3

Boolean Algebra

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand :

- *Basic, derived properties of Boolean functions.*
- *Boolean multiplication and addition.*
- *Electrical switching systems and circuits with composite operations.*

3·0 INTRODUCTION

Boolean Algebra is a two-valued algebra, applied earlier to statements and sets which were either true or false and now to switches which are either closed or open, *i.e.*, ON or OFF respectively. George Boole developed this branch of mathematics in his book "An Investigation of the Laws of Thought" now known as *symbolic logic*. This provided the basic logic for operations on binary numbers (1 and 0). Since modern business machines are based on binary system, the symbolic logic of George Boole was found extremely useful and is being considered as the base of *Modern Mathematics*.

However, while symbolic logic was invented in the 19th century, it was used much later when in the 20th century Claude Shannon discovered he similarity of structures between it and telephone switching circuits. His paper "A Symbolic Analysis of Relay and Switching Circuits" made an important contribution to the use of Boolean Algebra towards the designing of modern business machines based on binary numbers.

There are three basic operations in the Boolean Algebra AND, OR and NOT. These were symbolised by $\wedge$, and $\vee$ and $\sim$ respectively in the case of logical statements and by $\cap$, $\cup$ and $[\ ]^c$ respectively in case of the theory of sets. In this chapter, the more common symbolic plus $+$, dot '.' and prime (′) would be used for the three operations respectively. The similarity would become obvious in the way the present chapter would synthesise and generalise what we have studied earlier and apply it to the end purpose of the designing of the electric circuits. Given below is a table showing the operations of symbolic logic to the three more or less corresponding systems.

		Logical statements	*Theory of sets*	*Electric circuits*
1.	Elements	$[p, q, r, s]$	$\{x, y, \ \}$ $\{1, 2, 3, \ldots, n\}$	$\{x, y, \ldots\}$ $\{a, b, c, \ldots\}$
2.	Tautology	T	U	1
3.	Fallacy	F	ϕ	0
4.	Operator 'AND'	$\wedge$	$\cap$	'.'
5.	Operator 'OR'	$\vee$	$\cup$	$+$
6.	Operator 'NOT'.	$\sim$	A^c	a'
7.	Implication	$p \rightarrow q$	mapping	$a'+b$
8.	Equivalence	$p \leftrightarrow q$	one-to-one mapping	$a'b'+ba$

3·1. BASIC PROPERTIES

A set of elements in the Boolean system indicated by $B=\{a, b, c, \ldots\}$ has two binary operations AND (.), OR (+) and one unary operator NOT (′). The four basic properties of the system are :

I. Both the operations are commutative,

(*i*) $a+b=b+a$

(*ii*) $a \cdot b=b \cdot a$

II. Identity elements are there in both the operations,

(*i*) $a+0=a$

(*ii*) $a \cdot 1=a$

III. Each operation is distributive with respect to the other.

(*i*) $a+(b \cdot c)=(a+b) \cdot (a+c)$

(*ii*) $a \cdot (b+c)=(a \cdot b)+(a \cdot c)$

IV. There exists a' for each $a \in B$ such that

(*i*) $a+a'=1$

(*ii*) $a \cdot a'=0$

Example 1. *Given the set $\{0, 1\}$ of two elements, where the elemnts have been denoted by the symbols 0 and 1 as is customary and they have no relation with the numbers 0 and 1 used in arithmetic. Let the twobienary operations be denoted by + known as logical addition and (.) known as logical multiplication which have no relation to the operations of addition and multiplication used in arithmetic. In tables 1 and 2 are given the* *logical sums and logical products, i.e., the results of the above operations on the elements of the set.*

Table 1

+	0	1
0	0	1
1	1	1

Table 2

(.)	0	1
0	0	0
1	0	1

Prove that the set (0, 1) with the operations defined in the tables is Boolean.

Solution. Both the operations are Boolean because of the following properties :

1. *Closure.* Tables 1 and 2 ensure the closure property for both the operations + and (.).

2. *Commutative.* Since there is symmetry about the leading diagonals, both the operations + and (.) are commutative. Also,

(*i*) $0+1=1+0=1$

(*ii*) $0 \,.\, 1=1 \,.\, 0=0$

3. *Associative.* These operations are associative, *e.g.*,

$(1+0)+1=1+1=1$ and $1+(0+1)=1+1=1$

So that

$(1+0)+1=1+(0+1)$ and $(1 \,.\, 0) \,.\, 1=0 \,.\, 1=0$

and $1 \,.\, (0 \,.\, 1)=1 \,.\, 0=0$

So that

$(1.0).1=1.(0.1)$

The reader is advised to verify this property in the remaining cases.

4. *Distributive.* Each operation is distributive with respect to the other. For example

(*i*) $1+(0 \,.\, 1)=(1+0) \,.\, (1+1)=1$

so that + is distributive with respect to (.) in this case:

(*ii*) $1 \,.\, (0+1)=(1 \,.\, 0)+(1 \,.\, 1)=1$

so that '.' is distributive with respect to + in this case.

