OBJECTIVES

After studying this chapter, you should be able to understand

- types of matrices, scalar multiplication of a matrix, equality of matrices, addition, subtraction, multiplication of matrices.
- determinants, properties of determinants, Cramer's rule, solution of linear equations.
- inverse of a matrix, solution of equations by matrix method.
- rank of a matrix.

20.0 INTRODUCTION

A matrix consists of a rectangular presentation of symbols or numerical elements arranged systematically in rows and columns describing various aspects of a phenomenon inter-related in some manner.

It is a powerful tool in modern mathematics having wide applications. Sociologists use matrices to study the dominance within a group. Demographers use matrices in the study of births and survivals, marriage and descent, class structure and mobility, etc. Matrices are all the more useful for practical business purposes and, therefore, occupy an important place in Business Mathematics. Obviously, because business problems can be presented more easily in distinct finite number of gradations than in infinite gradations as we have in calculus. The matrix form therefore suits very well for games theory, allocation of expenses, budgeting for by products, etc. Economists now use matrices very extensively in 'social accounting, 'input-output tables' and in the study of 'inter-industry economics.

There is not mere presentation, matrix algebra provides a system of operations on well ordered set of numbers. The common operations are addition, multiplication, inversion, transposition, etc. A most significant contribution of matrix algebra is its extensive use in the solution of a system of large number of simultaneous linear equations. The widely used 'Linear Programming' has its basis in matrix algebra It is on this account, matrix algebra is defined at times as linear algebra.

In the study of communication theory and in electrical engineering the 'net work analysis' is greatly aided by the use of matrix representations. Statistics and particularly the 'design of experiments', and 'multivariate analysis' heavily rely on the use of matrix algebra. Above all, the matrix form is amenable to machine operations. Even if the operations are somewhat lengthy, these are worked out by electronic speed and the final results are both quick and reliable.

A matrix to put in simple language is a rectangular array of numbers. Now what is a rectangular array ? For this, we consider the following illustrations :

I. In an elocution contest, a participant can speak either of the five languages : Hindi, English, Punjabi, Gujarati and Tamil. A college (say No. 1) sent 30 students of which 10 offered to speak in Hindi, 9 in English, 6 in Punjabi, 3 in Gujarati and the rest in Tamil. Another college (say No. 2) sent 25 students of which 7 spoke in Hindi. 8 in English and 10 in Punjabi. Out of 22 students from the third college (say No. 3), 12 offered to speak in Hindi, 5 in English and 5 in Gujarati.

The information furnished in the above manner is somewhat cumbersome. It can be written in a more compact manner if we consider the following tabular form :

	Hindi	English	Punjabi	Gujarati	Tamil
College 1	10	9	6	3	2
College 2	7	8	10	0	0
College 3	12	5	0	5	0

The number in the above data are said to form a *rectangular array*. In any such array, lines across the page are called *rows* and lines down the page are called *columns*. Any one number within the arrangement is called an entry or an element. Thus in the above data there are 3 rows and 5 columns and hence $3 \times 5 = 15$ elements. If it is enclosed by a pair of square brackets then

$$\begin{bmatrix} 10 & 9 & 6 & 3 & 2 \\ 7 & 8 & 10 & 0 & 0 \\ 12 & 5 & 0 & 5 & 0 \end{bmatrix}$$

is called a *matrix*.

Since it has 3 rows and 5 columns it is said to be a matrix of order 3×5 or simply a 3×5 (read as '3 by 5') matrix. It may be noted that a matrix can have any number of rows and any number of columns. Thus in the above illustration if there are entries from 12 colleges and if the competition is held in 8 languages then we can construct a 12×8 matrix.

2. Consider a system of two linear equations in three unknown, *viz.*,

$$2x - 3y + z = 7$$

$$4x + 5y - 3z = 5$$

The co-efficients of x, y, z in the first equation are 2, -3, 1 and those in the second are 4, 5, -3 respectively. They form the matrix (called the co-efficient matrix)

$$\begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{pmatrix}$$

which is a 2×3 matrix.

Remark The reason for enclosing a rectangular array by a pair of brackets is that hereafter we shall treat a rectangular array (and hence a matrix) as a single entity. In fact, we shall develop a new algebra which may be called 'Algebra of Matrices' where operations are performed on the whole array of numbers and not on a single number. It will be seen that this algebra bears a close resemblance to the Algebra of Sets.

20·1. DEFINITION

A matrix is a rectangular array of numbers arranged in rows and columns enclosed by a pair of brackets and subject to certain rules of presentation. The numbers can be substituted by symbols, with appropriate suffixes indicating the row and column numbers. It will be possible to identify the exact location of a number or a symbol in the whole arrangement of a matrix. We will find that through a matrix form of presentation, the complex phenomena with various characteristics or relations would be presented in a very concise manner.

Sometimes a pair of brackets [ ], or a pair of double bars ‖ ‖ are used instead of a pair of parentheses, *e.g.*, the matrix

$$\begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{pmatrix}$$

may also be written as

$$\begin{bmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{bmatrix} \text{ or } \begin{Vmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{Vmatrix}$$

Notations. A matrix is usually denoted by a capital letter and its elements by corresponding small letters followed by two suffixes, the first one indicating the row and the second one the column in which the element appears.

For example, in the first illustration just as the colleges were numbered from 1 to 3, let the languages be numbered from 1 to 5. Then the matrix can be written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}$$

where a_{11} = the number of students from College No. 1 who offered language No. 1 (*i.e.*, Hindi) = 10, a_{12} = those from College No. 1 offering language No. 2 (*i.e.*, English) = 9 and so on.

It should be noted that all the elements in the 1st row have 1 as the first suffix, those in the 2nd and 3rd rows have respectively 2 and 3 as the first suffix. Also all the elements in the 1st column have 1 as the second suffix, those in the 2nd, 3rd, 4th and 5th columns have respectively 2, 3, 4 and 5 as the second suffix.

A general form of a matrix. A matrix of order $m \times n$ (*i.e.*, one having m rows and n columns) can be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1j} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2j} \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{ij} & a_{i.} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mj} \dots & a_{mn} \end{bmatrix}_{m \times n}$$

where $a_{11}, a_{12}...$ stand for real numbers. The above matrix can also be written in a more concise form as :

$$A=[a_{ij}]_{m\times n}$$

where $i=1, 2,..., m$; $j=1, 2,..., n$ and where a_{ij} is the element in the ith row and jth column and is referred as (i, j)th element.

Illustration. *Read the elements $a_{24}, a_{41}, a_{13}, a_{22}$ and the corresponding 'b' elements in the following matrices.*

$$A=\begin{bmatrix} 3 & 4 & 5 & 9 \\ 2 & 0 & -6 & 2 \\ 1 & 3 & 7 & 8 \\ 3 & -6 & -2 & -4 \end{bmatrix} \quad B=\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 7 & 8 \\ -1 & 2 & 6 \end{bmatrix}$$

Solution. (*i*)

Let
$$A=\begin{bmatrix} 3 & 4 & 5 & 9 \\ 2 & 0 & -6 & 2 \\ 1 & 3 & 7 & 8 \\ 3 & -6 & -2 & -4 \end{bmatrix}_{4\times}$$

Now a_{24} indicates the element which appears in the second row and fourth column.

$$\therefore \qquad a_{24}=2$$

Again a_{41} indicates the element which appears in the fourth row and first column.

$$\therefore \qquad a_{41}=3$$

Similarly $a_{13}=5$, $a_{22}=0$

(*ii*) Here

$$B=\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 7 & 8 \\ -1 & 2 & 6 \end{bmatrix}_{4\times 3}$$

b_{24}= not possible, $b_{41}=-1$, $b_{13}=1$ and $b_{22}=2$.

20·2. TYPES OF MATRICES

I. Square Matrix. A matrix in which the number of rows is equal to the number of columns, is called a square matrix. Thus a $m\times n$ matrix will be a square matrix if $m=n$ and it will be referred as a square matrix of order n or n-rowed matrix. Thus

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}, \begin{bmatrix} a_{11} & a_{12} \cdot a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{bmatrix}_{n\times n}$$

are square matrices.

Remark. In a square matrix all those elements a_{ij} for which $i=j$, *i.e.*, those which occur in the same row and same column namely a_{11}, a_{22}, ..., a_{nn} are called the diagonal elements. A square matrix has of course two diagonals. Diagonal extending from the upper left to the lower right is more important than the other diagonal. This is known as the *principal diagonal* or the *main diagonal* and its elements are called the diagonal elements.

Illustration. $\begin{bmatrix} 1 & 2 & -3 \\ 6 & 8 & 5 \\ 2 & -1 & 6 \end{bmatrix}$ 3×3 (square) Matrix

Principal Diagonal is (1, 8, 6)

II. Row and Column Matrices. A row matrix is defined as a matrix having a single row and a column matrix is one having a single column, *e.g.*,

$[a_{11} a_{12} \ldots a_{1n}]_{1\times n}$ is a row matrix

$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{m1} \end{bmatrix}_{m\times 1}$ is a column matrix

Remark. Row and column matrices are sometimes called the row and column vectors. The latter names are also used to designate any row or column of a $m\times n$ matrix.

III. Diagonal Matrix. A square matrix all of whose elements, except those in the leading diagonal, are zero is called a diagonal matrix. Thus the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{bmatrix}_{n\times n}$$

is a diagonal matrix and may be written as

$$\mathbf{A} = \text{diag}\,(a_{11}\, a_{22} \ldots a_{nn})$$

Remarks. 1. The square matrix **A** will be a diagonal matrix if all elements a_{ij} for which $i \neq j$ are zero.

2. A diagonal matrix whose all the diagonal elements are equal is called a scalar matrix, *e.g.*,

$$A=\begin{bmatrix} a & 0 & 0 & \dots & 0 \\ 0 & a & 0 & \dots & 0 \\ 0 & 0 & a & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a \end{bmatrix}=\text{diag. }(a, a, \dots, a)$$

IV. Unit Matrix. A scalar matrix each of whose diagonal element is unity (or one) is called a unit matrix or an identity matrix. A unit matrix of order n is written as I. Thus

$$I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are unit matrices of order two and three respectively.

Remarks. In general for a unit matrix

$$\begin{cases} a_{ij}=0, & i\neq j \\ a_{ij}=1, & i=j \end{cases}$$

V. Zero Matrix or a Null Matrix. A matrix, rectangular or square, each of whose elements are zero is called a zero matrix or a null matrix and is denoted by **0**. Thus

$$\mathbf{0}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a zero (null) matrix of order 4×4.

VI. Triangular Matrices. A square matrix $A=(a_{ij})_{n\times n}$ is called upper triangular matrix if $a_{ij}=0$ for $i>j$ and is called lower triangular matrix if $a_{ij}=0$ for $i<j$.

Thus
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

are upper and lower triangular matrices.

VII. Sub Matrix. A matrix obtained by deleting some rows or columns or both of a given matrix is called a sub matrix of a given matrix.

Let
$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times 4}$$

If we delete the first row and first column, the sub-matrix of A is

$$\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}_{3\times 3}$$

VIII. Scalar Matrix. A square matrix when given in the form of a scalar multiplication to an identity matrix is called a scalar matrix. For example

(*i*) $$3\mathbf{I} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(*ii*) $$a\mathbf{I} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are scalar matrices.

IX. Symmetric Matrices. A symmetric matrix is a special kind of a square matrix $A=[a_{ij}]$ for which

$$a_{ij}=a_{ji} \text{ for all } i \text{ and } j$$

i.e., the (i, j)th element $=(j, i)$th element. For example the matrices.

$$\begin{pmatrix} 5 & 2 & 1 \\ 2 & 6 & -1 \\ 1 & -1 & 5 \end{pmatrix} \quad \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

are symmetric matrices.

X. Complex Conjugate of a Matrix. It is a matrix obtained by replacing all its elements by their respective complex conjugates.

For example

If $$A=\begin{bmatrix} 2+3i & 4 \\ 5-3i & 7 \end{bmatrix} \text{ then } \bar{A}=\begin{bmatrix} 2-3i & 4 \\ 5+3i & 7 \end{bmatrix}$$

XI. Skew-symmetric Matrix. It is square matrix A if

$$A^t=-A,$$

i.e., the transpose of a square matrix is equal to the negative of that matrix. For example the following matrix

$$A=\begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}$$

is skew symmetric.

Or

A square matrix A is called a skew-symmetric matrix if $a_{ij}=-a_{ji}$ for all i and j. In a skew-symmetric matrix all the diagonal elements are zeros.

20·3. SCALAR MULTIPLICATION OF A MATRIX

A real number is referred to as a scalar when it occurs in operations involving matrices. The scalar multiple $k\mathbf{A}$ of a matrix $\mathbf{A}$ by scalar k, is a matrix obtained by multiplying every element of $\mathbf{A}$ by the scalar k, *i.e.*, the scalar multiple of the matrix $\mathbf{A}=[a_{ij}]_{m\times n}$ by scalar k is the matrix $\mathbf{C}=[c_{ij}]_{m\times n}$, where $c_{ij}=ka_{ij}$, $i=1, 2, \ldots, m$; $j=1, 2, \ldots, n$. Thus if

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \text{ then } k\mathbf{A}=\begin{pmatrix} ka_{11} & ka_{12} & \ldots & ka_{1n} \\ ka_{21} & ka_{22} & \ldots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \ldots & ka_{mn} \end{pmatrix}$$

Illustrations :

1. $$3\begin{pmatrix} 4 & -3 \\ 8 & -2 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} 12 & -9 \\ 24 & -6 \\ -3 & 0 \end{pmatrix}$$

2. $$5\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$$

3. If $$\mathbf{A}=\begin{pmatrix} 3 & 7 & 6 & -5 \\ 2 & -6 & 0 & 4 \\ 5 & 2 & 8 & 8 \\ -1 & 6 & 5 & -3 \end{pmatrix}$$

hen $$-\mathbf{A}=\begin{pmatrix} -3 & -7 & -6 & 5 \\ -2 & 6 & 0 & -4 \\ -5 & -2 & -8 & -8 \\ 1 & -6 & -5 & 3 \end{pmatrix} \quad 4\mathbf{A}=\begin{pmatrix} 12 & 28 & 24 & -20 \\ 8 & -24 & 0 & 16 \\ 20 & 8 & 32 & 32 \\ -4 & 24 & 20 & -12 \end{pmatrix}$$

!0 4. EQUALITY OF MATRICES

Two matrices are said to be equal if and only if

(*i*) they are comparable, *i.e.*, they are of the same order, if one is 3×2, the other one is also 3×2 and not 2×3.

(*ii*) each element of one is equal to the corresponding element of the other, *i.e.*, if

$$\mathbf{A}=[a_{ij}]_{m\times n} \quad \text{and} \quad \mathbf{B}=[b_{ij}]_{m\times n}, \text{ then}$$
$$\mathbf{A}=\mathbf{B} \text{ iff } a_{ij}=b_{ij} \ \forall \ \left.\begin{matrix} i=1, 2, \ldots, m \\ j=1, 2, \ldots, n \end{matrix}\right\}$$

Illustrations. 1. If

$$\mathbf{A}=\begin{pmatrix} 4 & 7 & 0 \\ 7 & -2 & 5 \end{pmatrix}, \mathbf{B}=\begin{pmatrix} 4 & 7 & 0 \\ 7 & -2 & 5 \end{pmatrix}$$

then $\mathbf{A}=\mathbf{B}$.

2. If

$$A=\begin{pmatrix} 3 & 4 \\ 4 & -2 \end{pmatrix} \text{ and } B=\begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}$$

then $A \neq B$ (since $a_{22}=-2$ and $b_{22}=1$).

3. If $A \begin{pmatrix} 3 & 4 & 7 \\ 2 & 8 & 6 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 4 & 7 \\ 2 & 8 & 6 \\ 1 & 2 & 5 \end{pmatrix}$

A B because first they are not comparable, matrix A being 2×3 and B being 3×3. Second, the elements are not the same in respective columns and rows.

4. The following is a statement of matrix equality given the values of the components.

$$\begin{pmatrix} x+y & 2z+w \\ x-y & z-w \end{pmatrix}=\begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$$

if $x=2$, $y=1$, $z=3$ and $w=-1$.

