } x=0$$

$$\therefore \quad y=0+\frac{a}{m}=\frac{a}{m}$$

Hence the foot of the perpendicular is $\left(0, \frac{a}{m}\right)$.

EXERCISE (II)

1. (*a*) Find the equation of a straight line parallel to the x-axis and at distance (*i*) 4 units above it, (*ii*) 5 units below it.

(*b*) Find the equation of a straight line parallel to the y-axis and at a distance (*i*) 5 units to the right, (*ii*) 6 units to the left of it.

2. (*a*) Find the equation of a straight line parallel to the x-axis and passing through (*i*) $(-5, -7)$, (*ii*) $(-8, 5)$.

(*b*) Find the equation of a straight line parallel to the y-axis and passing through (*i*) $(-3, -2)$, (*ii*) $(-7, 6)$.

3. Find the equation of a straight line making an angle of 120° with the x-axis and cutting the y-axis at a distance 3 below the origin.

(*b*) Find the equation of the straight line passing through the orign and making with the x-axis an angle of (*i*) 45°, (*ii*) 60° (*iii*) 90° and (*iv*) 135°.

(*c*) Find the equation of a straight line cutting off an intercept 2 from the positive direction of the axis of y and inclined at 45° to the x-axis.

(*d*) Find the equation of a line which makes an angle of 60° with the x-axis and passes through the point (0, 5).

4. (*a*) Find the equation of a straight line passing through the point (3, 4) such that the sum of its intercepts on the axes is 14.

(*b*) Find the equation to the straight line which passes through $(-5, 2)$ and is such that the portion of it between the axes is divided by the point in the ratio 2 : 3.

5. Find the equation of a line which passes through the point $(1, -2)$ and makes the intercepts on the axis equal in magnitude and opposite in sign.

6. Find the equation of the line passing through the points

(i) (a, b) and $(a+b, a-b)$

(ii) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

(iii) $(a \cos \theta_1, b \sin \theta_1)$ and $(a \cos \theta_2, b \sin \theta_2)$

7. (a) The vertices of a triangle are $(2, 0)$, $(0, 2)$ and $(4, 6)$. Find the median through the first vertex.

(b) Find the equations of the medians of the triangle given by the points $(10, 4)$, $(-4, 9)$ and $(-2. -1)$.

8. Find the equations of the diagonals of the rectangle whose sides are given by $x=2$, $x=-4$, $y=3$ and $y=-5$.

9. Prove that the three points $(-1, -1)$, $(5, 7)$ and $(8, 11)$ lie on a straight line. Find the intercepts it makes on the axis.

10. (a) In what ratio is the line joining $(3, 7)$ and $(6, 3)$ divided by the line joining $(9, 0)$ and $(17, -10)$?

(b) Find the ratio in which the join of $(-1, 0)$ and $(-2, 3)$ is divided by the line $x+2y-3=0$.

11. Find the equations of the sides of a rectangle whose vertices are $(3, 2)$, $(11, 8)$, $(8, 12)$ and $(0, 6)$.

12. Find the acute angle between the lines :

(a) $3x+2y-11=0$ and $2x+y+12=0$

(b) $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{a}+\frac{y}{b}=1$.

13. Find the equation of the straight line.

(i) parallel to $2x-3x-5=0$ and passing through $(4, 5)$,

(ii) perpendicular to $2x+3y+4=0$ and passing through $(3, -2)$.

14. Show that the line joining $(2, 1)$ and $(3, 4)$ is perpendicular to the line joining $(7, 5)$ and $(4, 6)$.

15. Find the equations of the two lines through

(a) the point $(2, -1)$ and making an angle of $45°$ with the line $6x+5y-1 \quad 0$,

(b) the point $(3, -2)$ and inclined at $60°$ to the straight line $x\sqrt{3}+y=1$.

16. (a) The vertices of a $\triangle ABC$ are $A\ (0, 0)$, $B\ (1, 5)$ and $C\ (-2, 2)$. Find the equation of altitude through A.

(b) The vertices of a $\triangle ABC$ are $A(6, 2)$, $B(-3, 8)$ and $C(-5, -3)$. Find the equation of the altitude through A.

17. (*a*) Find the equation of a line which is parallel to $2x-y-9=0$ and which passes through the intersection of

$$5x+y+4=0 \text{ and } 2x+3y-1=0$$

(*b*) Find the equation of that line which passes through the point of intersection of $5x+y+4=0$ and $2x+3y-1=0$ and is perpendicular to

$$2x-y=9$$

18. Find the value of k for which the three lines are concurrent.

(*a*) $x-y=6$, $4x-3y-20=0$ and $6x+5y+k=0$

(*b*) $3x+4y+1=0$, $kx+2y-3=0$ and $2x-y-3=0$

19. (*a*) Find the coordinates of the foot of the perpendicular from the point $(6, -2)$ to the straight line $4x-5y+7=0$.

(*b*) Find the length of the perpendicular and the coordinates of the foot of the perpendicular from the point $(3, 4)$ to the line $8x+15y+1=0$.

20. Find the equations of the altitudes of the triangle formed by the lines $4x-y=4$, $2x-y=6$, $x+y=6$ and show that the orthocentre of the triangle is at $(8, 1)$.

21. Find the equation of a straight line passing through the point of intersection of the lines $x-2y+3=0$, $2x-3y+4=0$ and parallel to the line joining the points $(1, 1)$ and $(0, -1)$.

22. A line passes through the point of intersection of the lines $x+2y-1=0$ and $2x+3y-4=0$ and makes equal intercepts on the coordinate axis, show that its equation is $x+y=3$.

23. Prove that the triangle formed by the lines $\sqrt{3}x+y-2=0$, $\sqrt{3}x-y+1=0$ and $y=0$ is an equilateral triangle.

24. (*a*) The line containing points $(-8, 3)$ and $(2, 1)$ is parallel to the line containing the points $(11, -1)$ and $(k, 0)$, show that $k=6$.

(*b*) Determine the values of k for which the line containing the points $(k, 3)$ and $(-2, 1)$ will be parallel to the line through $(5, k)$ and $(1, 0)$.

25. The total cost y, for x units of a certain product consists of fixed cost and the variable cost (proportional to the number of units produced). It is known that the total cost is Rs. 6000 for 500 units and Rs. 9000 for 1000 units.

(*i*) Find the linear relationship between x and y,

(*ii*) Find the slope of the line, what does it indicate ?

(*iii*) Find the number of units that must be produced so that

(*a*) There is neither profit nor loss,

(*b*) There is a profit of Rs. 1000.

(*c*) There is a loss of Rs. 300 ; it being given that the selling price is Rs. 8 per unit

26 A firm invests Rs. 10,000 in a business which has a net return of Rs. 500 per year. An investment of Rs. 20,000 would yield an income

of Rs. 2000 per year. What is the linear relationship between investment and annual income? What would be the annual return on an investment of Rs. 12,000?

27. The total cost y, for x units of a certain product consists of fixed costs and the variable cost (proportional to the number of units produced). It is known that the total cost is Rs. 1200 for 100 units and Rs. 2700 for 400 units.

(*i*) Find the linear relationship between x and y.

(*ii*) Find the slope of the line and what does it indicate.

(*iii*) If the selling price is Rs. 7 per unit, find the number of units that must be produced so that (*a*) there is neither profit nor loss, (*b*) there is a profit of Rs. 300, (*c*) there is a loss of Rs. 300

28. An investment of Rs. 90,000 in a certain business yields an income of Rs. 8,000. An investment of Rs. 50,000 yield an income of Rs. 5,000. If the income is a linear function of investment, determine the equation for this relation. What is the slope ? Interpret the slope in terms of the money involved.

29. An investment of Rs. 100 in a certain business yields an income of Rs. 20. An investment of Rs. 1000 yields an income of Rs. 90. If the income is a linear function of investment, find the equation for this relation. What is the slope? Interpret the slope in terms of the money involved.

30. M/s. R.K. Industry spends Rs. 4,000 to process 100 orders and Rs. 6,000 to process 200 orders. Find the linear relation between the processing money and the number of orders. Find the money spent for 300 orders?

31. A factory produces 200 bulbs for a total cost of Rs. 800 and 400 bulbs for a total cost of Rs. 1200. Given that the cost curve is a straight line, find the equation of the straight line and use it to find the cost of producing 300 bulbs.

32. For sending one wagon of wheat, Food Corporation of India spends Rs 300 for a distance of 20 kilometres and Rs. 500 for a distance of 200 kilometres. What is the linear relation between the amount spent and the number of kilometres covered? What are the slope and intercepts of the line? Also find the cost of sending through 400 kilometres.

33. The cost of producing 200 pens is Rs. 1000 and the cost of producing 400 pens is Rs. 1500. (*i*) Find the linear relation between the cost y of producing x pens, (*ii*) what number of pens must be produced and sold at Rs. 3 per pen, so that there is neither profit nor loss? (*iii*) what should be the selling price of a pen if 600 pens are produced and sold with a profit of Rs 400 ?

