6

Indices and Surds

Structure

Objectives

After studying this chapter, you should be able to understand :

- *Indices ; positive fractional indices, and operations on them.*
- *Surds, operations on surds, rationalising of surds and calculation of root of a surd.*

6·1. INDICES

We are aware of certain operations of addition and multiplication and now we take up certain higher order operations with powers and roots under the respective heads of indices and surds.

The knowledge of these rules is indispensable for any serious mathematical manipulation. We will deal with indices and surds in this chapter and the use of logarithms to help simplifying these operations in the next chapter.

We know that the result of a repeated addition can be had by multiplication, *e.g.*,

$$4+4+4+4+4=5(4)=20 \text{ or}$$

$$a+a+a+a+a=5(a)=5a.$$

Likewise the repeated multiplication can be reduced to a power function as follows :

$$4\times4\times4\times4\times4=4^5 \quad \text{...(1)}$$

$$a\times a\times a\times a\times a=a^5 \quad \text{...(2)}$$

It may be noticed that in the first case 4 is multiplied 5 times and in the second case 'a' is multiplied 5 times. In all such cases a factor which multiplies is called the "base" and the number of times it is multiplied is called the "power" or the "index". Therefore, "4" and "a" were the bases and "5" was the index for both. Remember that the base has to be same in order to convert a product function into a power function. In the above case the index was positive and integral but it can be negative or fractional which we shall consider later.

6·2 POSITIVE INDICES

In a positive index the base multiplies a given number of times depending on the power or the value of the index. In case of a negative index it is reciprocal of the base which multiplies a number of times depending on the value of the negative index. The formal definition and the fundamental rules of operations with positive index are given below which would be relevant in other cases also.

Definition. *If n is a positive integer, and 'a' a real number, i.e., $n \in N$ and $a \in R$, a^n is used to denote the continued product of n factors each equal to 'a' shown below :*

$$a^n = a \times a \times \ldots\ldots \text{ to } n \text{ factors.}$$

where a is called the index or the *exponent* of base a.

Laws of Indices. If $a, b \in R$; $m, n \in N$, then

I. $$\mathbf{a^m \times a^n = a^{m+n}}$$

Proof. $$a^m \times a^n = (a.a\,a.\ldots\text{to } m \text{ factors}) \times (a.a.a.\ldots \text{ to } n \text{ factors})$$
$$= a.\ a.\ a\ldots\text{to } (m+n) \text{ factors}$$
$$= a^{m+n} \text{ (by definition)}$$

This is called the *Fundamental Index Law*.

II. $$\mathbf{a^m \div a^n = a^{m-n}}$$

Proof. $$\frac{a^m}{a^n} = \frac{a.\ a.\ a\ldots m \text{ factors}}{a.\ a.\ a\ldots n \text{ factors}}$$

(*i*) If $m > n$, there will be $(m-n)$ factors of 'a', it being more in the numerator than in the denominator. Cancelling with n common factors from the numerator and the denominator, we are left with $(m-n)$ factors of a in the numerator.

$$a^m \div a^n = a \times a \times a\ldots(m-n) \text{ factors} = am-n$$

(*ii*) If $m = n$, there are same number of factors of a in the numerator and denominator which cancel away.

(*iii*) If $m < n$, there are $(n-m)$ extra factors of a in the denominator. Cancelling the common m factors from the numerator and the denominator, we are left with extra $(n-m)$ factors of a in the denominator.

$$\therefore \quad a^m \div a^n = \frac{1}{a\ .\ a\ .\ a\ldots(n-m) \text{ factors}} = \frac{1}{an-m}$$

III. $$(\mathbf{a}^m)^n = a^{mn}$$

Proof $(a^m)^n = a^m \cdot a^m \cdot a^m \ldots n$ factors

$= (a \cdot a \cdot a \ldots m \text{ factors}) \times (a \cdot a \cdot a \ldots m \text{ factors})$

$\ldots \times (a \cdot a \cdot a \ldots m \text{ factors})$

$= a \cdot a \cdot a \ldots (mn)$ factors

$= a^{mn}$

IV $$(ab)^m = a^m b^m$$

Proof. $(a \cdot b)^m = (a \cdot b) \times (a \cdot b) \times \ldots m$ factors

$= (a \cdot a \cdot a \ldots m \text{ factors}) \times (b \cdot b \cdot b \ldots m \text{ factors}$

$= a^m \times b^m$

V $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

Proof $\left(\frac{a}{b}\right)^m = \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \ldots m$ factors

$$= \frac{a \cdot a \cdot a \ldots m \text{ factors}}{b \cdot b \cdot b \ldots m \text{ factors}} = \frac{a^m}{b^m}$$

Remark. The above laws can be extended to the case involving three or more power functions. It should be remembered that powers are added for multiplication and subtracted for division. Simple addition or subtraction of power functions is not possible.

There can be a negative integral index to any base except 0 and 1 in a power function. When this is there, the power function becomes the reciprocal of the function having a positive index. For example

$$a^{-m} = \frac{1}{a^m}, \quad \frac{1}{a^m} = a^{-m}, \text{ where } a \neq 0 \text{ or } 1$$

Thus, a negative integral index makes the power function an inverse of the one with a positive integral index. The only restriction is that the base is not 0 or 1.

Illstrations. 1. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ 2. $\frac{1}{4^{-3}} = 4^3 = 64$

3. $a^{-m} \times a^n = a^{-m+n}$ 4. $a^n \times a^{-m} = \frac{a^n}{a^m} = a^{n-m}$

ZERO AND UNITY INDEX

The general principle is that anything other than zero raised to the power zero is one, *i.e.*,

$$a^0 = x^0 = 5^0 = 1, \ (a, x \neq 0)$$

Thus $a^0 \times a^n = a^{0+n} = a^n$; $a^n \times a^{-n} = a^{n-n} = a^0 = 1$

As a rule any base raised to unity or 1 is equal to the base itself

$$a^1 = a \ ; \ 5^1 = 5$$

POWER RAISED TO A POWER

A power function can be raised to a power as given below :

$$a^{3^2}=a^{(3)^2}=a^9$$

However, this will not be the same as the whole function being raised to a power, in that case the power will multiply as given below :

$$(a^3)^2=a^6,\ (4a^2)^3=4^3\ .\ a^6=64a^6$$

$$x^{m^n}=x^{(m)^n}\ ;\ \text{and } (x^m)^n=x^{mn}$$

6·3. FRACTIONAL INDEX

In a positive fractional index the numerator represents the power and the denominator, the root. For example

1. $x^{\frac{1}{2}}=\sqrt[2]{x}=\sqrt{x}$ 2. $x^{\frac{1}{3}}=\sqrt[3]{x}$ 3. $x^{\frac{1}{a}}=\sqrt[a]{x}$

4. $x^{\frac{p}{q}}=\sqrt[q]{x^p}$. In particular, we have

(*i*) $16^{\frac{1}{2}}=\sqrt{16}=4$,

(*ii*) $64^{\frac{1}{3}}=\sqrt[3]{64}=4$, and (*iii*) $16^{\frac{3}{4}}=(16^3)^{\frac{1}{4}}=\sqrt[4]{16^3}=8$

Note that in the (*iii*) above, the fraction has been broken into $\left(3\times\frac{1}{4}\right)$. This is necessary before transforming a power function with a fractional index in the radical form.

Meaning of $a^{\frac{p}{q}}$, where p and q are any two positive integers.

Since $a^m\times a^n=a^{m+n}$ holds true for all values of m and n, putting $m=n=\frac{p}{q}$, we have

$$a^{\frac{p}{q}}\times a^{\frac{p}{q}}=a^{\frac{p}{q}+\frac{p}{q}}=a^{2\frac{p}{q}}$$

Similarly $a^{\frac{p}{q}}\times a^{\frac{p}{q}}\times a^{\frac{p}{q}}\times\ldots q$ factors $=a^{\left(\frac{p}{q}\right)\times q}=a^p$

$$\therefore\quad a^{\left(\frac{p}{q}\right)q}=a^p$$

$$a^{p/q}=\sqrt[q]{a^p}$$

Therefore $a^{\frac{p}{q}}$ represents the q^{th} root of the p^{th} power of a. In a like manner, $a^{\frac{p}{q}}=(\sqrt[q]{a})^p$, represents the p^{th} power of the q^{th} root of a.

In case, the fractional index is negative, the function is transformed into the reciprocal of one with a positive fractional index as shown below :

1. $x^{-\frac{p}{q}}=\frac{1}{x^{p/q}}$ 2. $x^{-\frac{1}{2}}=\frac{1}{x^{1/2}}=\frac{1}{\sqrt{x}}$

3. $8^{-\frac{2}{3}}=\frac{1}{8^{2/3}}=\frac{1}{\sqrt[3]{8^2}}=\frac{1}{\sqrt[3]{64}}=\frac{1}{4}$

6·4. OPERATIONS WITH POWER FUNCTIONS

The two operations involved in power functions are multiplication and division. As indicated earlier power functions cannot be added or subtracted so as to derive a new resultant function.

Multiplication with Common Base. In the case of multiplication of two or more power functions with a common base, the powers are added, in other words, the base is raised to the sum of the indices. The formulae is

$$x^m \times x^n \times x^p = x^{m+n+p}$$

For example $a^6 \times a^4 = a^{6+4} = a^{10}$

This can be shown as follows :

Also $\{a\times a\times a\times a\times a\times a\} \times \{a\times a\times a\times a\}=a^{10}$

But remember $a^6+a^4 \neq a^{10}$

Division with a Common Base. In this case, the base will be raised by the difference of the indices. The formulae is

(*i*) $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$ (where $m>n$)

and $a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ (where $m<n$)

For example

$$\frac{a^7}{a^5}=\frac{a\times a\times a\times a\times a\times a\times a}{a\times a\times a\times a\times a}=a^{7-5}=a^2$$

and $$\frac{a^5}{a^7}=\frac{a\times a\times a\times a\times a}{a\times a\times a\times a\times a\times a\times a}=a^{5-7}=a^{-2}=\frac{1}{a^2}$$

Illustrations :

1. $2^{3^2}=2^9$.

2. (*i*) $$\frac{a^{-m}}{b^{-n}}=\frac{\frac{1}{a^m}}{\frac{1}{b^n}}=\frac{1}{a^m}\times\frac{b^n}{1}=\frac{b^n}{a^m}$$

(*ii*) $$\frac{1}{a^{m-n}}=\frac{1}{a^m\times a^{-n}}=a^{-m}\times a^n=a^{n-m}$$

3. Simplify $\frac{(4x^2)^3}{(2x^3)^2}+\frac{(6x^3)^2}{(3x^2)^3}$

which is $$\frac{4^3x^6}{2^2x^6}+\frac{6^2x^6}{3^3x^6}=\frac{64x^6}{4x^6}+\frac{36x^6}{27x^6}$$

$$=16x^{6-6}+\frac{4}{3}x^{6-6}=16x^0+\frac{4}{3}x^0$$

$$=16+\frac{4}{3}=\frac{52}{3} \qquad (\because \; x^0=1)$$

4. $\sqrt[3]{a^5}=a^{\frac{5}{3}}=(a^5)^{\frac{1}{3}}=(a^{\frac{1}{3}})^5=(\sqrt[3]{a})^5=\sqrt[3]{a^5}$

5. $16^{-\frac{3}{4}}=\frac{1}{\sqrt[4]{16^3}}=\frac{1}{8}$

6. $8^{-4/3}=\frac{1}{\sqrt[3]{8^4}}=\frac{1}{16}$ 7. $16^{1\cdot 25}=16^{\frac{5}{4}}=\sqrt[4]{16^5}=32$

8. If $a^2=\cdot 01$ then $a^3=a^2\,.\,a=\cdot 01\times\sqrt{\cdot 01}=\cdot 01\times\cdot 1=\cdot 001$

9. If $x^{\frac{1}{2}}=\cdot 2$ then $x^{\frac{3}{2}}=x\,.\,x^{\frac{1}{2}}=\cdot 04\times\cdot 2=\cdot 008$

10. If $a^2=8$ then $a^{-4}=\frac{1}{a^4}=\frac{1}{a^2\times a^2}=\frac{1}{8\times 8}=\frac{1}{64}$

11. If $2^m=a$ then $2^{m+2}=2^m\,.\,2^2=4a$

and $$2^{m-1}=2^m\,.\,2^{-1}=a\,.\,\frac{1}{2}=\frac{a}{2}$$

12. $x^{\frac{3}{2}}\,.\,x^{\frac{1}{2}}\,.\,x^{\frac{1}{2}}$ can also be expressed as

$$x^{\frac{3}{2}+\frac{1}{2}+\frac{1}{2}} \text{ or } x^{\frac{5}{2}}$$

Multiplication of Factors with Different Bases. The rules for this can be stated as follows :

(*i*) $a^m\times b^m=(ab)^m$ (*ii*) $a^m\,.\,b^m\,.\,c^m=(abc)^m$

$$\therefore \quad 3^4 \times 5^4 = (3 \times 5)^4 = (15)^4$$

We can prove the formula $a^m \,.\, b^m = (ab)^m$ as follows :

$$a^m \,.\, b^m = (a \times a \times \ldots\ldots \text{to } m \text{ factors}) \times (b \times b \times \ldots\ldots \text{to } m \text{ factors})$$

$$= (a \times b) \times (a \times b) \ldots\ldots \text{to } m \text{ factors}$$

$$= (ab)^m$$

Division of Factors with Different Bases. The rule for the purpose can be stated as follows :

$$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

We can prove the formula $\left(\frac{a^m}{b^m}\right) = \left(\frac{a}{b}\right)^m$ as follows :

$$\frac{a^m}{b^m} = \frac{a \times a \times a \ldots\ldots \text{ to } m \text{ factors}}{b \times b \times b \ldots\ldots \text{ to } m \text{ factors}}$$

$$= \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \ldots\ldots \text{to } m \text{ factors}$$

$$= \left(\frac{a}{b}\right)^m$$

Illustrations :

1. *To which power should we raise $x^{2/3}$ to get x ?*

$$(x^{2/3})^{\frac{3}{2}} = x^{\frac{2}{3} \times \frac{3}{2}} = x^1 = x$$

2. *If $a^3 = 4$ and $b^2 = a$, find b^6 and b^{12}.*

If $\quad b^2 = a \quad$ then $\quad b^6 = a^3 = 4$

and $\quad b^{12} = (a^3)^2 = (4)^2 = 16$

Example 1. *Find the value of (i) $(81)^{\cdot 5}$ and (ii) $(\cdot 001)^{1/3}$*

Solution. (*i*) $81^{1/2} = \sqrt{81} = 9$

(*ii*) $(\cdot 001)^{\frac{1}{3}} = \sqrt[3]{\cdot 001} = \sqrt[3]{\frac{1}{1000}} = \frac{1}{10} = 0\cdot 1$

Example 2. *Express* (*i*) ·00001 and (*ii*) $\sqrt[3]{100}$ into index form.