EXERCISE (I)

1. Read the elements a_{31}, a_{24}, a_{34}, a_{11} in each of the following matrices given below. Also give their diagonal element.

$$\begin{pmatrix} 8 & 7 & -4 & 2 \\ 3 & 2 & 0 & 5 \\ 7 & 6 & 3 & 1 \\ -5 & 12 & 5 & 9 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 3 \\ 3 & 2 & 5 \\ 7 & 0 & -6 \end{pmatrix}$$

2. Find x and y if

$$\begin{pmatrix} x+y & 2 \\ 1 & x-y \end{pmatrix}=\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix}$$

3. Classify the following matrices :

$$(i) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (ii) \begin{pmatrix} 3 & 0 & 0 \\ 1 & -4 & 0 \\ 9 & 5 & 10 \end{pmatrix}$$

$$(iii) \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \quad (iv) \ (-1 \ -2 \ -3), \quad (v) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(vi) \begin{pmatrix} 3 & -1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \end{pmatrix}, \quad (vii) \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

$$(viii) \begin{pmatrix} 3 & 8 & 7 \\ 4 & 3 & -1 \end{pmatrix} \quad (ix) \begin{pmatrix} 12 & 6 & 2 & 1 \\ 9 & 7 & 4 & -3 \\ 8 & -1 & 5 & 4 \end{pmatrix}$$

4. The following matrix shows the results of the college swimming meet :

$$\begin{bmatrix} 2 & 0 & 3 & 1 \\ 0 & 3 & 3 & 4 \\ 5 & 3 & 0 & 1 \\ 2 & 3 & 4 & 4 \end{bmatrix}$$

The rows represent the teams : Team A, B, C and D in that order. The columns represent the number of wins ; first place, second place, third place and fourth place scored by the teams.

(*a*) How many events did the team A win ?

(*b*) How many first places did team B win ?

(*c*) How many fourth places did team B win ?

(*d*) Write down the row vector which represents team B's result.

(*e*) Write down the column vectors which represent the results of first place and 4th places.

(*f*) Write down the 2×4 matrix which represents the results of team A and D.

(*g*) In the row vector (5 3 0 1) what does 0 represent ?

ANSWERS

1. 7, 5, 1, 8 and 7, ×, X, −1, the diogonal elements are 8, 2, 3, 9 and −1, 2, −6. 2. $x=5$, $y=-2$.

3. (*i*) Identity, (*ii*) lower triangular, (*iii*) column matrix, (*iv*) row matrix, (*v*) null, (*vi*) upper triangular, (*vii*) scalar, (*viii*) 2×3 matrix (*ix*) 3×4 matrix.

4. (*a*) 6, (*b*) 0, (*c*) 4. (*d*) (0 3 3 4), (*e*) $\begin{pmatrix} 2 \\ 0 \\ 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 1 \\ 4 \end{pmatrix}$

(*f*) $\begin{pmatrix} 2 & 0 & 3 & 1 \\ 2 & 3 & 4 & 4 \end{pmatrix}$, (*g*) Team C has not been placed at third place.

20·5 MATRIX OPERATIONS

In matrix algebra the elements are ordered numbers and therefore operations on them have to be done in a manner an army sergeant gives drill to the platoon. Every cadet has to maintain his position vis-a-vis his fellow cadets. Again the main operations are addition and multiplication while the subtraction and division is derived out of these operatiions.

20 6. ADDITION AND SUBTRACTION

(*i*) Matrices can be added or subtracted if and only if they are of the same order.

(*ii*) The sum or difference of two $(m \times n)$ matrices is another matrix $(m \times n)$ whose elements are the sum or differences of the *corresponding* elements in the component matrices.

Symbolically let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ be two matrices of order $m\times n$ each then their sum (difference) $A\pm B$ is the matrix $C=[c_{ij}]_{m\times n}$ where

$c_{ij}=a_{ij}\pm b_{ij}$; $\left\{\begin{array}{l} i=1, 2, \ldots, m \\ j=1, 2, \ldots, n \end{array}\right\}$ is the matrix each element of which is the sum (difference) of the corresponding element of **A** and **B**. Let

$$A=\begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \quad \vdots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix}_{m\times n}, \quad B=\begin{bmatrix} b_{11} & b_{12} \ldots b_{1n} \\ b_{21} & b_{22} \ldots b_{2n} \\ \vdots & \vdots \quad \vdots \\ b_{m1} & b_{m2} \ldots b_{mn} \end{bmatrix}_{m\times n}$$

$$A\pm B=\begin{bmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} \ldots a_{1n}\pm b_{1n} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} \ldots a_{2n}\pm b_{2n} \\ \vdots & \vdots \quad \vdots \\ a_{m1}\pm b_{m1} & a_{m2}\pm b_{m2} \ldots a_{mn}\pm b_{mn} \end{bmatrix}_{m\times n}$$

20·7. PROPERTIES

Commutative. If **A** and **B** are any two matrices of order $m\times n$ each, then

$$A+B=B+A$$

Proof. Let $A=[a_{ij}]_{m\times n}$ $B=[b_{ij}]_{m\times n}$

then, $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

and $B+A=[b_{ij}]_{m\times n}+[a_{ij}]_{m\times n}=[b_{ij}+a_{ij}]_{m\times n}$

But a_{ij} and b_{ij} are the corresponding elements of the matrices **A** and **B**, and by commutative law of real numbers

$$a_{ij}+b_{ij}=b_{ij}+a_{ij}$$

$\Rightarrow$ (i, j)th element of $A+B=(i, j)$th element of $B+A$

Hence $$A+B=B+A$$

II. If **A, B** and **C** are any three comparable matrices of the same type $m\times n$, then

$$(A+B)+C=A+(B+C)$$

Proof. Let $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ $C=[c_{ij}]_{m\times n}$

$\therefore$ $(A+B)+C=[(a_{ij}+b_{ij})+c_{ij}]_{m\times n}$

$A+(B+C)=[a_{ij}+(b_{ij}+c_{ij})]_{m\times n}$

But a_{ij}, b_{ij} and c_{ij} are elements of the matrices and by associative aw of numbers

$$a_{ij}+(b_{ij}+c_{ij})=(a_{ij}+b_{ij})+c_{ij}$$

$\Rightarrow$ (i, j)th element of $A+(B+C)=(i, j)$th element of $(A+B)+C$

Hence $$A+(B+C)=(A+B)+C$$

III. Distributive with respect to Scalar.

$$k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B}$$

Proof. $k(\mathbf{A}+\mathbf{B})=[k(a_{ij}+b_{ij})]_{m\times n}$

$$k\mathbf{A}+k\mathbf{B}=[ka_{ij}]_{m\times n}+[kb_{ij}]_{m\times n}$$
$$=[ka_{ij}+kb_{ij}]_{m\times n}$$

But by distributive law of numbers, we have

$$k(a_{ij}+b_{ij})=ka_{ij}+kb_{ij}$$

$\Rightarrow$ (i, j)th element of $k(\mathbf{A}+\mathbf{B})=(i, j)$th element of $[k\mathbf{A}+k\mathbf{B}]$

Hence $k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B}$

IV. Existence of An Additive Identity. $\mathbf{A}+\mathbf{O}=\mathbf{A}=\mathbf{O}+\mathbf{A}$

where $\mathbf{O}$ is the null matrix of the same type.

(*Proof is left as an exercise to the reader*)

V. Existence of An Inverse If $\mathbf{A}$ be any given matrix then the matrix $-\mathbf{A}$ which must exist, is the additive inverse of $\mathbf{A}$.

$$\therefore \qquad \mathbf{A}+(-\mathbf{A})=\mathbf{O}=(-\mathbf{A})+\mathbf{A}$$

IV Cancellation Law. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are matrices of the same order, then

$$\mathbf{A}+\mathbf{C}=\mathbf{B}+\mathbf{C} \Rightarrow \mathbf{A}=\mathbf{B}$$

Proof. $\mathbf{A}+\mathbf{C}=\mathbf{B}+\mathbf{C} \Rightarrow (\mathbf{A}+\mathbf{C})+(-\mathbf{C})=(\mathbf{B}+\mathbf{C})+(-\mathbf{C})$

$$\Rightarrow \mathbf{A}+(\mathbf{C}-\mathbf{C})=\mathbf{B}+(\mathbf{C}-\mathbf{C})$$
$$\Rightarrow \mathbf{A}+\mathbf{O}=\mathbf{B}+\mathbf{O}$$
$$\Rightarrow \mathbf{A}=\mathbf{B}$$

Example 1. *If* $\mathbf{A}=\begin{pmatrix}0 & 2 & 3\\ 2 & 1 & 4\end{pmatrix}$ *and* $\mathbf{B}=\begin{pmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{pmatrix}$ *find the value of* $2\mathbf{A}+3\mathbf{B}$.

Solution. $2\mathbf{A}=2\begin{bmatrix}0 & 2 & 3\\ 2 & 1 & 4\end{bmatrix}=\begin{bmatrix}0 & 4 & 6\\ 4 & 2 & 8\end{bmatrix}$

$$3\mathbf{B}=3\begin{bmatrix}7 & 6 & 3\\ 1 & 4 & 5\end{bmatrix}=\begin{bmatrix}21 & 18 & 9\\ 3 & 12 & 15\end{bmatrix}$$

$$2\mathbf{A}+3\mathbf{B}=\begin{bmatrix}0 & 4 & 6\\ 4 & 2 & 8\end{bmatrix}+\begin{bmatrix}21 & 18 & 9\\ 3 & 12 & 15\end{bmatrix}$$
$$=\begin{bmatrix}0+21 & 4+18 & 6+9\\ 4+3 & 2+12 & 8+15\end{bmatrix}$$
$$=\begin{bmatrix}21 & 22 & 15\\ 7 & 14 & 23\end{bmatrix}$$

20·8. MULTIPLICATION

Earlier we considered scalar product of a matrix. To recollect if

$$A=\begin{pmatrix}2 & 0\\ 1 & 4\end{pmatrix} \text{ then } 3A=\begin{pmatrix}3\times 2 & 3\times 0\\ 3\times 1 & 3\times 4\end{pmatrix}=\begin{pmatrix}6 & 0\\ 3 & 12\end{pmatrix}$$

Now, a step ahead we take a vector product of a matrix. If

$$\text{Vector } A=(1, 2, 3) \text{ and matrix } B=\begin{pmatrix}4 & 9\\ 6 & 3\\ 8 & 0\end{pmatrix}$$

then
$$AB=(1\ 2\ 3)\times\begin{pmatrix}4 & 9\\ 6 & 3\\ 8 & 0\end{pmatrix}$$

$$=[1.4+2.6+3.8 \quad 1.9+2.3+3.0]$$

$$=[4+12+24 \quad 9+6+0]$$

$$=[40 \quad 15]$$

It was a pre-multiplication of a matrix by a vector. A post-multiplication in the following form is not possible

$$\begin{pmatrix}4 & 9\\ 6 & 3\\ 8 & 0\end{pmatrix}\times(1\ 2\ 3)$$

The reason being whereas in the earlier case the columns in the vector were 3 which were equal to the number of rows of the matrix which were also 3. But, in the latter situation the matrix had 2 columns but the vector had only one row. *For matrix multiplication, the number of columns in the first matrix or vector must be equal to the number of rows in the second matrix or the vector.*

The rule is to multiply the first element in the first row of the first matrix with the first element in the first column of the second matrix, the second element in the first row of the first matrix with the second element in the first column of the second matrix, the nth element of the first row of the first matrix is multiplied by the nth element in the first column of the second matrix. This further proves the need of the number of columns in the first matrix to be equal to the number of rows in the second matrix. Now, these products are added together to give the first element of the first row and the first column of the product matrix. Next we multiply the elements of first row of the first matrix with the elements of the second column of the second matrix and obtain the second element of the first row of the product matrix and so on.

Thus the two matrices are conformable for multiplication if the *number of columns* of first matrix is equal to the *number of rows* of the second matrix. If the matrix **A** is of type $m\times n$, *i.e.*, has m rows and n columns, then **B** must be of the type $n\times p$ where n is the number of rows which are the same as number of columns in **A** and p is any number not necessarily m. Then the product **AB** is another matrix $C=A\times B$ of the type $m\times p$ (number of rows of **A** and number of columns of **B**).

Let $\mathbf{A}=[a_{ij}]_{m\times n}$ and $\mathbf{B}=[b_{jk}]_{n\times p}$ *be two matrices then the product AB is the matrix*

$$\mathbf{C}=[c_{ik}]_{m\times p}$$

where c_{ik} is obtained by multiplying the corresponding entries of the *i*th row of **A** and those of *k*th column of **C** and then adding the results. Thus

$$\begin{bmatrix} a_{11} & a_{12} \ldots a_{1j} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots a_{2j} \ldots & a_{2n} \\ \vdots & \vdots \quad \vdots & \vdots \\ a_{i1} & a_{i2} \ldots a_{ij} \ldots & a_{in} \\ \vdots & \vdots \quad \vdots & \\ a_{m1} & a_{m2} \ldots a_{mj} \ldots & a_{mn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \ldots & b_{1k} & \ldots b_{1p} \\ b_{21} & b_{22} \ldots & b_{2k} & \ldots b_{2p} \\ \vdots & & \vdots & \vdots \\ b_{j1} & b_{j2} & b_{jk} & b_{jp} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{nk} & \ldots b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} \ldots & c_{1k} \ldots & c_{1p} \\ c_{21} & c_{22} \ldots & c_{2k} \ldots & c_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} \ldots & c_{ik} \ldots & c_{ip} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} \ldots & c_{mk} & c_{mp} \end{bmatrix}_{m\times p}$$

where $$c_{ik}=a_{i1}\, b_{1k}+a_{i2}\, b_{2k}+a_{i3}\, b_{3k}+\ldots+a_{in}\, b_{nk}$$

Remarks. 1. The rule of multiplication of two matrices is *Row-Columnwise* (→ ↓ wise), *i.e.*, row of one matrix is multiplied with column of the second matrix to get the corresponding elements of the product. In short first row of **AB** is obtained by multiplying the first row of **A** with 1st, 2nd, 3rd column...of **B** respectively. Similarly the second row of **AB** is obtained by multiplying the second row **A** with 1st, 2nd, 3rd columns ..., of **B** respectively and so on.

2. The rule of multiplication (*viz.*,→ ↓ wise) is the same for matrices of any order provided the matrices are conformable for multiplication.

3. If

$$\mathbf{A}=\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix}_{m\times n}$$ where R_i denotes the *i*th row of matrix **A** and can be regarded as $m\times n$ matrix.

$$\mathbf{B}=[\mathbf{C}_1\ \mathbf{C}_2 \ldots \mathbf{C}_j \ldots \mathbf{C}_p]_{n\times p}$$ where $\mathbf{C}_j$ denotes the *j*th column of matrix **B** and can be regarded as an $n\times p$ matrix.

then $$\mathbf{AB}=\begin{bmatrix} R_1C_1 & R_1C_2 \ldots R_1C_p \\ R_2C_1 & R_2C_2 \ldots R_2C_p \\ \vdots & \vdots \quad\quad \vdots \\ R_mC_1 & R_mC_2 \ldots R_mC_p \end{bmatrix}_{m\times p}$$

4. In the product **AB**, **A** is said to have been post-multiplied by **B** and **B** is said to have been premultiplied by **A**, *i.e.*, **AB** is called the post-multiplication of **A** by **B** or premultiplication of **B** by **A**.

5. *Matrix multiplication in general is not commutative.* If **AB** is defined, it is not necessary that **BA** is also defined, *e.g.*, if **A** is of the type $m\times n$ and **B** of the type $n\times p$ then **AB** is defined but **BA** is not defined. Even if **AB** and **BA** are both defined, it is not necessary that they are equal *e.g.*, if **A** is $m\times n$ and **B** is $n\times m$ then **AB** is $m\times m$ and **BA** is $n\times n$ so that $AB\neq BA$ because they are not of the same order.

20·9. PROPERTIES

I. Multiplication is distributive w.r.t. addition.

If **A**, **B**, **C** are $m\times n$, $n\times p$ and $n\times p$ matrices respectively, then

$$A.(B+C)=AB+AC$$

II. Multiplication is associative if conformability is assured

If **A**, **B**, **C** are $m\times n$, $n\times p$ and $p\times q$ matrices respectively, then

$$(AB)C=A(BC)$$

III. If **A** is $n\times m$, **O** is $m\times n$, then

$$AO=O=OA$$

(Proof is left as an exercise to the reader.)

