18

Integral Calculus

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- to find the indefinite integral of a given function
- to state the standard indefinite integrals
- to evaluate definite integrals.

18·0. INTRODUCTION

In the previous chapter, we dealt with the methods of finding derivatives of a given function. We had noticed that the derivatives so obtained were also the functions. In this chapter, we propose to deal with the converse. Consider the following examples :

(*i*) If $f(x)=x$, then $f'(x)=1$

(*ii*) If $f(x)=x^{-3}$, then $f'(x)=-3x^{-1}$

(*iii*) If $f(x)=x^{5/2}$, then $f'(x)=\frac{5}{2}x^{3/2}$

(*iv*) If $f(x)=\sin x$, then $f'(x)=\cos x$

(*v*) If $f(x)=\sec x$, then $f'(x)=\sec x \tan x$

Now let us consider the questions :

(*i*) What is the function whose detrivative is 1 ?

(*ii*) What is the function whose derivative is $-3x^{-4}$?

(*iii*) What is the function whose derivative is $\frac{5}{2}x^{3/2}$?

(*iv*) What is the function whose derivative is $\cos x$?

(*v*) What is the function whose derivative is $\sec x \tan x$?

Clearly the answers to these questions are x, x^{-3}, $x^{5/2}$, $\sin x$ and $\sec x$ respectively. The functions which we find are called primitives or anti-derivatives or integrals of the given function. Thus we are given a function of x and we try to find another function whose derivative is always the given function. This is exactly the problem of integral calculus. Integration, therefore, is called the inverse process of differentiation. For example, the function whose derivative is $\cos x$ is $\sin x$.

$\therefore$ $\sin x$ is called the primitive or anti-derivative or the integral of $\cos x$.

Definition. If $\phi(x)$ be any differentiable function of x such that

$$\frac{d}{dx}[\phi(x)] \quad f(x)$$

then $\phi(x)$, is called an *anti-derivative* or *a primitive* or an *indefinite integral* or simply an *integral* of $f(x)$.

Symbolically this is written as

$$\phi(x) = \int f(x)dx$$

and is read as "*$\phi(x)$ is the integral of $f(x)$ w.r.t. x*".

The process of finding the integral of a given function is called *Integration* and the given function is called the *Integrand*.

Remarks. 1. The symbol $\int$ used for 'integral' is a distorted form of the letter S, the first letter of the word 'sum'. This is because originally 'integral' was defined as the sum of a certain infinite series.

2 The symbol $\int \quad dx$ is purely a symbol of operation, which means integral of with respect to x. $\int$ and dx mean nothing when taken separately.

3. From the definition of anti-derivative it is clear that

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

4. In the present chapter we shall study several methods of finding integrals of a function but this however does not mean that integral of any given function can always be obtained.

18·1. INDEFINITE INTEGRAL

Suppose $f(x)=x^2$, $\phi(x)=x^2+9$ and $\psi(x)=x^2+c$, where c is a constant. By differentiating these functions, we get

$$\frac{d}{dx}f(x)=\frac{d}{dx}\phi(x)=\frac{d}{dx}\psi(x)=2x$$

$$\therefore \qquad \int 2x\,dx=x^2 \text{ or } x^2+9 \text{ or } x^2+c$$

Hence $\int 2x\,dx$ does not give a definite value and is called an indefinite integral. The general value of $\int 2x dx=x^2+c$, where c is an arbitrary constant. The constant added with the integral is called the constant of integration.

For example

$$\frac{d}{dx}(\sin x)=\cos x, \quad \therefore \int \cos x\,dx=\sin x+c$$

From these considerations, we conclude that integral of a function is not unique and that if $f(x)$ be any one integral of $\phi(x)$, then (*i*) $f(x)+c$ is also its integral, c being any constant. (*ii*) every integral of $\phi(x)$ can be obtained from $f(x)+c$, by giving a suitable value to c.

18·2. RULES OF INTEGRATION

Rule I. *The integral of the product of a constant and a function is equal to the product of the constant and integral of the function, i.e.,*

$$\int k\,f(x)dx=k\int f(x)dx,$$

where k is some constant.

Proof $\frac{d}{dx}\left[k\int f(x)dx\right]=k\frac{d}{dx}\int f(x)dx=kf(x)$

$\therefore$ From the definition, we have

$$k\int f(x)dx=\int kf(x)dx$$

Rule II. *The integral of the sum or difference of functions is equal to the sum or difference of their integrals. Symbolically*

$$\int [f_1(x)+f_2(x)+\ldots+f_n(x)]dx$$

$$=\int f_1(x)dx+\int f_2(x)dx+\ldots+\int f_n(x)dx$$

where $f_1(x), f_2(x),\ldots, f_n(x)$ are functions of x.

Proof. $\frac{d}{dx}\left[\int f_1(x)dx+\int f_2(x)dx+\ldots+\int f_n(x)dx\right]$

$$=\frac{d}{dx}\int f_1(x)dx+\frac{d}{dx}\int f_2(x)dx+\ldots+\frac{d}{dx}\int f_n(x)dx$$

$$=f_1(x)+f_2(x)+\ldots+f_n(x)$$

$\therefore$ From the definition, we have

$$\int f_1(x)dx+\int f_2(x)dx+\ldots+\int f_n(x)dx$$

$$=\int\Big[\ f_1(x)+f_2(x)+\ldots+f_n(x)\ \Big]dx.$$

18·3. SOME STANDARD RESULTS

We give below some standard results by using the derivatives of some well-known functions.

I. $\because\ \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n,\qquad \therefore\ \int x^n\,dx=\frac{x^{n+1}}{n+1}$

II. $\because\ \frac{d}{dx}(\log x)=\frac{1}{x}\qquad \therefore\ \int\frac{1}{x}dx=\log x$

III. $\because\ \frac{d}{dx}(\sin x)=\cos x,\qquad \therefore\ \int\cos x\,dx=\sin x$

IV. $\because\ \frac{d}{dx}(\cos x)=-\sin x,\qquad \therefore\ \int\sin x\,dx=-\cos x$

V. $\because\ \frac{d}{dx}(\tan x)=\sec^2 x,\qquad \therefore\ \int\sec^2 x\,dx=\tan x$

VI. $\because\ \frac{d}{dx}(\cot x)=-\text{cosec}^2 x,\ \therefore\ \int\text{cosec}^2 x\,dx=-\cot x$

VII. $\because\ \frac{d}{dx}(\sec x)=\sec x\tan x,\ \therefore\ \int\sec x\tan x\,dx=\sec x$

VIII. $\because\ \frac{d}{dx}(\text{cosec}\,x)=-\text{cosec}\,x\cot x,$

$\therefore\ \int\text{cosec}\,x\cot x\,dx=-\text{cosec}\,x$

IX. $\because\ \frac{d}{dx}(e^x)=e^x,\qquad \therefore\ \int e^x\,dx=e^x$

X. $\because\ \frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}},\ \therefore\ \int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x$

XI. $\because\ \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2},\quad \therefore\ \int\frac{dx}{1+x^2}=\tan^{-1}x$

XII. $\therefore\ \frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}},\ \therefore\ \int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x$

XIII. $\because \ \frac{d}{dx}\left(\frac{a^x}{\log a}\right)=a^x, \qquad \therefore \ \int a^x \, dx=\frac{a^x}{\log a}$

Illustrations

1. $$\int 5x^2 \, dx=5\int x^2 \, dx=5\frac{x^{2+1}}{2+1}=\frac{5}{3}x^3$$

2. $$\int (3-2x-x^4) \, dx=3\int dx-2\int x \, dx-\int x^4 \, dx$$
$$=3x-2\frac{x^{1+1}}{1+1}-\frac{x^{4+1}}{4+1}$$
$$=3x-x^2-\frac{x^5}{5}$$

3. $$\int (4x^3+3x^2-2x+5) \, dx$$
$$=4\int x^3 \, dx+3\int x^2 \, dx-2\int x \, dx+5\int dx$$
$$=4\frac{x^{3+1}}{3+1}+3\frac{x^{2+1}}{2+1}-2\frac{x^{1+1}}{1+1}+5\frac{x^{0+1}}{0+1}$$
$$=x^4+x^3-x^2+5x.$$

4. $$\int (x^2-1)^2 \, dx=\int (x^4-2x^2+1) \, dx$$
$$=\int x^4 \, dx-2\int x^2 \, dx+\int dx$$
$$=\frac{x^5}{5}-\frac{2}{3}x^3+x$$

5. $$\int\left(\sqrt{x}-\frac{1}{2}x+\frac{2}{\sqrt{x}}\right) dx=\int \sqrt{x} \, dx-\frac{1}{2}\int x \, dx+2\int \frac{1}{\sqrt{x}} \, dx$$
$$=\frac{2}{3}x^{3/2}-\frac{1}{4}x^2+4x^{1/2}$$

6. $$\int (1-3x)(1+x) \, dx=\int (1-2x-3x^2) \, dx$$
$$=\int dx-2\int x \, dx-3\int x^2 \, dx$$
$$=x-x^2-x^3.$$

7. $$\int \frac{x^4+1}{x^2} \, dx=\int (x^2+x^{-2}) \, dx=\int x^2 \, dx+\int x^{-2} \, dx$$
$$=\frac{x^3}{3}-\frac{1}{x}.$$

8. *Evaluate* $\int\left(3x^{-1}+4x^2-3x+8\right)dx$

Solution. $I \int \frac{3}{x}\,dx+\int 4x^2\,dx-\int 3x\,dx+\int 8dx$

$$=3\int \frac{1}{x}\,dx+4\int x^2dx-3\int x\,dx+8\int dx$$

$$=3\log x+\frac{4x^3}{3}-\frac{3x^2}{2}+8x.$$

9. *Integrate* $\left(x-\frac{1}{x}\right)^3$ *w.r.t. x.*

Solution. $\int\left(x-\frac{1}{x}\right)^3 dx=\int\left(x^3-3x+\frac{3}{x}-\frac{1}{x^3}\right)dx$

$$=\int (x^3-3x+3x^{-1}-x^{-3})\,dx$$

$$=\int x^3\,dx-3\int x\,dx+3\int x^{-1}\,dx-\int x^{-3}\,dx$$

$$=\frac{x^{3+1}}{3+1}-3\,\frac{x^{1+1}}{1+1}-3\,.\,\log x-\frac{x^{-3+1}}{-3+1}$$

$$\frac{1}{4}x^4-\frac{3}{2}x^2+3\log x+\frac{1}{2}x^{-2}.$$

Example 1. *Integrate* $\frac{x^2-3x+\sqrt[3]{x}+7}{\sqrt{x}}$ *w.r.t. x.*

Solution. $I \int\left(x^{3/2}-3x^{1/2}+x^{-1/6}+7x^{-1/2}\right)dx$

$$=\frac{x^{5/2}}{\frac{5}{2}}-3\,.\,\frac{x^{3/2}}{\frac{3}{2}}+\frac{x^{5/6}}{\frac{5}{6}}+7\,.\,\frac{x^{1/2}}{\frac{1}{2}}$$

$$=\frac{2}{5}x^{5/2}-2x^{3/2}+\frac{6}{5}x^{5/6}+14x^{1/2}.$$

Example 2. *Evaluate* $\int\frac{ax+bx^{-3}+cx^{-7}}{kx^{-2}}dx.$

Solution. $I=\frac{a}{k}\int x^3\,dx+\frac{b}{k}\int\frac{1}{x}\,dx+\frac{c}{k}\int x^{-5}\,dx$

$$=\frac{ax^4}{4k}+\frac{b\log x}{k}-\frac{cx^{-4}}{4k}.$$

Example 3. *Evaluate* $\int\left(8e^x-4a^x+3\cos x+\sqrt[4]{x}\right)dx$

Solution. $I=8\int e^x\,dx-4\int a^x\,dx-3\int \cos x\,dx+\int x^{1/4}\,dx$

$$=8e^x-\frac{4a^x}{\log a}+3\sin x+\frac{4}{5}x^{5/4}$$

Example 4. *Evaluate* $\int\frac{a+b\sin x}{\cos^2 x}dx$

Solution. $\int\frac{a+b\sin x}{\cos^2 x}dx=\int a\sec^2 x\,dx\quad\int b\sec x\tan x\,dx$

$$=a\tan x+b\sec x.$$

Example 5. *Evaluate* $\int\frac{\sin x\,dx}{1+\sin x}$

Solution. $I=\int\frac{\sin x\,(1-\sin x)}{1-\sin^2 x}\,dx\quad\int\frac{\sin x-\sin^2 x}{\cos^2 x}\,dx$

$$=\int\frac{\sin x}{\cos^2 x}dx-\int\frac{\sin^2 x}{\cos^2 x}dx$$

$$=\int\sec x\tan x\,dx-\int\tan\ x\,dx$$

$$=\sec x-\int(\sec^2 x-1)\,dx$$

$$=\sec x-\tan x+x.$$

Example 6. *Integrate* $\frac{2\cos x}{5\sin^2 x}+\frac{1}{5\cos^2 x}$ *w.r.t. x.*