Solution. (*i*) $\cdot 00001 = \frac{1}{100000} = \frac{1}{10^5} = 10^{-5}$

(*ii*) $\sqrt[3]{100} = \sqrt[3]{10^2} = 10^{2/3}$

Example 3. *Simplify* $\frac{3^5 \,.\, 27^3 \,.\, 9^4}{3 \,.\, (81)^4}$

Solution. Given expression$=\dfrac{3^5 \,.\, (3^3)^3 \,.\, (3^2)^4}{3 \,.\, (3^4)^4}$

$$=\frac{3^5 \,.\, 3^9 \,.\, 3^8}{3 \,.\, 3^{16}}=\frac{3^{5+9+8}}{3^{1+16}}=\frac{3^{22}}{3^{17}}=3^{22-17}=3^5$$

Example 4. *Simplify* $\dfrac{9(4^x)^2}{16^{x+1}-2^{x+1} \,.\, 8^x}$

Solution. Given expression$=\dfrac{3^2 \,.\, [(2^2)^x]^2}{(2^4)^{x+1}-2^{x+1} \,.\, (2^3)^x}$

$$=\frac{3^2 \,.\, (2^{2x})^2}{2^{4x+4}-2^{x+1} \,.\, (2)^{3x}}=\frac{3^2 \,.\, 2^{4x}}{2^{4x+4}-2^{4x+1}}$$

$$=\frac{3^2 \,.\, 2^{4x}}{2^{4x} \,.\, 2^4-2^{4x} \,.\, 2^1}=\frac{3^2}{2^4-2^1}=\frac{9}{14}$$

Example 5. *Simplify* $\dfrac{x^{4/7} \,.\, \sqrt[5]{x^3} \,.\, \sqrt[7]{x^3}}{\sqrt[8]{x^{-3}} \,.\, \sqrt[5]{y^5 \,.\, x^8}} \cdot \dfrac{y^2}{(x^{1/8})^3}$

Solution. $\dfrac{x^{4/7} \,.\, x^{3/5} \,.\, x^{3/7}}{x^{-3/8} \,.\, y^{5/5} \,.\, x^{8/5}} \cdot \dfrac{y^2}{x^{3/8}}$

$$=\frac{x^{4/7} \,.\, x^{3/5} \,.\, x^{3/7} \,.\, x^{-\frac{8}{5}}}{y^{5/5} \,.\, y^{-2}}=\frac{x^0}{y^{-1}}=y \quad (\because\ x^0=1)$$

Example 6. *Simplify* $\dfrac{\sqrt[p]{x^n} \,.\, x^{\frac{m}{p}} \,.\, \sqrt[q]{x^n}}{\sqrt[q]{x^{-m}} \,.\, x^{n-1} \,.\, x^{m+1}}$

Solution. Given expression$=\dfrac{x^{\frac{n}{p}} \,.\, x^{\frac{m}{p}} \,.\, x^{\frac{n}{q}}}{x^{\frac{-m}{q}} \,.\, x^{n-1} \,.\, x^{m+1}}$

$$=x^{\frac{n}{p}+\frac{m}{p}+\frac{n}{q}+\frac{m}{q}-(n-1)-(m+1)}$$

$$=x^{n\left(\frac{1}{p}+\frac{1}{q}\right)+m\left(\frac{1}{p}+\frac{1}{q}\right)-(n+m)}$$

$$=x^{(n+m)\left(\frac{1}{p}+\frac{1}{q}\right)-(m+n)}$$

$$=x^{(m+n)\left(\frac{1}{p}+\frac{1}{q}-1\right)}$$

Example 7. *Divide* $x^{\frac{5}{8}} \,.\, y^{-\frac{3}{5}} \,.\, z^{\frac{3}{7}}$ *by* $\sqrt[5]{x^2} \,.\, \sqrt[3]{y^4} \,.\, \sqrt[7]{z^8}$.

and multiply the quotient by $x^{\frac{1}{40}} \cdot y^{\frac{5}{3}} \cdot z^{\frac{5}{7}}$ *Evaluate for* $x=16, y=2^{15}$

Solution. $$\frac{x^{\frac{5}{8}} \cdot y^{-\frac{3}{5}} \cdot z^{\frac{3}{7}}}{x^{\frac{2}{5}} \cdot y^{\frac{4}{3}} \cdot z^{\frac{8}{7}}} = x^{\frac{5}{8}-\frac{2}{5}} \cdot y^{-\frac{3}{5}-\frac{4}{3}} \cdot z^{\frac{3}{7}-\frac{8}{7}}$$

$$= x^{\frac{9}{40}} \cdot y^{-\frac{29}{15}} \cdot z^{-\frac{5}{7}} \qquad (*)$$

Now, multiplying (*) by $x^{\frac{1}{40}} \cdot y^{\frac{5}{3}} \cdot z^{\frac{5}{7}}$, we get the product as

$$x^{\frac{9}{40}+\frac{1}{40}} \cdot y^{-\frac{29}{15}+\frac{5}{3}} \cdot z^{-\frac{5}{7}+\frac{5}{7}}$$

$$= x^{\frac{10}{40}} \; y^{-\frac{4}{15}} z^0 = x^{\frac{1}{4}} \cdot y^{-\frac{4}{15}} \cdot z^0$$

If $x=16, y=2^{15}$, the given expression

$$\frac{16^{1/4}}{(2^{15})^{4/15}} = \frac{(2^4)^{1/4}}{2^4} = \frac{2}{16} = \frac{1}{8}.$$

Example 8. *Simplify* $\left(\frac{5^{-1} \cdot 7^2}{5^2 \cdot 7^{-4}}\right)^{\frac{7}{2}} \times \left(\frac{5^{-2} \cdot 7^3}{5^3 \cdot 7^{-5}}\right)^{-\frac{5}{2}}$.

Solution. Given expression $= \left(\frac{5^{-1} \cdot 7^2}{5^2 \cdot 7^{-4}}\right)^{\frac{7}{2}} \times \left(\frac{5^{-2} \cdot 7^3}{5^3 \cdot 7^{-5}}\right)^{-\frac{5}{2}}$

$$= [5^{-1-2} \; 7^{2-(-4)}]^{\frac{7}{2}} \times [5^{-2-3} \; 7^{3+5}]^{-\frac{5}{2}}$$

$$= (5^{-3} \cdot 7^6)^{\frac{7}{2}} (5^{-5} \cdot 7^8)^{-\frac{5}{2}}$$

$$= 5^{-\frac{21}{2}} \cdot 7^{21} \cdot 5^{\frac{25}{2}} \; 7^{-20} = 5^{-\frac{21}{2}+\frac{25}{2}} \; 7^{21-20}$$

$$= 5^2 \cdot 7^1 = 175$$

Example 9. *Find the value of*

$$\frac{(0{\cdot}3)^{1/3} \left(\frac{1}{27}\right)^{1/4} (9)^{1/6} (0{\cdot}81)^{2/3}}{(0{\cdot}9)^{2/3} (3)^{-1/2} \left(\frac{1}{3}\right)^{-2} (243)^{-1/4}}$$

Solution. Let

$$x=\frac{\left(\frac{3}{10}\right)^{1/3}\left(\frac{1}{27}\right)^{1/4}\left(9\right)^{1/6}\left(\frac{81}{100}\right)^{2/3}}{\left(\frac{9}{10}\right)^{2/3}\left(3\right)^{-1/2}\left(\frac{1}{3}\right)^{-2}\left(243\right)^{-1/4}}$$

$$=\frac{\left(\frac{3}{10}\right)^{1/3}\left(\frac{1}{3^3}\right)^{1/4}\left(3^2\right)^{1/6}\left(\frac{3^4}{10^2}\right)^{2/3}}{\left(\frac{3^2}{10}\right)^{2/3}\left(3\right)^{-1/2}\left(\frac{1}{3}\right)^{-2}\left(3^5\right)^{-1/4}}$$

$$=(3\times10^{-1})^{\frac{1}{3}}\times(3^{-3})^{\frac{1}{4}}\times(3^2)^{\frac{1}{6}}\times(3^4\times10^{-2})^{\frac{2}{3}}$$

$$\times(3^2\times10^{-1})^{-\frac{2}{3}}\times(3)^{\frac{1}{2}}\times(3^{-1})^2\times(3^5)^{\frac{1}{4}}$$

$$=3^{1/3}\times10^{-\frac{1}{3}}\times3^{-\frac{3}{4}}\times3^{1/3}\times3^{8/3}\times10^{-\frac{4}{3}}$$

$$\times3^{\frac{4}{3}}\times10^{2/3}\times3^{1/2}\times3^{-2}\times3^{5/4}$$

$$=3^{\frac{1}{3}-\frac{3}{4}+\frac{1}{3}+\frac{8}{3}-\frac{4}{3}+\frac{1}{2}+\frac{5}{4}-2}\times10^{-\frac{1}{3}-\frac{4}{3}+\frac{2}{3}}$$

$$=3\,.\;10^{-1}=\frac{3}{10}=0{\cdot}3$$

Example 10. *Simplify* $\dfrac{\sqrt[7]{x^2}}{y^{-\frac{1}{2}}}\times\dfrac{\sqrt[5]{x^2}}{\sqrt[3]{y^2}}\times\dfrac{x^{-\frac{9}{7}}}{\sqrt[3]{y}}\times\dfrac{\sqrt{y}}{x^{-\frac{3}{5}}}$.

Solution. Removing the radical signs, the given expression

$$=\frac{x^{2/7}}{y^{-\frac{1}{2}}}\times\frac{x^{2/5}}{y^{2/3}}\times\frac{x^{-\frac{9}{7}}}{y^{1/3}}\times\frac{y^{1/2}}{x^{-3/5}}$$

$$=x^{\frac{2}{7}+\frac{2}{5}-\frac{9}{7}+\frac{3}{5}}\,.\,y^{\frac{1}{2}+\frac{1}{2}-\frac{2}{3}-\frac{1}{3}}$$

$$=x^0y^0=1$$

Example 11. (*a*) *Simplify*

$$\frac{2^{m+3}\times3^{2m-n}\times5^{m+n+3}\times6^{n+1}}{6^{m+1}\times10^{n+3}\times15^m}$$

(*b*) *Obtain the simplest value of*

$$\frac{(2^{2n}-3 \,.\, 2^{2n-2})(3^n-2 \,.\, 3^{n-2})}{3^{n-4}(4^{n+3}-2^{2n})}$$

Solution. (*a*) Let the given expression be

$$x=\frac{2^{m+3}\times 3^{2m-n}\times 5^{m+n+3}\times 6^{n+1}}{6^{m+1}\times 10^{n+3}\times 15^m}$$

$$=\frac{2^{m+3}\times 3^{2m-n}\times 5^{m+n+3}\times (2\,.\,3)^{n+1}}{(2\,.\,3)^{m+1}\times (2\,.\,5)^{n+3}\times (3\,.\,5)^m}$$

$$=\frac{2^{m+3}\times 3^{2m-n}\times 5^{m+n+3}\times 2^{n+1}\times 3^{n+1}}{2^{m+1}\times 3^{m+1}\times 2^{n+3}\times 5^{n+3}\times 3^m\times 5^m}$$

$$=2^{m+3+n+1-m-1-n-3}\times 3^{2m-n+n+1-m-1-m}\times 5^{m+n+3-n-3-m}$$

$$=2^0 3^0 5^0=1$$

(*b*) The expression $=\dfrac{(2^{2n}-3\,.\,2^{2n}\,.\,2^{-2})(3^n-2\,.\,3^n\,.\,3^{-2})}{3^n\,.\,3^{-4}(2^{2n}\,.\,4^3-2^{2n})}$

$$=\frac{2^{2n}\left(1-\frac{3}{4}\right)3^n\left(1-\frac{2}{9}\right)}{3^n\,.\,(1/81)\,.\,2^{2n}(64-1)}$$

$$=\frac{\frac{1}{4}\times\frac{7}{9}}{63/81}=\frac{7}{36}\times\frac{81}{63}=0{\cdot}25$$

Example 12. (*a*) *Simplify* : $\dfrac{4^n\times 20^{m-1}\times 12^{m-n}\times 15^{m+n-2}}{16^m\times 5^{2m+n}\times 9^{m-1}}$

(*I.C.W.A., June 1990*)

(*b*) *Simplify* : $\dfrac{(3^{2n}-5\times 3^{2n-2})(5^n-3\times 5^{n-2})}{5^{n-4}(9^{n+3}-3^{2n})}$

[*I.C.W.A., December 1990*]

Solution. (*a*) We have $\dfrac{4^n\times 20^{m-1}\times 12^{m-n}\times 15^{m+n-2}}{16^m\times 5^{2m+n}\times 9^{m-1}}$

$$=\frac{(2^2)^n\times(2^2\times 5)^{m-1}\times(2^2\times 3)^{m-n}\times(3\times 5)^{m+n-2}}{(2^4)^m\times 5^{2m+n}\times(3^2)^{m-1}}$$

$$=\frac{2^{2n}\times 2^{2m-2}\times 5^{m-1}\times 2^{2m-2n}\times 3^{m-n}\times 3^{m+n-2}\times 5^{m+n-2}}{2^{4m}\times 5^{2m+n}\times 3^{2m-2}}$$

$$=2^{2n+2m-2+2m-2n-4m}\times 3^{m-n+m+n-2-2m+2}\times 5^{m-1+m+n-2-2m-n}$$

$$=2^{-2}\times 3^0\times 5^{-3}=\frac{1}{2^2\times 5^3}=\frac{1}{500}=0{\cdot}002$$

(*b*) We have

$$\frac{(3^{2n}-5\times 3^{2n-2})(5^n-3\times 5^{n-2})}{5^{n-4}(9^{n+3}-3^{2n})}=\frac{3^{2n-2}(3^2-5)\,.\,5^{n-2}(5^2-3)}{5^{n-4}\,.\,3^{2n}(3^6-1)}$$

$$=\frac{5^2\times 4\times 22}{3^2\times 728}=\frac{275}{819}=0{\cdot}34$$

Example 13. *If* $\dfrac{9^n\,.\,3^2\,.\,(3^{-n})^{-1}-27^n}{3^{3m}\,.\,2^3}=\dfrac{1}{27}$, *prove that* $m=1+n$.

Solution. We are given that

$$\frac{9^n\,.\,3^2\,.\,(3^{-n})^{-1}-27^n}{3^{3m}\ 2^3}=\frac{1}{27}$$

$$\Rightarrow\quad \frac{(3^2)^n\,.\,3^2\,.(3^{-n})^{-1}-(3^3)^n}{3^{3m}\,.\,2^3}=\frac{1}{27}$$

$$\Rightarrow\quad \frac{3^{2n}\,.\,3^{2+n}-3^{3n}}{3^{3m}\,.\,2^3}=\frac{1}{27},\ \ i.e.,\ \frac{3^{3n+2}-3^{3n}}{3^{3m}\,.\,2^3}=\frac{1}{27}$$

$$\Rightarrow\quad \frac{3^{3n}(9-1)}{3^{3m}\,.\,8}=\frac{1}{27}=\frac{1}{3^3}\ \Rightarrow\ 3^{3n-3m}=3^{-3}$$

$$\Rightarrow\quad 3n-3m=-3$$

$$\Rightarrow\quad n-m=-1$$

Hence $\quad m=n+1$

Example 14. *Show that*

(*a*) $$\left\{\frac{(9^{n+\frac{1}{4}}\sqrt{3\,.\,3^n}}{3\sqrt{3^{-n}}}\right\}^{\frac{1}{n}}=27$$

(*b*) $$\frac{16(32)^m-2^{3m-2}(4)^{m+1}}{15(2)^{m-1}(16)^m}-\frac{5(5)^{m-1}}{\sqrt{5^{2m}}}=1$$

Solution.