IV. **Multiplication of a Matrix by a Unit Matrix.** If **A** is a square matrix of order $n\times n$ and **I** is the unit matrix of the same order then

$$AI=A=IA$$

V. $AB=O$ (null matrix) does not necessarily imply that $A=O$ or $B=O$ or both$=O$, *e.g.*,

$$A=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\neq O \text{ and } B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\neq O$$

But

$$AB=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=0.$$

VI. Multiplication of Matrix by Itself. The product **A.A** is defined if the number of columns of **A** is equal to the number of rows of **A**, *i.e.*, if **A** is a square matrix and in that case **A.A** will also be a square matrix of the same order.

$$A^2A=(AA)A=A(AA) \qquad \text{[By Associative Law]}$$
$$A^2A=AAA=A^3$$

Similarly $A.A.A\ldots n$ times$=A^n$

Remark. If **I** is a unit matrix, then

$$I=I^2=I^3=\ldots=I^n$$

Example 2. *Write down the product* **AB** *of the two matrices* **A** *and* **B** *where*

$$A=(1\ 2\ 3\ 4) \text{ and } B=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

Solution. Since **A** is 1×4 matrix, **B** is 4×1 matrix, **AB** will be 1×1 matrix.

$$\therefore\quad AB=(1\ 2\ 3\ 4)_{1\times4}\times\begin{pmatrix}1\\2\\3\\4\end{pmatrix}_{4\times1}=[1.1+2.2+3.3+4.4]=[30]_{1\times1}$$

Example 3. *If* $A=\begin{pmatrix}2 & 5\\1 & 3\end{pmatrix}$ *and* $B=\begin{pmatrix}1 & -1\\-3 & 2\end{pmatrix}$, *find* **AB** *and* **BA**. *Is* **AB**=**BA** ?

Solution. Here

$$AB=\begin{pmatrix}2 & 5\\1 & 3\end{pmatrix}\times\begin{pmatrix}1 & -1\\-3 & 2\end{pmatrix}=\begin{pmatrix}2\times1+5\times(-3) & 2\times(-1)+5\times2\\1\times1+3\times(-3) & 1\times(-1)+3\times2\end{pmatrix}$$

$$=\begin{pmatrix}-13 & 8\\-8 & 5\end{pmatrix}$$

$$BA=\begin{pmatrix}1 & -1\\-3 & 2\end{pmatrix}\begin{pmatrix}2 & 5\\1 & 3\end{pmatrix}=\begin{pmatrix}1\times2+(-1)\times1 & 1\times5+(-1)\times3\\(-3)\times2+2\times1 & (-3)\times5+2\times3\end{pmatrix}$$

$$=\begin{pmatrix}1 & 2\\-4 & -9\end{pmatrix}$$

Thus $\qquad AB\neq BA$

Example 4 *Obtain the product*

$$\begin{pmatrix}2 & 1 & 0\\3 & 2 & 1\\1 & 0 & 1\end{pmatrix}\times\begin{pmatrix}1 & 2 & 3 & 4\\2 & 0 & 1 & 2\\3 & 1 & 0 & 5\end{pmatrix}$$

Solution. Let $A=\begin{pmatrix}2 & 1 & 0\\3 & 2 & 1\\1 & 0 & 1\end{pmatrix}_{3\times3}\quad B=\begin{pmatrix}1 & 2 & 3 & 4\\2 & 0 & 1 & 2\\3 & 1 & 0 & 5\end{pmatrix}_{3\times4}$

Since **A** is 3×3 and **B** is 3×4, product **AB** is valid and **AB** is 4×4.

$$AB=\begin{pmatrix}2 & 1 & 0\\3 & 2 & 1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\2 & 0 & 1 & 2\\3 & 1 & 0 & 5\end{pmatrix}$$

$$=\begin{pmatrix}2+2+0 & 4+0+0 & 6+1+0 & 8+2+0\\3+4+3 & 6+0+1 & 9+2+0 & 12+4+5\\1+0+3 & 2+0+1 & 3+0+0 & 4+0+5\end{pmatrix}$$

$$=\begin{pmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{pmatrix}$$

Example 5 *Find* (*a*) : $(x\ \ y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(*b*) $(x\quad y\quad z)\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Solution (*a*) $(x\ \ y)_{1\times 2}\left\{\begin{pmatrix} a & h \\ h & b \end{pmatrix}_{2\times 2}\begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1}\right\}$

$$=(x\quad y)_{1\times 2}\begin{pmatrix} ax+hy \\ hx+by \end{pmatrix}_{2\times 1}$$

$$=[x(ax+hy)+y(hx+by)]_{1\times 1}$$

$$=ax^2+hxy+hxy+by^2=ax^2+2hxy+by^2.$$

(*b*) $(x\quad y\quad z)_{1\times 3}\left\{\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}_{3\times 3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{3\times 1}\right\}$

$$=(x\quad y\quad z)_{1\times 3}\begin{pmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{pmatrix}_{3\times 1}$$

$$=[x(ax+hy+gz)+y(hx+by+fz)+z(gx+fy+cz)]_{1\times 1}$$

$$=ax^2+hxy+gxz+hxy+by^2+fyz+gzx+fyz+cz^2$$

$$=ax^2+by^2+cz^2+2(hxy+fyz+gzx).$$

Example 6. *If* $\mathbf{A}=\begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}$, *find the matrix* **X** *such that* $3\mathbf{A}+5\mathbf{B}+2\mathbf{X}=\mathbf{O}$

Solution. $3\mathbf{A}+5\mathbf{B}+2\mathbf{X}=\mathbf{O} \Rightarrow \mathbf{X}=-\frac{1}{2}[3\mathbf{A}+5\mathbf{B}]$

$$\Rightarrow \quad \mathbf{X}=-\frac{1}{2}\left\{3\begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix}+5\begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}\right\}$$

$$=-\frac{1}{2}\left\{\begin{pmatrix} 27 & 3 \\ 12 & 9 \end{pmatrix}+\begin{pmatrix} 5 & 25 \\ 35 & 60 \end{pmatrix}\right\}$$

$$=-\frac{1}{2}\left\{\begin{pmatrix} 27+5 & 3+25 \\ 12+35 & 9+60 \end{pmatrix}\right\}$$

$$=\begin{pmatrix} -\frac{32}{2} & -\frac{28}{2} \\ -\frac{47}{2} & -\frac{69}{2} \end{pmatrix} \quad \begin{pmatrix} -16 & -14 \\ -\frac{47}{2} & -\frac{69}{2} \end{pmatrix}$$

Example 7. *If* $\mathbf{A}=\begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix}$

and $\mathbf{B}=\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix}$

(*a*) *Find a* 2×4 *matrix* $\mathbf{X}$ *such that* $\mathbf{A}-\mathbf{X}=3\mathbf{B}$.

(*b*) *Find a* 2×4 *matrix* $\mathbf{Y}$ *such that* $\mathbf{A}+2\mathbf{Y}=4\mathbf{B}$.

Solution. (*a*) $\mathbf{A}-\mathbf{X}=3\mathbf{B}$

$\mathbf{X}=\mathbf{A}-3\mathbf{B}$

$$\Rightarrow \quad \mathbf{X}=\begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix}-3\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix}+\begin{pmatrix} -6 & -3 & 0 & -9 \\ -3 & +3 & -6 & -9 \end{pmatrix}$$

$$=\begin{pmatrix} 1-6 & 2-3 & 0 & 4-9 \\ 2-3 & 4+3 & -1-6 & 3-9 \end{pmatrix}=\begin{pmatrix} -5 & -1 & 0 & -5 \\ -1 & 7 & -7 & -6 \end{pmatrix}$$

(*b*) $\mathbf{A}+2\mathbf{Y}=4\mathbf{B} \quad \Rightarrow \quad \mathbf{Y}=2\mathbf{B}-\frac{1}{2}\mathbf{A}$

$$\Rightarrow \quad \mathbf{Y}=2\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix}-\frac{1}{2}\begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 2 & 0 & 6 \\ 2 & -2 & 4 & 6 \end{pmatrix}-\begin{pmatrix} \frac{1}{2} & \frac{2}{2} & 0 & \frac{4}{2} \\ \frac{2}{2} & \frac{4}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$=\begin{pmatrix} 4-\frac{1}{2} & 2-1 & 0 & 6-2 \\ 2-1 & -2-2 & 4+\frac{1}{2} & 6-\frac{3}{2} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{7}{2} & 1 & 0 & 4 \\ 1 & -4 & \frac{9}{2} & \frac{9}{2} \end{pmatrix}$$

Example 8. *When* $\mathbf{A}=\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ *and* $\mathbf{B}=\begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix}$

and $i=\sqrt{-1}$, *determine* **AB**. *Compute also* **BA**.

Solution. $\mathbf{AB}=\begin{pmatrix} 1\times i-i\times 1 & (-1)\times 1+(i)(-i) \\ i\times(-i)-1\times 1 & (-i)(-1)+1(-i) \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and $\mathbf{BA}=\begin{pmatrix} 2i & -2 \\ -2 & -2i \end{pmatrix}$

Example 9. *Given*

$$\mathbf{A}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathbf{B}=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \mathbf{C}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Prove the following relations :

$\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=\mathbf{I}$ (*unit matrix*)

$\mathbf{AB}=-\mathbf{BA},\ \mathbf{AC}=-\mathbf{CA},\ \mathbf{BC}=-\mathbf{CB}.$

Solution. $A^2=A.A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0+1 & 0+0\\ 0+0 & 1+0\end{pmatrix}$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

$$B^2=B.B=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}=\begin{pmatrix}0-i^2 & 0+0\\ 0+0 & -i^2+0\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

Similarly $C^2=I$

$$AB=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}=\begin{pmatrix}0+i & 0+0\\ 0+0 & -i+0\end{pmatrix}=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

$$-BA=-\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=-\begin{pmatrix}0-i & 0+0\\ 0+0 & i+0\end{pmatrix}$$

$$=-\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

$\therefore$ **AB = −BA.** Similarly we can prove the other relations.

Example 10. *If* $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ *and* $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, *show that*

$$A^2-(a+d)\,A=(bc-ad)\,I$$

Solution. We have

$$A^2=A.A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

$$=\begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix}=\begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & bc+d^2\end{pmatrix}$$

$$\therefore\ A^2-(a+d)\,A=\begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & bc+d^2\end{pmatrix}-(a+d)\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

or $$A^2-(a+d)\,A=\begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & bc+d^2\end{pmatrix}+\begin{pmatrix}-a(a+d) & -b(a+d)\\ -c(a+d) & -d(a+d)\end{pmatrix}$$

$$=\begin{pmatrix}a^2+bc-a(a+d) & b(a+d)-b(a+d)\\ c(a+d)-c(a+d) & bc+d^2-d(a+d)\end{pmatrix}$$

$$=\begin{pmatrix}bc-ad & 0\\ 0 & bc-ad\end{pmatrix}=(bc-ad)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=(bc-ad)\,I$$

Example 11. *If* $A=\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}$, $B=\begin{pmatrix}a & 1\\ b & -1\end{pmatrix}$

and $(A+B)^2=A^2+B^2$, *find a and b.*

Solution. $A+B=\begin{pmatrix}(1+a) & 0\\ (2+b) & -2\end{pmatrix}$

$$(A+B)^2=\begin{pmatrix}(1+a) & 0\\ (2+b) & -2\end{pmatrix}\begin{pmatrix}(1+a) & 0\\ (2+b) & -2\end{pmatrix}=\begin{pmatrix}(1+a)^2 & 0\\ 2a+ab-b-2 & 4\end{pmatrix}$$

$$A^2=\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

$$B^2=\begin{pmatrix}a & 1\\ b & -1\end{pmatrix}\begin{pmatrix}a & 1\\ b & -1\end{pmatrix}=\begin{pmatrix}a^2+b & a+1\\ ab-b & b+1\end{pmatrix}$$

$$A^2+B^2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}+\begin{pmatrix}a^2+b & a-1\\ ab-b & b+1\end{pmatrix}=\begin{pmatrix}a^2+b-1 & a-1\\ ab-b & b\end{pmatrix}$$

Now $$(A+B)^2=A^2+B^2$$

$$\Rightarrow\quad \begin{pmatrix}(1+a)^2 & 0\\ 2a-b+ab-2 & 4\end{pmatrix}=\begin{pmatrix}a^2+b-1 & a-1\\ ab-b & b\end{pmatrix}$$

$$\Rightarrow\quad a-1=0 \quad\text{or}\quad a=1 \quad\text{and}\quad b=4$$

Example 12. *Given the matrices* **A, B, C,**

$$A=\begin{pmatrix}2 & 3 & -1\\ 3 & 0 & 2\end{pmatrix},\quad B=\begin{pmatrix}1\\ 1\\ 2\end{pmatrix},\quad C=(1\quad -2)$$

verify that $(AB)C=A(BC)$

Solution. Clearly **AB** is defined and will be 2×1 matrix and hence $(AB)_{2\times 1}\,C_{1\times 2}$ is also defined and will be 2×2 matrix.

Also **BC** is defined and will be 3×2 matrix and hence $A_{2\times 3}\,(BC)_{3\times 2}$ is also defined and will be 2×2 matrix.

$$(AB)=\begin{pmatrix}2 & 3 & -1\\ 3 & 0 & 2\end{pmatrix}_{2\times 2}\times\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}_{3\times 1}=\begin{pmatrix}2.1+3.1-1.2\\ 3.1+0.1+2.2\end{pmatrix}$$

$$=\begin{pmatrix}3\\ 7\end{pmatrix}_{2\times 1}$$

$$(AB)C=\begin{pmatrix}3\\ 7\end{pmatrix}_{2\times 1}(1\quad -2)_{1\times 2}=\begin{pmatrix}3.1 & 3.(-2)\\ 7.1 & 7.(-2)\end{pmatrix}_{2\times 2}$$

$$=\begin{pmatrix}3 & -6\\ 7 & -14\end{pmatrix}_{2\times 2}\qquad \text{...(I)}$$

Again

$$BC=\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}_{3\times 1}(1\quad -2)_{1\times 2}=\begin{pmatrix}1.1 & 1.(-2)\\ 1.1 & 1.(-2)\\ 2.1 & 2.(-2)\end{pmatrix}_{3\times 2}=\begin{pmatrix}1 & -2\\ 1 & -2\\ 2 & -4\end{pmatrix}_{3\times 2}$$

$$\therefore\quad A(BC)=\begin{pmatrix}2 & 3 & -1\\ 3 & 0 & 2\end{pmatrix}_{2\times 3}\begin{pmatrix}1 & -2\\ 1 & -2\\ 2 & -4\end{pmatrix}_{3\times 2}$$

$$=\begin{pmatrix}2.1+3.1+(-1).2 & 2.(-2)+3(-2)+(-1)(-4)\\ 3.1+0.1+2.2 & 3.(-2)+0.(-2)+2(-4)\end{pmatrix}_{2\times 2}$$

$$=\begin{pmatrix} 3 & -6 \\ 7 & -14 \end{pmatrix}_{2\times 2} \qquad \ldots(II)$$

Thus we observe that

$$(AB)C=A(BC)$$

Example 13. *If*

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

show that $A^3-3A^2-A+9I=O$

Solution.

$$A^2=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}=\begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix}$$

$$A^3=A^2.A=\begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}=\begin{pmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{pmatrix}$$

Now $A^3-3A^2-A+9I=\begin{pmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{pmatrix}-\begin{pmatrix} 12 & 9 & 0 \\ -9 & 6 & -6 \\ 18 & 12 & 15 \end{pmatrix}$

$$-\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}+\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=0.$$

Example 14. *A finance company has offices located in every division, every district and every taluka in a certain State in India. Assume that there are five divisions, 30 districts and 200 talukas in the State. Each office has 1 head clerk, 1 cashier, 1 clerk and 1 peon. A divisional office has, in addition, an office superintendent, 2 clerks, 1 typist and 1 peon. A district office has, in addition, 1 clerk and 1 peon. The basic monthly salaries are as follows ; Office superintendent Rs. 500, Head clerk Rs. 200, Cashier Rs. 175, Clerks and typists Rs. 150, and peons Rs. 100. Using matrix notation find*

(i) *the total number of posts of each kind in all the offices taken together,*

(ii) *the total basic monthly salary bill of each kind of office, and*

(iii) *the total basic monthly salary bill of all the offices taken together.*

Solution. The number of offices can be arranged as elements of a row vector, say

$$A=\begin{matrix} \text{Division} & \text{District} & \text{Taluka} \\ (5 & 30 & 200) \end{matrix}$$

Staff composition can be arranged in 3×6 matrix **B**,

$$B=\begin{pmatrix} \overset{O}{1} & \overset{H}{1} & \overset{C}{1} & \overset{T}{1} & \overset{Cl}{3} & \overset{P}{2} \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

where O = Office superintendent, H = Head clerk, C = Cashier, T = Typist. Cl = Clerk, P = Peon.

