17

Differential Calculus

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- the derivative and write the derivatives of standard functions
- differentiate functions using standard derivatives and rules of differentiation

- higher-order derivatives of functions
- derivative as a rate measure
- points of inflexion, maxima and minima
- partial and total differentiation.

17·0. INTRODUCTION

The world calculus stands for the method of computation. There may be an arithmetic calculus or a probability calculus. The most common use of calculus is in regard to the computation of the rate of change in one variable with reference to an infinitesimal variation in the other variable. For example, we know that given the speed, the distance covered is a function of time or given the distance, the time taken is a function of speed. There is then a dependent variable which gets an impulse for change by a change in the independent variable. Calculus gives us the technique for measuring these changes in the dependent variable with reference to a very small change, approaching almost zero, in the independent variables or variables. The techniques concerning the calculation of the average rate of change are studied under differentiation or the Differential Calculus and the calculation of the total amount of change in the given range of values is studied under integration or Integral Calculus, which we shall study in the next chapter.

The usefulness of both these is very great in business. Given certain functional relations we can find out the average rate of change in the dependent variable with reference to a change in one or more independent variables. For example with a given demand function it would be possible to find the degree of change in demand with reference to a small change in price or income or both as the case may be and also the maximum and the minimum values of the function.

17·1. DIFFERENTIATION

To express the rate of change in any function we have the concept of derivative which involves infinitesimally small changes in the dependent variable with reference to a small change in independent variables.

Differentiation we can say is the process of finding out the derivative of a continuous function. A derivative is the limit of the ratio of the increment in the function corresponding to a small increment in the argument as the latter tends to zero.

Let us assume that y has been produced by labour x and that as we increase x (lavour) by one unit, the amount of y increases by four units. This relationship is shown by $y - 4x$; when x is increased by a small increment δx, then y increases by δy, and we have

$$y + \delta y = 4(x + \delta x) = 4x + 4\delta x$$

$$\delta y = 4\delta x \quad \Rightarrow \quad \frac{\delta y}{\delta x} = 4$$

$\frac{\delta y}{\delta x}$ is the incremental ratio of dependent variable y with respect to the independent variable x, *i.e.*, we can say $\frac{\delta y}{\delta x}$ is the change in y with respect to a small unit change in x. If the increments are very small tending to zero, we may write

$$\frac{d}{dx}(y) \text{ or } \frac{dy}{dx} \text{ or } \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$

Thus $\frac{dy}{dx}$ is the rate of change of y with respect to a change in x and is called the derivative of the function y with respect to x.

17·2. DERIVATIVE OF A FUNCTION OF ONE VARIABLE

Suppose that the function $f(x)$ denotes a continuous function of x. Let x receive an increment δx, then the function becomes

$$f(x+\delta x) \quad \text{...(1)}$$

Hence the corresponding increment of the function is

$$f(x+\delta x)-f(x) \quad \text{(2)}$$

The ratio of this increment of the function to the increment of the variable is

$$\frac{f(x+\delta x)-f(x)}{\delta x} \quad \text{...(3)}$$

The limit of this ratio when δx approaches zero, *i.e.*,

$$\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} \quad \text{...(4)}$$

is called the *derived function of* $f(x)$ with respect to x, or the *derivative of* $f(x)$ *w.r.t.* x, or the *x-derivative of* $f(x)$ *or the differential co-efficient of* $f(x)$ *w.r.t.* x.

If y is used to denote a function, *i.e.*,

$$y=f(x)$$

and x has an increment δx, then y will have a corresponding increment positive and negative), which may be denoted by δy so that

$$y+\delta y=f(x+\delta x)$$

$$\therefore \quad \delta y=f(x+\delta x)-f(x)$$

and
$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\therefore \quad \text{im}_{\delta x \to 0} \frac{\delta y}{\delta x}=\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

he phrase derivative of y with respect to x is symbolically equivalent to the $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ and is denoted by $\frac{dy}{dx}$. Thus

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \lim_{\delta x \to 0} \frac{\mathbf{f(x+\delta x)-f(x)}}{\delta \mathbf{x}} = \mathbf{f'(x)}$$

The process is quite general, as indicated in steps (1), (2), (3) and (4) above. These may be described in words, thus

(*a*) Let the independent variable have an increment,

(*b*) find the corresponding increment in the function,

(*c*) write the ratio of the increment in the function to the increment in the independent variable,

(*d*) find the limit of this ratio as the increment of the variable approaches zero.

It should be noted that $\frac{dy}{dx}$ does not mean the product of $\frac{d}{dx}$ with y. In fact dx is not a real number, so the notation $\frac{d}{dx}$ stands as a symbol to denote the operation of differentiation. The derivative or differential co-efficient of y w.r.t. x, $\frac{dy}{dx}$ is written in many other ways such as y', y_1, $\frac{d}{dx}[f(x)]$, $f'(x)$, Dy, etc.

17·3. DERIVATIVE OF A POWER FUNCTION

The most important rule is in regard to the differentiation of a power function. Let $y=f(x)=x^2$ and let there be increment in the function as follows :

$$y+\delta y=(x+\delta x)^2=x^2+2x\delta x+(\delta x)^2$$
$$\delta y=x^2+2x\,.\,\delta x+(\delta x)^2-x^2$$
$$\delta y=2x\,.\,\delta x+(\delta x)^2$$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x}=\lim_{\delta x \to 0}(2x+\delta x)$$

$$\therefore \quad f'(x) \text{ or } \frac{dy}{dx}=\lim_{\delta x \to 0}(2x+\delta x)=2x$$

Now for the general case where $y=x^n$, we have for an increment

$$y+\delta y=(x+\delta x)^n$$

Using the binomial expansion, we have

$$y+\delta y=x^n+{}^nC_1\,x^{n-1}(\delta x)+{}^nC_2\,x^{n-2}(\delta x)^2+\ldots+{}^nC_{n-1}\,x(\delta x)^{n-1}+(\delta x)^n$$

$$\therefore \quad \delta y=[x^n+{}^nC_1\,x^{n-1}(\delta x)+\ldots+(\delta x)^n]-x^n$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{[{}^nC_1\ x^{n-1}(\delta x) + \ldots + (\delta x)^n]}{\delta x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = {}^nC_1\ x^{n-1} + \ldots + (\delta x)^{n-1}$$

$$\Rightarrow \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} [{}^nC_1\ x^{n-1} + \ldots + (\delta x)^{n-1}]$$

$$\therefore \quad \mathbf{\frac{dy}{dx} = nx^{n-1}}$$

Thus the general formula is

$$\mathbf{\frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}}.$$

Illustrations. $y = x^3$, then $\frac{dy}{dx} = 3x^{3-1} = 3x^2$

$y = x^5$, then $\frac{dy}{dx} = 5x^4$

$y = x^{2/3}$, then $\frac{dy}{dx} = \frac{2}{3}\ x^{-1/3}$

$y = x^{-8}$, then $\frac{dy}{dx} = (-8)x^{-9}$.

17·4. DERIVATIVE OF A CONSTANT WITH ANY FUNCTION

Let $y = c\ f(x)$, where c is a constant.

Let x receive an increment δx, consequently y receives an incremen δy,

$$\therefore \quad y + \delta y = c\ f(x + \delta x)$$

$$\therefore \quad \delta y = c[f(x + \delta x) - f(x)]$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = c\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

$$\therefore \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} c\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] = cf'(x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx} f(x)\right] = cf'(x) \quad \ldots(1)$$

Thus the derivative of the product of a constant and a function is the product of the constant and the derivative of the function.

If $f(x)$ is denoted by u, then (1) is written as

$$\mathbf{\frac{d}{dx}(cu) = c\frac{du}{dx}} \quad \ldots(2)$$

Illustration. If $y = 5x^2$, then

$$\frac{dy}{dx} = 5\left[\frac{d}{dx}(x^2)\right] = 10x$$

Remark. When a function is equal to a constant say $y=a$, where a is constant, then $\frac{dy}{dx}=0$.

17·5. DERIVATIVE OF A SUM OF FUNCTIONS (SUM RULE)

Let $$y=f(x)+\phi(x)+\ldots$$

Then on giving x an increment δx, we have

$$y+\delta y=f(x+\delta x)+\phi(x+\delta x)+\ldots$$

$$\therefore \quad \delta y=f(x+\delta x)-f(x)+\phi(x+\delta x)-\phi(x)+\ldots$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{\phi(x+\delta x)-\phi(x)}{\delta x}+\ldots$$

Hence on letting δx approach zero, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}+\lim_{\delta x\to 0}\frac{\phi(x+\delta x)-\phi(x)}{\delta x}+\ldots$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{d}{dx}f(x)+\frac{d}{dx}\phi(x)+\ldots \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{d}{dx}[f(x)+\phi(x)+\ldots]=f'(x)+\phi'(x)+\ldots \qquad \ldots(2)$$

Thus the derivative of a sum of finite number of functions is the sum of their derivatives.

If the functions be denoted by $u, v, w, \ldots$, *i.e.*, if

$$y=u+v+w+\ldots$$

the result (1) may be expressed thus

$$\mathbf{\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\ldots}$$

Illustrations. 1. *If* $y=2x+x^2$, *then*

$$\frac{dy}{dx}=2\cdot\frac{d}{dx}(x)+\frac{d}{dx}(x^2)=2+2x$$

2. *If* $y=4x^3-7x^4$, *then*

$$\frac{dy}{dx}=4\cdot\frac{d}{dx}(x^3)-7\cdot\frac{d}{dx}(x^4)=4\cdot 3x^2-7\cdot 4x^3=12x^2-28x^3$$

3. *If* $y=\frac{4}{3}x^3-\frac{6}{7}x^7+4x^{-3}$, *then*

$$\frac{dy}{dx}=\frac{4}{3}\cdot\frac{d}{dx}(x^3)-\frac{6}{7}\cdot\frac{d}{dx}(x^7)+4\cdot\frac{d}{dx}(x^{-3})$$

$$=\frac{4}{3}\cdot 3x^2-\frac{6}{7}\cdot 7x^6+4(-3)x^{-4}$$

$$=4x^2-6x^6-12x^{-4}$$

4. *Find the differential coefficient of*

$$9x^4-7x^3+8x^2-\frac{8}{x}+\frac{10}{x^3} \text{ w.r.t. } x$$

Solution. $y=9x^4-7x^3+8x^2-\dfrac{8}{x}+\dfrac{10}{x^3}$

$$\frac{dy}{dx}=\frac{d}{dx}\left(9x^4-7x^3+8x^2-\frac{8}{x}+\frac{10}{x^3}\right)$$

$$=\frac{d}{dx}(9x^4)-\frac{d}{dx}(7x^3)+\frac{d}{dx}(8x^2)-\frac{d}{dx}\left(\frac{8}{x}\right)+\frac{d}{dx}\left(\frac{10}{x^3}\right)$$

$$=9.\frac{d}{dx}(x^4)-7.\frac{d}{dx}(x^3)+8.\frac{d}{dx}(x^2)-8.\frac{d}{dx}(x^{-1})+10.\frac{d}{dx}(x^{-3})$$

$$=9.4x^{4-1}-7.3x^{3-1}+8.2x^{2-1}-8(-1)x^{-1-1}+10.(-3)x^{-3-1}$$

$$=36x^3-21x^2+16x+\frac{8}{x^2}-\frac{30}{x^4}.$$

5. *Let* $y=\dfrac{(1-x)^2}{x^2}$, *find* $\dfrac{dy}{dx}$.

Solution. $y=\dfrac{(1-x)^2}{x^2}=\dfrac{1-2x+x^2}{x^2}$

$$=\frac{1}{x^2}-\frac{2}{x}+1=x^{-2}-2x^{-1}+1$$

$$\therefore \frac{dy}{dx}=(-2)x^{-3}-2(-1)x^{-2}+0=\frac{-2}{x^3}+\frac{2}{x^2}.$$

17·6 DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS (*Product rule*)

Let $y=f(x)\,\phi(x)$

Then, on giving x an increment δx, we have

$$y+\delta y=f(x+\delta x)\phi(x+\delta x)$$

$$\therefore \quad \delta y=f(x+\delta x)\phi(x+\delta x)-f(x)\phi(x)$$

$$\therefore \quad =f(x+\delta x)[\phi(x+\delta x)-\phi(x)]+\phi(x)[f(x+\delta x)-f(x)]$$

$$\therefore \quad \frac{\delta y}{\delta x}=f(x+\delta x).\frac{\phi(x+\delta x)-\phi(x)}{\delta x}+\phi(x)\,\frac{f(x+\delta x)-f(x)}{\delta x}$$

Hence on taking the limits as δx approaches zero, we have

$$\frac{dy}{dx}=f(x)\,\frac{d}{dx}[\phi(x)]+\phi(x)\frac{d}{dx}[f(x)] \quad \ldots(1)$$

Thus the derivative of the product of two functions is equal to the product of the first and the derivative of the second plus the product of the second and the derivative of the first.

If the functions be denoted by u and v, *i.e.*, if $y=uv$

then (1) may be expressed as

$$\frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{u}\frac{\mathbf{dv}}{\mathbf{dx}}+\mathbf{v}\frac{\mathbf{du}}{\mathbf{dx}} \quad ...(2)$$

Remark. The derivative of the product of any finite number of functions can be obtained by an extension of (2). For example, if

$$y=uvw$$

then, on regarding vw as a single function

$$\frac{dy}{dx}=(vw)\frac{du}{dx}+u\frac{d}{dx}(vw)$$

$$=vw\frac{du}{dx}+u\left(w\frac{dv}{dx}+v\frac{dw}{dx}\right)$$

$$=vw\frac{du}{dx}+wu\frac{dv}{dx}+uv\frac{dw}{dx}$$

Similarly if $y=uvwz$ then

$$\frac{dy}{dx}=vwz\frac{du}{dx}+uwz\frac{dv}{dx}+uvz\frac{dw}{dx}+uvw\frac{dz}{dx}$$

In general, to find the derivative of a product of several functions, multiply the derivative of each function in turn by all the other functions and add the results.

Illustrations 1. *Let* $y=(3x^2+1)(x^3+2x)$, *find* $\frac{dy}{dx}$.

Solution. Let us take $u=(3x^2+1)$ and $v=(x^3+2x)$ then the derivative of the product function $y=uv$ is

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=(3x^2+1)(3x^2+2)+(x^3+2x)(6x)$$

$$=9x^4+9x^2+2+6x^4+12x^2=15x^4+21x^2+2.$$

2. *Differentiate* $(3x^2+5)(2x^3+x+7)$ *w.r.t.* x.

Solution. Let $y=(3x^2+5)(2x^3+x+7)$

By using rule III, regarding the derivative of the product of two functions, we have

$$\frac{dy}{dx}=(3x^2+5)\frac{d}{dx}(2x^3+x+7)+(2x^3+x+7)\frac{d}{dx}(3x^2+5)$$

$$=(3x^2+5)\left[2.\frac{d}{dx}(x^3)+\frac{(dx)}{dx}+\frac{d}{dx}(7)\right]$$
$$+(2x^3+x+7)\left[3.\frac{d}{dx}(x^2)+\frac{d}{dx}(5)\right]$$
$$=(3x^2+5)(6x^2+1)+(2x^3+x+7)(6x)$$
$$=30x^4+39x^2+42x+5.$$

3 *Differentiate* $(\sqrt{x}+2.\sqrt[3]{x})(\sqrt[4]{x}-2.\sqrt[5]{x})$ *w.r.t.* x

Solution. Let $y=(\sqrt{x}+2.\sqrt[3]{x})\times(\sqrt[4]{x}-2.\sqrt[5]{x})$

$$=\left(x^{\frac{1}{2}}+2x^{\frac{1}{3}}\right)\left(x^{\frac{1}{4}}-2x^{\frac{1}{5}}\right)$$

$$\therefore \frac{dy}{dx}=\left(x^{\frac{1}{2}}+2x^{\frac{1}{3}}\right)\frac{d}{dx}\left(x^{\frac{1}{4}}-2x^{\frac{1}{5}}\right)+\left(x^{\frac{1}{4}}-2x^{\frac{1}{5}}\right)\frac{d}{dx}\left(x^{\frac{1}{2}}+2x^{\frac{1}{3}}\right)$$

$$=\left(x^{\frac{1}{2}}+2x^{\frac{1}{3}}\right)\left[\frac{1}{4}x^{-\frac{3}{4}}-\frac{2}{5}x^{-\frac{4}{5}}\right]$$
$$+\left(x^{\frac{1}{4}}-2x^{\frac{1}{5}}\right)\left[\frac{1}{2}x^{-\frac{1}{2}}+\frac{2}{3}x^{-\frac{2}{3}}\right]$$

4. *If* $f(x)=2x^{\frac{3}{2}}\left(\sqrt{x}+2\right)\left(\sqrt{x}-1\right)$ *find* $\frac{df(x)}{dx}$.

Solution.

$$f'(x)=\left(\sqrt{x}+2\right)\left(\sqrt{x}-1\right)\frac{d}{dx}\left(2x^{\frac{3}{2}}\right)+2x^{\frac{3}{2}}\left(\sqrt{x}-1\right)\frac{d}{dx}\left(\sqrt{x}+2\right)$$
$$+2x^{\frac{3}{2}}\left(\sqrt{x}+2\right)\frac{d}{dx}\left(\sqrt{x}-1\right)$$

$$=\left(\sqrt{x}+2\right)\left(\sqrt{x}-1\right)2.\tfrac{3}{2}x^{\frac{1}{2}}+2x^{\frac{3}{2}}\left(\sqrt{x}-1\right)\tfrac{1}{2}x^{-\frac{1}{2}}$$
$$+2x^{\frac{3}{2}}\left(\sqrt{x}+2\right)\frac{1}{2}x^{-\frac{1}{2}}$$

$$=(\sqrt{x}+2)(\sqrt{x}-1)3\sqrt{x}+x(\sqrt{x}-1)+x(\sqrt{x}+2)$$
$$=4x+5x\sqrt{x}-6\sqrt{x}.$$

17·7. DERIVATIVE OF THE QUOTIENT OF TWO FUNCTIONS

Let $$y=\frac{f(x)}{\phi(x)}$$

Then on proceeding as before, we have

$$y+\delta y=\frac{f(x+\delta x)}{\phi(x+\delta x)}$$

$$\therefore \quad \delta y=\frac{f(x+\delta x)}{\phi(x+\delta x)}-\frac{f(x)}{\phi(x)}$$

$$=\frac{f(x+\delta x)\,\phi(x)-f(x)\,\phi(x+\delta x)}{\phi(x)\,\phi(x+\delta x)}$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{f(x+\delta x)\,\phi(x)-f(x)\,\phi(x+\delta x)}{\phi(x)\,\phi(x+\delta x)\,\delta x} \qquad ...(1)$$

On letting $\delta x \to 0$, right hand side approaches the form $\frac{0}{0}$. In order to evaluate, introduce

$$\phi(x)\,f(x)-\phi(x)\,f(x)$$

in the numerator of right hand side. Then, on combining and arranging terms, (1) becomes

$$\frac{\delta y}{\delta x}=\frac{\phi(x)\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]-f(x)\left[\frac{\phi(x+\delta x)-\phi(x)}{\delta x}\right]}{\phi(x)\,\phi(x+\delta x)}$$

Hence on proceeding to limit as δx approaches zero, we have

$$\frac{dy}{dx}=\frac{\phi(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[\phi(x)]}{[\phi(x)]^2} \qquad ...(1)$$

Thus if one function be divided by another, then the derivative of the fraction thus formed is equal to the product of the denominator and the derivative of the numerator minus the product of the numerator and the derivative of the denominator, all divided by the square of the denominator.