ANSWERS

1. (*a*) (*i*) $y=4$, (*ii*) $y+5=0$ 2. (*a*) (*i*) $y+7=0$, (*ii*) $y-5=0$ (*b*) (*i*) $x+3=0$, (*ii*) $x+7=0$ 3. (*a*) $\sqrt{3}x+y+3=0$, (*b*) (*i*) $y=x$, (*ii*) $y=\sqrt{3}x$, (*iii*) $x=0$, (*iv*) $x+y=0$, (*c*) $x-y+2=0$, (*d*) $y=\sqrt{3}x-5$, 4. (*a*) $4x+3y=24$ $x+y=7$, (*b*) $3x-5y+25=0$ 5. $x-y=3$,

6. (*i*) $(a-2b)x-by+b^2+2ab-a^2=0$, (*ii*) $y(t_1+t_2)-2x=2at_1t_2$,

(*iii*) $\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}$

7. (*a*) $x=2$, (*b*) $y=4$, $15x+16y=84$, $3x-2y+4=0$

8. $4x-3y+1=0$, $4x+3y+7=0$ 9. $-\frac{1}{4}, \frac{1}{3}$ 10. (*a*) 2 : 3 externally (*b*) 4 : 1 (internally) 11. $3x-4y-1=0$, $3y+4x-68=0$, $3x-4y+24=0$, $3y+4x-18=0$.

12. (*a*) $\tan^{-1}\frac{1}{8}$, (*b*) $\tan^{-1}\frac{ab'-a'b}{aa'+bb'}$, 13. (*i*) $2x-3y+7=0$, (*ii*) $3x-2y-13=0$ 15. (*a*) $x+11y+9=0$, $11x-y-23=0$, (*b*) $y+2=0$, $\sqrt{3}x-y-(3\sqrt{3}+2)=0$ 16. (*a*) $x=y$, (*b*) $2x+11y=34$ 17. (*a*) $y=2x+3$, (*b*) $x+2y=1$ 18. (*a*) $K=8$, (*b*) $K=5$ 19. (*a*) (2, 3) (*b*) 5, $\left(\frac{11}{17}, \frac{-7}{17}\right)$ 21. $y=2x$ 25. (*i*) $y=6x+3000$, (*ii*) slope $=6$, it tells us that Rs. 6 is added to the total cost, if one additional unit is produced. (*a*) When 1500 units are produced, there is neither profit nor loss, (*b*) 2000 units, (*c*) 1350 units. 26. $20y=3x-20{,}000$, income is 800 27. (*i*) $y=5x+700$, (*ii*) $m=5$, (*iii*) 350 units, (*iv*) 500 units, (*v*) 200 units,

28. (*i*) $40y=3x+50{,}000$, (*ii*) $m=\frac{3}{40}$ 29. $7x-90y+1100=0$, slope $=\frac{7}{90}$ 30. $20x-y+2000=0$, Rs. 8000 31. $2x-y+400=0$, Rs. 1000

32. $10x-9y+2500=0$, slope $=\frac{10}{2}$, intercept on x-axis $=-250$, intercept on y-axis $=\frac{2500}{9}$, Rs. $\frac{6500}{9}$ 33. (*i*) $5x-2y+1000=0$, (*ii*) 1000 pens, (*iii*) Rs. 4 pen.

15·21. TANGENT AND NORMAL

If P, Q are any two points on a curve, their join PQ is called a chord. The chord produced both ways is called a secant. Let P be a given point on a curve and Q be any other point on it. PQ produced both ways is called the secant, As Q tends to P, the straight line PQ tends in general, to a definite straight line PT which is called the tangent to the curve at P. P is called the point of contact.

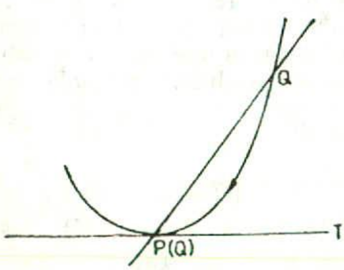


Fig. 29.

Any line perpendicular to the tangent at the point of contact is called a normal.

15·22. CIRCLE

The circle is the locus of a point which moves in such a way that its distance from a fixed point always remains constant.

The fixed point is called the centre of the circle and the constant distance is termed as the radius of the circle.

15·23. THE EQUATION OF A CIRCLE

Let the moving point be $P(x, y)$, the centre $C(h, k)$ and the fixed distance, the radius be r, then by definition, we have

$$CP=r$$

In terms of coordinates, this can be expressed as

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

$$\Rightarrow \quad (x-h)^2+(y-k)^2=r^2$$

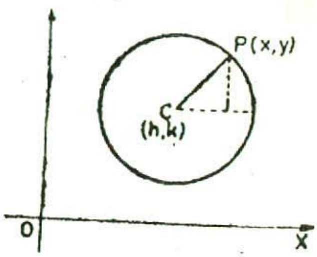


Fig. 30.

Illustration. *Find the equation of the circle whose centre is (4, 5) and the radius is 7.*

Solution. Using the formula given above, we have

$$(x-4)^2+(y-5)^2=7^2$$

$$\Rightarrow \quad x^2+y^2-8x-10y-8=0$$

We now find some particular forms of the equation of a circle corresponding to the different sets of conditions determining the position of the circle in a plane.

I. *Equation of a circle with centre as the origin.*

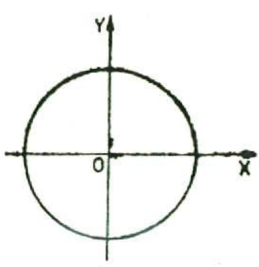


Fig. 31.

The equation of the circle is

$$(x-0)^2+(y-0)^2=r^2$$

$$\Rightarrow \quad x^2+y^2=r^2$$

II. *The equation of a circle passing through the origin.*

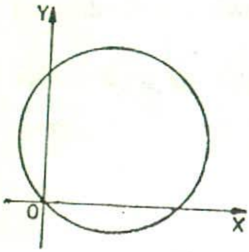


Fig. 32.

The equation of the circle is

$$(0-h)^2+(0-k)^2=r^2$$

$$\Rightarrow \quad h^2+k^2=r^2$$

III. *The equation of a circle of radius r, passing through the origin and having its centre on the (i) x-axis, (ii) y-axis.*

(*i*) It can be seen from the figure that the centre of the circle is $(r, 0)$ and its radius is equal to r.

$\therefore$ The equation of the required circle is

$$(x-r)^2+(y-0)^2=r^2$$

$$\Rightarrow \quad x^2+y^2-2rx=0.$$

(*ii*) Similarly, if the centre lies on the y-axis and the origin lies on the circle, the equation of the circle would be

$$(x-0)^2+(y-r)^2=r^2$$

$$\Rightarrow \quad x^2+y^2-2ry=0.$$

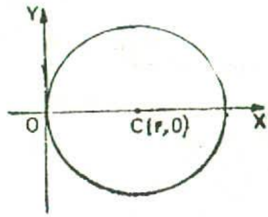


Fig. 33.

IV. *Equation of a circle of radius r and touching the (i) x-axis, (ii) y-axis.*

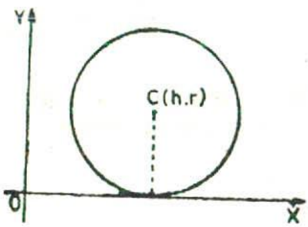


Fig. 34.

(*i*) Since the circle is to touch the x-axis, the ordinate of the centre must be equal to r.

Hence the equation of such a circle is

$$(x-h)^2+(y-r)^2=r^2$$

$$\Rightarrow \quad x^2+y^2-2hx-2ry+h^2=0$$

(*ii*) If the circle touches the y-axis, the abscissa of the centre will be equal to r

The equation of the circle would be

$$(x-r)^2+(y-k)^2=r^2$$

$$\Rightarrow \quad x^2+y^2-2rx-2ky+k^2=0$$

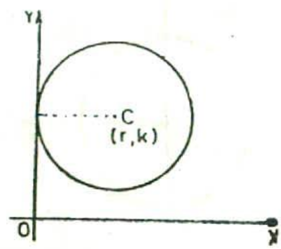


Fig. 35.

V. *Equation of a circle of radius r and touching both the axis.*

If the circle touches both the axis in the positive quadrant, the coordinates of the centre will be (r, r). The required equation is of the form

$$(x-r)^2+(y-r)^2=r^2$$

$$\Rightarrow \quad x^2+y^2-2rx-2ry+r^2=0$$

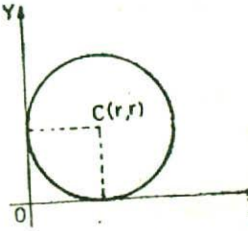


Fig. 36.

VI. *Equation of the circle on the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ as a diameter.*

Let $P(x, y)$ be any point on the circle.

Join AP and BP.

$\angle APB$, being in a semi-circle, is equal to one rt. angle.

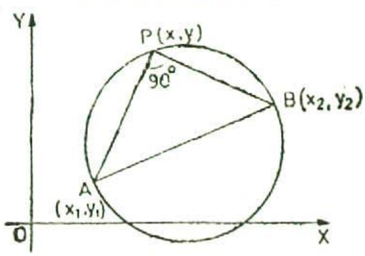


Fig. 37.

Thus the two lines AP and BP are perpendicular to each other.

$\therefore$ Slope of $AP \times$ Slope of $BP=-1$ ...(1)

Now the slope of $AP=\dfrac{y-y_1}{x-x_1}$

and the slope of $BP=\dfrac{y-y_2}{x-x_2}$

$\therefore$ From (1), we have

$$\frac{y-y_1}{x-x_1}\times\frac{y-y_2}{x-x_2}=-1$$

$\therefore$ The required equation is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Since (x_1, y_1) and (x_2, y_2) are the extremities of a diameter, this form may also be called the diameter form of the equation of a circle.

