4

Real Number System

STRUCTURE

Objectives

After studying this chapter, you should be able to understand :

- *natural numbers, integers, rational numbers, irrational numbers, real numbers, imaginary/complex numbers.*
- *properties and operations of these numbers.*

4·1 NUMBER SYSTEM

It is composed of various numbers, symbols or figures representing numbers and certain rules governing operations on them. The numbers can be represented by {1, 2, 3,...}, {I, II, III, } or $\{K, K^*, K^{**}, \ldots\}$, what is of importance is the nature and characteristic of these numbers, whether they are capable of performing one or more operations of addition, multiplication, subtraction and division. It is because of this that instead of attaching any undue importance to any system, *e.g.*, decimal, binary, a good deal of importance is being attached to the properties of the system. It will at times be explained by symbols only. The conceptual clarity or the logic of the system is emphasised more and not mere familiarity with the numbers. We initiate the number system by natural numbers and then proceed on to other systems.

4·2 THE NATURAL NUMBERS (N)

The numbers 1, 2, 3, 4,..., which are used for counting, are natural numbers. Thus while 618 is a natural number ; 0, -7, 13·2 and $\frac{7}{8}$ are not natural numbers.

Italian mathematician Peano has given five postulates (axioms) called Peano Postulates (P) as the properties of natural numbers. To speak in the language of modern mathematics, we say let there be a non-empty set N such that

P I : 1 is natural number; $1 \in N$.

P II : For each $n \in N$, there exists a unique natural number $n \in N$, called the successor of n, we can write it as $(n+1)$ also.

P III : For each $n \in N$, we have n^* or $n+1 \neq 1$.

P IV : If $m, n \in N$ and $m^* = n^*$, then $m = n$.

Thus we have successors for each number.

P V : Any subset S of N is equal to N if

(*i*) $1 \in S$

(*ii*) $m \in S \Rightarrow m^* \in S$

By postulate V we reach any natural number starting with 1 and counting consecutive successive numbers.

4·3. ADDITION ON N

The operation of addition on N can be defined as follows :

(*i*) $n+1 = n^*$ for every $n \in N$

(*ii*) $n+m^* = (n+m)^*$ wherever $n+m$ is defined.

This can be grasped be recollecting that $4+5=(4+4)+1=9$. The basic laws of addition composition are :

A$_1$. *Closure Law.*

$$\text{For} \quad m, n \in N, m+n \in N$$

A$_2$. *Commutative Law.*

$$m+n=n+m \ \forall \, m, n \in N$$

A$_3$. *Associative Law.*

$$m+(n+p)=(m+n)+p \ \forall \, m, n, p \in N$$

A$_4$. *Cancellation Law.*

$$m+p=n+p \Rightarrow m=n \ \forall \, m, n, p \in N$$

4·4. MULTIPLICATION ON N

The operation of multiplication on N is defined as follows :

(*i*) $n \,.\, 1 = n$ for every $n \in N$

(*ii*) $n \,.\, m = (n \,.\, m)+n$ whenever $n \,.\, m$ is defined.

This can be grasped if you recollect that $4 \,.\, 5=(4 \,.\, 4)+4=20$. The basic laws governing multiplication composition are :

M_1. *Closure Law.* For all $m, n \in N$; $m \,.\, n \in N$, *i.e.*, the product of two natural numbers is also a natural number.

M_2. *Commutative Law.* $m \,.\, n=n \,.\, m \;\forall\; m, n \in N$

M_3. *Associative Law.*

$$m \,.\, (n \,.\, p)=(m \,.\, n) \,.\, p \;\forall\; m, n, p \in N$$

M_4. *Cancellation Law.*

$$m \,.\, p=n \,.\, p \Rightarrow m=n \;\forall\; m, n, p \in N$$

M_5. *Existence of Identity.* There exists an element $1 \in N$ such that

$$m \,.\, 1=m=1 \,.\, m \;\forall\; m \in N$$

The laws governing multiplication and addition composition are :

D_1. *Left Distributive Law.*

$$m \,.\, (n+p)=m \,.\, n+m \,.\, p \;\forall\; m, n, p \in N$$

D_2. *Right Distributive Law.*

$$(n+p) \,.\, m=n \,.\, m+p \,.\, m \;\forall\; m, n, p \in N$$

4·5. ORDER RELATIONS ON N

There are two types of order relations in N *viz.*, greater than $(>)$ and less than $(<)$. The relation "$a<b$" is read as "a is less than b." It can be stated also as "$b>a$" read as "b is greater than a."

Greater than $(>)$. A natural number $m \in N$ is said to be greater than a natural number $n \in N$ if and only if there exists $p \in N$ such that

$$m=n+p$$

Smaller than $(<)$. A natural number $m \in N$ is said to be lesser than a natural number $n \in N$ (symbollically $m<n$) if there exists $p \in N$ such that $m+p=n$.

The lows governing order relations are :

Q_1. *Trichotomy Law.* Given any two natural numbers m and n then one and only one of the following three possibilities hold.

(*i*) $m=n$, (*ii*) $m>n$, (*iii*) $m<n$

Q_2. *Transitive Law*

$$m>n \text{ and } n>p \Rightarrow m>p \;\forall\; m, n, p \in N$$

Q_3. *Anti-symmetric Law*

$$m>n \text{ and } n>m \Rightarrow n=m \;\forall\; m, n \in N$$

Q_4. *Monotone Property of Addition,*

$$m>n \Rightarrow m+p>n+p \;\forall\; m, n, p \in N$$

Q_5. *Monotone Property of Multiplication,*

$$m>n \Rightarrow mp>np \;\forall\; m, n, p \in N$$

Therefore the relation "less than or equal to" and "greater than or equal to" and denoned as "$\leqslant$" and "$\geqslant$" respectively are defined as

(*i*) $m \leqslant n$ if $m=n$ or $m<n$

(*ii*) $m \geqslant n$ if $m=n$ or $m>n$

PROOFS

Example 1. $m+(n+p)=(m+n)+p \ \forall m, n, p \in N$

Solution. Let us treat m and n as fixed natural numbers and put 1 for p which is the first element of natural numbers as per postulate I.

$$\therefore \quad m+(n+1)=(m+n)+1$$

Let us first take the L.H.S.

$$m+(n+1)=m+n^* \qquad \text{[addition rule } (i)]$$
$$=(m+n)^* \qquad \text{[addition rule } (ii)]$$
$$=(m+n)+1$$

Now, by placing $k \in N$ for p, we have $m+(n+k)=(m+n)+k$ which would mean that

$$m+(n+k^*)=(m+n)+k^*$$

Now $\quad m+(n+k^*)=m+(n+k)^*$

$$=[m+(n+k)]^*$$
$$=[(m+n)+k]^*$$
$$=(m+n)+k^*$$

Thus the associative property in addition is proved.