If the functions be denoted by u and v, *i.e.*, if

$$y=\frac{u}{y}$$

then (1) has the form

$$\frac{\mathbf{dy}}{\mathbf{dx}}=\frac{\mathbf{v}\frac{\mathbf{du}}{\mathbf{dx}}-\mathbf{u}\frac{\mathbf{dv}}{\mathbf{dx}}}{\mathbf{v}^2} \quad v\neq 0 \qquad ...(2)$$

Illustrations 1. *Find the differential co-efficient of* $\frac{x^2-1}{x^2+1}$ *w.r.t. x.*

Solution. Let $\quad y=\frac{x^2-1}{x^2+1}$

$$\therefore \quad \frac{dy}{dx}=\frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$=\frac{(x^2+1)\,2x-(x^2-1)\,2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

2. *Differentiate w.r.t. x, the function*

$$\frac{(x+1)(2x-1)}{(x-3)}$$

Solution. Let $y=\dfrac{(x+1)(2x-1)}{x-3}=\dfrac{2x^2+x-1}{x-3}$

$$\therefore \quad \frac{dy}{dx}=\frac{(x-3)\dfrac{d}{dx}(2x^2+x-1)-(2x^2+x-1)\dfrac{d}{dx}(x-3)}{(x-3)^2}$$

$$=\frac{(x-3)(4x+1)-(2x^2+x-1)\,.\,1}{(x-3)^2}$$

$$=\frac{2(x^2-6x-1)}{(x-3)^2}$$

3. *Find the differential co-efficient of* $\dfrac{x^{\frac{1}{2}}+2}{x^{\frac{1}{2}}}$

Solution. Let $y=\dfrac{x^{\frac{1}{2}}+2}{x^{\frac{1}{2}}}$

$$\frac{dy}{dx}=\frac{x^{\frac{1}{2}}\dfrac{d}{dx}\left(x^{\frac{1}{2}}+2\right)-\left(x^{\frac{1}{2}}+2\right)\dfrac{d}{dx}\left(x^{\frac{1}{2}}\right)}{\left(x^{\frac{1}{2}}\right)^2}$$

$$=\frac{x^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\left(x^{\frac{1}{2}}+\frac{1}{2}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}$$

$$=-\frac{1}{x^{\frac{3}{2}}}.$$

17·8. DERIVATIVE OF A FUNCTION OF A FUNCTION

Here we deal with derivative of a composite function (function of a function). If y is a function of u, say $y=f(u)$, where u itself is a function of x say $u=\phi(x)$, then y is called a function of a function or a composite function of x.

Suppose derivative of y with respect to x is required. [Here $f(u)$ and $\phi(x)$ are differentiable functions]. The method, which naturally comes first to mind, is to substitute $\phi(x)$ for u in first equation, thus getting $y=f[\phi(x)]$, and then to proceed according to preceding articles. This method, however, is often more tedious and difficult than the one now explained.

Let x receive an increment δx, accordingly, u receives an increment δu and y receives an increment δy. Then

$$y+\delta y=f(u+\delta u)$$

$$\therefore \quad \delta y=f(u+\delta u)-f(u)$$

$$\therefore \quad \frac{\delta y}{\delta x}=\frac{f(u+\delta u)-f(u)}{\delta u}\cdot\frac{\delta x}{\delta x}$$

Assume $\delta u \neq 0$ when $\delta x \neq 0$. When δx approaches zero, δu also approaches zero and this relation becomes

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}=\lim_{\delta u \to 0} \frac{f(u+\delta u)-f(u)}{\delta u}\cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{d}{du}[f(u)].\frac{du}{dx}$$

Hence $$\mathbf{\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dx}}.$$

Illustrations. 1. *Differentiate* $\sqrt{(3x^2-7)}$ *w.r.t. x.*

Solution. Let $y=\sqrt{(3x^2-7)}=(3x^2-7)^{\frac{1}{2}}$

Put $u=(3x^3-7)$ then $y=u^{\frac{1}{2}}$

Now $$\frac{du}{dx}=\frac{d}{dx}(3x^2-7)=6x$$

and $$\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2}.(3x^2-7)^{-\frac{1}{2}}$$

Hence $$\frac{dy}{dx}=\frac{dy}{dx}\cdot\frac{du}{dx}$$

$$=\frac{1}{2}(3x^2-7)^{-\frac{1}{2}}.6x=\frac{3x}{\sqrt{3x^2-7}}$$

Aliter $$\frac{dy}{dx}=\frac{1}{2}(3x^2-7)^{\frac{1}{2}-1}\frac{d}{dx}(3x^2-7)$$

$$=\frac{1}{2}(3x^2-7)^{-\frac{1}{2}}.6x=\frac{3x}{\sqrt{3x^2-7}}.$$

2. *Find the differential coefficient of* $(3x^3-5x^2+8)^3$ *w.r.t. x.*

Solution. Let $y=(3x^3-5x^2+8)^3=u^3$,

where $u=3x^3-5x^2+8$

Now $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \quad ..(1)$$

But $$\frac{dy}{du}=\frac{d}{du}\ (u^3)=3u^2$$

$$=3(3x^3-5x^2+8)^2$$

and $$\frac{du}{dx}=\frac{d}{dx}\ (3x^3-5x^2+8)=9x^2-10x$$

Substituting these values in (1), we get

$$\frac{dy}{dx}=3(3x^3-5x^2+8)^2(9x^2-10x)$$

3 *Differentiate w.r.t. x the following function :*

$$\frac{1}{\sqrt[3]{6x^5-7x^3+9}}$$

Solution. Let $y=\frac{1}{\sqrt[3]{6x^5-7x^3+9}}=(6x^5-7x^3+9)^{-1/3}$

Put $u=6x^5-7x^3+9$, then $y=u^{-1/3}$

Now $\frac{dy}{du}=-\frac{1}{3}\ u^{-4/3}=-\frac{1}{3}\ (6x^5-7x^3+9)^{-4/3}$

and $$\frac{du}{dx}=30x^4-21x^2$$

Hence $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=-\frac{1}{3}\ (6x^5-7x^5-9)^{-4/3}(30x^4-21x^2)$$

Aliter. $$\frac{dy}{dx}=-\frac{1}{3}\ (6x^5-7x^3+9)^{-4/3}\ \frac{d}{dx}\ (6x^5-7x^3+9)$$

$$=-\frac{1}{3}\ (6x^5-7x^3+9)^{-4/3}(30x^4-21x^2).$$

4. *Differentiate* $\frac{1}{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}$ *w.r.t. x.*

Solution. Let $y=\frac{1}{\sqrt{x^2+a^2}+\sqrt{x^3+b^2}}$

Rationalising the denominator, we get

$$y=\frac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{a^2-b^2}=\frac{1}{a^2-b^2}\left[(x^2+a^2)^{1/2}-(x^2+b^2)^{1/2}\right]$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{a^2-b^2}\left[\frac{1}{2}\ (x^2+a^2)^{\frac{1}{2}-1}\ \frac{d}{dx}\ (x^2+a^2)\right.$$

$$\left.-\frac{1}{2}\ (x^2+b^2)^{\frac{1}{2}-1}\ \frac{d}{dx}\ (x^2+b^2)\right]$$

$$=\frac{1}{a^2-b^2}\left[\frac{2x}{2\sqrt{x^2+a^2}}-\frac{2x}{2\sqrt{x^2+b^2}}\right]$$

$$=\frac{x}{a^2-b^2}\left[\frac{1}{\sqrt{x^2+a^2}}-\frac{1}{\sqrt{x^2+b^2}}\right]$$

17·9. DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

I. Derivative of sin u. Let $y=\sin u$

Then $$y+\delta y=\sin(u+\delta u)$$

$\therefore$ $$\delta y=\sin(u+\delta u)-\sin u$$

$$=2\cos\left(u+\frac{\delta u}{2}\right)\sin\frac{\delta u}{2}$$

$\therefore$ $$\frac{\delta y}{\delta x}=2\cos\left(u+\frac{\delta u}{2}\right)\sin\frac{\delta u}{2}\cdot\frac{1}{\delta x}$$

$$=\cos\left(u+\frac{\delta u}{2}\right)\cdot\frac{\sin\frac{\delta u}{2}}{\frac{\delta u}{2}}\cdot\frac{\delta u}{\delta x}$$

Let $\delta x\to 0$, then also $\delta u\to 0$ and

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta u\to 0}\cos\left(u+\frac{\delta u}{2}\right)\lim_{\delta u\to 0}\frac{\sin\frac{\delta u}{2}}{\frac{\delta u}{2}}\cdot\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

$\Rightarrow$ $$\frac{dy}{dx}=\cos u\,.\,1\cdot\frac{du}{dx}$$

$\Rightarrow$ $$\frac{d}{dx}(\sin u)=\cos u\,.\,\frac{du}{dx}$$

In particular, if $u=x$

$$\mathbf{\frac{d}{dx}(\sin x)=\cos x}$$

Thus the rate of change of the sine of an angle with respect to the angle is equal to the cosine of the angle.

II. Derivative of cos u. Let $y=\cos u$, then $y=\sin\left(\frac{\pi}{2}-u\right)$

$\therefore$ $$\frac{dy}{dx}=\cos\left(\frac{\pi}{2}-u\right)\cdot\frac{d}{dx}\left(\frac{\pi}{2}-u\right)=-\sin u\,.\,\frac{du}{dx}$$

$\therefore$ $$\frac{d}{dx}(\cos u)=-\sin u\quad\frac{du}{dx}$$

In particular, if $u=x$

$$\frac{\mathbf{d}}{\mathbf{dx}}\,(\mathbf{cos\ x})=-\mathbf{sin\ x}.$$

III. Derivative of tan u. Let $y=\tan u$, then $y=\dfrac{\sin u}{\cos u}$

$$\therefore \qquad \frac{dy}{dx}=\frac{\cos u\,\dfrac{d}{dx}\,(\sin u)-\sin u\dfrac{d}{dx}(\cos u)}{\cos^2 u}$$

$$=\frac{(\cos^2 u+\sin^2 u)}{\cos^2 u}\cdot\frac{du}{dx}$$

$$=\frac{1}{\cos^2 u}\cdot\frac{du}{dx}=\sec^2 u.\frac{du}{dx}$$

$$\Rightarrow \qquad \frac{d}{dx}(\tan u)=\sec^2 u\,.\,\frac{du}{dx}$$

If $u=x$, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{tan\ x})\ \mathbf{sec^2\ x}.$

IV. Derivative of cot u. Substituting $\dfrac{1}{\tan u}$ for cot u and differentiating, it can be found that

$$\frac{d}{dx}\,(\cot u)=-\operatorname{cosec}^2 u\,\frac{du}{dx}$$

If $u=x$, $\qquad \dfrac{\mathbf{d}}{\mathbf{dx}}\,(\mathbf{cot\ x})=-\mathbf{cosec^2\ x}.$

V. Derivative of sec u. Let $y=\sec u=\dfrac{1}{\cos u}$, then

$$\frac{dy}{dx}=\frac{\sin u}{\cos^2 u}\cdot\frac{du}{dx}=\frac{1}{\cos u}\cdot\frac{\sin u}{\cos u}\cdot\frac{du}{dx}$$

$$\Rightarrow \qquad \frac{d}{dx}\,(\sec u)=\sec u\tan u\,\frac{du}{dx}$$

If $u=x$, $\qquad \dfrac{\mathbf{d}}{\mathbf{dx}}\,(\mathbf{sec\ x})=\mathbf{sec\ x\ tan\ x}.$

VI. Derivative of cosec u. Let $y=\operatorname{cosec} u=\dfrac{1}{\sin u}$, then

$$\frac{dy}{dx}=-\frac{\cos u}{\sin^2 u}\,\frac{du}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\operatorname{cosec} u\cot u\,\frac{du}{dx}$$

If $u=x$, $\quad \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cosec\ x})=-\mathbf{cosec\ x\ cot\ x}$

VII. Derivative of $\sin^{-1} u$. Let $y=\sin^{-1} u$ so that $\sin y=u$

On differentiation, we get $\cos y \frac{dy}{dx}=\frac{du}{dx}$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\cos y}\cdot\frac{du}{dx}=\frac{1}{\sqrt{1-\sin^2 y}}\cdot\frac{du}{dx}$$

$$\Rightarrow \quad \frac{d}{dx}(\sin^{-1} u)=\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

If $u=x$, $\quad \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\sin^{-1} x})=\frac{1}{\sqrt{1-\mathbf{x}^2}}$

VIII. Derivative of $\cos^{-1} u$. Let $y=\cos^{-1} u$ so that $\cos y=u$

On differentiation, $-\sin y \frac{dy}{dx}=\frac{du}{dx}$

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{\sin y}\cdot\frac{du}{dx}=-\frac{1}{\sqrt{1-\cos^2 y}}\cdot\frac{du}{dx}$$

$$\Rightarrow \quad \frac{d}{dx}(\cos^{-1} u)=-\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

If $u=x$, $\quad \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cos^{-1} x})=-\frac{1}{\sqrt{1-\mathbf{x}^2}}$

IX. Derivative of $\tan^{-1} u$. Let $y=\tan^{-1} u$ so that $\tan y=u$

On differentiation, we get $\sec^2 y \frac{dy}{dx}=\frac{du}{dx}$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sec^2 y}\cdot\frac{du}{dx}=\frac{1}{1+\tan^2 y}\cdot\frac{du}{dx}$$

$$\Rightarrow \quad \frac{d}{dx}(\tan^{-1} u)=\frac{1}{1+u^2}\cdot\frac{du}{dx}$$

In particular if $u=x$,

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\tan^{-1} x})=\frac{\mathbf{1}}{\mathbf{1+x^2}}.$$

X. Derivative of $\cot^{-1} u$. On proceeding in a manner similar to that in (IX), it can be shown that

$$\frac{d}{dx}(\cot^{-1} u)=-\frac{1}{1+u^2}\cdot\frac{du}{dx}$$

If $u=x$, $\quad \frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cot^{-1} x})=-\frac{\mathbf{1}}{\mathbf{1+x^2}}$

XI. Derivative of $\sec^{-1} u$. Let $y=\sec^{-1} u$ so that $\sec y=u$

On differentiation, we get $\sec y \tan y \dfrac{dy}{dx}=\dfrac{du}{dx}$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sec y \tan y}\cdot\frac{du}{dx}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}\cdot\frac{du}{dx}$$

i.e.,
$$\frac{d}{dx}(\sec^{-1} u)=\frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

If $u=x$, then $\mathbf{\dfrac{d}{dx}(\sec^{-1} x)=\dfrac{1}{x\sqrt{x^2-1}}}$.

XII. Derivative of $\operatorname{cosec}^{-1} u$. On proceeding in a manner similar to (XI), it can be shown that

$$\frac{d}{dx}(\operatorname{cosec}^{-1} u)=-\frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

If $u=x$, then $\mathbf{\dfrac{d}{dx}(\operatorname{cosec}^{-1} x)=-\dfrac{1}{x\sqrt{x^2-1}}}$.

17·10. DERIVATIVE OF LOGARITHMIC FUNCTIONS

XIII. Derivative of $\log_a u$. Let $y=\log_a u$ and let x receive an increment δx, then u and y consequently receive increments δu and δy respectively.

Then
$$y+\delta y=\log_a(u+\delta u)$$

$$\therefore \quad \delta y=\log_a(u+\delta u)-\log_a u$$

$$=\log_a\left(\frac{u+\delta u}{u}\right)=\log_a\left(1+\frac{\delta u}{u}\right)$$

$$\therefore \quad \frac{\delta y}{\delta x}=\log_a\left(1+\frac{\delta u}{u}\right)\cdot\frac{1}{\delta x}$$

On introducing $\dfrac{1}{u}\cdot\dfrac{u}{\delta u}\cdot\delta u$ in the second member, we have

$$\frac{\delta y}{\delta x}=\frac{1}{u}\cdot\frac{u}{\delta u}\cdot\log_a\left(1+\frac{\delta u}{u}\right)\frac{\delta u}{\delta x}$$

$$=\frac{1}{u}\log_a\left(1+\frac{\delta u}{u}\right)^{\frac{u}{\delta u}}\cdot\frac{\delta u}{\delta x}$$

From this, on letting δx approach zero and remembering that δu and δy approach zero with δx, it follows that

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{1}{u}\lim_{\delta u\to 0}\log_a\left(1+\frac{\delta u}{u}\right)^{\frac{u}{\delta u}}\cdot\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{u}\cdot \log_a e \cdot \frac{du}{dx} \qquad \left[\because \lim_{m\to 0}(1+m)^{1/m}=e\right]$$

If $u=x$, then $\frac{d}{dx}(\log_a x)=\frac{1}{x}(\log_a e)$

If $a=e$, then $\frac{d}{dx}(\log u)=\frac{1}{u}\ \frac{du}{dx}$

If $u=x$ and $a=e$, then $\frac{d}{dx}(\log x)=\frac{1}{x}$

XIV. Derivative of e^x. If $y=e^u$, then

$$\frac{dy}{dx}=e^u\ \frac{du}{dx}$$

If $u=x$, then $\frac{dy}{dx}=e^x$.

Illustrations. 1. *Find the differential coefficients of the following functions :*

(*a*) $\sin 6x$, (*b*) $\tan(5x+7)$, (*c*) $\sec^3 4x$.

Solution. (*a*) Let $y=\sin 6x=\sin u$, where $u=6x$.

Then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ ...(1)

But $\frac{dy}{du}=\cos u$ and $\frac{du}{dx}=6$

$\therefore$ $\frac{dy}{dx}=\cos u.\ 6=6\cos 6x$

Aliter. $\frac{dy}{dx}=\frac{d}{dx}(\sin 6x)=\cos 6x\ \frac{d}{dx}\ (6x)=6\cos 6x.$

(*b*) Let $y=\tan(5x+7)=\tan u$, where $u=5x+7$

Then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ ...(1)

But $\frac{dy}{du}=\sec^2 u$ and $\frac{du}{dx}=5$

$\therefore$ $\frac{dy}{dx}=\sec^2 u.\ 5=5\sec^2(5x+7).$

(c) Let $y=\sec^3 4x$

$$\frac{dy}{dx}=\frac{d}{dx}(\sec^3 4x)=3\sec^2 4x\ \frac{d}{dx}(\sec 4x)$$

$$=3\sec^2 4x\ .\ \sec 4x\tan 4x.\ \frac{d}{dx}(4x)$$

$$=12\sec^3 4x\tan 4x.$$

2. *If* $y=\frac{1-\sin x}{1+\cos x}$, *find* $\frac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{(1+\cos x)\ \frac{d}{dx}(1-\sin x)-(1-\sin x)\ \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(-\cos x)-(1-\sin x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x-\cos x-(\sin^2 x+\cos^2 x)}{(1+\cos x)^2}=\frac{\sin x-\cos x-1}{(1+\cos x)^2}.$$

3. *Find the differential co-efficients of* :

(i) $y=\log 5x$, (ii) $y=\log(\sin x)$, *and* (iii) $y=\log(x\cos x)$

Solution. Let $5x=u$ so that $y=\log u$ and $\frac{du}{dx}=5$

But $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$\therefore$ $$\frac{dy}{dx}=\frac{1}{u}\cdot\frac{du}{dx}=\frac{1}{5x}\ .\ 5=\frac{1}{x}.$$

(ii) Let $\sin x=u$ so that $\frac{du}{dx}=\cos x$ and $y=\log u$

But $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{1}{u}\cdot\frac{du}{dx}=\frac{1}{\sin x}\cdot\cos x=\cot x.$$

(iii) Let $u=x\cos x$ so that

$$\frac{du}{dx}=\cos x-x\sin x \text{ and } y=\log u$$

But $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{u}\cdot\frac{du}{dx}$$

$$=\frac{1}{x\cos x}\cdot(\cos x-x\sin x)=\frac{(\cos x-x\sin x)}{x\cos x}$$

More about logarithmic differentiation will be discussed in the later section of this chapter.

4. *Find the differential co-efficient of the following functions :*

(*a*) $cos^{-1}\,5x$, (*b*) $tan^{-1}\sqrt{x}$, (*c*) $log\,(sec^{-1}\,x)$.

