19

Vector Algebra

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- Addition, subtraction, scalar product and vector product and solve problems based on the above.

19·1. VECTORS

In mathematics we generally deal with two kinds of quantities. Those which are specified by a single real number called the magnitude, in other words, those which measure quantities but not related to any direction in space. Such quantities are called *scalars*. The examples of such quantities used are in measurement of mass, volume, electric charge, temperature, sales, production, etc. The other types of physical quantities are those which have got magnitude as well as a definite direction in space. Such quantities are called vector quantities or simply *vectors*. The most familiar examples of this type are for the measurement of velocity, acceleration, etc

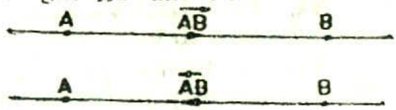
A vector is often denoted as a directed line or line segment. Such a line has an initial point or the origin and a terminal point or the terminus, and whose direction is indicated by an arrow. Such a directed line in geometric expression has the following three attributes.

(*i*) length or magnitude,

(*ii*) support or inclination, and

(*iii*) direction or sense.

Thus in vector $\overrightarrow{AB}$, A is called the origin and B the terminus. The magnitude of the vector is given by the length AB and its direction is from A to B or B to A depending on the arrow indication. In addition to the above notation of vectors giving their origin and terminus, we shall use bold face type **a**, **b**, **c**. The corresponding italic letters a, b, c denote only the magnitude of the vectors. We also use



$\vec{i}, \vec{j}, \vec{k}$ vector bases or just **i**, **j**, **k** depending on the context in which these have been employed.

19·2. TYPES OF VECTORS

I. Free and Localised Vectors. When we are at liberty to choose the origin of the vector at any point, it is said to be a free vector but where it is restricted to a certain specified point, the vector is said to be a localised vector.

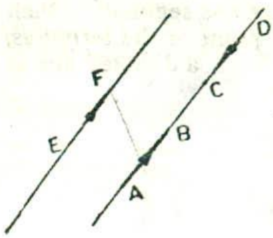
II. Null Vector. If the origin and terminal points of a vector coincide, it is said to be a *zero or null* vector. Evidently its length is zero and its direction is indeterminate. A null vector is denoted by the bold faced type **o**. All zero vectors are equal and they can be expressed as $\overrightarrow{AA}$, $\overrightarrow{BB}$, etc.

III. Unit Vector. A vector whose modulus is unity is called a unit vector. If there be any vector **a** whose modulus is a, then the corresponding unit vector in that direction is denoted by $\hat{\mathbf{a}}$ which has its parallel supports.

$$\mathbf{a}=a\,\hat{\mathbf{a}} \quad \text{or} \quad \hat{\mathbf{a}}=\frac{\mathbf{a}}{a}$$

IV. Reciprocal Vector. A vector whose direction is the same as that of a given vector **a** but whose magnitude is the reciprocal of the magnitude of the given vector, it is called the reciprocal of **a** and is written as $\mathbf{a}^{-1}$

Thus if $\mathbf{a}=a\,\hat{\mathbf{a}}$, then $\mathbf{a}^{-1}=\frac{1}{a}\hat{\mathbf{a}}=\frac{\mathbf{a}}{a^2}$



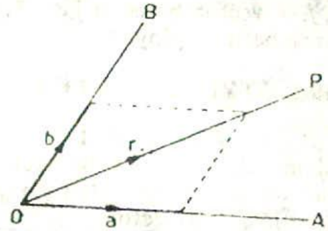
V. Equal Vectors. Two vectors are said to be equal if they have (*i*) the same length (magnitude), (*ii*) the same sense and (*iii*) the same or parallel supports. The equality is symbolically denoted by $\mathbf{a}=\mathbf{b}$. Thus equal vectors may be represented by parallel lines of equal length drawn in the same sense or direction irrespective of the origin.

$\overrightarrow{AB}=\overrightarrow{EF}=\overrightarrow{CD}\neq\overleftarrow{CD}$

VI. Co-initial Vectors. Vectors having the same initial point or origin are called co-initial vectors.

VII. Collinear Vectors. Any number of vectors are said to be collinear when they are parallel to the same line whatever their magnitudes may be.

VIII. Coplanar Vectors. Vectors whose supports are parallel to the same plane are called coplanar vectors. Any plane which is parallel to this plane is called the *plane of such vectors.*



$x\mathbf{a}+y\mathbf{b}$ is coplanar with the vectors $\mathbf{a}$, $\mathbf{b}$, whatever be the values of the scalars x and y. If $\mathbf{a}$, $\mathbf{b}$ be two non-collinear vectors then every vector $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be represented as a linear combination.

If three vectors are coplanar, then any one of them can be expressed in terms of the other two. Converse also holds good, *i.e.*, if there are three vectors and any one of them can be expressed in terms of the other two, then the vectors are coplanar.

Remarks. 1. If $\mathbf{a}$, $\mathbf{b}$ be two non-collinear vectors, then any vector $\vec{r}$, coplanar with $\mathbf{a}$, $\mathbf{b}$ can be expressed as $\mathbf{r}=x\mathbf{a}+y\mathbf{b}$ and x, y being scalars.

2. *Non-coplanar Vectors.* If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non-collinear and non-coplanar vectors, then any vector $\mathbf{r}$ can be expressed as

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$$

where x, y, z are scalars.

IX. Linear Combination. The linear combination is the addition of two or more vectors multiplied by the respective scalars. For example, we have the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and the respective scalars x, y and z. The linear combination of the above vectors will then be expressed

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}$$

With numerical expression of scalars, vector combinations can be

$$2\mathbf{a}+\mathbf{b},\ 3\mathbf{a}+2\mathbf{b}-4\mathbf{c},\ -4\mathbf{a}+3\mathbf{b}+\sqrt{2}\mathbf{c}$$

X. Linear Dependence. Now a linear combination r may be of linear dependence or independence. The vectors **a**, **b** and **c** are said to be dependent if there are scalars x, y and z not all zero such that their linear combination with scalar multiplications yield a zero vector as indicated below :

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=\mathbf{0} \qquad [\ \mathbf{0} \text{ is a zero vector }]$$

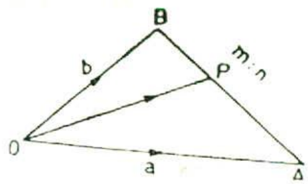
If there is no such set of scalars that the linear combination of vectors with scalar multiplications yield a zero vector then the vectors **a**, **b** and **c** are linearly independent. In such a case the linear combination **r** will be a zero vector only if all the scalars x, y, z are zero.

XI. Position Vector. Any vector joining an arbitrary point of origin O and the terminal point P can be the position vector in a place which can be used to explain vector from other origins but terminating at P or those from other origins in the same plane. In a way this concept of position vector is the basis of vector geometry we are discussing here.