VII *Equation of a circle passing through the origin and making intercepts a and b on the coordinate axis.*

If the circle intersects the axes in A and B, then the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

Since $\angle AOB=90°$ and AB is a diameter, the required equation of the circle is

$$(x-a)(x-0)+(y-0)(y-b)=0$$

$$\Rightarrow \quad x^2+y^2-ax-by=0$$

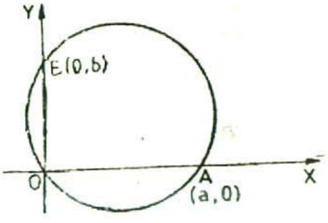


Fig. 38.

15·24. GENERAL EQUATION OF THE CIRCLE

We have obtained the equation of the circle in the form

$$(x-h)^2+(y-k)^2=r^2$$

which can be written as

$$x^2+y^2-2hx-2ky+(h^2+k^2-r^2)=0$$

The equation represents the same circle even if we multiply it throughout by a constant A, a non-zero number.

$$\therefore \quad A(x^2+y^2)-2Ahx-2Aky+A(h^2+k^2-r^2)=0$$

Writing $-Ah=G$, $-Ak=F$ and $A(h^2+k^2-r^2)=C$, the equation takes the form

$$A(x^2+y^2)+2Gx+2Fy+C=0$$

On dividing by A throughout, we get

$$x^2+y^2+2\frac{G}{A}x+2\frac{F}{A}y+\frac{C}{A}=0$$

If $\frac{G}{A}=g$, $\frac{F}{A}=f$ and $\frac{C}{A}=c$, the above equation becomes

$$x^2+y^2+2gx+2fy+c=0$$

which is referred to as the general form of the equation of the circle.

Conversely any equation of the form

$$x^2+y^2+2gx+2fy+c=0$$

represents a circle as we can write the equation in the form

$$(x^2+2gx)+(y^2+2fy)=-c$$

$$\Rightarrow \quad (x+g)^2+(y+f)^2=g^2+f^2-c$$

$$\Rightarrow \quad [x-(-g)]^2+[y-(-f)]^2=(\sqrt{g^2+f^2-c})^2$$

which is of the form

$$(x-h)^2+(y-k)^2=r^2$$

$\therefore$ Equation represents a circle with its centre at $(-g, -f)$

and $\qquad$ radius $=\sqrt{g^2+f^2-c}$.

Observing the general form of the equation of a circle, we conclude that the conditions for an equation to represent a circle are

(*i*) the equation should be of the second degree in x and y,

(*ii*) the co-efficient of x^2 and y^2 should be equal,

(*iii*) there should be no term involving the product xy.

Aid to memory :

Coordinates of centre $=(-\frac{1}{2}$ coeff. of x, $-\frac{1}{2}$ coeff. of $y)$

Radius $=\sqrt{(-\frac{1}{2}\text{ coeff. of }x)^2+(-\frac{1}{2}\text{ coeff. of }y)^2-\text{constant term}}$.

Example 37. *Find the coordinates of the centre and the radius of the circle given by $4x^2+4y^2+16x-24y+3=0$.*

Solution. Let the equation be rewritten in the general form where the co-efficients of x^2 and y^2 are each equal to unity. Thus we have the equation

$$x^2+y^2+4x-6y+\frac{3}{4}=0$$

$\therefore$ The x-coordinate of the centre, *viz.*, $-g=-\frac{1}{2}$ coeff. of $x=-2$ and the y-coordinate of the centre, *viz.*, $-f=-\frac{1}{2}$ coeff. of $y=3$

$$\text{Radius}=\sqrt{g^2+f^2-c}=\sqrt{(-2)^2+(3)^2-\left(\frac{3}{4}\right)}$$

$$=\sqrt{3+9-\frac{3}{4}}=\sqrt{\frac{49}{4}}=\frac{7}{2}$$

Example 38. *State the values of a and b if the equation*

$$ax^2+2bxy-2y^2+8x+12y+6=0$$

represents a circle. Substituting the values of a and b in the equation, find the centre and radius of the circle.

Solution. If the given equation is to be a circle, then $a=-2$, $b=0$, the equation of circle then becomes

$$-2x^2-2y^2+8x+12y+6=0$$

or

$$x^2+y^2-4x-6y-3=0$$

$$\text{Centre}=(-\tfrac{1}{2}\text{ coeff. of } x, -\tfrac{1}{2}\text{ coeff. of } y)=(2, 3)$$

$$\text{Radius}=\sqrt{2^2+3^2-(-3)}=4$$

Example 39. *Find the equation of the circle whose centre is (4, 5) and which passes through the centre of the circle*

$$x^2+y^2+4x+6y-12=0.$$

Solution. The centre of the given circle

$$x^2+y^2+4x+6y-12=0 \text{ is } (-2, -3)$$

Since the required circle passes through the centre of the given circle, radius of the required circle will be equal to distance between (4, 5) and (−2, −3).

$$\therefore \qquad r=\sqrt{(4+2)^2+(5+3)^2}=10$$

Hence the equation of the required circle is

$$(x-4)^2+(y-5)^2=100$$

Example 40. *Find the equation of the circle passing through the points (0, 0), (1, 2) and (2, 0).*

Solution. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(1)$$

Since this circle passes through the points (0, 0), (1, 2) and (2, 0), we have

$$c=0 \quad ...(2)$$

$$1+4+2g+4f+c=0 \text{ or } 2g+4f+5=0 \quad ...(3)$$

$$4+0+4g+0+c=0 \text{ or } 4g+4=0 \quad ...(4)$$

Solving (2), (3) and (4), we get

$$g=-1, f=-\frac{3}{4} \text{ and } c=0$$

$\therefore$ The equation of the required circle is

$$x^2+y^2+2(-1)x+2\left(-\frac{3}{4}\right)y+0=0$$

or $$x^2+y^2-2x-\frac{3}{2}y=0$$

or $$2x^2+2y^2-4x-3y=0.$$

Example 41. *Find the equation of the circle which passes through the points (4, 1) and (6, 5) and has its centre on the line $4x+y=16$.*

Solution. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad ...(1)$$

Since the circle passes through the points (4, 1) and (6, 5), therefore, we have

$$16+1+8g+2f+0=0 \text{ or } 8g+2f+c+17=0 \quad ...(2)$$

and $$36+25+12g+10f+c=0 \text{ or } 12g+10f+c+61=0 \quad ...(3)$$

Also the centre $(-g, -f)$ lies on the line $4x+y=16$, therefore, we have

$$-4g-f=16 \quad ...(4)$$

Solving (2), (3) and (4), we get

$$g=-3, f=-4 \text{ and } c=15$$

$\therefore$ The equation of the required circle is

$$x^2+y^2+2(-3)x+2(-4)y+15=0.$$

or $$x^2+y^2-6x-8y+15=0.$$

Example 42. *Find the equations of the circles passing through the origin, having its centre on x-axis and radius equal to 2 units.*

Solution. As the circle passes through the origin, its centre lies on x-axis and its radius is equal to 2, therefore, its centre is (2, 0) or $(-2, 0)$.

Hence the required circles are

$$(x-2)^2+(y-0)^2=2^2 \quad \text{or} \quad x^2+y^2-4x=0$$

and $$(x+2)^2+(y-0)^2=2^2 \quad \text{or} \quad x^2+y^2+4x=0$$

Example 43. *Find the equation of the line joining the centres of the two circles.*

$$x^2+y^2-2x+4y-1=0 \qquad ...(I)$$

$$x^2+y^2+2x-4y+1=0 \qquad ..(II)$$

Solution. Here

Centre of circle I, $C_1=(-\frac{1}{2}$ coeff. of x, $-\frac{1}{2}$ coeff. of $y)=(1, -2)$

Centre of circle II, $C_2=(-1, 2)$

$\therefore$ Equation of the line joining C_1 and C_2 is

$$y+2=\frac{2+2}{-1-1}(x-1), \text{ i.e., } y+2=-2x+2$$

$$\therefore \qquad y+2x=0$$

We now take up the general equation of circle through the intersection of two circles.

Let the given equations of circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad ...(1)$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad ...(2)$$

respectively.

Consider the equation

$$(x^2+y^2+2g_1x+2f_1y+c)+\lambda(x^2+y^2+2g_2x+2f_2y+c)=0 \qquad ...(*)$$

where λ is an arbitrary constant.

It is a second degree equation in x and y in which the co-efficients of x^2 and y^2 are equal and there is no term of xy.

$\therefore$ It represents a circle.

Let (x_1, y_1) be one of the points of intersection of (1) and (2).

$\because$ (x_1, y_1) satisfies both (1) and (2), we have

$$x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1=0 \qquad ...(3)$$

and

$$x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2=0 \qquad ...(4)$$

Adding λ times (4) to (3), we have

$$x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1+\lambda(x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2)=0$$

This equation shows that (x_1, y_1) lies on the locus of (*).

Similarly, if (x_2, y_2) be the other point of intersection of (1) and (2), it also satisfies (*).

$\therefore$ (*) represents the general equation of a circle passing through the points of intersection of (1) and (2).

It may be noted that in equation (*), λ is determined from the additional condition given in the problem.

Example 44. *Find the equation of the circle drawn on the line segment joining the points of the intersection of the circle $x^2+y^2=a^2$ and the straight line $x \cos \alpha+y \sin \alpha=p$ as a diameter.*

Solution. Equation of any circle passing through the ends of the chord intercepted on $x \cos \alpha+y \sin \alpha=p$ by the circle $x^2+y^2-a^2=0$ is

$$(x^2+y^2-a^2)+\lambda(x \cos \alpha+y \sin \alpha-p)=0 \qquad \ldots(*)$$

The centre of the circle is $\left(-\frac{\lambda}{2} \cos \alpha, -\frac{\lambda}{2} \sin \alpha\right)$.