Solution. (*a*) Let $y=\cos^{-1}(5x)=\cos^{-1}u$, where $u=5x$

Then $$\frac{dy}{dx}=\frac{-1}{\sqrt{1-u^3}}\text{ and }\frac{du}{dx}=5$$

Now $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{-1}{\sqrt{1-u^2}}\cdot 5=\frac{-5}{\sqrt{1-25x^2}}.$$

(*b*) Let $y=\tan^{-1}\sqrt{x}=\tan^{-1}u$, where $u=x^{1/2}$

Then $$\frac{dy}{du}=\frac{1}{1+u^3}\text{ and }\frac{du}{dx}=\frac{1}{2}\,x^{-\frac{1}{2}}$$

∴ $$\frac{dy}{dx}=\frac{1}{1+u^2}\cdot\frac{1}{2}\,x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}\,(1+x)}$$

(*c*) Let $y=\log(\sec^{-1}x)=\log u$, where $u=\sec^{-1}x$

Then $$\frac{dy}{du}=\frac{1}{u}\text{ and }\frac{du}{dx}=\frac{1}{x\sqrt{x^2-1}}$$

∴ $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{u}\cdot\frac{1}{x\sqrt{x^2-1}}$$

$$=\frac{1}{\sec^{-1}x\,.\,x\sqrt{x^2-1}}$$

Aliter. $$\frac{dy}{dx}=\frac{d}{dx}\,(\log\sec^{-1}x$$

$$=\frac{1}{\sec^{-1}x}\cdot\frac{d}{dx}\,(\sec^{-1}x)$$

$$=\frac{1}{\sec^{-1}x}\cdot\frac{1}{x\sqrt{x^2-}}$$

Example 1. *Find the derivative of* $(x^4-4x^3+9)\tan x\, e^x$.

Solution. Let $y=(x^4-4x^3+9)\tan x\, e^x$.

By using rule III, regarding the derivative of the product of three functions, we have

$$\frac{dy}{dx}=(x^4-4x^3+9)\tan x\,\frac{d}{dx}(e^x)+(x^4-4x^3+9)\,e^x\,\frac{d}{dx}(\tan x)$$
$$+\tan x\, e^x\,\frac{d}{dx}(x^4-4x^3+9)$$

$$=(x^4-4x^3+9)\tan x\, e^x+(x^4-4x^3+9)e^x\sec^2 x+\tan x\, e^x(4x^3-12x^2)$$

$$=(x^4-12x^2+9)\tan x\, e^x+(x^4-4x^3+9)\,e^x\sec^2 x.$$

Example 2. *Find the differential co-efficient of*

$$7\,(\cos x)\,(\log x)+3\,\operatorname{cosec} x+\frac{\tan x}{(x+e^x)}\ \text{w.r.t. } x.$$

Solution. Let $y=7\,(\cos x)(\log x)+3\operatorname{cosec} x+\dfrac{\tan x}{(x+e^x)}$.

The first term on the right hand side is the product of two functions $\cos x$ and $\log x$, where 7 is only a constant, therefore, we can apply the product rule. The second term is the product of a constant and a function and the third term is the quotient of two functions $\tan x$ and $(x+e^x)$. In this we shall have to apply the quotient rule.

$$\therefore\ \frac{dy}{dx}=7\left[\cos x.\,\frac{d}{dx}(\log x)+\log x.\,\frac{d}{dx}(\cos x)\right]+3\,\frac{d}{dx}(\operatorname{cosec} x)$$
$$+\frac{(x+e^x).\,\frac{d}{dx}(\tan x)-\tan x.\,\frac{d}{dx}\,x+e^x)}{(x+e^x)^2}$$

$$=7\left[\frac{\cos x}{x}-\log x\sin x\right]-3\operatorname{cosec} x\cot x$$
$$+\frac{(x+e^x)\sec^2 x-\tan x\,(1+e^x)}{(x+e^x)^2}.$$

Example 3. *Differentiate the following function w.r.t.* x :

(*a*) $\sin\,(x^2+2x-5)^7$, (*b*) $\log\sin x^3$.

Solution. (*a*) Let $y=\sin\,(x^2+2x-5)^7$

Put $u=(x^2+2x-5)$, $v=u^7$, then $y=\sin v$

$$\therefore\quad \frac{du}{dx}=2x+2,\ \frac{dv}{du}=7u^6=7(x^2+2x-5)^6$$

and $$\frac{dy}{dv}=\cos v=\cos u^7=\cos\,(x^2+2x-5)^7$$

By using the chain rule, we get

$$\frac{dy}{dx}=\frac{dy}{dv}\times\frac{dv}{du}\times\frac{du}{dx}$$

$$=\cos(x^2+2x-5)^7\times 7(x^2+2x-5)^6\times(2x+2)$$

Aliter. $$\frac{dy}{dx}=\cos(x^2+2x-5)^7.\frac{d}{dx}[(x^2+2x-5)^7]$$

$$=\cos(x^2+2x-5)^7.7(x^2+2x-5)^6\frac{d}{dx}[(x^2+2x-5)]$$

$$=7\cos(x^2+2x-5)^7(x^2+2x-5)^6(2x+2).$$

(*b*) Let $y=\log\sin x^2$

Put $u=x^2$, $v=\sin u$, then $y=\log v$

$\therefore$ $$\frac{du}{dx}=2x,\ \frac{dv}{du}=\cos u=\cos x^2$$

and $$\frac{dy}{dv}=\frac{1}{v}=\frac{1}{\sin u}=\frac{1}{\sin x^2}$$

$\therefore$ $$\frac{dy}{dx}=\frac{dy}{dv}\times\frac{dv}{du}\times\frac{du}{dx}$$

$$=\frac{1}{\sin x^2}\cdot\cos x^2\,.\,2x=2x\cot x^2.$$

Aliter. $$\frac{dy}{dx}=\frac{1}{\sin x^2}\cdot\frac{d}{dx}(\sin x^2)$$

$$=\frac{1}{\sin x^2}\cdot\cos x^2\ \frac{d}{dx}(x^2)$$

$$=\frac{2x}{\sin x^2}\cdot\cos x^2=2x\cot x^2.$$

Example 4. *Find the differential co-efficient of*

(*a*) $\tan(\log\tan^{-1}\sqrt{x})+\tan^{-1}(\sin e^{ax})$ *w.r.t.* x,

(*b*) $e^{\tan x}$ *w.r.t.* $\sin x$.

Solution. (*a*) We shall apply sum rule, chain rule and the standard results.

Let $y=\tan(\log\tan^{-1}\sqrt{x})+\tan^{-1}(\sin e^{ax})$

$$\frac{dy}{dx}=\frac{d}{dx}[\tan(\log\tan^{-1}\sqrt{x})]+\frac{d}{dx}[\tan^{-1}(\sin e^{ax})]$$

$$=\sec^2[\log\tan^{-1}\sqrt{x})\frac{d}{dx}[\log\tan^{-1}\sqrt{x}]$$

$$+\frac{1}{1+\sin^2 e^{ax}}\cdot\frac{d}{dx}(\sin e^{ax})$$

$$=\sec^2(\log\tan^{-1}\sqrt{x})\frac{1}{\tan^{-1}\sqrt{x}}\cdot\frac{d}{dx}(\tan^{-1}\sqrt{x})$$

$$+\frac{1}{1+\sin^2 e^{ax}}\cos e^{ax}\frac{d}{dx}(e^{ax})$$

$$=\sec^2(\log\tan^{-1}\sqrt{x})\frac{1}{\tan^{-1}\sqrt{x}}\cdot\frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x})$$

$$+\frac{1}{1+\sin^2 e^{ax}}\cdot\cos e^{ax}\cdot e^{ax}\frac{d}{dx}(ax)$$

$$=\frac{\sec^2(\log\tan^{-1}\sqrt{x})}{2(\tan^{-1}\sqrt{x})\sqrt{x}}\cdot\frac{1}{1+x}+\frac{ae^{ax}\cos(e^{ax})}{1+\sin^2(e^{ax})}$$

(*b*) Let $y=e^{\tan x}$, $z=\sin x$, then we need $\frac{dy}{dz}$, for which we shall use chain rule. Now

$$\frac{dy}{dx}=e^{\tan x}\frac{d}{dx}(\tan x)=\sec^2 x\, e^{\tan x}\text{ and }\frac{dz}{dx}=\cos x$$

$$\therefore\quad \frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}=e^{\tan x}\cdot\frac{\sec^2 x}{\cos x}=e^{\tan x}\cdot\sec^3 x.$$

Example 5. *Find* $\frac{dy}{dx}$ *when* $x=a\cos^3 t$, $y=a\sin^3 t$.

Solution.
$$\frac{dx}{dt}=-3a\cos^2 t\sin t$$

$$\frac{dy}{dt}=3a\sin^2 t\cos t$$

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\frac{3a\sin^2 t\cos t}{3a\cos^2 t\sin t}=-\tan t.$$

Example 6. *Find* $\frac{dy}{dx}$ *when* $x=a(t-\sin t)$, $y=a(1-\cos t)$.

Solution. Differentiating w.r.t. t, we obtain

$$\frac{dx}{dt}=a(1-\cos t),\ \frac{dy}{dt}=a\sin t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{a \sin t}{a(1-\cos t)}$$

$$=\frac{2a \sin \frac{t}{2} \cos \frac{t}{2}}{2a \sin^2 \frac{t}{2}}=\cot \frac{t}{2}.$$

EXERCISE (I)

Find the derivatives of the following w.r.t. x :

1. (a) $\frac{1}{\sqrt{x}}$, (b) $\frac{-3}{\sqrt[3]{x^7}}$

2. (a) $7x^4+3x^3-9x+5$, (b) $x+\frac{4}{x}-\frac{2}{x^7}$

3. (a) $\left(x-\frac{1}{x}\right)^2$, (b) $\left(x^{\frac{1}{3}}+\frac{1}{x^{\frac{1}{3}}}\right)^3$

4. (a) $(x+a)(x+b)(x+c)$, (b) $\sqrt{x}\,(ax^2+bx+c)(lx^2+mx+n)$

5. (a) $\frac{3x^2+5x}{7x+4}$, (b) $\frac{(2x+1)(3x+1)}{4x+1}$, (c) $\frac{5x^4-6x^2-7x+8}{5x-6}$

6. (a) $(5x^3+6x^2+11x+7)^{11}$, (b) $\sqrt{ax^2+bx+c}$

   (c) $\frac{1}{\sqrt[3]{2x^4+3x^3-5x+6}}$

7. (a) $(1-x^2)\tan x$, (b) $3\sqrt{x}\sin x$, (c) $x^3 \tan x$

8. (a) $\frac{1+\cos x}{\sin x}$, (b) $\frac{\tan x^2}{ax+b}$, (c) $\frac{\log \cos x}{\tan (\log x)}$

9. (a) $\frac{\sin 2x}{1+\cos 2x}$, (b) $\frac{1-\cos 2x}{\sin 2x}$, (c) $\frac{\cos x}{\cos x+\sin x}$

10. If $y=\sin(2\sin^{-1} x)$, show that $\frac{dy}{dx}=2\sqrt{\frac{1-y^2}{1-x^2}}$.

11. (a) $\log \sin x$, (b) $e^{\log \sin x}$, (c) $\log(\sqrt{x-1}-\sqrt{x+1})$

12. (a) $\log \sin \sqrt{x}$, (b) $\sqrt{\sin \sqrt{x}}$, (c) $e^{\log \tan^{-1} x}$

    (d) $\log\sqrt{x+\sqrt{x^2+a^2}}$

13. (a) $\frac{1+\tan x}{1-\tan x}$, (b) $\frac{a+b \sin x}{a \sin x+b}$, (c) $\frac{\cos x+\sin x}{\cos x-\sin x}$

14. (*a*) $\cos\{2\sin^{-1}(\cos x)\}$, (*b*) $\tan^{-1}\dfrac{2+\cos x}{1+2\cos x}$

(*c*) $b\tan^{-1}\left(\dfrac{x}{a}\tan^{-1}\dfrac{x}{a}\right)$, (*d*) $\sqrt{\sin(m\sin^{-1}x)}$

(*e*) $e^{\tan^{-1}x}\log(\sec^2 x^3)$.

15. $\log\sec^{-1}x^3+\log(\log\cos x^2)$

16. (*a*) If $y=\dfrac{1}{6}\log\dfrac{(x-1)^2}{x^2+x+1}+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$,

prove that $\dfrac{dy}{dx}=\dfrac{x}{x^3-1}$

(*b*) If $y=\dfrac{1}{3}\log\dfrac{x+1}{\sqrt{(x^2-x+1)}}+\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}$,

prove that $\dfrac{dy}{dx}=\dfrac{1}{1+x^3}$

17. Differentiate $\sin(\log x)$ w.r.t. $\tan(e^x)$.

18. Differentiate $(1+x^2)\cot^{-1}x$ w.r.t. $\log\{e^x(1+x)\}$

19. Differentiate $(x^2+ax+a^2)^n\log\cot\dfrac{x}{2}$ w.r.t. $\tan^{-1}(a\cos bx)$

20. Differentiate $x^n\log\tan^{-1}x$ w.r.t. $\dfrac{\sin\sqrt{x}}{x^{3/2}}$

Find $\dfrac{dy}{dx}$ in the following cases :

21. $x=a(t-\sin t)$, $y=a(1-\cos t)$

22. $x=\dfrac{3at}{(1+t^3)}$, $y=\dfrac{3at^2}{(1+t^3)}$

23. $x=\log t+\sin t$, $y=e^t+\cos t$

24. $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\cos\theta)$

25. $x=3\cos t-2\cos^3 t$, $y=3\sin t-2\sin^3 t$.

ANSWERS

1. (*a*) $-\frac{1}{2}x^{-3/2}$, (*b*) $7x^{-10/3}$. 2. (*a*) $28x^3+9x^2-9$,

(*b*) $1-\dfrac{4}{x^3}+\dfrac{14}{x^8}$. 3. (*a*) $2x-\dfrac{2}{x^3}$, (*b*) $1-\dfrac{1}{x^2}+\dfrac{1}{x^{2/3}}-\dfrac{1}{x^{4/3}}$

4. (*a*) $3x^2+2ax+2bx+2cx+ab+bc+ca$,

(*b*) $\frac{1}{2\sqrt{x}}(lx^2+mx+n)(ax^2+bx+c)+\sqrt{x}(lx^2+mx+n)(2ax+b)$
$+\sqrt{x}(ax^2+bx+c)(2lx+m)$

5. (*a*) $\frac{21x^2+24x+20}{(7x+4)^2}$, (*b*) $\frac{24x^2+12x+1}{(4x+1)^2}$

(*c*) $\frac{75x^4-120x^3-30x^2+72x+2}{(5x-6)^2}$

6. (*a*) $11(5x^3+6x^2+11x+7)^{10}(15x^2+12x+11)$, (*b*) $\frac{2ax+b}{2\sqrt{ax^2+bx+c}}$

(*c*) $-\frac{8x^3+9x^2-5}{3\cdot\sqrt[3]{(2x^4+3x^3-5x+6)^4}}$ 7. (*a*) $(1-x^2)\sec^2 x-2x\tan x$,

(*b*) $\frac{3}{2\sqrt{x}}(2x\cos x+\sin x)$, (*c*) $x^2(3\tan x+x\sec^2 x)$

8. (*a*) $-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$, (*b*) $\frac{2x(ax+b)\sec^2 x^2-a\tan x^2}{(ax+b)^2}$,

(*c*) $-\frac{[\tan(\log x)\tan x+x^{-1}\log\cos x\sec^2(\log x)]}{\tan^2(\log x)}$

9. (*a*) $\sec^2 x$, (*b*) $\sec^2 x$, (*c*) use $\cos 2x=(\cos x-\sin x)\times(\cos x+\sin x)$,

$-(\sin x+\cos x)$ **11.** (*a*) $\cot x$, (*b*) $\cos x$, (*c*) $\frac{-1}{2\sqrt{x^2-1}}$

12. (*a*) $\frac{(\cot\sqrt{x})}{2\sqrt{x}}$, (*b*) $\frac{\cos\sqrt{x}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}$, (*c*) $\frac{e^{\log(\tan^{-1})x}}{\tan^{-1}x(1+x^2)}$

(*d*) $\frac{1}{2\sqrt{x^2+a^2}}$ 13. (*a*) $\frac{2\sec^2 x}{(1-\tan x)^2}$, (*b*) $\frac{(b^2-a^2)\cos x}{(a\sin x+b)^2}$

(*c*) $\frac{2}{(1-\sin 2x)}$ **14.** (*a*) $2\sin\{2\sin^{-1}(\cos x)\}$

(*b*) $\frac{3\sin x}{5\cos^2 x+8\cos x+5}$ (*c*) $\frac{ab}{a^2+x^2\left(\tan^{-1}\frac{x}{a}\right)^2}\left[\tan^{-1}\frac{x}{a}+\frac{ax}{a^2+x^2}\right]$

(*d*) $\frac{m\cos(m\sin^{-1}x)}{2\sqrt{\sin(m\sin^{-1}x)}}\cdot\frac{1}{\sqrt{1-x^2}}$ (*e*) $e^{\tan^{-1}x}\left[\frac{\log(\sec^2 x^3)}{1+x^2}+6x^2\tan x^3\right]$

15. $\frac{2}{2\sqrt{x^4-1}\sec^{-1}x^2}-\frac{2x\sin x^2}{\cos x^2\log\cos x^2}$.

17. $\dfrac{\cos(\log x)/x}{\sec^2(e^x)\, e^x}$ 18. $\dfrac{(-1+2x\cot^{-1}x)(x+1)}{(2+x)}$

19. $$\frac{-(1+a^2\cos^2 bx)(x^2+ax+a^2)^{n-1} \times\left[n(2x+a)\log\cot\frac{x}{2}-\operatorname{cosec}x\,(x^2+ax+a^2)\right]}{ab\sin bx}$$

20. $2.\ \dfrac{n(1+x^2)\tan^{-1}x\log\tan^{-1}x+x}{(1+x^2)\tan^{-1}x\,(\sqrt{x}\cos\sqrt{x}-3\sin\sqrt{x})}\cdot x^{2n+3/2}$

21. $\cot\dfrac{t}{2}$ 22. $\dfrac{(2t-t^4)}{(1-2t^3)}$ 23. $\dfrac{t(e^t-\sin t)}{1+t\cos t}$ 24. $\tan\theta$ 25. $\cot t$

17·11. DIFFERENTIATION BY THE METHOD OF SUBSTITUTION

Sometimes we can reduce the given expression to be differentiated to a much simpler form by making a suitable substitution. To hit at the proper substitution, a fair knowledge of trigonometry and algebra, together with a good amount of practice is needed. We illustrate the method by the following solved examples :

Example 7. *Find the derivative of* $\sin^{-1}\dfrac{2x}{1+x^2}$.

Solution. Let $y=\sin^{-1}\dfrac{2x}{1+x^2}$ and put $x=\tan\theta$.

then $$y=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}(\sin 2\theta)=2\theta$$

Now $x=\tan\theta$ gives $\theta=\tan^{-1}x$.

$$\therefore \qquad 2\theta=2\tan^{-1}x \qquad (1)$$

Therefore, instead of finding the derivative of the given expression, we find the derivative of $2\tan^{-1}x$ which is in much simpler form than the given expression. Differentiating (1) with respect to x, we get

$$\frac{dy}{dx}=2\cdot\frac{1}{1+x^2}$$

Hence $$\frac{d}{dx}\left(\sin^{-1}\frac{2x}{1+x^2}\right)=\frac{2}{1+x^2}.$$

Example 8. *If* $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, *prove that* $\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$

Solution. Put $x=\tan\theta$ so that $\theta=\tan^{-1}x$

$$y=\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

then $y=2\tan^{-1}x$. Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{2}{1+x^2}.$$

Example 9. *Find the derivative of* $\tan^{-1}\frac{3x-x^3}{1-3x^2}$.

Solution. Let $y=\tan^{-1}\frac{3x-x^3}{1-3x^2}$. Put $x=\tan\theta$ so that $\theta=\tan^{-1}x$

then $$y=\tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]=\tan^{-1}(\tan 3\theta)=3\theta$$

$\therefore$ $$y=3\tan^{-1}x$$

Differentiating w.r.t. x, we get $\frac{dy}{dx}=\frac{3}{1+x^2}$.