Solution. Let $I=\int\left\{\frac{2}{5}\ \frac{\cos x}{\sin^2 x}+\frac{1}{5}\cdot\frac{1}{\cos^2 x}\right\}dx$

$$\int\frac{2}{5}\ \frac{\cos x}{\sin^2 x}dx+\int\frac{1}{5}\cdot\frac{1}{\cos^2 x}dx$$

$$=\frac{2}{5}\int\operatorname{cosec} x\cot x\,dx+\frac{1}{5}\int\sec^2 x\,dx$$

$$-\frac{2}{5}\operatorname{cosec} x+\frac{1}{5}\tan x.$$

Example 7. *Integrate* $\sqrt{1+\sin 2x}$ *w.r.t. x.*

Solution. $I=\int\sqrt{1+\sin 2x}\,dx$

$$=\int\sqrt{\cos^2 x+\sin^2 x+2\sin x\cos x}\,dx$$

$$=\int\sqrt{(\cos x+\sin x)^2}\,dx+\int(\cos x+\sin x)\,dx$$

$$=\int\cos x\,dx+\int\sin x\,dx+\sin x+\cos x.$$

EXERCISE (I)

Integrate the following functions :

1. (i) $x^{6/5}$, (ii) $\sqrt[3]{x^4}$, (iii) $\frac{1}{\sqrt{x}}$

2. (i) $\sqrt{x}-\frac{1}{\sqrt{x}}$

   (ii) $7x^2-3x+8-\frac{1}{\sqrt{x}}+\frac{1}{x}+\frac{1}{x^2}$

   (iii) $\frac{ax^3+bx^2+cx+d}{x}$

   (iv) $\frac{4x^6+3x^5+2x^4+x^3+x^2+1}{x^3}$

3. $\left(2^x+\frac{1}{2}e^{-x}+\frac{4}{x}-\frac{1}{\sqrt[3]{x}}\right)$

4. $\frac{x^4+x^2+1}{2(x^2+1)}$ 5. $\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}$ 6. $\frac{\cos 2x}{\cos^2 x\sin^2 x}$

ANSWERS

1. (i) $\frac{5}{11}x^{11/5}$, (ii) $\frac{3}{7}x^{7/3}$, (iii) $2x^{1/2}$

2. (i) $\frac{2}{3}x^{3/2}-2x^{1/2}$, (ii) $\frac{7}{3}x^3-\frac{3}{2}x^2+8x-2\sqrt{x}+\log x-\frac{1}{x}$

   (iv) $x^4+x^3+x^2+x+\log x-\frac{1}{2x^2}$ 3. $\frac{2^x}{\log 2}-\frac{1}{2}e^{-x}+4\log x-\frac{3}{2}x^{2/3}$,

4. (i) $\frac{1}{2}\left(\frac{x^3}{3}+\tan^{-1}x\right)$

5. $\sec x-\operatorname{cosec} x$. 6. $-\sec x\operatorname{cosec} x$.

18·4. INTEGRATION BY SUBSTITUTION

Integration can often be facilitated by the substitution of a new variable for the given independent variable, in other words, by changing the independent variable. Experience is the best guide as to what substitution is likely to transform the given expression into another that is more readily integrable.

Illustrations :

1. $$\int \cos^3 x\,dx$$

Let $\sin x=t \Rightarrow \cos x\frac{dx}{dt}=1$

$$\therefore \quad \int \cos^3 x\,dx=\int \cos^3 x\frac{dx}{dt}\,.\,dt$$

$$=\int \cos^2 x \,.\, \cos x \,\frac{dx}{dt} \,.\, dt$$

$$=\int (1-\sin^2 x)\,.\,dt=\int dt-\int t^2\,dt=t-\frac{t^3}{3}$$

$$=\sin x-\tfrac{1}{3}\sin^3 x.$$

2. *Evaluate* $\int (4x+5)^6\,dx.$

Let $4x+5=t \Rightarrow 4\,\frac{dx}{dt}=1$ or $\frac{dx}{dt}=\frac{1}{4}$

$$\therefore \quad \int (4x+5)^6\,dx=\int (4x+5)^6\,\frac{dx}{dt}\,.\,dt$$

$$=\int t^6\,.\,\frac{1}{4}\,dt=\frac{1}{4}\int t^6\,dt=\frac{1}{4}\,.\,\frac{t^7}{7}$$

$$=\frac{1}{28}\,.\,(4x+5)^7$$

3. *Evaluate* $\int x\,(x^2+4)^5\,dx.$

Put $x^2+4=t \Rightarrow 2x\,\frac{dx}{dt}=1$ or $x\,\frac{dx}{dt}=\frac{1}{2}$

$$\therefore \quad \int x(x^2+4)^5\,dx=\int (x^2+4)^5\,x\,\frac{dx}{dt}\,.\,dt$$

$$=\int t^5\,.\,\frac{1}{2}\,dt=\frac{1}{12}\,t^6$$

$$=\frac{1}{12}\,(x^2+4)^6.$$

4. *Evaluate* $\int \sin x\,\cos^3 x\,dx.$

Put $\cos x=t \Rightarrow -\sin x\,\frac{dx}{dt}=1$

$$\therefore \quad \int \sin x\,\cos^3 x\,dx=-\int \cos^3 x\,(-\sin x)\,\frac{dx}{dt}\,dt$$

$$=-\int t^3\,.\,1\,dt=-\frac{t^4}{4}=-\frac{\cos^4 x}{4}$$

Example 8. *Integrate* $(x+a)^n$ *w.r.t.* x.

Solution. We put $(x+a)=t \Rightarrow \frac{dx}{dt}=1 \Rightarrow dx=dt$

$$\therefore \qquad \int (x+a)^n\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1} = \frac{(x+a)^{n+1}}{n+1}$$

Example 9. *Integrate the following functions w.r.t. x.*

(*a*) $(x^3+2)^2 \,.\, 3x^2$, (*b*) $(x^3+2)^{1/2}\, x^2$, (*c*) $\dfrac{8x^3}{(x^3+2)^3}$

and (*d*) $\dfrac{x^2}{(x^3+2)^{1/4}}$

Solution, Let us put $(x^3+2)=t$ then $dt=3x^2\, dx$

(*a*) $$\int (x^3+2)^2 \,.\, 3x^2\, dx = \int t^2\, dt = \frac{t^3}{3} = \frac{1}{3}(x^3+2)^3.$$

(*b*) $$\int (x^3+2)^{1/2}\, x^2\, dx = \frac{1}{3}\int (x^3+2)^{1/2} \,.\, 3x^2\, dx$$

$$= \frac{1}{3}\int t^{1/2}\, dt = \frac{1}{3} \,.\, \frac{t^{3/2}}{3/2}$$

$$= \frac{2}{9}\,(x^3+2)^{3/2}$$

(*c*) $$\int \frac{8x^2}{(x^3+2)^3}\, dx = 8 \,.\, \frac{1}{3}\int (x^3+2)^{-3}\, 3x^2\, dx$$

$$= \frac{8}{3}\int t^{-3}\, dt = \frac{8}{3}\left(-\frac{1}{2}t^{-2}\right)$$

$$= -\frac{4}{3(x^3+2)^2}$$

(*d*) $$\int \frac{x^2\, dx}{(x^3+2)^{1/4}} = \frac{1}{3}\int (x^3+2)^{-\frac{1}{4}}\, 3x^2\, dx$$

$$= \frac{1}{3}\int t^{-\frac{1}{4}}\, dt = \frac{1}{3} \,.\, \frac{4}{3}\, t^{3/4}$$

$$= \frac{4}{9}\,(x^3+2)^{3/4}.$$

Example 10. *Integrate* $e^{\tan x} \sec^2 x$ *w.r.t. x.*

Solution. We put $\tan x = t$

$$\Rightarrow \qquad \sec^2 x\, \frac{dx}{dt} = 1,\ i.e.,\ \sec^2 x\, dx = dt$$

$$\therefore \qquad \int e^{\tan x} \sec^2 x\, dx = \int e^t\, dt = e^t = e^{\tan x}$$

Example 11. *Integrate* $\dfrac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$ *w.r.t. x.*

Solution. Here $\sin^{-1} x$ is involved in the integrand and its derivative $\dfrac{1}{\sqrt{1-x^2}}$ is factor of the integrand.

This suggests the substitution $\sin^{-1} x = t \Rightarrow \dfrac{1}{\sqrt{1-x^2}}\, dx = dt$

$$\therefore \qquad I = \int e^t\, dt = e^t = e^{\sin^{-1} x}$$

Example 12. *Evaluate* $\displaystyle\int \frac{x^5 dx}{1+x^{12}}$

Solution. We put $x^6 = t \Rightarrow 6x^5 \dfrac{dx}{dt} = 1$, *i.e.* $6x^5 dx = dt$

$$\therefore \qquad \int \frac{x^5}{1+x^{12}}\, dx = \int \frac{dt}{6(1+t^2)} = \frac{1}{6} \tan^{-1} t = \frac{1}{6} \tan^{-1} x^6$$

Example 13. *Integrate the following w.r.t. x :*

(*i*) *$\sin^3 x$*, (*ii*) *$\sin 4x \cos 2x$.*

Solution. (*i*) We know $\sin 3x = 3 \sin x - 4 \sin^3 x$

$$\Rightarrow \qquad \sin^3 x = \frac{3 \sin x - \sin 3x}{4} = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

$$\therefore \qquad \int \sin^3 x\, dx = \frac{3}{4} \int \sin x\, dx - \frac{1}{4} \int \sin 3x\, dx$$

$$= -\frac{3}{4} \cos x + \frac{1}{4} \cdot \frac{\cos 3x}{3}$$

(*ii*) $$\sin 4x \cos 2x = \frac{1}{2} \left[2 \sin 4x \cos 2x \right] = \frac{1}{2} \left[\sin 6x + \sin 2x \right]$$

$$= \frac{1}{2} \sin 6x + \frac{1}{2} \sin 2x$$

$$\therefore \int \sin 4x \cos 2x\, dx = \frac{1}{2} \int \sin 6x\, dx + \frac{1}{2} \int \sin 2x\, dx$$

$$= -\frac{1}{2} \cdot \frac{\cos 6x}{6} - \frac{1}{2} \cdot \frac{\cos 2x}{2}.$$

Example 14. *Integrate* $\dfrac{e^{m \tan^{-1} x}}{1+x^2}$ *w.r.t. x.*

Solution. Put $m \tan^{-1} x = t$ so that $\frac{m}{1+x^2} dx = dt$. Therefore

$$\int \frac{e^{m \tan^{-1} x} dx}{1+x^2} = \frac{1}{m} \int e^t \, dt = \frac{1}{m} . e^t = \frac{1}{m} e^{m \tan^{-1} x}$$

Example 15. *Integrate* $\frac{1}{\sqrt{x}} \sin \sqrt{x}$ *w.r.t. x.*

Solution. Put $\sqrt{x} = t$, then $\frac{1}{2\sqrt{x}} dx = dt$, *i.e.*, $\frac{dx}{\sqrt{x}} = 2dt$

$$\therefore \quad I = \int 2 \sin t \, dt = -2 \cos t = -2 \cos \sqrt{x}.$$

Example 16. *Evaluate* $\int \frac{e^x (1+x)}{\cos^2 (xe^x)} dx$.

Solution. Put $xe^x = t$ so that $(xe^x + e^x)\, dx = dt$

$$I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t \, dt = \tan t = \tan (xe^x).$$

Example 17. *Obtain* $\int \frac{3x \, dx}{(x^2+k^2)^n}$.