Column vector D will have the elements that correspond to basic monthly salaries.

$$D=\begin{matrix} O \\ H \\ C \\ T \\ Cl \\ P \end{matrix}\begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix}$$

(*i*) total number of posts of each kind in all the offices are the elements of the matrix

$$AB, \text{ i.e., } \begin{pmatrix} O & H & C & T & Cl & P \\ 5 & 235 & 235 & 5 & 275 & 270 \end{pmatrix}$$

(*ii*) Total basic monthly salary bill of each kind of offices is the elements of matrix

$$BD=\begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{pmatrix}=\begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix}$$

(*iii*) Total bill of all these offices is the element of the matrix

$$(5 \quad 30 \quad 200)\begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix}=1{,}59{,}625.$$

EXERCISE (II)

1. Find $(x \ y)$ if

(*i*) $(4 \quad 5)+(x \quad y)=(7 \quad 3)$

(*ii*) $(1 \quad -9)-(2 \quad -3)=(x \quad y)$

(*iii*) $(x \quad y)-(0 \quad -1)=(5 \quad 4)$

(*iv*) $\left(\frac{1}{2} \quad \frac{1}{3}\right)-\left(\frac{2}{3} \quad \frac{3}{4}\right)=\left(x \quad y\right)$

2. Given

$$A=\begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix},\quad B=\begin{pmatrix} 8 & 4 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 6 \end{pmatrix},\quad C=\begin{pmatrix} 8 & 2 & 0 \\ 0 & 2 & -6 \\ -8 & 4 & -10 \end{pmatrix}$$

Compute the following :

(*a*) $\mathbf{A+B}$, (*b*) $\mathbf{A-B}$, (*c*) $\mathbf{A+(B+C)}$, (*d*) $\mathbf{(A+B)+C}$
(*e*) $\mathbf{(A-B)+C}$, (*f*) $\mathbf{A-B-C}$, (*g*) $\mathbf{2(A+B)}$
(*h*) $\mathbf{2A+2B}$, (*i*) $\mathbf{3A+2B-3C}$, (*j*) $\mathbf{3B+2A}$
and (*k*) $\mathbf{2B+3A}$.

3. $\mathbf{A}=\begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 8 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 3 & 5 & 9 \\ 6 & -2 & 1 \end{pmatrix}$

(*a*) Write down the order of the matrices **A** and **B**.
(*b*) Write down the order of the product **AB**. (*c*) Calculate **AB**.
(*d*) Is it possible to calculate **BA** ? (*e*) Is $\mathbf{AB=BA}$?
(*f*) Are the following possible ? $\mathbf{A+B}$, $\mathbf{A-B}$, $\mathbf{2B}$, $\mathbf{A}^2$.

4. $\mathbf{A}=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$, $\mathbf{C}=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Show that

(*i*) $\mathbf{A(B+C)=AB+AC}$, (*ii*) $\mathbf{(AB)C=A(BC)}$

5 If $\mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\mathbf{i}=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\mathbf{j}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathbf{k}=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Show that

(*i*) $\mathbf{ij=k}$, $\mathbf{jk=i}$, $\mathbf{ki=j}$, $\mathbf{ji=-k}$, $\mathbf{kj=-i}$, $\mathbf{ik=-j}$
(*ii*) $\mathbf{i^2=j^2=k^2=-I}$.

6. Find the matrice **B** if

(*a*) $\mathrm{A}=\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $\mathrm{A+2B=A^2}$

(*b*) $\mathrm{A}=\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ and $\mathrm{A^2+3A+B}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

7. (*a*) $\mathbf{A}=\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$. Find the matrix **C**

if (*i*) $\mathbf{2C=A+B}$ (*ii*) $\mathbf{C+A}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (*iii*) $\mathbf{5C+2B=A}$

(*b*) If $\mathbf{A}=\begin{pmatrix} 4 & 1 & 0 \\ 1 & -2 & 2 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 4 \end{pmatrix}$, $\mathbf{C}=\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Find a matrix **X** such that $\mathbf{(3B-2A)C+2X=O}$.

(*c*) If $\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}\mathbf{A}=\begin{pmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{pmatrix}$, find **A** (*d*) If $\mathbf{A}=\begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{C}=\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$

Find $(\mathbf{AB})\mathbf{C}$, hence or otherwise write down the value of $\mathbf{A}(\mathbf{BC})$.

8. If $\mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{A}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, prove that

$$(a\mathbf{I}+b\mathbf{A})^3=a^3\mathbf{I}+3a^2b\mathbf{A}$$

9. If $\mathbf{A}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, prove that

$$\mathbf{A}^2=\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \text{ and } \mathbf{A}^3=\begin{pmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{pmatrix}$$

What do you suppose is the general result ?

10. If $\mathbf{A}=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, show that $\mathbf{A}^2=2\mathbf{A}$ and $\mathbf{A}^3=4\mathbf{A}$

11. If $\mathbf{A}=\begin{pmatrix} 0 & -\tan\frac{1}{2}\alpha \\ \tan\frac{1}{2}\alpha & 0 \end{pmatrix}$ and $\mathbf{I}$ is a unit matrix, show that

$$\mathbf{I}+\mathbf{A}=(\mathbf{I}-\mathbf{A})\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

12. (*a*) If $\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$, show that $\mathbf{A}^2=0$

(*b*) $\mathbf{A}=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, compute $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$.

(*c*) $\mathbf{A}=\begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$, show that $\mathbf{A}^3=\mathbf{A}$

13. (*a*) If $\mathbf{A}=\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{pmatrix}$

Show that $\mathbf{AB}=0$

(*b*) If $\mathbf{A}=\begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}$,

$$\mathbf{C}=\begin{pmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{pmatrix}$$

Show that $\mathbf{AB}$ and $\mathbf{CA}$ are null matrices but $\mathbf{BA}\neq 0$, $\mathbf{AC}\neq 0$.

(*c*) If $\mathbf{M}=\begin{pmatrix} 1 & a & b & c \\ a^{-1} & 1 & a^{-1}b & a^{-1}c \\ b^{-1} & ab^{-1} & 1 & b^{-1}c \\ c^{-1} & ac^{-1} & bc^{-1} & 1 \end{pmatrix}$

Prove that $\mathbf{M}^2=4\mathbf{M}$.

14. (*a*) If $\mathbf{A}=\begin{pmatrix}1 & 5 & 1 & 3\\ 2 & 1 & 0 & 5\\ 7 & 1 & 8 & -7\\ 0 & 2 & 1 & 6\end{pmatrix}$

and $\mathbf{B}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

Show that $\mathbf{AB}=\mathbf{BA}$. Also compute $4\mathbf{A}^2-6\mathbf{B}^2$.

(*b*) For the following matrices, find **AB** or **BA** whichever is defined.

$$\mathbf{A}=\begin{pmatrix}3 & 2 & -1 & 2\\ 7 & -6 & 0 & 8\\ 9 & 5 & 6 & -5\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}3 & 2\\ 5 & 0\\ -7 & 3\\ 5 & 9\end{pmatrix}$$

(*c*) $\mathbf{A}=\begin{pmatrix}3 & -2 & -2\\ 4 & -5 & 8\\ 5 & -6 & -3\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}2 & 3\\ 0 & -5\\ -7 & 6\end{pmatrix}$

and $\mathbf{C}=\begin{pmatrix}1 & 3 & 5 & 0\\ 2 & 5 & 7 & -4\end{pmatrix}$, show that $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$

(*d*) If $\mathbf{A}=\begin{pmatrix}-3 & 2 & 5\\ 1 & 5 & 0\\ 5 & 3 & 6\end{pmatrix}\quad \mathbf{B}=\begin{pmatrix}1 & 4 & 0 & 3\\ 2 & -1 & 3 & -2\\ 3 & 2 & 5 & -5\end{pmatrix}$

and $\mathbf{C}=\begin{pmatrix}2 & -2 & -7 & 0\\ 3 & -1 & -5 & 4\\ -5 & 0 & -2 & 3\end{pmatrix}$

show that $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$.

15. (*a*) For which values of x, will

$$(x \quad 4 \quad 1)\begin{pmatrix}2 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & 4\end{pmatrix}\begin{pmatrix}x\\ 4\\ -1\end{pmatrix}=0\ ?$$

(*b*) If the numbers p, q and r satisfy the equation $p^2+q^2+r^2=1$, show that the matrix

$$\mathbf{A}=\begin{pmatrix}0 & r & q\\ -r & 0 & p\\ -q & -p & 0\end{pmatrix}$$

satisfies the equation $\mathbf{A}^3+\mathbf{A}=\mathbf{0}$.

(c) Prove that the matrix **A**, given by

$$\mathbf{A}=\begin{pmatrix} 0 & 1 & 2 \\ 2 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

satisfies the relation $\mathbf{A}^3+4\mathbf{A}^2-\mathbf{A}=12\ \mathbf{I}_3$, where $\mathbf{I}_3$ is a unit matrix of order three.

16. (a) A shopkeeper sold 8 kg of peas, 20 kg. of potatoes, 12 kg of tomatoes and 4 kg of onions on Monday. On Tuesday he sold 10 kg of peas, 15 kg of potatoes, 6 kg of tomatoes and 8 kg of onions. Describe by means of 2×4 matrix, the position of sales on the two different days of different vegetables.

The prices of different items per kg were Rs. 2·50 for peas, Rs. 1·25 for potatoes, Rs. 2·25 for tomatoes and Re. 1 for onions on Monday. The rates on Tuesday per kg were Rs. 0·25 more than on Monday for each item. Express the prices on the two days through a 4×2 matrix.

Express also his total sales position of Monday sales at Monday rates, Tuesday sales at Tuesday rates and likely sales on Monday at Tuesday rates, Tuesday sales at Monday rates by a 2×2 matrix.

[Hint.

$$\mathbf{P}=\begin{matrix} & \text{Peas} & \text{Potatoes} & \text{Tomatoes} & \text{Onions} \\ \text{Monday} & 8 & 20 & 12 & 4 \\ \text{Tuesday} & 10 & 15 & 6 & 8 \end{matrix}$$

$$\mathbf{Q}=\begin{matrix} & \text{Monday} & \text{Tuesday} \\ \text{Peas} & 2{\cdot}50 & 2{\cdot}75 \\ \text{Potatoes} & 1{\cdot}25 & 1{\cdot}50 \\ \text{Tomatoes} & 2{\cdot}25 & 2{\cdot}50 \\ \text{Onions} & 1{\cdot}00 & 1{\cdot}25 \end{matrix}$$

$$\mathbf{PQ}=\begin{matrix} & \text{Mondy rates} & \text{Tuesday rates} \\ \text{Monday} & 76 & 87 \\ \text{Tuesday} & 65{\cdot}25 & 75 \end{matrix}\ ]$$

(b) A manufacturer produces three products : P, Q and R which he sells in two markets. Annual sale volumes are indicated as follows :

Markets	*Products*		
	P	*Q*	*R*
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(i) If unit sale prices of P, Q and R are Rs. 2·50, 1·25 and 1·50 respectively, find the total revenue in each market with the help of matrix algebra.

(ii) If the unit costs of the above 3 commodities are Rs. 1·80, 1·20 and 0·80 respectively, find the gross profit.

[**Hint.** (*i*) Total revenue in each market is obtained from the product matrix :

$$(2{\cdot}50 \quad 1{\cdot}25 \quad 1{\cdot}50)\begin{pmatrix} 10,000 & 6,000 \\ 2,000 & 20,000 \\ 18,000 & 8,000 \end{pmatrix} = (54,500 \quad 52,000)$$

(*ii*) Total cost $=(1{\cdot}80 \quad 1\ 20 \quad 0{\cdot}80)\begin{pmatrix} 10,000 & 6,000 \\ 2,000 & 20,000 \\ 18,000 & 8,000 \end{pmatrix}$

$=(34,800 \quad 41,200)$

Now find the profit.]

17. The matrix $\mathbf{A}=\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

represents the number of instruments **P** and **Q**, two factories **X** and **Y** can produce in a day, according to the table shown below :

	Factory **X**	Factory **Y**
Instrument **P**	2 per day	1 per day
Instrument **Q**	4 per day	3 per day

Let $\mathbf{B}=\begin{pmatrix} 5 \\ 6 \end{pmatrix}$

represent the number of days the two factories operate per week, *i.e.*, **X** operate 5 days per week and **Y** six days a week. Find **AB** and state what it represents.

18. A company is marketing 4 different types of pumps. Although the four models have the same rating, the principal difference between them lies in the combination of accessories produced. For example one type may not have automatic shut off control and another may be without mounting brackets. Five parts are required in various quantities depending upon the model and the following tabulation shows the requirements.

Pump Model	*Parts Required*				
	A	**B**	**C**	**D**	**E**
I	1	2	0	5	2
II	0	3	0	1	5
III	1	1	4	2	2
IV	1	2	4	5	5

What will be requirements of the parts **A, B, C, D, E** if the company has to supply 3 model I pump, 5 model **II** pumps, 2 model III pumps, and 10 model IV pumps ? If the cost of parts **A, B, C, D, E** be Rs. 30, Rs. 12, Rs. 5, Rs. 4 and Rs. 7 respectively, find the amount spent on purchasing the parts.

19.

$$\mathbf{D} = \begin{matrix} \\ m \\ t \\ w \\ th \\ f \end{matrix} \begin{matrix} \textit{Tea} & \textit{Coffee} & \textit{Chocolate} \\ \left(\begin{matrix} 33 \\ 28 \\ 56 \\ 36 \\ 41 \end{matrix}\right. & \begin{matrix} 42 \\ 35 \\ 64 \\ 49 \\ 53 \end{matrix} & \left.\begin{matrix} 55 \\ 43 \\ 41 \\ 38 \\ 28 \end{matrix}\right) \end{matrix}, \qquad \mathbf{C} = \begin{matrix} \text{Tea} \\ \text{Coffee} \\ \text{Choc.} \end{matrix} \begin{pmatrix} 2p \\ 3p \\ 3p \end{pmatrix}$$

Matrix **D** shows the daily sales of drinks from a hot drinks machine for each of the 5 days of one week.

Matrix **C** shows the cost of each type of drink.

(*a*) Calculate $(1\ \ 1\ \ 1\ \ 1\ \ 1)\ \mathbf{D}$ and say what information this gives.

(*c*) Calculate $\mathbf{D}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and state what information this gives.

(*d*) Find $(1\ \ 1\ \ 1\ \ 1\ \ 1)\ \mathbf{D}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. What does this represent ?

10. Three persons buy cold drinks of different brands **A**, **B**, **C**. The first person buys 12 bottles of **A**, 5 bottles of **B**, 3 bottles of **C**. The second person buys 4 bottles of **A**, 6 bottles of **B**, and 10 bottles of **C**. The third person buys 6 bottles of **A**, 7 bottles of **B** and 9 bottles of **C**. Represent the information in the form of a matrix. If each bottle of brand **A** costs Rs. 4, each bottle of **B** costs Rs. 5 and each bottle of **C** costs Rs. 6. then using matrix operations find the total sum of money spent individually by the three persons for the purchase of cold drinks.