Example 10. *Find the derivative of* $\tan^{-1}\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}$.

Solution. Let $y=\tan^{-1}\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}$. Put $\sqrt{x}=\tan\theta$ and $\sqrt{a}=\tan\alpha$

Then $$y=\tan^{-1}\left[\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}\right]=\tan^{-1}[\tan(\theta+\alpha)]=\theta+\alpha$$

Now $\sqrt{x}=\tan\theta \Rightarrow \theta=\tan^{-1}\sqrt{x}$ and $\sqrt{a}=\tan\alpha\Rightarrow\alpha=\tan^{-1}\sqrt{a}$

$\therefore$ $y=\tan^{-1}\sqrt{x}+\tan^{-1}\sqrt{a}$. Differentiating *w.r.t.* x, we get

$$\frac{dy}{dx}=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}(1+x)}.$$

Example 11. *Find the derivative of* $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$.

Solution Let $y=\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Put $x=\tan\theta$ so hat $\theta=\tan^{-1}x$ then we have

$$y=\tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]=\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$$=\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]=\tan^{-1}\left[\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]=\frac{\theta}{2}$$

$\therefore \qquad y=\frac{1}{2}\tan^{-1}x$

Differentiating, we get $\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{1+x^2}$.

Example 12. *If* $y=\sec^{-1}\frac{1+x^2}{1-x^2}$, *show that* $\frac{dy}{dx}=\frac{2}{1+x^2}$.

Solution. Put $x=\tan\theta$ so that $\theta=\tan^{-1}x$

$$y=\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)=\sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)=\sec^{-1}(\sec 2\theta)=2\theta$$

then $y=2\tan^{-1}x$. Differentiating, we get

$$\frac{dy}{dx}=\frac{2}{1+x^2}.$$

Example 13. *If* $y=\tan^{-1}\left(\frac{a\sin x+b\cos x}{a\cos x-b\sin x}\right)$, *find* $\frac{dy}{dx}$.

Solution. Let $a=r\cos\alpha$, $b=r\sin\alpha$ so that

$$\frac{a\sin x+b\cos x}{a\cos x-b\sin x}=\frac{\sin(x+\alpha)}{\cos(x+\alpha)}=\tan(x+\alpha)$$

$$\therefore \qquad y=\tan^{-1}[\tan(x+\alpha)]=x+\alpha$$

Hence $\qquad \frac{dy}{dx}=1$, α being a constant.

Example 14. *Differentiate with respect to x the function :*

$$\tan^{-1}\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$

Solution. Put $x=\cos 2\theta$, then

$$\sqrt{1+x}=\sqrt{1+\cos 2\theta}=\sqrt{2}\cos\theta,\ \sqrt{1-x}=\sqrt{1-\cos 2\theta}=\sqrt{2}\sin\theta$$

$$\therefore \quad y=\tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1-x}+\sqrt{1-x}}=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)=\tan^{-1}\left\{\tan^{-1}\left(\frac{\pi}{4}-\theta\right)\right\}=\frac{\pi}{4}-\theta$$

$$y=\tfrac{1}{4}\pi-\tfrac{1}{2}\cos^{-1}x$$

Hence $\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}$

Example 15. *Differentiate*

$$\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} \text{ w.r.t. } \cos^{-1} x^2.$$

Solution. Let $\cos^{-1} x^2 = u$ so that $x^2 = \cos u$, hence

$$\sqrt{1+x^2}=\sqrt{1+\cos u}=\sqrt{2}\cos \tfrac{1}{2}u$$

$$\sqrt{1-x^2}=\sqrt{1-\cos u}=\sqrt{2}\sin \tfrac{1}{2}u$$

Now
$$y=\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\tan^{-1}\left[\frac{\cos\frac{1}{2}u-\sin\frac{1}{2}u}{\cos\frac{1}{2}u+\sin\frac{1}{2}u}\right]$$

$$=\tan^{-1}\left(\frac{1-\tan\frac{1}{2}u}{1+\tan\frac{1}{2}u}\right)=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{1}{2}u\right)\right\}$$

$$=\frac{1}{4}\pi-\frac{1}{2}u$$

$$\therefore \quad \frac{dy}{du}=-\tfrac{1}{2}.$$

EXERCISE (II)

Differentiate with respect to x the following :

1. (*a*) $\sin^{-1}(2x\sqrt{1-x^2})$, (*b*) $\cos^{-1}(1-2x^2)$, (*c*) $\sin^{-1}(3x-4x^3)$

2. (*a*) $\sin^{-1}(\sqrt{1-x^2})$, (*b*) $\cos^{-1}(2x^2-1)$, (*c*) $\cos^{-1}(4x^3-3x)$

3. (*a*) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, (*b*) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, (*c*) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

4. (*a*) $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, (*b*) $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$, (*c*) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

5. (*a*) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$, (*b*) $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

   (*c*) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\}$

6. (*a*) $\sec^{-1}\frac{\sqrt{a^2+x^2}}{a}$, (*b*) $\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)$ 7. $\tan^{-1}\frac{3a^2x-x^3}{a^3-3ax^2}$

ANSWERS

1 **Hint.** Put x sin 0 ; (*a*) and (*b*) $\frac{2}{\sqrt{1-x^2}}$, (*c*) $\frac{3}{\sqrt{1-x^2}}$

2. **Hint.** Put $x=\cos 0$; (*a*) $\frac{-1}{\sqrt{1-x^2}}$ (*b*) $\frac{-2}{\sqrt{1-x^2}}$, (*c*) $\frac{-3}{\sqrt{1-x^2}}$

3. **Hint.** Put $x=\tan\theta$; $\frac{2}{1+x^2}$ in each case.

4. **Hint** Put $x=\tan\theta$; (*a*) $\frac{2}{1+x^2}$. (*b*) $\frac{-2}{1+x^2}$, (*c*) $\frac{3}{1+x^2}$

5. (*a*) **Hint** Express $\frac{\sin x}{1+\cos x}=\tan\frac{x}{2}$; $\frac{1}{2}$

(*b*) **Hint.** Express $\sqrt{\frac{1-\cos x}{1+\cos x}}=\sqrt{\frac{2\sin^2(x/2)}{2\cos^2(x/2)}}=\tan\frac{x}{2}$; $\frac{1}{2}$

(*c*) **Hint.** Put $x^2=\cos\theta$.

The bracketed exp. $=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$; $\frac{-x}{\sqrt{1-x^4}}$

6. (*a*) $\frac{a}{a^2+x^2}$, (*b*) $\frac{6}{1+4x^2}$ 7. **Hint.** $x=a\tan\theta$, $\frac{3a}{a^2+x^2}$

17·12. LOGARITHMIC DIFFERENTIATION

In order to find the derivative of (*i*) a function which is the product or quotient of a number of factors or (*ii*) a function of the form $(\text{variable})^{\text{variable}}$, *i.e.*, of the form $[f(x)]^{g(x)}$, where $f(x)$ and $g(x)$ are both derivable, it is often advisable to take the logarithms of the function first and then differentiate. The process of taking logarithm and then differentiating is known as logarithmic differentiation. The following examples will illustrate the method of logarithmic differentiation.

Example 16. *Differentiate* $\log\left[e^{3x}.\left(\frac{5x-3}{4x+2}\right)^{1/3}\right]$ *w.r.t. x.*

Solution. Let
$$y=\log\left[e^{3x}.\left(\frac{5x-3}{4x+2}\right)^{1/3}\right]$$
$$=\log e^{3x}+\log\left(\frac{5x-3}{4x+2}\right)^{1/3}$$
$$=3x\log e+\frac{1}{3}\log\frac{5x-3}{4x+2}$$
$$=3x+\tfrac{1}{3}[\log(5x-3)-\log(4x+2)]$$

Differentiating *w.r.t. x*, we get
$$\frac{dy}{dx}=\frac{d}{dx}\left[3x+\frac{1}{3}\{\log(5x-3)-\log(4x+2)\}\right]$$
$$=\frac{d}{dx}(3x)+\frac{1}{3}\left[\frac{d}{dx}\log(5x-3)-\frac{d}{dx}\log(4x+2)\right]$$
$$=3+\frac{1}{3}\left\{\frac{1}{5x-3}\cdot\frac{d}{dx}(5x-3)-\frac{1}{4x+2}\cdot\frac{d}{dx}(4x+2)\right\}$$
$$=3+\frac{1}{3}\left\{\frac{5}{5x-3}-\frac{4}{4x+2}\right\}$$

Example 17. *Differentiate* $(ax^2+bx+c)^n\, e^{cx} \tan(lx+m) \cos^{-1} x$ *w.r.t. x.*

Solution. Let $y=(ax^2+bx+c)^n\, e^{cx} \tan(lx+m) \cos^{-1} x$ ...(1)

Taking logarithms of both sides, we get

$$\log y = n \log(ax^2+bx+c)+cx+\log \tan(lx+m)+\log \cos^{-1} x$$

Differentiating *w.r.t. x*, we get

$$\frac{1}{y}\frac{dy}{tx}=\frac{n}{ax^2+bx+c}\cdot\frac{d}{dx}(ax^2+bx+c)+c$$

$$+\frac{1}{\tan(lx+m)}\frac{d}{dx}\tan(lx+m)+\frac{1}{\cos^{-1}x}\frac{d}{dx}(\cos^{-1}x)$$

$$=\frac{n}{ax^2+bx+c}\cdot(2ax+b)+c+\frac{1}{\tan(lx+m)}\cdot\sec^3(lx+m)\,.\,l$$

$$+\frac{1}{\cos^{-1}x}\cdot\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore\quad \frac{dy}{dx}=y\left\{\frac{n(2ax+b)}{ax^2+bx+c}+c+\frac{l\sec^2(lx+m)}{\tan(lx+m)}-\frac{1}{\cos^{-1}x\sqrt{1-x^2}}\right\}$$

where y is given in (1).

Example 18. *Differentiate* $\dfrac{(x^2-1)^{4/5}(3x+5)^{2/7}e^{3x}}{(x-9)^{1/2}(2x-7)^4}$ *w.r.t. x.*

Solution. Let $y=\dfrac{(x^2-1)^{4/5}(3x+5)^{2/7}e^{3x}}{(x-9)^{1/2}(2x-7)^4}$

Taking logarithms of both sides and using the theorems on logarithms, we get

$$\log y=\tfrac{4}{5}\log(x^2-1)+\tfrac{2}{7}\log(3x+5)+3x-\tfrac{1}{2}\log(x-9)-4\log(2x-7)$$

Differentiating *w.r.t. x*, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{4}{5}\cdot\frac{1}{x^2-1}\,.\,2x+\frac{2}{7}\,\frac{1}{3x+5}\cdot 3+3-\frac{1}{2}\cdot\frac{1}{x-9}$$

$$-4\cdot\frac{1}{2x-7}\,.\,2$$

$$\therefore\quad \frac{dy}{dx}=\frac{(x^2-1)^{4/5}(3x+5)^{2/7}e^{3x}}{(x-9)^{1/2}(2x-7)^4}$$

$$\times\left[\frac{8x}{5(x^2-1)}+\frac{6}{7(3x+5)}+3-\frac{1}{2(x-9)}-\frac{8}{(2x-7)}\right]$$

Example 19. (*a*) *If* $y=x^x$, *find* $\dfrac{dy}{dx}$.

Solution. We have $y=x^x$

Taking logarithms of both sides, we have

$$\log y=x \log x$$

Differentiating *w.r.t.* x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx}=\log x+x.\frac{1}{x}=1+\log x$$

$$\therefore \quad \frac{dy}{dx}=y(1+\log x)=x^x(1+\log x)$$

(*b*) *If* $y=x^{x^x}$, *find* $\frac{dy}{dx}$.

Solution. Taking logarithms of both sides, we have

$$\log y=\log x^{x^x}=x^x \log x$$

Differentiating *w.r.t.* x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx}=x^x \frac{d}{dx}(\log x)+\log x \frac{d}{dx}(x^x)$$

But $\quad \frac{d}{dx}(x^x)=x^x(1+\log x) \quad$ [From Example 19(*a*)]

$$\therefore \quad \frac{1}{y} \cdot \frac{dy}{dx}=x^x.\frac{1}{x}+\log x \,.\, x^x(1+\log x)$$

$$\Rightarrow \quad \frac{dy}{dx}=x^{x^x}[x^{x-1}+\log x \,.\, x^x(1+\log x)]$$

Example 20. *If* $x^y=e^{x-y}$, *prove that* $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$

[*I.C.W.A., June 1985*]

Solution. Taking logarithms of both sides, we have

$$y \log x=(x-y)\log e \quad \Rightarrow \quad y(1+\log x)=x \qquad (\because \log e=1)$$

$$\therefore \quad y=\frac{x}{1+\log x}$$

$$\therefore \quad \frac{dy}{dx}=\frac{(1+\log x)\,.\,1-x\times\frac{1}{x}}{(1+\log x)^2}=\frac{\log x}{(1+\log x)^2}.$$

Example 21. *Differentiate* $x^{\tan x}+(\sin x)^{\cos x}$ *w.r.t.* x.

Solution. Let $\quad y=x^{\tan x}+(\sin x)^{\cos x}$

Here we cannot take logarithms directly because

$$\log(m+n)\neq\log m+\log n$$

We now find the differential coefficient of each term on R.H.S. separately.

Let $u = x^{\tan x}$ and $v = (\sin x)^{\cos x}$ so that $y = u + v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Now $u = x^{\tan x}$ Taking logarithms of both sides, we get

$$\log u = \log (x^{\tan x}) = \tan x \, . \log x$$

Differentiating *w.r.t.* x, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = \tan x \, . \, \frac{1}{x} + \log x \, . \sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \log x \, . \sec^2 x\right] \quad \ldots(1)$$

Again $v = (\sin x)^{\cos x}$. Taking logarithms of both sides, we get

$$\log v = \log (\sin x)^{\cos x} = \cos x \log \sin x$$

Differentiating, $\frac{1}{v} \cdot \frac{dv}{dx} = \cos x \, . \, \frac{1}{\sin x} . \cos x - \log \sin x \, . \sin x$

$$\Rightarrow \quad \frac{dv}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \log \sin x \, . \sin x\right] \quad \ldots(2)$$

Hence $\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \log x \, . \sec^2 x\right]$

$$+ (\sin x)^{\cos x}\left[\frac{\cos x^2}{\sin x} - \log \sin x \, . \sin x\right]$$

EXERCISE (III)

1. If $y = \sin x \cos x (\log x) \, . \, e^x \, . \tan^{-1} x \, . \, x^n$, find $\frac{dy}{dx}$.

2. Find the derivative of (*a*) $\frac{(x+a)(x+b)(x+c)(x+d)}{(x-a)(x-b)(x-c)(x-d)}$,

   (*b*) $\frac{x\sqrt{x^2-4a^2}}{\sqrt{x^2-a^2}}$

3. If $y = (2-x)\left(\frac{3-x}{1+x}\right)^{1/2}$, find $\frac{dy}{dx}$.

Differentiate the following functions *w.r.t.* x :

4. (*a*) $\log \left[e^x \left(\frac{x-2}{x+3}\right)^{3/4}\right]$, (*b*) $e^{5/x} . (\sqrt{2x^2-1})$,

   (*c*) $\frac{x^2 \, e^{5x}}{(3x+1)^{1/2} (2x-1)^{1/3}}$

5. $\dfrac{x^{1/2}(5-2x)^{2/3}}{(4-3x)^{3/4}(7-4x)^{4/5}}$ 6. x^x.

7. (a) $(1+x)^{2x}$, (b) $x^{1/x}$, (c) $(x^x)^x$

8. (a) $x^{\log x}$, (b) $x^{\log(\log x)}$

9. $x^a+a^x+x^x+a^a$, (a is constant)

10. (a) $(\sin x)^{\log x}+(x)^{\sin x}$, (b) $(\cot x)^{\sin x}+(\tan x)^{\cos x}$

ANSWERS

1. $y\left\{\cot x-\tan x+\dfrac{1}{x\log x}+1+\dfrac{1}{\tan^{-1}x}\cdot\dfrac{1}{1+x^2}+\dfrac{n}{x}\right\}$

2. (a) $y\left[\dfrac{1}{x+a}+\dfrac{1}{x+b}+\dfrac{1}{x+c}+\dfrac{1}{x+d}-\dfrac{1}{x-a}-\dfrac{1}{x-b}-\dfrac{1}{x-c}-\dfrac{1}{x-d}\right]$

(b) $\dfrac{(x^4-2a^2x^2+4a^4)}{(x^2-a^2)^{3/2}(x^2-4a^2)^{1/2}}$

3. $(2-x)\left[\dfrac{3-x}{1+x}\right]^{1/2}\left\{\dfrac{1}{x-2}+\dfrac{1}{2(x-3)}-\dfrac{1}{2(1+x)}\right\}$

4. (a) $1+\dfrac{3}{4(x-2)}-\dfrac{3}{4(x+3)}$, (b) $e^{5/x}\cdot\dfrac{2x^3-10x^2+5}{x^2\sqrt{2x^2-1}}$

(c) $\dfrac{x^2e^{5x}}{(3x+1)^{1/2}(2x-1)^{1/3}}\left[5+\dfrac{2}{x}-\dfrac{3}{2(3x+1)}-\dfrac{2}{3(2x-1)}\right]$

5. $\dfrac{x^{1/2}(5-2x)^{2/3}}{(4-3x)^{3/4}(7-4x)^{4/5}}\left[\dfrac{1}{2x}-\dfrac{4}{3(5-2x)}+\dfrac{9}{4(4-3x)}+\dfrac{16}{3(7-4x)}\right]$

6. $x^x(1+\log x)$

7. (a) $2(1+x)^{2x}\left[\dfrac{x}{x+1}+\log(x+1)\right]$, (b) $x^{1/x}\cdot\dfrac{1-\log x}{x^2}$

(c) $x^{x^2+1}(1+2\log x)$ 8. $(2x^{\log x-1}.\log x$,

(b) $x^{\log(\log x)}\left(\dfrac{1+\log(\log x)}{x}\right)$

9. $ax^{a-1}+a^x\log a+x^x(1+\log x)$

10. (a) $(\sin x)^{\log x}\left(\cot x\,.\log x+\dfrac{\log\sin x}{x}\right)$

$+(x)^{\sin x}\left[\cos x\log x+\dfrac{\sin x}{x}\right]$

(b) $(\cot x)^{\sin x}[\cos x(\log\cot x)-\sec x]$

$+(\tan x)^{\cos x}[\operatorname{cosec} x-\sin x\log\tan x]$.

17·13. DIFFERENTIATION OF IMPLICIT FUNCTIONS

Sometimes y is not given directly in terms of x, the value of $\frac{dy}{dx}$ can be found by differentiating the given equation term by term and then separating $\frac{dy}{dx}$.

Example 22. *Find* $\frac{dy}{dx}$ *if* $x^3+y^3=3axy$.