Example 2. *Prove that* $m+n=n+m$ *for all* $m, n \in N$.

Solution. Let us treat n as a fixed natural number and take any $k \in N$ such that

$$k+n=n+k$$

$$\therefore \quad k^*+n=(k+1)+n=k+(1+n)$$
$$=k+(n+1)=k+n^*$$
$$=(k+n)^*$$
$$=n+k^*$$

Thus if $\quad k^*+n=n+k^*$ then $k+n=n+k$

which proves the commutative property of addition.

Example 3. *Prove that*

$$(n+p) \,.\, m=n \,.\, m+p \,.\, m, \text{ for all } m, n, p \in N$$

Solution. Let us take n and p as fixed and substitute 1 and then k and k^* for m so that, we have

$$(n+p) \,.\, 1=n+p=n \,.\, 1+p \,.\, 1$$

Now if $\quad (n+p) \,.\, k=n \,.\, k+p \quad k$

then $\quad (n+p) \,.\, k^*=n \,.\, k^*+p \,.\, k^*.$

Let us take the L.H.S.

$$\therefore \quad (n+p) \,.\, k^* = (n+p) \,.\, k + (n+p), \text{ [rule (ii) of multiplication]}$$
$$= n \,.\, k + p \,.\, k + n + p$$
$$= n \,.\, k + (p \,.\, k + n) + p$$
$$= n \,.\, k + (n + p \cdot k) + p$$
$$= (n \,.\, k + n) + (p \,.\, k + p)$$
$$= n \,.\, k^* + p \,.\, k^*$$

Hence $(n+p) \,.\, m = n \,.\, m + p \,.\, m$. for all m, $n\ p \in N$.

4·6. THE INTEGERS (I)

The integers are whole numbers positive, negative or zero. We can also define them as ratios of two numbers which do not have a remainder. On a number line they range between $-\infty$ to 0 and 0 to $+\infty$. Thus -15 -207, 0, -9 are all integers but $\sqrt{7}$, 0·392, $-0·76$ and $\frac{7}{8}$ are not integers.

The set $\quad I = \{x \mid x = 0, x \in N \text{ or } -x \in N\}$

$$= \{\ldots, -4, -3, -2, -1, 0, 1, 0, 2, 3, 4, \ldots\}$$

are called the set of integers. You may note

(*i*) The numbers $-1, -2, -3, -4, \ldots$ are negative integers.

(*ii*) The numbers $+1, +2, +3, +4, \ldots$ are positive integers.

They are generally written without any sign.

(*iii*) The number 0 is the only integer that has no sign.

Integers thus fulfil a gap of zero and negative numbers in natural numbers. For example, the natural numbers do not provide answers to

(*i*) $a + x = a$ (*ii*) $a + x = y$ [when $a > y$]

for which we need a zero and a negative number respectively.

The operations of addition and multiplication on integers thus satisfy all the properties of natural numbers with a modification in the cancellation law as follows :

Cancellation Law. If $m \,.\, p = n \,.\, p$ and if $p \neq 0 \in I$ then $m = n$ for all $m, n \in I$.

The two additional properties for the opertion of addition are

A_6. There exists an identity element $0 \in I$ for the operation of addition such that

$$n + 0 = 0 + n = n \text{ for every } n \in I$$

A_6. There exists an additive inverse $-n \in I$ such that

$$n + (-n) = (-n) + n = 0$$

4·8. PRIME NUMBERS (P)

An integer other than 0 or 1 is a prime number if and only if its only divisors are 1 and the number itself. We can write

$$p \neq 0, \pm 1 \text{ whose divisors are } \pm 1 \text{ and } \pm p \text{ only.}$$

Properties of prime numbers :

(*i*) If p is prime and if p is a factor of ab where $a, b \in I$ then p is a factor of a or p is a factor b.

(*ii*) It p is a prime and if p is a divisor of the product of $a. b. c \ldots\ldots r$ of integers then p is a divisor of at least one of these.

MODULO (m)

It is a positive integer often indicated by m and defined by the following expression

$$a \equiv b \pmod{m} \qquad \text{where} \qquad a, b \in I$$

and m is a factor of $(a-b)$.

For example :

(*a*) $25 \equiv 1 \pmod{4}$ since 4 is a factor of of 24

(*b*) $89 \equiv 1 \pmod{4}$ since 4 is a factor of 88

(*c*) $24 \not\equiv 3 \pmod{5}$ since 5 is not a factor of 21

(*d*) $24 \equiv 4 \pmod{5}$ since 5 is a factor of 20.

The concept of modulo helps in having residue classes in case of operations on integers $I/(4)$ as follows :

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

4·9. RATIONAL NUMBERS (Q)

The number which can be expressed in the form $\frac{p}{q}$ where p is any integer and q is a integer not equal to zero is called a rational number. To state it formally

The set $Q = \{p/q : p, q \in I \text{ and } q \neq 0\}$ is called the set of rational numbers.

Thus $4 \in Q$, $-5 \in Q$, $\frac{4}{5} \in Q$, $-\frac{7}{11} \in Q$, $0 \in Q$ etc. It is also clear that $N \subset I$, $I \subset Q$.

