16

Functions, Limits and Continuity

STRUCTURE

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OBJECTIVES

After studying this chapter, you should be able to understand :

- functions, mapping, notations for functions and types of function
- the concept of limit of a function
- the continuous functions at a point and in an interval.

16·1. FUNCTIONS

A function is a technical term used to symbolise relationship between variables. When two variables are so related that, for any arbitrarily assigned value to one of them, there corresponds a definite value (or a set of definite values) for the other, the second variable is said to be the function of the first.

For example, the distance covered is a function of time and speed, the railway freight charged is a function of weight or volume, the quantity demanded or supplied is a function of price, etc. The area of a circle depends upon the length of its radius and so the area is said to be the function of radius.

Thus, a function explains the nature of correspondence between variables indicated by some formula, graph or a mathematical equation. We can say that a function is some sort of a mould which gives some unique shape to the material poured into it. It should be remembered that it is not the degree of variation but the way the variables move in a given function which is of significance However, a functional relationship should not be confused with any causal (cause and effect) relationship. It is purely a mathematical relationship.

The idea of function is sometimes expressed as :

The relationship between two real variables say x and y which are so related that corresponding to every value 'a' of x defined is the domain, we get a finite value 'b' of y defined as the range then y is said to be the function of x.

The domain of variation of x in a function is called the domain of the definition.

The fact that y is a function of a variable x is expressed symbolically by equations :

$$y=f(x),\ y=F(x),\ y=\varphi(x), \text{ etc.}$$

The set of the values of x belonging to the domain of a function generate another set which consists of values of $y\{$ or $f(x)\}$. This generated set is called the range of the function $f(x)$.

If a is any particular value of x, the value of the function $f(x)$ for $x=a$ is denoted by $f(a)$.

If the relation between x and y is such that the value of y is nxed as soon as a definite value is assigned to x then the variable y is said to be dependent variable and x is the independent variable. The independent variable is sometimes called the argument of the function.

We shall now clarify the above concept of function by following examples.

(*i*) If y is always equal to x^2, then y is a function of x and we write

$$y=x^2 \qquad [\text{here } f(x)=x^2]$$

Similarly $\cos x$, e^x, $\log x$, $(x+a)^n$ etc., are all functions of x.

(*ii*) If v is defined by saying that

$$\begin{cases} y=x^2, & \text{when } x>2 \\ y=x-1, & \text{when } x\leqslant 2 \end{cases}$$

Here y is defined as a function of x but two formulae have been used to define the function, one holds for one part of the domain and the other for the remaining part of the domain.

(*iii*) Consider the two numbers x and y with their relationship defined by the equations :

$$\begin{cases} y=x^2, \text{ when } x<0 \\ y=x, \text{ when } 0\leqslant x\leqslant 1 \\ y=\dfrac{1}{x} \text{ when } x>1 \end{cases}$$

The domain of definition of this function which is expressed by three formulae is the whole set of real numbers. The first formula is used for the domain of all real numbers less than 0. The second formula is used for the domain of all real numbers lying between 0 and 1. The third formula is used for the domain of all those real numbers which are greater than 1.

(*iv*) Let $y=x\,!$

Here y is a function of x defined for aggregate of positive integers only.

(*v*) Let $y=|x|$

Here y is a function of x defined for the entire field of real numbers. The same function can also be defined as follows :

$$y=x, \text{ when } x\geqslant 0$$
$$y=-x, \text{ when } x<0$$

16·2. MAPPING

In modern mathematics, the equivalent expression for a function is *mapping*.

Definition. *If f is a rule which associates every element of set X with one and only one element of set Y, then the rule f is said to be the function or mapping from the set X to the set Y. This we write symbolically as*

$$f: X \to Y$$

If y is the element of Y, corresponding to an element x of X, given by rule f, we write this as yfx or $y=f(x)$ and read as 'y is the value of at x'.

16·3. NOTATIONS FOR FUNCTIONS

The mere fact that a quantity is a function of a single variable, say, x, is indicated by writing the function in one of the forms $f(x)$, $F(x)$, $\phi(x)$, $g(x)$...$f_1(x)$, $f_2(x)$,... If one of these occurs alone, it is read "a function of x" or "some function of x", if some of these are together, they are read "the f-function of x", "the F function of x", "the phi-function of x",...... The latter y is often used to denote a function of x.

The fact that a quantity is a function of several variables, say $x, y, z,$... it is indicated by means of several symbols, $f(x, y)$, $\phi(x, y)$, $F(x, y, z)$, $\psi(x, y, z, u,......)$ These are read as "the f function of x and y", "the phi-function of x and y", "the F-function of x, y and z," etc.

Sometimes the exact relation between the function and the dependent variable (or variables) is stated, as for example

$$f(x)=x^2+3x-7, \text{ or } y=x^2+3x-7,\ F(x, y)=2e^x+7e^y+xy-1$$

In such cases the f-function of any other number is obtained by substituting that number for x in $f(x)$, and the F-function of any two numbers is obtained by substituting those for x and y respectively in $F(x, y)$. Thus

$$f(z)=z^2+3z-7, f(4)=4^2+3\times4-7=21$$
$$f(t, z)=2e^t+7e^z+tz-1,\ F(2, 3)=2e^2+7e^3+5$$

(*i*) **Constants.** *The symbols which retain the same value throughout a set of mathematical operations are called constants.*

It has become customary to use initial alphabets a, b, c,...as symbols for constants. These are of two types :

(*a*) *Absolute constants.* Those which have the same value in all operations and discussions For example π, $\sqrt{2}$,etc., e are absolute constants.