[*I.C.W.A., December, 1990*]

Solution. We have

$$x^3+y^3=3axy$$

Differentiating with respect to x, we have

$$\therefore \quad 3x^2+3y^2\frac{dy}{dx}=3a\left(y+x\frac{dy}{dx}\right)$$

or $$\frac{dy}{dx}(3y^2-3ax)=3ay-3x^2$$

or $$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}.$$

Example 23. *If* $x\sqrt{1+y}+y\sqrt{1+x}=0$, *prove that*

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

Solution. From the given equation, we get

$x\sqrt{1+y}=-y\sqrt{1+x}$, which on squaring and rearranging gives

$x^2-y^2+x^2y-y^2x=0$, *i.e.*, $(x-y)\{x+y+xy\}=0$

Thus $$x+xy+y=0 \qquad [\because\ x\neq y \Rightarrow x-y\neq 0]$$

$$\Rightarrow \quad y=-\frac{x}{1+x}$$

$$\therefore \quad \frac{dy}{dx}=-\frac{(1+x)\cdot 1-x\cdot 1}{(1+x)^2}=\frac{-1}{(1+x)^2}.$$

Example 24. *Find* $\frac{dy}{dx}$ *from the following equation :*

$$x^2-y^2+3x=5y.$$

Solution. Differentiating, we get $2x-2y\cdot\frac{dy}{dx}+3=5\frac{dy}{dx}$

$$\Rightarrow \quad 2y\cdot\frac{dy}{dx}+5\frac{dy}{dx}=2x+3$$

$$\Rightarrow \quad \frac{dy}{dx}(2y+5)=2x+3$$

$$\therefore \quad \frac{dy}{dx}=\frac{2x+3}{2y+5}$$

Example 25. *Find* $\frac{dy}{dx}$ *if* $x^3-xy^2+3y^2+2=0.$

Solution. Differentiating each term w.r.t. x, we get

$$3x^2+\left(-x\ .\ 2y\frac{dy}{dx}-y^2\right)+6y\,\frac{dy}{dx}=0.$$

$$\Rightarrow \quad 3x^2-2xy\,\frac{dy}{dx}-y^2+6y\,\frac{dy}{dx}=0$$

$$\Rightarrow \quad (6y-2xy)\frac{dy}{dx}=y^2-3x^2$$

Hence $$\frac{dy}{dx}=\frac{y^2-3x^2}{2y(3-x)}.$$

Example 26. *Find* $\frac{dy}{dx}$ *if* $ax^2+2hxy+by^2+2gx+2fy+c=0.$

Solution. Differentiating the equation w.r.t. x, we get

$$2ax+2h\left(x\,\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}+2g+2f\,\frac{dy}{dx}=0$$

$$\Rightarrow \quad (hx+by+f)\,\frac{dy}{dx}=-(ax+hy+g)$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{ax+hy+g}{hx+by+f}$$

Example 27. *If* $y=x^{x^{x\ldots\infty}}$, *prove that*

$$x\,\frac{dy}{dx}=\frac{y^2}{1-y\log x}$$

Solution. $$y=x^{x^{x\ldots\infty}}=x^y$$

$$\therefore \quad \log y=y\log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\,\frac{dy}{dx}=y\,.\,\frac{1}{x}+\log x\,.\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y}-\log x\right)=\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y^2}{x(1-y\log x)}$$

Hence $$x\,\frac{dy}{dx}=\frac{y^2}{1-y\log x}.$$

EXERCISE (IV)

Find $\frac{dy}{dx}$ of the following :

1. (a) $x^2+y^2-2x=0$, (b) $x^2+3xy+y^2=4$.
2. $x^3+5x^4y+yx=5$ 3. $\sin y=x\sin(a+y)$.
4. $(x+y)^{m+n}=x^m y^n$ 5. $y^x=x^{\sin y}$
6. If $y=\sqrt{[\sin x+\sqrt{\sin x+\sqrt{\sin x}+\ldots\infty}}$, prove that

$$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

ANSWERS

1. (a) $\frac{1-x}{y}$, (b) $-\frac{2x+3y}{3x+2y}$
2. $\frac{dy}{dx}=-\frac{(3x^2+10xy+y)}{x(5x+1)}$ 3. $\frac{\sin^2(a+y)}{\sin a}$
4. $\frac{y}{x}$ 5. $\frac{-\log y+x^{-1}\sin y}{\cos y\log x-y^{-1}x}$

17·14. DERIVATIVE AS A RATE MEASURE

Differentiation is employed to measure the rate of change in a dependent variable with reference to a minute change in the independent variable. Let, the relation between two variable x and y be $y=f(x)$ and let δx represent a given increase in x, then δy will be consequent increase in y.

$\therefore$ For a unit change in x, the change in y is $\frac{\delta y}{\delta x}$

$\therefore$ $\frac{\delta y}{\delta x}$ represents the 'average' change in y per unit change in x in the interval $(x, x+\delta x)$.

Now, as δx approaches zero, the average rate $\frac{\delta y}{\delta x}$ in the interval $(x, x+\delta x)$ becomes the actual rate at x.

i.e., $\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x}$ becomes the actual rate at x.

i e. $\frac{dy}{dx}$ becomes the actual rate at x.

Hence $\frac{dy}{dx}$ represents the actual rate of change in y per unit change in x for the particular value of x.

Or $\frac{dy}{dx}$ is the rate at which y is changing with respect to x.

Now, to find out a change in the dependent variable, *i.e.*, δy when x, need not approach 0, we can use differentiation as an approximate measure of change so that

$$\delta y = \frac{dy}{dx} \cdot \delta x$$

This is because, $\frac{\delta y}{\delta x} = \frac{dy}{dx} + \varepsilon$

where ε is a small quantity which vanishes in the limit.

Proof.

$$\delta y = \left(\frac{dy}{dx} + \varepsilon\right)\delta x$$

$$= \frac{dy}{dx} \quad \delta x + \varepsilon \delta x$$

$$= \frac{dy}{dx} \cdot \delta x$$

This is because $\varepsilon \delta x$ approach zero as $\delta x \to 0$.

Hence $\delta y = \frac{dy}{dx} \cdot \delta x$

Velocity. It is defined as a change in a given phenomenon with respect to time. In business economics there is use of income velocity, money velocity, credit velocity, etc. In science, this refers to the rate of displacement or change of position with respect to time. For example,

If $f(V) = 3t + t^3$ ...(*i*)

Then, $f'(V)$ or $\frac{df(V)}{dt} = 3 + 3t^2$ ...(*ii*)

(*ii*) above refers to velocity.

Again if, $f(V) = 2t^2 + 3t$ (where t stands for time units)

Then $f'(V)$ or $\frac{df(V)}{dt} = 4t + 3$

Therefore, the velocity after 4 units of time is $4(4)+3=19$ per unit of time.

Acceleration. It refers to the rate of change in velocity or a change in the rate of change. Therefore, if velocity is expressed by

$\frac{dV}{dt}$, acceleration is expressesd

$$\text{by } \frac{d^2V}{dt^2}$$

This is because

$$\frac{dV}{dt}=\frac{d}{dt}(V) \text{ and } \frac{d}{dt}\left(\frac{dV}{dt}\right)=\frac{d^2V}{dt^2}$$

Illustration. *Given the function of speed as $f(S)=3t+t^3$ in t seconds, calculate both velocity and acceleration after 2 seconds.*

Solution. If

$$f(S)=3t+t^3$$

$$\frac{df(S)}{dt}=3+3t^2$$

and

$$\frac{d^2f(S)}{dt^2}=6t$$

Therefore, after 2 seconds

$$\text{Velocity}=3+3(2)^2=15$$

and

$$\text{Acceleration}=6\times 2 \quad =12$$

17·15. SUCCESSIVE DIFFERENTIATION

As observed in many of the preceding examples, the derivative of a function of x is, in general, also a function of x. This derivative, which may be called the *first* derived function, or *the first derivative* (of the function). may itself be differentiated, the result is accordingly called the *second* derived function or *the second derivative* (of the original function). If the second derivative is differentiated, the result is called the *third* derived function, or *the third derivative* and so on. If the operation of differentiation is performed on a function n times in succession, the final result is called the nth derived function or *the nth derivative* of the function.

NOTATIONS

I. If y denotes the function of x, then

the first derivative, namely $\frac{d}{dx}(y)$, is denoted by $\frac{dy}{dx}$,

the second derivative, namely $\frac{d}{dx}\left(\frac{d}{dx}\right)$, is denoted by $\frac{d^2y}{d^2x}$,

the third derivative, namely $\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$, is denoted by $\frac{d^3y}{dx^3}$

and so on. On this way of writing,

the nth derivative is denoted by $\dfrac{d^n y}{dx^n}$.

II. The letter D is frequently used to denote both the operations and the result of the operation indicated by the symbol $\dfrac{d}{dx}$. The successive derivatives of y are then Dy, $D(Dy)$, $D[D(Dy)]$, ..., these are respectively denoted by

$$Dy,\ D^2y,\ D^3y,\ ...,\ D^ny.$$

III. Instead of the symbols shown in I and II, for the successive derivatives of y, the following are sometimes used, namely

$$y',\ y'',\ y''',\ ...,\ y^{(n)}$$

IV. If the function be denoted by $f(x)$, its first, second, third, ..., and nth derivatives (with respect to x) are generally denoted by

$$f'(x), f''(x), f'''(x), ..., f^{(n)}(x) \text{ respectively,}$$

also by $\dfrac{d}{dx} f(x), \dfrac{d^2}{dx^2} f(x), \dfrac{d^3}{dx^3} f(x), ..., \dfrac{d^n}{dx^n} f(x)$.

THE nth DERIVATIVE OF SOME SPECIAL FUNCTIONS

I. $$\mathbf{y=x^n}$$

Then $y_1=nx^{n-1},\ y_2=n(n-1)x^{n-2},\ ...$

and $y_n=n(n-1)(n-2)...3.2.1\ x^{n-n}=n\,!$

II. $$\mathbf{y=(ax+b)^n}$$

Then $y_1=n(ax+b)^{n-1}\,a,$

$y_2=n(n-1)\ (ax+b)^{n-2}\,a^2,\ ....$

and $y_n=n(n-1)(n-2)...3.2.1\ (ax+b)^{n-n}a^n=n\,!\ a^n$

III. $$\mathbf{y=e^{ax}}$$

Then $y_1=ae^{ax},\ y_2=a^2\,e^{ax},\ y_3=a^3\,e^{ax},\ ...,\ y_n=a^n\,e^{ax}$

IV. $$\mathbf{y=\frac{1}{ax+b}}$$

$$y_1=\frac{(-1)a}{(ax+b)^2},\quad y_2=\frac{(-1)(-2)a^2}{(ax+b)^3},\ ....$$

$$y_n=\frac{(-1)(-2)(-3)...(-n)a^n}{(ax+b)^{n+1}}=\frac{(-1)^n\,n\,!\,a^n}{(ax+b)^{n+1}}$$

V. $$\mathbf{y=\sin\,(ax+b)}$$

$$y_1=a\cos\,(ax+b)=a\sin\left[\frac{1}{2}\,\pi+(ax+b)\right]$$

$$y_2 = a^2 \cos\left[\frac{1}{2}\pi + (ax+b)\right] = a^2 \sin\left[2.\frac{1}{2}\pi + (ax+b)\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$y_n = \frac{d^n}{dx^n}\left[\sin(ax+b)\right] = a^n \sin\left[n.\frac{1}{2}\pi + (ax+b)\right]$$

VI. $\quad \mathbf{y = e^{ax} \cos (bx+c)}$

Here $\quad \dfrac{dy}{dx} = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$

$$= e^{ax}[a\cos(bx+c) - b\sin(bx+c)]$$

Put $\quad a = r\cos\theta,\ b = r\sin\theta,$

then $\quad r = \sqrt{a^2+b^2}$ and $\tan\theta = \dfrac{b}{a}$

$\therefore \quad \dfrac{dy}{dx} = e^{ax}[r\cos\theta\cos(bx+c) - r\sin\theta\sin(bx+c)]$

$$= re^{ax}\cos(bx+c+\theta)$$

$$= e^{ax}(a^2+b^2)^{1/2}\cos\left(bx+c+\tan^{-1}\frac{b}{a}\right)$$

We see that to differentiate once, we multiply with $(a^2+b^2)^{1/2}$ and add $\tan^{-1}\dfrac{b}{a}$ to $(bx+c)$. To differentiate second time we shall have to multiply with the square of $(a^2+b^2)^{1/2}$ and add $2\tan^{-1}\dfrac{b}{a}$ to $(bx+c)$ so that we get

$$\frac{d^2y}{dx^2} = e^{ax}\left(a^2+b^2\right)^{2\cdot 1/2}\cos\left(bx+c+2\tan^{-1}\frac{b}{a}\right)$$

Therefore, to differentiate n times, we have to multiply by $(a^2+b^2)^{1/2}$ n times and add $n\tan^{-1}\dfrac{b}{a}$ to $(bx+c)$.

Thus $\quad \dfrac{d^ny}{dx^n} = e^{ax}\left(a^2+b^2\right)^{n\cdot 1/2}\cos\left(bx+c+n\tan^{-1}\dfrac{b}{a}\right)$

Example 28. *If $y = a\sin nx + b\cos nx$, prove that*

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

Solution. We have $\dfrac{dy}{dx} = an\cos nx - bn\sin nx$

Differentiating again, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = -an^2\sin nx - bn^2\cos nx$$

$$= -n^2(a\sin nx + b\cos nx) = -n^2y$$

$\Rightarrow \quad \frac{d^2y}{dx^2}+n^2y=0$, the required result.

Example 29. *Find the fourth derivative of* $\log\sqrt{3x+4}$.

Solution. Let $y=\log\sqrt{3x+4}=\frac{1}{2}\log(3x+4)$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{3x+4}\cdot 3=\frac{3}{2}(3x+4)^{-1}$$

Again differentiating, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=\frac{3}{2}\cdot(-1)(3x+4)^{-2}\cdot 3=-\frac{9}{2}(3x+3)^{-2}$$

Differentiating again, we have

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}=-\frac{9}{2}(-2)(3x+4)^{-3}\cdot 3=27(3x+4)^{-3}$$

Differentiating again, we have

$$\frac{d^4y}{dx^4}=27(-3)(3x+4)^{-4}\cdot 3=-243(3x+4)^{-4}=-\frac{243}{(3x+4)^4}.$$

Example 30. *Find* $\frac{d^2y}{dx^2}$ *when* $x=a\cos\theta$, $y=b\sin\theta$.

Solution. We have $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{b\cos\theta}{-a\sin\theta}=-\frac{b}{a}\cot\theta$

$$\frac{d^2y}{dx^2}=-\frac{b}{a}(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}=\frac{b}{a\sin^2\theta}\cdot\frac{1}{\frac{dx}{d\theta}}$$

But $\quad \frac{dx}{d\theta}=-a\sin\theta$, therefore $\frac{d^2y}{dx^2}=-\frac{b}{a^2\sin^3\theta}$.

Example 31. *If* $x=a(\theta-\sin\theta)$ *and* $y=a(1-\cos\theta)$, *find* $\frac{d^2y}{dx^2}$.

Solution. We have $\frac{dx}{d\theta}=a(1-\cos\theta)$ and $\frac{dy}{d\theta}=a\sin\theta$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\sin^2\frac{\theta}{2}}=\cot\frac{\theta}{2}$$

Then $\dfrac{d^2y}{dx^2}=-\text{cosec}^2\dfrac{\theta}{2}\ \dfrac{1}{2}\ \dfrac{d\theta}{dx}=-\dfrac{1}{2\sin^2\dfrac{\theta}{2}}\ \dfrac{1}{\dfrac{dx}{d\theta}}$

$$=-\frac{1}{2\sin^2\dfrac{\theta}{2}\ 2a\sin^2\dfrac{\theta}{2}}=-\frac{1}{4a\sin^4\dfrac{\theta}{2}}$$

Example 32. *If* $y=\dfrac{1}{2}(\sin^{-1}x)^2$, *show that*

$$(1-x^2)y_2,-xy_1=1$$

Solution. We have $y_1=\dfrac{1}{2}\ .\ 2\sin^{-1}x\ .\ \dfrac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \qquad \sqrt{1-x^2}\,y_1=\sin^{-1}x$$

Squaring both sides, we have

$$(1-x^2)y_1^{\,2}=(\sin^{-1}x)^2=2y$$

Differentiating both sides, we get

$$(1-x^2)\ .\ 2y_1y_2-2xy_1^{\,2}=2y_1$$

Cancelling out the common factor $2y_1$, we get

$$(1-x^2)y_2-xy_1=1.$$

Example 33. *If* $y=(x+\sqrt{1+x^2})^m$, *show that*

$$(1+x^2)y_2+xy_1=m^2y.$$

Solution. We have $y_1=m(x+\sqrt{1+x^2})^{m-1}\left[1+\dfrac{x}{\sqrt{1+x^2}}\right]$

$$=m(x+\sqrt{1+x^2})^{m-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$=m\ \frac{(x+\sqrt{1+x^2})^m}{\sqrt{1+x^2}}=\frac{my}{\sqrt{1+x^2}}$$

$$\Rightarrow \qquad \sqrt{1+x^2}\,y_1=my$$

Squaring both sides, we have $(1+x^2)y_1^{\,2}=m^2y^2$

Differentiating, we have $(1+x^2)2y_1y_2+2xy_1^{\,2}=m^2\ .\ 2yy_1$

Cancelling out the common factor $2y_1$, we get

$$(1+x^2)y_2+xy_1=m^2y.$$

Example 34. *If* $y=2x+\frac{4}{x}$, *prove that*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0.$$

[*I.C.W.A., June 1990*]

Solution. We have

$$y=2x+\frac{4}{x} \quad ...(1)$$

$$\frac{dy}{dx}=2-\frac{4}{x^2} \quad ...(2)$$

or $$x^2\frac{dy}{dx}=2x^2-4$$

Differentiating again, we have

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=4x$$

or $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x\frac{dy}{dx}-4x=0$$

or $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x\left(2-\frac{4}{x^2}\right)-4x=0 \quad \text{[From (2)]}$$

or $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2x-\frac{4}{x}-4x=0$$

or $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(2x+\frac{4}{x}\right)=0$$

or $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0. \quad \text{[From (1)]}$$

Example 35. *If* $2x=y^{1/4}+y^{-1/4}$, *then prove that*

$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-16y=0.$$

[*C.A., May 1991*]

Solution. We have

$$2x=y^{1/4}+y^{-1/4} \quad ...(1)$$

$\therefore$ $$2=\frac{1}{4}.\, y^{-3/4}\,.\,\frac{dy}{dx}-\frac{1}{4}\,.\,y^{-5/4}\,.\,\frac{dy}{dx}$$

or $$2=\frac{1}{4}\,.\,y^{-1}\,.\,(y^{1/4}-y^{-1/4})\,.\,\frac{dy}{dx}$$

or $$8y=(y^{1/4}-y^{-1/4})\,.\,\frac{dy}{dx} \quad ...(2)$$

But $(y^{1/4}-y^{-1/4})^2=(y^{1/4}+y^{-1/4})^2-4=4x^2-4$ [From (1)]

$\therefore \quad y^{1/4}-y^{-1/4}=2\sqrt{x^2-1}$

Substituting this in (2), we have

$$8y=2\sqrt{x^2-1}\,.\,\frac{dy}{dx}$$

or
$$4y=\sqrt{x^2-1}\,.\,\frac{dy}{dx}$$

or
$$16y^2=(x^2-1)\left(\frac{dy}{dx}\right)^2$$

Differentiating again, we have

$$32y\frac{dy}{dx}=2x\left(\frac{dy}{dx}\right)^2+(x^2-1).2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$$

or
$$16y=x\frac{dy}{dx}+(x^2-1)\frac{d^2y}{dx^2}$$

$$\therefore \quad (x^2-1)\cdot\frac{d^2y}{dx^2}+x\frac{dy}{dx}-16y=0.$$

Example 36. *If* $y=x^3\log\frac{1}{x}$, *then prove that*

$$\frac{d^2y}{dx^2}-\frac{2}{x}\cdot\frac{dy}{dx}+3x=0.$$ *[C.A., November 1991]*

Solution. We have

$$y=x^3\log\frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx}=3x^2\,.\,\log\frac{1}{x}+x^3\,.\,x.\left(-\frac{1}{x^2}\right)$$

or
$$\frac{dy}{dx}=3x^2\log\frac{1}{x}-x^2 \quad ...(1)$$

$$\therefore \quad \frac{d^2y}{dx^2}=6x\log\frac{1}{x}+3x^2\,.\,x\,.\left(-\frac{1}{x^2}\right)-2x$$

$$=6x\log\frac{1}{x}-3x-2x$$

$$=6x\log\frac{1}{x}-5x$$

$$=\frac{2}{x}\left[3x^2\log\frac{1}{x}-x^2\right]-3x$$

$$=\frac{2}{x}\cdot\frac{dy}{dx}-3x \quad \text{[From (1)]}$$

$$\therefore \quad \frac{d^2y}{dx^2}-\frac{2}{x}\cdot\frac{dy}{dx}+3x=0$$

Example 37. *If* $y=\log(x+\sqrt{l^2+x^2})$, *show that*
$(l^2+x^2)\,y_2+xy_1=0.$ *[I.C.W.A., December 1990]*

Solution. We have

$$y=\log(x+\sqrt{l^2+x^2})$$

$$\therefore \quad y_1=\frac{1}{x+\sqrt{l^2+x^2}}\times[1+\tfrac{1}{2}(l^2+x^2)^{-1/2}\,.\,2x]$$

$$=\frac{1}{x+\sqrt{l^2+x^2}}\times\left[1+\frac{x}{\sqrt{l^2+x^2}}\right]$$

$$=\frac{1}{x+\sqrt{l^2+x^2}}\times\left[\frac{\sqrt{l^2+x^2}+x}{\sqrt{l^2+x^2}}\right]=\frac{1}{\sqrt{l^2+x^2}}$$

or $$\sqrt{l^2+x^2}\,y_1=0$$

or $$(l^2+x^2)\,y_1^2=0$$

Differentiating again, we have

$$(l^2+x^2)\,.\,2y_1y_2+2xy_1^2=0$$

$$\therefore \quad (l^2+x^2)y_2+xy_1=0.$$

Example 38. *If* $y=e^{a\sin^{-1}x}$, *then show that*
$(1-x^2)y_2-xy_1=a^2y.$

Solution. We have $y_1=e^{a\sin^{-1}x}\dfrac{a}{\sqrt{1-x^2}}$

$$\Rightarrow \quad \sqrt{1-x^2}\,y_1=ae^{a\sin^{-1}x}=ay$$

$$\Rightarrow \quad (1-x^2)y_1^2=a^2y^2$$

$$\therefore \quad (1-x^2)2y_1y_2-2xy_1^2=2a^2yy_1$$

Hence $$(1-x^2)y_2-xy_1=a^2y$$

Example 39. *If* $y=\sin(m\sin^{-1}x)$, *then*
$(1-x^2)y_2-xy_1+m^2y=0$

Solution We have $y_1=\cos(m\sin^{-1}x)\,.\,m\,.\,\dfrac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \quad \sqrt{1-x^2}\,y_1=m\cos(m\sin^{-1}x)$$

$$\Rightarrow \quad (1-x^2)y_1^2=m^2\cos^2(m\sin^{-1}x)$$

$$=m^2[1-\sin^2(m\sin^{-1}x)]=m^2(1-y^2)$$

$\therefore \qquad (1-x^2)2y_1y_2-2xy_1^2=-2m^2yy_1$

Hence $\qquad (1+x^2)y_2-xy_1+m^2y=0.$

EXERCISE (V)

1. Find $\frac{d^2y}{dx^2}$ in the following :

   (*a*) $y=(2+x)\sin^{-1}x$, (*b*) $y=x^m e^{nx}$, (*c*) $y=x^2\tan^{-1}x$

2. (*a*) If $y=ax^3+bx^2+cx+d$, find y_3 and y_4.

