Multiply (1) and (2), find y in terms of x, cube both sides of (1), find x.]

32. Find the fourth root of

(*i*) $56-24\sqrt{5}$, (*ii*) $193+132\sqrt{2}$.

ANSWERS

1. (*i*), (*iv*) are surds

2. (*a*) (*i*) $\sqrt[3]{9}+\sqrt[3]{3}+1$, (*ii*) $\sqrt[3]{x}-\sqrt[3]{y}$, (*iii*) $\sqrt[3]{11}-5\sqrt{6}$

(*iv*) $(\sqrt{7}+\sqrt{5}+\sqrt{2})(10-2\sqrt{35})$ (*b*) (*i*) 0

(*ii*) $6\sqrt{5}-7\sqrt{3}+\frac{17}{2}\sqrt{2}$, (*iii*) $17\sqrt{7}$. **3.** (*a*) $\sqrt{3}$ (*b*) $\frac{13\sqrt{2}+9\sqrt{3}}{5}$

4. (*a*) (*i*) $16-\sqrt{3}=14{\cdot}268$ (*ii*) 3, (*b*) 0, (*c*) 1.

5. $\frac{9}{11}\sqrt{5}-\frac{19}{11}$ **6.** $a=11, b=6$ **7.** $a=0, x=1$ **8.** $a=\frac{13}{7}$, $x=\frac{9}{7}$

9. $\sqrt{5}-\sqrt{6}$ **13.** 194. **14.** (*i*) 10, (*ii*) 98, (*iii*) 98, **15.** 0

16. $126\sqrt{15}$ **17.** (*a*) 4, (*b*) $2\sqrt{3}$ **18.** (*i*) 0, (*ii*) $26-12\sqrt{2}$.

(*iii*) 10, (*iv*) $\sqrt{7}$, (*v*) $\frac{3-2\sqrt{2}}{9}$, (*vi*) 1, **19.** $\sqrt{6}$ **22.** 970 **23.** $x=28, y=3$

24. (*i*) $4-\sqrt{3}$ (*ii*) $\frac{3}{2}-\sqrt{\frac{5}{4}}$ (*iii*) $\sqrt{15}+\sqrt{3}$ (*iv*) $3\sqrt{2}+2\sqrt{3}$

(*v*) $\frac{1}{2}[\sqrt{x+y}+\sqrt{x-y}]$ **26.** 1 **29.** 7 **30.** $a=2, b=0$

31. $\sqrt{3}[1+\sqrt{\frac{2}{3}}]$ **32.** (*i*) $\sqrt{5}-1$, (*ii*) $(3+\sqrt{2})$.

7

Logarithms

STRUCTURE

7·0 INTRODUCTION

7·1 LAWS OF OPERATIONS

7·2 LOGARITHMS TABLES

7·3 OPERATIONS WITH LOGARITHMS

7·4 COMPOUND INTEREST

7·5 DEPRECIATION

7·6 ANNUITIES

OBJECTIVES

After studying this chapter, you should be able to understand :

- *Logarithms, its laws, tables and calculations with the help of logarithms*
- *Calculations with logarithms in case of compound interest, depreciation and annuities.*

7·0. INTRODUCTION

Logarithms are simply the powers or the indices to a given base. Once we have converted values into logarithms to a given base we can perform difficult mathematical operations of multiplication by addition of logarithms and division by subtraction of logarithms and like that the other higher order operations. Similarly problems on involution and evolution are reduced to those of ordinary multiplication and division. This is because, as explained in the preceding chapter, the indices of power functions are added in case of the multiplication and subtracted in case of the division.

The conversion of values into logarithms is, however, no problem since readymade conversion tables are available for the purpose. With little practice the use is very easy. The first such use was suggested by John Napier (1557—1610) who invented logarithms to the base $e = 2{\cdot}71828$

which is used mostly for theoretical mathematical purposes. But for common calculations, logaritms to the base 10 are used. These common logarithms given by Henry Briggs (1561—1630), have further simplified the work. So much is the importance of logarithms that we cannot think of tedious power or root calcuiations without the use of logarithms; the slide rule for calculation is also based on logarithms.

Logarithms are in fact corollary to the theory of indices. In order to understand the theory and the application of logarithms, a thorough knowledge of indices and laws governing them is essential.

Definition. *The logarithm of a number to a given base is the index or the power to which the base must be raised to produce the number, i.e., to make it equal to the given number.* If there are three quantities indicated by say a, x and n, they are related as follows :

$$a^x = n$$

then x is said to be the logarithm of the number n to the base 'a'. Symbolically it can be expressed as follows :

$$\log_a n = x$$

i.e., the logarithm of n to the base 'a' is x. We give some illustrations below :

(*i*) $2^4 = 16 \Rightarrow \log_2 16 = 4,$

i.e., the logarithm of 16 to the base 2 is equal to 4.