(*b*) *Arbitrary Constants.* Those which may have any *assigned* value throughout a set of mathematical operations.

For example, the radius of a circle or the sides of a right angled triangle in forming the trigonometric ratios are the arbitrary constants.

(*ii*) **Variables.** If a symbol x denotes any element of a given set of numbers, then x is said to be a variable.

The last few alphabets x, y, z or u, v, w,... are generally used to denote variables.

The variables which can take arbitrarily assigned values are usually termed as *independent variables.* The other variables whose values must be determined in order that they may correspond to these assigned values are usually termed as *dependent* variables. It will be seen later that a "function" and "dependent variable" are synonymous terms.

(*iii*) **A Continuous Real Variable.** If x assumes successively every numerical value of an aggregate of all real numbers from a given number 'a' to another given number 'b', then x is called a 'continuous real variable'.

(*iv*) **A Domain Interval.** If a variable x which can take only those numerical values which lie between two given numbers a and b then all the numerical values between a and b taken collectively is called domain or interval of the variable x and is usually denoted by (a, b).

If the set of values say x is such that $a \leqslant x \leqslant b$ then the domain or interval (a, b) is called a closed domain or interval. In this case the number a and b are also included in the domain.

If the set of x is such that $a<x<b$ then it is called an open domain interval which is denoted by $[a, b]$ to distinguish it from (a, b). Here the numbers a and b do not belong to the domain.

We may also have semi-closed intervals like.

$$(a, b] \text{ or } a \leqslant x < b\ ;\ [a, b) \text{ or } a < x \leqslant b$$

The first interval is closed on the left and the second one is closed on the right.

We may have domains of variation extending without bound in one or the other direction, which we write

$$(-\infty, b) \text{ or } x \leqslant b \text{ ; } (a, \infty) \text{ or } x \geqslant a \text{ ; } (-\infty, \infty) \text{ or any } x.$$

16·4 TYPES OF FUNCTIONS

We shall now introduce some different types of functions which are particularly useful in calculus.

I One Valued Functions. When a function has only one value corresponding to each value of the independent variable, the function is called a *one valued fuuction*. If it has two values corresponding to each value it is called a *two valued function*. In case a function has several values corresponding to each value of the independent variable, it is called a *multiple valued function*, or a *many valued function, e.g.*,

(*i*) If $y=x$, y is a single valued function of x,

(*ii*) If $y=\sqrt{x}$, y is a two valued function of x ($+\sqrt{x}$ and $-\sqrt{x}$).

II. Explicit Functions. A function expressed directly in terms of the dependent variable is said to be an *explicit function, e.g.*, $y=x^2+2x-5$.

In it one of the variables is dependent on the other and the relationship is not mutual so that the other could be expressed as a dependent variable.

The function which is not expressed directly in terms of the dependent variable there is a mutual relationship between two variables and either variable determines the other, is said to be an implicit function, *e.g.*,

(*i*) $2x-3y=0$ then $x=\frac{3}{2}\,y$ and $y=\frac{2}{3}\,x$

(*ii*) $x^2+y^2=16$ so that $x=\pm\sqrt{16-y^2}$ and $y=\pm\sqrt{16-x^2}$

(*iii*) $x^2-y^2-6x-8y-7=0$ so that

$$y=\pm(x-3)-4 \text{ and } x=\pm(y+4)+3.$$

III. Algebraic and Transcende tal Functions. Functions may also be classified according to the operations involved in the relation connecting a function and its dependent variable (or variables). When the relation which involves only a finite number of terms and the variables are affected only by the operations of addition, subtraction, multiplication, division, powers and roots, the function is said to be an *algebraic function*.

Thus $2x^3+3x^2-9$, $\sqrt{x}-\frac{1}{x^3}$ are algebraic functions of x.

All the functions of x which are not algebraio are called transcendental functions. We have the following sub-classes of transcendental functions ·

(*i*) Exponential Functions. (*ii*) Logarithmic Functions,

(*iii*) Trigonometric Functions. (*iv*) Inverse Trigonometric Functions.

Functions, *e.g.*, cos x, tan $(x+x)$, $\sin^{-1} xe^{2x}$ log x and log $(4x+5)$ are transcendental functions of x.

IV. Rational and Irrational Functions. Expressions involving x which consist of a finite number of terms of the form ax^n, in which 'a' is a constant and n a positive integer, *e.g.*,

$$4x^4+3x^3-2x^2+9x-7$$

is called a *rational integral function* of x.

When an expression having more than two terms but only one variable it is called *polynomial* in x. For example in

$$a_0x^m+a_1x^{m-1}+\ldots\ldots+a_m$$

where a_0, a_1, a_2,......, a_m are constants and m is the degree of the polynomial.

If an expression in x, in which x has positive integral exponents only and a finite number of terms including the division by a rational integral function of x, it is a called a rational function of x, *e.g.*,

$$\frac{x^2+6}{3x^2+9}, \quad \frac{7x^2-4x+7}{(x+5)(2x-3)}, \frac{\frac{x-2}{x+7}}{4x^3+5}+8x-9$$

Rational integral functions and rational functions are included in *rational* functions.

An expression involving x which involves root extraction of terms is called an *irrational function*, *e.g.*,

$$\sqrt{x}, \quad \sqrt{x^2+4x+5}+9x-7$$

We can say, a rational function is an algebraic expression which involves no variable in an irreducible radical form (or under a fractional exponent) ; a function which can be written as a quotient of polynomials. The expressions $2x^2+1$ and $2+\frac{1}{x}$ are rational but $\sqrt{x+1}$ and $x^{3/2}+1$ are not.