The reader is advised to verify this property in the remaining cases.

5. *Idempotent.*

$0+0=0,\ 1+1=1,\ 0 \,.\, 0=0,\ 1 \,.\, 1=1$

6. *Identity elements.* We have

(*i*) $0+0=0,\ 1+0=1$

so that 0 is the identity element for +.

(*ii*) $0 \,.\, 1=0,\ 1 \,.\, 1=1$

so that 1 is the identity element for (.).

7. *Complementation.* We note that

$$0'=1 \text{ and } 1'=0$$

Since $0+0'=0+1=1$

$1+1'=1+0=1$

and $0 \,.\, 0'=0 \,.\, 1=0$

$1 \,.\, 1'=1 \,.\, 0=0$

8. *Involution.* In view of property (7)

$$(0')'=(1)'=0 \text{ and } (1')'=(0)'=1$$

9. *De Morgan's Laws.* We have

$$\begin{cases}(0+1)'=1'=0 \text{ and } 0' \,.\, 1'=1 \,.\, 0=0\\(0 \,.\, 1)'=0 \;=1 \text{ and } 0'+1'=1+0=1\end{cases}$$

$$\begin{cases}(0+0)'=0'=1 \text{ and } 0' \,.\, 0'=1 \,.\, 1=1\\(0 \,.\, 0)'=0'=1 \text{ and } 0'+0 \;=1+1=1\end{cases}$$

$$\begin{cases}(1+1)'=1'=0 \text{ and } 1' \,.\, 1'=0 \,.\, 0=0\\(1 \,.\, 1)'=1'=0 \text{ and } 1'+1'=0+0=0\end{cases}$$

so that De-Morgan's laws hold.

10. *Absorption Laws.* These laws hold good, *e.g.*,

$$0+(0 \,.\, 1)=0+0=0,\ 1+(1 \,.\, 0)=1+0=1$$

$$0 \,.\, (0+1)=0 \,.\, 1=0,\ 1 \,.\, (1+0)=1 \,.\, 1=1$$

The reader is advised to verify these laws for the remaining cases.

11. We have

$$(0'+1')'+(0'+1)'=(1+0)'+(1+1)'=1'+1'=0+0=0. \text{ etc.}$$

In view of these properties of the set {0, 1} and the definition of + and ' ' as given by the tables 1 and 2, we conclude that it is Boolean.

Example 2. *State if the set* {*a, b, c, d*} *with the operations defined in the tables is Boolean* :

(*i*)

+	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*d*
b	*b*	*b*	*d*	*d*
c	*c*	*d*	*c*	*d*
d	*d*	*d*	*d*	*d*

(*ii*)

	a	*b*	*c*	*d*
a	*a*	*a*	*a*	*a*
b	*a*	*b*	*a*	*b*
c	*a*	*a*	*c*	*c*
d	*a*	*b*	*c*	*d*

Solution. Identity element in (*i*) is '*a*' such that

$$a+a=a,\ b+a=b,\ c+a=c,\ d+a=d$$

[see first row and first column of table (*i*).

(*ii*) '*d*' is the identity element in (*ii*) such that

$$a \,.\, d=a,\ b \,.\, d=b,\ c \,.\, d=c,\ d \,.\, d=d$$

Commutative : in (*i*) $a+b=b+a$ etc. and

(*ii*) $a \cdot b=b \cdot a$ etc.

Inverse : in (*i*) $b+c=d$

(*ii*) $b \cdot c=a$ [where $c=b'$ or the inverse of b]

Distributive : Each operation distributes over the other.

$$b+(c \cdot d)=d=(b+c) \cdot (b+d)$$

$$b \cdot (c+d)=b=(b \cdot c)+(b \cdot d)$$

3·2. DERIVED PROPERTIES

As a result of the properties given above we find that the laws applicable to the algebra of sets are also valid here. These are restated here in the context of Boolean expressions :

I. *Complement Law* : $a+a'=1$

$a \cdot a'=0$

Also $(a')'=a$

$0'=1$

$1'=0$

Let us prove that $(a')'=a$, for $a \in B$

$$\begin{aligned}(a')' &= 1 \cdot (a')' \\ &=(a+a') \cdot (a')' \\ &=a \cdot (a')'+a' \cdot (a')' \\ &=a \cdot (a')'+0 \\ &=a. (a')'+a \cdot a' \\ &=a \cdot [(a')'+a'] \\ &=a \cdot 1 \\ &=a\end{aligned}$$

II. *Identity Law* : $a+0=a$

$a+1=1$

$1 \cdot 1=1$ or $a \cdot 0'=a$

$a \cdot 0=0$ or $a \cdot 1'=0$

III. *Idempotent law* : $a+a=a$

$a \cdot a=a$

This can be proved by the application of the above laws. Let there by any $a \in B$, suppose we want to show that