(*ii*) $10^3 = 1000 \Rightarrow \log_{10} 1000 = 3,$

i.e., the logarithm of 1000 to the base 10 is 3.

(*iii*) $5^{-3} = \frac{1}{125} \Rightarrow \log_5 \frac{1}{125} = -3,$

i.e., the logarithm of $\frac{1}{125}$ to the base 5 is -3.

(*iv*) $2^3 = 8 \Rightarrow \log_2 8 = 3$

i.e., the logarithm of 8 to the base 2 is 3.

Remarks 1. It should be noted that the two equations $a^x = n$ and $x = \log_a n$ are only transformations of each other and should be remembered to change one form of the relation into the other.

2. The logarithm of 1 to any base is zero. This is because any number raised to the power zero is one. Since $a^0 = 1 \Rightarrow \log_a 1 = 0$.

3. The logarithm of any quantity to the same base is unity. This is because any quantity raised to the power 1 is that quantity only.

Since $a^1 = a \Rightarrow \log_a a = 1$

Restrictions. There are a few restrictions on the base : it should not be taken as 0 or 1 because a zero raised to any power is meaningless and 1 raised to any power is one only. Nor can the base be a negative number otherwise certain values will become imaginary.

As regards the number n for which we find the logarithm, the restriction is that is should be a positive value not equal to 1.

Illustrations :

1. (a) *If $\log_a \sqrt{2}=\frac{1}{6}$, find the value of a.*

We have $a^{1/6}=\sqrt{2} \Rightarrow a=(\sqrt{2})^6=2^3=8.$

2. *Find the logarithm of 5832 to the base $3\sqrt{2}$.*

Let us take $\log_{3\sqrt{2}} 5832=x$

$\therefore \quad (3\sqrt{2})^x=5832=8\times 729=2^3\times 3^6=(\sqrt{2})^6\times(3)^6=(3\sqrt{2})^6$

Hence $x=6$

7·1. LAWS OF OPERATIONS

We shall now proceed to prove the laws of logarithm which are valid for any base a (>0 but$\neq 1$). Since the term "logarithm" is merely a substitute for the term "index", the laws of logarithms can be easily deduced from the laws of indices.

I. Logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers to the same base, i.e.,

$$\log_a mn=\log_a m+\log_a n$$

Proof. Let $\log_a m=x$ so that $a^x=m$ ...(1)

and $\log_a n=y$ so that $a^y=n$ ...(2)

Multiplying (1) and (2), we get

$$m\times n=a^x\times a^y=a^{x+y}$$

$\Rightarrow \quad \log_a mn=x+y$ (by def.)

$\Rightarrow \quad \log_a mn=\log_a m+\log_a n$

Remark. This formula can be extended in a similar way to the product of any number of quantities, *i.e.*,

$$\log_a (mnpqr\ldots)=\log_a m+\log_a n+\log_a p+\log_a q+\log_a r+\ldots$$

It should be remembered that $\log(m+n)\neq\log m+\log n$ unless $m+n=mn$.

II. The logarithm of the quotient of two numbers is equal to the difference of their logarithms to the same base, i.e.,

$$\log_a \frac{m}{n}=\log_a m-\log_a n$$

Proof. Dividing (1) and (2) of (I), we get

$$\frac{m}{n}=\frac{a^x}{a^y}=a^{x-y}$$

then by the definition of logarithms, we get

$$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$$

Remark. $\log_a \left[\frac{1}{n}\right] = \log_a 1 - \log_a n = -\log_a n$. [$\because$ $\log_a 1 = 0$]

III. Logarithm of the number raised to a power is equal to the index of the power multiplied by the logarithm of the number to the same base, i.e.,

$$\log_a m^n = n \log_a m$$

Proof. Let $\log_a m = x$ so that $a^x = m$ ...(*)

Raising the power n on both sides of (*), we get

$$(a^x)^n = (m)^n$$

$\Rightarrow$ $$a^{nx} = m^n$$

$\Rightarrow$ $$\log_a m^n = nx$$ (by def.)

$\Rightarrow$ $$\log_a m^n = n \log_a m$$

Remarks. 1. The logarithm of a number to the base 'e' ($e = 2{\cdot}718$ approx.) is called '*Natural logarithm*' or '*The Napierian Logarithm*'.

2. The logarithm of a number to the base 10 is called '*Common Logarithm*' or '*Briggsian Logarithm*'.

3. In theoretical calculations, the base 'e' is used whereas for numerical calculations, the base '10' is most convenient. The tables that are given at the end of the book are all calculated with '10' as base.

4. When no base is mentioned, it is understood to be 10, *i.e.*, by the word 'logarithm' we generally mean '*Common Logarithm*'.

Illustrations.