V. Monotone Functions. When the dependent variable increases with an increase in the independent variable, the function is called a monotonically increasing function. For example, the function of supply is a monotonically increasing function of price. As against this a demand functions is a monotonically decreasing function of price because the quantity demanded decreases with every increase in price. A function $y=f(x)$ is called a monotone increasing function in an interval if a larger value of x gives a larger value of y, *i.e.*, an increase in x causes an increase in y in an interval. Similarly a function is called a monotone decreasing function in an interval if an increase in the value of x always brings out a decrease in the value the function in the interval. Thus, if x_1 and x_2 are only two numbers in the interval such that $x_2 > x_1$, then

$f(x)$ is monotonically increasing if $f(x_2) > f(x_1)$

and $f(x)$ is monotonically decreasing if $f(x_2) < f(x_1)$.

VI. Even and Odd Functions. If a function $f(x)$ is such that

$$f(-x) = f(x)$$

then it is said to be an *even function* of x, *e.g.*, x^4, $5x^2$, $7x^2 + \cos x$ are all even functions of x.

Now if a function $f(-x)$ is such that

$$f(-x) = -f(x)$$

then it is said to be an *odd function* of x, *e.g.*, x^3, $5x + 6x^3$, $\sin x$, are all odd functions of x.

VII. Periodic Functions. A function such that the range of the independent variable can be separated into equal sub-intervals such that the graph of the function is the same in each part interval. The length of the smallest such part is the period. Technically if $f(x+p) = f(x)$ for all x or $f(x)$ and $f(x+p)$ are both undefined, then p is the period of f. For example, the trigonometric function of sine has period 2π radians, since

$$\sin (x + 2\pi) = \sin x \text{ of all } x$$

VIII. Composite Functions. If $y = g(u)$ and $u = f(x)$ then $y = g\{f(x)\}$ is called a function of function or a *composite function*. For example the volume of a cylindrical water tank is the function of area and depth. while, area itself is a function of the radius ($\because A = \pi r^2$), so the function of the volume is the function of the area.

IX. Inverse Functions. If $y = f(x)$, defined in an interval (a, b), is a function such that we can express x as a function of y, say $x = \phi(y)$, then $\phi(y)$ is called the inverse of $f(x)$, *e.g.*,

(*i*) If $y = \dfrac{5x+3}{2x+9}$, then $x = \dfrac{3-9y}{2y-5}$ is the inverse of the first function.

(*ii*) $y = \sin^{-1} x$ is the inverse function of $x = \sin y$

(*iii*) $x = \sqrt[3]{y}$ is inverse function of $y = x^3$.

X. Continuous and Discontinuous Functions. A discussion on an exceedingly important classification of functions is given in the next section on limits of a function.

16·5. LIMIT OF A FUNCTION

The limit of a function is that fixed value to which a function approaches as the variable approaches a given value. The function approaches this fixed constant in such a way that the absolute value of the difference between the function and the constant may be made smaller and smaller than any positive number, however small. This difference continues to remain less than this assigned number say ε when the variable approaches still nearer to the particular value chosen for it.

The limit of a function say l is then that value to which a function $f(x)$ approaches, as x approaches a given value say a. In other words, as x reaches closer and closer to a, the function $f(x)$ reaches closer and closer to l so that given a positive number ε (epsilon), however small, we can find a number $\delta = | f(x) - l |$ such that $\delta < \varepsilon$ as x approaches closer and closer to a.

Def. *If corresponding to a positive number ε, however small, we are able to find a number δ such that*

$f(x) - l | < \varepsilon$ for all values of x satisfying $| x - a | < \delta$ then we say that $f(x) \to l$ as $x \to a$ and write this symbolically as

$$\lim_{x \to a} f(x) = l$$

It should be remembered that the function may not actually reach the limit l but it may get closer and closer to l as x approaches a so that $| f(x) - l |$ is less than any given value. For example, let us have a function $f(x) = x^2 - 2$. The function approaches the limit 7 as x approaches 3, we can express it is $\lim_{x \to 3} (x^2 - 2) = 7$ This can be shown below first with x approaching closer and closer to 3 from the lower side :

when $x = 2{\cdot}99$	$f(x) = 6{\cdot}9401$
when $x = 2{\cdot}999$	$f(x) = 6{\cdot}994001$
when $x = 2{\cdot}9999$	$f(x) = 6{\cdot}99940001$

Now when x approaches 3 from the higher side, we have

when $x = 3{\cdot}01$	$f(x) = 7{\cdot}0601$
when $x = 3{\cdot}001$	$f(x) = 7{\cdot}006001$
when $x = 3{\cdot}0001$	$f(x) = 7{\cdot}00060001$

It is evident from the above that as x is taken closer and closer to 3, $f(x)$ moves closer and closer to 7.

16·6. METHODS OF EVALUATING LIMIT OF A FUNCTION

In this section we shall give the various methods of finding the limits. The following are some theorems on the limits which are often used for evaluating the limits of a function. The proofs are, however, beyond the scope of the book.

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} \phi(x) = B$, then

I. $\lim_{x \to a} [f(x) \pm \phi(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} \phi(x) = A \pm B$

This can be extended to any finite number of functions.

II. $\lim_{x \to a} [f(x) \cdot \phi(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} \phi(x) = AB$

This can also be extended to any finite number of functions.