Solution. Put $x^2 + k^2 = t$ so that $2x \, dx = dt$

$$I = \int \frac{\frac{3}{2} dt}{t^n} = \frac{3}{2} \int t^{-n} \, dt = \frac{3}{2} \quad \frac{t^{-n+1}}{-n+1}$$

$$= \frac{3 (x^2+k^2)^{1-n}}{2 (1-n)}$$

Example 18. *Integrate* $\frac{x^3}{(x^2+1)^3}$ *w.r.t. x.*

Solution. Put $x^2 + 1 = t$ so that $2x \, dx = dt$

$$\therefore \quad I = \frac{1}{2} \int \frac{x^2 . 2x \, dx}{(x^2+1)^3} = \frac{1}{2} \int \frac{(t-1) \, dt}{t^3}$$

$$= \frac{1}{2} \int \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt = \frac{1}{2} \left(-\frac{1}{t} + \frac{1}{2t^2} \right)$$

$$= \frac{1}{2} \left(\frac{1-2t}{2t^2} \right) = \frac{1}{2} \left[\frac{1-2(x^2+1)}{2(x^2+1)^2} \right]$$

$$= -\frac{1}{4} \cdot \frac{2x^2+1}{(x^2+1)^2}$$

Example 19. *Integrate* $\frac{\sin x \cos x}{a^2 \sin^2 x + b^2 \cos^2 x}$ *w.r.t. x.*

Solution. Put $a^2 \sin^2 x + b^2 \cos^2 x = t$, so that

$$2(a^2 - b^2) \sin x \cos x \, dx = dt \quad \Rightarrow \quad \sin x \cos x \, dx = \frac{dt}{2(a^2 - b^2)}$$

$$\therefore \qquad I = \frac{1}{2(a^2-b^2)} \int \frac{dt}{t} = \frac{1}{2(a^2-b^2)} \log t$$

$$= \frac{1}{2(a^2-b^2)} \cdot \log (a^2 \sin^2 x + b^2 \cos^2 x)$$

Example 20. *Integrate* $\dfrac{1}{x \log x \log (\log x)}$ *w.r.t. x.*

Solution. Put $\log (\log x) = t$ so that $\dfrac{1}{\log x} \cdot \dfrac{1}{x} \, dx = dt$

$$\therefore \qquad I = \int \frac{1}{t} \, dt = \log t = \log [\log (\log x)].$$

Example 21. *Integrate* $\dfrac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta}$ *w.r.t.* θ.

Solution. $$\int \frac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta} \, d\theta = \int \frac{\frac{d}{d\theta} (\sin \theta - \cos \theta)}{\sin \theta - \cos \theta} \, d\theta$$

$$= \log (\sin \theta - \cos \theta).$$

Example 22. *Integrate* $\dfrac{\sec x \operatorname{cosec} x}{\log \tan x}$ *w.r.t. x.*

Solution. $$\int \frac{\sec x \operatorname{cosec} x \, dx}{\log \tan x} = \int \frac{\frac{d}{dx} (\log \tan x)}{\log \tan x} \, dx$$

$$= \log (\log \tan x).$$

Example 23. *Integrate* $\dfrac{1 + \sin 2x}{x + \sin^2 x}$ *w.r.t. x.*

Solution $$\int \frac{1 + \sin 2x}{x + \sin^2 x} \, dx = \int \frac{(1 + 2 \sin x \cos x)}{x + \sin^2 x} \, dx$$

$$= \int \frac{\frac{d}{dx} (x + \sin^2 x)}{x + \sin^2 x} \, dx = \log (x + \sin^2 x)$$

Example 24. *Integrate* $\dfrac{\cot x}{\log \sin x}$ *w.r.t. x.*

Solution. $$\int \frac{\cot x}{\log \sin x} \, dx = \int \frac{\frac{d}{dx} (\log \sin x)}{\log \sin x} \, dx$$

$$= \log (\log \sin x).$$

EXERCISE (II)

Find the integrals of the following functions *w.r.t.* x :

1. (*i*) $\sin x \cos x$, (*ii*) $\sec^n x \tan x$, (*iii*) $\tan^3 x \sec^2 x$.

2. (*i*) $e^{3+2\sin x} \cos x$, (*ii*) $(ax^2+2bx+c)^n (ax+b)$, (*iii*) $\dfrac{1}{x(\log x)^2}$

3. (*i*) $\dfrac{1}{(3\tan x+1)\cos^2 x}$ (*ii*) $e^x \sin(e^x)$, (*iii*) $\dfrac{x}{(x^2+3)^2}$

4. (*i*) $\dfrac{\sqrt{\tan^{-1} x}}{2(1+x^2)}$, (*ii*) $\dfrac{\cos x}{1+\sin^2 x}$

5. (*i*) $\dfrac{3\cos x}{1+\sin^2 x}$, (*ii*) $\dfrac{3}{x[1+(\log x)^2]}$, (*iii*) $\dfrac{4\cos x}{(1+\sin x)^2}$

   (*iv*) $\dfrac{\sin(2+3\log x)}{x}$

6. (*i*) $\dfrac{\sin x}{3+7\cos x}$ (*ii*) $\dfrac{\cos x}{9-2\sin x}$

   (*iii*) $\dfrac{\sec^2 x}{\sqrt{4-3\tan x}}$

7. (*i*) $\left[\dfrac{\sec x}{(1-\tan x)^3}\right]^2$

   (*ii*) $\dfrac{\sec 3x \tan 3x}{2\sec 3x-5}$, (*iii*) $\dfrac{\cos x \sin x}{5+\cos^2 x}$, (*iv*) $\dfrac{\sin 2x}{1+\sin^2 x}$

ANSWERS

1. (*i*) $-\dfrac{1}{4}\cos 2x$, (*ii*) $\dfrac{\sec^n x}{n}$, (*iii*) $\dfrac{\tan^4 x}{4}$

2. (*i*) $\dfrac{1}{2} e^{3+2\sin x}$, (*ii*) $\dfrac{(ax^2+2bx+c)^{n+1}}{2(n+1)}$, (*iii*) $-\dfrac{1}{\log x}$

3. (*i*) $\dfrac{1}{3}\log(3\tan x+1)$, (*ii*) $-\cos(e^x)$, (*iii*) $-\dfrac{1}{2(x^2+3)}$

4. (*i*) $\frac{1}{3}(\tan^{-1} x)^{3/2}$, (*ii*) $\tan^{-1}(\sin x)$,

5. (*i*) $3\tan^{-1}(\sin x)$,

   (*ii*) $3\tan^{-1}(\log x)$, (*iii*) $\dfrac{4}{1+\sin x}$, (*iv*) $-\dfrac{1}{3}\cos(2+3\log x)$

6. (*i*) $-\frac{1}{7}\log(3+7\cos x)$, (*ii*) $-\frac{1}{2}\log(9-2\sin x)$,

   (*iii*) $-\frac{2}{3}\sqrt{4-3\tan x}$

7. (*i*) $\dfrac{1}{5(1-\tan x)^5}$ (*ii*) $\frac{1}{6}\log(2\sec 3x-5)$

   (*iii*) $-\frac{1}{2}\log(5+\cos^2 x)$, (*iv*) $\log(1+\sin^2 x)$.

18·5. INTEGRATION OF TRIGONOMETRIC FUNCTIONS

In standard forms, we have already seen that

$$\int \sin x\, dx = -\cos x \text{ and } \int \cos x\, dx = \sin x$$

We now derive the following standard formulae for the primitives of other circular or trigonometric functions.

I. $$\int \tan x\,dx=\int \frac{\sin x}{\cos x}\,dx=-\int \frac{-\sin x}{\cos x}\,dx$$

$$=-\int \frac{\frac{d}{dx}(\cos x)}{\cos x}\,dx=-\log\cos x=\log\sec x$$

$\therefore$ $$\int \mathbf{\tan x\,dx=\log\sec x}$$

II. $$\int \cot x\,dx=\int \frac{\cos x}{\sin x}\,dx=\int \frac{\frac{d}{dx}(\sin x)}{\sin x}\,dx$$

$$=\log\sin x$$

$\therefore$ $$\int \mathbf{\cot x\,dx=\log\sin x}$$

III. $$\int \operatorname{cosec} x\,dx=\int \frac{dx}{\sin x}=\int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\int \frac{\sec^2\frac{x}{2}\,dx}{\sec^2\frac{x}{2}\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}=\int \frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{\tan\frac{x}{2}}$$

$$=\int \frac{\frac{d}{dx}\left(\tan\frac{x}{2}\right)}{\tan\frac{x}{2}}\,dx=\log\tan\left(\frac{x}{2}\right).$$

Second Method. $$\int \operatorname{cosec} x\,dx=\int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx$$

$$=\int \frac{-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x}{\operatorname{cosec} x-\cot x}\,dx$$

$$=\int \frac{\frac{d}{dx}(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx$$

$$=\log(\operatorname{cosec} x-\cot x)=\log\left(\frac{1-\cos x}{\sin x}\right)$$

$$=\log\frac{\left(2\sin^2\frac{x}{2}\right)}{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}=\log\tan\left(\frac{x}{2}\right).$$

$\therefore \quad \int \textbf{cosec x dx} = \textbf{log (cosec x} - \textbf{cot x)} = \textbf{log tan} \left(\frac{\textbf{x}}{\textbf{2}}\right).$

IV. $\int \sec x \, dx = \int \operatorname{cosec}\left(x + \frac{\pi}{2}\right) dx$

$$= \int \operatorname{cosec} t \, dt, \text{ putting } x + \frac{\pi}{2} = t, \text{ so that } dx = dt$$

$$= \log \tan \frac{t}{2} = \log \tan \left(\frac{x}{2} + \frac{\pi}{4}\right).$$

Second Method. $\int \sec x \, dx = \int \frac{\sec x \, (\sec x + \tan x)}{(\sec x + \tan x)} \, dx$

$$= \log (\sec x + \tan x)$$

Now $\sec x + \tan x = \frac{1 + \sin x}{\cos x}$

$$= \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}$$

$$= \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} = \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} = \tan\left(\frac{\pi}{2} + \frac{x}{2}\right).$$

$\therefore \quad \int \textbf{sec x dx} = \textbf{log (sec x} + \textbf{tan x)} = \textbf{log tan} \left(\frac{\pi}{4} + \frac{\textbf{x}}{\textbf{2}}\right)$

18·6. SOME STANDARD INTEGRALS

I. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos \theta \, d\theta}{a \cos \theta}$, putting $x = a \sin \theta$

$$= \int d\theta = \theta = \sin^{-1} \frac{x}{a}$$

Then $\int \frac{\mathbf{dx}}{\sqrt{\mathbf{a^2 - x^2}}} = \mathbf{\sin^{-1} \frac{x}{a}}$

II. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta \, d\theta}{a \tan \theta}$, putting $x = a \sec \theta$

$$= \int \sec \theta \, d\theta = \log (\sec \theta + \tan \theta)$$

$$= \log (\sec \theta + \sqrt{\sec^2 \theta - 1})$$

$$= \log \left(\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right) = \log \{(x + \sqrt{x^2 - a^2})/a\}$$

Thus $$\int\frac{\mathbf{dx}}{\sqrt{\mathbf{x^2-a^2}}}=\mathbf{\log}\ \{(\mathbf{x}+\sqrt{\mathbf{x^2-a^2}})/\mathbf{a}\}$$

III $$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{a\sec^2\theta\,d\theta}{a\sec\theta},\ \text{putting } x=a\tan\theta$$

$$=\int\sec\theta\,d\theta=\log(\sec\theta+\tan\theta)$$

$$=\log(\sqrt{1+\tan^2\theta}+\tan\theta)$$

$$=\log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)=\log\{(x+\sqrt{x^2+a^2})/a\}$$

Thus $$\int\frac{\mathbf{dx}}{\sqrt{\mathbf{x^2+a^2}}}=\mathbf{\log}\{(\mathbf{x}+\sqrt{\mathbf{x^2+a^2}})/\mathbf{a}\}$$

IV. Since $\frac{1}{x^2-a^2}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$, we have

$$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\left[\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right]$$

$$=\frac{1}{2a}\left[\log(x-a)-\log(x+a)\right]$$

$$=\frac{1}{2a}\log\frac{x-a}{x+a}$$

Thus $$\int\frac{\mathbf{dx}}{\mathbf{x^2-a^2}}=\frac{\mathbf{1}}{\mathbf{2a}}\mathbf{\log}\frac{\mathbf{x-a}}{\mathbf{x+a}}$$