1. (*a*) *Find the logarithm of 1728 to the base $2\sqrt{3}$.*

(*b*) *Find the logarithm of $\frac{1}{324}$ to the base $3\sqrt{2}$.*

Solution. (*a*) We have $1728 = 2^6 \times 3^3 = 2^6 \times (\sqrt{3})^6 = (2\sqrt{3})^6$

$\Rightarrow$ $$\log_{2\sqrt{3}} 1728 = 6$$

(*b*) We have

$$\frac{1}{324} = \frac{1}{3^4 \cdot 2^2} = \frac{1}{3^4(\sqrt{2})^4} = (3\sqrt{2})^{-4}$$

$\Rightarrow$ $$\log_{3\sqrt{2}}\left[\frac{1}{324}\right] = -4$$

2. *Simplify* : $\frac{1}{2}\log_{10} 25 - 2\log_{10} 3 + \log_{10} 18$.

Solution. The given expression

$$= \log_{10}(25)^{\frac{1}{2}} - \log_{10} 3^2 + \log_{10} 18$$

$$= \log_{10} 5 - \log_{10} 9 + \log_{10} 18$$

$$= \log_{10} \frac{5 \times 18}{9} = \log_{10} 10 = 1$$

3. (*i*) $\log_a 210 = \log_a (2 \times 3 \times 5 \times 7)$

$$= \log_a 2 + \log_a 3 + \log_a 5 + \log_a 7$$

(*ii*) $\log_{10} \frac{45}{77} = \log_{10} \frac{5 \times 3 \times 3}{11 \times 7}$

$$= \log_{10} 5 + \log_{10} 3 + \log_{10} 3 - \log_{10} 11 - \log_{10} 7$$

(*iii*) $\log_a x^3 = 3 \log_a x$

(*iv*) $\log_a \sqrt{(x+y)} = \log_a (x+y)^{\frac{1}{2}} = \frac{1}{2} \log_a (x+y)$

(*v*) $\log_a 81 = \log_a (3^4) = 4 \log_a 3$

4. *Without using log-tables, find x if*

$$\tfrac{1}{2} \log_{10} (11 + 4\sqrt{7}) = \log_{10} (2+x)$$

Solution. $\frac{1}{2} \log_{10} (11 + 4\sqrt{7}) = \log_{10} (11 + 4\sqrt{7})^{\frac{1}{2}}$

$$= \log_{10} (\sqrt{7} + \sqrt{4})$$

$\therefore \quad \log_{10} (\sqrt{7} + \sqrt{4}) = \log_{10} (2+x)$

$\Rightarrow \quad \sqrt{7} + \sqrt{4} = 2 + x$

$\Rightarrow \quad \sqrt{7} + 2 = 2 + x$

$\Rightarrow \quad x = \sqrt{7}$

CHANGE OF BASE

If the logarithm of a number to any base is given, then the logarithm of the same number to any other base can be determined from the following relation :

$$\log_a m = \log_b m \times \log_a b \quad \Rightarrow \quad \log_b m = \frac{\log_a m}{\log_a b}$$

Proof. Let $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$.

Then by def.

$$a^x = m, \; b^y = m \text{ and } a^z = b$$

Also $\quad a^x = b^y = (a^z)^y = a^{yz}$, therefore $x = yz$

$\Rightarrow \quad \log_a m = \log_b m \times \log_a b$

$$\Rightarrow \qquad \log_b m = \frac{\log_a m}{\log_a b}.$$

Remarks. Putting $m=a$, we have

$$\log_a a = \log_b a \times \log_a b$$

$$\Rightarrow \qquad \log_b a \times \log_a b = 1$$

This result can also be shown otherwise.

Let $\log_b a = x$ and $\log_a b = y$, then by def.,

$$b^x = a \text{ and } a^y = b$$

$$\therefore \qquad b = a^{\frac{1}{x}} = a^y$$

$$\Rightarrow \qquad \frac{1}{x} = y \text{ or } xy = 1$$

Hence $\log_b a \times \log_a b = 1$

Illustrations.

1. *Change the base of $\log_5 31$ into the common logarithmic base.*

Solution. Since $\log_a x = \dfrac{\log_b x}{\log_b a}$,

$$\therefore \qquad \log_5 31 = \frac{\log_{10} 31}{\log_{10} 5}$$

2. *Prove that*

$$\frac{\log_3 8}{\log_9 16 \log_4 10} = 3 \log_{10} 2$$

Solution. Change all logarithms on L.H.S. to the base 10 by using the formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$\log_3 8 = \frac{\log_{10} 8}{\log_{10} 3} = \frac{\log_{10} 2^3}{\log_{10} 3} = \frac{3 \log_{10} 2}{\log_{10} 3}$$

$$\log_9 16 = \frac{\log_{10} 16}{\log_{10} 9} = \frac{\log_{10} 2^4}{\log_{10} 3^2} = \frac{4 \log_{10} 2}{2 \log_{10} 3}$$

$$\log_4 10 = \frac{\log_{10} 10}{\log_{10} 4} = \frac{1}{\log_{10} 2^2} = \frac{1}{2 \log_{10} 2}$$