Since the given chord is a diameter of the circle, the centre of the circle lies on the chord.

$$\therefore \qquad -\left(\frac{\lambda}{2} \cos^2 \alpha+\frac{\lambda}{2} \sin^2 \alpha\right)=p$$

$$\Rightarrow \qquad \lambda=-2p$$

Substituting the value of λ in (1), we get

$$(x^2+y^2-a^2)--2p(x \cos \alpha+y \sin \alpha-p)=0$$

as the equation of the required circle.

Example 45. *Find the equation of the circle passing through the points of intersection of the circles*

$$x^2+y^2=2ax \textit{ and } x^2+y^2=2by$$

and having its centre on the line $\frac{x}{a}-\frac{y}{b}=2.$

Solution. Any circle through the points of intersection of the given circle is

$$(x^2+y^2-2ax)+\lambda(x^2+y^2-2by)=0 \qquad \ldots(1)$$

$$\Rightarrow \qquad (1+\lambda)x^2+(1+\lambda)y^2-2ax-2by\lambda=0$$

$$\Rightarrow \qquad x^2+y^2-\frac{2ax}{1+\lambda}-\frac{2b\lambda}{1+\lambda}y=0$$

Centre of this circle is $\left(\frac{a}{1+\lambda}, \frac{b\lambda}{1+\lambda}\right)$

This centre will lie on the line $\frac{x}{a}-\frac{y}{b}=2$ if

$$\frac{a}{(1+\lambda)a}-\frac{b\lambda}{(1+\lambda)b}=2 \qquad \Rightarrow \lambda=-\frac{1}{3}$$

Hence the equation of the required circle is

$$x^2+y^2-\frac{2ax}{1-\frac{1}{3}}-\frac{2by}{1-\frac{1}{3}}\left(-\frac{1}{3}\right)=0$$

$$\Rightarrow \qquad x^2+y^2-3ax+by=0.$$

15·25. EQUATION OF A TANGENT

The equation of a tangent at any point (x_1, y_1) *on the circle*

$$x^2+y^2+2gx+2fy+c=0$$

is
$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

Proof. Let (x_2, y_2) be any other point on the circle.

(*i*) The equation of the line passing through (x_1, y_1) and (x_2, y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}\ (x-x_1) \qquad \ldots(1)$$

(*ii*) Since (x_1, y_1), (x_2, y_2) lie on the circle,

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \ldots(2)$$

and
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \ldots(3)$$

Subtracting (2) from (3), we get

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)+2g(x_2-x_1)+2f(y_2-y_1)=0$$

$$\Rightarrow \quad (x_2-x_1)(x_1+x_2+2g)+(y_2-y_1)(y_1+y_2+2f)=0$$

$$\Rightarrow \quad \frac{y_2-y_1}{x_2-x_1}=-\frac{x_1+x_2+2g}{y_1+y_2+2f}$$

(*iii*) Substituting this value of $\frac{y_2-y_1}{x_2-x_1}$ in (1), the equation of the chord through (x_1, y_1), (x_2, y_2) is

$$y-y_1=-\frac{x_1+x_2+2g}{y_1+y_2+2f}\ (x-x_1) \qquad \ldots(4)$$

(*iv*) Let $x_2 \to x_1$ and $y_2 \to y_1$, then from (4), the equation of tangent at (x_1, y_1) is

$$y-y_1=-\frac{x_1+g}{y_1+f}\ (x-x_1)$$

$$\Rightarrow \quad (y-y_1)(y_1+f)=-(x_1+g)(x-x_1)$$

$$\Rightarrow \quad yy_1+yf-y_1^2-y_1f=-xx_1+x_1^2-xg+x_1g$$

$$\Rightarrow \quad xx_1+yy_1+yf+xg=x_1^2+y_1^2+x_1g+y_1f$$

Adding y_1f+x_1g+c on both sides, we get

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=x_1^2+y_1^2+2x_1g+2y_1f+c=0$$

which is the required equation.

Example 46. *If the tangents at* (x_r, y_r) ; $r=1, 2$ *on the circles*

$$x^2+y^2+2gx+2fy+c=0$$

are perpendicular, prove that

$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2=0$$

Solution. The equation of the tangent at (x_1, y_1) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

$$\Rightarrow \quad x(x_1+g)+y(y_1+f)+(gx_1+fy_1+c)=0$$

$$\text{Its slope} = -\left(\frac{x_1+g}{y_1+f}\right)$$

The equation of tangent at (x_2, y_2) is

$$xx_2+yy_2+g(x+x_2)+f(y+y_2)+c=0$$

$$\text{Its slope} = -\left(\frac{x_2+g}{y_2+f}\right)$$

Since these tangents are perpendicular,

$$\frac{x_1+g}{y_1+f} \times \frac{x_2+g}{y_2+f} = -1$$

$$\Rightarrow \quad x_1x_2+gx_2+gx_1+g^2=-y_1y_2-y_2f-y_1f-f^2$$

$$\Rightarrow \quad x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2=0$$

Remark. *Case of standard circle.* The tangent at (x_1, y_1) of the circle

$$x^2+y^2=a^2 \text{ is } xx_1+yy_1=a^2$$

15·26. EQUATION OF A NORMAL

A normal line to a curve at a point is the line perpendicular to the tangent line at the point of contact.

The equation of tangent at (x_1, y_1) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

$$\therefore \quad \text{Slope of the tangent} = -\frac{x_1+g}{y_1+f}$$

$$\therefore \quad \text{The slope of the normal} = \frac{y_1+f}{x_1+g}$$

Hence the equation of normal at (x_1, y_1) is

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$$

$$\Rightarrow \quad y(x_1+g)-x(y_1+f)+x_1f-y_1g=0$$

15·27. EQUATION OF A TANGENT IN SLOPE FORM

Let $\quad y=mx+c \quad \ldots(1)$

be a tangent to the circle $x^2+y^2=a^2 \quad \ldots(2)$

Then we have to find c in terms of m from the condition that (1) is a tangent to (2).

Now, the abscissae of the points of intersection of (1) and (2) are given by the equation obtained by eliminating y between (1) and (2), *viz.*, by

$$x^2+(mx+c)^2=a^2$$

$$\Rightarrow \quad (1+m^2)x^2+2mcx+(c^2-a^2)=0 \qquad ...(3)$$

which is a quadratic equation in x showing that in general the line meets the curve in two distinct points, real or imaginary.

If the line is a tangent, the roots are equal since the tangent meets the curve in two coincident points.

$$\therefore \quad m^2c^2-(1+m^2)(c^2-a^2)=0, \qquad \text{(Discriminant should vanish)}$$

$$\Rightarrow \quad m^2a^2-c^2+a^2=0, \quad i.e., \quad c^2=a^2(1+m^2)$$

$$\therefore \quad c=\pm a\sqrt{1+m^2}$$

which is a condition for tangency.

$$\therefore \quad y=mx\pm a\sqrt{1+m^2}$$

are the required equations of the tangents, whatever be the value of m.

Alternative Method. The equation of the line is

$$y=mx+c \qquad ...(1)$$

and of the circle $\quad x^2+y^2=a^2 \qquad ...(2)$

If the line touches the circle, the length of the perpendicular from the centre (0, 0) on the line (1) is equal to the radius

$$\therefore \quad \pm\frac{c}{\sqrt{m^2+1}}=a \quad \Rightarrow \quad c=\pm a\sqrt{1+m^2}$$

Example 47. *Show that the line $3x+4y-20=0$ touches the circle $x^2+y^2=16$ and find the point of contact.*

Solution. Since $3x+4y-20=0$, $y=\dfrac{20-3x}{4}$ ...(1)

Substituting (1) in the equation of the circle, we have

$$x^2+\left(\frac{20-3x}{4}\right)^2-16=0, \; i.e., \; 25x^2-120x+144=0$$

$$\therefore \quad (5x-12)^2=0$$

which has two roots each equal to $\dfrac{12}{5}$

$\therefore$ The line touches the circle at the point where $x=\dfrac{12}{5}$

Substituting this value of x in (1), we get

$$y=\frac{20-\dfrac{36}{5}}{4}=\frac{16}{5}$$

$\therefore$ The required point of contact is $\left(\dfrac{12}{5}, \dfrac{16}{5}\right)$

Example 48 *Show that $8x+5y-34=0$ is a tangent to the circle $x^2+y^2+10x+6y-55=0$.*

Solution. *Method I.* Since $8x+5y-34=0$,

$$x=\frac{34-5y}{8}$$

Substituting (1) in the equation of the circle, we have

$$\left(\frac{34-5y}{8}\right)^2+y^2+\frac{10(34-5y)}{8}+6y-55=0$$

$$\Rightarrow \quad 89y^2-356y+356=0, \text{ i.e., } y^2-4y+4=0$$

$$\therefore \quad (y-2)^2=0.$$

Hence the line meets the circle in two coincident points and is, therefore, a tangent.

Method II. The circle can be written as

$$(x+5)^2+(y+3)^2=89$$

$\therefore$ The centre is $(-5, -3)$ and radius$=\sqrt{89}$

Now the perpendicular from $(-5, -3)$ on to $8x+5y-34=0$ is of length

$$\left|\frac{8(-5)+5(-3)-34}{\sqrt{8^2+5^2}}\right|=\sqrt{89}$$

$\therefore$ The distance of the line from the centre is equal to the radius hence the line is a tangent.