(*a*) $$\text{L.H.S.}=\left\{\frac{(3^2)^{n+\frac{1}{4}}\,.\,(3^{n+1})^{\frac{1}{2}}}{3\,.\,(3^{-n})^{\frac{1}{2}}}\right\}^{\frac{1}{n}}$$

$$=\left[\frac{3^{2n+\frac{1}{2}}\,.\,3^{\frac{n+1}{2}}}{3^{1-\frac{n}{2}}}\right]^{\frac{1}{n}}$$

$$=\left[3^{2n+\frac{1}{2}}\,.\,3^{\frac{n+1}{2}}\,.\,3^{-\left(1-\frac{n}{2}\right)}\right]^{\frac{1}{n}}$$

$$=\left[3^{2n+\frac{1}{2}+\frac{n+1}{2}-1+\frac{n}{2}}\right]^{\frac{1}{n}}$$

$$=[3^{3n}]^{\frac{1}{n}}=3^3=27=\text{R.H.S.}$$

(*b*) $$\text{L.H.S.}=\frac{2^4(2^5)^m-2^{3m-2}(2^2)^{m+1}}{15(2^{m-1})(2^4)^m}-\frac{5(5^m \cdot 5^{-1})}{(5^{2m})^{\frac{1}{2}}}$$

$$=\frac{2^{5m+4}-2^{3m-2+2m+2}}{15(2^{m-1+4m})}-\frac{5^{m-1+1}}{5^m}$$

$$=\frac{2^{5m}(2^4-1)}{15(2^{5m} \cdot 2^{-1})}-5^{m-1+1-m}$$

$$=2^{5m-5m} \cdot 2-5^0=2^0 \cdot 2-5^0=2-1=1=\text{R.H.S.}$$

Example 15. *Show that*

(*i*) $$\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{a+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}=1$$

(*ii*) $$\sqrt[m+n]{\frac{x^{m^2}}{x^{n^2}}}\times\sqrt[n+p]{\frac{x^{n^2}}{x^{p^2}}}\times\sqrt[p+m]{\frac{x^{p^2}}{x^{m^2}}}=1$$

(*iii*) $$\left(\frac{q+r}{x^{r-p}}\right)^{\frac{1}{p-q}}\times\left(\frac{r+p}{x^{p-q}}\right)^{\frac{1}{q-r}}\times\left(\frac{p+q}{x^{q-r}}\right)^{\frac{1}{r-q}}=1$$

Solution.

(*i*) $$\text{L.H.S.}=\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}$$

$$=(x^{a-b})^{a+b}\times(x^{b-c})^{b+c}\times(x^{c-a})^{c+a}$$

$$=x^{a^2-b^2}\times x^{b^2-c^2}\times x^{c^2-a^2}$$

$$=x^{a^2-b^2+b^2-c^2+c^2-a^2}=x^0=1=\text{R.H.S.}$$

(*ii*) $$\text{L.H.S.}=\left[\frac{x^{m^2}}{x^{n^2}}\right]^{\frac{1}{m+n}}\times\left[\frac{x^{n^2}}{x^{p^2}}\right]^{\frac{1}{n+p}}\times\left[\frac{x^{p^2}}{x^{m^2}}\right]^{\frac{1}{p+m}}$$

$$=[x^{m^2-n^2}]^{\frac{1}{m+n}}\times[x^{n^2-p^2}]^{\frac{1}{n+p}}\times[x^{p^2-m^2}]^{\frac{1}{p+m}}$$

$$=x^{m-n}\times x^{n-p}\times x^{p-m}$$

$$=x^{m-n+n-p+p-m}=x^0=1=\text{R.H.S.}$$

(iii) $$\text{L.H.S.}=\left[x^{\frac{q+r}{(r-p)(p-q)}}\times x^{\frac{r+p}{(p-q)(q-r)}}\times x^{\frac{p+q}{(q-r)(r-p)}}\right]$$

$$=\left[x^{\frac{(q-r)(p+r)+(r+p)(r-p+(p\div q)(p-q)}{(p-q)(q-r)(r-p)}}\right]$$

$$=\left[x^{\frac{q^2-r^2+r^2-p^2+p^2-q^2}{(p-q)(q-r)(r-p)}}\right]=x^0=1=\text{R.H.S.}$$

Example 16. *If $x+y+z=0$, show that*

$$a^{x^2 \cdot y^{-1} \cdot z^{-1}} \cdot a^{x^{-1} \cdot y^2 \cdot z^{-1}} \cdot a^{x^{-1} \cdot y^{-1} \cdot z^2}=a^3.$$

(*I.C.W.A., December 1989*)

Solution. We have

$$\text{L.H.S.}=a^{x^2 y^{-1} z^{-1}} \cdot a^{x^{-1} y^2 z^{-1}} \cdot a^{x^{-1} y^{-1} z^2}$$

$$=a^{(x^2/yz)+(y^2/xz)+(z^2/xy)}$$

$$=a^{\frac{x^3+y^3+z^3}{xyz}} \quad \ldots(1)$$

Now $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$

$=0$ as $x+y+z=0$ (given)

$\therefore \quad x^3+y^3+z^3=3xyz$

Substituting the value of $x^3+y^3+z^3$ in (1), we get

$$\text{L.H.S.}=a^{\frac{3xyz}{xyz}}=a^3=\text{R.H.S.}$$

Example 17. *Find x, if*

$$x^{\sqrt[x]{x}}=(x\sqrt{x})^x$$

Solution. $$x^{\sqrt[x]{x}}=(x\sqrt{x})^x$$

$$x^{x \cdot x^{\frac{1}{2}}}=(x \cdot x^{1/2})^x$$

$$\Rightarrow \quad x^{x^{3/2}}=(x^{3/2})^x=x^{\frac{3}{2}x}$$

$$\Rightarrow \quad x^{3/2}=\frac{3}{2}x$$

$$\Rightarrow \quad x^{3/2} \cdot x^{-1}=\frac{3}{2}$$

$$\Rightarrow \quad x^{\frac{3}{2}-1}=\frac{3}{2}, \text{ i.e., } x^{\frac{1}{2}}=\frac{3}{2}$$

$$\Rightarrow \quad x=\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

Example 18. (*a*) *Show that*

$$\frac{1}{1+x^{a-b}+x^{a-c}}+\frac{1}{1+x^{b-c}+x^{b-a}}+\frac{1}{1+x^{c-a}+x^{c-b}}=1$$

(*b*) *Simplify the expression*

$$\frac{1}{x^b+x^{-c}+1}+\frac{1}{x^c+x^{-a}+1}+\frac{1}{x^a+x^{-b}+1}$$

given that $a+b+c=0$.

Solution. (*a*) First term $=\frac{1}{1+x^{a-b}+x^{a-c}}=\frac{1}{x^{a-a}+x^{a-b}+x^{a-c}}$

$$=\frac{1}{x^a \cdot x^{-a}+x^a \cdot x^{-b}+x^a \cdot x^{-c}}$$

$$=\frac{1}{x^a[x^{-a}+x^{-b}+x^{-c}]}$$

Second term $=\frac{1}{1+x^{b-c}+x^{b-a}}=\frac{1}{x^{b-b}+x^{b-c}+x^{b-a}}$

$$=\frac{1}{x^b[x^{-a}+x^{-b}+x^{-c}]}$$

Similarly the third term $=\frac{1}{x^c[x^{-a}+x^{-b}+x^{-c}]}$

By taking sum of the three terms, we get

$$\text{L.H.S.}=\frac{1}{(x^{-a}+x^{-b}+x^{-c})}\left[\frac{1}{x^a}+\frac{1}{x^b}+\frac{1}{x^c}\right]$$

$$=\frac{x^{-a}+x^{-b}+x^{-c}}{x^{-a}+x^{-b}+x^{-c}}=1=\text{R.H.S.}$$

(*b*) First term $=\frac{1}{x^b+x^{-c}+1}=\frac{x^c}{x^{b+c}+x^{c-c}+x^c}$

(multiplying denominator and numerator by x^c)

$$=\frac{x^c}{x^{-a}+1-x^c} \qquad (\because\ b+c=-a)$$

Third term $=\frac{1}{x^a+x^{-b}+1}=\frac{x^{-a}}{x^{a-a}+x^{-b-a}+x^{-a}}$

(multiplying denominator and numerator by x^{-a})

$$=\frac{x^{-a}}{1+x^c+x^{-a}} \qquad (\because\ -b-a=c)$$

∴ The given expression

$$=\frac{x^{c}}{x^{-a}+x^{c}+1}+\frac{1}{x^{-a}+x^{c}+1}+\frac{x^{-a}}{x^{-a}+x^{c}+1}$$

$$=\frac{x^{-a}+x^{c}+1}{x^{-a}+x^{c}+1}=1.$$

Example 19. (*a*) *If* $x=3^{\frac{1}{3}}+3^{-\frac{1}{3}}$, *prove that* $3x^3-9x=10$

(*b*) *If* $a\,.\sqrt[3]{x^2}+b\,.\sqrt[3]{x}+c=0$, *then prove that*

$$a^3x^2+b^3x+c^3=3abcx.$$

Solution. (*a*) $x=3^{\frac{1}{3}}+3^{-\frac{1}{3}}$

Cubing both sides, we get

$$x^3=\left(3^{\frac{1}{3}}\right)^3+\left(3^{-\frac{1}{3}}\right)^3+3.3^{\frac{1}{3}}.3^{-\frac{1}{3}}.x$$

[∵ $(a+b)^3=a^3+b^3+3ab(a+b)$]

$$=3+3^{-1}+3\,.\,3^{\circ}\,.\,x$$

$$=3+\tfrac{1}{3}-3x \qquad (\because\ a^{\circ}=1,\ a\neq 0)$$

∴ By transposition, we get $3x^3-9x=10$.

(*b*) $a\,.\sqrt[3]{x^2}+b\,.\sqrt[3]{x}=-c$

$\Rightarrow$ $(a\,.\sqrt[3]{x^2}+b\,.\sqrt[3]{x})^3=(-c)^3$

$\Rightarrow$ $a^3x^2+b^3x+3ab\,.\sqrt[3]{x^2}\,.\sqrt[3]{x}\,(a\,.\sqrt[3]{x^2}+b\,.\sqrt[3]{x})=-c^3$

$\Rightarrow$ $a^3x^2+b^3x+3abx(-c)=-c^3$

$\Rightarrow$ $a^3x^2+b^3x+c^3=3abcx.$

Example 20. (*a*) *If* $a^x=b$, $b^y=c$, $c^z=a$, *prove that* $xyz=1$.

(*b*) *If* $a^x=b^y=c^z$ *and* $b^2=ac$, *prove that*

$$y=\frac{2xz}{x+z}$$

(*c*) *If* $(2{\cdot}381)^x=({\cdot}2381)^y=10^z$, *prove that*

$$\frac{1}{x}=\frac{1}{y}+\frac{1}{z}$$

Solution. (*a*) $a^{xyz}=(a^x)^{yz}=b^{yz}=(b^y)^z=c^z=a=a^1$

$\Rightarrow$ $xyz=1$

(*b*) Let each of the given ratio be equal to k, so that

$$a^x = k \Rightarrow a = k^{1/x}$$

$$b^y = k \Rightarrow b = k^{1/y}$$

$$c^z = k \Rightarrow c = k^{1/z}$$

Substituting these values of a, b, c in terms of k, x, y and z in $b^2 = ac$, we have

$$(k^{1/y})^2 = k^{1/x} \cdot k^{1/z}$$

$$\Rightarrow \quad k^{2/y} = k^{(1/x)+(1/z)}$$

$$\Rightarrow \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$$\Rightarrow \quad \frac{2}{y} = \frac{z+x}{xz}$$

$$\Rightarrow \quad y = \frac{2xz}{z+x}$$

(*c*) $\quad (2{\cdot}381)^x = 10^z \quad \Rightarrow \quad 2{\cdot}381 = 10^{z/x}$

i.e., $\quad \dfrac{2{\cdot}381}{10} = \dfrac{10^{z/x}}{10}$

$$\Rightarrow \quad 0{\cdot}2381 = 10^{z/x} \cdot 10^{-1}$$

$$\Rightarrow \quad {\cdot}2381 = 10^{(z/x)-1}$$

Also $\quad ({\cdot}2381)^y = 10^z \quad \Rightarrow \quad {\cdot}2381 = 10^{z/y}$

$$\therefore \quad 10^{(z/x)-1} = 10^{z/y}$$

$$\Rightarrow \quad \frac{z}{x} - 1 = \frac{z}{y}$$

$$\Rightarrow \quad \frac{z}{x} - \frac{z}{y} = 1$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{y} + \frac{1}{z}$$

Example 21. (*a*) *If* $2^x = 3^y = 12^z$, *prove that* $xy = z(x+2y)$

(*b*) *If* $2^x = 4^y = 8^z$ *and* $xyz = 288$, *prove that*

$$\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{11}{96}$$

Solution. (*a*) The given relation can be written as

$$4^{x/2} = 3^y = 12^z = k \text{ (say)}$$

$$\therefore \quad 4 = k^{2/x},\ 3 = k^{1/y},\ 12 = k^{1/z}$$

Also $\quad 4 \times 3 = 12$

$$\therefore \quad k^{2/x} \cdot k^{1/y} = k^{1/z}$$

$$\Rightarrow \quad k^{(2/x)+(1/y)} = k^{1/z}$$

$\Rightarrow \quad \frac{2}{x}+\frac{1}{y}=\frac{1}{z}$

$\Rightarrow \quad \frac{2y+x}{xy}=\frac{1}{z}$

$\Rightarrow \quad xy=z(x+2y)$

(*b*) We have $2^x=4^y=8^z$,

i.e., $2^x=2^{2y}=2^{3z}$

$\therefore \quad x=2y=3z=k$, say

Also $xyz=288$

$\Rightarrow \quad k \cdot \frac{k}{2} \cdot \frac{k}{3}=288$

$\Rightarrow \quad k^3=6 \cdot 288=6\times 144\times 2$

$=6\times 12\times 12\times 2$

$=6\times 6\times 2\times 6\times 2\times 2$

$=6^3 \cdot 2^3=(2.6)^3$

$\Rightarrow \quad k=12$

$\therefore \quad x=12,\ y=6,\ z=4$

Hence $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z}=\frac{1}{24}+\frac{1}{24}+\frac{1}{32}=\frac{11}{96}$

Example 22. *If* $\frac{x^{n+1}}{a^n}=\frac{y^{n+1}}{b^n}=\frac{z^{n+1}}{c^n}=a+b+c$, *where n is a positive integer, show that*

$$\left[x^{\frac{(n+1)}{n}}+y^{\frac{(n+1)}{n}}+z^{\frac{(n+1)}{n}}\right]^{\frac{n}{n+1}}=a+b+c.$$

Solution. We are given

$$\frac{x^{n+1}}{a^n}=\frac{y^{n+1}}{b^n}=\frac{z^{n+1}}{c^n}=a+b+c$$

$\therefore \quad x^{n+1}=(a+b+c)a^n \quad \ldots(1)$

$y^{n+1}=(a+b+c)b^n \quad \ldots(2)$

$z^{n+1}=(a+b+c)c^n \quad \ldots(3)$

Now $x^{\frac{(n+1)}{n}}=[(a+b+c)a^n]^{\frac{1}{n}}=(a+b+c)^{\frac{1}{n}}\cdot a$

Similarly, $y^{\frac{(n+1)}{n}}=(a+b+c)^{\frac{1}{n}}\cdot b$

and $$z^{\frac{(n+1)}{n}}=(a+b+c)^{\frac{1}{n}}\,.\,c$$

$$\therefore \quad x^{\frac{(n+1)}{n}}+y^{\frac{(n+1)}{n}}+z^{\frac{(n+1)}{n}}$$

$$=(a+b+c)^{\frac{1}{n}}a+(a+b+c)^{\frac{1}{n}}b+(a+b+c)^{\frac{1}{n}}c$$

$$=(a+b+c)^{\frac{1}{n}}[a+b+c]$$

$$=(a+b+c)^{\frac{(1+n)}{n}}$$

Raising both the sides to power $\frac{n}{n+1}$, we get

$$\left[x^{\frac{(n+1)}{n}}+y^{\frac{(n+1)}{n}}+z^{\frac{(n+1)}{n}}\right]^{\frac{n}{(n+1)}}$$

$$=\left[(a+b+c)^{\frac{(n+1)}{n}}\right]^{\frac{n}{(n+1)}}=a+b+c.$$

EXERCISE (I)