$$[\because \log_{10} 10 = 1]$$

$$\text{L.H.S.} = \frac{3 \log_{10} 2}{\log_{10} 3} \times \frac{2 \log_{10} 3}{4 \log_{10} 2} \times \frac{2 \log_{10} 2}{1}$$

$$= 3 \log_{10} 2 = \text{R.H.S.}$$

Example 1. *Prove that*

$$2 \log x + 2 \log x^2 + 2 \log x^3 + \ldots + 2 \log x^n = n(n+1) \log x$$

Solution. $\text{L.H.S.} = \log x^2 + \log x^4 + \log x^6 + \ldots + \log x^{2n}$

$$= \log (x^2 \,.\, x^4 \,.\, x^6 \ldots x^{2n})$$

$$= \log(x^{2+4+6+\cdots+2n})$$

$$= \log(x^{2(1+2+3+\cdots+n)}) \qquad \left[\because \quad \Sigma n = \frac{n(n+1)}{2}\right]$$

$$= \log x^{n(n+1)} = n(n+1) \log x = \text{R.H.S.}$$

Example 2. *Show that*

$$\log_3\left(1+\frac{1}{3}\right) + \log_3\left(1+\frac{1}{4}\right) + \log_3\left(1+\frac{1}{5}\right) + \ldots + \log_3\left(1+\frac{1}{242}\right) = 4.$$

Solution. $\text{L.H.S.} = \log_3\left(\frac{4}{3}\right) + \log_3\left(\frac{5}{4}\right) + \log_3 \frac{6}{5} + \ldots + \log_3 \frac{243}{242}$

$$= \log_3\left(\frac{4}{3} \times \frac{5}{4} \times \frac{6}{5} \times \ldots \times \frac{242}{241} \times \frac{243}{242}\right)$$

$$= \log_3\left(\frac{243}{3}\right) = \log_3 81 = \log_3 3^4 = 4 \log_3 3 = 4 = \text{R.H.S.}$$

Example 3. *Show that*

$$\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$$

[I.C.W.A. June 1990]

Solution. $\text{L.H.S.} = \log 2 + 16 \log\left(\frac{2^4}{3 \times 5}\right) + 12 \log\left(\frac{5^2}{2^3 \times 3}\right) + 7 \log\left(\frac{3^4}{2^4 \times 5}\right)$

$$= \log 2 + 16(\log 2^4 - \log 3 - \log 5) + 12(\log 5^2 - \log 2^3 - \log 3) + 7(\log 3^4 - \log 2^4 - \log 5)$$

$$= \log 2 + 16\,(4 \log 2 - \log 3 - \log 5) + 12\,(2 \log 5 - 3 \log 2 - \log 3) + 7(4 \log 3 - 4 \log 2 - \log 5)$$

$$= (1 + 64 - 36 - 28) \log 2 + (-16 - 12 + 28) \log 3 + (-16 + 24 - 7) \log 5$$

$$= \log 2 \quad 0 \quad \log 3 + \log 5 = \log 2 + \log 5$$

$$= \log 10 \quad \log_{10} 10 = 1 = \text{R.H.S.}$$

Second Method

$$\text{L.H.S.} = \log 2 + \log\left(\frac{16}{15}\right)^{16} + \log\left(\frac{25}{24}\right)^{12} + \log\left(\frac{81}{80}\right)^{7}$$

$$= \log\left\{2 \quad \left(\frac{16}{15}\right)^{16} \quad \left(\frac{25}{24}\right)^{12} \quad \left(\frac{81}{80}\right)^{7}\right\}$$

$$=\log\left\{2\times\left(\frac{2^4}{3\times5}\right)^{16}\times\left(\frac{5^2}{2^3\times3}\right)^{12}\times\left(\frac{3^4}{2^4\times5}\right)^{7}\right\}$$

$$=\log\left\{2\times\frac{2^{64}}{3^{16}\times5^{16}}\times\frac{5^{24}}{2^{36}\times3^{12}}\times\frac{3^{28}}{2^{28}\times5^{7}}\right\}$$

$$=\log\left[\frac{2^{65}\times5^{24}\times3^{28}}{2^{64}\times5^{23}\times3^{28}}\right]=\log(2\times5)=\log 10=1=\text{R.H.S.}$$

Remark. Since no base of logarithm is mentioned, it is taken to be 10.

Example 4. *Find the value of* $\dfrac{2\log 6+6\log 2}{4\log 2+\log 27-\log 9}$.