V. Since $\frac{1}{a^2-x^2}=\frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right)$, we have

$$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\left[\int\frac{dx}{a+x}+\int\frac{dx}{a-x}\right]$$

$$=\frac{1}{2a}\left[\log(a+x)-\log(a-x)\right\}=\frac{1}{2a}\log\frac{a+x}{a-x}$$

$$\int\frac{\mathbf{dx}}{\mathbf{a^2-x^2}}=\frac{\mathbf{1}}{\mathbf{2a}}\mathbf{\log}\frac{\mathbf{a+x}}{\mathbf{a-x}}$$

VI. $$\int\frac{dx}{x^2+a^2}=\int\frac{a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)},\ \text{by putting } x=a\tan\theta$$

$$=\frac{1}{a}\int d\theta=\frac{1}{a}\theta=\frac{1}{a}\tan^{-1}\frac{x}{a}.$$

$\therefore$ $$\int\frac{\mathbf{dx}}{\mathbf{x^2+a^2}}=\frac{\mathbf{1}}{\mathbf{a}}\mathbf{\tan^{-1}}\left(\frac{\mathbf{x}}{\mathbf{a}}\right).$$

Example 25. *Evaluate* $\int\frac{dx}{2x^2-2x+1}$

Solution. $\int \frac{dx}{2x^2-2x+1}=\frac{1}{2}\int \frac{dx}{x^2-x+\frac{1}{2}}=\int \frac{dx}{\{(x^2-x+\frac{1}{4})+\frac{1}{4}\}}$

$$=\frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{1}{2})^2}$$

$$=\frac{1}{2}\int \frac{dt}{t^2+(\frac{1}{2})^2}, \text{ where } t=x-\tfrac{1}{2}$$

$$=\frac{1}{2}\left[\frac{1}{(1/2)}\tan^{-1}\frac{x-\frac{1}{2}}{\frac{1}{2}}\right]=\tan^{-1}(2x-1).$$

Example 26. *Evaluate* $\int \frac{dx}{(1-x)\sqrt{1-x^2}}$ *[C.A., May 1991]*

Solution. Put $1-x=\frac{1}{t}$ so that $-dx=-\frac{1}{t^2}\,dt$

or $$dx=\frac{dt}{t^2}.$$

$$1-x^2=1-\left(1-\frac{1}{t}\right)^2=\frac{2}{t}-\frac{1}{t^2}=\frac{2t-1}{t^2}$$

$\therefore$ $$\int \frac{dx}{(1-x)\sqrt{1-x^2}}=\int \frac{(dt/t^2)}{(1/t)\sqrt{(2t-1)/t^2}}$$

$$=\int \frac{dt}{\sqrt{2t-1}}=\int (2t-1)^{-1/2}\,dt$$

$$=\frac{(2t-1)^{1/2}}{(\frac{1}{2})(2)}=\sqrt{2t-1}=\sqrt{\frac{2}{1-x}-1}=\sqrt{\frac{1+x}{1-x}}$$

Example 27. *Integrate* $\frac{1}{\sqrt{7+5x-3x^2}}$ *w.r.t. x.*

Solution. $\int \frac{dx}{\sqrt{7+5x-3x^2}}=\frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{(\frac{7}{3}+\frac{5}{3}x-x^2)}}$

$$=\frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{7}{3}+\frac{25}{36}\right)-\left(x^2-\frac{5}{3}x+\frac{25}{36}\right)}}$$

$$=\frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(\frac{\sqrt{109}}{6}\right)^2-\left(x-\frac{5}{6}\right)^2}}$$

$$=\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(\frac{\sqrt{109}}{6}\right)^2-t^2}}, \text{ where } t=x-\frac{5}{6}$$

$$=\frac{1}{\sqrt{3}}\cdot\sin^{-1}\frac{t}{\sqrt{109/6}}=\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-5}{\sqrt{109}}\right)$$

Example 28. *Evaluate* $\int\frac{(3x+7)\,dx}{2x^2+3x-2}$

Solution. Let $(3x+7)\equiv\lambda\frac{d}{dx}(2x^2+3x-2)+\mu$

$\therefore\qquad (3x+7)=\lambda(4x+3)+\mu$

Then $4\lambda=3$ and $3\lambda+\mu=7$ giving $\lambda=\frac{3}{4},\ \mu=\frac{19}{4}$

$\therefore\quad 3x+7=\frac{3}{4}(4x+3)+\frac{19}{4}$. (This can be done by inspection also)

$$\therefore\qquad I=\int\frac{\left\{\frac{3}{4}(4x+3)+\frac{19}{4}\right\}}{2x^2+3x-2}dx$$

$$=\frac{3}{4}\int\frac{(4x+3)\,dx}{2x^2+3x-2}+\frac{19}{4}\int\frac{dx}{2x^2+3x-2}$$

$$=\frac{3}{4}\log(2x^2+3x-2)+\frac{19}{8}\int\frac{dx}{x^2+\frac{3}{2}x-1}$$

$$=\frac{3}{4}\log(2x^2+3x-2)+\frac{19}{8}\int\frac{dx}{\left(x+\frac{3}{4}\right)^2-\frac{25}{16}}$$

$$=\frac{3}{4}\log(2x^2+3x-2)+\frac{19}{8}\cdot\frac{1}{2.(\frac{5}{4})}\log\frac{\left(x+\frac{3}{4}-\frac{5}{4}\right)}{\left(x+\frac{3}{4}+\frac{5}{4}\right)}$$

$$=\frac{3}{4}\log(2x^2+3x-2)+\frac{19}{20}\log\frac{2x-1}{2(x+2)}.$$

Example 29. *Evaluate* $\int\frac{(4x+1)\,dx}{\sqrt{3+4x-4x^2}}$

Solution. Let $4x+1\equiv\lambda\frac{d}{dx}(3+4x-4x^2)+\mu$

$\Rightarrow\qquad 4x+1\equiv\lambda(4-8x)+\mu$

Equating co-efficients of x and constant term, we have

$4=-8\lambda$ and $1=4\lambda+\mu$, *i.e.*, $\lambda=-\frac{1}{2},\ \mu=3$

$\therefore\qquad 4x+1=-\frac{1}{2}(4-8x)+3$

$$\therefore\qquad I=\int\frac{\{-\frac{1}{2}(4-8x)+3\}}{\sqrt{3+4x-4x^2}}\,dx$$

$$=-\tfrac{1}{2}\int\frac{(4-8x)\,dx}{\sqrt{3+4x-4x^2}}+3\int\frac{dx}{\sqrt{3+4x-4x^2}}$$

$$=-\frac{1}{2}\int\frac{dt}{\sqrt{t}}+\frac{3}{2}\int\frac{dx}{\sqrt{\frac{3}{4}-(x^2-x)}}$$

$$=-t^{1/2}+\frac{3}{2}\int\frac{dx}{\sqrt{1-\left(x^2-x+\frac{1}{4}\right)}}$$

$$=-\sqrt{3+4x-4x^2}+\frac{3}{2}\sin^{-1}\left(\frac{2x-1}{2}\right)$$

Example 30. *Evaluate* $\int\frac{x^2\,dx}{x^4+1}$

Solution. $I=\frac{1}{2}\int\frac{(x^2+1)+(x^2-1)}{x^4+1}\,dx$ [Note this step]

$$=\frac{1}{2}\int\frac{x^2+1}{x^4+1}\,dx+\frac{1}{2}\int\frac{x^2-1}{x^4+1}\,dx$$

Now $\int\frac{x^2+1}{x^4+1}\,dx=\int\frac{\left(1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}\,dx=\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2}$

$$=\int\frac{dt}{t^2+2},\text{ where } t=x-\frac{1}{x} \text{ so that } dt=\left(1+\frac{1}{x^2}\right)dx$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}=\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right)\right]$$

Also $\int\frac{x^2-1}{x^4+1}\,dx=\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2}$

$$=\int\frac{du}{u^2-2},\text{ where } u=x+\frac{1}{x}$$

$$=\frac{1}{2\sqrt{2}}\log\frac{u-\sqrt{2}}{u+\sqrt{2}}=\frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}$$

$$\therefore\quad I=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)-\frac{1}{4\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}$$

Remark. The above technique can also be applied to evaluate integrals of the type $\int\frac{x^2+1}{x^4+kx^2+1}\,dx$.

Example 31. *Find* $\int \frac{dx}{4+5\sin^2 x}$.

Solution. $\int \frac{dx}{4+5\sin^2 x}=\int \frac{\sec^2 x\,dx}{4\sec^2 x+5\sin^2 x\sec^2 x}$ (Note this step)

$$=\int \frac{\sec^2 x\,dx}{4(1+\tan^2 x)+5\tan^2 x}=\int \frac{\sec^2 x\,dx}{4+9\tan^2 x}$$

$$=\frac{1}{3}\int \frac{dt}{4+t^2}, \text{ putting } 3\tan x=t$$

$$=\frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\frac{t}{2}=\frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right)$$

18·7. INTEGRALS REDUCIBLE TO SOME STANDARD FORM

When functions are of the type $\frac{1}{a+b\cos x}$, $\frac{1}{a+b\sin x}$ and $\frac{1}{a+b\cos x+c\sin x}$, then substitution $\tan\frac{x}{2}=t$ is most useful. All integrals of the above type can by reduced by this substitution to algebraic forms where the integrals can be evaluated by available methods of integration of algebraic functions.

We also make use of following results from trigonometry.

$$\sin x=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2} \text{ etc.}$$

Also $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$

$$\Rightarrow \frac{1}{2}(1+t^2)dx=dt, \text{ i.e., } dx=\frac{2dt}{1+t^2}$$

$$\therefore \int \frac{dx}{a+b\cos x}=\int \frac{2\cdot\frac{dt}{1+t^2}}{a+b\,\frac{1-t^2}{1+t^2}}=\int \frac{2dt}{(a-b)\,t^2+(a+b)}$$

$$=\frac{2}{a-b}\int \frac{dt}{t^2+\frac{a+b}{a-b}}$$

Case I. Let $a^2>b^2$, so that $\frac{a+b}{a-b}$ is positive say k^2. Then

$$I=\frac{2}{a-b}\int \frac{dt}{t^2+k^2}=\frac{2}{(a-b)k}\tan^{-1}\left(\frac{t}{k}\right)$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

Case II. Let $a^2 < b^2$, so that $\dfrac{a+b}{a-b}$ is negative say $-p^2$. Then

$$I=\frac{2}{a-b}\int\frac{dt}{t^2-p^2}=\frac{2}{a-b}\cdot\frac{1}{2p}\log\frac{t-p}{t+p}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\tan\frac{x}{2}-\sqrt{\frac{b+a}{b-a}}}{\tan\frac{x}{2}+\sqrt{\frac{b+a}{b-a}}}$$

Case III. Let $a^2=b^2$, so that $b=\pm a$

$$I=\int\frac{dx}{a\pm a\cos x}=\frac{1}{a}\int\frac{dx}{1\pm\cos x}$$

$$=\frac{1}{a}\int\frac{dx}{2\cos^2\frac{x}{2}}\text{ or }\frac{1}{a}\int\frac{dx}{2\sin^2\frac{x}{2}},$$

according as $b=a$ or $b=-a$.

$$=\frac{1}{2a}\int\sec^2\frac{x}{2}\,dx\text{ or }\frac{1}{2a}\int\operatorname{cosec}^2\frac{x}{2}\,dx$$

$$=\frac{1}{a}\tan\frac{x}{2}\text{ or }-\frac{1}{a}\cot\frac{x}{2}$$

Example 32. *Evaluate* $\displaystyle\int\frac{dx}{5+4\cos x}$.