If $\frac{p}{q}$, $\frac{r}{s}$ are two rational numbers so that $p, r \in I$ and q $s \in N$, we have

(*i*) $\dfrac{p}{q} \pm \dfrac{r}{s} = \dfrac{ps \pm qr}{qs}$, (*ii*) $\dfrac{p}{q} \times \dfrac{r}{s} = \dfrac{pr}{qs}$,

(*iii*) $\dfrac{p}{q} \div \dfrac{r}{s} = \dfrac{ps}{qr}$,

and (*iv*) $\left(\dfrac{p}{q}\right)^n = \dfrac{p^n}{q^n}$, n being a positive integer, $q \neq 0$.

An important characteristic of rational numbers is that when expressed as decimal fractions they are either terminating or non-terminating recurring decimals. For example.

$$\frac{2}{5} = 0.4 \quad \frac{35}{16} = 2.1875$$

$$\frac{1}{6} = 0{\cdot}1666\ldots \text{(to be written as } 0{\cdot}16\text{)}$$

$$\frac{3}{11} = 0{\cdot}272727\ldots = 0{\cdot}\overline{27}$$

$$\frac{29}{7} = 4{\cdot}142057,\ 142057,\ 142057\ldots = 4{\cdot}\overline{142057}$$

Conversely, we may show that any non-terminating recurring decimal represents a rational number. For example

$$x = 1{\cdot}344 \qquad \text{or} \qquad x = 1{\cdot}34$$

As the repeating cycle contains one digit, it should be multiplied by 10 and then the original quantity be deducted from the new one as shown below :

$$\begin{aligned} 10x &= 13.44\ldots \\ x &= 1.34\ldots \\ \hline 10x - x &= 12.1 \\ \Rightarrow \quad 9x &= 12.1 \\ \Rightarrow \quad x &= \frac{12.1}{9} = \frac{121}{90}, \text{ a rational number} \end{aligned}$$

However, if the repeating cycle is of two digits then the original quantity will be multiplied by 100 in place of 10 above and so on.

4·10 PROPERTIES OF Q

We now indicate the properties of rational numbers, $p/q = a$ and $r/s = b$ where $p, q, r, s \in I$ and $q \neq 0$ and $s \neq 0$ under various operations as follows :

I. Addition :

(*i*) *Closure*. If a and b are rational numbers, then $a+b$ is one and only one, *i.e.*, it is a unique rational number.

(*ii*) *Commutative*. $a+b=b+a$

(*iii*) *Associative.* $(a+b)+c=a+(b+c)$

(*iv*) *Identity* (*zero*). $a+0=a=0+a$

(*v*) *Inverse.* For every rational number 'a' there is a rational number $(-a)$ such that $a+(-a)=0$, $-a$ is called the additive inverse of a.

(*vi*) *Cancellation.* If a, b, c are rational numbers such that

$$a+c=b+c \quad \text{then} \quad a=b$$

II. Multiplication.

(*i*) *Closure.* If a and b are any rational numbers then $a\times b$ or ab is a unique rational number.

(*ii*) *Commutative.* $ab=ba$

(*iii*) *Associative.* $(ab)c=a(bc)$

(*iv*) *Identity* (1). $a\times 1=1\times a=a$

(*v*) *Inverse.* For every rational number a $(\neq 0)$ there is a rational number $(1/a)$ such that $a\times\left(\frac{1}{a}\right)=1$. Thus, for every rational number a, the multiplicative inverse is $\frac{1}{a}$.

(*vi*) *Distributive.* Only multiplication distributes over addition, *i.e.*

$$a(b+c)=ab+ac$$

(*vii*) *Cancellation.* If $ac=bc$, then $a=b$ iff $c\neq 0$.

III. Order Relation :

If $a=p/q$, $b=r/s$

(*i*) then $a-b=\dfrac{ps-qr}{qs}$.

We say that $a=b$ or $a>b$ or $a<b$ according as $ps-qr=0$, >0, or <0, respectively.

(*ii*) If $a>b$ and $b>c$ then $a>c$.

(*iii*) If $a>b$ then $a+c>b+c$

If $a<b$ then $a+c<b+c$.

(*iv*) If $a>b$, then $ac>bc$ $(c>0)$ and $ac<bc$ $(c<0)$.

IV. Equality with Zero :

If $ab=0$ then either a is zero or b is zero.

V. Density :

If a and b are distinct rational numbers then $\frac{a+b}{2}$ is a rational number lying between a and b. In other words,

$$a>\frac{a+b}{2}>b$$

or $$a < \frac{a+b}{2} < b$$

We can then say that there can be several intermediate rational numbers between two different rational numbers stated as follows :

$$a > a_1 > a_2 > a_3 > a_4 > b$$
$$a < a_1 < a_2 < a_3 < a_4 < b$$

4·11 IRRATIONAL NUMBERS (R_i)

We can define an irrational number with the help of a rational number. Now if there is a rational number a but there no rational number b such that $(b)^n = a$, then we write it as "n root of the equation $b^n = a$ as the irrational number $\sqrt[n]{a}$".

If $\sqrt[n]{a}$ is not equal to x, an integer, then it is called irrational number.

Or, the numbers which cannot be expressed in the form p/q, where $q \neq 0$ and p, q are both integers, are called irrational numbers and are denoted by R_i.

Examples of such numbers are $\sqrt{5}$, $\sqrt[3]{8}$, $2+\sqrt{5}$ etc. which are represented by non-terminating, non-recurring decimals as shown below :

$$\sqrt{2} = 1.1414...$$
$$\pi = 3.14159...$$
$$\sqrt{7} = 2.645751...$$
$$\sqrt{15} = 3.872983...$$

Example 8. *Prove that $\sqrt{2}$ is an irrational number.*

Solution. If possible, let $\sqrt{2}$, be a rational number so that

$$\sqrt{2} = \frac{p}{q},\ q \neq 0$$

and p and q are integers. Further suppose that p and q have no common factors.