Solution. $$\text{Expression}=\frac{2\log(2\times3)+6\log 2}{4\log 2+\log 3^3-\log 3^2}$$

$$=\frac{2(\log 2+\log 3)+6\log 2}{4\log 2+3\log 3-2\log 3}$$

$$=\frac{8\log 2+2\log 3}{4\log 2+\log 3}=\frac{2(4\log 2+\log 3)}{4\log 2+\log 3}$$

$$=2$$

Aliter. $$\text{Expression}=\frac{\log 6^2+\log 2^6}{\log 2^4+\log 27-\log 9}$$

$$=\frac{\log(6^2\times2^6)}{\log\left(\frac{2^4\times27}{9}\right)}=\frac{\log 48^2}{\log 48}$$

$$=\frac{2\log 48}{\log 48}=2$$

Example 5. *Without using tables, show that*

$$\frac{\log\sqrt{27}+\log\sqrt{8}-\log\sqrt{125}}{\log 6-\log 5}=\frac{3}{2}.$$

(C.A. Intermediate May 1982)

Solution.

$$\text{L.H.S.}=\frac{\log\sqrt{27}+\log\sqrt{8}-\log\sqrt{125}}{\log 6-\log 5}$$

$$=\frac{\frac{1}{2}\log 27+\frac{1}{2}\log 8-\frac{1}{2}\log 125}{\log(2\times3)+\log 5}$$

$$=\frac{\frac{3}{2}\log 3+\frac{3}{2}\log 2-\frac{3}{2}\log 5}{\log 2+\log 3-\log 5}$$

$$=\frac{3}{2}\cdot\frac{\log 3+\log 2-\log 5}{\log 2+\log 3-\log 5}$$

$$=\frac{3}{2}=\text{R.H.S.}$$

Example 6. *Prove that*

$$x^{\log y-\log z}\times y^{\log z-\log x}\times z^{\log x-\log y}=1$$

Solution. Let

$$u=x^{\log y-\log z}\times y^{\log z-\log x}\times z^{\log x-\log y}$$

$$\Rightarrow \quad \log u=\log [x^{\log y-\log z}\times y^{\log z-\log x}\times z^{\log x-\log y}]$$

$$=\log (x^{\log y-\log z})+\log (y^{\log z-\log x})+\log (z^{\log x-\log y})$$

$$=(\log y-\log z)\log x+(\log z-\log x)\log y+(\log x-\log y)\log z$$

$$=0$$

$\therefore \quad u=a^0=1$ (a being any base$\neq 0$)

Example 7. (*a*) *Find $\log_8 25$ given that $\log_{10} 2=0{\cdot}3010$.*
[*I.C.W.A., June, 1975 ; December 1989*]

(*b*) *If $\log 2=0{\cdot}3010$ and $\log 3=0{\cdot}4771$, find the value of*

$$\log \frac{(16)^{\frac{1}{5}}(5)^2}{(108)^3}$$

Solution. (*a*) $\log_8 25=\frac{\log_{10} 25}{\log_{10} 8}=\frac{\log_{10}\left(\frac{100}{4}\right)}{\log_{10} 2^3}$

$$=\frac{\log_{10} 100-\log_{10} 4}{3\log_{10} 2}=\frac{2\log 10-2\log 2}{3\log 2}$$

$$=\frac{2-2\times 0{\cdot}3010}{3\times 0{\cdot}3010}=\frac{2\quad 0{\cdot}6020}{0{\cdot}9030}=\frac{1{\cdot}3980}{0{\cdot}9030}=1{\cdot}55$$

(*b*) Let $\quad x=\log \frac{(16)^{\frac{1}{5}}(5)^2}{(108)^3}$

$$\log \frac{(2^4)^{\frac{1}{5}}\left(\frac{10}{2}\right)^2}{(2^2\times 3^3)^3}$$

$$=\log \frac{2^{\frac{4}{5}}\times(10)^2\times(2)^{2}}{(2^2\times 3^3)}$$

$$=\frac{4}{5}\log 2+2\log 10-2\log 2-6\log 2-9\log 3$$

$$=\frac{4}{5}\times 0{\cdot}3010+2\times 1-2\times 0{\cdot}3010-6\times 0{\cdot}3010$$
$$-9\times 0{\cdot}4771$$

$$=0{\cdot}2408+2-0{\cdot}6020-1{\cdot}8060-4{\cdot}2939$$

$$=-4{\cdot}4611$$

Example 8. *Find the value of the following*

$$\log_{729}\left(9^{\frac{5}{2}+\frac{7}{2}}\right)+\log_{729}\left(27^{\frac{9}{3}+\frac{11}{3}+\frac{13}{3}}\right)+\log_{729}\left(81^{\frac{15}{4}+\frac{17}{4}+\frac{19}{4}+\frac{21}{4}}\right)$$
$$+\log_{729}\left(243^{\frac{23}{5}+\frac{25}{5}+\frac{27}{5}+\frac{29}{5}+\frac{31}{5}}\right)$$

Solution. Let the given expression be equal to x, then

$$x=\log_{729}(9^{6})+\log_{729}(27^{11})+\log_{729}(81^{18})+\log_{729}[243^{27}]$$
$$=\log_{729}[(3^2)^6]+\log_{729}[(3^3)^{11}]+\log_{729}[(3^4)^{18}]$$
$$+\log_{729}[(3^5)^{27}]$$
$$=\log_{729}(3^{12})+\log_{729}(3^{33})+\log_{729}(3^{72})+\log_{729}(3^{135})$$
$$=\log_{729}(3^{12}\times 3^{33}\times 3^{72}\times 3^{135})$$
$$x=\log_{729}(3^{252})$$