Example 49. *Show that the line $y=m(x-a)+a\sqrt{1+m^2}$ touches the circle $x^2+y^2=2ax$ for all values of m.*

Solution. The circle is

$$x^2+y^2-2ax=0 \qquad \ldots(1)$$

and the line is

$$y=m(x-a)+a\sqrt{1+m^2} \qquad \ldots(2)$$

The centre of (1) is $(a, 0)$ and its radius is a. (2) touches (1) if perpendicular distance from the centre is equal to radius.

$$=\left|\frac{m(a-a)+a\sqrt{1+m^2}}{\sqrt{1+m^2}}\right|=\left|\frac{a\sqrt{1+m^2}}{\sqrt{1+m^2}}\right|=a=\text{radius}.$$

Hence (2) touches (1) for all values of m.

Example 50. *Find the equation of tangent and normal at the point $(-2, 5)$ on $x^2+y^2+3x-8y+17=0$.*

Solution. The equation of the tangent at the point $(-2, 5)$ to the circle $x^2+y^2+3x-8y+17=0$

is
$$-2x+5y+\frac{3}{2}(x-2)-4(y+5)+17=0$$

$$\Rightarrow \quad x-2y+12=0$$

Now Slope of the tangent $= \frac{1}{2}$

$\therefore$ Slope of the normal $= -2$

Hence equation of the normal is

$$y-5=-2(x+2)$$

$$\Rightarrow \quad 2x+y=1$$

Example 51. *Find the equations of the tangents drawn from $(3, -1)$ to the circle $x^2+y^2=5$.*

Solution. Let the equation of any tangent to the circle $x^2+y^2=5$ be

$$y=mx \pm \sqrt{5}\,.\,\sqrt{1+m^2} \qquad \ldots(1)$$

Since the point $(3, -1)$ lies on (1),

$$1=3m \pm \sqrt{5}\,.\,\sqrt{1+m^2}$$

$$\Rightarrow \quad 1-3m=\pm\sqrt{5}\,.\,\sqrt{1+m^2}$$

$$\Rightarrow \quad 1+9m^2-6m=5+5m^2$$

$$\Rightarrow \quad 4m^2-6m-4=0, \text{ i.e., } 2m^2-3m-2=0$$

$$m=\frac{3 \pm \sqrt{9+16}}{4}=2, -\frac{1}{2}$$

Hence the equations of the tangents are

$$\therefore \quad y-1=2(x-3) \quad \text{or } y=2x-5 \qquad \ldots(2)$$

$$\text{and} \quad y-1=-\tfrac{1}{2}(x-3) \quad \text{or } x+2y=5 \qquad \ldots(3)$$

For finding the points of contact, we solve $y=2x-5$ and $x^2+y^2=5$ in case of (2) and solve $x+2y=5$ and $x^2+y^2=5$ in case of (3).

In case (2) : $$x^2+(2x-5)^2=5$$

$$\Rightarrow \quad x^2+4x^2-20x+25=5$$

$$\Rightarrow \quad x^2-4x+4=0, \text{ i.e., } x=2 \text{ and } y=-1$$

$\therefore$ The point of contact is $(2, -1)$

In case (3) ; $$(5-2y)^2+y^2=5$$

$$\Rightarrow \quad 25-20y+4y^2+y^2=5$$

$$\Rightarrow \quad y^2-4y+4=0$$

$$\Rightarrow \quad x=1 \text{ and } y=2$$

$\therefore$ The point of contact in this case is $(1, 2)$.

Example 52. *Find the equations of the tangents to the circle $x^2+y^2=4$,* (*i*) *which are inclined at an angle of 45° to the axis of x,*

(*ii*) *which are parallel to the line $2x-y+4=0$,*

(*iii*) *which are perpendicular to the line $3x+2y-5=0$.*

Solution. The equations of the tangents to the circle $x^2+y^2=4$, in the slope form are

$$y=mx\pm 2\sqrt{1+m^2}$$

(*i*) Since the tangents are inclined at an angle of 45° to the axis of x,

$$m=\tan 45°=1$$

∴ The equations of the tangents are

$$y=1\,.\,x\pm 2\sqrt{1+1}=x\pm 2\sqrt{2}$$

(*ii*) The tangents will be parallel to the line $2x-y+4=0$, if

$$m=2=\text{slope of the given line.}$$

Hence the required tangents are

$$y=2x\pm 2\sqrt{1+4} \quad \text{or} \quad 2x-y\pm 2\sqrt{5}=0$$

(*iii*) The tangents will be perpendicular to the line $3x+2y-5=0$, if

$$m=\frac{2}{3}$$

Hence the required tangents are

$$y=\frac{2}{3}\,x\pm 2\sqrt{1+\frac{4}{9}},\ i.e.,\ 2x-3y\pm 2\sqrt{13}=0.$$

15·28. ELLIPSE

An ellipse is a sort of elongated circle, it is formed by a locus of point which moves in such a way that the sum of its distances from two fixed points is always constant ; these points are called the foci of the ellipse. An ellipse is symmetric with respect to the two lines called its axes. These axes refer usually to the segments cut off on the usual axes by the ellipse and are called the major for the longer one and the minor for the shorter one.

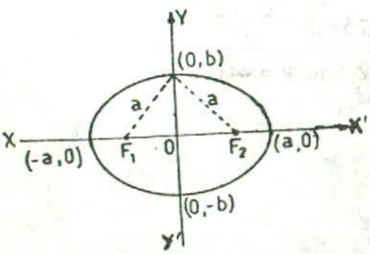


Fig. 39.

If the major and minor axes lie on the x-axis and y-axis respectively then the centre is at the origin and the equation in terms of cartesian coordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where a and b are the lengths of the semi-major and semi-minor axes.

It can be seen that the distance from an end of a minor axis to foci F_1 and F_2 is a. If c is the distance from the centre to any of the foci then the ratio c/a is the eccentricity of the ellipse. Two ellipse are similar if they have the same eccentricity.

The intersection of the axes is the centre of the ellipse. The vertices are the points where the ellipse cuts its major axis. The chord cutting the distance between foci and perpendicular to major axis are the latera recta (plural of latus rectum).

Example 53. *Construct the graph of an equation*

$$4x^2+9y^2=36.$$

Solution. Solving the equation for y, we have

$$y=\pm\tfrac{2}{3}\sqrt{9-x^2}$$

Assigning the values -3, -2, -1, 0 .. etc., to x we have the values of y tabulated below :

$x=$	-3	-2	-1	0	1	2	3
$y=$	0	$\pm1{\cdot}5$	$\pm1{\cdot}9$	±2	$\pm1{\cdot}9$	$\pm1{\cdot}5$	0

Plotting these points with reference to the two axis we get smooth curve as shown in the Fig. 40.

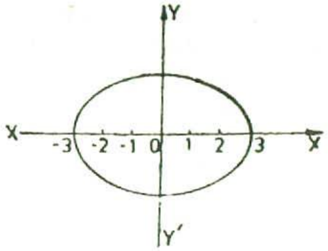


Fig. 40.

EXERCISE (III)

1. (a) $A(2, 0)$, $B(0, 3)$ are two points. Find the equation of the locus of P if $AP=3PB$. What does the locus represent geometrically ?

(b) The coordinates of A and B are $(3, -1)$ and $(2, 4)$ respectively. Find the equation of the locus of P if $2\,PA=3PB$.

2. (a) If $a\neq0$, show that

$$ax^2+ay^2+2gx+2fy+c=0$$

represents a circle. Find its centre and radius.

(a) State the conditions under which the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a circle and determine the centre and radius of the circle.

3. Obtain an equation of a circle of radius r touching both the coordinate axes. How many such circles are possible ? If the circle $x^2+y^2+2gx+2fy+c=0$ touches both the axes of coordinates, show that its radius is equal to $\sqrt{3}$.

4. If $x^2+y^2+kxy+8x-6y+9=0$ represents a circle, state the value of k. Substituting this value of k in the equation, find the centre and radius of the circle.

5. Write down the coordinates of the centre and the length of the radius of each of the following circles :

(i) $x^2+y^2+7x-9y-20=0$

(ii) $4(x^2+y^2)+12ax-6ay-a^2=0$

(iii) $\sqrt{1+a^2}\,(x^2+y^2)-2b(x+ay)=0.$

6. (a) Find the equation of the circle whose centre is $(2, -3)$ and passing through the point $(5, 1)$.

(b) Find the equation of the circle which is concentric with $x^2+y^2-8x+12y+43=0$ and (i) which passes through $(6, 2)$; (ii) has its radius equal to 7.

7. Find the equation of the circle

(i) whose centre lies on the x-axis and which passes through the points $(-1, 0)$ and $(5, 0)$

(ii) whose centre lies on the y-axis and which passes through the points $(0, 3)$ and $(0, -7)$.

8. (a) Find the equations of the circles passing through the following sets of points :

(i) $(1, 2)$ $(5, 7)$ and $(8, 6)$,

(ii) $(6, -8)$, $(-2, 9)$ and $(2, 1)$.

(a) Find the equation of the circle circumscribing the triangle formed by the lines $2x+y-3=0$, $x+y-1=0$ and $3x+2y-5=0$.

9. Show that the points $(2, 0)$, $(-1, 3)$, $(-2, 0)$ and $(1, -1)$ are concyclic, and determine the centre and the radius of the circle passing through them.