To illustrate, we express the position vector OP in the following figure in terms of OA and OB which are from the same origin as

$$OP=\frac{m\overrightarrow{OA}+n\overrightarrow{OB}}{m+n}$$

$$=\frac{m\mathbf{a}+n\mathbf{b}}{m+n}$$



where the point P divides the vector AB in the ratio $m : n$.

This result we had obtained earlier in the chapter on Coordinate Geometry.

Example 1. *The position vectors of four points, P, Q, R, S are* **a**, **b**, $2\mathbf{a}+3\mathbf{b}$, $2\mathbf{a}-3\mathbf{b}$ *respectively. Express the vectors* $\overrightarrow{PR}$, $\overrightarrow{RS}$ *and* $\overrightarrow{PQ}$ *in terms of* **a** *and* **b**.

Solution. Let O be the origin.

$$\therefore \quad \overrightarrow{OP}=\mathbf{a},\ \overrightarrow{OQ}=\mathbf{b},\ \overrightarrow{OR}=2\mathbf{a}+3\mathbf{b},\ \overrightarrow{OS}=2\mathbf{a}-3\mathbf{b}$$

Now $$\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=2\mathbf{a}+3\mathbf{b}-\mathbf{a}=\mathbf{a}+3\mathbf{b}$$

$$\overrightarrow{RS}=\overrightarrow{OS}-\overrightarrow{OR}=2\mathbf{a}-3\mathbf{b}-2\mathbf{a}-3\mathbf{b}=-6\mathbf{b}$$

and $$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\mathbf{b}-\mathbf{a}$$

Example 2. *Show that the points* $\mathbf{a}-2\mathbf{b}+3\mathbf{c}$, $2\mathbf{a}+3\mathbf{b}-4\mathbf{c}$ *and* $-7\mathbf{b}+10\mathbf{c}$ *are collinear.*

Solution. Let the given points be denoted by A, B and C. Let C be the origin of reference, then

$$\vec{AB}=\vec{OB}-\vec{OA}=2\mathbf{a}+3\mathbf{b}-4\mathbf{c}-\mathbf{a}+2\mathbf{b}-3\mathbf{c}=\mathbf{a}+5\mathbf{b}-7\mathbf{c}$$

$$\vec{AC}=\vec{OC}-\vec{OA}=-7\mathbf{b}+10\mathbf{c}-\mathbf{a}+2\mathbf{b}-3\mathbf{c}$$

$$=-\mathbf{a}-5\mathbf{b}+7\mathbf{c}=-(\mathbf{a}+5\mathbf{b}-7\mathbf{c})$$

$$\therefore \quad \vec{AB}=-\vec{AC}$$

Thus the vectors $\vec{AB}$ and $\vec{AC}$ are either parallel or collinear.

Further because these vectors are coterminus, hence the points A, B, C are collinear.

Example 3. *Show that the veotors* $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{i}-3\mathbf{j}-5\mathbf{k}$, $3\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ *are coplanar.*

Solution. Let $2\mathbf{i}-\mathbf{j}+\mathbf{k}\equiv x\,(\mathbf{i}-3\mathbf{j}-5\mathbf{k})+y(3\mathbf{i}-4\mathbf{j}-4\mathbf{k})$

$$\Rightarrow \quad 2\mathbf{i}-\mathbf{j}+\mathbf{k}\equiv(x+3y)\,\mathbf{i}-(3x+4y)\,\mathbf{j}-(5x+4y)\mathbf{k}$$

$$\therefore \quad x+3y=2 \qquad ...(1)$$

$$3x+4y=1 \qquad ...(2)$$

$$5x+4y=-1 \qquad ...(3)$$

From (1) and (2), we get $x=-1$, $y=1$.

These values of x and y satisfy the equation (3). Hence the vectors are coplanar.

Example 4. *Show that the points.* $-6\mathbf{a}+3\mathbf{b}+2\mathbf{c}$, $3\mathbf{a}-2\mathbf{b}+4\mathbf{c}$, $5\mathbf{a}+7\mathbf{b}+3\mathbf{c}$, $-13\mathbf{a}+17\mathbf{b}-\mathbf{c}$ *are coplanar,* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *being three non-coplanar vectors.*

Solution. Let the given points be A, B, C and D.

Let O be the origin of reference, then

$$\vec{OA}=-6\mathbf{a}+3\mathbf{b}+2\mathbf{c},\ \vec{OB}=3\mathbf{a}-2\mathbf{b}+4\mathbf{c},$$

$$\vec{OC}=5\mathbf{a}+7\mathbf{b}+3\mathbf{c},\ \vec{OD}=-13\mathbf{a}+17\mathbf{b}-\mathbf{c}$$

Then $\vec{AB}=\vec{OB}-\vec{OA}=3\mathbf{a}-2\mathbf{b}+4\mathbf{c}+6\mathbf{a}-3\mathbf{b}-2\mathbf{c}$

$$=9\mathbf{a}-5\mathbf{b}+2\mathbf{c}$$

$$\vec{AC}=\vec{OC}-\vec{OA}=5\mathbf{a}+7\mathbf{b}+3\mathbf{c}+6\mathbf{a}-3\mathbf{b}-2\mathbf{c}$$

$$=11\mathbf{a}+4\mathbf{b}+\mathbf{c}$$

$$\vec{AD}=\vec{OD}-\vec{OA}=-13\mathbf{a}+17\mathbf{b}-\mathbf{c}+6\mathbf{a}-3\mathbf{b}-2\mathbf{c}$$
$$=-7\mathbf{a}+14\mathbf{b}-3\mathbf{c}$$

Let us first prove that the vectors $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ are linearly connected.

Let $l\vec{AB}+m\vec{AC}=\vec{AD}$, then

$$l\,[9\mathbf{a}-5\mathbf{b}+2\mathbf{c}]+m\,[11\mathbf{a}+4\mathbf{b}+\mathbf{c}]=-7\mathbf{a}+14\mathbf{b}-3\mathbf{c}$$

$\Rightarrow$ $$(9l+11m)\,\mathbf{a}+[-5l+4m]\,\mathbf{b}+(2l+m)\,\mathbf{c}=-7\mathbf{a}+14\mathbf{b}-3\mathbf{c}$$

$\therefore$ $$9l+11m=-7 \quad \ldots(1)$$
$$-5l+4m=14 \quad \ldots(2)$$
$$2l+m=-3 \quad \ldots(3)$$

Solving (1) and (2), we get

$$l=-2,\ m=1$$

These values of l and m satisfy the equation (3) also. Hence vectors $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ are coplanar.

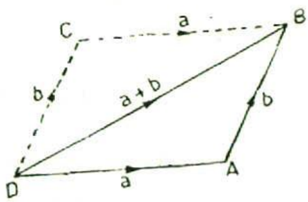
Because these vectors are coterminus, hence the four points A, B, C and D are coplanar.