Solution. $I=\displaystyle\int\frac{\frac{2dt}{1+t^2}}{5+4.\frac{1-t^2}{1+t^2}}$, putting $\tan\dfrac{x}{2}=t$

$$=\int\frac{2dt}{9+t^2}=\frac{2}{3}\tan^{-1}\frac{t}{3}$$

$$=\frac{2}{3}\tan^{-1}\left[\frac{1}{3}\tan\frac{x}{2}\right]$$

Example 33. *Evaluate* $\displaystyle\int\frac{dx}{13+3\cos x+4\sin x}$

Solution $I=\displaystyle\int\frac{\frac{2dt}{1+t^2}}{13+\frac{3(1-t^2)}{1+t^2}+\frac{8t}{1+t^2}}$, putting $\tan\dfrac{x}{2}=t$

$$=\int\frac{dt}{5t^2+4t+8}=\frac{1}{5}\int\frac{dt}{t^2+\frac{4}{5}t+\frac{8}{5}}$$

$$=\frac{1}{5}\int\frac{dt}{\left(t+\frac{2}{5}\right)^2+\left(\frac{6}{5}\right)^2}=\frac{1}{5}\cdot\frac{5}{6}\tan^{-1}\frac{\left(t+\frac{2}{5}\right)}{\frac{6}{5}}$$

$$=\frac{1}{6}\tan^{-1}\left[\frac{1}{6}\left(5\tan\frac{x}{2}+2\right)\right]$$

EXERCISE (III)

Evaluate the following :

1. (i) $\int\frac{dx}{3x^2+2x+5}$, (ii) $\int\frac{dx}{2x^2-x-1}$

2. (i) $\int\frac{3x+1}{2x^2-2x+3}\,dx$, (ii) $\int\frac{5x-2}{1+2x+3x^2}\,dx$, (iii) $\int\frac{x+1}{3+2x-x^2}\,dx$

3. (i) $\int\frac{dx}{\sqrt{5x^2+8x+4}}$, (ii) $\int\frac{3dx}{\sqrt{15-6x-x^2}}$, (iii) $\int\frac{dx}{\sqrt{2+3x-2x^2}}$

4. (i) $\int\frac{x+1}{\sqrt{5x^2+8x-4}}\,dx$, (ii) $\int\frac{x}{\sqrt{9+8x-x^2}}\,dx$, (iii) $\int\frac{a-x}{\sqrt{2ax-x^2}}\,dx$

5. (i) $\int\frac{x\,dx}{x^4+x^2+1}$, (ii) $\int\frac{x^2\,dx}{x^4+x^2+1}$, (iii) $\int\frac{x^2-1}{x^4-x^2+1}\,dx$

6. (i) $\int x\sqrt{\frac{1+x}{1-x}}\,dx$, (ii) $\int\frac{(x+1)\sqrt{x+2}}{\sqrt{x-2}}\,dx$

7. (i) $\int\frac{e^x\,dx}{3e^{2x}+3e^x+1}$, (ii) $\int\frac{\cos x\,dx}{4\sin^2 x+4\sin x+5}$

(iii) $\int\frac{dx}{2\cos^2 x+\sin x\cos x+\sin^2 x}$

8. (i) $\int\frac{dx}{5-4\cos x}$, (ii) $\int\frac{dx}{4+5\sin x}$, (iii) $\int\frac{dx}{2-3\sin 2x}$

9. $\int\frac{dx}{2+\cos x+\sin x}$ 10. $\int\frac{dx}{3\cos x+4\sin x+13}$

ANSWERS

1. (i) $\frac{1}{\sqrt{14}}\tan^{-1}\frac{3x+1}{\sqrt{14}}$, (ii) $\frac{1}{3}\log\left\{\frac{2(x-1)}{2x+1}\right\}$

2. (i) $\frac{3}{4}\log(2x^2-2x+3)+\frac{\sqrt{5}}{2}\tan^{-1}\left\{\frac{(2x-1)}{\sqrt{5}}\right\}$

(ii) $\frac{5}{6}\log(1+2x+3x^2)-\frac{1}{3\sqrt{2}}\tan^{-1}\left\{\frac{3x+1}{\sqrt{2}}\right\}$

(iii) $-\frac{1}{2}\log(3+2x-x^2)+\frac{1}{2}\log\left(\frac{x+1}{x-3}\right)$

3. (i) $\frac{1}{\sqrt{5}} \log \left\{\sqrt{5}x + \frac{4}{\sqrt{5}} + \sqrt{5x^2+8x+4}\right\}$

(ii) $3 \sin^{-1}\left\{\frac{x+3}{\sqrt{24}}\right\}$, (iii) $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x-3}{5}\right)$

4. (i) $\frac{1}{5}\sqrt{5x^2+8x-4} + \frac{1}{5\sqrt{5}} \log \left\{ 5\left(x+\frac{4}{5}+\sqrt{x^2+\frac{5}{8}x-\frac{4}{5}}\right)\Big/ 6 \right\}$

(ii) $-\sqrt{9+8x-x^2} + 4 \sin^{-1}\left(\frac{x-4}{5}\right)$, (iii) $\sqrt{2ax-x^2}$

5. (i) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{3}\right)$

(ii) $\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right) + \frac{1}{4}\log\left(\frac{x^2-x+1}{x^2+x+1}\right)$

(iii) $\frac{1}{2\sqrt{3}} \log\left(\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1}\right)$

6. (i) $\frac{1}{2}\left[\sin^{-1} x-(x+2)\sqrt{1-x^2}\right]$

(ii) $\frac{x}{2}\sqrt{x^2-4} + \log(x+\sqrt{x^2-4}) + 3\sqrt{x^2-4}$

7. (i) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2e^x+1}{\sqrt{3}}\right)$, (ii) $\frac{1}{4} \tan^{-1}\left(\frac{2 \sin x+1}{2}\right)$

(iii) $\frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{2 \tan x+1}{\sqrt{7}}\right)$ 8. (i) $\frac{2}{3} \tan^{-1}\left(3 \tan \frac{x}{2}\right)$

(ii) $\frac{1}{3} \log\left(\frac{1+2 \tan \frac{x}{2}}{4+2 \tan \frac{x}{2}}\right)$ (iii) $\frac{1}{2\sqrt{5}}\log\left[\frac{2 \tan x-(3+\sqrt{5})}{2 \tan x-(3-\sqrt{5})}\right]$

9. $\sqrt{2} \tan^{-1}\left[\frac{1}{\sqrt{2}}\left(\tan \frac{x}{2}+1\right)\right]$

10. $\frac{1}{6} \tan^{-1}\left(\frac{5}{6} \tan \frac{x}{2}+\frac{1}{3}\right)$

18·8. INTEGRATION BY PARTS

Let u and v be two functions of x. From differential calculus, we have

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

Integrating both sides w.r.t. x, we have

$$uv=\int u\frac{dv}{dx}\,dx+\int v\frac{du}{dx}\,dx$$

Transposing, we get $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$

If $u=f(x)$ and $\frac{dv}{dx}=\phi(x)$, so that $v=\int \phi(x)dx$,

the above rule may be written as

$$\int f(x)\,\phi(x)dx = f(x)\left(\int \phi(x)dx\right) - \int\left\{\frac{d}{dx} f(x)\right\}\left\{\int \phi(x)dx\right\}dx$$

Thus

The integral of the product of two functions = first function × integral of the second function — integral of (the derivative of the first × integral of the second function).

Example 34. *Integrate $x^2 \sin x$, w.r.t. x.*

Solution. Let x^2 be the first function and $\sin x$ be the second function. Then

$$I = x^2 \int \sin x\, dx - \int\left(\frac{d}{dx} x^2\right) . \left(\int \sin x\, dx\right) dx$$

$$= -x^2 \cos x + \int 2x \cos x\, dx$$

Integrating by parts, the second term of the R.H.S., we get

$$I = -x^2 \cos x + 2x \int \cos x\, dx - \int\left(\frac{d}{dx}(2x) \int \cos x\, dx\right) dx$$

$$= -x^2 \cos x + 2x \sin x - 2\int \sin x\, dx$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x.$$

Example 35. *Evaluate* $\int x^2 e^{3x} dx$

Solution. Let x^2 be the first function and e^{3x} be the second one. Then

$$I = x^2 \int e^{3x}\, dx - \int\left(\frac{d}{dx}(x^2) . \int e^{3x}\, dx\right) dx$$

$$= \frac{x^2 e^{3x}}{3} - \int 2x . \frac{e^{3x}}{3}\, dx$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int x e^{3x}\, dx$$

Integrating by parts the second member on the R.H.S., taking x as the first function, we have

$$I = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left\{x \int x^{3x}\, dx - \int\left(\frac{d}{dx}(x) . \int e^{3x}\, dx\right) dx\right\}$$

$$=\frac{x^2e^{3x}}{3}-\frac{2}{3}\left\{x.\frac{e^{3x}}{3}-\frac{1}{3}\int e^{3x}\,dx\right\}$$

$$=\frac{x^2e^{3x}}{3}-\frac{2xe^{3x}}{9}+\frac{2}{27}e^{3x}.$$

Example 36. *Evaluate* (*a*) $\int \log x\,dx$, (*b*) $\int x^n \log x\,dx$.

Solution. (*a*) Take $\log x$ as the first function and 1 as the second.

$$\int \log x\,.\,1\,dx=\log x\int 1\,.\,dx-\int\left[\frac{d}{dx}(\log x)\int 1\,.\,dx\right]dx$$

$$=x\log x-\int\frac{1}{x}\,.\,x\,dx$$

$$=x\log x-x.$$

(*b*) Taking $\log x$ as the first function, we have

$$I=\log x\int x^n\,dx-\int\left[\frac{d}{dx}(\log x)\,.\int x^n\,dx\right]dx$$

$$=\log x.\frac{x^{n+1}}{n+1}-\int\frac{1}{x}\,.\,\frac{x^{n+1}}{n+1}\,dx$$

$$=\log x\,.\,\frac{x^{n+1}}{n+1}-\frac{1}{n+1}\int x^n\,dx$$

$$=\log x\,.\,\frac{x^{n+1}}{n+1}-\frac{x^{n+1}}{(n+1)^2}$$

Example 37. *Evaluate* $\int x\tan^{-1}x\,dx$.

Solution. Let $\tan^{-1}x$ be the first function, then

$$I=\tan^{-1}x\int x\,dx-\int\left[\frac{d}{dx}(\tan^{-1}x)\,.\int x\,dx\right]dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\int\frac{1}{1+x^2}\,.\,\frac{x^2}{2}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{(x^2+1)-1}{x^2+1}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{1}{x^2+1}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x$$

Example 38. *Evaluate* $\int e^{ax}\sin(bx+c)\,dx$.

Solution. Taking e^{ax} as the first function, we have

$$I=e^{ax}\int\sin(bx+c)\,dx-\int\left\{\frac{d}{dx}(e^{ax})\int\sin(bx+c)\,dx\right\}dx$$

$$=-e^{ax}\cdot\frac{\cos(bx+c)}{b}+\int a\cdot e^{ax}\frac{\cos(bx+c)}{b}\,dx$$

$$=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b}\int e^{ax}\cos(bx+c)\,dx$$

Integrating the second member again by parts, we have

$$I=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b}\Big[e^{ax}\int\cos(bx+c)\,dx$$

$$-\int\Big\{\frac{d}{dx}(e^{ax})\int\cos(bx+c)\,dx\Big\}\,dx\Big]$$

$$=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b}\Big\{\frac{e^{ax}\sin(bx+c)}{b}$$

$$-\int ae^{ax}\cdot\frac{\sin(bx+c)}{b}\,dx\Big\}$$

$$=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$-\frac{a^2}{b^2}\int e^{ax}\sin(bx+c)\,dx$$

$\Rightarrow$ $$I+\frac{a^2}{b^2}I=\frac{e^{ax}}{b^2}\Big[-b\cos(bx+c)+a\sin(bx+c)\Big]$$

Hence $$I=\frac{e^{ax}}{a^2+b^2}\Big[a\sin(bx+c)-b\cos(bx+c)\Big]$$

Example 39. *Integrate the function $x\cos^2 x$ w.r.t. x.*

Solution. From trigonometry, $\cos^2 x=\dfrac{1+\cos 2x}{2}$

$\therefore$ $$I=\int x\left(\frac{1+\cos 2x}{2}\right)dx=\frac{1}{2}\int x\,dx+\frac{1}{2}\int x\cos 2x\,dx$$

The second integral is integrable by parts.