Now $$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2}$$

i.e. $$p^2 = 2q^2$$

$\because$ p^2 is even so that p is also even. ...(1)

Let $$p = 2m$$

$\because$ $$p^2 = 4m^2$$

$\therefore$ $$4m^2 = 2q^2$$

$\Rightarrow$ $$q^2 = 2m^2$$

$\therefore$ q^2 is even, *i.e.*, q is also even ...(2)

From (1) and (2), we find that p and q are both even, *i.e.*, they have a common factor 2 which contradicts our assumption that p and q have no common factors.

Hence it follows that $\sqrt{2}$ is not a rational number, *i.e.*, $\sqrt{2}$ is an irrational number.

4.12 THE REAL NUMBERS (R)

It comprises a set of all rational and irrational numbers. We generally denote this by R which will have either the rational numbers (Q) or irrational numbers (R_i) formally.

$$R=\{x : x \in Q \cup x \in R_i\}$$

Now, in relation to natural numbers etc., we have

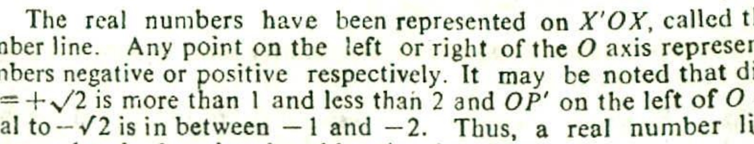
$$N \subset I \subset Q \subset R$$

Thus, natural numbers constitute a proper subset of integers and the integers constitute a proper subset of rational numbers and the latter constitutes a proper subset of real numbers.

The positive and negative real numbers are shown by R^+ and R^- respectively and the non-negative real numbers by R_0 as in the case of integers and rational numbers given earlier.

A real number system is a complete order field with zero, minus infinity, and plus infinity including their infinitesimal parts, with the exception of imaginary numbers singly or along with real numbers called the complex numbers. However, the real numbers are not divisible by zero.

We can represent the order field of real numbers by the following real line axis :



The real numbers have been represented on $X'OX$, called the real number line. Any point on the left or right of the O axis represents real numbers negative or positive respectively. It may be noted that distance $OP=+\sqrt{2}$ is more than 1 and less than 2 and OP' on the left of O axis is equal to $-\sqrt{2}$ is in between -1 and -2. Thus, a real number line can accommodate both rational and irrational numbers.

4·13. PROPERTIES OF R

We summarise below the fundamental properties of the real numbers to emphasize their basic importance which the students should remember.

Let us take the set R of real numbers with $a, b, c \in \mathbf{R}$ and then define the two algebraic operations of addition and multiplication, *i.e.*, '+' and '.' through the following properties.

I. Addition Operation :

A$_1$. *Closure Law.* If a and b are any two real numbers, their sum $(a+b)$ is also a real number. This can be expressed symbolically as

$$a+b \in R \; \forall a, b \in R$$

A$_2$. *Commutative Law.* If a and b are two real numbers, then

$$a+b=b+a, \; \forall a, b \in R$$

A$_3$. *Associative Law.* If a, b, c are any there real numbers, then

$$(a+b)+c=a+(b+c) \; \forall a, b, c \in R$$

A$_4$. *Existence of Identity.* There exists a real number 0 (zero) such that

$$a+0=a=0+a \; \forall a \in R$$

This real number '0' is known as additive identity and the property is known as property of zero.

A$_5$. *Existence of Inverse.* For every real number a there exists another real number b such that

$$a+b=0=b+a$$

The real number b is called additive inverse of a and is usually written as $-a$.

II. Multiplication Operation :

M$_1$. *Closure Law.* If a and b are any two real numbers, their product ab is also a real number. This can be expressed symbolically as

$$a \,.\, b \in R \; \forall a, b \in R$$

M$_2$. *Commutative Law.*

$$a \,.\, b=b \,.\, a \; \forall a, b \in R$$

M$_3$. *Associative Law.*

$$(a \,.\, b) \,.\, c=a \,.\, (b \,.\, c) \; \forall a, b, c \in R$$

M$_4$. *Existence of Identity.* There exists a real number 1 such that

$$a \,.\, 1=a=1 \,.\, a \; \forall a \in R$$

The real number '1' is called multiplicative identity and the property is called the property of 1.

M$_5$. *Existence of Inverse.* Corresponding to each real number 'a' $(a \neq 0)$, there exists a real number 'b' such that

$$a \,.\, b=b \,.\, a=1$$

b is called the reciprocal or multiplicative inverse of a and is usually written as $\frac{1}{a}$ or a^{-1}

III. Relation between the two Algebraic Operations :

Distributive Laws. Multiplication is distributive over addition. For any three real numbers $a, b, c \in R$ we have

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a$$

These are known as Right Distributive and Left Distributive laws respectively.

IV. Order Relation :

O_1. *Trichotomy Law.* If we are given two real numbers $a, b \in R$, then one and only one of the following three holds good :

(*i*) $a=b$, (*ii*) $a>b$, (*iii*) $a<b$

O_2. *Transitivity.*

$$a>b \text{ and } b>c \Rightarrow a>c \ \forall\, a, b, c \in R$$

O_3. *Anti-symmetry.*

$$a>b \text{ and } b>a \Rightarrow a=b \ \forall\, a, b \in R$$

O_4. *Order relation is compatible with addition.*

$$a>b \Rightarrow a+c>b+c \ \forall\, a, b, c \in R$$

O_5. *Order relation is compatible with multiplication.*

$$a>b \Rightarrow a \cdot c>b \cdot c \ \forall\, a, b, c \in R$$

V. Density Property. Between two real numbers there lie infinite number of real numbers. For any two distinct numbers $a, b \in R$, there is $\frac{a+b}{2}$ such that

$$a < \frac{a+b}{2} < b$$

or $$b > \frac{a+b}{2} > a$$

We can also state this property as

$$a > a_1 > a_2 > a_3 \ldots a_n > b$$

$$b < a_1 < a_2 < a_3 \ldots a_n < a$$

Some theorems based on above axioms :