19·3. OPERATIONS ON VECTORS

The two important operations are addition and multiplication whereas subtraction and division follows from them. It should be remembered that the operations on vectors are element by element because a vector is an ordered set. We have discussed with two dimensions but a vector point can be in n-dimensional space indicated by the n number of elements in a vector.

19·4. ADDITION

If **a** is a vector represented by $\vec{OA}$ and **b** is a vector represented by $\vec{AB}$ so that the terminus of **a** is the initial point of **b**, then their sum or resultant $(\mathbf{a}+\mathbf{b})$ is defined to be the vector represented in magnitude and direction by $\vec{OB}$, where OB is the third side of the triangle OAB. This method of addition is called the *triangle law of addition.*



If **a** is a vector represented by $\vec{OA}$ and **b** is a vector represented by $\vec{OC}$ (*i.e.*, they have a common origin) then their sum $(\mathbf{a}+\mathbf{b})$ is defined to be the vector represented in magnitude and direction by $\overrightarrow{OB}$, where $OABC$ is the completed parallelogram. This method of addition is called the *parallelogram law of addition*. It may be noted that these two methods are identical as is obvious from the definition of the equality of vectors.

19 5. PROPERTIES OF THE OPERATION OF ADDITION

I. *Inclusion Property*. If $\mathbf{a} \in V$ and $\mathbf{b} \in V$ then $\mathbf{a}+\mathbf{b} \in V$.

II. *Commutative Property*. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ for all $\mathbf{a}, \mathbf{b} \in V$.

Proof. Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{AB}=\mathbf{b}$ be two given vectors.

$\therefore$ By triangle law of addition, $\overrightarrow{OB}=\mathbf{a}+\mathbf{b}$

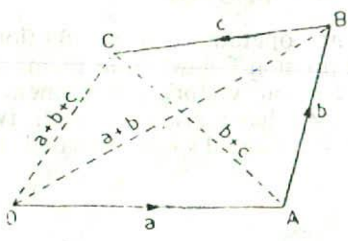
Complete the parallelogram $OABC$.

Then $\overrightarrow{OA}=\overrightarrow{CB}=\mathbf{a}$ and $\overrightarrow{OC}=\overrightarrow{AB}=\mathbf{b}$. Now

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}=\mathbf{a}+\mathbf{b} \text{ and } \overrightarrow{OC}+\overrightarrow{CB}=\overrightarrow{OB}=\mathbf{b}+\mathbf{a}$$

Hence $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$

III. *Associative Property*. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$



Proof. Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{AB}=\mathbf{b}$ and $\overrightarrow{BC}=\mathbf{c}$ be any three vectors.

Then

$$\mathbf{a}+(\mathbf{b}+\mathbf{c})=\overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC})=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC} \quad \ldots(1)$$

Again $(\mathbf{a}+\mathbf{b})+\mathbf{c}=(\overrightarrow{OA}+\overrightarrow{AB})+\overrightarrow{BC}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}$ $\quad \ldots(2)$

From (1) and (2), we have

$$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$$

IV. *Identity or Zero Property*. There exists a zero vector or identity vector for addition such that

$$\mathbf{a}+\mathbf{o}=\mathbf{a}=\mathbf{o}+\mathbf{a} \text{ for all } \mathbf{a} \in V$$

Because of this property the vector **o** is called the additive identity or the netural element for addition.

V. *Negative Property.* For any vector **a** there is a vector $-\mathbf{a}$ with the property that

$$\mathbf{a}+(-\mathbf{a})=\mathbf{o} \text{ and } (-\mathbf{a})+\mathbf{a}=\mathbf{o}$$

On account of this property, the vector $(-1)\,\mathbf{a}$ is called the negative of the vector **a** and we write

$$-\mathbf{a}=(-1)\,\mathbf{a}$$

It should be noted that the existence of the four properties of addition composition is referred to by saying that the set of vectors is a commutative group for the addition composition.

19·6. SUBTRACTION

Subtraction is the inverse of the operation of addition as shown below :

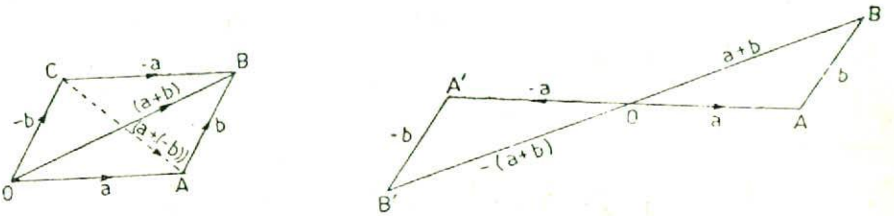
$$\mathbf{a}-\mathbf{a}=\mathbf{a}+(-\mathbf{a})=\mathbf{a}+(-1)\,\mathbf{a}=\mathbf{0}$$

It can also be defined as a difference of two positive vectors or a sum of a positive and a negative vector as shown below.

$$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$$

also $$-(\mathbf{a}+\mathbf{b})=-\mathbf{a}-\mathbf{b}$$

Diagramatically we can show these results as follows :



19·7. MULTIPLICATION BY A SCALAR.

If m is any scalar, then the product $m\mathbf{a}$ of a vector **a** and the real number m is defined as a vector whose magnitude is m times the modulus of **a** and whose direction is the same as that of **a** or the opposite direction according as the scalar m is positive or negative.

The division of a vector **a** by a real number m is defined as the multiplication of the vector **a** by $\frac{1}{m}$.

PROPERTIES OF SCALAR MULTIPLICATION

I. *It has an internal composition only, i.e.*, if

$$\mathbf{a} \in V \text{ then } m\,\mathbf{a} \in V$$

II. *It has an identity element* such that

$$1 \,.\, \mathbf{a} = \mathbf{a} = \mathbf{a} \,.\, 1 \text{ for all } \mathbf{a} \in V$$

III. *It has a zero element* such that

$$0 \,.\, \mathbf{a} = 0 = \mathbf{a} \,.\, 0$$

IV. *It has an associative property*

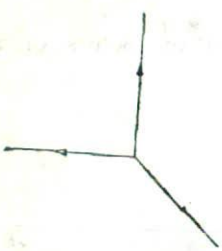
$$m\,(n\,\mathbf{a}) = (mn)\,\mathbf{a}, \text{ for all } \mathbf{a} \in V$$

V. *It distributes over addition of two vectors, i.e.,*

$$\left.\begin{array}{l} (a)\ m\,(\mathbf{a}+\mathbf{b}) = m\,\mathbf{a} + m\,\mathbf{b} \\ (b)\ (m+n)\,\mathbf{a} = m\,\mathbf{a} + n\,\mathbf{a} \end{array}\right\} \text{ for all } \mathbf{a}, \mathbf{b} \in V$$

19·8. ORTHONORMAL BASES

A set of vectors such that the length of each vector is unity and any two vectors are orthogonal then they form orthonormal bases. The vectors are orthogonal when their inner product is zero. Let us take three orthonormal bases i, j, k as shown in the adjoining figure :



Now, their relations are of the type that

$i \,.\, i = 1,\ j \,.\, j = 1,\ k \,.\, k = 1$

and $i \,.\, j = 0,\ j \,.\, k = 0,\ i \,.\, k = 0.$

The above vectors are coplanar, being in the same plane. Since their inner product is zero, they are orthogonal and therefore they constitute orthonormal bases.