[*C.A., November 1991*]

21. $$\mathbf{K} = \begin{matrix} \\ S_1 \\ S_2 \\ S_3 \end{matrix} \begin{matrix} R_1 & R_2 & R_3 & R_4 \\ \left(\begin{matrix} 6 \\ 5 \\ 8 \end{matrix}\right. & \begin{matrix} 8 \\ 7 \\ 3 \end{matrix} & \begin{matrix} 3 \\ 4 \\ 5 \end{matrix} & \left.\begin{matrix} 4 \\ 5 \\ 1 \end{matrix}\right) \end{matrix} \qquad \mathbf{V} = \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \begin{pmatrix} 18 \\ 20 \\ 24 \\ 35 \end{pmatrix}$$

$$\mathbf{W} = \begin{matrix} \\ S_1 \\ S_2 \\ S_3 \end{matrix} \begin{matrix} R_1 & R_2 & R_3 & R_4 \\ \left(\begin{matrix} 2 \\ 1 \\ 4 \end{matrix}\right. & \begin{matrix} 2 \\ 3 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} & \left.\begin{matrix} 3 \\ 2 \\ 0 \end{matrix}\right) \end{matrix}$$

Matrix **K** shows the stock of four types of record players R_1, R_2, R_3 and R_4 in three shops S_1, S_2 and S_3.

Matrix **V** shows the value of the record players in ('00 rupees).

Matrix **W** gives the week's sales. Find

(*a*) the stock at the end of the week.

(*b*) the order matrix to bring the stock of each of the cheaper pair of record players to 8 and the dearer pair to 5.

(*c*) the value of the sales, (*d*) the value of the order.

22. $$\mathbf{S}=\begin{matrix} & S_1 & S_2 & S_3 & S_4 \\ C_1 & 2 & 1 & 3 & 0 \\ C_2 & 4 & 2 & 1 & 4 \\ C_3 & 3 & 2 & 1 & 4 \end{matrix},\quad \mathbf{D}=\begin{pmatrix} 3 & 4 & 2 & 5 \\ 1 & 3 & 4 & 3 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\mathbf{L}=\begin{pmatrix} 2 & 1 & 2 & 4 \\ 1 & 3 & 1 & 2 \\ 2 & 3 & 4 & 2 \end{pmatrix}$$

Matrix **S** shows the stock of 3 types of cooker C_1, C_2 and C_3 in 4 showrooms S_1, S_2 and S_3 and S_4. Matrix **D** shows the deliveries of new cookers at the beginning of a week.

Matrix **L** shows the sales during that week. Find

(*a*) the stock immediately after delivery **D**.

(*b*) the stock at the end of the week.

(*c*) the order matrix to bring stocks of all cookers in all showrooms up to 6.

ANSWERS

1. (*a*) (*i*) $(3 \quad -2)$, (*ii*) $(-1 \quad -6)$ (*iii*) $(5 \quad 3)$

(*iv*) $\left(-\frac{1}{6} \quad -\frac{11}{12}\right)$

2. (*a*) $\begin{pmatrix} 10 & 4 & 2 \\ 6 & 0 & 8 \\ 4 & 6 & 12 \end{pmatrix}$, (*b*) $\begin{pmatrix} -6 & -4 & 6 \\ 6 & 4 & 8 \\ 0 & 2 & 0 \end{pmatrix}$,

(*c*) $\begin{pmatrix} 18 & 6 & 2 \\ 6 & 2 & 2 \\ 4 & 10 & 2 \end{pmatrix}$, (*d*) $\begin{pmatrix} 18 & 6 & 2 \\ 6 & 2 & 2 \\ -4 & 10 & 2 \end{pmatrix}$

(*e*) $\begin{pmatrix} 2 & 2 & 6 \\ 6 & 6 & 2 \\ -8 & 6 & -10 \end{pmatrix}$, (*f*) $\begin{pmatrix} -14 & -6 & 6 \\ 6 & 2 & 14 \\ 8 & -2 & 10 \end{pmatrix}$

(*g*) $\begin{pmatrix} 20 & 8 & 4 \\ 12 & 0 & 16 \\ 8 & 12 & 24 \end{pmatrix}$, (*h*) $\begin{pmatrix} 20 & 8 & 4 \\ 12 & 0 & 16 \\ 8 & 12 & 24 \end{pmatrix}$

(*i*) $\begin{pmatrix} 2 & 2 & 8 \\ 18 & -4 & 42 \\ 34 & 4 & 60 \end{pmatrix}$, (*j*) $\begin{pmatrix} 28 & 12 & 2 \\ 12 & -2 & 16 \\ 10 & 14 & 30 \end{pmatrix}$

(*k*) $\begin{pmatrix} 22 & 8 & 8 \\ 18 & 2 & 24 \\ 10 & 16 & 30 \end{pmatrix}$

3. (*a*) 3×2, 2×3, (*b*) 3×3, (*d*) yes, (*e*) no, (*f*) No, only 2**B** is possible.

6. (*a*) $\begin{pmatrix}7 & 3\\6 & 4\end{pmatrix}$ (*b*) $\begin{pmatrix}-20 & 20\\10 & -30\end{pmatrix}$

7. (*a*) (*i*) $\begin{pmatrix}0 & \frac{9}{2}\\3 & 2\end{pmatrix}$ (*ii*) $\begin{pmatrix}-3 & -7\\-2 & -5\end{pmatrix}$

(*iii*) $\frac{1}{5}\begin{pmatrix}9 & 3\\-6 & 7\end{pmatrix}$ (*b*) $=\mathbf{X}\begin{pmatrix}\frac{3}{2}\\-\frac{13}{2}\end{pmatrix}$ (*c*) $\mathbf{A}=(-1\ \ 2\ \ 1)$

15. $-2\pm\sqrt{6}$.

17. $\mathbf{AB}=\begin{pmatrix}16\\38\end{pmatrix}$ The number of instruments **P** produced by the factories is 16 and of the instruments **Q** is 38.

18. Part **A**=15, Part **B**=43, Part **C**=48, Part **D**=74, Part **E**=85. Amount spent is Rs. 2097.

19. (*a*) (194 243 205). Total sales of each drink.

(*b*) 17·32, (*c*) $\begin{pmatrix}130\\106\\161\\123\\122\end{pmatrix}$ Total daily sales. (*d*) (642), Total weekly sales.

20.
$$\begin{matrix} & A & B & C\\ 1 & 12 & 5 & 3\\ 2 & 4 & 6 & 10\\ 3 & 6 & 7 & 9\end{matrix}$$
Rs. 91, Rs. 106, Rs. 113.

21. (*a*) $\begin{pmatrix}4 & 6 & 2 & 1\\4 & 4 & 3 & 3\\5 & 2 & 3 & 1\end{pmatrix}$, (*b*) $\begin{pmatrix}4 & 2 & 3 & 4\\4 & 4 & 2 & 2\\3 & 6 & 2 & 4\end{pmatrix}$

(*c*) 49,900 (*d*) 95,600.

22. (*a*) $\begin{pmatrix}5 & 5 & 5 & 5\\5 & 5 & 5 & 7\\5 & 5 & 5 & 5\end{pmatrix}$, (*b*) $\begin{pmatrix}3 & 4 & 3 & 1\\4 & 2 & 4 & 5\\3 & 2 & 1 & 3\end{pmatrix}$, (*c*) $\begin{pmatrix}3 & 2 & 3 & 5\\2 & 4 & 2 & 1\\3 & 4 & 5 & 3\end{pmatrix}$

20·10. TRANSPOSE OF A MATRIX

The matrix obtained by interchanging rows and columns of the matrix **A** is called the transpose of **A** and is denoted by **A**′ or **A**ᵗ (read as **A** transpose), *e.g.*, if

$$\mathbf{A}=\begin{pmatrix}3 & 2\\4 & 1\\7 & -5\end{pmatrix}_{3\times2} \text{ then } \mathbf{A}'=\begin{pmatrix}3 & 4 & 7\\2 & 1 & -5\end{pmatrix}_{2\times3}$$

Symbolically if

$$\mathbf{A}=(a_{ij})_{m\times n} \text{ then } \mathbf{A}'=(a_{ji})_{n\times m}$$

i.e., the (i, j)th element of $\mathbf{A}=(j, i)$th element of $\mathbf{A}'$. In other words, if

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1i} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2j} & \dots a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots a_{ij} & \dots a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots a_{mj} & a_{mn} \end{bmatrix}_{m\times n} \quad \mathbf{A}'=\begin{bmatrix} a_{11} & a_{21} & \dots a_{i} & \dots a_{m1} \\ a_{12} & a_{22} & \dots a_{i2} & \dots a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1i} & a_{2j} & \dots a_{ij} & \dots a_{mj} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots a_{in} & a_{mn} \end{bmatrix}_{n\times m}.$$

Remarks. 1. If $\mathbf{A}$ is $m\times n$ matrix, then $\mathbf{A}'$ will be a $n\times m$ matrix.

2. The transpose of a row (column) matrix is a column (row) matrix

3. $(\mathbf{A}')'=\mathbf{A}$.

4. The transpose of the sum of two matrices is the sum of their transposes, *i.e.*, $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$

5. The transpose of the product $\mathbf{AB}$ is equal to the product of the transposes taken in the reverse order, *i.e.*, $(\mathbf{AB})'=\mathbf{B}'\mathbf{A}'$.

Example 15. *Let*

$$\mathbf{A}=\begin{pmatrix} 2 & -3 & 1 \\ 4 & 2 & 3 \end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & -5 \end{pmatrix}$$

Show that $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$.

Solution.

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix} 2+3 & (-3)+(-2) & 1+4 \\ 4+1 & 2+3 & 3+(-5) \end{pmatrix}=\begin{pmatrix} 5 & -5 & 5 \\ 5 & 5 & -2 \end{pmatrix}$$

$$\therefore\ (\mathbf{A}+\mathbf{B})'=\begin{pmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{pmatrix} \quad \dots\text{(I)}$$

Now $$\mathbf{A}'=\begin{pmatrix} 2 & 4 \\ -3 & 2 \\ 1 & 3 \end{pmatrix} \text{ and } \mathbf{B}'=\begin{pmatrix} 3 & 1 \\ -2 & 3 \\ 4 & -5 \end{pmatrix}$$

$$\therefore\ \mathbf{A}'+\mathbf{B}'=\begin{pmatrix} 2+3 & 4+1 \\ (-3)+(-2 & 2+3 \\ 1+4 & 3+(-5) \end{pmatrix}=\begin{pmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{pmatrix} \quad \dots\text{(II)}$$

Hence $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$

Symmetric Matrix. A square matrix is said to be symmetric if the transpose of a matrix is equal to the matrix itself, *e.g.*,

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \begin{pmatrix} 1 & 4 & 9 \\ 4 & 7 & 5 \\ 9 & 5 & 8 \end{pmatrix}$$ are symmetric matrices.

Symbolically $\mathbf{A}=(a_{ij})_{n\times n}$ is said to be symmetric if $a_{ij}=a_{ji}$ for all i and j.

Skew Symmetric Matrix. A square matrix $A=(a_{ij})_{n\times n}$ is said to be skew matrix if

$$a_{ij}=-a_{ji} \text{ for all } i \text{ and } j$$

$$e.g., \begin{pmatrix} 0 & h & g \\ -h & 0 & f \\ -g & f & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 8 \\ -6 & 0 & 9 \\ -8 & -9 & 0 \end{pmatrix}$$

are skew symmetric matrices.

Orthogonal Matrix. $A'A=I=AA'$

EXERCISE (III)

1. For each of the following matrices verify that $(\mathbf{A}')'=\mathbf{A}$

$$(a) \begin{pmatrix} 2 & 8 & 4 \\ 8 & 6 & -1 \\ 4 & -1 & 0 \end{pmatrix} \quad (b) \begin{bmatrix} 3 & 9 & 2 & -7 & 2 \\ 7 & 8 & 5 & 6 & 0 \end{bmatrix}$$

2. If $\mathbf{A}=\begin{pmatrix} 3 & -3 & 0 \\ 6 & 3 & 9 \\ 12 & 3 & 24 \end{pmatrix}$, $\mathbf{B}=\begin{bmatrix} 2 & 3 & 0 \\ 6 & -9 & 3 \\ 3 & 3 & -3 \end{bmatrix}$,

verify that $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$, $(\mathbf{AB})'=\mathbf{B}'\mathbf{A}'$

3. If $\mathbf{A}=\begin{bmatrix} 8 & 16 & -4 \\ -4 & 0 & 8 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} 12 & 16 & 20 \\ -4 & 8 & 28 \\ 8 & 4 & 0 \end{bmatrix}$

Compute $(\mathbf{AB})'$ and $\mathbf{B}'\mathbf{A}'$

4. (a) For the matrix $\mathbf{A}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$,

show that $\mathbf{AA}'=\mathbf{I}_2$.

(b) If $\mathbf{A}=\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$, verify that

$$\mathbf{AA}'=\mathbf{A}'\mathbf{A}=\mathbf{I}_3$$

(c) If $\mathbf{A}=\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ verify that

$$\mathbf{AA}'=\mathbf{A}'\mathbf{A}=\mathbf{I}_4$$

5. Find x and y so that the matrix

$$\mathbf{P}=\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$$

may satisfy the condition $\mathbf{P}'\mathbf{P}=\mathbf{PP}'=\mathbf{I}_3$.

Ans. 5. $x=-2, y=-1$

20·11. DETERMINANTS OF A SQUARE MATRIX

Let $\mathbf{A}=[a_{ij}]$ be a square matrix. We can associate with the square matrix **A** a determinant which is formed by exactly the same array of elements of the matrix **A**. A determinant formed by the same array of elements of the same square matrix **A** is called the determinant of the square matrix **A** and is denoted by the symbol det. **A** or | **A** | . It should be remembered that the determinant of a square matrix will be a scalar quantity, *i.e.*, with a determinant we associate some value whereas a matrix is essentially an arrangement of numbers and as such has no value.

For example, let a matrix $A=\begin{pmatrix}6 & 5\\ 3 & 2\end{pmatrix}$ so that $|\mathbf{A}|=\begin{vmatrix}6 & 5\\ 3 & 2\end{vmatrix}$

$$=6\times2-5\times3=-3.$$
$$=-3$$

Here $|A|=-3$ whereas **A** is a matrix giving only an arrangement of the four numbers 6, 5, 3, 2 in two rows and two columns. It should be noted that the positions occupied by the elements of a matrix are important. A change in the positions of the elements of a matrix gives rise to a different matrix.

For example $\begin{pmatrix}6 & 5\\ 3 & 2\end{pmatrix}$ and $\begin{pmatrix}2 & 5\\ 3 & 6\end{pmatrix}$ are different matrices, although formed by the same elements of a number 6, 5, 3 and 2. However, the determinants of these two square matrices are

$$\begin{vmatrix}6 & 5\\ 3 & 2\end{vmatrix} \text{ and } \begin{vmatrix}2 & 5\\ 3 & 6\end{vmatrix}$$

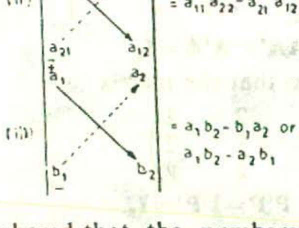
and have the same value, namely -3.

We will now take up determinants of various orders, *viz.*, two three and higher order.

20·12. DETERMINANTS OF ORDER TWO

The determinant of a 2×2 matrix is denoted by any of the following ways :

(*i*) $$\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-cb \text{ or } ad-bc$$



It should be remembered that the numbers enclosed by straight lines do not constitute a matrix—they are the coefficients or the numbers assigned to a square matrix. We will now illustrate its use in solution of simultaneous equations.

20·13 CRAMER'S RULE

It is a simple rule using determinants to express the solution of a system of linear equations for which the number of equations is equal to the number of variables.

We shall now show how the second order determinants can be used to give the solution of two simultaneous linear equations in a convenient form. Students are already familiar with the method of solving two simultaneous linear equations in two unknowns.