1. State with reasons, whether the following statements are true or false.

(*i*) $a^p=a^q \Rightarrow p=q$

(*ii*) $a^m=b^m \Rightarrow a=b$

(*iii*) $2^{3^2}=(2^3)^2$, (*iv*) $3^{3^2}=(3^3)^2$

(*v*) $a>b \Rightarrow a^3>b^3$

(*vi*) $a>b \Rightarrow a^{-1}>b^{-1}$

2. Simplify

(*i*) $(625)^{-\frac{1}{4}}$ (*ii*) $\left(\frac{32}{243}\right)^{\frac{4}{5}}$

(*iii*) $\left(\frac{81}{256}\right)^{-\frac{5}{4}}$ (*iv*) $\frac{\sqrt[3]{(343)^{-2}}}{\sqrt[5]{(32)^{-3}}}$

(*v*) $\sqrt[5]{(32)^{-3}}$

3. Prove that

$$\frac{2^n+2^{n-1}}{2^{n+1}-2^n}=\frac{3}{2}$$

4. (*a*) Find the value of

$$\frac{2^{\frac{1}{2}}\times 3^{\frac{1}{3}}\times 4^{\frac{1}{4}}}{10^{-\frac{1}{5}}\times 5^{\frac{3}{5}}}\div\frac{3^{\frac{4}{3}}\times 5^{-\frac{7}{5}}}{4^{-\frac{3}{5}}\times 6}$$

(*b*) Express in fractional indices and then simplify

(*i*) $\sqrt[3]{a^4} \times \sqrt[4]{a^2} \times \sqrt[6]{a^7}$, (*ii*) $y^{-5/3} \times \sqrt[4]{y^6} \times \sqrt[6]{y}$

5. Simplify

(*i*) $\dfrac{\sqrt[3]{135} - (8/25)^{-2/3}}{(3.645)^{1/3} + (200)^{-1/3}}$

(*ii*) $\sqrt{3 \times 5^{-3}} \div \sqrt[3]{3^{-1}}\sqrt{5} \times \sqrt[6]{3 \times 5^4}$

(*iii*) $\sqrt{81^{-\frac{3}{4}} \div \frac{32}{6^{-2}}^{\frac{1}{5}} \times \left(\frac{1}{27}\right)^{-\frac{4}{3}} + 12\sqrt{16^{-1}}}$

(*iv*) $\dfrac{3^{-3}(6)^2 \times \sqrt{98}}{(5)^2 \times \sqrt[3]{\frac{1}{25}} \times (15)^{-\frac{4}{3}} \times (3)^{\frac{1}{3}}}$

6. Simplify

(*i*) $\dfrac{x^{m+2n} . x^{3m-8n}}{x^{5m-6n}}$, (*ii*) $\dfrac{5^{2x+3} . 10^{x+1}}{25^{3x+2}\ 16^{x-\frac{1}{2}}}$

(*iii*) $\dfrac{7^{n+2} - 35 . 7^{n-1}}{7^n \times 11}$, and (*iv*) $\dfrac{9^n \times 3^2 \times \frac{1}{3^{-n}} - (27)^n}{3^{3n} \times 9}$

7. Show that

(*i*) $\dfrac{3 . 2^{n+1} + 2^n}{2^{n+2} - 2^{n-1}} = 2$, (*ii*) $\dfrac{2 . 3^{n+1} + 7 . 3^{n-1}}{3^{n+2} - 2(\frac{1}{3})^{1-n}} = 1$

(*iii*) $\dfrac{(3)(3^n)}{(3^n)^{n-1}} \div \dfrac{9^{n+1}}{(3^{n-1})^{n-1}} = \dfrac{1}{9^n}$

(*iv*) $\dfrac{(81)^n . 3^5 - 3^{4n-1}(243)}{9^{2n} . 3^3} - \dfrac{4 . 3^n}{3^{n+1} - 3^n} = 4$

8. If $\dfrac{9^n . 3^2(3^{-n/2})^{-2} - (27)^n}{3^{3m} . 2^3} = \dfrac{1}{27}$, show that $m = 1 + n$.

9. Show that

(*i*) $\left(\dfrac{x^b}{x^c}\right)^a \times \left(\dfrac{x^c}{x^a}\right)^b \times \left(\dfrac{x^a}{x^b}\right)^c = 1$

(*ii*) $\left(\dfrac{x^b}{x^c}\right)^{\frac{1}{bc}} \times \left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}} \times \left(\dfrac{x^a}{x^b}\right)^{\frac{1}{ab}} = 1$

(*iii*) $\left(\dfrac{x^c}{x^b}\right)^{a^2+ab+b^2} \times \left(\dfrac{x^b}{x^c}\right)^{b^2+bc+c^2} \times \left(\dfrac{x^c}{x^a}\right)^{c^2+ca+a^2} = 1$

(iv) $\left(x^{\frac{1}{a-b}}\right)^{\frac{1}{a-c}} \times \left(x^{\frac{1}{b-c}}\right)^{\frac{1}{b-a}} \times \left(x^{\frac{1}{c-a}}\right)^{\frac{1}{c-b}} = 1$

(v) $\left(\frac{x^b}{x^c}\right)^{b+c-a} \times \left(\frac{x^c}{x^a}\right)^{c+a-b} \times \left(\frac{x^a}{x^b}\right)^{a+b-c} = 1$

10. Simplify

$$\left(\frac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \times \left(\frac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \times \left(\frac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$$

11. Simplify

(a) $$\left[\left\{\frac{x^{\frac{a}{a-b}}}{x^{\frac{a}{a+b}}}\right\} \div \left\{\frac{x^{\frac{b}{b-a}}}{x^{\frac{b}{b+a}}}\right\}\right]^{a+b}$$

(b) $$\left[\frac{x^{ab}}{x^{a^2+b^2}}\right]^{a+b} \times \left[\frac{x^{b^2+c^2}}{x^{bc}}\right]^{b+c} \times \left[\frac{x^{ca}}{x^{c^2+a^2}}\right]^{c+a}$$

12. Simplify

$$\left(\frac{a^x}{a^y}\right)^{x+y} \times \left(\frac{a^y}{a^z}\right)^{y+z} \div 3(a^x a^z)^{x-z}$$

13. (a) Show that

$$\frac{1}{1+a^{n-m}} + \frac{1}{1+a^{m-n}} = 1$$

(b) If $pqr=1$, show that

$$\frac{1}{1+p+q^{-1}} + \frac{1}{1+q+r^{-1}} + \frac{1}{1+r+p^{-1}} = 1$$

[**Hint.** L.H.S. $= \frac{q}{q+pq+1} + \frac{1}{1+q+pq} + \frac{p}{p+pr+1}$

$= \frac{q}{q+pq+1} + \frac{1}{1+q+pq} + \frac{pq}{q(p+pr+1)}$

$= \frac{q}{q+pq+1} + \frac{1}{1+q+pq} + \frac{pq}{pq+1+p}$]

14. (i) If $2^x=3^y=(12)^z$, show that $\frac{1}{z} = \frac{1}{y} + \frac{2}{x}$.

(ii) If $2^x=3^y=6^{-z}$; show that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

(iii) If $3^x=5^y=(75)^z$, show that $xy=z(2x+y)$.

15. If $a^x = \left(\frac{a}{k}\right)^y = k^m$, prove that $\frac{1}{x} - \frac{1}{y} = \frac{1}{m}$.

16. If $2^x=4^y=8^z$ and $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z}=\frac{22}{7}$, show that

$$x=\frac{7}{16},\ y=\frac{7}{32},\ z=\frac{7}{48}$$

17. If $a^x=b^y=c^z=d^w$ and $ab=cd$, show that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z}+\frac{1}{w}$$

18. If $x^y=y^x$, show that $\left(\frac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$

and if $x=2y$, prove that $y=2$.

19. If $m=a^x$, $n=a^y$ and $(m^y \cdot n^x)^z=a^2$, show that $xyz=1$.

20. (*i*) If $a=xy^{p-1}$, $b=xy^{q-1}$, $c=xy^{r-1}$, show that

$$a^{q-r}\times b^{r-p}\times c^{p-q}=1$$

(*ii*) If $a=x^{q+r}y^p$, $b=x^{r+p}y^q$, $c=x^{p+q}y^r$, show that

$$a^{q-r}\times b^{r-p}\times c^{p-q}=1.$$

21. Obtain the simplest value of

$$[1-\{1-(1-x^3)^{-1}\}^{-1}]^{-1/3},\ \text{when } x=0{\cdot}1$$

22. If $x=2^{\frac{1}{3}}+2^{\frac{2}{3}}$, show that $x^3-6x=6$

23. (*i*) If $x=3^{\frac{2}{3}}+3^{-\frac{2}{3}}$, show that $9x^3-27x=82$

(*ii*) If $x=\sqrt[3]{\sqrt{2}+1}-\sqrt[3]{\sqrt{2}-1}$, show that $x^3+3x=2$

(*iii*) If $x=a^{\frac{1}{3}}+a^{-\frac{1}{3}}$, show that $x^3-3x=a+\frac{1}{a}$

24. If $x=3^{\frac{1}{4}}+3^{-\frac{1}{4}}$, $y=3^{\frac{1}{4}}-3^{-\frac{1}{4}}$, show that

$$3(x^2+y^2)^2=64$$

25. Simplify

$$\sqrt[5]{\{x^2\times\sqrt[3]{(x^{-2}y^{3/2})}\div\sqrt[5]{(x^{-3}y^{-4})}}$$

ANSWERS

1. (*iii*) $2^9\neq 2^6$, (*iv*) $3^9\neq 3^6$, (*vi*) $a>b \Rightarrow a^{-1}<b^{-1}$

2. (*i*) $\frac{1}{5}$, (*ii*) $\frac{16}{81}$, (*iii*) $\frac{1024}{243}$, (*iv*) $\frac{8}{49}$, (*v*) $\frac{1}{8}$

4. (*a*) 10, (*b*) (*i*) a^3, (*ii*) 1

5. (*i*) $\frac{7}{4}$, (*ii*) $\frac{3}{5}$, (*iii*) 3·204, (*iv*) $28\sqrt{2}$

6. (*i*) x^{-m}, (*ii*) 8, (*iii*) 4, (*iv*) $\frac{8}{9}$

10. $x^{2(a^3+b^3+c^3)}$

11. (*a*) 1, (*b*) $\frac{1}{x^{2a^3}}$ 12. $\frac{1}{3}$ 21. 0·1

25. $x^{\frac{29}{75}}, y^{\frac{13}{50}}$

6·5. SURDS

In the discussion on the theory of real number system we have seen that the numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$ etc. do not belong to the set of rational numbers and are known as irrational numbers. An irrational number we recall is one which cannot be expressed as the ratio of two integers in the form p/q, where p and q are the integers. For example $\frac{1{\cdot}5}{2}$, 3·14 are rational as they are equal to $\frac{3}{4}$ and $\frac{314}{100}$ respectively. However $\sqrt{2}$ is not a rational number as it is equal to 1·4142 to the nearest ten-thousand. Similarly $\pi=3{\cdot}1416$, $\frac{\sqrt{5}}{2}$, $\frac{2}{\sqrt{2}}$ and $2+\sqrt{7}$ are irrational numbers.

In this chapter we shall discuss the particular types of irrational numbers called surds. In other words, we shall confine to a subset of the set of irrational numbers.

Definition 1. *A surd is defined as the irrational root of a rational number, of the type* $\sqrt[n]{a}$, *where it is not possible to extract exactly the* n^{th} *root of "a"*. In other words, a real number $\sqrt[n]{a}$ is called a surd, if and only if

(*i*) it is an irrational number, and

(*ii*) it is a root of a rational number.

In the surd $\sqrt[n]{a}$, the index n is called the order of the surd and 'a' the radicand.

Illustration. $\sqrt{3}$ is a surd, since $\sqrt{3}$ is the irrational root of the rational number 3.

2. $\sqrt[3]{5}$, $\sqrt[3]{\sqrt[3]{8}}$, $(32)^{1/4}$ are surds.

3. (*i*) $\sqrt[3]{8}$ is not a surd, its root 2 is rational.

(*iii*) $\sqrt[3]{\frac{27}{8}}$ $\sqrt{\{\sqrt[3]{729}\}}$ are not surds, their roots are rational.

4. $\sqrt{2+\sqrt{2}}$ although an irrational number is not a surd because it is the square root of an irrational number.

5. Similarly $\sqrt{\pi}$, $\sqrt[3]{3+\sqrt{5}}$, $\sqrt{\sqrt{5}-\sqrt{7}}$ are not surds as the radicand of each of these is not a rational number.

Order of a surd. The order of the surd is the number which indicates the root, *e.g.*, $\sqrt{48}$, $\sqrt[3]{17}$, $\sqrt[4]{21}$ and $\sqrt[n]{a}$ are second, third, fourth and n^{th} order respectively. However, surds of different order can be converted into same order as follows :

$$\sqrt{3},\ \sqrt[3]{2},\ \sqrt[4]{7} = 3^{1/2},\ 2^{1/3},\ 7^{1/4} = 3^{6/12},\ 2^{4/12},\ 7^{3/12}$$
$$= \sqrt[12]{3^6},\ \sqrt[12]{2^4},\ \sqrt[12]{7^3}$$

Surds of second, third and fourth order are called quadratic, cubic and quartio surds, respectively.

A surd may be with or without a coefficient, for example, $\frac{3}{5}\sqrt[4]{7}$ has a coefficient while $\sqrt[5]{4}$ has no coefficient. The former is called the mixed surd, the coefficient there could be positive or negative. In the latter case, where there is no coefficient, it is presumed to have ± 1 as the coefficient and it is called an entire surd, *e.g.*,

(*i*) $-5\sqrt[3]{4}$ is a mixed surd,

(*ii*) $\sqrt{2}$ is an entire surd.

But a mixed surd can also be written as an entire surd, *e.g.*, $-\sqrt[3]{48}$ as $-\sqrt[3]{8.6}$ or $2\sqrt[3]{6}$ and $\sqrt{27}$ as $\sqrt{9.3}$ or $3\sqrt{3}$.

The surds of the type $\sqrt[n]{a}$ where '*a*' is not a prime integer can be split into prime integers. For example

(*i*) $\sqrt[3]{16} = 2.\sqrt[3]{2}$ and (*ii*) $\sqrt{\frac{5}{3}} = \frac{1}{3}\sqrt{15} = \frac{1}{3}(\sqrt{5}\times\sqrt{3})$.

6·6. OPERATIONS ON SURDS

Surds can always be expressed with fractional indices. Therefore, the rules of indices given earlier will apply to them also. These are stated here in the radical form.

(*i*) $\sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab}$ or $a^{1/n}.b^{1/n} = (ab)^{1/n}$

(*ii*) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ or $a^{1/n} \div b^{1/n} = \left(\dfrac{a}{b}\right)^{1/n}$

(*iii*) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$ or $(a^{1/n})^{1/m} = a^{1/mn}$

(*iv*) $\sqrt[m]{a^b} = \sqrt[mn]{a^{bn}}$ or $(a^b)^{1/m} = (a^{bn})^{1/mn}$

(*v*) $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$ or $(a^{1/n})^m = a^{m/n}$.

In the above rules m and n are positive integers and a and b are rational numbers.