III. It obviously follows that

$$\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = k.A$$

IV. $\lim_{x \to a} \dfrac{f(x)}{\phi(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)} = \dfrac{A}{B}$, where $\lim_{x \to a} \phi(x) \neq 0$

V. $\lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{\lim.\ _{x \to a} f(x)} = \dfrac{1}{A}$ provided $\lim_{x \to a} f(x) \neq 0$

VI. $\lim_{x \to a} \log f(x) = \log \lim_{x \to a} f(x) = \log A$

16·7. SOME IMPORTANT LIMITS

I. $\lim_{x \to 0} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

II. $\lim_{n \to 0} (1+n)^{1/n} = \lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n = e$

III. $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x} = n$

IV. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

Left Hand Side and Right Hand Side Limits

$$\lim_{x \to a-} f(x) = \lim_{h \to 0} f(a-h) = \text{limit of } f(x),$$

when x approaches 'a' from the L.H.S.

also $$\lim_{x \to a+} f(x) = \lim_{h \to 0} f(a+h) = \text{limit of } f(x),$$

when x approaches 'a' from the R.H.S.

Therefore, to find the left hand side limit, we write $a-h$ for x in $f(x)$ and take the limit as $h \to 0$. Similarly to find the right hand side limit, we write $a+h$ for x in $f(x)$ and take the limit as $h \to 0$, where h is always positive.

Example 1. *Find the behaviour of* $\dfrac{1}{x}$ *as* $x \to 0$ *from the left hand side as well as from the right hand side.*

Solution. For L.H.S., we have

$$\lim_{x\to 0-} \frac{1}{x} = \lim_{h\to 0} \frac{1}{0-h} = \lim_{h\to 0} \left(\frac{-1}{h}\right) = -\infty$$

and
$$\lim_{x\to 0+} \frac{1}{x} = \lim_{h\to 0} \frac{1}{0+h} = \lim_{h\to 0} \left(\frac{1}{h}\right) = +\infty$$

Example 2. *Find* $\lim_{x\to a} e^{\frac{1}{x-a}}$ *when* $x\to a$ *from the left hand as well as from the right hand side.*

Solution. For L.H.S. limit, we have

$$\lim_{x\to a-} e^{\frac{1}{x-a}} = \lim_{h\to 0} e^{\frac{1}{a-h-a}} = \lim_{h\to 0} e^{-\frac{1}{h}}$$

$$= \lim_{h\to 0} \frac{1}{e^{\frac{1}{h}}} = 0 \text{ for } \frac{1}{h} \to \infty \text{ as } h\to 0$$

For the R.H.S. limit, we have

$$\lim_{x\to a+} e^{\frac{1}{x-a}} = \lim_{h\to 0} e^{\left(\frac{1}{a+h-a}\right)} = \lim_{h\to 0} e^{\frac{1}{h}} = \infty$$

Example 3. *Find the limit of* $\frac{4x^4+3x^2-1}{x^3+7}$ *when* $x\to 1$.

Solution. We have to find $\lim_{x\to 1} \frac{4x^4+3x^2-1}{x^3+7}$

Substituting $x=1$ in the expression we find that it comes out to be a definite number $\frac{6}{8}$. Hence the required limit is $\frac{3}{4}$.

Example 4. *Evaluate* $\lim_{x\to 3} \frac{x^2+2x-15}{x^2-9}$

Solution. Replacing x by 3 in the expression, we get $\frac{0}{0}$, which is indeterminate, $x-3$ must therefore be a factor of the numerator as well as of the denominator.

Factorising, we get

$$\lim_{x\to 3} \frac{x^2+2x-15}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x+5)}{(x-3)(x+3)}$$

$$= \lim_{x \to 3} \frac{x+5}{x+3} = \frac{8}{6} = \frac{4}{3}, \text{ by putting } x=3.$$

Example 5. *Prove that* $\lim_{x \to 0} \frac{\sqrt{a+x^2} - \sqrt{a-x^2}}{x^2} = \frac{1}{\sqrt{a}}$.

Solution. We find that if we put $x=0$, we get $\frac{0}{0}$. In such cases rationalising the numerator, we have

$$\lim_{x \to 0} \frac{\sqrt{a+x^2} - \sqrt{a-x^2}}{x^2} = \lim_{x \to 0} \left\{ \frac{a+x^2-a+x^2}{x^2 \left[\sqrt{(a+x^2)} + \sqrt{(a-x^2)}\right]} \right\}$$

$$= \frac{2}{\sqrt{a+0} + \sqrt{a-0}} = \frac{1}{\sqrt{a}}.$$

Example 6. *Evaluate* : $\lim_{x \to 0} \frac{\sqrt{1+x}}{x}$. *[C.A., May 1991]*

Solution. $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \left[\frac{\sqrt{1+x}-1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \right]$

$$= \lim_{x \to 0} \left[\frac{(1+x)-1}{x(\sqrt{1+x}+1)} \right] = \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

Example 7. *Evaluate* : $\lim_{h \to 0} \frac{F(x+h)-F(x)}{h}$

where $F(x) = \sin^2 x$. *[C.A., November 1991]*

Solution. $\lim_{h \to 0} \frac{\sin^2 (x+h) - \sin^2 x}{h} = \lim_{h \to 0} \frac{\sin (2x+h) \sin h}{h}$

$$= \lim_{h \to 0} \left[\sin (2x+h) \frac{\sin h}{h} \right] = \sin 2x.$$

Example 8. *Find* $\lim_{x \to 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2-1}$.