   (*b*) If $y=\frac{\log x}{x}$, show that $\frac{d^2y}{dx^2}=\frac{2\log x-3}{x^3}$

3. (*a*) If $y=ae^{mx}+be^{-mx}$. prove that $\frac{d^2y}{dx^2}=m^2y$

   (*b*) If $y=a\cos m\theta+b\sin m\theta$, prove that $\frac{d^2y}{d\theta^2}+m^2y=0$

4. If $y=\sin ax$, prove that $\frac{d^2y}{dx^2}+a^2y=0$

5. (*a*) If $y=A\sin(\log x)$, prove that $x^2y_2+xy_1+y=0$

   (*b*) If $y=\sin(\sin x)$, prove that $y_2+y_1\tan x+y\cos^2 x=0$

6. (*a*) If $y=e^x\cos x$, then $y_4+4y=0$

   (*b*) If $y=e^x\cos 2x$, then show that

   $$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=0$$

7. If $y=Ae^{2x}+Bxe^{2x}$, where A and B are any constants, then show that $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$

8. If $y=A(x+\sqrt{x^2-1})^n+B(x-\sqrt{x^2-1})^n$, then prove that

   $$(x^2-1)y_2+xy_1-n^2y=0$$

9. If $y=x\sin(\log x)+x\log x$, then

   $$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$$

10. If $y=\sqrt{x+1}+\sqrt{x-1}$, prove that

    $$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=\frac{1}{4}y$$

11. (*a*) $y=\sin^{-1}x$, show that

    $$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$$

(*b*) If $y=e^{\tan^{-1}x}$, prove that $(1+x^2)\,y_2+(2x-1)\,y_1=0$

12. If $y=\log(x+\sqrt{1+x^2})$, then $(1+x^2)\,y_2+xy_1=0$

13. If $y=\cos(m\sin^{-1}x)$, show that $(1-x^2)\,y_2+m^2y=xy_1$

14. (*a*) Find $\frac{d^2y}{dx^2}$ when $x=at^2$ and $y=2at$

(*b*) Given that $x=a\cos^3\theta$, $y=b\sin^3\theta$, find $\frac{d^2y}{dx^2}$

(*c*) If $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, find $\frac{d^2y}{dx^2}$

15. If $x=e^t(\cos t+\sin t)$, $y=e^t(\cos t-\sin t)$, show that

$$\frac{dy}{dx}=-\tan t,\ \frac{d^2y}{dx^2}=-\frac{1}{2}\sec^3 t\quad e^{-t}$$

16. If $x=\frac{1-t}{1+t}$, $y=\frac{2t}{1+t}$, find $\frac{d^2y}{dx^2}$

ANSWERS

1. (*a*) $y_2=\frac{2+2x-x^2}{(1-x^2)^{3/2}}$, (*b*) $y_2=m(m-1)x^{m-2}e^{nx}+2mnx^{m-1}e^{nx}+n^2x^me^{nx}$

(*c*) $y_2=\frac{2x(2+x^2)}{(1+x^2)^2}+2\tan^{-1}x$ 14. (*a*) $\frac{1}{2at^3}$ (*b*) $\frac{b}{3a^2\cos^4\theta\sin\theta}$

(*c*) $\frac{1}{4a\cos^4\frac{\theta}{2}}$ 16. 0.

17·16. MACLAURIN'S SERIES

I. Suppose $f(x)=a+bx$ ...(1)

and we wish to investigate whether the constants a and b can be represented in terms of the special values of $f(x)$ at say $x=0$.

If we put $x=0$ in (1), we obtain

$$f(0)=a$$

Now differentiate (1) ; $f'(x)=b$

To correspond with the above we will again put $x=0$ in this and obtain

$$f'(0)=b$$

$$\therefore \quad a+bx=f(0)+xf'(0)$$

II. Suppose $f(x)=a+bx+cx^2$ ...(2)

Again $f(0)=a$

Differentiate (2), $f'(x)=b+2cx$ ...(3)

$\therefore$ $f'(0)=b$

Differentiate (3), $f''(x)=2c$

$\therefore$ $f''(0)=2c$ or $c=\frac{1}{2}f''(0)$

$$a+bx+cx^2=f(0)+xf'(0)+\frac{x^2}{2}f''(0)$$

where $f''(0)$ means that we have differentiated the given function $a+bx+cx^2$ twice and then place $x=0$

III. The student should assume $f(x)=a+bx+cx^2+dx^3$ and show that in this case

$$f(x)=f(0)+xf'(0)+\frac{x^2}{1.2}f''(0)+\frac{x^3}{1.2.3}f'''(0).$$

IV. We may now prove the general theorem. Assuming that $f(x)$ is a function that can be expanded in ascending powers of x, let

$$f(x)=a_0+\frac{a_1}{1!}x+\frac{a_2}{2!}x^2+\frac{a^3}{3!}x^3+\ldots+\frac{a_n}{n!}x^n+\ldots \quad (4)$$

We wish to find the unknown co-efficients $a_0, a_1, a_2, \ldots, a_n, \ldots$, in terms of the value of $f(x)$ and its differential co-efficients at $x=0$.

Put $x=0$ in (4), we have

$$f(0)=a_0$$

Now differentiate (4) and note that

$$\frac{n}{n!}=\frac{1}{(n-1)!}$$

We have

$$f'(x)=a_1+\frac{a_2}{1!}x+\frac{a_3}{2!}x^2+\ldots+\frac{a_n}{(n-1)!}x^{n-1}+\ldots \quad \ldots(5)$$

Put $x=0$ in this :

$$f'(0)=a_1$$

Differentiating (5), we have

$$f''(x)=a_2+\frac{a_3}{1!}x+\ldots+\frac{a_n}{(n-2)!}x^{n-2}+\ldots \quad \ldots(6)$$

Put $x=0$ in this :

$$f''(0)=a_2$$

Differentiate (6), we have

$$f'''(x)=a_3+\ldots+\frac{a_n}{(n-3)!}x^{n-3}+\ldots \quad \ldots(7)$$

Put $x=0$ in this :

$$f'''(0)=a_3$$

Proceeding in this way, we will find that

$$f^n(0)=a_n$$

for $n=1, 2, 3, \ldots$

We have now obtained Maclaurin's series which states that if $f(x)$ can be expanded in ascending powers of x, then

$$f(x)=f(0)+\frac{x}{1\,!}\,f'(0)+\frac{x^2}{2\,!}\,f''(0)+\frac{x^3}{3\,!}\,f'''(0)+\ldots+\frac{x^n}{n\,!}\,f^n(0)+\ldots$$

Example 40. *By Maclaurin series expand e^x and prove that*

$$e=2+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\ldots$$

Solution. Here $f(x)=e^x$, $\quad f(0)=e^0=1$

$f'(x)=e^x$, $\quad f'(0)=1$

$f''(x)=e^x$, $\quad f''(0)=1$

$f'''(x)=e^x$, $\quad f'''(0)=1$ and so on.

By Maclaurin's series we know

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}\,f''(0)+\frac{x^3}{3\,!}\,f'''(0)+\ldots$$

$$\therefore \qquad e^x=1+\frac{x}{1\cdot !}\cdot 1+\frac{x^2}{2\,!}\cdot 1+\frac{x^3}{3\,!}\;1+\ldots+\frac{x^n}{n\,!}\cdot 1+\ldots$$

$$=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots$$

Putting $x=1$, we get

$$e=2+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots\ldots\ldots .$$

Example 41. *Find the coefficient of x^n in the expansion of*

$$(1+ax+bx^2)\,e^{-x}$$

Solution. We know that $e^{-x}=1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+\ldots\ldots\ldots\ldots$

$$\ldots\ldots\ldots+(-1)^{n-2}\frac{x^{n-2}}{(n-2)\,!}+(-1)^{n-1}\frac{x^{n-1}}{(n-1)\,!}+(-1)^n\cdot\frac{x^n}{n\,!}$$

$$\therefore\quad (1+ax+bx^2)\,e^x=(1+ax+bx^2)\left\{1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+\ldots\ldots\right.$$

$$\left.+(-1)^{n-2}\frac{x^{n-2}}{(n-2)\,!}+(-1)^{n-1}\frac{x^{n-1}}{(n-1)\,!}+(-1)^n\frac{x^n}{n\,!}\right\}$$

The term in the given expression involving the nth power of x is given by

$$(-1)^{n-2} bx^2 \cdot \frac{x^{n-2}}{(n-2)\,!}+(-1)^{n-1}ax \cdot \frac{x^{n-1}}{(n-1)\,!}+(-1)^n \frac{x^n}{n\,!}$$

$$=(-1)^n\left\{\frac{b}{(n-2)\,!}-\frac{a}{(n-1)\,!}+\frac{1}{n\,!}\right\} x^n,$$

Hence the required coefficient is

$$(-1)^n \left\{\frac{1}{n\,!}-\frac{a}{(n-1)\,!}+\frac{b}{(n-2)\,!}\right\}$$

Example 42. *Expand sin x in ascending powers of x applying Maclaurin's expansion. Hence obtain the expansion of cos x.*

Solution.

$$f(x)=\sin x, \qquad f(0)=\sin 0=0$$
$$f'(x)=\cos x, \qquad f'(0)=\cos 0=1$$
$$f''(x)=-\sin x, \qquad f''(0)=\sin 0=0$$
$$f'''(x)=-\cos x, \qquad f'''(0)=-\cos 0=-1$$
$$f^{iv}(x)=\sin x, \qquad f^{iv}(0)=\sin 0=0$$

Evidently $f^n(0)$ is zero whenever n is an even integer and $+1$ and -1 alternately when n is an odd integer.

Now Maclaurin's expansion is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots$$

Substituting the values of $f(x)$, $f(0)$, $f'(0)$, $f''(0)$,..., we get

$$\sin x=0+\frac{x}{1\,!}\cdot 1+\frac{x^2}{2\,!}\;0+\frac{x^3}{3\,!}\;(-1)+\frac{x^4}{4\,!}(0)+\frac{x^5}{5\,!}(1)+\ldots$$

$$=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}+\frac{x^7}{7\,!}+\ldots$$

Differentiating both sides, we get

$$\cos x=1-\frac{3x^2}{3\,!}+\frac{5x^4}{5\,!}-\ldots\ldots\ldots\ldots$$

$$=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\ldots\ldots\ldots\ldots$$

Example 43. *Apply Maclaurin's expansion for expanding log (1+x), log (1−x), in ascending powers of x and deduce the expansion of log* $\frac{1+x}{1-x}$.

Solution. Let $f(x)=\log(1+x)$, $f(0)=\log 1=0$

$$f'(x)=\frac{1}{1+x},\; f'(0)=1,$$

$$f''(x)=\frac{-1}{(1+x)^2}, f''(0)=-1,$$

$$f'''(x)=\frac{(-1)(-2)}{(1+x)^3}, f'''(0)=2$$

$$f^{iv}(x) = \frac{(-1)(-2)(-3)}{(1+x)^4}, f^{iv}(0)=-6 \text{ and so on.}$$

$$\therefore \quad \log(1+x)=0+\frac{x}{1\,!}+\frac{x^2}{2\,!}(-1)+\frac{x^3}{3\,!}(2)+\frac{x^4}{4\,!}(-6)+\ldots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots \quad \ldots(1)$$

Changing the sign of x in result (1), we have

$$\log(1-x)=-x-\frac{x}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\ldots \quad \ldots(2)$$

Subtracting (2) from (1), we get

$$\log(1+x)-\log(1-x)=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\ldots$$

$$\therefore \quad \log\left(\frac{1+x}{1-x}\right)=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right\}$$

The student may note that unfortunately log x cannot be expanded by Maclaurin's theorem, since if $f(x)=\log x$, $f(0)$ is infinite and so is $f'(0)$, $f''(0)$, $f^n(0)$.

EXERCISE (VI)

1. Use Maclaurin's theorem to expand the following functions :

   (*a*) e^{5x}, (*b*) $\log(1-2x)$, (*c*) $\sin ax$, (*d*) $\cos bx$, (*e*) $(1+x)^n$.

2. Prove by Maclaurin's series

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2}+\frac{x^4}{4\,!}+\ldots$$

3. Prove that $x\sin x+\cos x=1+\frac{x^2}{2}-\frac{x^4}{2\,!\,4}+\frac{x^6}{4\,!\,6}-\ldots$

4. Prove that $e^{\sin x}=1+x+\frac{1}{2}x^2-\frac{1}{8}x^4-\ldots$

5. Prove that $e^x \cos x=1+x-\frac{2}{3\,!}x^2-\frac{2^2}{4\,!}x^4+\ldots$

ANSWERS

1. (*a*) $e^{5x}=1+5x+\frac{(5x)^2}{2\,!}+\frac{(5x)^3}{3\,!}+\ldots$

(*b*) $\log\ (1-2x)=-2x-2x^2-\dfrac{8x^3}{3}-\ldots\ldots$

(*c*) $\sin\ ax=ax-\dfrac{a^3x^3}{3\,!}+\dfrac{a^5x^5}{5\,!}-\ldots$

(*d*) $\cos\ bx=1-\dfrac{b^2x^2}{2\,!}+\dfrac{b^4x^4}{4\,!}-\ldots$

(*e*) $(1+x)^n=1+nx+\dfrac{n(n-1)}{2\,!}x^2+\dfrac{n(n-1)(n-2)}{3\,!}x^3+\ldots$

17·17. INCREASING AND DECREASING FUNCTIONS

If $y=f(x)$, then y is said to be an increasing function of x at the point $x=x_1$ if

$$\frac{dy}{dx} \text{ at } x=x_1>0, \; i.e., \left(\frac{dy}{dx}\right)_{x=x_1}>0,$$

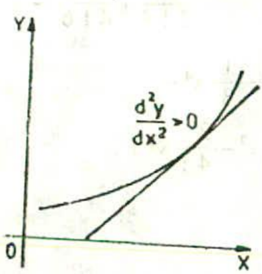
and it is said to be a decreasing function of x at the point $x=x_1$ if

$$\frac{dy}{dx} \text{ at } x \quad x_1<0, \; i.e., \left(\frac{dy}{dx}\right)_{x=x_1}<0$$

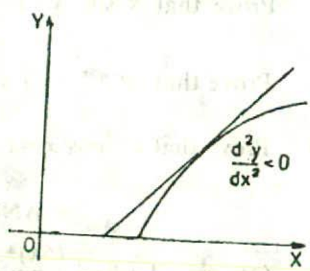
Convexity or Concavity of Curves. If a curve is a straight line then its first derivative or $\dfrac{dy}{dx}$ is equal to some value positive or negative depending on whether it is increasing or decreasing but it is the same at all points and its second derivative is equal to zero. Its rate of change does not change and is constant, therefore, acceleration rate is zero.

If the curve is concave upwards or convex downwards its rate of change will accelerate and $\dfrac{d^2y}{dx^2}$ will be positive or >0. If, the curve is concave downwards and convex upwards, its rate of change will decelerate and $\dfrac{d^2y}{dx^2}$ will be negative or <0. These two situations are shown below :

(*i*) *Concave upwards and convex downwards.*



(*ii*) *Convex upwards and concave downwards.*



17.18. POINTS OF INFLEXION

There can be a situation where the above two positions may be combined as in the following diagrams :

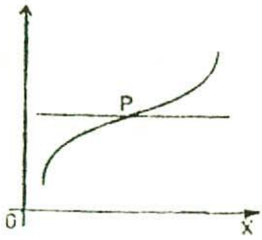


Fig. (*i*)

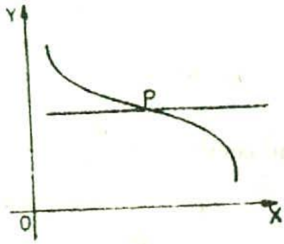


Fig. (*ii*)

At point P in both the curves $\frac{dy}{dx}$ is equal to zero, the tangents being perpendicular to y-axis. But $\frac{dy}{dx}$ at all other points is positive in figure (*i*) and negative in figure (*ii*).

As is evident $\frac{d^2y}{dx^2}$ is first positive and then negative in figure (*i*) and the reverse is the case in figure (*ii*).

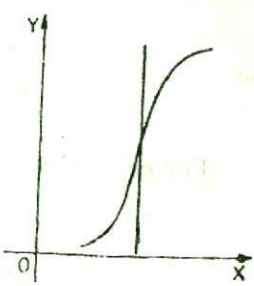
But $\frac{d^2y}{dx^2}$ is equal to zero at point P in both. Hence it is the point of inflexion. The conditions, therefore, are

(*i*) $\frac{dy}{dx}=0$, however, this is not a necessary condition.

(*ii*) $\frac{d^2y}{dx^2}=0$, this is a necessary condition.

(*iii*) $\frac{d^3y}{dx^3}\neq 0$, it may be positive or negative but it is a necessary condition.

$\frac{dy}{dx}$ is not equal to zero in the side diagram :



But the other two conditions will hold good in this case.