(*i*) $a+a=a$

Let'us have

$$\begin{aligned}a &= a+0 \\ &=a+aa' \\ &=(a+a)(a+a') \quad \text{[distributive law]} \\ &=(a+a) \cdot 1 \\ &=a+a\end{aligned}$$

Now, we want to show

$(ii)\ a\ .\ a=a$

Let us have
$$\begin{aligned} a &= a\ .\ 1 \\ &= a(a+a') \\ &= aa+aa' \quad \text{[distributive law]} \\ &= aa+0 \\ &= aa \\ &= a\ .\ a \end{aligned}$$

IV. *Associativity* : $(i)\ (a+b)+c=a+(b+c)$

$(ii)\ (a\ .\ b)\ .\ c=a\ .\ (b\ .\ c)$

V. *Commutativity* : Stated with the property I.

VI. *Distributivity* : Stated with the property II.

VII. *De Morgan's law* :

$(i)\ (a+b)'=a'\ .\ b'$

$(ii)\ (a\ .\ b)'=a'+b'$

$(iii)\ a(b+c)'=(ab')\ .\ (ac')$

$(iv)\ a(b\ .\ c)'=(ab')+(ac')$

VIII. *Duality* : The dual of

$a\ .\ (b+c)=(a\ .\ b)+(a\ .\ c)$ is $a+(b\ .\ c)=(a+b)\ .\ (a+c)$

and that of $(a\ .\ 0)+(1\ .\ a)=a$ is $(a+1)\ .\ (0+a)=a$

Let us take a few examples with the old symbols to reveal the identity of structure :

Example 3. *Let $x \in B$, prove that*

$(i)\ x \cup x=x$ *and* $(ii)\ x \cap x=x$

Solution. We prove these by taking the L.H.S. of the identity.

(i) Since
$$\begin{aligned} x \cup x &= (x \cup x) \cap 1 \ \text{Identity} \\ &= (x \cup x) \cap (x \cup x').\ \text{Complement} \\ &= x \cup (x \cap x'),\ \text{distributive} \\ &= x \cup 0\ \text{Complement} \\ &= x \end{aligned}$$

(ii) Since
$$\begin{aligned} x \cap x &= (x \cap x) \cup 0 \\ &= (x \cap x) \cup (x \cap x') \\ &= x \cap (x \cup x') \\ &= x \cap 1 \\ &= x \end{aligned}$$

Example 4. *Let $y \in B$, prove that*

$(i)\ y \cap 0=0$

$(ii)\ y \cup 1=1$

Solution. (*i*) $y \cap 0 = (y \cap 0) \cup 0$
$= (y \cap 0) \cup (y \cap y')$
$= y \cap (0 \cup y')$
$= (y \cap y')$
$= 0$

(*ii*) Left to the reader as an exercise.

Example 5. *Let* $x, y \in B$, *prove that*

(*i*) $x \cup (x \cap y) = x$

(*ii*) $x \cap (x \cup y) = x$

Solution. (*i*) Since $x \cup (x \cap y) = (x \cap 1) \cup (x \cap y) = x \cap (1 \cup y)$
$= x \cap 1$
$= x$

(*ii*) Left to the reader as an exercise.

Example 6. *Prove that* a' *is a unique element.*

Solution. Let us suppose that a' is not a unique element ; there exist two elements a' and a'' such that

$$a' \,.\, a = 0, \quad a + a' = 1$$
$$a'' \,.\, a = 0, \quad a + a'' = 1$$

Consider $a' = 1 \,.\, a' = (a + a'') \,.\, a' = a \,.\, a' + a'' \,.\, a'$
$= 0 + a'' \,.\, a'$
$= a \,.\, a'' + a'' \,.\, a'$
$= a'' \,.\, a + a'' \,.\, a'$
$= a'' \,.\, (a + a')$
$= a'' \,.\, 1 = a''$

Hence a' is a unique element.

Example 7. *Prove that*

(*i*) $(a \,.\, b) \,.\, c = a.\,(b \,.\, c)$,

(*ii*) $(a + b) + c = a + (b + c)$.