Theorem 1. (*Uniqueness of additive Identity*). ***There exists one and only one real number '0' such that***

$$a+0=a=0+a \ \forall\, a \in R$$

Proof. Let there be two additive identities of R say O and $0'$,

$$a+0=a \text{ and } a+0'=a \ \forall\, a \in R$$

We shall show that $0=0'$

Since $a+0=a \ \forall\, a \in R$, in particular for $a=0'$ also, so that

$$0'+0=0' \qquad \ldots(1)$$

Again as $a+0'=a \forall a \in R$, so it is true for $a=0$ also.

$$0+0'=0 \quad ...(2)$$

Thus $$0'=0'+0$$
$$=0+0' \quad \text{(Commutative Law)}$$
$$=0$$

There is thus a unique additive identity 0 satisfying

$$a+0=a=0+a \forall a \in R$$

Theorem 2. (*Uniqueness of the additive inverse*). *For every real number $a \in R$ there exists one and only one real number b such that*

$$a+b=0=b+a$$

Proof. Let us suppose that there exists two real numbers b and b' for every real number a such that

$$a+b=0 \quad ...(1)$$

and $$a+b'=0 \quad ...(2)$$

We shall show that $b=b'$

Now $$b'=b'+0 \quad \text{(Property of '0')}$$
$$=b'+(a+b)$$
$$=(b'+a)+b \quad \text{(Asso, of +)}$$
$$=(a+b')+b \quad \text{(Comm, of +)}$$
$$=0+b \quad \text{(Property of '0')}$$
$$=b$$

Hence, there exists a unique additive inverse for every number.

Theorem 3. (*Cancellation laws for Addition*). *If a, b and c are real numbers then*

$$a+b=a+c \Rightarrow b=c$$

Proof. $a+b=a+c$ (given)

By adding $-a$ on both sides, we have

$$(-a)+(a+b)=(-a)+(a+c)$$
$$\Rightarrow [(-a)+a]+b=[(-a)+a]+c \quad \text{(Asso. of +)}$$
$$\Rightarrow 0+b=0+c$$
$$\Rightarrow b=c \quad \text{(Property of 0)}$$

Theorem 4. (*Uniqueness of multiplicative identity*). *There exists one and only one real number 1 such that*

$$a \,.\, 1=1 \,.\, a \forall a \in R$$

proof. Let us suppose that there exists two real numbers, say, 1 and 1 such that

$$a \,.\, 1=a=1 \,.\, a \quad ...(1)$$
$$a \,.\, 1'=a=1' \,.\, a \quad ...(2)$$

Then we have to show that $1=1'$.

Since $a \cdot 1 = a \forall a \in R$, so in particular for $a = 1'$ also, *i.e.*,

$$1' \cdot 1 = 1'$$

Again, as $a \cdot 1' = a \forall a \in R$ so in particular it is true for $a = 1$ also, *i.e.*,

$$1 \cdot 1' = 1$$

Thus $\quad 1' = 1' \cdot 1 = 1 \cdot 1' = 1$

This shows that there is a unique multiplicative identity 1 satisfying

$$a \cdot 1 = a = 1 \cdot a \forall a \in R$$

Theorem 5. (*Uniqueness of multiplicative inverse*). *There exists one and only one real number b such that*

$$a \cdot b = 1 = b \cdot a$$

Proof. Let there be two multiplicative inverses b and b', for a non zero real number a, satistying

$$a \cdot b = 1 = b \cdot a \qquad \ldots(1)$$

and

$$a \cdot b' = 1 = b' \cdot a \qquad \ldots(2)$$

we shall show that $\quad b = b'$

Now
$$\begin{aligned} b' &= b' \cdot 1 && \text{(Property of '1')} \\ &= b' \cdot (a \cdot b) && \text{(using 1)} \\ &= (b' \cdot a) \cdot b && \text{(Assoc. of } \times) \\ &= (a \cdot b') \cdot b && \text{(Comm, of } \times) \\ &= 1 \cdot b \\ &= b \end{aligned}$$

Hence there is a unique multiplicative inverse for every non-zero real number.

Theorem 6. (*Cancellation Laws of multiplication*). *If a, b, c be any three real numbers, then*

$$a \cdot b = a \cdot c \Rightarrow b = c$$

Proof. Now
$$\begin{aligned} & a \cdot b = a \cdot c \\ \Rightarrow\ & a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c) && \text{(Closure Law)} \\ \Rightarrow\ & (a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c && \text{(by } M_3) \\ \Rightarrow\ & 1 \cdot b = 1 \cdot c && \text{(by } M_5) \\ \Rightarrow\ & b = c && \text{(by } M_4) \end{aligned}$$

Theorem 7. *For any real number a,*

$$a \cdot 0 = 0 \cdot a = 0.$$

Proof.
$$\begin{aligned} a \cdot 0 &= a \cdot (0+0) && \text{(Property of zero)} \\ &= a \cdot 0 + a \cdot 0 && \text{(Distributive law)} \end{aligned}$$

$$\Rightarrow \quad a \cdot 0 + a \cdot 0 = a \cdot 0 + 0$$

$$\Rightarrow \quad -(a \cdot 0) + a \cdot 0 + a \cdot 0 = -(a \cdot 0) + (a \cdot 0 + 0)$$

[adding—$(a \cdot 0)$ to both sides]

$\Rightarrow \quad [-(a\,.\,0)+a\,.\,0]+a\,.\,0=[-(a\,.\,0)+(a\,.\,0]+0$

$\Rightarrow \quad 0+a\,.\,0=0+0$

$a\,.\,0=0$

Theorem 8. *If a and b are any two real numbers then*

$$a\,.\,b=0 \Rightarrow a=0 \text{ or } b=0$$

Proof. We shall prove that for $ab=0$ at least one of them must be zero

Two cases may arise

(i) $a=0$, (ii) $a \neq 0$

Case (i), If $a=0$ then $a\,.\,b=0$ is obvious.