Further to illustrate the two vectors which are perpendicular (will have cos angle of 90° which is equal to zero) are orthogonal as shown below :

$$i \,.\, j = \left(\frac{1}{\sqrt{2}},\ \frac{1}{3\sqrt{2}},\ \frac{2}{3}\right) \begin{bmatrix} \dfrac{-1}{\sqrt{2}} \\ \dfrac{1}{3\sqrt{2}} \\ \dfrac{2}{3} \end{bmatrix} = \frac{-1}{2} + \frac{1}{18} + \frac{4}{9} = 0$$

showing that $i \,.\, j$ are perpendicular and therefore orthogonal and form orthonormal bases.

19·9. PRODUCT OF TWO VECTORS

There are two different ways by which vector quantities are multiplied, one is called *scalar or dot product* and the other is called *vector or cross product*. The former is a mere number and does not involve any direction whereas the latter is associated with a definite direction and as such is a vector quantity. However in each case the product is proportional to the products of the lengths of the two vectors and they also follow the distributive law just as in the product of ordinary numbers. The scalar or dot product of two vectors **a** and **b** is written as **a** . **b**, *i.e.*, by placing a dot (.) between **a** and **b** whereas the vector or cross product of vectors **a** and **b** is written as $\mathbf{a} \times \mathbf{b}$, *i.e.*, by placing a cross ($\times$) between **a** and **b**.

19·10. SCALAR PRODUCT OF TWO VECTORS OR DOT PRODUCT

The scalar or dot product of two vectors **a** *and* **b** *is defined to be the scalar*

$$|\mathbf{a}| \; |\mathbf{b}| \cos \theta$$

where θ is the angle between the vector quantities **a** and **b** and $|\mathbf{a}|$ $|\mathbf{b}|$ are the moduli of **a** and **b** respectively.

19·11. PROPERTIES OF SCALAR PRODUCT

I. *Commutative Property.* From the above we find that

$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = ab \cos \theta$, *i.e.*, scalar product is commutative.

$$\therefore \qquad \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \; |\mathbf{b}|} = \frac{\text{Scalar Product}}{\text{Product of moduli}}$$

II. The scalar product of two non-zero vectors is positive, zero or negative according as the angle between them is acute, a right angle or obtuse.

III. *Scalar product of a vector with itself, i.e.*, the square of a vector is equal to the square of its modulus.

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}| \; |\mathbf{a}| \cos 0 = |\mathbf{a}|^2 = a^2$$

IV. *Condition of Perpendicularity.* If two vectors **a** and **b** are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$ ($\because \mathbf{a} \cdot \mathbf{b} = ab \cos 90° = 0$), *i.e.*, for perpendicular vectors, the scalar product is zero. Conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then either $\mathbf{a} = 0$ or $\mathbf{b} = 0$ or **a** is perpendicular to **b**.

V. If two vectors have the same direction, $\theta = 0$ or $\cos \theta = 1$

$$\therefore \qquad \mathbf{a} \cdot \mathbf{b} = ab$$

and if two vectors have opposite directions, $\theta = \pi$ or $\cos \pi = -1$

VI. (*i*) $\qquad \mathbf{a} \cdot (-\mathbf{b}) = -\mathbf{a} \cdot \mathbf{b}$

(*ii*) $\qquad (-\mathbf{a}) \cdot \mathbf{b} = -\mathbf{a} \cdot \mathbf{b}$

(*iii*) $\qquad (-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$

Proof. (*i*) $\qquad \mathbf{a} \cdot (-\mathbf{b}) = ab \cos (\pi - \theta) = (-\mathbf{a}) \cdot \mathbf{b}$

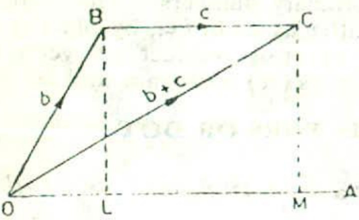
$$= -ab \cos \theta = -(\mathbf{a} \cdot \mathbf{b})$$

VII. *Orthonormal vector triads* **i**, **j**, **k**. We know that *i*, *j*, *k* are three mutually perpendicular unit vectors.

$$\therefore \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1 \text{ and } \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

VIII. *Distributive Law of Multiplication, i.e.*,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$



Proof. Taking O as origin, let $\vec{OA}$, $\vec{OB}$ and $\vec{BC}$ represent the vectors **a**, **b** and **c** respectively so that $\vec{OC}=\mathbf{b}+\mathbf{c}$. Draw BL and CM perpendiculars from B and C on OA.

$\therefore$ OL and OM are projections of OB and OC on OA and LM is the projection of BC on OA.

$$\therefore \quad OM=OL+LM$$

Thus the algebraic sum of the projections of OB and BC on OA

= the projection of OC on OA

$$\therefore \quad \mathbf{a}\,.\,(\mathbf{b}+\mathbf{c})=\mathbf{a}\,.\,(\vec{OC})=OA(OC\cos\theta)$$
$$=OA(OM)=OA(OL+LM)$$
$$=OA(OL)+OA(LM)$$
$$=\mathbf{a}\,.\,\mathbf{b}+\mathbf{a}\,.\,\mathbf{c}$$

because OL and LM are the projections of **b**, **c** on **a**.

On the same lines, we can prove that

$$(\mathbf{a}+\mathbf{b})\,.\,(\mathbf{c}+\mathbf{d})=(\mathbf{a}+\mathbf{b})\,.\,\mathbf{c}+(\mathbf{a}+\mathbf{b})\,.\,\mathbf{d}$$
$$=\mathbf{a}\,.\,\mathbf{c}+\mathbf{b}\,.\,\mathbf{c}+\mathbf{a}\,.\,\mathbf{d}+\mathbf{b}\,.\,\mathbf{d}$$

IX. *Scalar product in terms of components.*

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ be two vectors **a** and **b**, then

$$\mathbf{a}\,.\,\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\,.\,(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})=a_1b_1+a_2b_2+a_3b_3.$$

$$(\because \quad \mathbf{i}\,.\,\mathbf{i}=\mathbf{j}\,.\,\mathbf{j}=\mathbf{k}\,.\,\mathbf{k}=1 \text{ and } \mathbf{i}\,.\,\mathbf{j}=\mathbf{j}\,.\,\mathbf{k}=\mathbf{k}\,.\,\mathbf{i}=0)$$

i.e., the scalar product of two vectors is equal to the sum of the product of their corresponding components.