Solution. (*i*) Let

$(a \,.\, b) \,.\, c = x$ and $a \,.\, (b \,.\, c) = y$

$a + x = a + [(a \,.\, b) \,.\, c]$
$= [a + (a \,.\, b)] \,.\, (a + c)$
$= (a + a \,.\, b) \,.\, (a + c)$
$= [a \,.\, (1 + b)] \,.\, (a + c) = [a \,.\, (b' + b + b)] \,.\, (a + c)$
$= [a \,.\, (b' + b)] \,.\, (a + c) = [a \,.\, 1] \,.\, (a + c)$
$= a \,.\, (a + c) = a \,.\, a + a \,.\, c$
$= a + a \,.\, c = a \,.\, (1 + c) = a \,.\, (c' + c + c)$
$= a \,.\, (c' + c) = a \,.\, 1 = a$

$a + x = a$

Similarly $a+y=a$

Now $a'+x=a'+[(a\,.\,b)\,.\,c]=[a'+(a\,.\,b)]\,.\,(a'+c)$

$=[(a'+a)\,.\,(a'+b)]\,.\,(a'+c)=[1\,.\,(a'+b)]\,.\,(a'+c$

$=(a'+b)\,.\,(a'+c)=a'+b\,.\,c$

$=1\,.\,(a'+b\,.\,c)=(a'+a)\,.\,(a'+b\,.\,c)$

$=a'+a\,.\,(b\,.\,c)$

$=a'+y$

Thus $(a+x)\,.\,(a'+x)=(a+y)\,.\,(a'+y)$

$aa'+x=a\,.\,a'+y$

$0+x=0+y$

$x=y$

$(a\,.\,b)\,.\,c=a\,.\,(b\,.\,c)$

(*ii*) Students are advised to do the proof independently on the same lines as shown in the first case.

Example 8. *Prove that* $(a\,.\,b)'=a'+b'$

Solution. To prove this, we have to show that

$(a\,.\,b)+(a'+b')=1$ and $(a\,.\,a)\,.\,(a'+b')=0$

Consider

$$(a\,.\,b)+(a'+b')=\{a+(a'+b')\}\{b+(a'+b')\}$$
$$=\{(a+a')+b'\}\{(b+a')+b'\}$$
$$=(1+b')\{(a'+b)+b'\}$$
$$=(1+b')\{a'+(b+b')\}$$
$$=(1+b')(a'+1)=1.1=1$$

and

$$(a\,.\,b)(a'+b')=(a\,.\,b)a'+(ab)b'$$
$$=(ba)a'+a(bb')$$
$$=b(aa')+a.0$$
$$=b\,.\,0+a\,.\,0$$
$$=0+0=0$$

$\therefore \quad (a\,.\,b)'=a'+b'$

Example 9. *Define Boolean Algebra and establish the following results ;*

(*i*) $a\,.\,(a+b)=a$

(*ii*) $(a\,.\,b)\,.\,c=a\,.\,(b\,.\,c)=a\,.\,b\,.\,c$

Solution. (*i*) $a\,.\,(a+b)=(a+b)\,.\,a$

$$=(a+b)\,.\,(a+0)$$
$$=a+b\,.\,0$$
$$=a+0$$
$$=a$$

$\therefore \quad a\,.\,(a+b)=a$

(*ii*) Let $\quad (a\,.\,b)\,.\,c=p,\ a\,.\,(b\,.\,c)=q$ and $a\,.\,b\,.\,c=r$

Then

$$a+p=a+[(a.\,b)\,.\,c]=[a\,.\,(a\,.\,b)][a+c]$$
$$=(a+a\,.\,b)\,.\,(a+c)=[a\,.\,(1+b)]\,.\,(a+c)$$
$$=[a\,.\,(b'+b+b)]\,.\,(a+c)=[a\,.\,(b'+b)]\,.\,(a+c)$$
$$=(a\,.\,1)\,.\,(a+c)=a\,.\,a+a\,.\,c$$
$$=a+a\,.\,c=a\,.\,(1+c)=a\,.\,(c'+c+c)$$
$$=a\,.\,(c'+c)=a\,.\,1=a$$
$$a+p=a$$

Similarly $\quad a+q=a$ and $a+r=a$

Also

$$a'+p=a'+[(a\,.\,b)\,.\,c]=[a'+(a\,.\,b)]\,.\,[a'+c]$$
$$=[(a'+a)\,.\,(a'+b)]\,.\,(a'+c)$$
$$=[1\,.\,(a'+b)]\,.\,(a'+c)=(a'+b)\,.\,(a'+c)$$
$$=a'+b\,.\,c=1\,.\,(a'+b\,.\,c)=(a'+a)\,.\,(a'+b\,.\,c)$$
$$=a'+[a\,.\,(b\,.\,c)]=a'+q$$

Thus $(a+p)\,.\,(a'+p)=(a+q)\,.\,(a'+q)$

$$a\,.\,a'+p=a\,.\,a'+q$$
$$0+p=0+q$$
$$\therefore \quad p=q \qquad \text{...(1)}$$
$$a'+q=a'+[a\,.\,(b\,.\,c)]=(a'+a)\,.\,[a'+b\,.\,c]$$
$$=(a'+a)\,.\,(a'+b\,.\,c)$$
$$=a'+a\,.\,b\,.\,c=a'+r$$
$$\therefore \quad a'+q=a'+r$$

Thus $(a+p)\,.\,(a'+p)=(a+r)\,.\,(a'+r)$

$$a\,.\,a'+p=a\,.\,a'+r$$
$$0+p=0+r$$
$$\therefore \quad p=r \qquad \text{...(2)}$$