Let the given equations be written in the form

$$a_1x+b_1y=c_1 \qquad \ldots(1)$$

and

$$a_2x+b_2y=c_2 \qquad \ldots(2)$$

To find the value of x we eliminate y by multiplying (1) by b_2 and (2) by b_1 and then subtract the latter from the former, we then get

$$(a_1b_2-a_2b_1)x=c_1b_2-c_2b_1 \qquad \ldots(3)$$

Similarly to find the value of y we eliminate x by multiplying (1) by a_2 and (2) by a_1 and then subtract the latter from the former, we then get

$$(b_1a_2-a_1b_2)y=c_1a_2-c_2a_1 \qquad \ldots(4)$$

The values of x and y as given by (3) and (4) can be written as

$$\frac{x}{c_1b_2-c_2b_1}=\frac{y}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

This solution can be conveniently written in the determinant form as follows :

$$\frac{x}{\begin{vmatrix} c_1 & b_1 \\ c_1 & b_2 \end{vmatrix}}=\frac{y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\Rightarrow \qquad x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad \ldots(5)$$

We observe that the denominator for each unknown is the determinant in which the elements are the coefficients of x and y arranged as in the two given equations. The determinant, we shall call, as the determinant of co-efficients and will denote by D. Observe further that the numerator for any unknown is the same as D except that the column of co-efficients of that unknown is replaced by the column of constant terms. Let us call for convenience the determinant in the numerator for x by N_x and determinant in the numerator for y by N_y. The rule embodied in the solution given in terms of determinants as described above is known as *Cramer's rule.*

Remark. From Coordinate Geometry we know that equations (1) and (2) being linear in x, y represent two straight lines. The values of x, y given in solution (5) give the coordinates of the point of intersection of lines (1) and (2). If $a_1b_2-a_2b_1=0$, the equations (1) and (2) are not satisfied by finite values of x and y and the lines become parallel. However, if $a_1b_2-a_2b_1\neq 0$, the lines (1) and (2) intersect in a finite point whose co-ordinates are given by (5).

We, now, illustrate the use of Cramer's Rule for the solution of simultaneous linear equations in two unknowns.

Example 16. *Solve the following simultaneous linear equations using determinants :*

$$2x-y=5$$
$$3x+2y=-3$$

Solution. Let us first find out the denominator of the quotient of the value of x and y as follows :

$$D=\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}=2.2-3\,(-1)=4+3=7$$

$\therefore$ $D\neq 0$, the system has a unique solution.

$$N_x=\begin{vmatrix} 5 & -1 \\ -3 & 2 \end{vmatrix}=5.2(-3)\,(-1)=10-3=7$$

$$N_y=\begin{vmatrix} 2 & 5 \\ 3 & -3 \end{vmatrix}=2.(-3)-3.5=-6-15=-21$$

$$x=\frac{N_x}{D}=\frac{7}{7}=1 \text{ and } y=\frac{N_y}{D}=\frac{-21}{7}=-3$$

20 14. DETERMINANT OF ORDER THREE

In a 3 by 3 matrix, the determinants are defined as follows :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_2 & b_3 \end{vmatrix}$$
$$=a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$

It may be noticed that in each case a 2 by 2 determinant has been taken by omitting the row and column of a particular row element in order a_1, b_1 and c_1. Another thing to note is the alternating signs for this row elements.

Example 17. *Compute the determinant of the following matrix*

$$\mathbf{A}=\begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$$

Solution.

$$|\mathbf{A}| = \det(\mathbf{A}) = 2\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}$$

$$=2(-20+2)-3(0-2)-4(0+4)$$

$$=-36+6-16=-46$$

20 15. SOLUTION OF THREE LINEAR EQUATIONS

On the lines of the solution for the two equations, the solution for the three unknowns will be through the following quotients of determinants.

$$x=\frac{N_x}{D},\quad y=\frac{N_y}{D},\quad z=\frac{N_z}{D}$$

In order to illustrate we take the following system of three linear equations :

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1 \\ a_2x+b_2y+c_2z&=d_2 \\ a_3x+b_3y+c_3z&=d_3 \end{aligned}\right| \quad \begin{aligned} 2x+y-z&=3 \\ x+y+z&=1 \\ x-2y-3z&=4 \end{aligned}$$

The denominator D of each quotient is

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

In the above case

$$D=\begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$=2(-3+2)-1(-3-1)-1(-2-1)$$

$$=-2+4+3=5$$

$\therefore \quad D \neq 0$

Now

$$N_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

We have replaced the column of the coefficients of x by the column of the constant terms.

$$\therefore \quad N_x=\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix}=3(-3+2)-1(-3-4)-(-2-4)$$

$$=-3+7+6=10$$

Also $N_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ We have replaced the column of the coefficients of y by the column of the constant terms.

$$\therefore \quad N_y=\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix}=2(-3-4)-3(-3-1)-1(4-1)$$

$$=-14+12-3=-5$$

and $N_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ We have replaced the column of the coefficients of z by the column of the constant terms.

$$\therefore \quad N_z=\begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix}=2(4+2)-1(4-1)+3(-2-1)$$

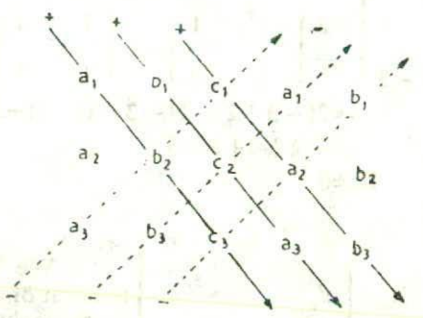
$$=12-3-9=0$$

The required solutions are

$$x=\frac{N_x}{D}=\frac{10}{5}=2,\ y=\frac{N_y}{D}=\frac{-5}{5}=-1,\ z=\frac{N_z}{D}=\frac{0}{5}=0$$

20·16. SARRUS DIAGRAM

We can find out determinants of a given matrix very conveniently if we extend the matrix by adding the first two columns and connect the elements by arrows downwards preceded by a plus sign and upwards by a minus sign as illustrated below :



The product of elements joined by downward arrows preceded by plus signs are

$$+a_1b_2c_3+b_1c_2a_3+c_1a_2b_3$$

And the products of each of three elements joined by upward arrows preceded by minus signs are

$$a_3b_2c_1-b_3c_2a_1-c_3a_2b_1$$

Example 18. *Find the value of the determinants*

$$\begin{vmatrix} 2x & 4y \\ x & 3y \end{vmatrix}, \quad \begin{vmatrix} x & x+1 \\ x+2 & x+3 \end{vmatrix}$$

Solution.

(*i*) $$\begin{vmatrix} 2x & 4y \\ x & 3y \end{vmatrix}=2x.3y-x.4y=2xy$$

(*ii*) $$\begin{vmatrix} x & x+1 \\ x+2 & x+3 \end{vmatrix}=x(x+3)-(x+1)(x+2)=-2$$

Example 19. *Find the value of*

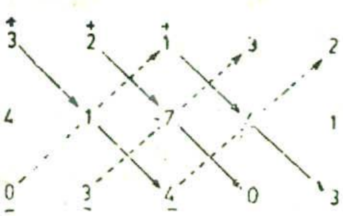
$$\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & -7 \\ 0 & 3 & 4 \end{vmatrix}$$

Solution. Since there is zero in the first column, we expand by the elements of the first column,

$$\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & -7 \\ 0 & 3 & 4 \end{vmatrix}=3\begin{vmatrix} 1 & -7 \\ 3 & 4 \end{vmatrix}-4\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}+0\begin{vmatrix} 2 & 1 \\ 1 & -7 \end{vmatrix}$$

$$=3(4+21)-4(8-3)+0=55$$

By Sarrus Diagram



$= 12+0+12-32+63+0$
$= 55$

20·17. PROPERTIES OF DETERMINANTS

I. *If the rows of a determinant are changed into columns and vice versa, the value of the determinant remains unchanged, i.e., det A = det A'*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

For example

$$\begin{vmatrix} 1 & 5 & 6 \\ 2 & 8 & 7 \\ 3 & -9 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 8 & -9 \\ 6 & 7 & 0 \end{vmatrix}.$$

II. *If any two rows (or columns) are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant, i.e.,*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

III. *If any two rows or any two columns of a determinant are identical the value of the determinant is zero.*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

IV. *If the elements of a row (column) of a determinant are added (subtracted) k-times the corresponding elements of another row (column), the value of the determinant remains unchanged.*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} & a_{13}+ka_{11}-ma_{12} \\ a_{21} & a_{22} & a_{23}+ka_{21}-ma_{22} \\ a_{31} & a_{32} & a_{33}+ka_{31}-ma_{32} \end{vmatrix}.$$

V. *If the elements of a row (column) of a matrix are multiplied by the same number, k say, the determinant of the matrix thus obtained is k times the determinant of the original matrix.*

$$\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

VI. *If the elements of any row or any column of a determinant is sum (difference) of two or more elements then the determinant can be expressed as sum (difference) of two or more determinants*

$$\begin{vmatrix} a_{11}+\alpha_{11} & a_{12} & a_{13} \\ a_{21}+\alpha_{21} & a_{22} & a_{23} \\ a_{31}+\alpha_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & a_{12} & a_{13} \\ \alpha_{21} & a_{22} & a_{23} \\ \alpha_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 20. *Prove that*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} \quad \text{Apply } \begin{matrix} c_1-c_2 \\ c_2-c_3 \end{matrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \left\{ \begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix} \right\}$$

$$= (a-b)(b-c)(b+c-a-b)$$

$$= (a-b)(b-c)(c-a)$$

Example 21. *Prove that*

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution.

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2 \end{vmatrix}$$

Apply $c_1+c_2+c_2$

$$=2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & +a+2b \end{vmatrix}$$

$$=2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \quad \text{Apply } \begin{matrix} R_2-R_1 \\ R_3-R_1 \end{matrix}$$

$$=2(a+b+c)\left\{\begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}\right\}$$

$$=2(a+b+c)^3$$

Example 22. *Evaluate*

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$$

Solution.

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} \quad abc\begin{vmatrix} 0 & b^2 & c^2 \\ a^2 & 0 & c^2 \\ a^2 & b^2 & 0 \end{vmatrix}$$

$$abc\,(a^2b^2c^2)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$=a^3b^3c^3[-1(0-1)+1(1-0)]=2a^3b^3c^3$$

EXERCISE (V)

1. Show that

$$\begin{vmatrix} 3 & -7 \\ 8 & 6 \end{vmatrix}=74, \quad \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}=0,$$

$$\begin{vmatrix} x & y \\ -1 & +1 \end{vmatrix}=x+y, \quad \begin{vmatrix} -4 & 2 \\ -3 & -4 \end{vmatrix}=22$$

2. (*a*) Show that

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \quad \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} \quad 4$$

(*b*) Show that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} b & q \\ p & c \end{vmatrix} + \begin{vmatrix} p & d \\ a & q \end{vmatrix} = 0$$

3. Show that

$$\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 6 & 8 & 0 \end{vmatrix} = -48, \quad \begin{vmatrix} 3 & 4 & 8 \\ 2 & 1 & 3 \\ 7 & -2 & 0 \end{vmatrix} \quad 14$$

4. Show that

$$\begin{vmatrix} 3 & 4 & 7 \\ 2 & 1 & 3 \\ -5 & -1 & 2 \end{vmatrix} = -40$$

5 Show that

$$\begin{vmatrix} 1 & 2 & 3 \\ a & -a & b \\ -a & 0 & -b \end{vmatrix} = ab - 3a^2$$

6. Show that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc - af^2 - bg^2 - ch^2 + 2fgh$$

7. Evaluate the following :

$$\begin{vmatrix} x & 1 & 2 \\ 2 & x & 2 \\ 3 & 1 & x \end{vmatrix}, \quad \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & b^1 & b^2 \end{vmatrix}, \quad \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$$

8 Show that

$$\begin{vmatrix} 2 & 45 & 55 \\ 1 & 92 & 32 \\ 3 & 68 & 87 \end{vmatrix} = 54$$

[**Hint.** [Apply $R_1 - 2R_2$, $R_3 - 3R_2$ and expand.]

9. Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (b-c)(c-a)(a-b)$$

(*b*) Show that

$$\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$$

10. Find the value of

$$\begin{vmatrix} 1 & \omega & \omega \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$, where ω is cube root of unity

[**Hint.** $1+\omega+\omega^2=0$]

11. Show that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

12. Prove that

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$$

[**Hint.** Apply $c_1+c_2+c_3$]

13.

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

[**Hint.** Apply $R_1+R_2+R_3$]

14.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

15. Show that :

$$\begin{vmatrix} x-y & 1 & x \\ y-z & 1 & y \\ z-1 & 1 & g \end{vmatrix} = \begin{vmatrix} x & 1 & y \\ y & 1 & z \\ z & 1 & x \end{vmatrix}$$

[*I.C.W.A., June 1991*]

16. Show that :

$$\begin{vmatrix} a^3 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} = (a^3+b^3)^2$$

[*I.C.W.A., December, 1990*]

ANSWERS

7. (*i*) $x^3+10x+10$, (*ii*) $(a-b)^2$, (*iii*) -8. **10.** 0.

20·18. EXPANSION OF THE DETERMINANTS

Determinants can be represented as linear combinations of order two with co-efficients from second row or third row or in terms of the elements of any column. The only thing to remember is that 2×2 determinant accompanying any co-efficient can be obtained by deleting the row and column containing the co-efficient in the original determinant. Further, the signs accompanying the co-efficient in the original determinant will follow the following checker board pattern :

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Example 23. *Give the determinants with co-efficients from (i) first column and (ii) the third row in the following co-efficients of the determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Solution.

(*i*) $\triangle = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

$$= a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$

$$= a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1$$

(ii) $\Delta = a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$= a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1)$
$= a_3b_1c_2 - a_3b_2c_1 - b_3a_1c_2 + b_3a_2c_1 + c_3a_1b_2 - c_3a_2b_1$

This aspect will be examined more extensively in the next article on minors of the matrix.

20 19. MINORS OF A MATRIX

Consider a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{5\times 3}$$

When we delete any one row and any one column of **A**, then we get a 2×2 matrix, which is called a submatrix of **A**, for example, if we strike off the first row and first column, we get the sub-matrix as

$$\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

The determinant of any such submatrix is called a minor of determinant **A**, thus

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ is minor of det. } \mathbf{A}.$$

The minor of a_{11}, a_{12}, a_{13} in $|\mathbf{A}|$ are

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ respectively.}$$

The minors of a_{21}, a_{22}, a_{23} in $|\mathbf{A}|$ are

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \text{ respectively,}$$

The minors of a_{31}, a_{32}, a_{33} in $|\mathbf{A}|$ are

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ respectively.}$$

In general, the minor obtained by striking off the ith row and jth column of a matrix $\mathbf{A} = [a_{ij}]_{n\times n}$ is called the minor of a_{ij} in $|\mathbf{A}|$

$$\begin{vmatrix} a_{11} & a_{12} \cdots\cdots a_{1j} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2j} \cdots\cdots a_{2n} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ -a_{i1} & --a_{i2} --- a_{ij} --- a_{in}- \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ a_{n1} & a_{n2} \cdots\cdots a_{nj} \cdots\cdots a_{nn} \end{vmatrix}$$

The minor of element a_{ij} is designated by $\mathbf{M}_{ij}$.

20·20. CO-FACTORS OF A MATRIX

If we multiply the minor of the element in the ith row and jth column of the determinant of the matrix by $(-1)^{i+j}$ the product is called he co-factor of the element. It is usual to denote the co-factor of an element by the corresponding capital letters. Symbolically

$$A_{ij}=(-1)^{i+j}\times \text{minor of } a_{ij} \text{ in } |\mathbf{A}|=(-1)^{i+j}|\mathbf{M}_{ij}|, \text{ e.g.,}$$

$$|\mathbf{A}|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A_{11}=(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad A_{12}=(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad A_{23}=(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and so on.

Example 24. *If*

$$\mathbf{A}=\begin{pmatrix} 3 & 4 & 7 \\ -2 & 5 & 6 \\ 7 & 3 & -9 \end{pmatrix}$$

find the co-factors of elements 6, −9.

Solution. The co-factor of element a_{23}, *i.e.*, 6 is

$$A_{23}=(-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 7 & 3 \end{vmatrix}=-(9-28)=+19$$

The co-factor of the element a_{33}, *i.e.*, −9 is

$$A_{33}=(-1)^{3+3}\begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix}=+(15+8)=+23$$

Remarks. 1. The sum of the products of the elements of any row (column) of a determinant with the corresponding co-factors is equal to the value of the determinant.