$\therefore$ $$\int x\cos^2 x\,dx=\frac{x^2}{4}+\frac{1}{2}\Big[x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\,dx\Big]$$

$$=\frac{x^2}{4}+\frac{x\sin 2x}{4}+\frac{1}{8}\cos 2x$$

Example 40. *Integrate $\sqrt{a^2-x^2}$ w.r.t. x.*

Solution. $I=\int\sqrt{a^2-x^2}\cdot 1\,dx$

$$=\sqrt{a^2-x^2}\cdot x-\int\frac{-2x}{2\sqrt{a^2-x^2}}\cdot x\,dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$2\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}$$

$$\therefore\quad\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Example 41. *Integrate* $\dfrac{x^2}{(x\sin x+\cos x)^2}$ *w.r.t. x.*

Solution.

$$\int\frac{x^2}{(x\sin x+\cos x)^2}\,dx=\int\frac{x}{\cos x}\cdot\frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$$

Since $\dfrac{d}{dx}(x\sin x+\cos x)=x\cos x$, integrating by parts, we get

$$I=\frac{x}{\cos x}\cdot\frac{-1}{(x\sin x+\cos x)}+\int\frac{d}{dx}\left(\frac{x}{\cos x}\right)\cdot\frac{1}{(x\sin x+\cos x)}\,dx$$

$$=\frac{-x}{\cos x\,(x\sin x+\cos x)}+\int\frac{\cos x+x\sin x}{\cos^2 x}\cdot\frac{1}{(x\sin x+\cos x)}\,dx$$

$$=\frac{-x}{\cos x\,(x\sin x+\cos x)}+\int\sec^2 x\,dx$$

$$=\frac{-x}{\cos x\,(x\sin x+\cos x)}+\tan x$$

Example 42. *Evaluate the following integrals :*

(*i*) $\displaystyle\int\frac{xe^x}{(x+1)^2}\,dx$ *[I.C W.A., June 1991]*

(*ii*) $\displaystyle\int e^x\cdot\frac{1+\sin x}{1+\cos x}\,dx.$

Solution. (*i*) $\displaystyle\int\frac{xe^x}{(x+1)^2}\,dx=\int\frac{(x+1)-1}{(x+1)^2}\,e^x\,dx$

$$=\int\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]e^x\,dx$$

Integrating $\dfrac{1}{x+1}\,e^x\,dx$ by parts, we get

$$\int \frac{1}{x+1} e^x \, dx = \frac{1}{x+1} . e^x - \int \frac{-1}{(x+1)^2} e^x \, dx$$

$$\Rightarrow \quad \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2} \right] e^x \, dx = \frac{1}{x+1} . e^x$$

(ii) $$e^x \left(\frac{1+\sin x}{1+\cos x} \right) = e^x \left\{ \frac{1+2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right\}$$

$$= e^x \left\{ \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right\}$$

Integrating $e^x \tan \frac{x}{2}$ by parts, we get

$$\int e^x \tan \frac{x}{2} \, dx = e^x \tan \frac{x}{2} - \int e^x \frac{1}{2} \sec^2 \frac{x}{2} \, dx$$

$$\Rightarrow \quad \int e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx = e^x \tan \frac{x}{2}$$

Example 43. *Integrate* $\frac{x \sin^{-1} x}{\sqrt{1-x^2}}$ *w.r.t.* x.

Solution. Let $\sin^{-1} x = \theta \Rightarrow x = \sin \theta$

so that $dx = \cos \theta \, d\theta$

$$\therefore \quad \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx = \int \frac{\sin \theta . \theta \cos \theta}{\cos \theta} \, d\theta$$

$$= \int \theta \sin \theta \, d\theta$$

$$= \theta (-\cos \theta) + \int \cos \theta \, d\theta$$

$$= -\theta \cos \theta + \sin \theta = \sin \theta - \theta \sqrt{1-\sin^2 \theta}$$

$$= x - \sqrt{1-x^2} . \sin^{-1} x$$

Example 44. *Evaluate* $\int \frac{x^2 \tan^{-1} x}{1+x^2} \, dx$.

Solution. Let $\tan^{-1} x = \theta$, *i.e.*, $x = \tan \theta$ so that $dx = \sec^2 \theta \, d\theta$

$$\therefore \quad I = \int \frac{\tan^2 \theta . \theta \sec^2 \theta}{1+\tan^2 \theta} \, d\theta = \int \frac{\theta \tan^2 \theta \sec^2 \theta}{\sec^2 \theta} \, d\theta$$

$$= \int \theta \tan^2 \theta \, d\theta = \int \theta (\sec^2 \theta - 1) . d\theta$$

$$= \int \theta \sec^2 \theta \, d\theta - \int \theta \, d\theta$$

$$= \theta \tan \theta - \int 1 \, . \tan \theta \, d\theta - \frac{\theta^2}{2}$$

$$= \theta \tan \theta - \log \sec \theta - \frac{1}{2} \theta^2$$

$$= \theta \tan \theta - \log \sqrt{1 + \tan^2 \theta} - \frac{1}{2} \theta^2$$

$$= x \tan^{-1} x - \frac{1}{2} \log (1 + x^2) - \frac{1}{2} [\tan^{-1} x]^2$$

EXERCISE (IV)

Integrate the following functions :

1. (*i*) xe^x, (*ii*) $x^2 e^x$ [*I.C.W.A., December 1990*]
2. (*i*) $x \sin x$, (*ii*) $x \sin (x/2)$, (*iii*) $x^3 \cos x$.
3. (*i*) $\sin^{-1} x$, (*ii*) $\cot^{-1} x$, (*iii*) $x \sec^{-1} x$.
4. (*i*) $x \log x$, (*ii*) $(\log x)^2$.
5. (*i*) $x \sec^2 x$, (*ii*) $x^2 \sin x \cos x$.
6. (*i*) $x \sin x \sin 2x \sin 3x$, (*ii*) $\sin x \log (\cos x)$.
7. (*i*) $\int \frac{x + \sin x}{1 + \cos x} \, dx$, (*ii*) $\int \frac{x - \sin x}{1 - \cos x} \, dx$
8. (*i*) $\int \frac{1 + x}{(2 + x)^2} e^x \, dx$, (*ii*) $\int \frac{1 + x \log x}{x} e^x \, dx$.

   (*iii*) $\int e^x \left(\frac{\cos x + \sin x}{\cos^2 x} \right) dx$
9. (*i*) $\int \frac{\sin^{-1} x}{(1 - x^2)^{3/2}} \, dx$, (*ii*) $\int \frac{x^3 \sin^{-1} x}{\sqrt{1 - x^2}} \, dx$,

   (*iii*) $\int \frac{x \tan^{-1} x}{(1 + x^2)^{3/2}} \, dx$.
10. $\int \cos 2\theta \, . \log \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \, d\theta$.

ANSWERS

1. (*i*) $(x-1)e^x$, (*ii*) $(x^2 - 2x + 2) e^x$. 2. (*i*) $\sin x - x \cos x$,

   (*ii*) $-2x \cos \left(\frac{x}{2}\right) + 4 \sin \left(\frac{x}{2}\right)$,

   (*iii*) $(x^3 - 6x) \sin x + 3(x^2 - 2) \cos x$.

3. (*i*) $x \sin^{-1} x + \sqrt{1 - x^2}$, (*ii*) $x \cot^{-1} x + \frac{1}{2} \log(1 + x^2)$,

   (*iii*) $\frac{1}{2} \left[x^2 \sec^{-1} x - \sqrt{(x^2 - 1)} \right]$

4. (*i*) $\frac{1}{4} x^2 \log \left(\frac{x^2}{e}\right)$, (*ii*) $x(\log x)^2 - 2x \log x + 2x$.

5. (i) $x \tan x - \log \sec x$, (ii) $\frac{1}{4}\quad(1-2x^2)\cos 2x + \frac{x}{2}\sin 2x$

6. (i) $-\frac{1}{8}\,x\left(\cos 2x + \frac{1}{2}\cos 4x - \frac{1}{3}\cos 6x\right)$

$$+\frac{1}{16}\left(\sin 2x + \frac{1}{4}\sin 4x - \frac{1}{9}\sin 6x\right)$$

(ii) $\cos x[1-\log(\cos x)]$.

7. (i) $x \tan \frac{x}{2}$, (ii) $-x \cot \frac{x}{2}$.

8. (i) $\frac{e^x}{2+x}$, (ii) $e^x \log x$, (ii) $e^x \sec x$, 9. (iii) $\frac{1}{\sqrt{1+x^2}}(x-\tan^{-1} x)$

10. **Hint.** $\frac{\sin 2\theta}{2} \cdot \log \frac{1+\tan\theta}{1-\tan\theta} - \int \frac{\sin 2\theta}{2} \cdot \frac{2}{\cos 2\theta}\, d\theta$

(Integrate by parts)

18·10. INTEGRATION USING PARTIAL FRACTIONS

Example 45. *Integrate* $\frac{x}{(x-1)(2x+1)}$ *w.r.t.* x.

Solution. Let $\frac{x}{(x-1)(2x+1)} = \frac{A}{(x-1)} + \frac{B}{(2x+1)}$

Multiplying both sides by $(x-1)\ (2x+1)$, we have

$$x = A(2x+1) + B(x-1)$$

Putting $x=1$ and $-\frac{1}{2}$, we get

$$1 = 3A \quad \Rightarrow \quad A = \frac{1}{3}$$

and $\quad -\frac{1}{2} = -\frac{3}{2}B \quad \Rightarrow \quad B = \frac{1}{3}$

$$\therefore \quad \int \frac{x}{(x-1)(2x+1)}\,dx = \int\left[\frac{1}{3(x-1)} + \frac{1}{3(2x+1)}\right]dx$$

$$= \frac{1}{3}\int \frac{1}{(x-1)}\,dx + \frac{1}{3}\int \frac{1}{(2x+1)}\,dx$$

$$= \frac{1}{3}\log(x-1) + \frac{1}{3}\cdot\frac{1}{2}\log(2x+1)$$

Example 46. *Evaluate* $\int \frac{dx}{x-x^3}$

Solution. $\int \frac{dx}{x-x^3} = \int \frac{dx}{x(1-x^2)} = \int \frac{dx}{x(1-x)(1+x)}$

Let $$\frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{1+x}$$

Multiplying both sides by $x(1-x)(1+x)$, we have

$$1=A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$$

Putting $x=0$, 1 and -1, we have

$$1=A \Rightarrow A=1$$

$$1=2B \Rightarrow B=\tfrac{1}{2}$$

and $$1=-2C \Rightarrow C=-\tfrac{1}{2}$$

$$\therefore \quad \int\frac{dx}{x-x^3}=\int\left\{\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}\right\}dx$$

$$=\log x-\tfrac{1}{2}\log(1-x)-\tfrac{1}{2}\log(1+x)$$

$$=\tfrac{1}{2}[2\log x-\log(1-x)-\log(1+x)]$$

$$=\frac{1}{2}\left\{\log\frac{x^2}{1-x^2}\right\}$$

Exmple 47. *Evaluate* $\int\frac{x^3}{(x-a)(x-b)(x-c)}\,dx.$

Solution. Let $$\frac{x^3}{(x-a)(x-b)(x-c)}=1+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$$

Multiplying both sides by $(x-a)(x-b)(x-c)$, we have

$$x^3=(x-a)(x-b)(x-c)+(x-b)(x-c)A+(x-c)(x-a)B+(x-a)(x-b)C$$

Putting $x=a$, b and c, we get

$$A=\frac{a^3}{(a-b)(a-c)},\ B=\frac{b^3}{(b-a)(b-c)},\ C=\frac{c^3}{(c-a)(c-b)}$$

$$I=\int dx+A\int\frac{dx}{x-a}+B\int\frac{dx}{x-b}+C\int\frac{dx}{x-c}$$

$$=x+A\log(x-a)+B\log(x-b)+C\log(x-c)$$

where A, B, C have the values obtained above.

Example 48. *Evaluate* $\int\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$.