$\therefore$ From (1) and (2), we have

$$p=q=r$$
$$(a\,.\,b)\,.\,c=a\,.\,(b\,.\,c)=a\,.\,b\,.\,c$$

Example 10. *Show that in a Boolean Algebra, B,*

$(a')'=a$ for all $a \in B$ (*C.A. Entrance June 1984*)

Solution. We have

$$\text{L.H.S.}=(a')'$$
$$=1\,.\,(a')'$$
$$=(a+a')\,.\,(a')'$$
$$=a\,.\,(a')'+a'\,.\,(a')'$$
$$=a\,.\,(a')'+0 \qquad [\because\ a\,.\,a'=0]$$

$= a \,.\, (a')' + a \,.\, (a')$

$= a \,.\, [(a')' + (a')]$

$= a \,.\, 1$ $\qquad [\because \; a + a' = 1]$

$= a$

$=$ R.H.S.

$\therefore \quad (a')' = a$ for all $a \in B$.

Example 11. *Prove that :*

$a + a \,.\, b = a.$

(C.A. Entrance December 1983)

Solution. We have

L.H.S. $= a + a \,.\, b$

$= a \,.\, 1 + a \,.\, b$

$= a \,.\, (1 + b)$

$= a \,.\, 1$

$= a$

$=$ R.H.S.

$\therefore \quad a + a \,.\, b = a.$

Example 12. *Show that* : $a' + ab = a' + b.$

(C.A. Intermediate November 1982)

Solution. We have

R.H.S. $= a' + b$

$= a' \,.\, 1 + b \,.\, 1$

$= a' \,.\, (a + 1) + b \,.\, (a + a')$

$= a' \,.\, a + a' \,.\, 1 + b \,.\, a + b \,.\, a'$

$= 0 + a' \, 1 + a \,.\, b + a' \,.\, b$

$= a' \,.\, 1 + a' \,.\, b + a \,.\, b$

$= a' \,.\, (1 + b) + a \,.\, b$

$= a' \,.\, 1 + a \,.\, b$

$= a' + ab$

$=$ L.H.S.

$\therefore \quad a' + ab = a' + b.$

Example 13. *Show that*

$pqr + pqr' + pq'r + p'qr = pq + qr + rp.$

Solution. We have

L.H.S. $= pqr + pqr' + pq'r + p'qr$

$= (pqr + pqr') + (pq'r + p'qr)$

$= pq(r + r') + p'qr)$

$= pq \,.\, 1 + pq'r + p'qr$ $\qquad [\because \; a + a' = 1]$

$pq+pq\ r+p'qr$ $\quad [\ a\ .\ 1=a]$

$=(pq+pq'r)+p'qr$

$=p\ (q+q'r)+p'qr$

$=p\ (q+r)+p'qr$ $\quad [\because\ a+a'\ .\ b=a+b]$

$=pq+pr+p'qr$

$=(pq+p'qr)+pr$

$=q(p+p'\ .\ r)+pr$

$=q\ (p+r)+pr$ $\quad [\because\ a+a'\ .\ b=a+b]$

$=pq+qr+rp$ $\quad [a\ .\ b=b\ .\ a]$

$=$ R.H.S.

$\therefore \quad pqr+pqr'+pq'r+p'qr=pq+qr+rp.$

3·3. BOOLEAN FUNCTIONS

A variable x which takes two distinct values symbolically denoted by 0 and 1 and for which the two binary operations $(+)$ and $(\ .\ )$ are defined by tables 1 and 2 of example 1, is called a binary Boolean variable. In Boolean functions we have 0 and 1 as the constants and a, b, c or x, y, z as some arbitrary variables. Sometimes the initial variables a, b, c etc., are used to express group relations with x, y, z. The Boolean functions are the relations, expressed in the above constants and variables with $+$ for $\cup$ and $(\ .\ )$ for $\cap$ and $(\ '\ )$ attached to any variable for 'not'.

We now take a Boolean function and show how it is verified.

Example 14. *Verify that* $(x\cap y)\cup[(x\cup y')\cap y]'=1$

Solution. Let us verify this by the old method of truth table taking $x\cap y=a$ and $[(x\cup y')\cap y]'=b'$.