Example 44 *Given the function $y=x^3-3x^2+3x$, find the point of inflexion.*

Solution. Now $\frac{dy}{dx}=3x^2-6x+3$

The first condition is $\frac{dy}{dx}=0$,

i.e., $$3x^2-6x+3=0$$

$$3(x^2-2x+1)=0,\ i.e.,\ 3(x-1)^2=0$$

$\therefore$ $$x=1$$

Thus $\frac{dy}{dx}=0$, when $x=1$.

The second condition is $\frac{d^2y}{dx^2}=0$

$\Rightarrow$ $$6x-6=0,\ i.e.,\ 6x=6$$

$\therefore$ $$x=1$$

Thus $$\frac{d^2y}{dx^2}=0 \text{ when } x=1$$

Now $$\frac{d^3y}{dx^3}=6 \text{ which is not zero.}$$

The point of inflexion is $x=1$ and $y=1$ or $(1, 1)$.

Example 45. *Show that the function $y=xe^{-x}$ has a point of inflexion at $x=2$.* [*C.A., November 1991*]

Solution. We have

$$y=xe^{-x}$$

$\therefore$ $$\frac{dy}{dx}=e^{-x}+x\,(-e^{-x})=e^{-x}\,(1-x)$$

$$\frac{d^2y}{dx^2}==e^{-x}\,(-1)+(1-x)(-e^{-x})$$

$$=e^{-x}\,(x-2)$$

and $$\frac{d^3y}{dx^3}=e^{-x}\,(1)+(x-2)(-e^{-x})$$

$$=e^{-x}\,(3-x)$$

For point of inflexion we must have

$$\frac{d^2y}{dx^2}=0 \text{ and } \frac{d^3y}{dx^3}\neq 0.$$

$\therefore$ $$\frac{d^2y}{dx^2}=0 \quad \Rightarrow e^{-x}\,(x-2)=0 \quad \Rightarrow x=2$$

and $$\left.\frac{d^3y}{dx^3}\right|_{x=2}>0.$$

Hence $x=2$ is the point of inflexion.

Example 46. *Show that the curve $y=x^2(3-x)$ has a point of inflexion at the point (1, 2).* [*C.A., May 1991*]

Solution. We have

$$y=x^2(3-x)=3x^2-x^3$$

$$\therefore \quad \frac{dy}{dx}=6x-3x^2$$

$$\frac{d^2y}{dx^2}=6-6x$$

$$\frac{d^3y}{dx^3}=-6$$

For point of inflexion, we must have

$$\frac{d^2y}{dx^2}=0 \text{ and } \frac{d^3y}{dx^3}\neq 0.$$

$$\frac{d^2y}{dx^2}=0 \quad \Rightarrow 6-6x=0 \quad \Rightarrow x=1$$

and $$\left.\frac{d^3y}{dx^3}\right|_{x=1}<0$$

$\therefore$ When $x=1$, $y=3(1)^2-1^3=2$.

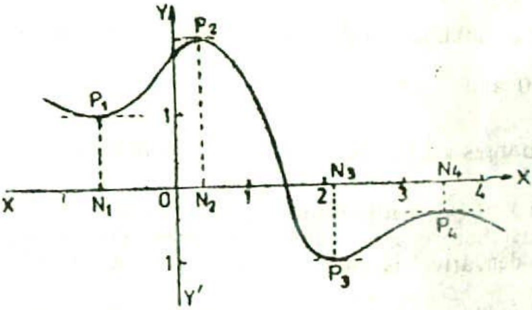
Hence (1, 2) is the point of inflexion.

17·19. MAXIMA AND MINIMA

(*a*) *A function $f(x)$ is said to have attained its maximum value at $x=a$ if the function ceases to increase and begins to decrease at $x=a$.*

(*b*) *A function $f(x)$ is said to have attained its minimum value at $x=b$, if the function ceases to decrease and begins to increase at $x=b$.*

Suppose that the following figure shows the graph of some function of x. The points P_2, P_4 are called *maximum points* of the graph, the points P_1, P_3 are called *minimum points*. The function has a maximum value



N_2P_2 when $x=ON_2$; and a maximum value N_4P_4 when $x=ON_4$. Again, the function has a minimum value N_1P_1 when $x=ON_1$ and a minimum value N_3P_3 when $x=ON_3$. Notice that in this case the minimum value at P_1 is greater than the maximum value at P_4.

It should be noted carefully that according to the definition given above it is clear that

(*i*) the maximum' and 'minimum' values of a function at a point does not mean the 'greatest' and the least' values of the function but only signifies that it is the greatest and the smallest value of the function in the immediate neighbourhood of that point,

(*ii*) the function may have several maximum and minimum values,

(*iii*) maximum and minimum values occur alternately,

(*iv*) some of the minimum values of the function can be greater than some of its maximum values.

(*v*) maxima are like mountain tops and minima like valley bottoms.

(*vi*) The maximum and minimum values of a function together are also called the extreme values of the function,

(*vii*) points at which a function has a minimum or a maximum value are classed together as *turning points*, and the maximum and minimum values are called *turning values*.

Criteria for Maxima and Minima. (*a*) When $y=f(x)$ is maximum at a point say $x=a$, by definition, it is an increasing function for values of x which just precede a and is a decreasing function for values of x which just follow a, *i.e.*, its derivative $\frac{dy}{dx}$ is positive just before $x=a$ and negative just after a. Thus at the point $x=a$. $\frac{dy}{dx}$ changes sign from positive to negative. But $\frac{dy}{dx}$ being a continuous function of x can change sign from positive to negative only by passing through zero value. Thus $\frac{dy}{dx}=0$.

Hence for a maximum value of the function at a point.

(*i*) $\frac{dy}{dx}=0$ and

(*ii*) $\frac{dy}{dx}$ changes sign from +ve to −ve at that point.

(*b*) When $y=f(x)$ is minimum at $x=a$, by definition, it is a decreasing function just before $x=a$ and an increasing function just after $x=a$, *i.e.*, its derivative is negative just before $x=a$ and is positive just after $x=a$, Thus at $x=a$, $\frac{dy}{dx}$ changes sign from negative to positive values. But $\frac{dy}{dx}$ being a continuous function can change sign from negative to positive values only by passing through zero value. Thus $\frac{dy}{dx}=0$.

Hence for a minimum value of the function at a point,

(*i*) $\frac{dy}{dx}=0$ and

(*ii*) $\frac{dy}{dx}$ changes sign from $-$ve to $+$ve at that point.

Modification of Second Condition. For a maximum point, $\frac{dy}{dx}$ changes sign from $+$ve to $-$ve. This means that $\frac{dy}{dx}$ is a decreasing function of x.

$\therefore$ Its differential coefficient, *i.e.* $\frac{d^2y}{dx^2}<0$

Similarly, for a minimum point, $\frac{dy}{dx}$ changes sign from $-$ve to $+$ve. This means that $\frac{dy}{dx}$ is an increasing function of x.

$\therefore$ Its differential coefficient, *i.e.* $\frac{d^2y}{dx^2}>0$.

Hence, the modified conditions for maximum and minimum points can be stated as follows :

For a maximum point :

(*i*) $\frac{dy}{dx}=0$, (*ii*) $\frac{d^2y}{dx^2}<0$

For a minimum point :

(*i*) $\frac{dy}{dx}=0$, (*ii*) $\frac{d^2y}{dx^2}>0$

Working Rule for finding Maximum and Minimum Values of a function :

First Method

Step I. Find $\frac{dy}{dx}$ for the given function $y=f(x)$

Step II. Find the value or values of x which make $\frac{dy}{dx}$ zero. Let these be a, b, c,...

We shall test these values of maxima and minima in turn.

Step III. To test $x=a$, study the signs of $\frac{dy}{dx}$ for values of x slightly $<a$ and for values of x slightly $>a$.

If $\frac{dy}{dx}$ changes sign from +ve to −ve, then $y=f(x)$ has maximum value at $x=a$ and *max* $y=f(a)$.

If on the other hand, $\frac{dy}{dx}$ changes sign from −ve to +ve, then it has a minimum value at $x=a$ and *min* $y=f(a)$.

If $\frac{dy}{dx}$ does not change sign, then $x=a$ is a point of inflexion.

Similarly test the other values of x found in step II.

Second Method :

Step I. Find $\frac{dy}{dx}$ for the given function $y=f(x)$.

Step II. Find the value or values of x which make $\frac{dy}{dx}$ zero. Let these be $a, b, c, \ldots$

Step III. Find $\frac{d^2y}{dx^2}$.

Step IV. Put $x=a$ in $\frac{d^2y}{dx^2}$ If the result is −ve, the function is maximum at $x=a$ and *max.* $y=f(a)$.

If by putting $x=a$ in $\frac{d^2y}{dx^2}$, the result is +ve, the function has minimum value at $x=a$ and *min.* $y=f(a)$.

Similarly test other values $b, c, \ldots$ of x found in step *II*.

Step V. When $\frac{d^2y}{dx^2}=0$ for a particular value $x=a$ (say), then we either employ the first method or find $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, ...and put $x=a$ successively in these derivatives.

We tabulate the result as :

	Maximum	*Minimum*
Necessary condition :	$\frac{dy}{dx}=0$	$\frac{dy}{dx}=0$
Sufficient condition :	$\frac{dy}{dx}=0,\ \frac{d^2y}{dx^2}<0$	$\frac{dy}{dx}=0,\ \frac{d^2y}{dx^2}>0$

Example 47. *Investigate the maxima and minima of the function :*

$$2x^3+3x^2-36x+10.$$

Solution. Let y denote the given function of x.

Then $$\frac{dy}{dx}=6x^2+6x-36$$

$$=6(x^2+x-6)=6(x-2)(x+3)$$

The function has turning values where $\frac{dy}{dx}=0$, *i.e.*, at $x=2$ and $x=-3$.

To find whether these values are maxima or minima, we must examine the sign of $\frac{dy}{dx}$ near these points.

If $x>2$, $6(x-2)(x+3)$ is +ve.

If $2>x>-3$, $6(x-2)(x+3)$ is −ve.

If $x<-3$, $6(x-2)(x+3)$ is +ve.

Therefore, when x is just less than 2, $\frac{dy}{dx}$ is −ive, and when x is just greater than 2, $\frac{dy}{dx}$ is +ve, *i.e.*, $x=2$ makes y a minimum.

At this point $y=16+12-72+10=-34$

Again when x is just less than -3, $\frac{dy}{dx}$ is + e, and when x is just greater than -3, $\frac{dy}{dx}$ is −ve, *i.e.*, $x=-3$ makes y a maximum.

At this point, $x=-54+27+108+10=91.$

Example 48. *Find the extreme values of the function* x^3e^{-x}.

Solution. $$y=x^3e^{-x}$$

$$\frac{dy}{dx}=\frac{d}{dx}(x^3e^{-x})=x^3\frac{d}{dx}(e^{-x})+e^{-x}\frac{d}{dx}(x^3)$$

$$=x^3[e^{-x}(-1)]+e^{-x}.3x^2$$

$$=e^{-x}x^2[3-x] \quad (1)$$

$\therefore$ $$\frac{dy}{dx}=0 \Rightarrow e^{-x}x^2.(3-x)=0$$

$\Rightarrow$ $$e^{-x}=0 \text{ or } x^2=0 \text{ or } 3-x=0$$

$\Rightarrow$ $$x=\infty \text{ or } x=0 \text{ or } x=3$$

For x=0

When x is slightly less than 0, $\frac{dy}{dx}=(+)\,(+)\,(+)=(+)$

When x is slightly greater than 0, $\frac{dy}{dx}=(+)\,(+)\,(+)=(+)$

Thus $\frac{dy}{dx}$ does not change sign as x passes through 0.

$\therefore$ $x=0$ gives neither a maximum nor a minimum value.

For x=3.

When x is slightly less than 3, $\frac{dy}{dx}=(+)\,(+)\,(+)=(+)$

When x is slightly greater than 3, $\frac{dy}{dx}=(+)\,(+)\,(-)=(-)$

So $x=3$ gives a maximum and the maximum value is

$$f(3)=3^3 \,.\, e^{-3}=27e^{-3}$$

For x=∞.

When x is slightly less than ∞, $\frac{dy}{dx}=(+)\,(+)\,(-)=(-)$

When x is slightly greater than ∞, $\frac{dy}{dx}=(+)\,(+)\,(-)=(-)$

Since $\frac{dy}{dx}$ does not change sign, hence $x=\infty$ does not give maximum or minimum value.

Example 49. *Given* $\frac{x}{a}+\frac{y}{b}=1$. *Prove that* xy *has a maximum value* $\frac{1}{4}ab$ *when* $x=\frac{a}{2}$ *and* $y=\frac{b}{2}$ $(a>0\,;\,b>0)$ *[C.A., May 1991]*

Solution. Let $F=xy$

$$=x\,.\,b\left(1-\frac{x}{a}\right) \qquad \left[\because \frac{x}{a}+\frac{y}{b}=1\right]$$

$$=b\left(x-\frac{x^2}{b}\right)$$

$$\therefore \qquad \frac{dF}{dx}=b\left(1-\frac{2x}{a}\right)$$

$$\frac{d^2F}{dx^2}=-\frac{2b}{a}<0$$

F will be maximum if $\frac{dF}{dx}=0$ and $\frac{d^2F}{dx^2}<0$.

$\therefore \qquad \frac{dF}{dx}=0 \Rightarrow b\left(1-\frac{2x}{a}\right)=0 \Rightarrow x=\frac{a}{2}$

$$y=b\left(1-\frac{x}{a}\right)$$

$\therefore$ when $x=\frac{a}{2}, \quad y=b\left(1-\frac{1}{2}\right)=\frac{b}{2}$

$\therefore \qquad F=xy=\frac{1}{4}ab$

Hence xy has a maximum value $\frac{1}{4}ab$ when

$$x=\frac{a}{2} \text{ and } y=\frac{b}{2}.$$

Example 50. *Find the maximum and minimum values of the function*

$$\frac{2}{3}x^3+\frac{1}{2}x^2-6x+8.$$

Solution. Let $f(x)=\frac{2}{3}x^3+\frac{1}{2}x^2-6x+8$

$\therefore \qquad f'(x)=2x^2+x-6=(x+2)(2x-3)$

$$=0, \text{ at } x=-2, \frac{3}{2}$$

$$f''(x)=4x+1$$

(*i*) at $x=-2$, $f''(x)=4(-2)+1$, *i.e.*, negative.

Hence $f(x)$ has a maximum at $x=-2$.

(*ii*) at $x=\frac{3}{2}$ $f''(x)=4.\frac{3}{2}+1$, *i.e.*, positive.

Hence $f(x)$ has a maximum at $x=\frac{3}{2}$.

Example 51. *Find the maximum and minimum values of the function*

$$x^4+2x^3-3x^2-4x+4.$$

Solution. Let $f(x)=x^4+2x^3-3x^2-4x+4$

$\therefore \qquad f'(x)=4x^3+6x^2-6x-4$

$$=2(x+2)(2x+1)(x-1)$$

Now $\qquad f'(x)=0 \Rightarrow x=-2, -\frac{1}{2}, 1$

To find the maximum and minimum values we have to test these values in the second derivative of the function which is

$$f''(x)=12x^2+12x-6$$

(*i*) When $x=-2, f''(x)$ is >0 or positive.

$\therefore$ $f(x)$ attains minimum at $x=-2$

(*ii*) When $x=-\frac{1}{2}, f''(x)$ is <0 or negative.

$\therefore$ $f(x)$ attains a maximum at $x=-\frac{1}{2}$.

When $x=1$, $\frac{d^2y}{dx^2}>0$ or positive

$\therefore$ $f(x)$ attains a minimum at $x=1$.

Hence the minimum values of the function are

$$f(-2)=(-2)^4+2(-2)^3-3(-2)^2-4(-2)+4=0$$

$$f(1)=1+2-3-4+4=0$$

and the maximum value is

$$f(-\tfrac{1}{2})=(-\tfrac{1}{2})^4+2(-\tfrac{1}{2})^3-3(-\tfrac{1}{2})^2-4(-\tfrac{1}{2})+4=\frac{81}{16}$$

Example 52. *A company has examined its cost structure and revenue structure and has determined that C the total cost, R total revenue, and x the number of units produced are related as :*

$$C=100+0{\cdot}015x^2 \text{ and } R=3x$$

Find the production rate x that will maximise profits of the company. Find that profit. Find also the profit when x= 120.

Solution. Let P denote the profit of the company, then

$$P=\text{Revenue}-\text{cost}=R-C$$

$$=3x-(100+0{\cdot}015x^2)=3x-100-\frac{15}{1000}x^2$$

$$\therefore \quad \frac{dP}{dx}=3-\frac{30x}{1000}$$

For max. min. values $\frac{dP}{dx}=0$

$$\Rightarrow \quad 3-\frac{30x}{1000}=0, \text{ i.e. } x=100 \text{ units.}$$

also $\frac{d^2P}{dx^2}=-\frac{3}{100}$, which is $-$ve when $x=100$.

$\therefore$ Profit is maximum when $x=100$.

Maximum profit$=3\times100-0{\cdot}015\times(100)^2-100$

$=300-100-150=50$ rupees.

Profit when $x=120$ is

$$P=3\times120-100-0{\cdot}015\times(120)^2$$

$$=360-100-216=44 \text{ rupees.}$$

Example 53. *By an 'Economic Order Quantity' we mean a quantity Q, which when purchased in each order, minimizes the total cost T incurred in obtaining and storing material for a certain time period to fulfil a given rate of demand for the material during the time period.*

The material demanded is 10,000 units per year ; the cost price of material Re. 1 per unit, the cost of replenishing the stock of material per order regardless of the size Q of the order is Rs. 25 ; and the cost of storing the material is $12\frac{1}{2}$ per cent per year on the rupee value of average quantity $Q/2$ on hand.

(*i*) *Show that* $T=10,000+\dfrac{2,50,000}{Q}+\dfrac{Q}{16}$.

(*ii*) *Find the Economic Order Quantity and the cost T corresponding to that.*

(*iii*) *Find the total cost when each order is placed for 2`00 units.*

Solution. Let x be the number of units made in ·ach production run. We shall assume that after a batch has been made, the Q units in batch are placed in inventory and then used up (withdrawn from inventory at a uniform rate such that inventory is zero when the next batch appears. This last assumption permits us to use the average $Q/2$ to formulate inventory cost.

The cost structure is the cost of obtaining (purchasing) 10,000 articles at Re. 1 each $=$ Rs. 10,000

Q number of articles being the lot size, the number of production runs (batches) per year $=\dfrac{10,000}{Q}$

The cost of replenishing the stock of material, *i.e.*, cost to make the factory ready for production

$$=25\times\frac{10,000}{Q}=\text{Rs. }\frac{2,50,000}{Q}$$

The average inventory $=\dfrac{Q}{2}$ number and its cost

$$=1\times\frac{Q}{2}=\text{Rs. }\frac{Q}{2}$$

The cost of storing material at $12\frac{1}{2}$ per cent on the rupee value per year $=\dfrac{0}{2}\times\dfrac{12{\cdot}5}{100}=\dfrac{Q}{16}$ rupees

Hence total cost $T=10,000+\dfrac{2,50,000}{Q}+\dfrac{Q}{16}$

(*ii*)
$$\frac{dT}{dQ}=-\frac{2,50,000}{Q^2}+\frac{1}{16}$$

But for max. or min. value $\frac{dT}{dQ}=0$

$$\Rightarrow \qquad -\frac{2,50,00}{Q^2}+\frac{1}{16}=0, \text{ i.e. } Q=\pm 2000$$

Q being quantity purchased cannot be negative, rejecting the negative value, we get $Q=2000$ units.

Also $\frac{d^2T}{dQ^2}=\frac{500,000}{Q^2}$ is positive, when $Q=2000$

Hence T is minimum when $Q=2000$ units.