[*C.A., November 1991*]

Solution. Put $\sin x=t$ so that $\cos x\,dx=dt$

$$\int\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}=\int\frac{dt}{(1+t)(2+t)}$$

$$=\int\left[\frac{1}{1+t}-\frac{1}{2+t}\right]dt$$

$$= \int \frac{dt}{1+t} - \int \frac{dt}{2+t}$$

$$= \log(1+t) - \log(2+t)$$

$$= \log\left(\frac{1+t}{2+t}\right) = \log\left(\frac{1+\sin x}{2+\sin x}\right)$$

Example 49. *Evaluate* $\int \frac{dx}{\sin x + \sin 2x}$

Solution. Let $I = \int \frac{dx}{\sin x + \sin 2x} = \int \frac{dx}{\sin x + 2\sin x \cos x}$

$$= \int \frac{dx}{\sin x(1+2\cos x)} = \int \frac{\sin x \, dx}{\sin^2 x(1+2\cos x)}$$

$$= \int \frac{\sin x \, dx}{(1-\cos^2 x)(1+2\cos x)}$$

$$= \int \frac{\sin x \, dx}{(1-\cos x)(1+\cos x)(1+2\cos x)}$$

Put $\cos x = t$ so that $\sin x \, dx = -dt$

$$\therefore \quad I = -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

Let $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t}$

$$1 = (1+t)(1+2t)A + (1-t)(1+2t)B + (1-t)(1+t)C$$

Putting $t = 1, -1$ and $-\frac{1}{2}$ respectively, we have

$$1 = 6A \qquad A = \frac{1}{6}$$

$$1 = -2B \Rightarrow B = -\frac{1}{2}$$

and $$1 = \frac{3}{4}C \Rightarrow C = \frac{4}{3}$$

$$\therefore \quad I = -\int \left[\frac{1}{6(1-t)} - \frac{1}{2(1+t)} + \frac{4}{3(1+2t)}\right] dt$$

$$= \frac{1}{6}\log(1-t) + \frac{1}{2}\log(1+t) - \frac{2}{3}\log(1+2t)$$

$$= \frac{1}{6}\log(1-\cos x) + \frac{1}{2}\log(1+\cos x) - \frac{2}{3}\log(1+2\cos x)$$

Example 50. *Integrate* $\frac{x+5}{(x+1)(x+2)^2}$ *w.r.t. x.*

Solution. Let $\frac{x+5}{(x+1)(x+2)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

$\Rightarrow \quad x+5 \equiv A(x+2)^2 + B(x+1)(x+2) + C(x+1)$

Putting $x=-2$ and $x=-1$, we get $A=4$, $C=-3$.

Comparing co-efficient of x^2, we get $A+B=0$, *i.e.*, $B=-4$

$$\therefore \quad I = 4\int \frac{dx}{x+1} - 4\int \frac{dx}{x+2} - 3\int \frac{dx}{(x+2)^2}$$

$$= 4 \log(x+1) - 4 \log(x+2) + \frac{3}{x+2}$$

Example 51. *Integrate* $\frac{x^2-1}{(x^2+1)(x^2+2)(x^2+3)}$ *w.r.t. x.*

Solution. Let x^2 be denoted by y (we are not substituting y for x). Then the integrand becomes

$$\frac{y-1}{(y+1)(y+2)(y+3)} = \frac{-1}{y+1} + \frac{3}{y+2} + \frac{-2}{y+3}$$

(Using partial fractions)

Writing back x^2 for y, we get

$$\int \frac{(x^2-1)\,dx}{(x^2+1)(x^2+2)(x^2+3)} = -\int \frac{dx}{x^2+1} + 3\int \frac{dx}{x^2+2} - 2\int \frac{dx}{x^2+3}$$

$$= -\tan^{-1} x + \frac{3}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} - \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}}$$

EXERCISE (V)

Integrate the following functions w.r.t. x

1. (*i*) $\frac{x}{(x-3)(x+1)}$, (*ii*) $\frac{1}{(x+1)(x+2)(x+3)}$

2. (*i*) $\frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}$, (*ii*) $\frac{x+12}{x^2-13x+42}$

   (*iii*) $\frac{2x-1}{(x+1)(x-1)(2x+1)}$.

3. (*i*) $\frac{x^2-4}{(x^2-1)(3x+2)}$, (*ii*) $\frac{x^2}{(x-a)(x-b)}$.

4. $\frac{1}{x[6(\log x)^2+7\log x+2]}$.

5. (*i*) $\frac{\sin 2x}{(\sin x+1)(\sin x+2)(\sin x+3)}$

(*ii*) $\dfrac{1}{3 \sin x + \sin 2x}$, (*ii*) $\dfrac{\sec^2 x}{(a+\tan x)(b+2 \tan x)}$

6. (*i*) $\dfrac{x}{(x-1)^2(x+2)}$, (*iii*) $\dfrac{3x+4}{(x+2)^2(x-6)}$

7. $\dfrac{x^2-1}{(x^2+1)(x^2+2)(x^2+3)}$

ANSWERS

1. (*i*) $\frac{1}{4} \log (x-3)^3 (x+1)$, (*ii*) $\frac{1}{2} \log [(x+1)(x+3)/(x+2)^2]$.

2. (*i*) $10 \log (x+3) - 30 \log (x+4) + 21 \log (x+5)$,

   (*ii*) $19 \log (x-7) - 18 \log (x-6)$

   (*iii*) $-\frac{3}{2} \log (x+1) + \frac{1}{6} \log (x-1) + \frac{4}{3} \log (2x+1)$.

3. (*i*) $-\frac{3}{8} \log (x-1) - \frac{3}{4} \log (x+1) + \frac{35}{24} \log (3x+1)$,

   (*ii*) $x + \frac{a^2}{a-b} \log (x-a) - \frac{b^2}{a-b} \log (x-b)$

4. (*i*) $\log \dfrac{2 \log x + 1}{3 \log x + 2}$.

5. (*i*) $4 \log (\sin x + 2) - \log (\sin x + 1) - 3 \log (\sin x + 3)$,

   (*ii*) $\frac{1}{10} \log (1-\cos x) - \frac{1}{2} \log (1+\cos x) + \frac{2}{5} \log (3+2 \cos x)$,

   (*iii*) $\frac{1}{b-2a} \log (a+\tan x) + \frac{1}{2a-b} \log (b-2 \tan x)$.

6. (*i*) $\frac{2}{9} \log \frac{x-1}{x+2} - \frac{1}{3(x-1)}$, (*ii*) $\frac{11}{32} \log \frac{x-6}{x+2} - \frac{1}{4(x+2)}$

7. $\frac{3}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} - \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - \tan^{-1} x$

8·10. DEFINITE INTEGRALS

In geometrical and other applications of Integral Calculus it becomes necessary to find the difference in the values of the integral of a function $f(x)$ between two assigned values of an independent variable x, say, a, b. The difference is called the *Definite Integral* $f(x)$, over the interval $[a, b]$ and is denoted by

$$\int_a^b f(x)\, dx$$

Thus $$\int_a^b f(x)\,dx=\phi(b)-\phi(a),$$

where $\phi(x)$ is an integral of $f(x)$.

The difference $[\phi(b)-\phi(a)]$ is sometimes denoted as

$$\Big[\phi(x)\Big]_a^b$$

Thus if $\phi(x)$ is an integral of $f(x)$, we write

$$\int_a^b f(x)\,dx=\Big[\phi(x)\Big]_a^b=\phi(b)-\phi(a)$$

The numbers a and b are respectively called the lower limit and the upper limit of definite integral. $\int_a^b f(x)\,dx$ is called the definite integral because the indefinite constant of integration does not appear in it. Since

$$\int_a^b f(x)\,dx=\Big[\phi(x)+c\Big]_a^b=\Big\{\phi(b)+c\Big\}-\Big\{\phi(a)+c\Big\}$$

$$=\phi(b)-\phi(a)$$

so that arbitrary constant c disappears in the process.

Illustrations :

1. $$\int_{-1}^{1}\left(2x^2-x^3\right)dx=\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_{-1}^{1}=\left(\frac{2}{3}-\frac{1}{4}\right)-\left(-\frac{2}{3}-\frac{1}{4}\right)$$

$$=\frac{4}{3}.$$

2. $$\int_2^4 (3x-2)^2\,dx=\int_2^4 (9x^2-12x+4)\,dx$$

$$=\left[9\,.\,\frac{x^3}{3}-12\,.\,\frac{x^2}{2}+4x\right]_2^4$$

$$=\left[3x^3-6x^2+4x\right]_2^4$$

$$=(192-96+16)-(24-24+8)=104$$

3. $\displaystyle\int_{-3}^{-1}\left[\frac{1}{x^2}-\frac{1}{x^3}\right]dx=\left[-\frac{1}{x}+\frac{1}{2x^2}\right]_{-3}^{-1}=\left[1+\frac{1}{2}\right]-\left[\frac{1}{3}+\frac{1}{18}\right]$

$$=\frac{10}{9}$$

4. $\displaystyle\int_1^4\frac{dx}{\sqrt{x}}=\left[2\sqrt{x}\right]_1^4$

$$=2(\sqrt{4}-\sqrt{1})=2$$

5. $\displaystyle\int_6^{10}\frac{dx}{x+2}=\left[\log(x+2)\right]_6^{10}$

$$=\log 12-\log 8=\log\left(\frac{12}{8}\right)=\log\left(\frac{3}{2}\right)$$

6. $\displaystyle\int_1^e\log x\,dx=\left[x\log x-x\right]_1^e$

$$=(e\log e-e)-(\log 1-1)=1$$

7. $\displaystyle\int_{\pi/2}^{3\pi/4}\sin x\,dx=\left[-\cos x\right]_{\pi/2}^{3\pi/4}$

$$=-\left(-\frac{1}{2}\sqrt{2}-0\right)=\frac{1}{2}\sqrt{2}$$

8. $\displaystyle\int_{-2}^{3}e^{-x/2}\,dx=\left[-2e^{-x/2}\right]_{-2}^{3}$

$$=-2\,(e^{-3/2}-e)=4{\cdot}9904$$

9. $\displaystyle\int_{-2}^{2}\frac{dx}{x^2+4}=\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2}$

$$=\frac{1}{2}\left[\frac{1}{4}\pi-\left(-\frac{1}{4}\pi\right)\right]=\frac{1}{4}\pi$$

18·11. PROPERTIES

In this article we establish some important properties which considerably facilitate the evaluation of many definite integrals. It is assumed throughout that

$$\phi'(x)=f(x).$$

Property I. *The value of definite integral is independent of the variable of integration, i.e.,*

$$\int_a^b f(t)\,dt=\int_a^b f(x)\,dx$$

Proof. $$\text{L.H.S.}=\Big[\,\phi(t)\,\Big]_a^b=\phi(b)-\phi(a)$$

$$=\int_a^b f(x)\,dx=\text{R.H.S.}$$

Property II. $$\int_a^a f(x)dx=0$$

Proof. $$\text{L.H.S.}=\Big[\,\phi(x)\,\Big]_a^a=\phi(a)-\phi(a)=0=\text{R.H.S.}$$

Property III. $$\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$$

Proof. $$\text{L.H.S.}=\Big[\,\phi(x)\,\Big]_a^b=\phi(b)-\phi(a)$$

$$=-\Big[\,\phi(a)-\phi(b)\,\Big]=-\Big[\,\phi(x)\,\Big]_b^a$$

$$=-\int_b^a f(x)\,dx=\text{R.H.S.}$$

Property IV $$\int_a^c f(x)dx+\int_c^b f(x)dx=\int_a^b f(x)dx$$

Proof $\quad$ L.H.S. $=[\phi(c)-\phi(a)]+[\phi(b)-\phi(c)]$

$$=\phi(b)-\phi(a)=\int_a^b f(x)dx=\text{R.H.S.}$$

Property V. $\displaystyle\int_0^a f(x)dx=\int_0^a f(a-x)dx$

Proof. In the integral on R.H.S., put $a-x=t$ so that $-dx=dt$. Also $t=a$ when $x=0$ and $t=0$ when $x=a$.

$$\therefore \qquad \text{R.H.S.}=-\int_a^0 f(t)dt$$

$$=\int_0^a f(t)dt \qquad \text{(by Property III)}$$

$$=\int_0^a f(x)dx \qquad \text{(by Property I)}$$

$$=\text{L.H.S.}$$

Cor. $\quad$ If $f(a-x)=-f(x)$ then $\displaystyle\int_0^a f(x)dx=0$

Property VI. (*i*) $\displaystyle\int_{-a}^a f(x)dx=2\int_0^a f(x)dx$, *if* $f(x)$ *is even function of* x, *i.e., if* $f(-x)=f(x)$ *and*

(*ii*) $\displaystyle\int_{-a}^a f(x)dx=0$, *if* $f(x)$ *is an odd function of* x, *i.e., if*

$$f(-x)=-f(x).$$

Proof. We can write

$$\therefore \qquad \int_{-a}^a f(x)dx=\int_{-a}^0 (x)dx+\int_0^a f(x)dx$$

In the first integral on the right hand side, put $x=-t$ so that $t=0$ when $x=0$ and $t=a$ when $x=-a$.

$$\therefore \quad \int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-t)dt$$

$$= \int_{0}^{a} f(-t)dt \qquad \text{(by Property III)}$$

$$= \int_{0}^{a} f(-x)dx \qquad \text{(by Property I)}$$

Therefore

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} [f(-x)+f(x)]\,dx$$

$$= 2\int_{0}^{a} f(x)dx \text{ if } f(-x)=f(x)$$

$$= 0, \text{ if } f(-x) = -f(x)$$

Property VII. $\int_{0}^{2a} f(x)dx = 2\int_{0}^{a} f(x)dx$ *if* $f(2a-x)=f(x)$

$$= 0 \text{ if } f(2a-x) = -f(x)$$

Proof. We have

$$\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{a}^{2a} f(x)dx \qquad \ldots(1)$$