Solution. Substituting $x=1$, we get $\frac{0}{0}$, hence by rationalising,

$$\lim_{x \to 1} \frac{\sqrt{3+x} - \sqrt{5-x}}{x^2-1}$$

$$= \lim_{x \to 1} \frac{[\sqrt{3+x} - \sqrt{5-x}][\sqrt{3+x} + \sqrt{5-x}]}{(x^2-1)[\sqrt{3+x} + \sqrt{5-x}]}$$

$$= \lim_{x \to 1} \frac{(3+x)-(5-x)}{(x+1)(x-1)[\sqrt{3+x} + \sqrt{5-x}]}$$

$$=\lim_{x\to 1}\frac{2}{(x+1)[\sqrt{3+x}+\sqrt{5-x}]}$$

$$=\frac{2}{(1+1)[\sqrt{3+1}+\sqrt{5-1}]}=\frac{1}{4}$$

Example 9. *Show that* $\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{1}{x^2-3x+2}\right]=1.$

Solution. $\lim_{x\to 1}\left[\frac{1}{x-2}-\frac{1}{x^2-3x+2}\right]$

$$=\lim_{x\to 2}\left[\frac{x^2-3x+2-(x-2)}{(x-2)(x^2-3x+2)}\right]$$

$$=\lim_{x\to 2}\frac{x^2-4x+4}{(x-2)(x-2)(x-1)}$$

$$=\lim_{x\to 2}\frac{1}{x-1}=\frac{1}{2-1}=1$$

Example 10. *Show that* $\lim_{n\to\infty}\frac{2^{-n}(n^2+5n+6)}{(n+4)(n+5)}=0.$

Solution. The given limit $=\lim_{n\to\infty} 2^{-n}\lim_{n-\infty}\frac{n^2+5n+6}{(x+4)(n+5)}$

$$=\lim_{n\to\infty}\frac{1}{2^n}\cdot\lim_{n\to\infty}\frac{n^2+5n+6}{n^2+9n+20}$$

$$=\lim_{n\to\infty}\frac{1}{2^n}\cdot\lim_{n\to\infty}\frac{1+\frac{5}{n}+\frac{6}{n^2}}{1+\frac{9}{n}+\frac{20}{n^2}}$$

$$=0\times 1=0.$$

Example 11. *Show that* $\lim_{x\to 1} f(x)$ *exists and is equal to* $f(1)$, *where*

$$f(x)=x+1 \text{ for } x\leqslant 1$$

$$=3-x^2 \text{ for } x>1.$$

[*I.C.W.A., June 1991*]

Solution. We have

$$f(1)=1+1=2.$$

$$\text{L.H.L.}=\lim_{x\to 1-} f(x)=\lim_{h\to 0} f(1-h),\ h>0$$

$$=\lim_{h\to 0}[(1-h)+1]=2$$

$$\text{R.H.L.} = \lim_{x \to 1+} f(x) = \lim_{h \to 0} f(1-h),\ h > 0$$

$$= \lim_{h \to 0} [3-(1+h)^2] = 2$$

Since L.H.L. = R.H.L., $\lim_{x \to 1} f(x)$ exists and is equal to $f(1)$.

Example 12. *Discuss the existence of* $\lim_{x \to \frac{3}{2}} f(x)$, *if*

$$f(x) = 3+2x \text{ for } -\frac{3}{2} \leqslant x < 0$$

$$= 3-2x \text{ for } 0 \leqslant x < \frac{3}{2}$$

$$= -3-2x \text{ for } x \geqslant \frac{3}{2}$$

[*I.C.W.A., June 1990*]

Solution.

$$\text{L.H.L.} = \lim_{x \to \frac{3}{2}+0} f(x) = \lim_{h \to 0} f\left(\frac{3}{2}+h\right),\ h > 0$$

$$= \lim_{h \to 0} \left[-3-2\left(\frac{3}{2}+h\right)\right] = -3-3 = -6.$$

$$\text{R.H.L.} = \lim_{x \to \frac{3}{2}-} f(x) = \lim_{h \to 0} f\left(\frac{3}{2}-h\right),\ h > 0$$

$$= \lim_{h \to 0} \left[3-2\left(\frac{3}{2}-h\right)\right] = 3-3 = 0$$

Since L.H.L. $\neq$ R.H.L., $\lim_{x \to \frac{3}{2}} f(x)$ does not exist.

16·8. CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x=a$, if corresponding to any arbitrarily assigned positive number ε, however small (but not zero) there exists a positive number δ such that

$$| f(x) - f(a) | < \varepsilon \text{ for all } | x-a | \leqslant \delta$$

We note that in order to obtain the above we have to replace l by $f(a)$ in the definition of '$f(x)$ tending to a limit l as $x \to a$'. Hence the above definition becomes :

A function $f(x)$ is said to be continuous at a point $x=a$, if $f(x)$ possesses a finite and definite limit as x tends to the value 'a' from either side and each of these limits is equal to $f(a)$ so that

$$\lim_{x\to a-} f(x)=f(a)=\lim_{x\to a+} f(x)$$

Thus the continuity of a function at point $x=a$ boils down to the determination of three numbers :

$$(i)\ f(a),\quad (ii)\ \lim_{x\to a-} f(x),\quad (iii)\ \lim_{x\to a+} f(x)$$

which involves only the simple process of

(*i*) replacing x by 'a' in $f(x)$ and then finding if $f(a)$ is finite and definite.

(*ii*) evaluating the left hand limit, }
(*iii*) evaluating the right hand limit } by methods already explained,

If all the three number so obtained are equal, then $f(x)$ is continuous at $x=a$ otherwise it is discontinuous.