Illustrations. 1. $\sqrt[3]{3}\times\sqrt[3]{4} = 3^{1/3}.4^{1/3} = (3.4)^{1/3} = (12)^{1/3} = \sqrt[3]{12}$

2. $\dfrac{\sqrt[4]{5}}{\sqrt[4]{3}} = \dfrac{5^{1/4}}{3^{1/4}} = \left(\dfrac{5}{3}\right)^{1/4} = \sqrt[4]{\dfrac{5}{3}}$

3. (i) $\sqrt[5]{\sqrt{a}}=[a^{1/2}]^{1/5}=(a)^{1/10}=\sqrt[10]{a}$

(ii) $\sqrt{\sqrt[5]{a}}=[a^{1/5}]^{1/2}=(a)^{1/10}=\sqrt[10]{a}$

4. (i) $\sqrt[4]{2^3}=(2^3)^{1/4}=(2)^{3/4}=\sqrt[4]{2^3}$

(ii) $\sqrt[4]{2^3}=\sqrt[4\times3]{2^{3\times3}}=\sqrt[12]{2^9}$

It may be noted that the order of the surd can be changed by multiplying the surd index and the index of the radicand by the same integer.

Remark. Two surds are similar if they can be reduced to the same irrational factors. For example $\sqrt{48}$ and $\sqrt{147}$ are similar surds because they can be put as $4\sqrt{3}$ and $7\sqrt{3}$.

Rules for Operations on Surds :

1. Surds of the same order can be multiplied as follows :

$$\sqrt[n]{a}\times\sqrt[n]{b}=a^{1/n}\times b^{1/n}=(ab)^{1/n}=\sqrt[n]{ab}$$

With rational coefficients also, they can be multiplied as follows ;

$$p\,.\sqrt[n]{a}\times q\,.\sqrt[n]{b}=pq\,.\sqrt[n]{ab}$$

Example 23. *Multiply* $6\,.\sqrt[3]{4}$ *by* $3\,.\sqrt[3]{2}$

Solution. $6\,.\sqrt[3]{4}\times3\,.\sqrt[3]{2}=6.3\sqrt[3]{4\times2}=18\,.\,\sqrt[3]{8}=18.2=36$

2. Surds of the same order can be divided as follows :

$$\sqrt[n]{a}\div\sqrt[n]{b}=a^{1/n}\div b^{1/n}=\left(\frac{a}{b}\right)^{1/n}=\sqrt[n]{\frac{a}{b}}$$

With rational coefficients they can be divided as follows :

$$p\cdot\sqrt[n]{a}\div q\cdot\sqrt[n]{b}=pa^{1/n}\div qb^{1/n}=\frac{p}{q}\left(\frac{a}{b}\right)^{1/n}=\frac{p}{q}\times\sqrt[n]{\frac{a}{b}}$$

3. Surds which are not of the same order can be reduced to the lowest possible common order by multiplying both the index and the radicand by the same number as per rule shown below :

$$\sqrt[m]{a^n}=\sqrt[pm]{a^{pn}}=a^{pn/pm}$$

Example 24. *Multiply* $5\sqrt[3]{6}$ *by* $3\sqrt{2}$.

Solution. It will be necessary to first reduce the two quantities to a common order.

Now $\qquad 5\,.\sqrt[3]{6}\times3\sqrt{2}$

$\Rightarrow \qquad 5\,.\sqrt[3\times2]{6^2}\times3\,.\sqrt[3\times2]{2^3}$

$\Rightarrow \qquad 5\times3\sqrt[6]{36\times8}$ or $15\,.\sqrt[6]{288}$.

Remark. If the surds are of the same order, multiply the rational and irrational factors separately. But, if they are of different order then first reduce them to the same order and then proceed.

Example 25. *Multiply* $\left(\sqrt{5}-\sqrt{3}+\sqrt{\frac{1}{2}}\right)$ *by* $\sqrt{3}+\sqrt{2}$

Solution. Required product is

$$=\sqrt{3}\left(\sqrt{5}-\sqrt{3}+\sqrt{\frac{1}{2}}\right)+\sqrt{2}\left(\sqrt{5}-\sqrt{3}+\sqrt{\frac{1}{2}}\right)$$

$$=\sqrt{15}-3+\sqrt{\frac{3}{2}}+\sqrt{10}-\sqrt{6}+1$$

$$=\sqrt{15}-3+\frac{1}{2}\sqrt{6}+\sqrt{10}-\sqrt{6}+1$$

$$=\sqrt{15}+\sqrt{10}-\frac{1}{2}\sqrt{6}-2.$$

Example 26. *Divide* $4\,.\,\sqrt[3]{10}$ *by* $5\,.\,\sqrt[6]{3}$

Solution. $4\,.\,\sqrt[3]{10}\div 5\,.\,\sqrt[6]{3}$

$$=4\,.\,\sqrt[3\times 2]{10^2}\div 5\,.\,\sqrt[6]{3}$$

$$=\frac{4}{5}\,.\,\sqrt[6]{\frac{100}{3}}$$

Simplification for Addition and Subtraction of Surds can be effected by taking the common factor out in the manner indicated below :

$$p\,.\,\sqrt[n]{a}+q\,.\,\sqrt[n]{a}=(p+q)\,.\,\sqrt[n]{a}$$

$$p\,.\,\sqrt[n]{a}-q\,.\,\sqrt[n]{a}=(p-q)\,.\,\sqrt[n]{a}$$

Example 27. *Simplify.* $2\sqrt{180}-7\sqrt{20}+10\sqrt{45}$

Solution. We can have

$$(2\sqrt{36}\times\sqrt{5})-(7\sqrt{5}\times\sqrt{4})+(10\sqrt{9}\times\sqrt{5})$$

$$=2\,.\,6\sqrt{5}-7\,.\,2\,.\,\sqrt{5}+10\,.\,3\sqrt{5}$$

$$=12\sqrt{5}-14\,.\,\sqrt{5}+30\sqrt{5}$$

$$=(12-14+30)\sqrt{5}$$

$$=28\sqrt{5}$$

Example 28. *Simplify* $3\sqrt{147}-\frac{7}{3}\sqrt{\frac{1}{3}}+7\sqrt{\frac{1}{3}}$

Solution. We have $3\sqrt{49}\,.\,\sqrt{3}-\frac{7\sqrt{3}}{3\sqrt{3}\sqrt{3}}+\frac{7\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$

$$=3\times 7\,.\,\sqrt{3}-\frac{7}{9}\,.\,\sqrt{3}+\frac{7}{3}\sqrt{3}$$

$$=\left(21-\frac{7}{9}+\frac{7}{3}\right)\sqrt{3}$$

$$=\frac{203}{9}\sqrt{3}.$$

Simplification for Arrangement. For arranging surds in an order it is always advisable to convert them into surds of lowest possible common index.

Example 29. *Arrange* $\sqrt[3]{4}, \sqrt[4]{5}, \sqrt[4]{7}$ *in an ascending order.*

Solution. First we have to make the index for all the quantities equal to 12 and then compare. These would be

$$\sqrt[12]{4^4}, \quad \sqrt[12]{5^3} \quad \text{and} \quad \sqrt[12]{7^3}$$

or

$$\sqrt[12]{256}, \quad \sqrt[12]{125}, \quad \text{and} \quad \sqrt[12]{343}$$

In ascending order the surds will be placed as

$\sqrt[12]{125}, \sqrt[12]{256}, \sqrt[12]{343}$, *i.e.*, $\sqrt[4]{5}, \sqrt[3]{4}, \sqrt[4]{7}$

The division of two surd expressions can be effected by rationalising as discussed below :

6·7. RATIONALISING FACTOR (R.F.)

When the product of two surds is rational then each one of them is called the rationalising factor of the other. The following are illustrations of rationalising factors :

(*a*) *R.F. of Monomial Surds.* The rationalising factors of monomial surds are obtained by inspection as follows :

(*i*) $a\sqrt{x} \times \sqrt{x} = ax$

(*ii*) $3 \cdot \sqrt[4]{8} \times \sqrt[4]{2} = 3(2^3)^{1/4}(2)^{1/4} = 3 \cdot 2^{4/4} = 6$

(*iii*) $(a^{3/2} \cdot b^{-1/3} \cdot c^{2/5}) \times (a^{1/2} \cdot b^{-2/3} \cdot c^{3/5}) = a^2 b^{-1} c = \frac{a^2 c}{b}$

(*b*) *R.F. of Binomial Quadratic Surds* :

(*i*) $(\sqrt{9} - \sqrt{7})(\sqrt{9} + \sqrt{7}) = (\sqrt{9})^2 - (\sqrt{7})^2 = 9 - 7 = 2$

(*ii*) $(\sqrt{x} - \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$

(*iii*) $(a\sqrt{x} + b\sqrt{y})(a\sqrt{x} - b\sqrt{y}) = a^2x - b^2y$

(*c*) *R.F. of Trinomial Quadratic Surds.* The surd expressions

(*i*) $(\sqrt{x} \pm \sqrt{y} \pm \sqrt{z})$, (*ii*) $(a\sqrt{x} \pm b\sqrt{y} \pm c\sqrt{z})$

are called trinomial quadratic surds. The following method is applied for finding a R.F. of (*ii*) above.

$$(a\sqrt{x} + b\sqrt{y} + c\sqrt{z})(a\sqrt{x} + b\sqrt{y} - c\sqrt{z}) = (a\sqrt{x} + b\sqrt{y})^2 - (c\sqrt{z})^2$$
$$= a^2x + b^2y + 2ab\sqrt{xy} - c^2z$$
$$= (a^2x + b^2y - c^2z) + 2ab\sqrt{xy}$$

Again

$$[(a^2x + b^2y - c^2z) + 2ab\sqrt{xy}]\,[(a^2x + b^2y - c^2z) - 2ab\sqrt{xy}]$$
$$= (a^2x + b^2y - c^2z)^2 - 4a^2b^2xy, \text{ which is rational}$$

Remarks. The quantities $a - \sqrt{b}$ and $a + \sqrt{b}$ are called the conjugate binomial surds. The additional feature of them is that the sum and product of these factors is also a rational quantity, *e.g.*, and $(a+\sqrt{b})+(a-\sqrt{b})=2a$

$$(a-\sqrt{b})\times(a+\sqrt{b})=a^2-b$$

Many times R.F.'s can be obtained by using algebraic identities such as

(i) $x^3+y^3=(x+y)(x^2-xy+y^2)$

(ii) $x^3-y^3=(x-y)(x^2+xy+y^2)$, (iii) $x^6-y^6=(x^3-y^3)(x^3+y^3)$

(iv) $x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)$

(v) $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

Example 30. *Rationalise* $\dfrac{7\sqrt{3}-5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$

Solution $\dfrac{7\sqrt{3}-5\sqrt{2}}{\sqrt{48}+\sqrt{18}}\times\dfrac{\sqrt{48}-\sqrt{18}}{\sqrt{48}-\sqrt{18}}$

$$=\frac{(7\sqrt{3}-5\sqrt{2})(4\sqrt{3}-3\sqrt{2})}{48-18}$$

$$=\frac{7\sqrt{3}(4\sqrt{3}-3\sqrt{2})-5\sqrt{2}(4\sqrt{3}-3\sqrt{2})}{30}$$

$$=\frac{84-21\sqrt{6}-20\sqrt{6}+30}{30}$$

$$=\frac{114-41\sqrt{6}}{30}$$

Example 31. *Rationalise* $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{10}}$

Solution. $=\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{10}}\times\dfrac{\sqrt{2}+\sqrt{3}-\sqrt{10}}{\sqrt{2}+\sqrt{3}-\sqrt{10}}$

$$=\frac{\sqrt{2}+\sqrt{3}-\sqrt{10}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{10})^2}=\frac{\sqrt{2}+\sqrt{3}-\sqrt{10}}{2+3+2\sqrt{6}-10}$$

$$=\frac{\sqrt{2}+\sqrt{3}-\sqrt{10}}{-5+2\sqrt{6}}$$ (Rationalise further)

$$=\frac{\sqrt{2}+\sqrt{3}-\sqrt{10}}{-5+2\sqrt{6}}\times\frac{-5-2\sqrt{6}}{-5-2\sqrt{6}}$$

$$=\frac{(\sqrt{2}+\sqrt{3}-\sqrt{10})(-5-2\sqrt{6})}{25-24}$$

$$=(\sqrt{2}+\sqrt{3}-\sqrt{10})(-5-2\sqrt{6})$$

$$=4\sqrt{15}+5\sqrt{10}-9\sqrt{3}-11\sqrt{2}.$$

Example 32. *Divide* $\sqrt{98}-\sqrt{50}$ *by* $\sqrt{12}$.

Solution. The required quotient is

$$=\frac{\sqrt{98}-\sqrt{50}}{\sqrt{12}}=\frac{\sqrt{49\times 2}-\sqrt{25\times 2}}{\sqrt{4\times 3}}$$

$$=\frac{7\sqrt{2}-5\sqrt{2}}{2\sqrt{3}}=\frac{2\sqrt{2}}{2\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\frac{1}{3}\sqrt{6}.$$

Properties of Bi-quadratic Surds :

1. If $a+\sqrt{b}=x+\sqrt{y}$ or $a-\sqrt{b}=x-\sqrt{y}$

then $a=x$ and $b=y$

where a and x are rational $\sqrt{b}$ and $\sqrt{y}$ are surds.

2. If $\sqrt{a}=b+\sqrt{c}$ then $b=0$ and $a=c$

where b is rational and $\sqrt{a}$ and $\sqrt{c}$ are surds.

3. If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ ...(*)

then $\sqrt{a-\sqrt{b}}=\pm(\sqrt{x}-\sqrt{y})$

where a, b, x, y are all positive rational numbers and $\sqrt{b}$ is the only surd.

Proof. Squaring both sides of (*), we get

$$a+\sqrt{b}=x+y+2\sqrt{xy}$$

$\Rightarrow$ $a=x+y$ and $\sqrt{b}=\sqrt{xy}$ (From first property)

$\Rightarrow$ $a-\sqrt{b}=(x+y)-2\sqrt{xy}=(\sqrt{x}-\sqrt{y})^2$

$\Rightarrow$ $\sqrt{a-\sqrt{b}}=\pm(\sqrt{x}-\sqrt{y})$

Example 33. *If* $x=\sqrt{3}+\frac{1}{\sqrt{3}}$, *calculate the value of*

$$\left(x-\frac{\sqrt{126}}{\sqrt{42}}\right)\left(x-\frac{1}{x-\frac{2\sqrt{3}}{3}}\right)$$

correct to two places of decimal.