Truth Table : $(x\cap y)\cup[(x\cup y')\cap y]'=1$

x	y	$a=x\cap y$	y'	$x\cup y'$	$b=(x\cup y')\cap y$	b'	$a\cup b'$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	1	1	0	1	1	0	1
1	0	0	1	1	0	1	1
0	1	0	0	0	0	1	1
0	0	0	1	1	0	1	1

By means of simplification we can prove the equality as follows :

$(x\cap y)\cup[x\cap y')\cap y]'=(x\cap y)\cup[(x\cup y')'\cup y']$

$=(x\cap y)\cup[(x'\cap y)\cup y']$

$=(x\cap y)\cup[cx'\cup y')\cap(y\cup y')]$

$=(x\cap y)\cup[(x'\cup y')\cap 1]$

$=(x\cap y)\cup(x'\cup y')$

$=1$

Example 14. *Simplify* $\{[a'\cap b')'\cup c]\cap(a\cup c)\}'$

Solution. $\{[(a'\cap b')'\cup c]\cap(a\cup c)\}'=[(a'\cap b')'\cup c]'\cup(a\cup c)'$

$$=[(a'\cap b')\cap c']\cup(a'\cap c')$$
$$=(a'\cap b'\cap c')\cup(a'\cap c')$$
$$=a'\cap c'$$

This is because $\{a'\cap c'\}\supset\{a'\cap b'\cap c'\}$

Example 16. *Simplify the following and show as a union of intersections and intersection of unions.*

$$[(x\cup y')\cap(x\cap y'\cap z)']'$$

Solution. $[(x\cup y')\cap(x\cap y'\cap z)']'=(x\cup y')'\cup(x\cap y'\cap z)$

$$=(x'\cap y)\cup(x\cap y'\cap z) \qquad ...(i)$$

Now, we proceed further for having intersection of unions :

$$=(x'\cup x)\cap(y\cup x)\cap(x'\cup y')\cap(y\cup y')\cap(x'\cap z)\cap(y\cup z)$$
$$=1\cap(y\cup x)\cap(x'\cup y')\cap 1\cap(x'\cup z)\cap(y\cap z)$$
$$=(y\cup x)\cap(x'\cup y')\cap(x'\cup z)\cap(y\cup z) \qquad ...(ii)$$

The two expressions could be written as

$$x'y+xy'z \qquad ...(i)$$
$$(y+x)(x'+y')(x'+y)(y+z) \qquad ...(ii)$$

It may be noted that expression (*i*) is shorter than (*ii*) therefore for saving the time of machine the former is sometimes preferred.

3·4. CANONICAL FORM

In this all the elements are expressed as x or x', y or y' and z or z' as union of intersections or intersection of unions. A complete canonical form with three variables will take the following form :

1	1	1	x	y	z
1	1	0	x	y	z'
1	0	1	x	y'	z
1	0	0	x	y'	z'
0	1	1	x'	y	z
0	1	0	x'	y	z'
0	0	1	x'	y'	z
0	0	0	x'	y'	z'

Now, a complete canonical form has a union of all the 8 (*i.e.*, 2^3) intersections :

$$(xyz)+(xyz')+(xy'z)+(xy'z')+$$
$$(x'yz)+(x'yz')+(x'y'z)+(x'y'z')$$

But a canonical form may be preferred in even few alternative forms having all elements. For example

$$(x'\cap y)\cup(x\cap y'\cap z)=(x'\cap y\cap 1)\cup(x\cap y'\cap z)$$
$$=[(x'\cap y)\cap(z\cup z')]\cup(x\cap y'\cap z)$$
$$=(x'\cap y\cap z)\cup(x'\cap y\cap z')\cup(x\cap y'\cap z)$$

It should be noted that all the expressions are in the form of union of intersections having all the three elements x, y, and z.

Example 17. *Convert the following expression in canonical form as intersection of unions and not as the union of intersections shown above.*

$$(x\cup y)\cap(y\cap z)\cap(x'\cup z)\cap(x'\cup y')$$

Solution

$$=[(x\cup y)\cup(z\cap z')]\cap[(y\cup z)\cup(x\cap x')]\cap(x'\cup z)\cup$$
$$(y\cap y')]\cap[(x'\cup y')\cup(z\cap z')]$$
$$=[(x\cup y)\cup z)\cap(x\cup y\cup z')\cap(x\cup y\cup z)\cap(x'\cup y\cup z')]$$
$$\cap[(x'\cup y\cup z)\cap(x'\cup y'\cup z)\cap(x'\cup y'\cup z)\cap(x'\cup y'\cup z')]$$

By eliminating the repeated unions, we have

$$(x\cup y\cup z)\cap(x\cup y\cup z')\cap(x'\cup y\cup z)\cap(x'\cup y'\cup z)\cap(x'\cup y'\cup z')$$

It is also possible to shorten the canonical form into a shorter but non-canonical form. Let the Boolean function be

$$F(x, y, z)=(x\cap y\cap z)\cup(x\cap y'\cap z)\cup(x'\cap y'\cap z)\cup(x'\cap y'\cap z')$$
$$=(x\cap z)\cup(x'\cap y').$$

The dual of the above can be :

$$F(x, y, z)=(x'\cup y'\cup z')\cap(x'\cup y\cup z')\cap(x\cup y\cup z')\cap(x\cup y\cup z)$$
$$=(x'\cup z')\cap(x\cup y)$$

3·5. ELECTRICAL SWITCHING SYSTEM

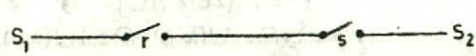
It has a network of electrical switches which is an example of the practical application of Boolean algebra. Let us take the switches r, s, t, q, etc. The value of a closed switch or when it is ON is equal to 1 and when it is is open or OFF is equal to 0.