10. (i) Find the equation of the circle whose centre is $(-2, -5)$ and which passes through the centre of the circle $3x^2+3y^2+6x-9y+16=0$.

(ii) Find the equation of the circle passing through the point $(6, -9)$ and having its centre at $(3, -5)$. Find also the coordinates of the points of intersection with the x-axis.

11. Find the equation of the circle

(i) which passes through the points $(3, 2)$ and $(5, 4)$ and having its centre on the line $3x+2y=12$.

(ii) which passes through the points $(4, 5)$, $(6, -4)$ and having its centre on the axis of x.

(iii) whose radius is 3 units and which passes through the origin and has its centre on the x-axis.

(iv) passing through the origin, whose radius is 5 and whose centre lies on $3x-4y+15=0$.

12. Find the equations of the tangent and normal to the circle.

(i) $x^2+y^2=16$ at the point $\left(-\frac{12}{5}, -\frac{16}{5}\right)$

(ii) $2x^2+2y^2-2x-5y+3=0$ at $(1, 1)$.

(iii) $3x^2+3y^2-4x-9y=0$ at the origin.

13. Find the equations of the tangents to

(*i*) the circle $x^2+y^2=7$ which makes an angle of 60° with the x-axis.

(*ii*) the circle $x^2+y^2=7$ which is inclined at 45° to the x-axis.

14. Tangents are drawn to the circle $x^2+y^2=169$ at the points $(5, 12)$ and $(12, -5)$. Prove that they are perpendicular and find the points of their intersection.

15. Prove that the straight line $y=x+c\sqrt{2}$ touches the circle $x^2+y^2=c^2$, and find its point of contact.

16. (*a*) Find the condition that the straight line $3x+4y=k$ may touch the circle $x^2+y^2=10y$.

(*b*) Find the equation of the circle which has its centre at the point $(4, 3)$ and touches the straight line $5x-12y-10=0$.

17. Find the equations of the tangents to the circle $x^2+y^2=9$

(*i*) which are parallel to $2x+y-3=0$,

(*ii*) which are parallel to the axis of x,

(*iii*) which are parallel to the axis of y,

(*iv*) which are perpendicular to the line $3x-4y+6=0$

18. Find the equation of the circle which has its centre at the origin and touches the line $5x-12y+13=0$.

19. Show that the circles $x^2+y^2=2$ and $x^2+y^2-6x-6y+10=0$ touch one another at $(1, 1)$.

20. Find the equation of the circle passing through the points of intersection of the circles

$$x^2+y^2-2x-4y+3=0$$

and

$$x^2+y^2+4x+6y-4=0$$

and whose centre lies on the line $x+y+1=0$.

ANSWERS

1. (*b*) $5x^2+5y^2-12x-80y+140=0$. 2. (*a*) $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ $\sqrt{\frac{g^3}{a^3}+\frac{f^2}{a^2}-\frac{c}{a}}$. 3. $x^2+y^2\pm 2rx\pm 2ry+r^2=0$.

4. $k=0$, centre$=(-4, 3)$ and radius$=4$ 5. (*i*) $\left(-\frac{7}{2}, \frac{9}{2}\right)$ $\frac{\sqrt{210}}{2}$, (*ii*) $\left(-\frac{3a}{2}, \frac{3a}{4}\right)$, $\frac{7a}{4}$, (*iii*) $\left(\frac{b}{\sqrt{1+a^2}}, \frac{ab}{\sqrt{1+a^2}}\right)$; b

6. (*a*) (*i*) $(x-2)^2+(y+3)^2=25$, (*b*) (*i*) $x^2+y^2-8x+12y-16=0$ (*ii*) $x^2+y^2-8x+12y+3=0$, 7. (*i*) $(x-2)^2+y^2=9$, (*ii*) $x^2+(y+2)^2=25$ 8. (*ii*) $x^2+y^2+370x+175y-920=0$ (*b*) $x^2+y^2-13x-5y+16=0$ 10. (*i*) $x^2+y^2+4x+10y-\frac{57}{5}=0$, (*ii*) $x^2+y^2-6x+10y+9=0$ 11. (*i*) $x^2+y^2+4x-18y+11=0$,

(*ii*) $16(x^2+y^2)-88x=304$,

(*iv*) $x^2+y^2+10x=0$ or $5x^2+5y^2-14x-48y=0$.

12. (*i*) $3x+4y+20=0$, (*ii*) $2x-y-1=0$, $x+2y=3$, (*iii*) $4x+9y=0$. $2y=3x$. 13. (*i*) $y=\sqrt{3}x\pm2\sqrt{7}$, (*ii*) $y=x\pm2\sqrt{37}$

14. $5x+12y=169$, $12x-5y=169$, $(17, 7)$. 15. $\left(-\dfrac{c}{\sqrt{2}}, \dfrac{c}{\sqrt{2}}\right)$

16. (*b*) $x^2+y^2-8x-6y+21=0$. 17. (*i*) $2x+y\pm3\sqrt{5}=0$.

18. $x^2+y^2=1$, (*ii*) $y=\pm3$, (*iii*) $x=\pm3$, (*iv*) $4x+3y\pm15=0$.

15·29. PARABOLA

The locus of point which moves in a plane so that its distance from a given point is equal to its perpendicular distance from a given straight line is defined as a parabola.

The fixed point is called the focus and the fixed straight line is called the directrix of the parabola.

15·30. STANDARD EQUATION OF A PARABOLA

Let the given fixed point, the focus be S and the directrix is the line ZM. Draw SZ perpendicular to ZM and bisect SZ at O. The point O is on the locus and is called the vertex of the parabola. Take axis as shown with OS as the x-axis, O being the origin.

Let the given distance SZ be $2a$ so that S is the point $(a, 0)$ and the equation of the directrix ZM is $x=-a$, *i.e.*, $x+a=0$.

Let $P(x, y)$ be any point on the parabola. Draw PM perpendicular upon the directrix from the point P.

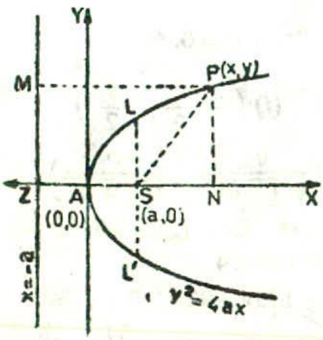


Fig. 41.

Now by def. of the parabola, $SP=PM$

$\Rightarrow \quad SP^2=PM^2$

$\Rightarrow \quad (x-a)^2+(y-0)^2=(x+a)^2$

$[\because \ PM=NZ=NA+AZ=x+a]$

$\Rightarrow \quad (x^2-2ax+a^2)+y^2=x^2+2ax+a^2$

$\Rightarrow \quad y^2=4ax$

which is the required equation.

We now give the following definitions :

(*i*) **Axis.** A straight line about which the curve is symmetrical is called an *axis* of the parabola.

(*ii*) **Vertex.** The point in which an axis of the parabola meets the curve is called a *vertex* of the parabola.

(*iii*) **Focal distance.** The distance of any point on the parabola from its focus is called the *focal distance.*

(*iv*) **Focal chord.** A chord of a curve passing through its focus is called the *focus chord.*

(*v*) **Focal axis.** The axis on which the focus of a curve lies is called *focal axis.*

(*vi*) **Double ordinate.** The double ordinate of a point P is the chord PP' of the curve, which is perpendicular to the axis.

(*vii*) **Latus rectum.** The focal chord of a curve perpendicular to its axis is called a *Latus Rectum.*

In Fig. 41, the double ordinate LSL' drawn through the focus is the latus rectum and SL, one half of it, the semi-latus rectum.

Since $AS=a$, from $y^2=4ax$, we get

$$SL^2=4a\ .\ a$$

$\therefore$ $\qquad SL=2a$ and the latus rectum is $4a$.

15 31. FORM OF PARABOLA

Form 1. $\mathbf{y^2=4ax}$

For this parabola, we have

(1) $S(a, 0)$ as the focus,

(2) directrix, the line is $x=-a$,

(3) axis, $y=0$

(4) vertex, $(0, 0)$

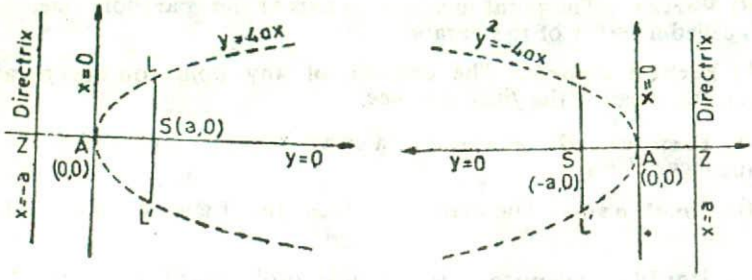
(5) tangent at the vertex, $x=0$,

(6) latus rectum, $4a$,

(7) extremities of Latus-Rectum $(a, \pm 2a)$.

Form II. $\mathbf{y^2=-4ax}$

If the directrix is a vertical straight line parallel to the y-axis and to the right side of the focus then the condition $SP=PM$, when expressed, gives the equation of the parabola in the form $y^2=-4ax$ and since it lies completely on the left hand side of the y-axis, it may be called the left handed parabola.