Solution. $x-\frac{\sqrt{126}}{\sqrt{42}}=x-\frac{\sqrt{3}\sqrt{42}}{\sqrt{42}}=x-\sqrt{3}$

$$=\sqrt{3}+\frac{1}{\sqrt{3}}-\sqrt{3}=\frac{1}{\sqrt{3}}$$

$$x-\frac{2\sqrt{3}}{3}=x-\frac{2}{\sqrt{3}}=\sqrt{3}+\frac{1}{\sqrt{3}}-\frac{2}{\sqrt{3}}=\sqrt{3}-\frac{1}{\sqrt{3}}$$

$\therefore$ Given expression$=\frac{1}{\sqrt{3}}\left[\sqrt{3}+\frac{1}{\sqrt{3}}-\frac{\sqrt{3}}{2}\right]$

$$=1+\frac{1}{3}-\frac{1}{2}=\frac{5}{6}=0.83$$

Example 34. (*a*) *Simplify*

$$\frac{4\sqrt{3}}{2-\sqrt{2}}-\frac{30}{4\sqrt{3}-\sqrt{18}}-\frac{\sqrt{18}}{3+2\sqrt{3}}$$

(*b*) *Simplify* :

$$\frac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}+\frac{2\sqrt{3}}{\sqrt{6}+2}.$$

(*C.A. Intermediate, November 1981*)

Solution. (*a*) Given expression$=\frac{4\sqrt{3}}{2-\sqrt{2}}-\frac{30}{4\sqrt{3}-\sqrt{9}\times\sqrt{2}}-\frac{\sqrt{9}\times\sqrt{2}}{3+2\sqrt{3}}$

$$=\frac{4\sqrt{3}}{2-\sqrt{2}}-\frac{30}{4\sqrt{3}-3\sqrt{2}}-\frac{3\sqrt{2}}{3+2\sqrt{3}}$$

Rationalising each term, we get

$$\frac{4\sqrt{3}}{2-\sqrt{2}}=\frac{4\sqrt{3}}{2-\sqrt{2}}\times\frac{2+\sqrt{2}}{2+\sqrt{2}}=\frac{8\sqrt{3}+4\sqrt{6}}{4-2}$$

$$=\frac{2(4\sqrt{3}+2\sqrt{6})}{2}=(4\sqrt{3}+2\sqrt{6})$$

$$\frac{30}{4\sqrt{3}-3\sqrt{2}}=\frac{30}{4\sqrt{3}-3\sqrt{2}}\times\frac{4\sqrt{3}+3\sqrt{2}}{4\sqrt{3}+3\sqrt{2}}$$

$$=\frac{120\sqrt{3}+90\sqrt{2}}{48-18}=4\sqrt{3}+3\sqrt{2}$$

and $\frac{3\sqrt{2}}{3+2\sqrt{3}}=\frac{3\sqrt{2}}{3+2\sqrt{3}}\times\frac{3-2\sqrt{3}}{3-2\sqrt{3}}=\frac{9\sqrt{2}-6\sqrt{6}}{9-12}$

$$=\frac{3(3\sqrt{2}-2\sqrt{6})}{-3}=-3\sqrt{2}+2\sqrt{6}$$

$\therefore$ $\frac{4\sqrt{3}}{2-\sqrt{2}}-\frac{30}{4\sqrt{3}-\sqrt{18}}-\frac{\sqrt{18}}{3+2\sqrt{3}}$

$$=4\sqrt{3}+2\sqrt{6}-4\sqrt{3}-3\sqrt{2}+3\sqrt{2}-2\sqrt{6}=0.$$

(*b*) We have

$$\frac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}+\frac{2\sqrt{3}}{\sqrt{6}+2}$$

$$=\frac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}\times\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}+\sqrt{3}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}\times\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}+\frac{2\sqrt{3}}{\sqrt{6}+2}\times\frac{\sqrt{6}-2}{\sqrt{6}-2}$$

$$=\frac{3\sqrt{2}(\sqrt{6}+\sqrt{3})}{6-3}-\frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{6-2}+\frac{2\sqrt{3}(\sqrt{6}-2)}{6-4}$$

$$=\sqrt{2}(\sqrt{6}+\sqrt{3})-\sqrt{3}(\sqrt{6}+\sqrt{2})+\sqrt{3}(\sqrt{6}-2)$$

$$=\sqrt{12}+\sqrt{6}-\sqrt{18}-\sqrt{6}+\sqrt{18}-2\sqrt{3}$$

$$=0.$$

Example 35. *If* $x=\frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$ *find the value of*

$$\frac{x+\sqrt{8}}{x-\sqrt{8}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}$$

Solution. Given $x=\frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$, $x^2=\frac{96}{5+2\sqrt{6}}$

Now

$$\frac{x+\sqrt{8}}{x-\sqrt{8}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}=\left(1+\frac{x+\sqrt{8}}{x-\sqrt{8}}\right)+\left(\frac{x+\sqrt{12}}{x-\sqrt{12}}-1\right)$$

$$=\frac{2x}{x-\sqrt{8}}+\frac{2\sqrt{12}}{x-\sqrt{12}}$$

$$=\frac{2x}{x-2\sqrt{2}}+\frac{4\sqrt{3}}{x-2\sqrt{3}}$$

$$=\frac{2x^2-8\sqrt{6}}{x^2-2x(\sqrt{2}+\sqrt{3})+4\sqrt{6}}$$

$$=\frac{\frac{192}{5+2\sqrt{6}}-8\sqrt{6}}{\frac{96}{5+2\sqrt{6}}-\frac{4\sqrt{24}}{\sqrt{2}+\sqrt{3}}(\sqrt{2}+\sqrt{3})+4\sqrt{6}}$$

$$=\frac{\{192-8\sqrt{6}(5+2\sqrt{6})\}}{96-4\sqrt{24}\,(5+2\sqrt{6})+4\sqrt{6}(5+2\sqrt{6})}$$

$$=\frac{192-40\sqrt{6}-96}{96-20\sqrt{24}-8\sqrt{144}+20\sqrt{6}+48}$$

$$=\frac{2(96-20\sqrt{6}-48)}{96-20\sqrt{6}-48}=2.$$

Example 36. *If* $2\sqrt{54}+5\sqrt{294}+\frac{19}{30}\sqrt{6}-\sqrt{\frac{27}{50}}-\sqrt{\frac{2}{3}}$ $=a\sqrt{6}$, *find a.*

Solution.

$$\text{L.H.S.}=2\sqrt{9\times6}+5\sqrt{49\times6}+\frac{19}{30}\sqrt{6}-\sqrt{\frac{27\times2}{50\times2}}-\sqrt{\frac{2\times3}{3\times3}}$$

$$=(2\times3\sqrt{6})+(5\times7\sqrt{6})+\frac{19}{30}\sqrt{6}-\frac{3}{10}\sqrt{6}-\frac{1}{3}\sqrt{6}$$

$$=6\sqrt{6}+35\sqrt{6}+\frac{19}{30}\sqrt{6}-\frac{3}{10}\sqrt{6}-\frac{1}{3}\sqrt{6}$$

$$=41\sqrt{6}+\frac{1}{30}[19\sqrt{6}-9\sqrt{6}-10\sqrt{6}]$$

$$=41\sqrt{6}+0=41\sqrt{6}$$

$\therefore \quad 41\sqrt{6}=a\sqrt{6}$

Hence $\quad a=41$

Example 37. (*a*) *If* $\dfrac{1+\sqrt{48}}{5\sqrt{3}+4\sqrt{2}-\sqrt{72}-\sqrt{108}+\sqrt{8}+2}$ $=a+b\sqrt{3}$, *find a and b.*

(*b*) *If* $\dfrac{4+\sqrt{18}}{4\sqrt{48}-\sqrt{128}+\sqrt{200}-8\sqrt{12}+5\sqrt{8}}=a+b\sqrt{2}$, *find a and b.*

Solution. (*a*) $\text{L.H.S.}=\dfrac{1+\sqrt{16\times3}}{5\sqrt{3}+4\sqrt{2}-\sqrt{36\times2}-\sqrt{36\times3}+\sqrt{4\times2}+2}$

$$=\frac{1+4\sqrt{3}}{5\sqrt{3}+4\sqrt{2}-6\sqrt{2}-6\sqrt{3}+2\sqrt{2}+2}$$

$$=\frac{1+4\sqrt{3}}{2-\sqrt{3}}=\frac{1+4\sqrt{3}}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$=\frac{2+8\sqrt{3}+\sqrt{3}+4\sqrt{3}\sqrt{3}}{4-3}$$

$$=2+9\sqrt{3}+4\times3=14+9\sqrt{3}$$

$\therefore \quad 14+9\sqrt{3}=a+b\sqrt{3}$

Hence $\quad a=14,\ b=9$

(*b*) $\text{L.H.S.}=\dfrac{4+3\sqrt{2}}{4\times4\sqrt{3}-8\sqrt{2}+10\sqrt{2}-8\times2\sqrt{3}+5\times2\sqrt{2}}$

$$=\frac{4+3\sqrt{2}}{12\sqrt{2}}\times\frac{12\sqrt{2}}{12\sqrt{2}}$$

$$=\frac{48\sqrt{2}+72}{288}$$

$$\therefore \qquad \frac{72}{288}+\frac{48}{288}\sqrt{2}=a+b\sqrt{2}$$

Hence $\qquad a=\frac{1}{4},\ b=\frac{1}{6}$

Example 38. *Find the values of a, b, c and d if*

$$\frac{1}{1+\sqrt{5}+\sqrt{3}}=a+b\sqrt{3}+c\sqrt{5}+d\sqrt{15}$$

Solution. $\dfrac{1}{1+\sqrt{5}+\sqrt{3}}=\dfrac{1}{1+\sqrt{5}+\sqrt{3}}\times\dfrac{1+\sqrt{5}-\sqrt{3}}{1+\sqrt{5}-\sqrt{3}}$

$$=\frac{1+\sqrt{5}-\sqrt{3}}{(1+\sqrt{5})^2-(\sqrt{3})^2}$$

$$=\frac{1+\sqrt{5}-\sqrt{3}}{3+2\sqrt{5}}\times\frac{3-2\sqrt{5}}{3-2\sqrt{5}}$$

$$=\frac{(1+\sqrt{5}-\sqrt{3})(3-2\sqrt{5})}{9-(2\sqrt{5})^2}$$

$$=\frac{3+3\sqrt{5}-3\sqrt{3}-2\sqrt{5}-10+2\sqrt{15}}{9-20}$$

$$=\frac{-7+\sqrt{5}-3\sqrt{3}+2\sqrt{15}}{-11}$$

$$=\frac{7}{11}-\frac{1}{11}\sqrt{5}+\frac{3}{11}\sqrt{3}-\frac{2}{11}\sqrt{15}$$

$$\therefore \qquad a=\frac{7}{11},\ b=\frac{3}{11},\ c=-\frac{1}{11} \text{ and } d=-\frac{2}{11}$$

Example 39. *If* $x=\dfrac{5-\sqrt{21}}{2}$, *prove that*

$$\left(x^3+\frac{1}{x^3}\right)-5\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=0$$

Solution. $\qquad x=\dfrac{5-\sqrt{21}}{2}$

$$\therefore \qquad \frac{1}{x}=\frac{2}{5-\sqrt{21}}$$

$$=\frac{2}{5-\sqrt{21}}\times\frac{5+\sqrt{21}}{5+\sqrt{21}} \qquad \text{(Rationalising)}$$

$$=\frac{2(5+\sqrt{21})}{25-21}=\frac{5+\sqrt{21}}{2}$$

$$\therefore \quad x+\frac{1}{x}=\frac{5-\sqrt{21}}{2}+\frac{5+\sqrt{21}}{2}=5 \qquad \ldots(1)$$

Squaring both sides, we get

$$\left(x+\frac{1}{x}\right)^2=25 \quad \Rightarrow \quad x^2+\frac{1}{x^2}+2=25$$

$$\Rightarrow \quad x^2+\frac{1}{x^2}=23 \qquad \ldots(2)$$

Cubing both sides of (1), we get

$$\left(x+\frac{1}{x}\right)^3=125$$

$$\Rightarrow \quad x^3+\frac{1}{x^3}+3\,.\,x\,.\,\frac{1}{x}\left(x+\frac{1}{x}\right)=125$$

$$\Rightarrow \quad x^3+\frac{1}{x^3}=125-3.5=110$$

Hence $\left(x^3+\frac{1}{x^3}\right)-5\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)$

$$=110-5\,.\,23+5=0$$

Example 40. *If $x=9+4\sqrt{5}$, find the value of*

(*i*) $x^2-\frac{1}{x^2}$, (*ii*) $x^3-\frac{1}{x^3}$

Solution. $x=9+4\sqrt{5}$

$$\frac{1}{x}=\frac{1}{9+4\sqrt{5}}=\frac{9-4\sqrt{5}}{(9+4\sqrt{5})(9-4\sqrt{5})}=9-4\sqrt{5}$$

$$\therefore \quad x+\frac{1}{x}=(9+4\sqrt{5})+(9-4\sqrt{5})=18 \qquad \ldots(1)$$

and $$x-\frac{1}{x}=(9+4\sqrt{5})-(9-4\sqrt{5})=8\sqrt{5} \qquad \ldots(2)$$

(*i*) Multiplying (1) and (2) together, we get

$$\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)=18\times 8\sqrt{5}$$

$$\Rightarrow \quad x^2-\frac{1}{x^2}=144\sqrt{5}=321{\cdot}9984$$

(*ii*) Cubing both sides of (2), we get

$$\left(x-\frac{1}{x}\right)^3=(8\sqrt{5})^3$$

$$\Rightarrow \qquad x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)=2560\sqrt{5}$$

$$\Rightarrow \qquad x^3-\frac{1}{x^3}-3\times 8\sqrt{5}=2560\sqrt{5}$$

$$\Rightarrow \qquad x^3-\frac{1}{x^3}=2584\sqrt{5}=5778{\cdot}0824$$

Example 41. *If $x=3+\sqrt{8}$, find the value of*

$$x^4+\frac{1}{x^4}$$

Solution. $x=3+\sqrt{8}$

$$\frac{1}{x}=\frac{1}{3+\sqrt{8}}=\frac{1}{3+\sqrt{8}}\times\frac{3-\sqrt{8}}{3-\sqrt{8}}=3-\sqrt{8}$$

$$\therefore \qquad x+\frac{1}{x}=3+\sqrt{8}+3-\sqrt{8}=6$$

Squaring both sides, we get

$$\left(x+\frac{1}{x}\right)^2=6^2 \quad \Rightarrow \quad x^2+\frac{1}{x^2}+2=36$$

$$\Rightarrow \qquad x^2+\frac{1}{x^2}=34$$

Squaring again, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=(34)^2$$

$$x^4+\frac{1}{x^4}=1156-2=1154$$

Example 42. *If $x=\frac{\sqrt{5}-2}{\sqrt{5}+2}$, prove that x^4+x^{-4} in an integer.*

(C.A. Intermediate November 1982)

Solution. We have

$$x=\frac{\sqrt{5}-2}{\sqrt{5}+2}=\frac{\sqrt{5}-2}{\sqrt{5}+2}\times\frac{\sqrt{5}-2}{\sqrt{5}-2}=\frac{9-4\sqrt{5}}{5-4}=9-4\sqrt{5}$$

$$\frac{1}{x}=\frac{\sqrt{5}+2}{\sqrt{5}-2}=\frac{\sqrt{5}+2}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}=\frac{9+4\sqrt{5}}{5-4}=9+4\sqrt{5}$$

$$\therefore \quad x+\frac{1}{x}=18$$

Squaring both sides, we get

$$x^2+2\,.\,x\,.\,\frac{1}{x}+\frac{1}{x^2}=324$$

or $$x^2+\frac{1}{x^2}=324-2=322$$

Squaring again both sides, we get

$$x+2\ .\ x^2\ .\ \frac{1}{x^2}+\frac{1}{x^4}=(322)^2$$

or $$x^4+\frac{1}{x^4}=(322)^2-2$$

$\therefore$ $$x^4+x^{-4}=(322)^2-2=\text{an integer.}$$

Example 43. *If $x=3+2\sqrt{2}$, find the value of*

$$(i)\ \sqrt{x}+\frac{1}{\sqrt{x}},\ (ii)\ \sqrt{x}-\frac{1}{\sqrt{x}}.$$

Solution. $$x=3+2\sqrt{2}$$

$\therefore$ $$\frac{1}{x}=\frac{1}{3+2\sqrt{2}}=\frac{1}{3+2\sqrt{2}}\times\frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$

$$=\frac{3-2\sqrt{2}}{9-8}=3-2\sqrt{2}$$

$\therefore$ $$x+\frac{1}{x}=3+2\sqrt{2}+3-2\sqrt{2}=6 \qquad ...(1)$$

Adding two on both sides, we get

$$x+\frac{1}{x}+2=6+2=8$$

$\Rightarrow$ $$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=(2\sqrt{2})^2$$