$\therefore$ $729^x=3^{252}$ (By definition)

$\Rightarrow$ $(3^6)^x=3^{252}$

$\Rightarrow$ $3^{6x}=3^{252}$

$\Rightarrow$ $6x=252$

Hence $x=42$

Example 9. *Without using a log table, prove that*

$$\log_2\left(\frac{75}{16}\right)-2\log_2\left\{\frac{\sqrt[4]{\left(\frac{25}{81}\right)^3}\cdot\sqrt[3]{\frac{25}{81}}}{\sqrt[12]{\left(\frac{25}{81}\right)^7}}\right\}+\frac{1}{3}\log_2(2^{15}\,3^{-15})=1$$

Solution. $\log_2\left(\frac{75}{16}\right)=\log_2 75-\log_2 16$

$$=\log_2(5^2\times 3)-\log_2 2^4$$
$$=2\log_2 5+\log_2 3-4\log_2 2$$

$$2\log_2\left\{\frac{\sqrt[4]{\left(\frac{25}{81}\right)^3}\times\sqrt[3]{\frac{25}{81}}}{\sqrt[12]{\left(\frac{25}{81}\right)^7}}\right\}$$

$$=2\log_2\left\{\left(\frac{25}{81}\right)^{\frac{3}{4}}.\left(\frac{25}{81}\right)^{\frac{1}{3}}.\left(\frac{25}{81}\right)^{-\frac{7}{12}}\right\}$$

$$=2\log_2\left(\frac{25}{81}\right)^{\frac{3}{4}+\frac{1}{3}-\frac{7}{12}}=2\log_2\left(\frac{25}{81}\right)^{\frac{1}{2}}$$

$$=2\log_2\left(\frac{5}{9}\right)=2\ (\log_2 5-\log_2 3^2)$$

$$=2\ (\log_2 5-2\log_2 3)=2\log_2 5-4\log_2 3$$

$$\tfrac{1}{3}\log_2\ (2^{16}\ 3^{-15})=\tfrac{1}{3}\ (15\log_2 2-15\log_2 3)=5\log_2 2-5\log_2 3$$

Substituting these in the L.H.S., we get

$$\text{L.H.S.}=2\log_2 5+\log_2 3-4\log_2 2-2\log_2 5+4\log_2 3+5\log_2 2-5\log_2 3$$

$$=\log_2 2=1=\text{R.H.S.}$$

Example 10. *Find the simplest value of*

$$\log_3 \sqrt[4]{729\ .\ \sqrt[3]{9^{-1}\ .\ 27^{-4/3}}}$$

Solution. Let $x=\log_3\sqrt[4]{729\ .\ \sqrt[3]{9^{-1}\ .\ 27^{-4/3}}}$

$$=\log_3\sqrt[4]{729\ .\ \sqrt[3]{3^{-2}\ .\ 3^{-4}}}$$

$$=\log_2\sqrt[4]{729\ .\ 3^{-2}}$$

$$=\log_3\sqrt[4]{3^6\ .\ 3^{-2}}=\log_3\sqrt[4]{3^4}$$

$$x=\log_3 3=1\qquad [\because \log_a a=1]$$

Example 11. (*a*) *If* $\log\dfrac{x+y}{7}=\dfrac{1}{2}(\log x+\log y)$. *show that*

$$\frac{x}{y}+\frac{y}{x}=47$$

(*b*) *Prove that* $\log\left[\tfrac{1}{3}(a+b)\right]=\tfrac{1}{2}(\log a+\log b)$, *if* $a^2+b^2=7ab$.

(c) *If* $x^2+y^2=11xy$, *show that*

(*i*) $\log\dfrac{x-y}{3}=\tfrac{1}{2}(\log x+\log y)$

(*ii*) $2 \log (x-y) = 2 \log 3 + \log x + \log y$

(*d*) *If* $a^2+b^2=7\,ab$, *show that*

$$2 \log (a+b) = \log a + \log b + 2 \log 3$$

[*I.C.W.A.. December 1990*]

Solution. (*a*) $\log \frac{x+y}{7} = \frac{1}{2} (\log x + \log y) = \frac{1}{2} \log (xy) = \log (xy)^{\frac{1}{2}}$