Case (ii), If $a \neq 0$, we shall show that $b=0$

Now $\quad ab=0$

$\Rightarrow \quad a^{-1}\,.\,(ab)=a^{-1}\,.\,0$

$\Rightarrow \quad (a^{-1}\,.\,a)b=0$

$\Rightarrow \quad 1\,.\,b=0$

$b=0$

Hence $\quad ab=0 \Rightarrow a=0$ or $b=0$

Theorem 9. *For any two real numbers a and b*

(i) $\quad a\,.\,(-b)=(-a)\,.\,b=-a\,.\,b$

(ii) $\quad (-a)(-b)=ab$

Proof. (i) We have

$$0=a\,.\,0=a\,.\,(-b+b)=a\,.\,(-b)+a\,.\,b$$

or $\quad 0+(-ab)=a\,.\,(-b)+a\,.\,b+(-ab)$

or $\quad -a\,.\,b=a\,.\,(-b)+[a\,.\,b+(-a\,.\,b)]$

$=a\,.\,(-b)+0=a\,.\,(-b)$

$\therefore \quad -a\,.\,b=a\,.\,(-b) \quad \ldots(1)$

Again $\quad 0=0\,.\,b=(-a+a)\,.\,b$

$=(-a)\,.\,b+a\,.\,b$

or $\quad 0+(-a\,.\,b)=(-a)\,.\,b+a\,.\,b+(-a\,.\,b)$

$-a\,.\,b=(-a)\,.\,b+[a\,.\,b+(-a\,.\,b)]$

$=(-a)\,.\,b+0=(-a)\,.\,b$

$\therefore \quad -a\,.\,b=(-a)\,.\,b \quad \ldots(2)$

From (1) and (2), we have

$$a\,.\,(-b)=(-a)\,.\,b=-a\,.\,b$$

(ii) $\quad (-a)\,.\,(-b)=(-a)\,.\,(-b)+(-a)\,.\,b-(-a)\,.\,b$

$=(-a)\,.\,[-b+b]-(-a)\,.\,b$

$=(-a)\,.\,0-(-a)\,.\,b$

$=0-(-a)\,.\,b$

$$= -(-a) \cdot b$$
$$= -(-a) \cdot b - a \cdot b + a \cdot b$$
$$= -[-a+a] \cdot b + a \cdot b$$
$$= -0 \cdot b + a \cdot b$$
$$= a \cdot b$$

$\therefore \quad (-a) \cdot (-b) = a \cdot b$

4·14. Modulus of real number (*i.e.*, the absolute numerical value).

The modulus of a real number a is defined as the real number a, $-a$, or 0 according as a is positive, negative or zero. We denote the modulus of a real number a by the symbol $|a|$ and define it by

$$|a| = \begin{cases} a, \text{ if } a \text{ is positive} \\ -a, \text{ if } a \text{ is negative} \\ 0, \text{ if } a \text{ is zero} \end{cases}$$

Following five results are evident from the definition which are found to be very useful.

1. The modulus of a real number is never negative, *i.e.*,
$$|a| \geqslant 0$$
2. For every real number a,
$$a \leqslant |a| \text{ and } -a \leqslant |a|$$
3. $$|a| = |-a|$$
4. $|a|$ denotes the greater of the two numbers a and $-a$, *i.e.*,
$$|a| = \max. \{a, -a\}$$

5. We may also define $|a|$ by a single equation. Since the positive root of a positive number is a positive number, it follows that we may define $|a|$ by

$$|a| = (a^2)^{1/2}$$

Theorem 10. *If a and b are any real numbers, then*

(*a*) $$|ab| = |a| \cdot |b|$$

(*b*) $$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \ (b \neq 0)$$

(*c*) $$\left|\frac{1}{a}\right| = \frac{1}{|a|}$$

(*d*) $$|a+b| \leqslant |a| + |b|$$

(*e*) $$|a-b| \geqslant |a| - |b|$$

Proof. We shall use the definition in the form

$$|a| = (a^2)^{1/2}$$

(*a*) $|ab| = [(ab)^2]^{1/2} = (a^2b^2)^{1/2} = (a^2)^{1/2} \cdot (b^2)^{1/2} = |a| \cdot |b|$

Remark. Putting $a = -1$, we obtain

$$|-b| = |-1| \, |b| = |b|$$

(*b*) $\left|\frac{a}{b}\right|=\left[\left(\frac{a}{b}\right)^2\right]^{1/2}=\left(\frac{a^2}{b^2}\right)^{1/2}=\frac{(a^2)^{1/2}}{(b^2)^{1/2}}=\frac{|a|}{|b|}$

(*c*) In (*b*), put $a=1$ and $b=a$, we get

$$\left|\frac{a}{b}\right|=\frac{1}{|a|}$$

(*d*) $|a+b|=[(a+b)^2]^{1/2}=(a^2+2ab+b^2)^{1/2}$

Since $|a|^2=a^2$, $|b|^2=b^2$, and $ab\leqslant|a|\,|b|$, the above equation yields

$$|a+b|\leqslant[|a|^2+2|a|\,|b|+|b|^2]^{1/2}$$
$$=[(|a|+|b|)^2]^{1/2}=|a|+|b|$$

Thus $|a+b|\leqslant|a|+|b|$

(*e*) We have $a=(a-b)+b$

$\therefore$ $|a|=|(a-b)+b|\leqslant|a-b|+|b|$

$\Rightarrow$ $|a-b|\geqslant|a|-|b|$

Example 5. *State if the statement $x>1\Leftrightarrow x^2>1$ is true where x is a real number.*

Solution. No, the statement is not true $x^2>1$ implies $x>1$ or $x<-1$.

4.15. IMAGINARY NUMBERS (i)

Square roots of negative numbers are called imaginary numbers because, the square of any number is positive only. This occurs in some quadratic equations and therefore has to be taken into account and properly defined. For example

(*i*) if $x^2-9=0$ then $x=\pm3$ but

(*ii*) if $x^2+4=0$ then $x=\sqrt{-4}$

The (*ii*) above shows that x is equal to an imaginary number.