Example 13. *Show that $f(x)=3x^2+2x-1$ is continuous at $x=2$. Hence prove that $f(x)$ is continuous for all values of x.*

Solution. The conditions to be satisfied by a function before we can say that it is continuous at a particular point say $x=a$ are

$$f(a),\ \lim_{x\to a-} f(x) \text{ and } \lim_{x\to a+} f(x)$$

should have definite and finite values and that

$$\lim_{x\to a-} f(x)=f(a)=\lim_{x\to a+} f(x)$$

Let us examine whether these conditions are satisfied by $f(x)=3x^2+2x-1$ for $x=2$. Here $a=2$, therefore, we have

(*i*) $f(2)=3.2^2+2.2-1=15$

Again by the method of finding the left hand and right hand side limits, we have

(*ii*) $\lim_{x\to 2-} (3x^2+2x-1)=\lim_{h\to 0} \{3(2-h)^2+2(2-h)-1\}=15$

$\therefore$ Left hand side limit$=15$

Also (*iii*) $\lim_{x\to 2+} (3x^2+2x-1)=\lim_{h\to 0} \{3(2+h)^2+2(2+h)-1\}=15$

$\therefore$ Right hand side limit$=15$.

We find that the value of the function at $x=2$, the left hand and the right hand limits all exist, and are finite and equal.

We shall show further that $f(x)=3x^2+2x-1$ is continuous for values of x. The method followed is quite general and the students are required to note it carefully.

Let $x=k$ be any value of x arbitrarily selected, and find out whetner the given function is continuous at $x=k$.

Here $a=k$, therefore $f(k)=3k^2+2k-1$ (finite number) ...(1)

Also $$\lim_{x\to k-}(3x^2+2x-1)=\lim_{h\to 0}\{3(k-h)^2+2(k-h)-1\}$$

$$=\lim_{h\to 0}(3k^2-6kh+3h^2+2k-2h-1)$$

$$=3k^2+2k-1 \qquad ...(2)$$

Similarly, we find that

$$\lim_{x\to k+}(3x^2+2x-1)=3k^2+2k-1 \qquad ...(3)$$

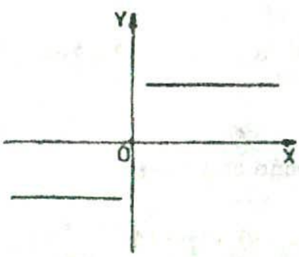
From (1), (2) and (3), we deduce that the given function is continuous at $x=k$. Since k is any arbitrary value of x, therefore, $f(x)$ is continuous for all values of x.

Example 14. *Discuss the continuity of*

$$f(x)=\frac{|x|}{x},\ x\neq 0$$
$$f(x)=0,\ x=0$$

Solution. $f(x)$ can be written as

$$f(x)=1,\ x>0$$
$$f(x)=0,\ x=0$$
$$f(x)=-1,\ x<0$$



Graphically the function is discontinuous at $x=0$. Analytically we have $f(0)=0$.

But $\lim f(x)=-1$, which is not equal to $f(0)$.

Hence the funetion is discontinuous at this point.

Example 15. *Show that*

$$f(x)=\frac{e^{-1/x}}{1+e^{1/x}},\ \textit{when}\ x\neq 0$$

$=0$, *when* $x=0$ *is not continuous at* $x=0$.

Solution. Here $f(0)=0$, given

$$\text{L.H.L.}=\lim_{x\to 0-}\frac{e^{-\frac{1}{x}}}{1+e^{\frac{1}{x}}}=\lim_{h\to 0}\frac{e^{-\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}} \qquad (1)$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{h}}}{1+e^{-\frac{1}{h}}} = \infty \qquad \ldots(2)$$

$$\left[\because \quad \frac{1}{h} \to \infty \text{ as } h \to 0, \text{ therefore } \frac{1}{e^{1/h}} \to 0 \text{ as } h \to 0 \right.$$
$$\left. \text{and } e^{1/h} \to \infty \text{ as } h \to 0 \right]$$

$$\text{R.H.L.} = \lim_{x \to 0+} \frac{e^{-1/x}}{1+e^{1/x}} = \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{1/h}} = 0 \qquad \ldots(3)$$

(2) and (3) $\Rightarrow$ R.H.L. $\neq$ L.H.L.

$\Rightarrow$ $\lim_{x \to 0} f(x)$ does not exist

$\Rightarrow$ $f(x)$ is discontinuous at $x=0$.

Example 16. *Consider the functions defined as follows :*

(*a*) $f(x) = \frac{x^2-4}{x-2}$, *for* $x < 2$
$f(x) = 4$, *for* $x = 2$
$f(x) = 2$, *for* $x > 2$

(*b*) $f(x) = \frac{x^2-4}{x-2}$, *for* $x < 2$.
$f(x) = 2$, *for* $x \geqslant 2$

(*c*) $f(x) = \frac{x^2-4}{x-2}$, *when* $0 \leqslant x < 2$
$f(x) = 2$, *when* $x = 2$
$f(x) = x+1$, *when* $x > 2$

(*d*) $f(x) = \frac{x^2-4}{x-2}$, *when* $0 \leqslant x < 2$
$f(x) = 2$, *when* $x = 2$
$f(x) = \frac{3x+2}{x}$, *when* $x > 2$

Discuss the continuity at $x=2$.

Solution. (*a*) Here $f(2)=4$, a finite and definite given number. The first condition is, therefore, satisfied. The function is defined for values of $x<2$ by $f(x) = \frac{x^2-4}{x-2}$.

$\therefore$ For the left hand limit we have to take up the first part of function. We then have $\lim_{x \to 2-} \frac{x^2-4}{x-2} = \lim_{x \to 2-} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2-} (x+2) = 4$

Again the function is defined for values of $x>2$ by $f(x)=2$, therefore the right hand limit is to be calculated from $f(x)=2$.