(i) (ii)

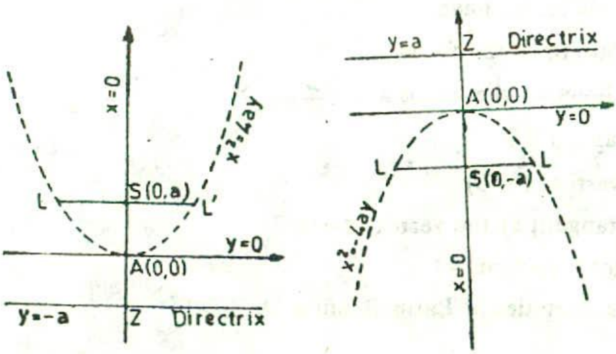
Fig. 42.

Form III. $x^2=4ay$

If the directrix is a horizontal straight line parallel to the x-axis and below the focus, then the condition $SP=PM$, when expressed, gives the equation of the parabola in the form $x^2=4ay$ and since it lies completely above the x-axis, it may be called the upward parabola.

Form IV. $x^2=-4ay$

If the directrix is a horizontal straight line parallel to the x-axis and above the focus, then the condition $SP=PM$, when expressed, gives the



(i) (ii)

Fig. 43.

equation of the parabola in the form $x^2=-4ay$ and since it lies completely below the x-axis, it may be called the down-ward parabola.

Parabola	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix	$x=-a$	$x=a$	$y=-a$	$y=a$
Axis	$y=0$	$y=0$	$x=0$	$x=0$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Tangent at the vertex	$x=0$	$x=0$	$y=0$	$y=0$
Latus Rectum	$4a$	$4a$	$4a$	$4a$

15·32. PARAMETRIC REPRESENTATION

The equation $y^2=4ax$ can be written as

$$\frac{y}{2a}=\frac{2x}{y}=t$$

$$\Rightarrow \qquad y=2at \text{ and } x=t\cdot\frac{y}{2}=\frac{t}{2}\cdot 2at=at^2$$

$\therefore$ the equations $x=at^2$, $y=2at$ taken together are called the parametric equations of the parabola $y^2=4ax$, t being the parameter.

Also $(at^2, 2at)$ are the parametric coordinates of any point on the parabola $y^2=4ax$.

Example 54. *A point moves in such a way that its distance from the point (2, 3) is equal to the distance from the line $4x+3y=5$. Find the equation of its path. What is the name of this curve ?*

Solution. Here the coordinates of focus are (2, 3) and the equation of directrix is $4x+3y=5$.

Let $P(x, y)$ be any point on the curve. Draw PM, a perpendicular from P on the directrix line.

Now $\qquad SP=PM$ (given)

i.e., $\qquad SP^2=PM^2 \qquad \ldots(1)$

We have $\qquad SP=\sqrt{(x-2)^2+(y-3)^2}$

$$PM=\frac{4x+3y-5}{\sqrt{4^2+3^2}}$$

$\therefore$ The equation of the locus of the point P is

$$(x-2)^2+(y-3)^2=\frac{(4x+3y-5)^2}{25}$$

$$\Rightarrow \quad 25(x^2-4x+4+y^2-6y+9)=16x^2+9y^2+25+24xy-40x-30y$$

$$\Rightarrow \quad 9x^2+16y^2-24xy-60x-120y+300=0$$

This equation is of the second degree wherein the terms second degree form a perfect square.

$\therefore$ The name of the curve is a *parabola.*

Example 55. *Find the equation of the parabola with its vertex at* (3, 2) *and its focus at* (5, 2).

Solution. We know that the vertex is the mid-point of the join of the foot of the directrix and the focus.

$\therefore$ If (x_1, y_1) is the foot of the directrix, then

$$3=\frac{x_1+5}{2} \text{ and } 2=\frac{y_1+2}{2} \Rightarrow x_1=1 \text{ and } y_1=2$$

Now the directrix is the line through (1, 2) and perpendicular to the axis.

$\therefore$ Equation of directrix is $x-1=0$

By definition, if $P(x, y)$ is any point on the parabola then PS is equal to the perpendicular distance of P from the directrix.

$$\therefore \quad \sqrt{(x-5)^2+(y-2)^2}=\frac{x-1}{1}$$

$$\Rightarrow \quad (x-5)^2+(y-2)^2=(x-1)^2$$

$$\Rightarrow \quad y^2-4y-8x+28=0$$

Example 56. *Find the focus, directrix, axis, vertex, tangent at the vertex and the latus rectum of the parabola.*

$$(y-k)^2=4a(x-h).$$

Solution. Let the origin be shifted to the point $A(h, k)$.

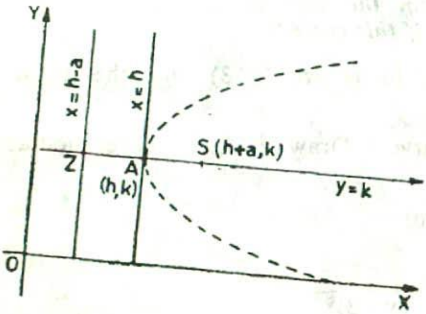


Fig. 44.

Let (X, Y) denote the current coordinates of a point with reference to the new set of axis. We have the transformation formulae

$$x=h+X,\ y=k+Y$$

The equation, with reference to the new set of axis, becomes

$$Y^2=4aX$$

With reference to the new set of axis

(*i*) Focus is given by

$$X=a,\ Y=0$$

(*ii*) directrix is $X=-a$

(*iii*) axis is $Y=0$,

(*iv*) vertex is $(0, 0)$

(*v*) tangent at the vertex is $X=0$,

(*vi*) latus rectum $=4a$.

$\therefore$ With reference to the original set of axis, we have

(*i*) focus : $(h+a, k)$, (*ii*) directrix : $x=h-a$, (*iii*) axis : $y=k$, (*iv*) vertex : (h, k), (*v*) tangent at the vertex : $x=h$, (*vi*) latus rectum : $4a$.

Example 57. *Find the focus directrix, axis, vertex and tangent at the vertex for the parabola* (*a*) $y^2+2y+4x+5=0$, (*b*) $x^2+20y+4x+56=0$.

Solution. (*a*) Completing the squares, the equation of the parabola can be written as

$$y^2+2y+1=-4x-5+1$$

$$\Rightarrow \quad (y+1)^2=-4(x+1)$$

Shifting the origin to the point $(-1, -1)$ and denoting the current coordinates with reference to the new set of axis by X, Y, we have $x+1=X$, $y+1=Y$ and the equation becomes

$$Y^2=-4X$$

For this parabola

(*i*) the latus rectum $4a=4$ and hence $a=1$

(*ii*) the coordinates of the focus are given by

$$Y=0,\ X=-a=-1$$

$$\Rightarrow \quad x=-2,\ y=-1$$

$\therefore$ The focus is at the point $(-2, -1)$.

(*iii*) the directrix is $X=a$

$$\Rightarrow \quad x+1=1,\ i.e.,\ x=0$$

$\therefore$ The equation of the directrix is $x=0$

(*iv*) the axis is $Y=0$, *i.e.*, $y+1=0$

(*v*) the vertex is given by $X=0$, $Y=0$

$$\Rightarrow \quad x+1=0,\ y+1=0$$

$\therefore$ The vertex is at the point $(-1, -1)$.

(*vi*) The tangent at the vertex is

$$X=0 \quad \Rightarrow \quad x+1=0$$

(*b*) Completing the squares, the equation of the parabola can be written as

$$x^2+4x+4=-20y+56+4$$

$$\Rightarrow \quad (x+2)^2=-20(y-3)$$

Shifting the origin to the point $(-2, 3)$ and denoting the current coordinates with reference to the new set of axis by X, Y, we have $x+2=X$, $y-3=Y$ and the equations becomes $X^2=-20Y$.

For this parabola

(*i*) the latus rectum $=4a=20$, *i.e.*, $a=5$.

(*ii*) The coordinates of the focus are given by

$$Y=-a,\ X=0$$

$$\Rightarrow \qquad y-3=-5,\ x+2=0$$

$\therefore$ The focus is at point $(-2, -2)$.

(*iii*) The directrix is $Y=a$, *i.e.*, $y-3=5$

$\therefore$ The equation of the directrix is $y-8=0$

(*iv*) The axis is $X=0 \quad \Rightarrow \quad x+2=0$

(*v*) The vertex is given by

$$X=0,\ Y=0$$

$$\Rightarrow \qquad x+2=0,\ y-3=0$$

$\therefore$ The vertex is at the point $(-2, 3)$

(*vi*) The tangent at the vertex is

$$Y=0 \quad \Rightarrow \quad y-3=0.$$

Example 58. *Find the coordinates of the points of intersection of the parabola $y^2=4x$ and the line $y+4=2x$. Also obtain the length of the chord so formed.* [*I.C.W.A., June 1990*]

Solution. The x-coordinates of the points of intersection are given by

$$(2x-4)^2=4x$$

or $$4x^2-16x+16=4x$$

or $$4x^2-20x+16=0$$

or $$x^2-5x+4=0$$

or $$(x-1)(x-4)=0$$

$$\therefore \qquad x=1, \quad \text{or} \quad x=4$$

When $x=1,\ y=2x-4=2-4=-2$

When $x=4,\ y=2x-4=8-4=4$

$\therefore$ $(1,-2)$ and $(4, 4)$ are the points of intersection.

$\therefore$ Length of the chord so formed

$$=\sqrt{(1-4)^2+(-2-4)^2}=\sqrt{9+36}=\sqrt{45}=3\sqrt{5}$$

15·33. EQUATION OF THE TANGENT

The equation of the tangent to the parabola $y^2=4ax$ at the point (x_1, y_1) is

$$yy_1=2a(x+x_1)$$

Let (x_2, y_2) be any other point on the parabola.