Further

$$x^2-4x+13=0$$
$$\Rightarrow \quad x^2-4x+4+9=0$$
$$\Rightarrow \quad (x-2)^2+9=0$$
$$\Rightarrow \quad x-2=\pm\sqrt{-9}$$
$$\Rightarrow \quad x=2\pm\sqrt{-9}=2\pm3i.$$

Now, the number of the form bi is an imaginary number where $i=\sqrt{-1}$ or $i^2=\sqrt{-1}\times\sqrt{-1}=-1$. So that we can indicate an imaginary number in real form as

$$\sqrt{-9}=\sqrt{-1}\,.\,\sqrt{9}=i\,.\,3 \text{ or } 3i$$
$$\sqrt{-121}=\sqrt{-1}\,.\,\sqrt{121}=i\,.\,11 \text{ or } 11i$$

In general

$$\sqrt{-a}=\sqrt{a\times(-1)}=\sqrt{a}\,.\,\sqrt{-1}=\sqrt{a}\,.\,i$$

4·16. COMPLEX NUMBERS (a+ib)

If a and b are real numbers then $a+ib$ is known as a complex number which has 'a' the real part and 'b' the imaginary part. Now

(i) If in the complex number $a+ib$, $a=0$, the number $0+ib$ is an imaginary number only.

(ii) If in the above $a+ib$, $b=0$ then the complex number reduces to a purely real number a.

(iii) The two complex numbers $a+ib$ and $a-ib$ are called the conjugates, *e.g.*,

$$2+3i \quad \text{and} \quad 2-3i$$
$$(-\sqrt{3})+5i \quad \text{and} \quad (-\sqrt{3})-5i$$

Addition and Subtraction. In these operations we add or subtract real part and imaginary part separately, *e.g.*

$$(a+ib)\pm(c+id)=(a\pm c)+i(b\pm d)$$

Multiplication. This opertion is done in a normal way taking

$$i=\sqrt{-1}$$

Such that

$$(a+ib)\,(c+id)=(ac-bd)+i(ad+bc)$$

Let us elaborate

$$\begin{aligned}(a+ib)\,(c+id)&=a(c+id)+ib(c+id)\\&=ac+aid+cib+ibid\\&=(ac+ibid)+(aid+cib)\\&=(ac-bd)+i(ad+bc) \qquad [\because \quad i^2=-1]\end{aligned}$$

Division : $$\frac{a+ib}{c+id}=\frac{a+ib}{c+id}\times\frac{c-id}{c-id}$$
$$=\frac{(ac+bd)+i(bc-ad)}{c^2-i^2d^2}$$
$$=\frac{ac+bd}{c^2+d^2}+i\,\frac{bc-ad}{c^2+d^2} \qquad [\because \quad i^2=-1]$$

(i) Complex numbers obey the laws of algebra :

$$\begin{aligned}i^2&=i\times i=\sqrt{-1}\times\sqrt{-1}=-1\\i^3&=i^2\times i=(-1)\,i=-i\\i^4&=(i^2)\,(i^2)=(-1)\times(-1)=1\\i^5&=(i^4)\,(i)=(1)\,i=i\end{aligned}$$

(ii) If $a+ib=0$, where a , $b\in R$ then $a=0$ and $b=0$. This we can prove as follow :

if $\quad a+ib=0$ then $a=-ib$

Squaring both sides, we have

$$a^2=+i^2b^2=-b^2$$

$\therefore \quad a^2+b^2=0$

But a^2+b^2 cannot be equal to zero unless a and b both are equal to zero.

(*iii*) Sum and product of two conjugate numbers are real :

$$(a+ib)+(a-ib)=2a$$

$$(a+ib)\times(a-ib)=a^2-i^2b^2=a^2+b^2$$

(*iv*) If $a+ib=c+id$ then $a=c$ and $b=d$.

We can prove this as follows :

$$a+ib=c+id$$

$$\therefore \quad (a+ib)-(c+id)=0$$

$$(a-c)+i\,(b-d)=0 \qquad \text{(See rule of subtraction)}$$

$$\therefore \quad a-c=0 \text{ and } b-d=0 \qquad \text{[See property } (ii) \text{ above]}$$

$$\therefore \quad a=c \text{ and } b=d$$

However, we cannot say in complex numbers that a given complex number is greater than or lesser than any other complex number.

Example 6. $\dfrac{3+2i}{5-3i}$

Solution.

$$\frac{3+2i}{5-3i}=\frac{(3+2i)\,(5+3i)}{(5-3i)\,(5+3i)}$$

$$=\frac{(15-6)+i\,(9+10)}{25-9i^2}$$

$$=\frac{9+19i}{25+9}=\frac{9}{34}+\frac{19}{34}\,i.$$

Example 7. *Find the square root of* $6+8\sqrt{-1}$.