Here we have $\lim_{x \to 2+} 2 = 2$

Also we see that $f(2) = \lim_{x \to 2-} f(x) \neq \lim_{x \to 2+} f(x)$

Hence the function is discontinuous at $x=2$.

(*b*) Here $f(2)=2$ and $\lim_{x\to 2+} f(x)=\lim_{x\to 2+} 2=2$

But $\lim_{x\to 2-} f(x)=\lim_{x\to 2-}\frac{x^2-4}{x-2}=\lim_{x\to 2-}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2-}(x+2)=4.$

Whence $f(2)=2\neq\lim_{x\to 2-}\frac{x^2-4}{x-2}$

$\therefore$ Hence $f(x)$ is discontinuous at $x=2$.

(*c*) Here $f(2)=2$ (given)

$$\lim_{x\to 2-}\frac{x^2-4}{x-2}=4, \text{ and } \lim_{x\to 2+}(x+1)=3$$

which show that

$$\lim_{x\to 2-}\frac{x^2-4}{x-2}\neq\lim_{x\to 2+}(x+1)\neq f(2)$$

Hence the function is discontinuous at $x=2$.

(*d*) Here $\lim_{x\to 2-}\frac{x^2-4}{x-2}=4,\ \lim_{x\to 2+}\frac{3x+2}{x}=4$

but $f(2)=2$ which means that

$$\lim_{x\to 2-} f(x)=\lim_{x\to 2+} f(x)\neq f(2)$$

Hence the function is discontinuous at $x=2$.

Example 17. *Show that the function $f(x)$ as defined below, is discontinuous at $x=\frac{1}{2}$.*

$$f(x)=x, \text{ when } 0\leqslant x<\tfrac{1}{2}$$
$$f(x)=1, \text{ when } x=\tfrac{1}{2}$$
$$f(x)=1-x, \text{ when } \tfrac{1}{2}<x<1$$

Solution. We are given that

$$f(x)=1 \text{ when } x=\tfrac{1}{2}$$

which means that $f(\frac{1}{2})=1$, *i.e.*, the value of the function at $x=\frac{1}{2}$ is 1.

$\therefore$ the first condition is satisfied.

Now let us find $\lim_{x\to\frac{1}{2}-} f(x)$ and $\lim_{x\to\frac{1}{2}+} f(x)$

In $\lim_{x\to\frac{1}{2}-} f(x)$, we have to find the limit when $x\to\frac{1}{2}$ through those values of x which are less than $\frac{1}{2}$.

Now $f(x)=x$ is the only part of the function which is defined for values of x such that $0\leqslant x<\frac{1}{2}$.

Therefore, we should find $\lim_{x \to \frac{1}{2}-} x$. By the method of finding the left hand limit, we have

$$\lim_{x \to \frac{1}{2}-} x = \lim_{h \to 0} \left(\frac{1}{2} - h \right) = \frac{1}{2}$$

at this stage we stop and say that $f(x)$ is discontinuous at $x = \frac{1}{2}$ because

$$f(\tfrac{1}{2}) \neq \lim_{x \to \frac{1}{2}-} x.$$

There is no need of finding the right hand limit. In case we want to find the right hand limit then we must select $f(x) = 1 - x$ as our function because this is the only part of the given function which is defined for values of x greater than $\frac{1}{2}$.

CONTINUITY IN AN INTERVAL

A function $f(x)$ is said to be continuous in the closed interval (a, b) if it is continuous for every value of x in $a < x < b$, and if $f(x)$ is continuous from the right at 'a' and from the left at 'b', *i.e.*, if $\lim_{x \to a+} f(x)$ exists and is equal to $f(a)$, and $\lim_{x \to b-} f(x)$ exists and is equal to $f(b)$.

It is easily deduced from the theorems on limits that the sum, product, difference or quotient of two functions which are continuous at a certain point are themselves continuous at that point, except that in the case of quotient in which the denominator must not vanish at the point in question. Further it is true that the function of a continuous function is a continuous function.

We now take up a few examples to illustrate the method of application of the set of conditions arrived at in the previous sections to prove the continuity of a function at a point as well as in an interval.

Example 18. *A function $f(x)$ is defined as follows :*

$$f(x) = \frac{9x}{x+2}, \text{ for } x < 1$$

$$f(1) = 3$$

and

$$f(x) = \frac{x+3}{x}, \text{ for } x > 1$$

Examine the continuity of $f(x)$ in the interval $(-3, 3)$.

Solution. $f(x) = \dfrac{9x}{x+2}$ is to be considered for values of x lying between -3 and 1 because this is the part of the function which is defined for value of $x < 1$.

The denominator of $\dfrac{9x}{x+2}$ becomes 0 when $x = -2$. But -2 is a point between -3 and 1. Hence $f(x)$ is discontinuous at $x=-2$ as the function is not defined at $x=-2$. Again $f(1)=3$.

and $$\lim_{x\to 1-} \frac{9x}{x+2} = \lim_{h\to 0} \frac{9(1-h)}{(1-h)+2} = 3$$

also $$\lim_{x\to 1+} \frac{x+3}{x} = \lim_{h\to 0} \frac{1+h+3}{1+h} = 4$$

Since the right hand limit is not equal to the left hand limit and is not equal to the value of the function at $x=1$, therefore, the given function is discontinuous at $x=1$.

Hence $f(x)$ is discontinuous at $x=-2$ and 1, for all other values of x it is continuous.