X. *Angle between two vectors in terms of the components of the given vectors.*

Let $\quad OA=\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $OB=\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ be two vectors and θ be the angle between them.

$$\therefore \quad OA=\sqrt{a_1^2+a_2^2+a_3^2} \text{ and } OB=\sqrt{b_1^2+b_2^2+b_3^2}$$

Now $\quad \mathbf{a}\,.\,\mathbf{b}=OA\,.\,OB\cos\theta$

$$\Rightarrow \quad \cos\theta=\frac{\mathbf{a}\,.\,\mathbf{b}}{OA\,.\,OB}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

Example 5. *Given that* $\vec{a}$, $\vec{b}$ *are two vectors and*

$$\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k} \text{ and}$$

$$\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$$ *where* i, j, k *are orthonormal or orthonormaltriad of vectors, show that*

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$$

Solution. We know that

$$\vec{i}\cdot\vec{i}=1,\quad \vec{j}\cdot\vec{j}=1,\quad \vec{k}\cdot\vec{k}=1$$

and $\quad \vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{i}=0,\quad \vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{j}=0,\quad \vec{k}\cdot\vec{i}=\vec{i}\cdot\vec{k}=0$

$$\therefore \quad \vec{a}\cdot\vec{b}=(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\cdot(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})$$

$$=a_1b_1(\vec{i}\cdot\vec{i})+a_2b_2(\vec{j}\cdot\vec{j})+a_3b_3(\vec{k}\cdot\vec{k})$$

$$+a_1b_2(\vec{i}\cdot\vec{j})+a_2b_1(\vec{j}\cdot\vec{i})+a_2b_3(\vec{j}\cdot\vec{k})$$

$$+a_3b_2(\vec{k}\cdot\vec{j})+a_3b_1(\vec{k}\cdot\vec{i})+a_1b_3(\vec{i}\cdot\vec{k})$$

$$=a_1b_1+a_2b_2+a_3b_3$$

19·12. VECTOR PRODUCT OR CROSS PRODUCT

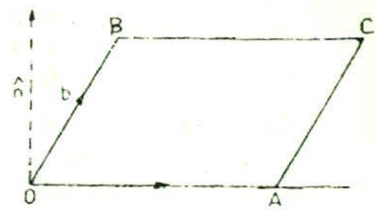
The vector (or cross) product of two vectors **a** *and* **b**, *written as* **a** × **b** *is a vector* **c**, *where*

(*i*) modulus $|\mathbf{c}| = |\mathbf{a}|\ |\mathbf{b}| \sin\theta$, where θ is the angle between vectors **a**, **b**, and $0 \leqslant \theta \leqslant 180°$.

(*ii*) the support of the vector **c**, is perpendicular to that of **a**, as well as of **b**.

(*iii*) the sense of the vector **c** is such that **a**, **b**, **c** is a right handed system.

Thus the vector product of two vectors **a** and **b** whose directions are inclined at an angle θ is defined as



$$\mathbf{a}\times\mathbf{b}=ab\sin\theta\ \mathbf{n}$$

where $a=|\mathbf{a}|$, $b=|\mathbf{b}|$, and **n** is a unit vector perpendicular to both **a** and **b** and the sense of **n** is such that **a**, **b**, **n** form a right-handed triad of vectors.

Also the modulus $ab \sin\theta$ of $\mathbf{a}\times\mathbf{b}$ is the area of parallelogram whose adjacent sides are $\mathbf{a}$ and $\mathbf{b}$.

19·13. PROPERTIES OF VECTOR PRODUCT

I. *The vector product is not commutative.* In fact

$$\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$$

This follows from the fact that the magnitude and support of $\mathbf{b}\times\mathbf{a}$ are the same as those of $\mathbf{a}\times\mathbf{b}$ but the senses are different.

II. $$-\mathbf{a}\times\mathbf{b}=-(\mathbf{a}\times\mathbf{b}),\ \mathbf{a}\times(-\mathbf{b})=-(\mathbf{a}\times\mathbf{b}),$$
$$(-\mathbf{a})\times(-\mathbf{b})=\mathbf{a}\times\mathbf{b}$$

Generally $m\mathbf{a}\times n\mathbf{b}=mn(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times mn\,\mathbf{b}$,
where m and n are any scalars, positive or negative.

III. *The vector product of two parallel or equal vectors is the zero vector*, for in this case, $\theta=0$ or $180°$ so that $\sin\theta=0$ and as such

$$\mathbf{a}\times\mathbf{b}=0$$

From here it also follows that $\mathbf{a}\times\mathbf{a}=0$.

Conversely, if $\mathbf{a}\times\mathbf{b}=0$, $ab\sin\theta\,\mathbf{n}=0$, then either $\mathbf{a}=0$ or $\mathbf{b}=0$, or $\sin\theta=0$, *i.e.*, either of the vectors is a zero or null vector, and in case neither of the vectors is a zero vector, then $\sin\theta$ being zero shows that they are parallel.

IV. In case the vectors are perpendicular, *i.e.*, $\theta=90°$, then $\sin 0=1$ so that $\mathbf{a}\times\mathbf{b}=ab\,.\,n$.

Thus the cross product of two perpendicular vectors is a vector whose modulus is equal to the product of the modulus of the given vectors and whose direction is such that $\mathbf{a}$, $\mathbf{b}$ *and* $\hat{\mathbf{n}}$ *form a right-handed system of mutually perpendicular vectors.*

V. *Vector product of unit vectors* $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

We have
$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$$
$$\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i}=-\mathbf{k}\times\mathbf{j}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k}.$$

VI. *To express the vector product as determinant.*

Let $\mathbf{a}$ and $\mathbf{b}$ be the two vectors. Let us express them in terms of orthonormal unit vector $\mathbf{i}, \mathbf{j}, \mathbf{k}$, *i.e.*,

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k} \text{ and } b=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}.$$

$$\mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$
$$=\{a_1b_1(\mathbf{i}\times\mathbf{i})+a_1b_2(\mathbf{i}\times\mathbf{j})+a_1b_3(\mathbf{i}\times\mathbf{k})\}$$
$$+\{a_2b_1(\mathbf{j}\times\mathbf{i})+a_2b_2(\mathbf{j}\times\mathbf{j})+a_2b_3(\mathbf{j}\times\mathbf{k})\}$$
$$+\{a_3b_1(\mathbf{k}\times\mathbf{i})+a_3b_2(\mathbf{k}\times\mathbf{j})+a_3b_3(\mathbf{k}\times\mathbf{k})\}$$

$$=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

VII. $\sin\theta=\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,.\,|\mathbf{b}|}$

$$=\frac{\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}}{\sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}}$$