$\Rightarrow$ $$\sqrt{x}+\frac{1}{\sqrt{x}}=2\sqrt{2}$$

Subtracting two from both sides of (1), we get

$$x+\frac{1}{x}-2=6-2=4$$

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2=(2)^2 \quad \Rightarrow \quad \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)=2$$

Example 44. *If $x=\sqrt{\frac{7+4\sqrt{3}}{7-4\sqrt{3}}}$, show that $x^2(x-14)^2=1$*

Solution. $$x=\sqrt{\frac{7+4\sqrt{3}}{7-4\sqrt{3}}}=\sqrt{\frac{(7+4\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})}}$$

$$=\sqrt{\frac{(7+4\sqrt{3})^2}{49-16\,(3)}}=(7+4\sqrt{3})$$

and $$x-14=7+4\sqrt{3}-14=4\sqrt{3}-7$$

$$\therefore \quad x^2(x-14)^2=(4\sqrt{3}+7)^2(4\sqrt{3}-7)^2$$
$$=\{(4\sqrt{3}+7)(4\sqrt{3}-7)\}^2$$
$$=\{16(3)-49\}^2=1$$

Example 45. *If $x=3+2\sqrt{2}$ and $y=\dfrac{1}{3+2\sqrt{2}}$, find the value of $5x^2+10xy+5y^2$.*

Solution. Here $x=3+2\sqrt{2}$

$$\therefore \quad x^2=(3+2\sqrt{2})^2=9+8+12\sqrt{2}=17+12\sqrt{2}$$

$$y=\frac{1}{3+2\sqrt{2}}=\frac{1}{3+2\sqrt{2}}\times\frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$
$$=\frac{3-2\sqrt{2}}{9-8}=3-2\sqrt{2}$$
$$y^2=(3-2\sqrt{2})^2=17-12\sqrt{2}$$

Also $$xy=(3+2\sqrt{2})\times\frac{1}{3+2\sqrt{2}}=1$$

$$\therefore \quad 5x^2+10xy+5y^2=5(17+12\sqrt{2})+10\,.\,(1)+5(17-12\sqrt{2})$$
$$=85+60\sqrt{2}+10+85-60\sqrt{2}=180.$$

Example 46. (*a*) *If $x=3-\sqrt{5}$, find the value of*

$$x^4-x^3-20x^2-16x+39$$

(*b*) *If* $x=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, *find the value of*

$$2x^4-21x^3+12x^2-x+10$$

Solution. (*a*) $x=3-\sqrt{5}$, *i.e.*, $x-3=-\sqrt{5}$

Squaring both sides, we get

$$x^2+9-6x=5$$

$$\Rightarrow \quad x^2-6x+4=0$$

Now divide $x^4-x^3-20x^2-16x+39$ by x^2-6x+4

$$
\begin{array}{r|l}
 & \quad x^2+5x+6 \\
x^2-6x+4 & x^4-x^3-20x^2-16x+39 \\
 & x^4-6x^3+4x^2 \\
 & -\quad +\quad - \\
\hline
 & \quad 5x^3-24x^2-16x \\
 & \quad 5x^3-30x^2+20x \\
 & \quad -\quad +\quad - \\
\hline
 & \qquad 6x^2-36x+39 \\
 & \qquad 6x^2-36x+24 \\
 & \qquad -\quad +\quad - \\
\hline
 & \qquad\qquad 15
\end{array}
$$

Thus we can write

$$x^4-x^3-20x^2-16x+39=(x^2-6x+4)(x^2+5x+6)+15$$
$$=(0)(x^2+5x+6)+15=15$$

(b) $x=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\times\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\dfrac{3+2+2\sqrt{6}}{3-2}$

$$=5+2\sqrt{6}$$

$\Rightarrow \qquad x-5=2\sqrt{6}$

$\Rightarrow \qquad x^2+25-10x=24$

$\Rightarrow \qquad x^2-10x+1=0$

$\therefore \qquad 2x^4-21x^3+12x^2-x+10$

$$=2x^4-20x^3+2x^2-x^3+10x^2-x+10$$
$$=2x^2(x^2-10x+1)-x(x^2-10x+1)+10$$
$$=2x^2(0)-x(0)+10=10.$$

Example 47. *If* $x=(4+\sqrt{15})^{1/3}+(4+\sqrt{15})^{-1/3}$
prove that $\qquad x^3-3x-8=0.$

Solution. Let $(4+\sqrt{15})^{1/3}=a \quad \Rightarrow \quad a^3=4+\sqrt{15}$

Then $\qquad (4+\sqrt{51})^{-1/3}=\dfrac{1}{a}$

$\Rightarrow \qquad \dfrac{1}{a}=\dfrac{1}{4+\sqrt{15}}=\dfrac{1}{4+\sqrt{15}}\times\dfrac{4-\sqrt{15}}{4-\sqrt{15}}=4-\sqrt{15}$

$\therefore \qquad x=a+\dfrac{1}{a}$

Cubing both sides, we get

$$x^3=a^3+a^{-3}-3a.\,\frac{1}{a}\left(a+\frac{1}{a}\right)=a^3+a^{-3}+3x$$
$$=(4+\sqrt{15})+(4-\sqrt{15})+3x=8+3x$$

$\therefore \qquad x^3-3x-8=0.$

6·8. ROOT OF A MIXED SURD

There are two methods of finding the square root of a mixed surd.

First Method. This method is known as *method of inspection.* In this case we try to express the given surd $a \pm \sqrt{b}$ in the form $(\sqrt{x} \pm \sqrt{y})^2$ then $\pm(\sqrt{x} \pm \sqrt{y})$ is the required root.

Illustration. *Consider the square*

$$(3+\sqrt{5})^2=9+5+6\sqrt{5}=14+6\sqrt{5}$$

Now the square root of $14+6\sqrt{5}$ is $\pm(3+\sqrt{5})$

Also $14+6\sqrt{5}=14+2\sqrt{45}$ since $\sqrt{45}=3\sqrt{5}$

Here $14=9+5$ and $45=9\times 5$

Hence find two numbers whose sum is 14 and product is 45. These are 9 and 5.

$$\therefore \quad 14+2\sqrt{45}=9+5+2\sqrt{9}.\sqrt{5}=(\sqrt{9}+\sqrt{5})^2=(3+\sqrt{5})^2$$

To find the square root of $x \pm 2\sqrt{y}$, find two numbers a and b, whose sum is x and the product is y, then the square root is $\pm(\sqrt{a} \pm \sqrt{b})$.

The first step, then, is to put the given surd in the form $x+2\sqrt{y}$. For example $9+4\sqrt{5}=9+2\sqrt{20}$, the two numbers whose sum is 9 and the product is 20 are 4 and 5.

In case 2 is not there, multiply and divide the surd by 2. For example

$$8+3\sqrt{7}=8+\sqrt{63}=\tfrac{1}{2}[16+2\sqrt{63}]$$

Now square root of $\frac{1}{2}$ is $\frac{1}{\sqrt{2}}$ and to find the square root of $16+2\sqrt{63}$, find two numbers whose sum is 16 and product is 63 which are 9 and 7.

Thus $$\sqrt{16+2\sqrt{63}}=\pm(\sqrt{9}+7)=\pm(3+\sqrt{7})$$

Second Method. This is a general method. Here we suppose that square root$=\pm(\sqrt{x} \pm \sqrt{y})$.

Now square both sides, equating rational and irrational parts and then find x and y.

Example 48. *Find the square root of $3+\sqrt{5}$.*

Solution. Let $\sqrt{3+\sqrt{5}}=\sqrt{x}+\sqrt{y}$

Squaring both sides, we get

$$3+\sqrt{5}=x+y+2\sqrt{xy}$$

Equating rational and irrational parts, we have

$$x+y=3 \quad \text{...(2)}$$

$$2\sqrt{xy}=\sqrt{5} \quad \text{...(3)}$$

Squaring both sides, we get

$$4xy=5$$

Also $\quad (x-y)^2=(x+y)^2-4xy=(3)^2-4\times\frac{5}{4}=4$

$$\Rightarrow \quad x-y=2 \quad \text{...(4)}$$

Adding (2) and (4), we get

$$2x=5 \quad \Rightarrow \quad x=\frac{5}{2}$$

$$\therefore \quad y=3-x=3-\frac{5}{2}=\frac{1}{2}$$

Hence $\quad \sqrt{3+\sqrt{5}}=\sqrt{5/2}+\sqrt{1/2}$

The second square root being $-\sqrt{5/2}-\sqrt{1/2}$

Example 49. *Evaluate* $\sqrt{28-5\sqrt{12}}$.

Solution. Let $\sqrt{28-5\sqrt{12}}=\sqrt{x}-\sqrt{y}$ ...(1)

Squaring both sides, we get

$$28-5\sqrt{12}=x+y-2\sqrt{xy}$$

$$\therefore \quad x+y=28 \quad \text{...(2)}$$

and

$$2\sqrt{xy}=5\sqrt{12} \quad \text{...(3)}$$

$$\Rightarrow \quad 4xy=25\times 12=300$$

Also $\quad (x-y)^2=(x+y)^2-4xy$

$$=(28)^2-300=784-300=(22)^2$$

$$\Rightarrow \quad x-y=22 \quad \text{...(4)}$$

Adding (2) and (4), we get

$$2x=28+22=50 \quad \Rightarrow x=25$$

Also $\quad y=28-25=3$

Hence $\quad \sqrt{28-5\sqrt{12}}=\sqrt{25}-\sqrt{3}=5-\sqrt{3}.$

Example 50. *Given* $\sqrt{5}=2{\cdot}23607$, *find the value of*

$$\frac{10\sqrt{2}}{\sqrt{8}-\sqrt{(3+\sqrt{5})}}-\frac{\sqrt{10}+\sqrt{18}}{\sqrt{8}+\sqrt{(3-\sqrt{5})}}$$

Solution. We have

$$3+\sqrt{5}=\tfrac{1}{2}(6+2\sqrt{5})=\tfrac{1}{2}(\sqrt{5}+1)^2$$

$$\therefore \quad \sqrt{3+\sqrt{5}}=\frac{\sqrt{5}+1}{\sqrt{2}} \text{ and } \sqrt{3-\sqrt{5}}=\frac{\sqrt{5}-1}{\sqrt{2}}$$

$$\therefore \text{ Expression}=\frac{10\sqrt{2}}{2\sqrt{2}-\frac{1}{\sqrt{2}}(\sqrt{5}+1)}-\frac{\sqrt{10}+3\sqrt{2}}{2\sqrt{2}+\frac{1}{\sqrt{2}}(\sqrt{5}-1)}$$

$$=\frac{10\times 2}{4-\sqrt{5}-1}-\frac{2\sqrt{5}+6}{4+\sqrt{5}-1}$$

$$=\frac{10\times 2}{3-\sqrt{5}}-\frac{2(3+\sqrt{5})}{(3+\sqrt{5})}$$

$$=\frac{20\,(3+\sqrt{5})-8}{4}=13+5\sqrt{5}=24{\cdot}18035$$

Example 51. *Prove that*

$$\frac{\sqrt{4-\sqrt{7}}}{\sqrt{8+3\sqrt{7}}-2\sqrt{2}}=1$$

Solution.
$$\text{L.H.S.}=\frac{\sqrt{\frac{1}{2}\;(8-2\sqrt{7})}}{\sqrt{\frac{1}{2}\,(16+2\sqrt{9\times 7})}-2\sqrt{2}}$$

$$=\frac{\sqrt{\frac{1}{2}\,(\sqrt{7}-\sqrt{1})^2}}{\sqrt{\frac{1}{2}(\sqrt{9}+\sqrt{7})^2}-2\sqrt{2}}$$

$$=\frac{\frac{1}{\sqrt{2}}\,(\sqrt{7}-1)}{\frac{3}{\sqrt{2}}+\frac{\sqrt{7}}{\sqrt{2}}-2\sqrt{2}}=\frac{\sqrt{7}-1}{3+\sqrt{7}-4}$$

$$=\frac{\sqrt{7}-1}{\sqrt{7}-1}=1=\text{R.H.S.}$$

Example 52. *If* $x=4(6+2\sqrt{5})^{-1/2}$, *find the value of* x^2-7x+5.

Solution. We have

$$x=\frac{4}{\sqrt{6+2\sqrt{5}}}=\frac{4}{\sqrt{(\sqrt{5}+1)^2}}=\frac{4}{\sqrt{5}+1}$$

$$=\frac{4}{\sqrt{5}+1}\times\frac{\sqrt{5}-1}{\sqrt{5}-1}=\sqrt{5}-1$$

$\Rightarrow \quad x+1=\sqrt{5}$

$\Rightarrow \quad x^2+2x+1=5$

$\Rightarrow \quad x^2+2x-9x-4+9=-9x+9$

$\Rightarrow \quad x^2-7x+5=-9(\sqrt{5}-1)+9=18-9\sqrt{5}.$

Example 53. *Simplify and show that*

$$\frac{\sqrt{\frac{5}{2}+\sqrt{7-3\sqrt{5}}}}{\sqrt{\frac{7}{2}+\sqrt{16-5\sqrt{7}}}}$$

is a rational number.

Solution. Let $\sqrt{7-3\sqrt{5}}=\sqrt{x}-\sqrt{y}$

$\therefore \quad 7-3\sqrt{5}=x+y-2\sqrt{xy}$

Equating rational and irrational parts, we have

$$x+y=7 \text{ and } 2\sqrt{xy}=3\sqrt{5}, \text{ i.e., } 4xy=45$$

Now $\quad (x-y)^2=(x+y)^2-4xy=7^2-45=4$

$\therefore \quad x-y=2$

Solving $\left.\begin{array}{l} x+y=7 \\ x-y=2 \end{array}\right\}$, we get $x=\frac{9}{2}$ and $y=\frac{5}{2}$

$\therefore \quad \sqrt{7-3\sqrt{5}}=\sqrt{x}-\sqrt{y}=\sqrt{\frac{9}{2}}-\sqrt{\frac{5}{2}}=\frac{3}{\sqrt{2}}-\sqrt{\frac{5}{2}}$

Similarly $\sqrt{16-5\sqrt{7}}=\sqrt{\frac{25}{2}}-\sqrt{\frac{7}{2}}=\frac{5}{\sqrt{2}}-\sqrt{\frac{7}{2}}$

$\therefore$ The given expression$=\dfrac{\sqrt{\frac{5}{2}}+\frac{3}{\sqrt{2}}-\sqrt{\frac{5}{2}}}{\sqrt{\frac{7}{2}}+\frac{5}{\sqrt{2}}-\sqrt{\frac{7}{2}}}$

$$=\frac{3/\sqrt{2}}{5/\sqrt{2}}=\frac{3}{5}, \text{ rational number.}$$

Example 54. *Evaluate*

$$\sqrt{16+4\sqrt{10}-2\sqrt{15}-4\sqrt{6}}$$

Solution.