Now in the second integral on the R.H.S. put $x=2a-t$ so that when $x=a$, $t=a$ and when $x=2a$, $t=0$.

$$\int_{a}^{2a} f(x)dx = -\int_{a}^{0} f(2a-t)dt = \int_{0}^{a} f(2a-t)dt = \int_{0}^{a} f(2a-x)dx$$

$\therefore$ From (1), we get

$$\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(2a-x)dx = \int_{0}^{a} [f(x)+f(2a-x)]dx$$

$$\therefore \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x)=f(x)$$

$$= 0, \text{ if } f(2a-x) = -f(x)$$

Property VIII. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof. In the integral on the R.H.S. put $a+b-x=t$ so that $-dx=dt$. Also when $x=a$, $t=b$ and when $x=b$, $t=a$.

$$\therefore \quad \text{R.H.S.} = -\int_b^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx = \text{L.H.S.}$$

Example 52. *Evaluate* $\int_1^2 \frac{t^2+2t+5}{t}\,dt.$

[*I.C.W.A., June 1990*]

Solution. $\int_1^2 \frac{t^2+2t+5}{t}\,dt = \int_1^2 \left(t+2+\frac{5}{t}\right)dt$

$$= \left| \frac{t^2}{2} + 2t + 5\log t \right|_1^2$$

$$= \frac{1}{2}(4-1) + 2(2-1) + 5(\log 2 - \log 1)$$

$$= \frac{3}{2} + 2 + 5\log 2 = 5\log 2 + \frac{7}{2}.$$

Example 53. *Evaluate* $\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}.$ [*C.A., May 1991*]

Solution. Put $x = a\tan\theta$ so that $dx = a\sec^2\theta\, d\theta$

When $x=0$, $\theta = \tan^{-1}\left(\frac{0}{a}\right) = 0.$

When $x=a$, $\theta = \tan^{-1}\left(\frac{a}{a}\right) = \tan^{-1} 1 = \frac{\pi}{4}$

$$\therefore \int_0^a \frac{dx}{(a^2+x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int_0^{\pi/4} \frac{d\theta}{\sec\theta}$$

$$=\frac{1}{a^2}\int_0^{\pi/4}\cos\theta\,d\theta=\frac{1}{a^2}\Big|\sin\theta\Big|_0^{\pi/4}$$

$$=\frac{1}{a^2}\left(\sin\frac{\pi}{4}-\sin 0\right)=\frac{1}{\sqrt{2}\,a^2}.$$

Example.54. *Evaluate the integral*

$$\int_0^{\pi}\frac{dx}{1+\frac{1}{2}\cos x}.$$ *[C.A., May 1991]*

Solution. Put $\tan\frac{x}{2}=t$ so that $dx=\frac{2\,dt}{1+t^2}$.

When $x=0$, $t=\tan\left(\frac{0}{2}\right)=0$

When $x=\pi$, $t=\tan\left(\frac{\pi}{2}\right)=\infty$

$$\therefore\quad \int_0^{\pi}\frac{dx}{1+\frac{1}{2}\cos x}=\int_0^{\infty}\frac{2dt/(1+t^2)}{1+\frac{1}{2}\cdot\frac{1-t^2}{1+t^2}}=\int_0^{\infty}\frac{4dt}{2+2t^2+1-t^2}$$

$$=\int_0^{\infty}\frac{4dt}{t^2+3}=4\int_0^{\infty}\frac{dt}{t^2+(\sqrt{3})^2}=4\cdot\frac{1}{\sqrt{3}}\left|\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right|_0^{\infty}$$

$$=\frac{4}{\sqrt{3}}\left(\tan^{-1}\infty-\tan^{-1}0\right)=\frac{4}{\sqrt{3}}\left(\frac{\pi}{2}-0\right)=\frac{2\pi}{\sqrt{3}}$$

Example 55. *Evaluate* $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx.$

[C.A., November 1991]

Solution. Put $1+x^2=t^2$ so that $x\,dx=t\,dt$

When $x=0$, $t=1$ and when $x=1$, $t=\sqrt{2}$.

$$\therefore\quad \int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx=\int_1^{\sqrt{2}}\frac{\sqrt{1-(t^2-1)}}{t}\,t\,dt$$

$$=\int_1^{\sqrt{2}}\sqrt{2-t^2}\,dt$$

$$\left[\because \int \sqrt{a^2-x^2}\ dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=\left|\frac{t\sqrt{2-t^2}}{2}+\frac{2}{2}\sin^{-1}\frac{t}{\sqrt{2}}\right|_1^{\sqrt{2}}$$

$$=\left\{\frac{\sqrt{2}\,.\sqrt{2-2}}{2}-\frac{1\,.\sqrt{2-1}}{2}\right\}+\left\{\sin^{-1}\frac{\sqrt{2}}{\sqrt{2}}-\sin^{-1}\frac{1}{\sqrt{2}}\right\}$$

$$=-\frac{1}{2}+\sin^{-1}1-\sin^{-1}\frac{1}{\sqrt{2}}=-\frac{1}{2}+\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}-\frac{1}{2}.$$

Example 56. *Evaluate the following integrals :*

(i) $\int_0^{\pi/2} \cos x\, dx$, *(ii)* $\int_2^3 e^{2x}\, dx$ *and (iii)* $\int_0^{\pi/2} \sqrt{1+\sin x}\, dx$

Solution. (*i*) $\int_0^{\pi/2} \cos x\, dx=\left[\sin x\right]_0^{\pi/2}=\sin\frac{\pi}{2}-\sin 0=1$

(*ii*) $\int_2^3 e^{2x}\, dx=\left[\frac{e^{2x}}{2}\right]_2^3=\frac{1}{2}(e^6-e^4)=\frac{1}{2}e^4(e^2-1)$

(*iii*) $$\int_0^{\pi/2}\sqrt{(1+\sin x)}\ dx=\int_0^{\pi/2}\sqrt{\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}\right)}dx$$

$$=\int_0^{\pi/2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx$$

$$=\left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}+\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$=2\left[\sin\frac{x}{2}-\cos\frac{x}{2}\right]_0^{\pi/2}$$

$$=2\left[\left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)-\left(\sin 0-\cos 0\right)\right]$$

$$=2\left[\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-\left(0-1\right)\right]=2$$

Example 57. *Evaluate the following integrals :*

(*i*) $\int_0^{\pi/2} \cos^4 x \sin x\, dx$, (*ii*) $\int_0^{\pi/3} \frac{\cos x\, dx}{3+4\sin x}$

(*iii*) $\int_0^{\pi/4} \frac{\sin 2\theta\, d\theta}{\sin^4\theta+\cos^4\theta}$

Solution. (*i*) Let $I=\int_0^{\pi/2} \cos^4 x \sin x\, dx$

Put $\cos x=t$, then $-\sin x\, dx=dt$ or $\sin x\, dx=-dt$

Also, when $x=0$, $t=1$ and when $x=\frac{\pi}{2}$, $t=0$

$$\therefore \quad I=-\int_1^0 t^4\, dt=-\frac{1}{5}\left[t^5\right]_1^0=-\frac{1}{5}\ (0-1)=\frac{1}{5}$$

(*ii*) Let $I=\int_0^{\pi/3} \frac{\cos x\, dx}{3+4\sin x}$

Put $\sin x=t$, then $\cos x\, dx=dt$

Also, when $x=0$, $t=0$ and when $x=\frac{\pi}{3}$, $t=\frac{\sqrt{3}}{2}$

$$\therefore \quad I=\int_0^{\sqrt{3}/2} \frac{dt}{3+4t}=\frac{1}{4}\left[\log(3+4t)\right]_0^{\sqrt{3}/2}$$

$$=\frac{1}{4}\left[\log(3+2\sqrt{3})-\log 3\right]$$

$$\frac{1}{4}\log\frac{3+2\sqrt{3}}{3}=\frac{1}{4}\log\left(1+\frac{2}{\sqrt{3}}\right)$$

(*iii*) $I=\int_0^{\pi/4} \frac{\sin 2\theta\, d\theta}{\sin^4\theta+\cos^4\theta}=\int_0^{\pi/4} \frac{2\sin\theta\cos\theta\, d\theta}{\sin^4\theta+\cos^4\theta}$

$$=\int_0^{\pi/4} \frac{2\tan\theta \,.\, \sec^2\theta \, d\theta}{\tan^4\theta+1}$$

[Dividing num. and den. by $\cos^4\theta$]

Put $\tan^2\theta = t$ then $2\tan\theta\sec^2\theta\, d\theta = dt$

Also, when $\theta=0$, $t=0$ and when $\theta=\frac{1}{4}\pi$, $t=1$.

$$\therefore \qquad I=\int_0^1 \frac{dt}{t^2+1}=\Big[\tan^{-1} t\Big]_0^1=\frac{\pi}{4}-0=\frac{\pi}{4}$$

Example 58. *Evaluate* $I=\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$

Solution. $$I=\int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx$$

[by Property (V)]

$$=\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$\therefore \quad 2I \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx+\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$=\int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx=\int_0^{\pi/2} 1 \quad dx \quad \frac{\pi}{2}$$

$$I=\frac{\pi}{4}.$$

Example 59. *Evaluate* $I=\int_0^{\pi/2} \log(\sin x)\, dx.$

Solution. $$I=\int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2}-x\right)\right] dx=\int_0^{\pi/2} \log\cos x\, dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\left[\log(\sin x)+\log(\cos x)\right]dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\log(\sin x\cos x)dx=\frac{1}{2}\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\log(\sin 2x)\,dx-\frac{1}{2}\int_0^{\pi/2}\log 2\,dx$$

$$=\frac{1}{2}I_1-\frac{1}{2}\left(\frac{\pi}{2}\log 2\right)$$

where $$I_1=\int_0^{\pi/2}\log(\sin 2x)\,dx$$

$$I_1=\frac{1}{2}\int_0^{\pi}\log\sin t\,dt,\text{ putting } t=2x$$

$$=\frac{1}{2}\cdot 2\int_0^{\pi/2}\log\sin t\,dt \qquad \text{[by Property (VII)]}$$

$$=\int_0^{\pi/2}\log\sin x\,dx=I$$

$\therefore$ $$I=\frac{1}{2}I-\frac{\pi}{4}\log 2,\text{ whence } I=-\frac{\pi}{2}\log 2$$

EXERCISE (VI)

Evaluate the following definite integrals

1. $\displaystyle\int_{-1}^{1} x^2\sqrt{4-x^2}\,dx$

2. $\displaystyle\int_{3}^{4}\frac{dx}{25-x^2}$

3 $\displaystyle\int_{3}^{11}\sqrt{2x+3}\,dx.$

4. Prove that (*i*) $\int_3^8 \frac{dx}{x-3} - \log 5,$

(*ii*) $\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)} = \log \frac{4}{3}$, (*iii*) $\int_0^{\pi} \frac{dx}{5+3\cos x} = \frac{\pi}{4}$.

5. $\int_0^{\pi/2} \frac{dx}{3+\cos 2x}$, 6. $\int_0^{\pi/4} \frac{dx}{2+\tan x}$.

7. Prove that $\int_0^{\pi/2} \frac{dx}{1+\tan x} = \frac{\pi}{4}$. [*C.A., May 1984*]

$\left[\right.$**Hint.** Write $\tan x = \frac{\sin x}{\cos x}\left.\right]$

8. Prove that $\int_0^{\pi} \frac{x\, dx}{1+\sin x} = \pi$ [*C.A., November 1981*]

9. Evaluate (*i*) $\int_0^2 \frac{\sqrt{x}\, dx}{\sqrt{x}+\sqrt{2-x}}$ (*ii*) $\int_0^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$.

10. Evaluate $\int_2^5 \frac{\sqrt{x}}{\sqrt{7-x}+\sqrt{x}}\, dx$.

ANSWERS

1. $\frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$ 2. $\frac{1}{5}\log\frac{3}{2}$

3. $\frac{98}{3}$. 5. $\frac{\sqrt{2}}{8}\pi$

9. (*i*) 1 (*ii*) $\frac{1}{\sqrt{3}}$

10. $\frac{3}{2}$.