VIII. *The magnitude of* $\mathbf{a}\times\mathbf{b}$ *can be expressed in terms of scalar products, i.e.,*

$$(\mathbf{a}\times\mathbf{b})^2=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\,.\,\mathbf{b})^2$$

Proof. $(\mathbf{a}\times\mathbf{b})^2=(\mathbf{a}\times\mathbf{b})\,.\,(\mathbf{a}\times\mathbf{b})=|\mathbf{a}\times\mathbf{b}|^2$

$$=|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta$$
$$=|\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta)$$
$$=\mathbf{a}^2\mathbf{b}^2-|\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$$
$$=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\,.\,\mathbf{b})^2$$

IX. *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be three vectors, then*

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$$

Example 6. *Find the angle between the vectors*

(*i*) $\mathbf{a}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $\mathbf{b}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$

(*ii*) $\mathbf{a}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

(*iii*) $\mathbf{p}=a_1\mathbf{i}+b_1\mathbf{j}+c_1\mathbf{k}$, $\mathbf{q}=a_2\mathbf{i}+b_2\mathbf{j}+c_2\mathbf{k}$,

and find the condition that they are perpendicular to each other.

Solution. (*i*) $\mathbf{a}\,.\,\mathbf{b}=(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\,.\,(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$

$$=(1)(1)+(2)(-2)+(2)(2)$$
$$=1-4+4=1$$

$$a=\sqrt{1^2+2^2+2^2}=3,\quad b=\sqrt{1^2+(-2)^2+2^2}=3$$

$\therefore$ $\cos\theta=\dfrac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}=\dfrac{1}{9}$

$\Rightarrow$ $\theta=\cos^{-1}(1/9)$

(*ii*) $\mathbf{a}\,.\,\mathbf{b}=2(-2)+1.2+1.2=0$

$\therefore$ $\cos\theta=0 \quad\Rightarrow\quad \theta=\dfrac{\pi}{2}$

i.e., vectors a and **b** are perpendicular to each other.

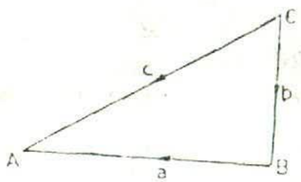
(*iii*) $\mathbf{p}\,.\,\mathbf{q}=a_1a_2+b_1b_2+c_1c_2$

$$p=\sqrt{a_1^2+b_1^2+c_1^2},\ q=\sqrt{a_2^2+b_2^2+c_2^2}$$

$$\cos\theta=\frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}|\ |\mathbf{q}|}=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

The condition that $\theta=\frac{\pi}{2}$ provides $a_1a_2+b_1b_2+c_1c_2=0$.

Example 7. *Show that the vectors* $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{i}-3\mathbf{j}-5\mathbf{k}$ *and* $3\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ *form the sides of a right-angled triangle. Also find the remaining angles of the triangle.*



Solution. Let us suppose that

$\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{b}=\mathbf{i}-3\mathbf{j}-5\mathbf{k}$

and $\mathbf{c}=3\mathbf{i}-4\mathbf{j}-4\mathbf{k}$

Let $\mathbf{a}$ and $\mathbf{b}$ represent the vectors BA and CB respectively.

Then $\vec{CA}=\vec{CB}+\vec{BA}$

$$=(\mathbf{i}-3\mathbf{j}-5\mathbf{k})+(2\mathbf{i}-\mathbf{j}+\mathbf{k})$$
$$=3\mathbf{i}-4\mathbf{j}-4\mathbf{k}=\mathbf{c}$$

$\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ form the sides of a triangle.

Again $\mathbf{a}\cdot\mathbf{b}=(2\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-3\mathbf{j}-5\mathbf{k})$

$$=(2)(1)+(-1)(-3)+(1)(-5)$$
$$=2+3-5$$
$$=0$$

$\therefore$ $\mathbf{a}$ and $\mathbf{b}$, *i.e.*, $\vec{BA}$ and $\vec{CB}$ are perpendicular to each other.

Also $\cos A=\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\ |\mathbf{c}|}$

$$=\frac{(2\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}-4\mathbf{k})}{\sqrt{(2^2+1^2+1^2)}\ \sqrt{(3^2+4^2+4^2)}}$$

$$=\frac{(2)(3)+(-1)(-4)+(1)(-4)}{\sqrt{6}\cdot\sqrt{41}}=\frac{6+4-4}{\sqrt{6}\sqrt{41}}=\sqrt{\frac{6}{41}}$$

or $A=\cos^{-1}\sqrt{\frac{6}{41}}$

and $\cos C=\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\ |\mathbf{c}|}$

$$=\frac{(\mathbf{i}-3\mathbf{j}-5\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}-4\mathbf{k})}{\sqrt{(1^2+3^2+5^2)}\ \sqrt{(3^2+4^2+4^2)}}$$

$$=\frac{(1)(3)+(-3)(-4)+(-5)(-4)}{\sqrt{35}\ \sqrt{41}}$$

$$=\frac{3+12+20}{\sqrt{35}\sqrt{41}}=\sqrt{\left(\frac{35}{41}\right)}$$

or
$$C=\cos^{-1}\sqrt{\left(\frac{35}{41}\right)}$$

Example 8. *Given three vectors* $\vec{a}$, $\vec{b}$, $\vec{c}$, *such that*

$$7\vec{a}=2\vec{i}+3\vec{j}+6\vec{k}$$

$$7\vec{b}=3\vec{i}-6\vec{j}+2\vec{k}$$

$$7\vec{c}=6\vec{i}+2\vec{j}-3\vec{k}$$

Show that $\vec{a}$, $\vec{b}$, $\vec{c}$, *are each of unit length and are mutually perpendicular*

Solution. We are given the vectors

$$\vec{a}=\frac{2}{7}\vec{i}+\frac{3}{7}\vec{j}+\frac{6}{7}\vec{k}$$

$$\vec{b}=\frac{3}{7}\vec{i}-\frac{6}{7}\vec{j}+\frac{2}{7}\vec{k}$$

$$\vec{c}=\frac{6}{7}\vec{i}+\frac{2}{7}\vec{j}-\frac{3}{7}\vec{k}$$

Magnitude of $\vec{a}=|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{49}{49}}=1$

Magnitude of $\vec{b}=|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{49}{49}}=1$

Magnitude of $\vec{c}=|\vec{c}|\quad\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{49}{49}}=1$

$\therefore$ $\vec{a}$, $\vec{b}$, $\vec{c}$ are all of magnitude equal to 1, *i.e.*, they are unit vectors

Now to prove that they are mutually perpendicular, let us take their dot products.