Let $\sqrt{16+4\sqrt{10}-2\sqrt{15}-4\sqrt{6}}=\pm(\sqrt{x}+\sqrt{y}-\sqrt{z})$

Squaring both sides, we get

$$16+4\sqrt{10}-2\sqrt{15}-4\sqrt{6}=x+y+z+2\sqrt{xy}-2\sqrt{xz}-2\sqrt{yz}$$

Equating rational and irrational parts on both sides, we get

$$x+y+z=16 \quad \ldots(1)$$

$$2\sqrt{xy}=4\sqrt{10} \quad \Rightarrow \quad xy=40 \quad \ldots(2)$$

$$2\sqrt{xz}=2\sqrt{15} \quad \Rightarrow \quad xz=15 \quad \ldots(3)$$

$$2\sqrt{yz}=4\sqrt{6} \quad \Rightarrow \quad yz=24 \quad \ldots(4)$$

Multiplying (2), (3) and (4), we get

$$x^2y^2z^2=40\times 15\times 24$$

$$\Rightarrow \quad xyz=\pm 120$$

$\therefore$ Either $xyz=+120$

Dividing this by (2), (3) and (4) turn by turn, we get

$x=5,\ y=8,\ z=3$

or $xyz=-120$

Dividing this by (2), (3) and (4) turn by turn, we get

$x=-5,\ y=-8,\ z=-3$

Equation (1) is satisfied only by positive values of x, y, z.

$\therefore \quad x=5,\ y=8,\ z=3.$

Hence

$$\sqrt{16+4\sqrt{10}-2\sqrt{15}-4\sqrt{6}}=\pm(\sqrt{5}+\sqrt{8}-\sqrt{3})$$

$$=\pm(\sqrt{5}+2\sqrt{2}-\sqrt{3}).$$

Example 55. *Find the square root of*

$$5-\sqrt{10}-\sqrt{15}+\sqrt{6}.$$

(*C.A. Intermediate May 1981*)

Solution. Let $\sqrt{5-\sqrt{10}-\sqrt{15}+\sqrt{6}}=\pm(\sqrt{x}-\sqrt{y}+\sqrt{z})$

Squaring both sides, we get

$$5-\sqrt{10}-\sqrt{15}+\sqrt{6}=x+y+z-2\sqrt{xy}-2\sqrt{yz}+2\sqrt{xz}$$

Equating rational and irrational parts on both sides, we get

$$x+y+z=5 \quad \ldots(1)$$

$$2\sqrt{xy}=\sqrt{10} \quad \Rightarrow \quad 4xy=10 \quad \ldots(2)$$

$$2\sqrt{yz}=\sqrt{15} \quad \Rightarrow \quad 4yz=15 \quad \ldots(3)$$

$$2\sqrt{xz}=\sqrt{6} \quad \Rightarrow \quad 4xz=6 \quad \ldots(4)$$

Multiplying (2), (3) and (4), we get

$$64\,x^2 y^2 z^2 = 10 \times 15 \times 6$$

$$\Rightarrow \quad 4\,xyz = \pm 15$$

$\therefore$ Either $4xyz = +15$

Dividing this by (2), (3) and (4) turn by turn, we get

$$x=1,\ y=\tfrac{5}{2},\ z=\tfrac{3}{2}$$

or $4xyz = -15$

Dividing this by (2), (3) and (4) turn by turn, we get

$$x=-1,\ y=-\tfrac{5}{2},\ z=-\tfrac{3}{2}$$

Equation (1) is satisfied only by positive values of x, y, z.

$$\therefore \quad x=1,\ y=\tfrac{5}{2},\ z=\tfrac{3}{2}.$$

Hence $\sqrt{5-\sqrt{10}-\sqrt{15}+\sqrt{6}} = \pm(\sqrt{1}-\sqrt{\tfrac{5}{2}}+\sqrt{\tfrac{3}{2}})$

$$= \pm(1-\sqrt{\tfrac{5}{2}}+\sqrt{\tfrac{3}{2}}).$$

Example 56. *Find the fourth root of $137-36\sqrt{14}$.*

Solution. We have

$$137-36\sqrt{14} = 137-2\sqrt{324\times 14}$$

$$= 137-2\sqrt{4536}$$

$$= 81+56-2\sqrt{81\times 56} = (9-\sqrt{56})^2$$

$$\therefore \quad \sqrt{137-36\sqrt{14}} = (9-\sqrt{56})$$

Now $\quad 9-\sqrt{56} = 9-2\sqrt{14}$

$$= 7+2-2\sqrt{7\times 2} = (\sqrt{7}-\sqrt{2})^2$$

$$\therefore \quad \sqrt{9-\sqrt{56}} = (\sqrt{7}-\sqrt{2})$$

Hence $\quad \sqrt[4]{137-36\sqrt{14}} = \sqrt{9-\sqrt{56}} = \sqrt{7}-\sqrt{2}.$

Example 57. *Prove that*

$$\sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+\sqrt{14-6\sqrt{5}}}} = 2$$

Solution. L.H.S. $= \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}-\sqrt{(9+5-2\sqrt{9\times 5})}}}$

$$= \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+(3-\sqrt{5})}}$$

$$= \sqrt{2+\sqrt{5}-\sqrt{9-4\sqrt{5}}}$$

$$= \sqrt{2+\sqrt{5}-\sqrt{5+4-2\sqrt{5\times 4}}}$$

$$= \sqrt{2+\sqrt{5}-(\sqrt{5}-2)} = \sqrt{4} = 2 = \text{R.H.S.}$$

EXERCISE (II)

1. State which of the following are surds.

(i) $\sqrt[3]{81}$, (ii) π (iii) $\sqrt[5]{32}$, (iv) $\sqrt{1+\sqrt{3}}$, (v) $\sqrt{\frac{9}{169}}$

2. (a) Find the rationalising factors :

(i) $\sqrt[3]{3}-1$, (ii) $\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}$

(iii) $(3\sqrt{11}+5\sqrt{6})$, and (iv) $(\sqrt{5}-\sqrt{2}+\sqrt{7})$

(b) Simplify

(i) $\sqrt{63}+\sqrt{28}-\sqrt{175}$

(ii) $2\sqrt{18}+\sqrt{20}-\sqrt{147}+\frac{1}{2}\sqrt{50}+\frac{4}{3}\sqrt{45}$

(iii) $\sqrt{112}-\sqrt{63}+\dfrac{224}{\sqrt{28}}$

3. Simplify

(a) $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$

(b) $\dfrac{5+\sqrt{6}}{15\sqrt{3}-2\sqrt{32}+2\sqrt{50}-8\sqrt{12}}$

4. Simplify

(a) (i) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{3}}{2+\sqrt{3}}+\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$,

(ii) $\dfrac{7+3\sqrt{5}}{3+\sqrt{5}}+\dfrac{7-3\sqrt{5}}{3-\sqrt{5}}$

(b) $\dfrac{6}{2\sqrt{3}-\sqrt{6}}+\dfrac{4\sqrt{6}}{4\sqrt{3}+4\sqrt{2}}-\dfrac{4\sqrt{15}}{\sqrt{30}-\sqrt{10}}$

(c) $\dfrac{7\sqrt{3}}{\sqrt{10}+\sqrt{3}}-\dfrac{2\sqrt{5}}{\sqrt{6}+\sqrt{5}}-\dfrac{3\sqrt{2}}{\sqrt{15}+3\sqrt{2}}$

5. Express $\dfrac{3-\sqrt{5}}{3+2\sqrt{5}}$ in the form $(a\sqrt{5}-b)$, where a and b are simple fractions.

6. Find the value of a and b if both are rational numbers and

$$\frac{5+2\sqrt{3}}{7+4\sqrt{3}}=a-b\sqrt{3}$$

7. If $\frac{3+\sqrt{6}}{5\sqrt{3}-4\sqrt{2}-2\sqrt{12}+\sqrt{50}}=a+x\sqrt{3}$, find a and x.

8. If $\frac{(\sqrt{2}+1)^2}{3-\sqrt{2}}=a+x\sqrt{2}$,

find a and x.

9. Express $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ in the form $a\sqrt{5}+b\sqrt{6}$.

10 Show that

(*a*) $\frac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}+\frac{1}{\sqrt{2}-\sqrt{3}-\sqrt{5}}=\frac{1}{\sqrt{2}}$

(*b*) $\frac{7+3\sqrt{5}}{\sqrt{2}+\sqrt{7+3\sqrt{5}}}+\frac{7-3\sqrt{5}}{\sqrt{2}+\sqrt{7-3\sqrt{5}}}=2\sqrt{2}$

11. Show that

$$\frac{3}{1-\sqrt{2}+\sqrt{3}}+\frac{1}{1-\sqrt{2}-\sqrt{3}}-\frac{2}{1+\sqrt{2}-\sqrt{3}}+\frac{3}{\sqrt{2}}=1$$

12. Prove that

$$\frac{5}{\sqrt[3]{16}-\sqrt[3]{4}+1}-\frac{3}{\sqrt[3]{16}+\sqrt[3]{4}+1}=2$$

13. If $x=7+4\sqrt{3}$, $y=7-4\sqrt{3}$, find the value of

$$\frac{1}{x^2}+\frac{1}{y^2}$$

14. If $x=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ and $y=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, find the value of

(*i*) $x+y$, (*ii*) x^2+y^2, (*iii*) $\frac{1}{x^2}+\frac{1}{y^2}$

15. If $x=7-\sqrt{48}$, find the value of

$$\left(x^3+\frac{1}{x^3}\right)-15\left(x^2+\frac{1}{x^2}\right)+20\left(x+\frac{1}{x}\right)-72$$

16. If $x=4+\sqrt{15}$, find the value of

$$\left(x^3-\frac{1}{x^3}\right)$$

17. (*a*) $x=7-4\sqrt{3}$, find the value of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

(*b*) If $x=5+2\sqrt{6}$, find the value of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

18. Find the value of

(*i*) $x^2-10x+1$, if $x=\frac{1}{5+2\sqrt{6}}$

(*ii*) x^3-6x^2+7x+8, if $x=3-2\sqrt{2}$

(*iii*) $x^4-4x^3-2x^2-4x+31$, if $x=3+\sqrt{2}$

(*iv*) $2x^4-9x^3-14x^2+7x-3$, if $x=\frac{2}{3-\sqrt{7}}$

(*v*) $\frac{1}{5x^2-2x+1}$, if $x=2+\sqrt{2}$

(*vi*) $10x-x^2$, if $x=(5+2\sqrt{6})^{-1}$

19. Find the value of

$$\frac{x+1}{x-1}+\frac{x-1}{x+1}, \text{ if } x=\sqrt{3}+\sqrt{2}$$

20. If $x=\frac{1}{2-\sqrt{3}}$ and $y=\frac{1}{2+\sqrt{3}}$, show that

$$7x^2+11xy-7y^2=11+56\sqrt{3}.$$

21. If $\frac{3x+5}{x+2}=2\sqrt{2}$, show that $x^2-2x+3=10$

22. If $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ and $y=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, find the value of x^3+y^3.

[Hint. $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}=5-2\sqrt{6}$

Similarly $y=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=5+2\sqrt{6}$

$$x^3+y^3=(x+y)^3-3xy(x+y)=(10)^3-3\,.\,1\,.\,10=970.]$$

23. If $\sqrt{31+4\sqrt{21}}=\sqrt{x}+\sqrt{y}$, find x and y.

24. Find the square root of

(*i*) $19-8\sqrt{3}$, (*ii*) $\frac{7}{2}-\frac{3}{2}\sqrt{5}$

(*iii*) $18+6\sqrt{5}$

(*iv*) $6(5+2\sqrt{6})$

(*v*) $x+\sqrt{x^2-y^2}$.

25. Show that

(*a*) $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}=\sqrt{10}$

(*b*) $(28-10\sqrt{3})^{1/2}-(7+4\sqrt{3})^{-1/2}=3.$

26. Find the value of

$$\sqrt{-\sqrt{3}+\sqrt{4+\sqrt{5}+\sqrt{17-4\sqrt{15}}}}$$

27. Prove that

(*a*) $\dfrac{1}{\sqrt{11-2\sqrt{30}}}-\dfrac{3}{\sqrt{7-2\sqrt{10}}}-\dfrac{4}{\sqrt{8+4\sqrt{3}}}=0$

(*b*) $\dfrac{1}{\sqrt{3}-\sqrt{3-\sqrt{8}}}+\dfrac{1}{\sqrt{3}+\sqrt{3+\sqrt{8}}}=1$

28. Show that

$$\sqrt{19+4\sqrt{21}}+\sqrt{7}-\sqrt{30-2\sqrt{56}}-\sqrt{12}=\sqrt{2}$$

29. Find x and y if $(\sqrt{x}+\sqrt{y})^2=4+\sqrt{15}$

Deduce that

$$\sqrt{4+\sqrt{15}}=\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}$$

Hence find k, If

$$(4+\sqrt{15})^{3/2}+(4-\sqrt{15})^{3/2}=k\sqrt{10}$$

30. If $\left[\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}\right]^2=a+b\sqrt{3},$

find the value of a and b.

31. Find the cube root of $9\sqrt{3}+11\sqrt{2}$

[**Hint.** $\sqrt[3]{9\sqrt{3}+11\sqrt{2}}=(\sqrt{3})\sqrt[3]{3+\frac{11}{3}\sqrt{\frac{2}{3}}}$

Let $\sqrt[3]{3+\frac{11}{3}\sqrt{\frac{2}{3}}}=x+\sqrt{y}$ ...(1)

so that $\sqrt[3]{3-\frac{11}{3}\sqrt{\frac{2}{3}}}=x-\sqrt{y}$ ...(2)

Multiply (1) and (2), find y in terms of x, cube both sides of (1), find x.]

32. Find the fourth root of

(*i*) $56-24\sqrt{5}$, (*ii*) $193+132\sqrt{2}$.

ANSWERS

1. (*i*), (*iv*) are surds

2. (*a*) (*i*) $\sqrt[3]{9}+\sqrt[3]{3}+1$, (*ii*) $\sqrt[3]{x}-\sqrt[3]{y}$, (*iii*) $\sqrt[3]{11}-5\sqrt{6}$

(*iv*) $(\sqrt{7}+\sqrt{5}+\sqrt{2})(10-2\sqrt{35})$ (*b*) (*i*) 0

(*ii*) $6\sqrt{5}-7\sqrt{3}+\frac{17}{2}\sqrt{2}$, (*iii*) $17\sqrt{7}$. **3.** (*a*) $\sqrt{3}$ (*b*) $\frac{13\sqrt{2}+9\sqrt{3}}{5}$

4. (*a*) (*i*) $16-\sqrt{3}=14{\cdot}268$ (*ii*) 3, (*b*) 0, (*c*) 1.

5. $\frac{9}{11}\sqrt{5}-\frac{19}{11}$ **6.** $a=11, b=6$ **7.** $a=0, x=1$ **8.** $a=\frac{13}{7}$, $x=\frac{9}{7}$

9. $\sqrt{5}-\sqrt{6}$ **13.** 194. **14.** (*i*) 10, (*ii*) 98, (*iii*) 98, **15.** 0

16. $126\sqrt{15}$ **17.** (*a*) 4, (*b*) $2\sqrt{3}$ **18.** (*i*) 0, (*ii*) $26-12\sqrt{2}$.

(*iii*) 10, (*iv*) $\sqrt{7}$, (*v*) $\frac{3-2\sqrt{2}}{9}$, (*vi*) 1, **19.** $\sqrt{6}$ **22.** 970 **23.** $x=28$, $y=3$

24. (*i*) $4-\sqrt{3}$ (*ii*) $\frac{3}{2}-\sqrt{\frac{5}{4}}$ (*iii*) $\sqrt{15}+\sqrt{3}$ (*iv*) $3\sqrt{2}+2\sqrt{3}$

(*v*) $\frac{1}{2}[\sqrt{x+y}+\sqrt{x-y}]$ **26.** 1 **29.** 7 **30.** $a=2$, $b=0$

31. $\sqrt{3}[1+\sqrt{\frac{2}{3}}]$ **32.** (*i*) $\sqrt{5}-1$, (*ii*) $(3+\sqrt{2})$.