$\Rightarrow \quad \frac{x+y}{7} = (xy)^{1/2}$

$\Rightarrow \quad (x+y)^2 = 49\,xy$

$\Rightarrow \quad x^2+y^2+2xy = 49\,xy$

$\Rightarrow \quad x^2+y^2 = 47\,xy$

Dividing both sides by xy, we get

$$\frac{x}{y} + \frac{y}{x} = 47$$

(*b*) Here $\quad a^2+b^2+2\,ab = 9\,ab$

$\Rightarrow \quad \frac{(a+b)^2}{9} = ab$

$\therefore \quad \log [\frac{1}{3} (a+b)]^2 = \log ab$

$\Rightarrow \quad 2 \log [\frac{1}{3} (a+b)] = \log a + \log b$

$\Rightarrow \quad \log [\frac{1}{3} (a+b)] = \frac{1}{2} (\log a + \log b)$

(*c*) (*i*) $\quad x^2+y^2 = 11\,xy$

$\Rightarrow \quad x^2+y^2-2\,xy = 9\,xy$

$\Rightarrow \quad (x-y)^2 = 9\,xy \qquad \ldots(*)$

$\Rightarrow \quad \frac{(x-y)}{3} = (xy)^{1/2}$

Taking logarithm of both sides, we get

$$\log\left(\frac{x-y}{3}\right) = \frac{1}{2} (\log x + \log y)$$

(*ii*) Taking logarithm of both sides of (*), we get

$\log (x-y)^2 = \log 9 + \log x + \log y$

$= \log 3^2 + \log x + \log y$

$\therefore \quad 2 \log (x-y) = 2 \log 3 + \log x + \log y$

(*d*) We have $\quad a^2+b^2 = 7\,ab$

or $\quad a^2+b^2+2ab = 9\,ab$

or $\quad (a+b)^2 \quad 9\,ab \quad 3^2 \quad ab$

Taking logarithms of both sides, we have

$$2 \log (a+b) = 2 \log 3 + \log a + \log b.$$

Example 12. (*a*) *If* $\log_2 [\log_3 (\log_2 x)] = 1$, *find* x.

(*b*) *Find the value of* $\log_2 [\log_2 \{\log_3 (\log_3 27^3)\}]$

Solution. (*a*) Using the definition of logarithms, we get

$$\log_3 (\log_2 x) = 2^1 = 2$$

$$\Rightarrow \quad \log_2 x = 3^2 = 9$$

$$\Rightarrow \quad x = 2^9 = 512$$

(*b*) Let $\log_2 [\log_2 \{\log_3(\log_3 27^3)\}] = u$

Assuming $x = \log_3 27^3 = \log_3 3^9 = 9 \log_3 3 = 9$

$$y = \log_3 9 = \log_3 3^2 = 2 \log_3 3 = 2$$

$$z = \log_2 2 = 1$$

$$u = \log_2 1 = 0$$

Example 13. (*a*) *If* $\dfrac{\log a}{y-z} = \dfrac{\log b}{z-x} = \dfrac{\log c}{x-y}$, *show that*

$$a^x b^y c^z = a^{y^2+yz+z^2}\, b^{z^2+xz+x^2}\, c^{x^2+xy+y^2} = 1$$

(*b*) *If* $\dfrac{\log x}{l+m-2n} = \dfrac{\log y}{m+n-2l} = \dfrac{\log z}{n+l-2m}$,

show that $xyz = 1$

Solution. (*a*) Let $\dfrac{\log a}{y-z} = \dfrac{\log b}{z-x} = \dfrac{\log c}{x-y} = k$, (say)

$\therefore$ $\log a = k(y-z)$, $\log b = k(z-x)$, and $\log c = k(x-y)$

If the common base is, say, e, then

$$a = e^{k(y-z)},\ b = e^{k(z-x)} \text{ and } c = e^{k(x-y)}$$

(*i*) $a^x b^y c^z = e^{xk(y-z)} \times e^{yk(z-x)} \times e^{zk(x-y)}$

$$= e^{xk(y-z)+yk(z-x)+zk(x-y)} = e^0 = 1$$

(*ii*) $a^{y^2+yz+z^2} \quad b^{z^2+xz+x^2} \quad c^{x^2+xy+y^2}$

$$= e^{k(y-z)(y^2+yz+z^2)} \times e^{k(z-x)(x^2+xz+x^2)} \times e^{k(x-y)(x^2+xy+y^2)}$$

$$= e^{k(y^3-z^3)+k(z^3-x^3)+k(x^3-y^3)} = e^0 = 1$$

(*b*) Let $\dfrac{\log x}{l+m-2n} = \dfrac{\log y}{m+n-2l} = \dfrac{\log z}{n+l-2m} = k$ (say)

$\therefore$ $\log x = k(l+m-2n)$, $\log y = k(m+n-2l)$

and $\log z = k(n+l-2m)$

Let e be taken as base of logarithms, we have

$$x=e^{k(l+m-2n)},\ y=e^{k(m+n-2l)},\ z=e^{k(n+l-2m)}$$

$$xyz=e^{k(l+m-2n)+k(m+n-2l)+k(n+l-2m)}=e^{0}=1$$

Example 14 *Simplify* : $\log_a b.\ \log_b c.\ \log_c d.\ \log_d a.$

Solution. Changing the base to a common base, e, the given expression becomes

$\log_a b.\ \log_b c.\ \log_c d.\ \log_d a$

$$=\frac{\log_e b}{\log_e a}\times\frac{\log_e c}{\log_e b}\times\frac{\log_e d}{\log_e c}\times\frac{\log_e a}{\log_e d}=1$$