$$\vec{a}\cdot\vec{b}=\left(\frac{2}{7}\vec{i}+\frac{3}{7}\vec{j}+\frac{6}{7}\vec{k}\right)\cdot\left(\frac{3}{7}\vec{i}-\frac{6}{7}\vec{j}+\frac{2}{7}\vec{k}\right)$$

$$=\frac{2\cdot3}{7\cdot7}-\frac{3\cdot6}{7\cdot7}+\frac{2\cdot6}{7\cdot7}=\frac{6-18+12}{49}=0$$

$\Rightarrow$ Vectors $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\vec{b}\cdot\vec{c}=\left(\frac{3}{7}\vec{i}-\frac{6}{7}\vec{j}+\frac{2}{7}\vec{k}\right)\cdot\left(\frac{6}{7}\vec{i}+\frac{2}{7}\vec{j}-\frac{3}{7}\vec{k}\right)$$

$$=\frac{3.6}{7.7}-\frac{6.2}{7.7}-\frac{2.3}{7.7}=\frac{18-12-6}{49}=0$$

$\Rightarrow$ Vectors $\vec{b}$ and $\vec{c}$ are perpendicular.

$$\vec{a}\cdot\vec{c}=\left(\frac{2}{7}\vec{i}+\frac{3}{7}\vec{j}+\frac{6}{7}\vec{k}\right)\cdot\left(\frac{6}{7}\vec{i}+\frac{2}{7}\vec{j}-\frac{3}{7}\vec{k}\right)$$

$$=\frac{2.6}{7.7}+\frac{3.2}{7.7}-\frac{3.6}{7.7}=\frac{12+6-18}{49}=0$$

Hence vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular.

Example 9. *If* $\mathbf{a}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{c}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, *find* $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ *and* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *and hence show that* $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

Solution.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & 2\\ 2 & 1 & -1\end{vmatrix}=-\mathbf{i}+7\mathbf{j}+5\mathbf{k}$$

$\therefore$ $$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(-\mathbf{i}+7\mathbf{j}+5\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 7 & 5\\ 1 & -2 & 2\end{vmatrix}=24\mathbf{i}+7\mathbf{j}-5\mathbf{k} \qquad \ldots(1)$$

Similarly we can show that

$$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -1\\ 2 & -2 & 2\end{vmatrix}=-5\mathbf{j}-5\mathbf{k}$$

$\therefore$ $$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(3\mathbf{i}-\mathbf{j}+2\mathbf{k})\times(-5\mathbf{j}-5\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & 2\\ 0 & -5 & -5\end{vmatrix}=15(\mathbf{i}+\mathbf{j}-\mathbf{k}) \qquad \ldots(2)$$

From (1) and (2), we conclude

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c}).$$

Example 10. *If* $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ *and* $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}$, *prove that* $\mathbf{a}\times\mathbf{b}$ *epresents a vector which is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$.

Solution.

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}=-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}$$

$$\therefore \quad (\mathbf{a}\times\mathbf{b}) \,.\, \mathbf{a}=(-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}) \,.\, (2\mathbf{i}-\mathbf{j}+\mathbf{k})$$
$$=-6-5+11$$
$$=0$$

Here $(\mathbf{a}\times\mathbf{b})$ is perpendicular to $\mathbf{a}$. Similarly we can prove that $(\mathbf{a}\times\mathbf{b})$ is perpendicular to $\mathbf{b}$ also.

Example 11. *Two vectors $\vec{a}$ and $\vec{b}$ are expressed in terms of unit vector as follows :*

$$\vec{a}=2\,\vec{i}-6\,\vec{j}-3\,\vec{k} \text{ and } \vec{b}=4\,\vec{i}+3\,\vec{j}-\vec{k}.$$

What is the unit vector perpendicular to each of the vectors. Also determine the sine of the angle between the given vectors.

Solution. $$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}=15i-10j+30k$$

$$|\,\mathbf{a}\times\mathbf{b}\,|=\sqrt{15^2+10^2+30^2}=\sqrt{1225}=35$$

$\therefore$ Unit vector perpendicular to each of the vectors $\mathbf{a}$ and $\mathbf{b}$

$$=\frac{\mathbf{a}\times\mathbf{b}}{|\,\mathbf{a}\times\mathbf{b}\,|}=\frac{15i-10j+30k}{35}$$

$$=\frac{3}{7}i-\frac{2}{7}j+\frac{6}{7}k.$$

$$\sin\theta=\frac{|\,\mathbf{a}\times\mathbf{b}\,|}{|\,\mathbf{a}\,|\,|\,\mathbf{b}\,|}=\frac{35}{\sqrt{2^2+6^2+3^2}\sqrt{4^2+3^2+1^2}}$$

$$=\frac{35}{\sqrt{49}\,\sqrt{26}}\quad\frac{5}{\sqrt{26}}\quad\sqrt{\frac{25}{26}}$$

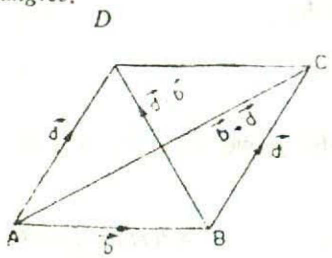
Example 12. *Prove that*

$$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=0$$

Solution. $$\text{L.H.S.}=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}$$
$$+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$$

$$=(\vec{a}\times\vec{c})-(\vec{c}\times\vec{a})+(\vec{b}\times\vec{c})-(\vec{a}\times\vec{b})$$
$$+(\vec{c}\times\vec{a})-(\vec{b}\times\vec{c})$$
$$=0=\text{R.H.S.}$$

Example 13 *Show that the diagonals of a rhombus are at right angles.*



Solution. With A as origin let $\vec{b}, \vec{d}$ be the position vectors of B, D then $\vec{b}+\vec{d}$ is the position vector of C.

Now $\vec{AC}=\vec{b}+\vec{d}$,

$\vec{BD}=\vec{d}-\vec{b}$.

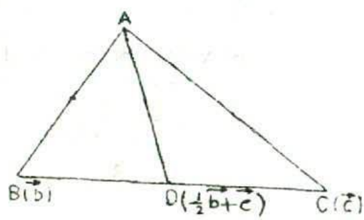
$\therefore \quad \vec{AC} \,.\, \vec{BD} \quad (\vec{b}+\vec{d}) \quad (\vec{d}-\vec{b})=d^2-b^2=0$

$[\because \quad AB=AD, \text{ i.e., } b=d]$

Since the scalar product of $\vec{AC}$ and $\vec{BD}$ is zero it follows that AC and BD are at right angles.