Example 19. *Show that the function $f(x)=x \sin(1/x)$, $x\neq 0$ is continuous at $x=0$, where $f(0)=0$.*

Solution. Here $f(0)=0$

and $$\lim_{x\to 0-} x \sin\frac{1}{x} = \lim_{h\to 0} (-h) \sin(1/-h) = \lim_{h\to 0} h \sin(1/h) = 0$$

Also $$\lim_{x\to 0+} x \sin\frac{1}{x} = \lim_{h\to 0} h \sin\frac{1}{h} = 0$$

Hence the function is continuous at $x=0$.

Example 20. *Show that the function defined as under is continuous at $x=0$:*

$$f(x)=x^2 \cos e^{1/x} \text{ for all } x \text{ excepting } x=0$$
$$f(x)=1 \text{ for } x=0$$

Render it continuous by changing its definition.

Solution. Here $f(0)=1$

$$\lim_{x\to 0-} x^2 \cos x^{1/x} = \lim_{h\to 0} h^2 \cos\frac{1}{e^{1/h}} = 0$$

Also $$\lim_{x\to 0+} x^2 \cos e^{1/x} = \lim_{h\to 0} h^2 \cos e^{1/h} = 0$$

$$\therefore \quad \lim_{x\to 0} x^2 \cos e^{1/x} = 0$$

Since $\lim_{x\to 0} f(x) \neq f(0)$, the function is not continuous at $x=0$.

The function becomes continuous at $x=0$ if we define the function as follows :

$$f(x)=x^2 \cos e^{1/x},\ x\neq 0$$
$$f(x)=0, \text{ when } x=0.$$

EXERCISES

Evaluate the following limits :

1. (a) $\lim_{x \to 2} \frac{2x^2 - 7x + 6}{5x^2 - 11x + 2}$, (b) $\lim_{x \to 1} \frac{x^3 - 5x^2 + 2x + 2}{x^3 + 2x^2 - 6x + 3}$,

   (c) $\lim_{x \to 0} \frac{4x^4 + 5x^3 + 7x^2 + 6x}{5x^5 + 7x^2 + x}$

2. (a) $\lim_{x \to 2} \frac{(x^2 - 5x + 6)(x^2 - 3x + 2)}{x^3 - 3x^2 + 4}$ (b) $\lim_{x \to \infty} \frac{\sqrt{3x^4 - 5x^3 + 7x + 5}}{4x^2}$

3. (a) $\lim_{x \to 0} \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x}$, (b) $\lim_{x \to a} \frac{\sqrt{x + a} - \sqrt{2a}}{x - a}$

   (c) $\lim_{x \to 2} \frac{\sqrt{x^2 + x - 3} - \sqrt{x + 1}}{x - 2}$

4. (a) $\lim_{x \to 2} \left[\frac{1}{x - 2} - \frac{2}{x(x - 1)(x - 2)}\right]$

   (b) $\lim_{x \to 3} \left[\frac{1}{x - 3} - \frac{3}{x(x^2 - 5x + 6)}\right]$

   (c) $\lim_{x \to a} \frac{x^{-3} - a^{-3}}{x^{-2} - a^{-2}}$

5. Prove that the function $x^2 + 4x - 2$ is continuous at $x = 1$.

6. Prove that the function $\frac{x^2 - 9}{x - 3}$ is discontinuous at $x = 3$.

7. Discuss the continuity of the following functions :

(i) $f(x) = \frac{1}{x}$ at $x = 0$, (ii) $f(x) = \frac{1}{x^2}$ at $x = 0$.

8. Show that the function defined as

$$f(x) = 1 \text{ for } x = 0$$
$$f(x) = x \text{ for } x > 0$$

is discontinuous at the end point $x = 0$.

9. (a) A function $f(x)$ is defined as follows :

$$f(x) = \tfrac{1}{2} - x, \text{ when } 0 < x < \tfrac{1}{2}$$
$$f(x) = 0, \text{ when } x = \tfrac{1}{2}$$
$$f(x) = \tfrac{3}{2} - 3x, \text{ when } \tfrac{1}{2} < x < 1$$

Show that $f(x)$ is continuous at $x = \frac{1}{2}$.

(*b*) A function $f(x)$ is defined as follows:

$$f(x)=3+2x, \text{ for} -\tfrac{3}{2} \leqslant x < 0$$

$$f(x)=3-2x, \text{ for } 0 \leqslant x < \tfrac{3}{2}$$

$$f(x)=-3-2x, \text{ for } x \geqslant \tfrac{3}{2}$$

Show that $f(x)$ is continuous at $x=0$ and is discontinuous at $x=\frac{3}{2}$.

10. A function $f(x)$ is defined in the interval (0, 3) in the following way:

$$f(x)=x^2, \text{ when } 0 < x < 1$$

$$f(x)=x, \text{ when } 1 \leqslant x < 2$$

$$f(x)=\tfrac{1}{4}x^3, \text{ when } 2 \leqslant x < 3$$

Show that $f(x)$ is continuous at $x=2$ and $x=1$.

ANSWERS

1. (*a*) $\frac{1}{9}$, (*b*) -5, (*c*) 6 **2.** (*a*) $-\frac{1}{3}$, (*b*) $\frac{\sqrt{3}}{4}$ **3.** (*a*) 0, (*b*) $\frac{1}{2\sqrt{2a}}$, (*c*) $\frac{2}{\sqrt{3}}$ **4.** (*a*) $\frac{3}{2}$, (*b*) $\frac{4}{3}$ (*c*) $\frac{3}{2a}$