An open switch r is indicated in the diagram as follows :

A closed switch r is indicated in the diagram as follows :

3·6. BOOLEAN MULTIPLICATIONS

The two switches r and s in the series will preform the operation of Boolean multiplication. See the circuit.



Obviously the current will not pass from point S_1 to S_2 when either or both are open, it will pass only when both are closed. Please recollect the truth table given in the first chapter reproduced here with new symbols :

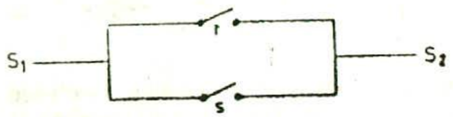
Truth Table : $r \cdot s$

r	s	$r \cdot s$
1	1	1
1	0	0
0	1	0
0	0	0

The operation is true in only one of the four cases, *i.e.*, when both the switches are closed.

3·7. BOOLEAN ADDITION

In the case of an operation of addition the two switches will be in the parallel series, and not in the same series. See the circuit below :



The circuit shows that the current will pass when either or both the switches are closed. It will not pass only when both are open. Please recollect the turth table for this operation given in the first chapter which is reproduced here with the new symbols :

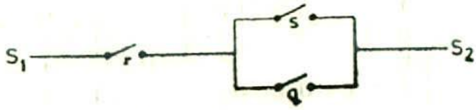
Truth Table : $r+s$

r	s	$r+s$
1	1	1
1	0	1
0	1	1
0	0	0

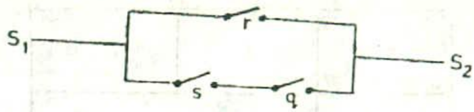
The operation is not true only in one of the four situations when both r and s are open.

3·8. CIRCUITS WITH COMPOSITE OPERATIONS

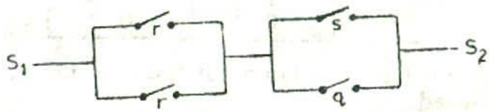
(*i*) Circuit showing : $r(s+q)=rs+rq$



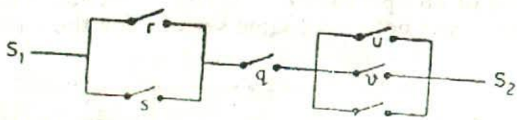
(*ii*) Circuit showing : $r+(s \cdot q)$



(*iii*) A corresponding circuit simplifying (*ii*) above into $rs+rq$ is :



(*iv*) A circuit for $(r+s)q(u+v+w)$ is :

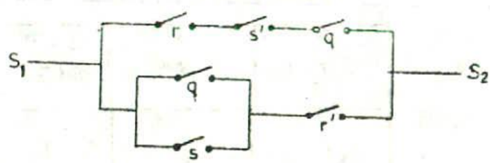


So far we have seen that all the switches are closed or open independently of one another and both the states were shown by one letter symbol. But now we want to show the closed state by the letter say r, s and open state by r', s' (both with prime).

Therefore the function :

$$F=(r \cdot s' \cdot q)+[(q+s)r']$$

will be shown by the circuit as follows :



The closed properties of the above function are nicely shown by the following truth table :

Truth Table : $(r \cdot s' \cdot q)+[(q+s)r']$

r (1)	s (2)	q (3)	s' (4)	$(r \cdot s' \cdot q)$ 5	r' (6)	$(q+s)$ (7)	$(q+s)r'$ (8)	col. 5+col.8 (9)
1	1	1	0	0	0	1	0	0
1	1	0	0	0	0	1	0	0
1	0	1	1	1	0	1	0	1
1	0	0	1	0	0	0	0	0
0	1	1	0	0	1	1	1	1
0	1	0	0	0	1	1	1	1
0	0	1	1	0	1	1	1	1
0	0	0	1	0	1	0	0	0

The closed properties of the function, therefore, are (*i*) $rs'q$ (*ii*) $r'sq$ (*iii*) $r'sq'$ and (*iv*) $r'sq$ For these see the row numbers 3, 4, 6 and 7.