Example 14. *D is the mid-point of the side BC of a triangle ABC, show that*

$AB^2+AC^2=2\,(AD^2+BD^2)$



Solution. With A as origin let $\vec{b}, \vec{c}$ be the position-vectors of B and C so that the position-vector of D is $\frac{1}{2}(\vec{b}+\vec{c})$

Now $\vec{BD}=\vec{AD}-\vec{AB}$

$$(\vec{b}+\vec{c})-\vec{b}=\tfrac{1}{2}(\vec{c}-\vec{b}).$$

Again $\quad AB^2+AC^2=b^2+c^2, \quad (\because \; AB=b,\; AC=c)$

Thus $AD^2+BD^2+\frac{1}{4}(\vec{b}+\vec{c})^2=\frac{1}{4}(\vec{c}-\vec{b})^2$

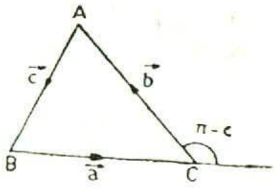
$$=\tfrac{1}{4}[(b^2+c^2)+2(\vec{b}\,.\,\vec{c})+c^2+b^2-2(\vec{b}\,.\,\vec{c})]$$
$$=\tfrac{1}{2}(b^2+c^2)$$

$\therefore \qquad 2(AD^2+BD^2)=AB^2+AC^2$

Example 15. *In any triangle ABC, show that*

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

Solution. Let $\vec{a}, \vec{b}, \vec{c}$ represent the sides of the $\triangle ABC$.



$$\therefore \quad \vec{a}+\vec{b}+\vec{c}=0$$

$$\therefore \ \vec{a}\times(\vec{a}+\vec{b}+\vec{c})=0$$

i.e., $(\vec{a}\times\vec{a})+(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})=0$

But $(\vec{a}\times\vec{a})=0$

$$\therefore \quad (\vec{a}\times\vec{b})=-(\vec{a}\times\vec{c})=(\vec{c}\times\vec{a})$$

$$\therefore \quad |\vec{a}\times\vec{b}|=|\vec{c}\times\vec{a}|$$

Similarly $|\vec{c}\times\vec{a}|=|\vec{b}\times\vec{c}|$

Now $|\vec{a}\times\vec{b}|=[ab\sin(\pi-C)]=ab\sin C$ etc.

Now $ab\sin C=ca\sin B=bc\sin A$

or $$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

EXERCISES

1. The position vectors of the four points A, B, C, D are **a, b, 2a+3b** and **a−2b** respectively. Express $\vec{AC}, \vec{BD}, \vec{BC}$ *and* $\vec{DA}$ *in terms* of **a** and **b**.

2. If **a, b, c** are non-coplanar vectors, show that

   (*i*) **2a−3b+c, a+b−2c, a+2b+4c,**

   (*ii*) **b+c, c+a, a+b**

are also non-coplanar.

3. If **a, b, c** are non-coplanar vectors, show that the points :

   6a+2b−c, 2a−b+3c, −a+2b−4c, −12a−b−3c

are coplanar.

4 If $\mathbf{a}=2i+3j-4k$, $\mathbf{b}=3i+j-2k$, $\mathbf{c}=-2i-j+k$, find :

   (*i*) **a . b**, (*ii*) **a × b**,

   (*iii*) **a . (b × c)**.

5. A vector is a linear combination of the vectors $3i+4j+5k$ and $6i+7j-3k$ and is perpendicular to the vector $i+j-k$. Find the vector.

6. The dot product of a vector with the three vectors $2i$, $3j$ and $4k$ are respectively 6, 18 and 16. Find the vector.

7. If $\mathbf{a}=2i+j+k$ and $\mathbf{b}=-2i+2j+2k$, find the angle between the vectors **a** and **b**. Also find a unit vector perpendicular to the plane of these two vectors.

8. If $\mathbf{a}=4i+j-k$, $\mathbf{b}=3i-2j+2k$ and $\mathbf{c}=-i-2j+k$, calculate a unit vector parallel to $2\mathbf{a}-\mathbf{b}-\mathbf{c}$ but in the opposite sense.

9. Three vectors are given below :

$$\mathbf{a}=2i+5j+3k,\ \mathbf{b}=3i+3j+6k,\ \mathbf{c}=2i+7j+4k$$

Find the magnitudes of the vectors $(\mathbf{a}-\mathbf{b})$ and $(\mathbf{c}-\mathbf{a})$ and also find their inner product $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a})$.

10. Prove that the three vectors $i-2j+3k$, $-2i+3j-4k$ and $-j+2k$ form a linearly dependent system. [*C.A., November 1991*]

11. Given three vectors :

$$a_1=5i+7j+11k,\ a_2=2i+j+3k,\ a_3=3i+6j+8k$$

Find *a* vector $k_1a_1+k_2a_2+k_3a_3$ where the scalars are $k_1=-1$, $k_2=1$, $k_3=1$. Are the three vectors linearly dependent or independent ?

[*C.A., May 19*

12. If θ is the angle between two unit vectors **a** and **b**, show that

$$\frac{1}{2}\ |\mathbf{a}+\mathbf{b}|=\cos\frac{\theta}{2}.$$

13. If **a** and **b** are two vectors, then show that

$$(\mathbf{a}\times\mathbf{b})^2+(\mathbf{a}\cdot\mathbf{b})^2=|\mathbf{a}|^2\cdot|\mathbf{b}|^2.$$

14. If $\mathbf{a}=2i-j+2k$, $\mathbf{b}=10i-2j+7k$, find the value of $\mathbf{a}\times\mathbf{b}$. Also find a unit vector perpendicular to the given vectors.

15. If $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{d}$ and $\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}$, show that $\mathbf{a}-\mathbf{d}$ is parallel to $\mathbf{b}-\mathbf{c}$.

[**Hint.**

$$\begin{aligned}&(\mathbf{a}-\mathbf{d})\times(\mathbf{b}-\mathbf{c})\\ &=\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{d}\times\mathbf{b}+\mathbf{d}\times\mathbf{c}\\ &=(\mathbf{a}\times\mathbf{b})-(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{d})-(\mathbf{c}\times\mathbf{d})=0\end{aligned}$$

$\therefore$ Vector $(\mathbf{a}-\mathbf{d})$ is parallel to vector $(\mathbf{b}-\mathbf{c})$.]

ANSWERS

1. $\mathbf{AC}=a+3b$, $\mathbf{DB}=-a-b$, $\mathbf{BC}=2a+2b$, $\mathbf{DA}=2b$.

4. (*i*) -1, (*ii*) $-10i-16j-7k$, (*iii*) -11

5. $-3i+11j+8k$. 6. $3i+6j+4k$.

7. $\frac{\pi}{2}$, $-\frac{1}{\sqrt{2}}j+\frac{1}{\sqrt{2}}k$. 8. $\frac{1}{\sqrt{97}}(-6i-6j+5k)$.

14. $\mathbf{a}\times\mathbf{b}=-3i+6j+6k$; $-\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k$.

20

Matrix Algebra

STRUCTURE