(*i*) The equation of the line passing through (x_1, y_1) and (x_2, y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \ldots(1)$$

(*ii*) $\because$ (x_1, y_1), (x_2, y_2) lie on the parabola

$$\therefore \qquad y_1^2=4ax_1 \qquad \ldots(2)$$

$$y_2^2=4ax_2 \qquad \ldots(3)$$

Subtracting (2) from (3), we get

$$y_2^2-y_1^2=4a(x_2-x_1)$$

$$\Rightarrow \qquad (y_2+y_1)(y_2-y_1)=4a(x_2-x_1)$$

$$\frac{y_2-y_1}{x_2-x_1}=\frac{4a}{y_2+y_1}$$

(*iii*) Substituting this value of $\frac{y_2-y_1}{x_2-x_1}$ in (1), the equation of the chord through (x_1, y_1), (x_2, y_2) is

$$y-y_1=\frac{4a}{y_2+y_3}(x-x_1) \qquad \ldots(4)$$

(*iv*) Let $x_2 \to x_1$, $y_2 \to y_1$, then from (4), the equation of the tangent at (x_1, y_1) is

$$y-y_1=\frac{2a}{y_1}(x-x_1)$$

$$\Rightarrow \qquad yy_1-y_1^2=2ax-2ax_1$$

$$\Rightarrow \qquad yy_1=4ax_1+2ax-2ax_1 \qquad (\because \; y_1^2=4ax_1)$$

$$\Rightarrow \qquad yy_1=2a(x-x_1)\text{, which is the required equation.}$$

Also the equation of tangent in the *parametric form* is

$$y \,.\, 2at=2a(x+at^2)$$

$$yt=x+at^2$$

15·34. EQUATION OF THE TANGENT IN SLOPE FORM

Let the straight line $y=mx+c$ ...(1)

meets the parabola $y^2=4ax$ ...(2)

Solving the two equations, we get

$$(mx+c)^2=4ax$$

$$\Rightarrow \qquad m^2x^2+2(mc-2a)x+c^2=0,$$

a quadratic equation in x showing that in general the line meets the curve in two distinct points real or imaginary.

If the line is a tangent, the roots are equal since a tangent meets the curve in two coincident points.

$$\therefore \qquad 4(mc-2a)^2=4m^2c^2,$$

This reduces to

$$mc=a, \quad i.e., \quad c=\frac{a}{m}$$

Hence the condition that the line $y=mx+c$ be a tangent to the parabola

$$y^2=4ax \text{ is } c=\frac{a}{m}$$

We can therefore substitute for c in $y=mx+c$ and say that

$$y=mx+\frac{a}{m}$$

is always a tangent to $y^2=4ax$ for all values of m.

15·35. EQUATION OF NORMAL

Equation of tangent at (x_1, y_1) is

$$yy_1=2a(x+x_1)$$

$$\text{Slope of tangent}=-\frac{-2a}{y_1}=\frac{2a}{y_1}$$

$$\therefore \quad \text{Slope of normal}=-\frac{y_1}{2a}$$

Since the normal must pass through (x_1, y_1), we have the required equation of the normal as

$$y-y_1=-\frac{y_1}{2a}\,(x-x_1)$$

or

$$xy_1+2ay=x_1y_1+2ay_1$$

15·36. EQUATION OF THE NORMAL IN SLOPE FORM

We know that the equation of normal at (x_1, y_1) is

$$y-y_1=-\frac{y_1}{2a}\,(x-x_1)$$

Let the slope of normal $=-\dfrac{y_1}{2a}=m$, we get

$$y_1=-2am \text{ and therefore } x_1=\frac{y_1^2}{4a}=am^2$$

On substituting for x_1 and y_1, the equation of the normal reduces to

$$y+2am=m(x-am^2) \quad \Rightarrow \quad y=mx-2am-am^3$$

which is the equation of the normal at the point $(am^2, -2am)$ on the parabola $y^2=4ax$.

Example 59. *Prove that $y=2x+2$ touches $y^2=16x$.*

Solution. Comparing $y^2=16x$ with $y^2=4ax$, we note that $a=4$.

Also $y=mx+\frac{4}{m}$ is always a tangent to $y^2=16x$.

Since $m=2$, we find that $y=2x+2$ is a tangent.

The student should, however, note the general method.

Since $\quad y=2x+2$ cuts $y^2=16x$

$$(2x+2)^2=16$$

$$\therefore \quad x^2-2x+1=0$$

$$\Rightarrow \quad (x-1)^2=0$$

The two values of x are equal and, therefore, the line is a tangent.

Example 60. *Find the tangents common to $x^2+y^2=8$ and $y^2=16x$.*

Solution. Any tangent to $y^2=16x$ is $y=mx+\frac{4}{m}$

when this meets $x^2+y^2=8$, we get

$$x^2+\left(mx+\frac{4}{m}\right)^2=8$$

$$\Rightarrow \quad x^2(1+m^2)+8x+8\left(\frac{2}{m^2}-1\right)=0$$

For tangency, the roots of this are equal, therefore

$$64=4(1+m^2)\times 8\left(\frac{2}{m^2}-1\right)$$

$$m^4+m^2-2=0, \text{ i.e., } (m^2+2)(m^2-1)=0$$

$$\Rightarrow \quad m^2=1 \text{ and } -2 \text{ (inadmissible)}$$

$$\therefore \quad m=\pm 1$$

Hence there are only two real tangents common to both curves

$$y=x+4 \text{ and } y=-x-4$$

EXERCISE (IV)

1. Define a parabola, its focus and directrix. Find the equation of the parabola whose focus is the point (5, 1) and whose directrix is the line $3x-4y+5=0$.

2. A point moves in such a way that its distance from the point (2, 5) is equal to the distance from the line $2x+4y-3=0$. Find the equation of its path. What is the name of the curve ?

3. (*a*) Find equation of the parabola whose focus is the point (a, b) and the directrix is the line $\frac{x}{a}+\frac{y}{b}=1$.

(*b*) Find the equation of the parabola with vertex at the origin having its axis along the x-axis and passing through the point (2, 3).

4. (*a*) Find the equation of the parabola whose focus is $(1, -1)$ and vertex (2, 1).

(*b*) Obtain the equation of the parabola whose vertex is at $(a, 0)$ and focus at $(b, 0)$; $b \neq a$.

5. Find the coordinates of the focus, vertex, the equation of the directrix and axis of the parabola :

(*a*) $y^2-4y-6x+22=0$

(*b*) $3x^2+12x-8y=0$

(*c*) $5x^2+30x+2y+59=0$

(*d*) $(y-\beta)^2=4a(x-\alpha)$

(*e*) $(x-\alpha)^2=4a(y-\beta)$

6. Write down the equations of tangent and normal

(*a*) at the point (6, 6) on $y^2=6x$

(*b*) at the ends of the latus rectum on $y^2=3x$.

7. Find the tangent to $y^2=2x$ which is parallel to the line $y=x+3$. Find also the point of contact.

8. Find the tangents to $y^2=9x$ which pass through the point (4, 10).

9. Find the equation of normal to $y^2=8x$, perpendicular to the line $2x+6y-5=0$. Find the foot of normal also.

10. Find the equation of the tangent to the parabola $y^2=8x$ which makes an angle of 45° with the x-axis.

ANSWERS

1. $16x^2+9y^2+24xy-280x-10y+625=0$.

2. $(2x+4y-3)^2=20[(x-2)^2+(y-5)^2]$, equation of a parabola.

3. (*a*) $(ax-by)^2-2a^3x-2b^3y+a^4+a^2b^2+b^4=0$, (*b*) $2y^2-9x=0$

4. (*a*) $4x^2+y^2-4xy+8x+46y-71=0$

(*b*) **Hint.** The distance between focus and the vertex is $b-a$. Then the directrix is the line perpendicular to the axis of the parabola and is at the same distance from the vertex as is the focus.

Let x_1 be a point on the x-axis and at a distance of $b-a$ from a, then

$$(a-x_1)=b-a \quad \Rightarrow \quad x_1=2a-b$$

$\therefore$ The equation of the directrix is $x=2a-b$

Then by focus-directrix property,

$\Rightarrow \quad \sqrt{(x-b)^2+y^2}=(x-2a+b)$

$\Rightarrow \quad (x-b)^2+y^2=(x-2a+b)^2$

$\Rightarrow \quad x^2-2bx+b^2+y^2=x^2+4a^2+b^2-4ax+2bx-4ab$

$\Rightarrow \quad y^2=4(b-a)(x-a)$ (on simplification)

5. (*a*) $\left(\frac{9}{2}, 0\right)$, $(3, 2)$, $2x-3=0$ and $y-2=0$

(*c*) $\left(-3, \frac{71}{10}\right)$, $(-3, 7)$, $y-\frac{69}{10}=0$, $x+3=0$

(*d*) $(a+\alpha, \beta)$, (α, β), $x=\alpha-a$, $y=\beta$

(*e*) $(\alpha, a+\beta)$, (α, β), $y=\beta-a$, $x=\alpha$.

6 (*a*) $2y=x+6$, $y+2x=18$, (*b*) $y=x+\frac{3}{4}$, $y+x=\frac{9}{4}$;

$y+x+\frac{3}{4}=0$, $y=x-\frac{9}{4}$.

7. $y=x+\frac{1}{2}$, $(\frac{1}{2}, 1)$.

8. $4y=x+36$, $4y=9x+4$.

9. $y=3x-66$, $(18, -12)$.

10. $y=x+2$.