2. The sum of the products of the elements of any row (column) with the co-factors of the corresponding elements of any other row (column) is zero.

20·21. ADJOINT OF A SQUARE MATRIX

Let $\mathbf{A}=[a_{ij}]_{n\times n}$ be a square matrix of order n, then adjoint of **A** is defined to be transpose of matrix $[A_{ij}]_{n\times n}$, where A_{ij} is co-factor of a_{ij} in $|\mathbf{A}|$. In other words, let

$$\mathbf{A}=\begin{vmatrix} a_{11} & a_{12}\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots a_{2n} \\ \vdots & \vdots \qquad \vdots \\ a_{n1} & a_{n2}\ldots\ldots a_{nn} \end{vmatrix}_{n\times n}$$

$$\text{Adj } \mathbf{A}=\text{transpose of } \begin{vmatrix} A_{11} & A_{12}\ldots\ldots A_{1n} \\ A_{21} & A_{22}\ldots\ldots A_{2n} \\ \vdots & \vdots \qquad \vdots \\ A_{n1} & A_{n2}\ldots\ldots A_{nn} \end{vmatrix}_{n\times n}$$

$$=\begin{vmatrix} A_{11} & A_{21}\ldots\ldots A_{n1} \\ A_{12} & A_{22}\ldots\ldots A_{n2} \\ \vdots & \vdots \qquad \vdots \\ A_{1n} & A_{2n} \qquad A_{nn} \end{vmatrix}_{n\times n}$$

Here A_{11}=co-factor of a_{11} in $|\mathbf{A}|$

A_{12}= ,, ,, a_{12} in $|\mathbf{A}|$ and so on.

Remarks 1. If **A** be an n-rowed (*viz.*, $n\times n$) square matrix, then **A** (adj **A**)=(adj **A**) **A**= $|\mathbf{A}|$ $\mathbf{I}_n$, where $\mathbf{I}_n$ is a unit matrix of order n.

2. Adj (**AB**)=(Adj **B**)(Adj **A**)

Example 25. *Find the adjoint of the matrix*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$$

and verify the theorem

$$\mathbf{A}\ (Adj\ \mathbf{A})=(Adj\ \mathbf{A})\ \mathbf{A}=|\mathbf{A}|\ \mathbf{I}_3$$

Solution. Adj **A**=transpose of $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{3\times 3}$

where

$$A_{11}=\text{co-factor of } a_{11} \text{ in } |\mathbf{A}|=(-1)^{1+1}\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix}=3$$

$$A_{12}=\text{co-factor of } a_{12} \text{ in } |\mathbf{A}|=(-1)^{1+2}\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix}=-9$$

$$A_{13}=\text{co-factor of } a_{13} \text{ in } |\mathbf{A}|=(-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}=-5$$

Similarly

$$A_{21}=(-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}=-4,\quad A_{22}=(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=1$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}=3,\quad A_{31}=(-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}=-5$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}=4,\quad A_{33}=(-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=1$$

Therefore

$$\text{Adj } \mathbf{A}=\begin{pmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{pmatrix}'=\begin{pmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{pmatrix}$$

Also $\quad |\mathbf{A}| = 1.3-1.(4)+2.(-5)=-11$

Now

$$\mathbf{A}\ (\text{Adj } \mathbf{A})=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{pmatrix}=-11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \end{pmatrix}$$

$$=|\mathbf{A}|\ \mathbf{I}_3 \qquad \ldots(1)$$

$$\text{Also } (\text{Adj } \mathbf{A})\ \mathbf{A}=\begin{pmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} -11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}=|\mathbf{A}|\ \mathbf{I}_3 \qquad \ldots(2)$$

From (1) and (2), we get

$$\mathbf{A}\ (\text{Adj } \mathbf{A})=(\text{Adj})\ \mathbf{A}=|\mathbf{A}|\ \mathbf{I}_3$$

EXERCISE (VI)

1. Find the adjoint of the matrix

$$\mathbf{A}=\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$$

Verify $\quad \mathbf{A}(\text{Adj } \mathbf{A})=(\text{Adj } \mathbf{A})\ \mathbf{A}=|\mathbf{A}|\ \mathbf{I}_2$

2. Find the adjoint of the matrices

$$(i)\ \mathbf{A}=\begin{pmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & -3 \end{pmatrix},\quad (ii)\ \begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$$

and verify that

$$\mathbf{A}\ (\text{Adj}\ \mathbf{A})=(\text{Adj}\ \mathbf{A})\ \mathbf{A}=|\ \mathbf{A}\ |\ \mathbf{I}_3$$

3. If $\mathbf{A}=\begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$, show that Adj A=3 $\mathbf{A}'$

4. If $\mathbf{A}=\begin{bmatrix} 1 & 2 & 1 \\ 5 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, verify

$$\mathbf{A}\ (\text{Adj}\ \mathbf{A})=|\ \mathbf{A}\ |.\mathbf{I}=(\text{Adj}\ \mathbf{A}).\mathbf{A},$$ [*I.C.W.A., Dec. 1990*]

20·22. INVERSE OF A MATRIX

The operation of dividing one matrix directly by another does not exist in matrix theory but equivalent of division of a unit matrix by any square matrix can be accomplished (in most cases) by a process known as "Inversion of Matrix".

In ordinary algebra if $x \times y=1$, then $x=1/y$ or we say that y is inverse of x or x is inverse of y. The product of quantity x and its inverse is one.

Definition. Let **A** be any $n \times n$ matrix. The n-square matrix **B** is called inverse of **A** if

$$\mathbf{AB}=\mathbf{BA}=\mathbf{I}_n$$

The inverse of A is denoted by $\mathbf{A}^{-1}$, *i.e.*, $\mathbf{B}=\mathbf{A}^{-1}$ so that

$$\mathbf{AA}^{-1}=\mathbf{A}^{-1}=\mathbf{I}_n$$

The concept of inverse matrix is useful in solving simultaneous equations, input-output analysis and regression analysis. There are three methods of finding the inverse of a given square matrix.

(*i*) Using adjoint matrices—co-factor method.

(*ii*) Using linear equations.

(*iii*) Gauss Elimination Method.

Remark. A square matrix **A** has an inverse if and only if $|\ \mathbf{A}\ | \neq 0$, *i.e.*, only non-singular matrix possesses an inverse.

Co-factor Method. The inverse of **A** is given by

$$\mathbf{A}^{-1}=\frac{1}{|\ \mathbf{A}\ |}\ \text{Adj}\ \mathbf{A}$$

Example 26. *Find the inverse of the matrix :*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Solution. $$A^{-1}=\frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|}$$

$$|\mathbf{A}|=(ad-bc)$$

$$\text{Adj } \mathbf{A}=\text{transpose of } \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}=\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$$

Now

$$A_{11}=(-1)^{1+1}\, d=d,\ A_{12}=(-1)^{1+2}\, c=-c$$

$$A_{21}=(-1)^{2+1}\, b=-b,\ A_{22}=(-1)^{2+2}\, a=a$$

$\therefore$ $$\text{Adj } \mathbf{A}=\begin{pmatrix} d & -b \\ -c, & a \end{pmatrix}$$

Hence $$\mathbf{A}^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Example 27. *Compute the inverse of the matrix :*

$$\begin{pmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{pmatrix}$$

Solution We know $\mathbf{A}^{-1}=\frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|}$

$$|\mathbf{A}|=\begin{vmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{vmatrix}$$

$$=1\ (4+5)-0\ (-4-15)-4\ (2-6)=25$$

$$\text{Adj } \mathbf{A}=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}'=\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

The co-factors of the elements of **A** are

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix}=9,\ A_{12}=(-1)^{1+2}\begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix}=19$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix} -2 & 2 \\ 3 & -1 \end{vmatrix}=-4,\ A_{21}=(-1)^{2+1}\begin{vmatrix} 0 & -4 \\ -1 & 2 \end{vmatrix}=4$$

$A_{22}=14,\ A_{23}=1,\ A_{31}=8,\ A_{32}=3,\ A_{33}=2$

$$\text{Adj } \mathbf{A}=\begin{pmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{25}\begin{bmatrix} & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}=\begin{bmatrix} \frac{9}{25} & \frac{4}{25} & \frac{8}{25} \\ \frac{19}{25} & \frac{14}{25} & \frac{3}{25} \\ -\frac{4}{25} & \frac{1}{25} & \frac{2}{25} \end{bmatrix}$$

Remark. The students are advised to verify that

$$\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}_3$$

EXERCISE (VII)

1. $\mathbf{B}=\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{C}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Find **X** if **BX**=**C**.

2. If $\mathbf{A}=\begin{pmatrix} -1 & -5 \\ -2 & 3 \end{pmatrix}$, then show that $\mathbf{A}^{-1}=-\frac{1}{13}\begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}$

3. (*a*) Find the inverse of

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

(*b*) Show that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}=\begin{pmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{pmatrix}^{-1}$$

[**Hint.** Let $\mathbf{A}=\begin{pmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{pmatrix}$

Then show that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{A}=\begin{pmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{AA}^{-1}=\begin{pmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{pmatrix}\mathbf{A}^{-1}$$

Hence the result. $(\because\ \mathbf{AA}^{-1}=\mathbf{I})$]

4. $\mathbf{A}=\begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}$. Find matrix **B** if **AB** equals

(*i*) $\begin{pmatrix} 22 & 6 \\ 11 & 3 \end{pmatrix}$, (*ii*) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, (*iii*) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

5. Find matrix **B** if $\mathbf{B}^2$ equals

(*i*) $\begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$, (*ii*) $\begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$

6. Compute the adjoint and inverse of the matrices :

$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, (*ii*) $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

7. Verify that $\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}_3$

$$\text{If } \mathbf{A}=\begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}, \text{ then } \mathbf{A}^{-1} \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{3} \\ -\frac{14}{15} & \frac{8}{15} & \frac{-1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$$

8. If $\mathbf{A}=\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Show that

$$\mathbf{A}^{-1}=\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Find the reciprocal of the matrix

$$\mathbf{S}=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and show that the transform of the matrix

$$\mathbf{A}=\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$$

by **S**, *i.e.*, $\mathbf{SAS}^{-1}$ is a diagonal matrix $\begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{pmatrix}$

ANSWERS

1. $\mathbf{X}=\begin{pmatrix} 5/3 \\ 7/3 \end{pmatrix}$ 4. (*i*) $\begin{pmatrix} 33 & 9 \\ -110 & -30 \end{pmatrix}$ (*ii*) $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$

(*iii*) $\begin{pmatrix} -4 & 2 \\ 14 & -8 \end{pmatrix}$ 5. (*i*) $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$

(*ii*) $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ or $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$

6 (*i*) $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$ (*ii*) $\frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$

20·23. SIMUUTANEOUS FQUATIONS

Suppose we are given data on prices in (Rs. per kg.) of wheat and rice in the months of August and Sept.

	Wheat	*Rice*
August	3	2
Sept.	4	3

The family can spend Rs. 80 and Rs. 90 in August and Sept. respectively on wheat and rice. Now if the family wants to purchase the same combination of wheat and rice in August and Sept., the question is "how much wheat and how much rice it can buy in each month?"

Assuming they spent x kg. of wheat and y kg. of rice in each month. Then the amount spent are

$$3x+2y \quad \text{in August}$$

and

$$4x+3y \quad \text{in Sept.}$$

Since the family can spent Rs. 80 in August and Rs. 90 in Sept., we must have

$$\left.\begin{array}{l} 3x+2y=80 \\ 4x+3y=90 \end{array}\right\} \qquad \ldots(*)$$

Solving these equations for x and y, we get the required combination. The given data on prices can be written in the matrix form as

$$\mathbf{A}=\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \text{ the price matrix}$$

The purchase of the family may be expressed as

$$\mathbf{X}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ the required matrix.}$$

Then $\mathbf{AX}=\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 3x_1+2x_2 \\ 4x_1+3x_2 \end{pmatrix}$

Writing $\mathbf{B}=\begin{pmatrix} 80 \\ 90 \end{pmatrix}$, the equations (*) can now be written as

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x^2 \end{pmatrix}=\begin{pmatrix} 80 \\ 90 \end{pmatrix}$$

$$\mathbf{AX}=\mathbf{B}$$

In general, the two simultaneous equations in the two variables x_1 and x_2 are

$$a_{11}x_1+a_{12}x_2=b_1$$
$$a_{21}x_1+a_{22}x_2=b_2$$

can be written in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or

$$\mathbf{AX}=\mathbf{B} \qquad (*)$$

Similarly the three simultaneous equations

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$$
$$a_{31}x_1+a_{32}x_2+a_{33}x_3 \quad b_3$$

can be written in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\Rightarrow$ $$\mathbf{AX}=\mathbf{B}$$

where $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $\mathbf{X}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Since $|\mathbf{A}| \neq 0$, $\mathbf{A}^{-1}$ exists

Multiply (*) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\,\mathbf{AX}=\mathbf{A}^{-1}\,\mathbf{B}, \text{ i.e.,}$$

$\Rightarrow$ $$\mathbf{IX}=\mathbf{A}^{-1}\,\mathbf{B}$$

$\Rightarrow$ $$\mathbf{X}=\mathbf{A}^{-1}\,\mathbf{B}$$

Remarks. By elementary algebra, we can conveniently express $x_1, x_2 .., x_n$ in terms of $b_1, b_2, \ldots$, then the co-efficient matrix of this latter system is the inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$.

Illustration 1.

$$x+2y-z=5$$
$$3x-y+2z=9$$
$$5x+3y+4z=15$$

is equivalent to

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 5 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 9 \\ 15 \end{pmatrix}$$

2. $$\begin{pmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

gives the equation

$$\begin{pmatrix} 3x-y+5z \\ 5x+3y-z \\ -x+5y+3z \end{pmatrix}=\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

From this, we get the simultaneous equations as

$$3x-y+5z=4$$
$$5x+3y-z=-3$$
$$-x+5y+3z=2$$

Example 28. *Solve completely the following equations :*

$$2x-3y=3$$

and

$$4x-y=11$$

using matrices.

Solution. The above equations can be written in the matrix orm as

$$\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3 \\ 11 \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 11 \end{pmatrix} \qquad \ldots(1)$$

Now $\qquad A^{-1}=\dfrac{\text{Adj } A}{|A|}$

$$|A|=\begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix}=(-2+12)=10$$

$$\text{Adj } A=\text{transpose of }\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}=\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$$

$$A_{11}=(-1)^{1+1}(-1)=-1, \quad A_{12}=(-1)^{1+2}\,4=-4$$
$$A_{21}=(-1)^{2+1}(-3)=3, \qquad A_{22}=(-1)^{2+2}\,2=2$$

$$\therefore \qquad \text{Adj } A=\begin{pmatrix} -1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$A^{-1}=\frac{1}{10}\begin{pmatrix} -1 & 3 \\ -4 & 2 \end{pmatrix}=\begin{pmatrix} -\frac{1}{10} & \frac{3}{10} \\ -\frac{4}{10} & \frac{2}{10} \end{pmatrix}$$

From (*), we get

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{bmatrix} -\frac{1}{10} & \frac{3}{10} \\ -\frac{4}{10} & \frac{2}{10} \end{bmatrix}\begin{pmatrix} 3 \\ 11 \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \qquad x=3,\ y=1$$

Example 29. *Solve the following equations :*

$$5x-6y+4z=15$$
$$7x+4y-3z=19$$
$$2x+y+6z=46$$

Solution. The above system in the matrix notation is

$$\begin{pmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 15 \\ 19 \\ 46 \end{pmatrix}$$

$$\Rightarrow \qquad \mathbf{AX}=\mathbf{B}$$

$$\Rightarrow \qquad \mathbf{X}=\mathbf{A}^{-1}\mathbf{B} \qquad \ldots(*)$$

Now $\mathbf{A}^{-1}=\dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|}$, where $|\mathbf{A}|=\begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}=419$

Adj $\mathbf{A}$ = transpose of $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{23} & A_{33} & A_{33} \end{pmatrix}=\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$

where

$$A_{11}=(-1)^{1+1}\begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix}=24+3=27$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix}=-(42+6)=-48$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix}=(7-8)=-1$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix} -6 & 4 \\ 1 & 6 \end{vmatrix}=-(-36-4)=40$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix}=(30-8)=22$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix} 5 & -6 \\ 1 & 1 \end{vmatrix}=-(5+12)=-17$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix} -6 & 4 \\ 4 & -3 \end{vmatrix}=(18-16)=2$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}5 & 4\\ 7 & -3\end{vmatrix}=-(-15-28)=43$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}5 & -6\\ 7 & 4\end{vmatrix}=(20 \quad 42)=62$$

$$\therefore \quad \text{Adj } \mathbf{A}=\begin{pmatrix}27 & 40 & 2\\ -48 & 22 & 43\\ -1 & -17 & 62\end{pmatrix}$$

$$\therefore \quad \mathbf{A} \quad \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|}=\frac{1}{419}\begin{bmatrix}27 & 40 & 2\\ -48 & 22 & 43\\ -1 & -17 & 62\end{bmatrix}$$

From (+), ve get

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{419}\begin{pmatrix}27 & 40 & 2\\ -48 & 22 & 43\\ -1 & -17 & 63\end{pmatrix}\begin{pmatrix}15\\ 19\\ 46\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{419}\begin{pmatrix}27.15+40.19+2.46\\ -48.15+22.19+43.46\\ -1.15-17.19+62.46\end{pmatrix}$$

$$=\frac{1}{419}\begin{pmatrix}1257\\ 1676\\ 2514\end{pmatrix}=\begin{pmatrix}3\\ 4\\ 6\end{pmatrix}$$

$$\Rightarrow \quad x=3,\ y=4 \text{ and } z=6$$

Example 30. *The daily cost of operating a hospital C is a linear function of the number of in-patients I, and out-patients, P plus a fixed cost a, i.e.,*

$$C=a+bP+dI$$

Given the following data from 3 days, find the value of a, b and d by setting up a linear system of equations and using the matrix inverse.