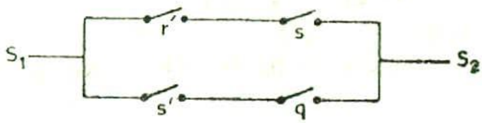
However, the above function can be simplified and presented by a simple network :

First, we show the process of simplification and then the simplified circuit.

$$F=(r\ .\ s'\ .\ q)+[(q+s)r']$$
$$=(r\ .\ s'\ .\ q)+[(r'\ .\ q)+(r'\ .\ s)]$$

We transform the latter part in canonical form as follows :

$$=(r\ .\ s\ .\ q)+(r'\ .\ q\ .\ s)+(r'\ .\ q\ .\ s')+(r'\ .\ s\ .\ q)+(r'\ .\ s\ .\ q')$$
$$=(r\ .\ s'\ .\ q)+(r'\ .\ s\ .\ q)+(r'\ .\ s\ .\ q')+(r'\ .\ s'\ .\ q)$$
$$=(r'\ .\ s)+(s'\ .\ q)$$ The circuit for this is also given.



EXERCISES

1. Define Boolean algebra.

2. Indicate whether the following subsets S_1 S_2 of the set N of natural numbers are Boolean for the operations indicated

$S_1=\{1, 2, 3, 6, 7, 21, 42,\}$ for least common multiple.
$S_2=\{1, 2, 3, 4, 6, 8, 12\}$ for greater common divisor,

3. Prove that for every $a, b \in B$

(*i*) $a\cap(a\cup b)=a\cup(a\cap b)$

(*ii*) $a\cup(a'\cap b)=a\cup b$

(*iii*) $(a+b)'=a'b'$

(*iv*) $(a\ .\ b)'=a'+b'$

4. Simplify :

(*i*) $[(x\cap y')'\cup z]\cap(x\cup y')'$

(*ii*) $(a\cup b')\cap(a'\cup b)\cap(a'\cup b')$

(*iii*) $[(a\cup b)\cap(c\cup b')]\cup[b\cap(a'\cup c')]$

5. Simplify :

(*i*) $(a\cup b)\cap(a'\cap b')$

(*ii*) $(a\cap b\cap c)\cup(a'\cup b'\cup c')$.

6. Express the following in canonical form :

(*i*) $x'\cup y'$

(*ii*) $(x\cap y')\cup(x'\cap y)$

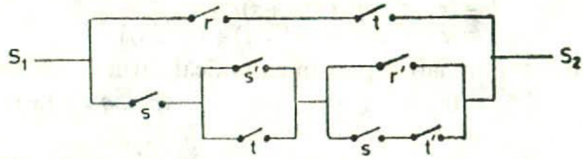
7. Rewrite the expressions in canonical form :

(i) $x=y=1$, $w=z=0$

(ii) $x=0$, $y=z=1$

8. Give the switches r, s, t in different forms.

9. Indicate the network, simplify and give a simpler network.



10. Give the circuits :

(i) $(x \cup y') \cap (x' \cup y) \cap (x' \cup y')$

(ii) $(x \cap y) \cup z \cap (x' \cup y')$

11. Give simple circuits for those in Q. No. 10.

ANSWERS

1. See the text. Give also the main properties.

2. S_1 is Boolean, 42 is the least common multiple for all elements in the set.

S_2 is not a Boolean, 3 is not a common divisor of 8

4. (i) $x' \cap y$ (ii) $a' \cap b'$ (iii) $a \cup b$

5. (i) 0 (ii) 1

6. (i) $(x' \cup y' \cup z) \cap (x' \cup y' \cup z')$

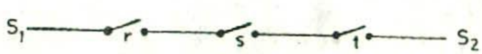
(ii) $(x \cap y' \cap z) \cup (x \cap y' \cap z') \cup (x' \cap y \cap z) \cup (x' \cap y \cap z')$

(iii) $(x \cup y \cup z) \cap (x \cup y \cup z') \cap (x' \cup y' \cup z) \cap (x' \cup y' \cup z')$

7. (i) $x \cap y \cap w' \cap z'$

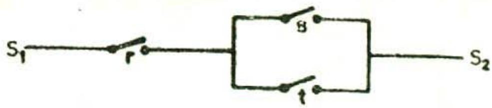
(ii) $x' \cap y \cap w \cap z$

8. (i)

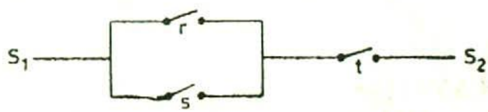


(ii)

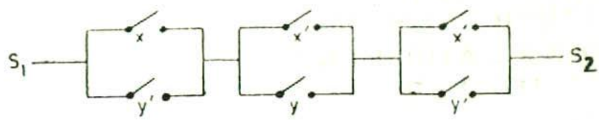
(*iii*)



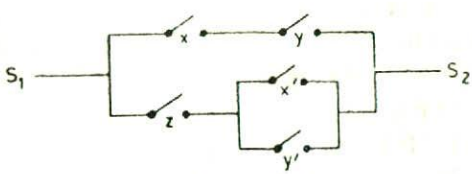
9. $(r \cap t) \cup [s \cap (s' \cup t) \cap \{r' \cup (s \cap t')\}]$ after simplification $(r \cup s)$ with a simpler network as follows :



10. (*i*)



(*ii*)



11. (*i*)

(*ii*)