Solution. Let

$$\sqrt{6+8\,i}=a+ib$$

$$\therefore \quad 6+8\,i=(a^2-b^2)+2\,iab$$

$$\Rightarrow \quad a^2-b^2=6 \qquad \ldots(i)$$

$$ab=4 \qquad \ldots(ii)$$

and

$$b^2=\frac{16}{a^2} \qquad \ldots(iii)$$

Now substituting (*iii*) in (*i*), we have

$$a^2-\frac{16}{a^2}=6$$

$$a^4-6\,a^2-16=0$$

or

$$(a^2-8)\,(a^2+2)=0$$

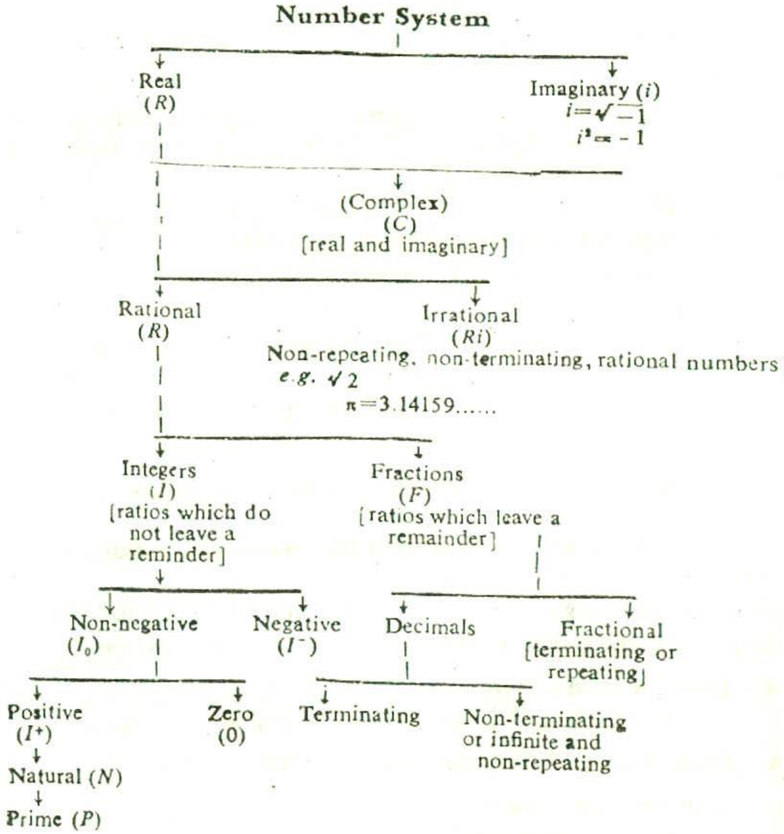
$$\therefore \quad a^2=8 \quad \text{or } a^2=-2$$

Since $a^2=-2$ is inadmissible, $a^2=8$. By substituting this in (*iii*) above, we have $b^2=2$.

$$\therefore \quad \sqrt{6+8\,i}=\pm(\sqrt{8}+i\sqrt{2})$$

Now, the square of any complex number is in the form of a complex number.

We now present the various number systems in the form of a chart :



EXCERCISES

1. (*a*) State which of the following statements are true and which are false :

(*i*) Every real number is a rational number.

(*ii*) Every irrational number is a real number.

(*iii*) A real number is either rational or irrational.

(*iv*) There can be a real number which is both rational and irrational.

(*b*) State the following in fractional form

(*i*) 1·2 (recurring decimal 1·2222...

(*ii*) $1\dot{\cdot}6$ (recurring decimal 1·666... ...)

(*c*) State the following in decimal form or fractional form

(*i*) $\sqrt{5}$

(*ii*) $5, \dot{2}\dot{3}$

(*iii*) $1/7$

2. State whether the following statements are true or false. It the statement is true, prove it; if you consider a statemnt to be false, give an example in support of your answer :

(*i*) The product of two rational numbers is rational.

(*ii*) The sum of two irrational numbers is irrational.

(*iii*) The product of two odd integers is an odd integer.

(*iv*) $x<y \Leftrightarrow x^2<y^2$.

(*v*) For any real number x, we can find a real number y such that

$$xy=1$$

(*vi*) If x is rational and y is irrational then xy is irrational.

(*vii*) If $x>0$ then $x^2>x$.

3. (*i*) Show that the sum of two rational numbers is a rational number.

(*ii*) Give an example to show that the quotient of natural numbers need not be a natural number.

(*iii*) Give two integers whose quotient is not a rational number

(*iv*) Show that there is no rational number whose square is 2.

4. Define a rational number.

Show that $\sqrt{3}$ and $\sqrt{7}$ are not rational numbers.

5. State if the following statements are true :

(*i*) $a>b$ and $c>0$ then $ac>bc$

(*ii*) $a\leqslant b$ and $b\leqslant a$ then $a=b$

(*iii*) $a>b$ then $a=b+c$ if c is some possible number.

(*iv*) If $a<b$ then $a<\dfrac{a+b}{2}<b$

(*v*) If $a>0$ and $b>0$ then $a^2>b^2$ in all cases.

(*vi*) $\dfrac{a}{b}<\dfrac{c}{d} \Leftrightarrow ad<bc$

6. Prove that

$\forall\ x, y \in Q,\ -(x+y)=(-x)+(-y)$

7. Show that $\dfrac{-x}{-y}=\dfrac{x}{y},\ (y\neq 0)$

8. Show that $\frac{x}{z} = \frac{y}{z} (z \neq 0) \Rightarrow x = y$

9. Show that $\frac{a}{x} + \frac{b}{y} = \frac{(ay + bx)}{xy}$, if $x \neq 0$, $y \neq 0$

10. (*i*) Multiply $4 - 3i$ by $5 + 7i$

(*ii*) Simplify and show if

$$\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i} \text{ is a rational number}$$

(*iii*) Simplify $\frac{9-7i}{2-3i}$

ANSWERS

1. (*a*) (*i*) False, (*ii*) True, (*iii*) True, (*iv*) False.

(*b*) (*i*) $\frac{11}{9}$ (*ii*) $\frac{5}{3}$

(*c*) (*i*) 2.23607...(*ii*) $\frac{518}{99}$ (*iii*) $0.\dot{1}4285\dot{7}$

2. (*i*) True, $\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$

(*ii*) False, $(p + \sqrt{q}) + (p - \sqrt{q}) = 2p$, rational number

(*iii*) True, $(n+1).(m+1) = mn + m + n + 1$ (n, m being even numbers)

(*iv*) False, $-1 < 1$ but $(-1)^2 \nless (1)^2$

(*v*) False, if $y = 0$ then $xy \neq 1$

(*vi*) True, $\sqrt{2}\ \sqrt{4} = \sqrt{8}$ which is irrational.

(*vii*) False, $\frac{1}{2} > 0$ but $(\frac{1}{2})^2 \ngtr \frac{1}{2}$.

3. (*i*) Closure property

(*ii*) $\frac{n}{n+1}$ [n is a natural number]

(*iii*) $\frac{p}{q}$ [$q = 0$] (*iv*) $1 < \sqrt{2} < 2$

4. See text.

5. (*i*) to (*iii*) are true, (*iv*) is true if $a > b$,

(*v*) is true if $a > b$, (*vi*) is true.

10. (*i*) $41 + 13i$, (*ii*) $-\frac{8}{29}$, (*iii*) $3 + i$.