Total minimum cost $T=10,000+\frac{250,00}{2000}+\frac{2000}{16}$

(by putting $Q=2000$)

$$=10,000+125+125$$
$$=10,250 \text{ rupees}$$

(iii) Now cost when each order is placed by 2500 units is given as follows :

$$T=10,000+\frac{250,000}{25000}+\frac{2500}{16}=\text{Rs. } 10265{\cdot}25$$

Example 54. *The demand function for a particular commodity is* $y=15e^{-x/3}$ *for* $0\leqslant x\leqslant 8$, *where y is the price per unit and x is the number of units demanded. Determine the price and the quantity for which the revenue is maximum.*

(**Hint.** *Revenue* : $R=y \,.\, x$)

Solution. Demand, $y=15 \,.\, e^{-x/3}$, for $0\leqslant x\leqslant 8$

Revenue, $R=xy=15x \,.\, e^{-x/3}$

For maximisation of revenue, we have

$$\frac{dR}{dx}=15e^{-x/3}+\left(-\frac{15}{3}\right)xe^{-x/3}$$
$$=15e^{-x/3}+5xe^{-x/3}$$

$$\frac{dR}{dx}=0 \Rightarrow 3e^{x/3}-xe^{-x/3}=0$$

$\therefore$ Either $x=3$ or $e^{-x/3}$ 0. *i.e.* $x-\infty$(absurd)

Also $\frac{d^2R}{dx^2}<0$ or negative when $x=3$.

Hence the maximum profit is yielded by substituting $x=3$ in the revenue equation.

$$\therefore \qquad R=15 \,.\, x \,.\, e^{-x/3}=45e^{-1}=\frac{45}{2{\cdot}72}=16{\cdot}54.$$

EXERCISE (VII)

Find the maximum and minimum values of the following functions :

1. x^3-2x^2-4x-1

2. $2x^3-15x^2+36x+12$

3. $\frac{x^3}{3}+x^2.a-3xa^2$

4. $(x-2)^6(x-3)^5$

5. (*a*) Show that $x^5-5x^4+5x^3-1$, has maximum value when $x=1$ and $x=0$, a minimum value when $x=3$.

(*b*) Show that x^3-3x^2+3x+7 has neither *a* maximum nor *a* minimum value.

6. Prove that the curve given by $3y-x^3-3x^2-9x+11$ has a maximum at $x=-1$, minimum at $x=3$ and *a* point of inflexion at $x=1$.

7. If $y=x^4-3x^3+3x^2+5x+1$, prove that $\frac{d^2y}{dx^2}$ is negative when x lies between $\frac{1}{2}$ and 1. What happens if $x=\frac{1}{2}$ or $x=1$

8. Find the maximum and the minimum values of the function $x^5-5x^4+5x^3-1$. Discuss its nature at $x=0$.

9. The difference of two numbers is 100. The square of the larger one exceeds five times the square of the smaller one by an amount which is maximum. Find the numbers. [*C.A., November 1988*]

[**Hint.** Let the numbers be x and $y(x>y)$, then $x-y=100$ and $x^2-5y^2=h$ or $h=x^2-5(x-100)^2$.

$$\frac{dh}{dx}=2x-10(x-100) \text{ and } \frac{d^2h}{dx^2}=-8 \quad 0, \quad \frac{dh}{dx}=0 \Rightarrow x=125,$$

$\therefore \qquad y=25$]

10. State whether $y=x^2-6x+13$ has maximum or a minimum value. Find the value.

11. The cost C of manufacturing a certain article is given by the formula

$$C=5+\frac{48}{x}+3x^2$$

where x is the number of articles manufactured. Find minimum value of C

12. A company finds that it can sell out a certain product that it produces, at the rate of Rs. 2 per unit. It estimates the cost function of the product to be Rs. $\left[1000+\frac{1}{2}\cdot\left(\frac{q}{50}\right)^2\right]$ for q units produced.

(*i*) Find the expression for the total profit, if q units are produced and sold.

(*ii*) Find the number of units produced that will maximize profit.

(*iii*) What is the amount of this maximum profit ?

(*iv*) What would be the profit if 6000 units are produced ?

13. By an economic lot size', we mean a lot size (x) which minimizes the total cost (T) incurred in obtaining and storing material for a certain period to fulfil a given rate of demand for the material during the time period.

The material demanded is 10,000 units per year, the cost of material is Rs. 2 per unit, the cost of replenishing the stock of material per order, regardless of the size order (x), is Rs. 40 per order, and the cost of storing material is 10 per cent per year on the rupee value of the average inventory $\left(\frac{x}{2}\right)$ on hand.

(*i*) Show that $T=20,000+\frac{4,00,000}{x}+\frac{x}{10}$

(*ii*) Find the economic lot size.

14. A firm has to produce 144,000 units of an item per year. It costs Rs. 60 to make the factory ready for a product run of the item regardless of units x produced in a run. The cost of material is Rs. 5 per unit and the cost of storing the material is 50 paise per item per year on the average inventory $\left(\frac{x}{2}\right)$ in hand. Show that the total cost C is given by

$$C=720,000+\frac{23,040,000}{x}+\frac{x}{4}$$

Find also the economic lot size, *i.e.*, value of x for which C is minimum.

15. A company has to manufacture 36,000 units of an item per year. It costs Rs. 250 to make the factory ready for production run of the item regardless of units x produced in a run. The cost of material per unit made is Rs. 5 and it costs 50 paise per year for each unit for storing on an average inventory $\frac{x}{2}$ in hand. Show that total cost C is given by

$$C=\frac{250\times 36000}{x}+1,80,000+\frac{x}{4}$$

Find also the economic lot size, *i.e.*, value of x for which C is minimum.

16. A company notices that higher sales of a particular item which it produces are achieved by lowering the price charged. As a result the total revenue from the sales at first rises as the number of units sold increases, reaches the highest point and then falls off. This pattern of total revenue is described by the relation

$$y=40{,}00{,}000-(x-2000)^2$$

where y is the total revenue and x the number of units sold.

(*i*) Find, what number of units sold maximizes total revenue ?

(*ii*) What is the amount of this maximum revenue ?

(*iii*) What would be the total revenue if 2500 units were sold ?

17. A sitar manufacturer notices that he can sell x sitars per week at p rupees each where $5x=375-3p$. The cost of production is $(500+13x+\frac{1}{5}x^2)$ rupees. Show that the maximum profit is obtained when the production is 30 sitars per week.

18. If the demand function of the monopolist is $3q=98\ \ 4p$ and average cost is $3q+2$ where q is output and p is the price, find maximum profit of the monopolist.

ANSWERS

1. Max. value $\frac{13}{27}$ and Min. value -9.

2. Max. value 40 at $x=2$ and min. value 39 at $x=3$.

3. Max. value $9a^3$ at $x=-3a$, min. value $\frac{-5a^3}{a}$ at $x=a$

4. Max. value 0 at $x=2$, min. value $\frac{-6^6 5^5}{11^{11}}$ at $x=\frac{28}{11}$. Neither max. nor min. value at $x=3$. It is a point of inflexion.

10. Min. value of $y=4$ at $x=3$.

11. 2 **12.** (*i*) Rs. $\left[2q-\frac{q^2}{5000}-1000\right]$, (*ii*) 50000 (*iii*) Rs. 4000 (*iv*) Rs. 3800 **13.** (*ii*) 2000 **14.** 4000 **15.** 6000 **16.** (*i*) 2000, (*ii*) Rs. 4,000,000, (*iii*) Rs. 3,750,000 **18.** Rs. 33·75.

17·20. PARTIAL DIFFERENTIATION

In earlier sections, we considered a function in the form,

$$y=f(x)$$

There y was a function of the single independent variable x. However in practice, very rarely we come across such a situation where a variable is a function of a single independent variable. Generally it is found to be a function of several variables. For example production may be treated as a function of labour and capital, price may be a function of supply and price of the substitutes, etc.

Consider the following example :

If V is the volume of a right circular cylinder of radius r and height h, we know

$$V = \pi r^2 h$$

Now suppose that the height h remains constant while the radius r changes. Since h is constant it may be considered as another constant like π and on differentiating w.r.t. r, we have

$$\left(\frac{dV}{dr}\right)_{h \text{ constant.}} = (\pi h) \,.\, 2r,$$

giving the rate of change in V with respect to r when h remains constant.

Similarly, if r is a constant while h varies

$$\left(\frac{dV}{dh}\right)_{r \text{ constant.}} = (\pi r^2) \,.\, 1$$

This notation, which shows precisely what has been done, is rather clumsy, therefore a special notation is introduced. We write

$$\frac{\partial V}{\partial r} = \pi h \,.\, 2r$$

and

$$\frac{\partial V}{\partial h} = \pi r^2 \,.\, 1$$

The 'curly' d is used to show that the expression to be differentiated contains more than one variable, and that we regard as constant all but the one used in the denominator on the left-hand side. We differentiate in the usual way with respect to this stated variable, treating all the others as constant.

Let u be a function of two independent variables x and y, we write this symbolically as

$$u = f(x, y)$$

Now we may consider a change in u corresponding to a change in x, keeping y as constant. Or, we may consider change in u corresponding to a change in y, x being kept fixed. Under these assumptions

$$f \text{ or } \frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \left[\frac{f(x+\delta x, y) - f(x, y)}{\delta x}\right]$$

is known as partial derivative of u w.r.t. x,

and

$$f_y \text{ or } \frac{\partial u}{\partial y} = \lim_{\delta y \to 0} \left[\frac{f(x, y+\delta y) - f(x, y)}{\delta y}\right]$$

is known as partial derivative of u w.r.t. y.

Illustrations. 1. *Find the first order partial derivatives of*

$$x^2+6xy+y^2=0$$

Solution. Let $z=f(x, y)=x^2+6xy+y^2$

Treating y as constant and differentiating partially with respect to x, we have

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(x^2+6xy+y^2)$$

$$=\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial x}(6xy)+\frac{\partial}{\partial x}(y^2)$$

$$=\frac{\partial}{\partial x}(x^2)+6y\frac{\partial}{\partial x}(x)+0$$

$$=2x+6y$$

Again $$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}(x^2+6xy+y^2)$$

$$=\frac{\partial}{\partial y}(x^2)+\frac{\partial}{\partial y}(6xy)+\frac{\partial}{\partial y}(y^2)$$

$$=0+6x\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial y}(y^2) \qquad [\because x \text{ is constant}]$$

$$=6x+2y.$$

2. *If* $u=e^{xy}$, *find* $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$.

Solution. $$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(e^{xy})=e^{xy}\cdot\frac{\partial}{\partial x}(xy)$$

$$=e^{xy}\cdot y\frac{\partial}{\partial x}(x)=y\,e^{xy}$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(e^{xy})=e^{xy}\cdot\frac{\partial}{\partial y}(xy).$$

$$=e^{xy}\cdot x\cdot\frac{\partial}{\partial y}(y)=xe^{xy}.$$

3. *If* $u=x^2y^3z^4+6x+7y+9z$, *find* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

Solution. $$\frac{\partial u}{\partial x}=y^3z^4.2x+6$$

$$\frac{\partial u}{\partial y}=x^2z^4.3y^2+7$$

$$\frac{\partial u}{\partial z}=x^2y^3.4z^3+9.$$

4. *Given below is a function of profit with two variables Q_1 and Q_2*

$$f(P) = -60 + 140Q_1 + 100Q_2 - 10Q_1^2 - 8Q_2^2 - 6Q_1Q_2$$

Determine the optimum values of Q_1 and Q_2.

Solution. Treating Q_2 as a constant and differentiating w.r.t. Q_1 we have

$$\frac{\partial f(P)}{\partial Q_1} = 0 + 140 + 0 - 20Q_1 - 0 - 6Q_2$$

$$= 140 - 20Q_1 - 6Q_2$$

Also
$$\frac{\partial f(P)}{\partial Q_2} = 0 + 0 + 100 - 0 - 16Q_2 - 6Q_1$$

$$= 100 - 16Q_2 - 6Q_1$$

Now taking the two partial derivatives equal to zero, *i.e.*,

$$140 - 20Q_1 - 6Q_2 = 0 \quad ...(1)$$

$$100 - 6Q_1 - 16Q_2 = 0 \quad ...(2)$$

Solving (1) and (2) for Q_1 and Q_2, we will have the optimum quantities assuring maximum profit as

$$Q_1 = 5{\cdot}77 \text{ and } Q_2 = 4{\cdot}08.$$

PARTIAL DERIVATIVES OF HIGHER ORDER

Higher partial derivatives are obtained in the same manner as higher derivatives.

For the function $u = f(x, y)$ we have four second order partial derivatives.

The direct partial derivatives are defined as :

(*i*) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = f_{xx}$$

(*ii*) $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = f_{yy}$$

Apart from these two second-order partial derivatives, there are also the mixed (or cross) partial derivatives defined as :

(*i*) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = f_{yx}$$

(*ii*) $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f_{xy}$$

Illustration 1. *Find second order partial derivatives of*

$$u = 4x^2 + 9xy - 5y^2.$$

Solution. Let $u = 4x^2 + 9xy - 5y^2$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(4x^2 + 9xy - 5y^2)$$

$$=\frac{\partial}{\partial x}(4x^2)+\frac{\partial}{\partial x}(9xy)-\frac{\partial}{\partial x}(5y^2)$$

$$=4\frac{\partial}{\partial x}(x^2)+9y\frac{\partial}{\partial x}(x)-5\cdot\frac{\partial}{\partial x}(y^2)$$

$$=4.2x+9y.1-5.0=8x+9y$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(4x^2+9xy-5y^2)$$

$$=\frac{\partial}{\partial y}(4x^2)+\frac{\partial}{\partial y}(9xy)-\frac{\partial}{\partial y}(5y^2)$$

$$=4\frac{\partial}{\partial y}(x^2)+9x\frac{\partial}{\partial y}(y)-5\frac{\partial}{\partial y}(y^2)$$

$$=4.0+9x.1-5.2y=9x-10y$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}(8x+9y)$$

$$=8\frac{\partial}{\partial x}(x)+9\frac{\partial}{\partial x}(y)=8.1+9.0=8$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}(9x-10y)$$

$$=9\frac{\partial}{\partial y}(x)-10\frac{\partial}{\partial y}(y)=9.0-10.1=-10$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}(9x-10y)$$

$$=9\frac{\partial}{\partial x}(x)-10\frac{\partial}{\partial x}(y)=9.1-10.0=9$$

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}(8x+9y)$$

$$=8\frac{\partial}{\partial y}(x)+9\frac{\partial}{\partial y}(y)=8.0+9.1=9$$

$$\therefore\quad \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$$

2. *Find first and second order partial derivatives of log (x^2+y^2)*

Solution. Let $u=\log(x^2+y^2)$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\log(x^2+y^2)=\frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2)$$

$$=\frac{1}{x^2+y^2}\cdot 2x=\frac{2x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\log(x^2+y^2)=\frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial y}(x^2+y^2)$$

$$=\frac{1}{x^2+y^2}\cdot 2y=\frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$=\frac{(x^2+y^2)\frac{\partial}{\partial x}\ (2x)-2x\frac{\partial}{\partial x}\ (x^2+y^2)}{(x^2+y^2)^2}$$

$$=\frac{(x^2+y^2)\ .\ 2-2x\ (2x)}{(x^2+y^2)^2}=\frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$=\frac{(x^2+y^2)\ \frac{\partial}{\partial y}\ (2y)-2y\ \frac{\partial}{\partial y}\ (x^2+y^2)}{(x^2+y^2)^2}$$

$$=\frac{(x^2+y^2)\ .\ 2-2y\ .\ 2y}{(x^2+y^2)^2}=\frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right)=2y\ \frac{\partial}{\partial x}\ (x^2+y^2)^{-1}$$

$$=2y(-1)(x^2+y^2)^{-2}\frac{\partial}{\partial x}\ (x^2+y^2)$$

$$=\frac{-2y}{(x^2+y^2)^2}\cdot 2x=-\frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right)=2x\ \frac{\partial}{\partial y}\ (x^2+y^2)^{-1}$$

$$=2x\ (-1)\ (x^2+y^2)^{-2}\ \frac{\partial}{\partial y}\ (x^2+y^2)$$

$$=\frac{-2x}{(x^2+y^2)^2}\cdot 2y=\frac{-4xy}{(x^2+y^2)^2}$$

$$\therefore\quad \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$$

17·21. TOTAL DIFFERENTIAL

From a practical point of view, the partial derivatives gave us a small change in $u=f\ (x, y)$ when there is a small change in either x or y. The total differential will give us a linear approximation of the small change in $u=f\ (x, y)$ when there is a small change in both x and y.

If δx and δy be the increments of x and y respectively and let δu be the corresponding increment of u.

Then $$u+\delta u=f\ (x+\delta x,\ y+\delta y)$$

$$\therefore\quad \delta u=f\ (x+\delta x,\ y+\delta y)-f(x,\ y) \tag{1}$$

Adding and subtracting $f(x, y+\delta y)$ on the R.H.S. of (1), we have

$$\delta u = f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y)$$

$$\Rightarrow \quad \delta u = \left[\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \right] \times \delta x$$

$$+ \left[\frac{f(x, y+\delta y) - f(x, y)}{\delta y} \right] \times \delta y \qquad \text{...(2)}$$

Let dx, dy and du be the limiting values of δx, δy and δu. Then from (2), we have

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

where du is known as the total differential of u.

Illustration. *Find the total differentials of the following functions :*

(*i*) $u = x^3y + x^2y^2 + xy^3$, (*ii*) $u = x \sin y - y \sin x$

Solution. (*i*) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3y + x^2y^2 + xy^3)$

$$= 3x^2y + 2xy^2 + y^3$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3y + x^2y^2 + xy^3)$$

$$= x^3 + 2x^2y + 3xy^2$$

The total differential is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= (3x^2y + 2xy^2 + y^3)\, dx + (x^3 + 2x^2y + 3xy^2) dy$$

(*ii*) $\frac{\partial u}{\partial x} = \sin y - y \cos x$

$$\frac{\partial u}{\partial y} = x \cos y - \sin x$$

The total differential is

$$du = (\sin y - y \cos x)\, dx + (x \cos y - \sin x)\, dy$$

EXERCISE (VIII)

1. In each of the following functions, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$.

   (*a*) $u = 3x^2 + 2xy + 4y^2$, (*b*) $u = \frac{x}{y}$,

   (*c*) $u = \frac{2x-y}{x+y}$, (*d*) $u = \tan^{-1}\left(\frac{y}{x}\right)$

2. Show that if

   (*a*) $u = f(x+y)$, then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

(*b*) $u=f(x-y)$, then $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$

(*c*) $u=f\left(\frac{x}{y}\right)$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

(*d*) $u=x^2+y^2+z^2$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u$

3. Find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\partial y}, \frac{\partial^2 u}{\partial y\partial x}, \frac{\partial^2 u}{\partial y^2}$ for the functions

(*a*) $u=x^2y^3$, (*b*) $u=ax^3+hx^2y+by^3$,

(*c*) $u=x\cos y-y\cos x$

4. If $u=\log(x^2+y^2)$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

5. If $u=\frac{y}{x}\log x$, verify that $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$.

6. If $u=x^2y+y^2z+z^2x$ show that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=(x+y+z)^2.$$

7. If $u=x\log y (y>0)$ show that

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}.$$

8. If $u=f[(z-x), (x-y), (y-z)]$, prove that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

[**Hint.** $u=f(a, b, c)$ where $a=z-x$, $b=x-y$, $c=y-z$]

9. If $z=\log\left(\frac{x^3+y^3}{x+y}\right)$, prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2.$$

10. If $u=\log(x^2+y^2+z^2)$, prove that

$$x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial z\partial x}=z\frac{\partial^2 u}{\partial x\partial y}.$$

ANSWERS

1. (*a*) $6x+2y$, $2x+8y$, (*b*) $\frac{1}{y}$, $-\frac{x}{y^2}$, (*c*) $\frac{3y}{(x+y)^2}$, $\frac{-3x}{(x+y)^2}$,

(*d*) $\frac{-y}{x^2+y^2}$, $\frac{x}{x^2+y^2}$.

3. (*a*) $2y^3$, $6xy^2$, $6xy^2$, $6x^2y$

(*b*) $6ax+2hy$, $2hx$, $2hx$, $6by$

(*c*) $y\cos x$, $-\sin y+\sin x$, $-\sin y+\sin x$, $-x\cos y$.