Day	*Cost in Rs.*	*No. of Inpatients, I*	*No. of out-patients, P*
1	*6,950*	*40*	*10*
2	*6,725*	*35*	*9*
3	*7,100*	*40*	*12*

Solution. Substituting the tabulated values in $C=a+bP+dI$, we obtain the following set of simultaneous linear equations

$$a+10b+40d=6{,}950$$
$$a+\ 9b+35d=6{,}725$$
$$a+12b+40d=7{,}100$$

The above system in the matrix notation is

$$\begin{bmatrix}1 & 10 & 40\\ 1 & 9 & 35\\ 1 & 12 & 40\end{bmatrix}\times\begin{bmatrix}a\\ b\\ d\end{bmatrix}=\begin{bmatrix}6{,}950\\ 6{,}725\\ 7{,}100\end{bmatrix}$$

$$\Rightarrow\quad \begin{bmatrix}a\\ b\\ d\end{bmatrix}=\begin{bmatrix}1 & 10 & 40\\ 1 & 9 & 35\\ 1 & 12 & 40\end{bmatrix}^{-1}\times\begin{bmatrix}6{,}950\\ 6{,}725\\ 7{,}100\end{bmatrix} \qquad \ldots(*)$$

Now $\mathbf{A}^{-1}=\dfrac{\text{Adj }\mathbf{A}}{|\mathbf{A}|}$ where $|A|=\begin{vmatrix}1 & 10 & 40\\ 1 & 9 & 35\\ 1 & 12 & 40\end{vmatrix}=-10$

and $\quad \text{Adj }\mathbf{A}=\begin{bmatrix}60 & -80 & 10\\ -5 & 0 & -5\\ -3 & 2 & 1\end{bmatrix}$

$\therefore$ From (*), we get

$$\begin{bmatrix}a\\ b\\ d\end{bmatrix}=-\frac{1}{10}\begin{bmatrix}60 & -80 & 10\\ 5 & 0 & -5\\ -3 & 2 & 1\end{bmatrix}\times\begin{bmatrix}6{,}950\\ 6{,}725\\ 7{,}100\end{bmatrix}$$

$$=-\frac{1}{10}\begin{bmatrix}60\times 6950-80\times 6725+7100\times 10\\ 5\times 6950-\ 0\times 6725-5\times 7100\\ -\ 3\times 6950+\ 2\times 6725+1\times 7100\end{bmatrix}$$

$$=-\frac{1}{10}\begin{bmatrix}-50{,}000\\ -\ 750\\ -\ 300\end{bmatrix}=\begin{bmatrix}5000\\ 75\\ 30\end{bmatrix}$$

Hence $\quad a=5000, \quad b=75$ and $d=30$

Example 31. *Show that the equations*

$$2x+6y+11=0$$
$$6x+20y-6z-3=0$$
$$6y-18z+1=0$$

are not consistent.

Solution. The above system of equations may be written as

$$\begin{pmatrix}2 & 6 & 0\\ 6 & 20 & -6\\ 0 & 6 & -18\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-11\\ 3\\ -1\end{pmatrix}$$

$$\Rightarrow \qquad \mathbf{AX}=\mathbf{B}$$

$$\Rightarrow \qquad \mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$$

But $\mathbf{A}^{-1}$ does not exist, since

$$|\mathbf{A}|=\begin{vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{vmatrix}=0$$

Hence the equations are inconsistent.

20·24. GAUSS ELIMINATION METHOD

This method is also called the pivotal reduction method. Taking three equations with three unknowns an attempt is made to reduce them to the following form :

$$\begin{aligned} x+b_1y+c_1z&=d_1 \\ y+c_2z&=d_2 \\ z&=d_3 \end{aligned}$$

The following example makes the point clear.

Example 32. *Find the solution of the following equations by means of an inverse matrix (Gauss Elimination Method).*

$$x-2y+3z=4$$

$$2x+y-3z=5$$

$$-x+y+2z=3$$

Solution. Let us have an extended matrix for the L.H.S. of the three equations and then perform elementary row operations to get the inverse of the matrix on the R.H.S. of it as follows :

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad [\text{since } \mathbf{AI}=\mathbf{A}]$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 1 & 0 \\ 0 & -1 & 5 & 1 & 0 & 1 \end{array}\right] R_1+R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 5 & -9 & -2 & 1 & 0 \\ 0 & -1 & 5 & 1 & 0 & 1 \end{array}\right] -2R_1+R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{9}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & -1 & 5 & 1 & 0 & 1 \end{array}\right] \frac{1}{5}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{9}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{16}{5} & \frac{3}{5} & \frac{1}{5} & 1 \end{array}\right] R_2+R_3$$

We have now got the L.H.S. in the upper triangular form. We can continue the operations to get it as an identity matrix and R.H.S. as an inverse matrix of the given matrix.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{9}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{16} & \frac{1}{16} & \frac{5}{16} \end{array}\right] \frac{5}{16} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{7}{16} & -\frac{3}{16} & -\frac{15}{16} \\ 0 & 1 & -\frac{9}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{16} & \frac{1}{16} & \frac{5}{16} \end{array}\right] R_1-3R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{7}{16} & -\frac{3}{16} & -\frac{15}{16} \\ 0 & 1 & 0 & -\frac{1}{16} & \frac{5}{16} & \frac{9}{16} \\ 0 & 0 & 1 & \frac{3}{16} & \frac{1}{16} & \frac{5}{16} \end{array}\right] R_2+\frac{9}{5} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{16} & \frac{7}{16} & \frac{3}{16} \\ 0 & 1 & 0 & -\frac{1}{16} & \frac{5}{16} & \frac{9}{16} \\ 0 & 0 & 1 & \frac{3}{16} & \frac{1}{16} & \frac{5}{16} \end{array}\right] R_1+2R_2$$

The inverse matrix can help in finding a solution of the set of equations as follows :

$$X=A^{-1}B=\begin{bmatrix} \frac{5}{16} & \frac{7}{16} & \frac{3}{16} \\ -\frac{1}{16} & \frac{5}{16} & \frac{9}{16} \\ \frac{3}{16} & \frac{1}{16} & \frac{5}{16} \end{bmatrix}\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}=\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

$\therefore$ $x=4$, $y=3$ and $z=2$

EXERCISE (VII)

1. If $A=\begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$, $X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $B=\begin{pmatrix} 7 \\ 1 \end{pmatrix}$

and $AX=B$, find x_1 and x_2.

2. Solve simultaneously for matrices **X** and **Y** from the equations

$$2(X-Y)+\tfrac{1}{2}(3X+2Y)=\begin{pmatrix} -2 & 5 \\ -3 & 6 \\ 0 & 2 \end{pmatrix}$$

and $3(X+2Y)+2(2X+3Y)+\begin{pmatrix} -4 & 2 \\ 5 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}=0$

where 0 denotes the 3×2 zero matrix.

3. Solve following system of equations using matrix method.

(*a*)
$$2x-3y+5z=11$$
$$5x+2y-7z=-12$$
$$-4x+3y+z=5$$

(*b*)
$$3x_1+x_2+x_3=1$$
$$2x_1+2x_3=0$$
$$5x_1+x_2+2x_3=2$$

(*c*)
$$x+y+z=7$$
$$x+2y+3z=16$$
$$x+3y+4z=22$$

4. Solve the equations

$$x+y+z=\alpha$$
$$x+2y+2z=\beta$$
$$2x+3y+8z=\gamma$$

by evaluating the inverse of the co-efficient matrix on the left

5. Solve the system of equations

$$x+y+z=6$$
$$x-y+2z=5$$
$$3x+y+z=8$$
$$2x-2y+3z=7$$

6. A trucking company owns three types of trucks X, Y, Z which are equipped to carry three different types of machines per load as shown below :

	Trucks		
	Type X	*Type Y*	*Type Z*
Machine I	2	3	4
Machine II	1	1	2
Machine III	3	2	1

How many trucks of each type should be used to carry exactly 29 of type I machines, 13 of type II machines and 16 of type III machines ? Assume that each truck is fully loaded ?

7. The prices of three commodities X, Y and Z are as x, y and z per unit respectively. A purchases 4 units of z and sells 3 units of x and 5 units of y. B purchases 3 units of y and sells 2 units of x and 1 unit of z, C purchases 1 unit of x and sells 4 units of y and 6 units of z In the process, A, B and C earn Rs. 6000, 5000 and 13,000 respectively. Find the prices per unit of the three commodities.

[**Hint.** The above data can be written in the form of simultaneous equations as

$$-3x-5y+4z=6000$$
$$-2x+3y-z=5000$$
$$x-4y-6z=13,000$$

and the equations can be written in the matrix form as

$$\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6000 \\ 5000 \\ 13000 \end{bmatrix}$$

8. Solve the matrix equation $AX=B$,

where $$A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix},\quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},\quad B=\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

[*C.A., November, 1991*]

9. A manufacturer produces two types of products X and Y. Each product is first processed in a machine M_1 and then sent to another machine M_2 for finishing. Each unit of X requires 20 minutes' time on M_1 and 10 minutes' time on M_2 whereas each unit of Y requires 10 minutes' time on M_1 and 20 minutes' time in M_2. The total time available on each machine is 600 minutes. Calculate the number of units of two types of products produced by constructing a matrix equation of the form $AX=B$ and then solving it by the matrix inversion method. [*C.A., May, 1991*]

10. Consider the following National Income Model :—

$$Y=C+I+G$$
$$C=a+b(Y-T)$$
$$T=d+tY$$

where Y=National income, C=consumption expenditure, T=tax collection, t=income-tax rate.

Write down the above system of equations in matrix form and solve for the endogenous variables Y, C and T. [*D. U. B.A. (Hons.) Eco. 1991*]

11. Calculate PQ and QP where

$$P=\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix},\quad Q=\begin{bmatrix} 1/3 & -1/6 \\ 0 & 1/2 \end{bmatrix}.$$

Also verify that : $(PQ)^{-1}=Q^{-1}, P^{-1}$

[*I.C.W.A., June, 1991*]

12. Show that the matrix

$$X=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

satisfies the equation $X^2-5X-5I=0$, where I is the unit matrix of order 3. Hence find X^{-1}. [*I.C.W.A., June 1990*]

ANSWERS

1. $x_1=-1\cdot7,\ x_2=2\cdot6$

2. $X=\frac{1}{49}\begin{bmatrix} -20 & 58 \\ -41 & 71\cdot5 \\ 0 & 25 \end{bmatrix},\quad Y=\frac{1}{15}\begin{bmatrix} 8 & -12 \\ 1 & -12\cdot5 \\ 0 & -3 \end{bmatrix}$

3. (*a*) $x=1, y=2, z=3$ (*b*) $x_1=1, x_2=-1, x_3=-1$
(*c*) $x=1, y=3, z=3$.

4. $x=\frac{1}{4}(7\alpha-5\beta+\gamma),\ y=\frac{1}{2}(3\beta-\alpha-\gamma),\ z=\frac{1}{4}(\gamma-\alpha-\beta)$.

5. $x=1, y=2, z=3$. 6. 2 in type X, 3 in type Y, 5 in type Z.

7. $x=3{,}000$; $y=1{,}000$ and $z=2{,}000$

8. $x=1, y=4, z=4$ 9. $x=20, y=20$

10. $$\begin{bmatrix} 1 & -1 & 0 \\ b & -1 & -b \\ t & 0 & -1 \end{bmatrix}\begin{bmatrix} Y \\ C \\ T \end{bmatrix}=\begin{bmatrix} I+G \\ -a \\ -d \end{bmatrix}$$

20·25. RANK OF A MATRIX

A non-zero matrix is said to have a rank say r if at least one of its minor (r-square) is different from zero while $(r+1)$ square minor, if any, is zero. For example the rank of the following matrix is 2 because with

$$\mathbf{A}=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \text{ its minor } \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}=-1 \text{ which is } \neq 0$$

while $|\mathbf{A}|=0$

It should be remembered that if $|\mathbf{A}|\neq0$ the above n-square matrix would have been called a non-singular matrix where its rank (r) is equal to its order n. But, since $|\mathbf{A}|$ above is equal to zero it is a singular matrix.

Illustrations

1. The rank of the matrix

$$A=\begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 5 \end{bmatrix} \text{ is 2 since } \begin{vmatrix} 1 & 2 \\ -4 & 0 \end{vmatrix} \neq 0$$

and there in no minor of order three.

2. $$B=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 8 \end{bmatrix} \text{ is 2}$$

Since while $|B|=0$, $\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} \neq 0$

3. $$C=\begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix} \text{ is 1 since } |C|=0$$

while each of the nine 2-square minors are equal to zero even when every element is not zero.

The following elementary transformations on a matrix do not change either its order or its rank. These are :

1. The interchange of *i*th and *j*th rows and *i*th and jth columns.

2. The multiplication of every element of the *i*th row or jth column by a non-zero scalar.

3. The addition to the elements of the *i*th row of *k*, a scalar times the corresponding elements of the jth row. Similarly, for the *i*th column.

EXERCISE (VIII)

1. What do you understand by the term rank of a matrix ? Find out the rank of the following matrix :

$$A=\begin{bmatrix} 7 & -1 & 0 \\ 1 & 1 & 4 \\ 13 & -3 & -4 \end{bmatrix}$$

[*D. U., B.A.* (*Hons.*) *Eco.*, *1991*]

2. Find the value of *x* such that the rank of the following matrix is less than 3 :

$$\begin{bmatrix} 3 & 5 & 0 \\ 3 & x & 2 \\ 9 & -1 & 8 \end{bmatrix}$$

[*D. U.*, *B.A.* (*Hons.*) *Eco. 1990*]

[**Hint.** For the rank to be less than 3 :

$$\begin{bmatrix} 3 & 5 & 0 \\ 3 & x & 2 \\ 9 & -1 & 8 \end{bmatrix} = 0.]$$

3. Define the term 'rank of a matrix'. Is 'rank' defined only for square matrices ? What is the rank of the identity matrix $\mathbf{I}_3$?

[*D. U. B.A. (Hons.) Econ. 1988*]

4. Find the rank of the following matrices ;

(*i*) $\begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & 10 \end{bmatrix}$, (*ii*) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

ANSWERS

1\. 2. 2 $x=1$ 3. 3 4. (*i*) 2 (*ii*) 3