Example 15. *If* $x=\log_{2a} a,\ y=\log_{3a} 2a,\ z=\log_{4a} 3a,$

prove that $xyz-1=2yz.$

Solution. L.H.S. $=xyz+1$

$$=\log_{2a} a\times\log_{3a} 2a\times\log_{4a} 3a+1$$

$$=\frac{\log_e a}{\log_e 2a}\times\frac{\log_e 2a}{\log_e 3a}\times\frac{\log_e 3a}{\log_e 4a}+1$$

[Changing to common base e]

$$=\frac{\log_e a}{\log_e 4a}+1$$

$$=\frac{\log_e a+\log_e 4a}{\log_e 4a}=\frac{\log_e (a\times 4a)}{\log_e 4a}$$

$$=\frac{\log_e (2a)^2}{\log_e 4a}=\frac{2\log_e 2a}{\log_e 4a}$$

$$=\frac{2\log_e 2a}{\log_e 3a}\times\frac{\log_e 3a}{\log_e 4a}$$

$$=2\log_{3a} 2a\times\log_{4a} 3a$$

$$=2yz=\text{R.H.S.}$$

Example 16. *Prove the following* :

(*a*) $$\frac{1}{\log_{xy}(xyz)}+\frac{1}{\log_{yz}(xyz)}+\frac{1}{\log_{zx}(xyz)}=2$$

(*b*) $$\frac{1}{(\log_a bc)+1}+\frac{1}{(\log_b ca)+1}+\frac{1}{(\log_c ab)+1}=1$$

(c) $\dfrac{1}{log_{\frac{p}{q}}(x)} + \dfrac{1}{log_{\frac{q}{r}}(x)} + \dfrac{1}{log_{\frac{r}{p}}(x)} = 0.$

Solution. (a) Using the rule $\log_a b = \dfrac{1}{\log_b a}$, we have

$$\text{L.H.S.} = \frac{1}{\log_{xy}(xyz)} + \frac{1}{\log_{yz}(xyz)} + \frac{1}{\log_{zx}(xyz)}$$

$$= \log_{xyz}(xy) + \log_{xyz}(yz) + \log_{xyz}(zx)$$

$$= \log_{xyz}(xy \times yz \times zx) = \log_{xyz}(xyz)^2$$

$$= 2\log_{xyz}(xyz) = 2 = \text{R.H.S.}$$

(b) $$\text{L.H.S.} = \frac{1}{(\log_a bc)+1} + \frac{1}{(\log_b ca)+1} + \frac{1}{(\log_c ab)+1}$$

$$= \frac{1}{\log_a bc + \log_a a} + \frac{1}{\log_b ca + \log_b b} + \frac{1}{\log_c ab + \log_c c}$$

$$= \frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}$$

Now proceed in the same way as part (a).

(c) $$\text{L.H.S.} = \frac{1}{\log_{\frac{p}{q}}(x)} + \frac{1}{\log_{\frac{q}{r}}(x)} + \frac{1}{\log_{\frac{r}{p}}(x)}$$

$$= \log_x\left(\frac{p}{q}\right) + \log_x\left(\frac{p}{r}\right) + \log_x\left(\frac{r}{p}\right)$$

$$= \log_x\left[\frac{p}{q} \times \frac{q}{r} \times \frac{r}{p}\right]$$

$$= \log_x 1 = 0 = \text{R.H.S.}$$

Example 17. *If* $a^{3-x} \, . \, b^{5x} = a^{x+5} \, . \, b^{3x}$, *show that*

$$x \log\left(\frac{b}{a}\right) = \log a.$$

[*C.A. Intermediate November, 1981*]

Salution. We have

$$a^{3-x} \, . \, b^{5x} = a^{x+5} \, . \, b^{3x}$$

Taking logarithms of both sides, we have

$$\log(a^{3-x} \, . \, b^{5x}) = \log(a^{x+5} \, . \, b^{3x})$$

$\Rightarrow \quad (3-x)\log a + 5x\log b = (x+5)\log a + 3x\log b$

$\Rightarrow \quad 3\log a - x\log a + 5x\log b = x\log a + 5\log a + 3x\log b$

$\Rightarrow \quad 2x\log b - 2x\log a = 2\log a$

$\Rightarrow \quad x \log b - x \log a = \log a$

$\Rightarrow \quad x(\log b - \log a) = \log a$

$\Rightarrow \quad x \log\left(\frac{b}{a}\right) = \log a.$

Example 18. *If* $a^x = b^y = c^z = d^w$ *show that*

$$\log_a (bcd) = x\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right)$$

Solution. Taking logarithms to the base a, we get

$$\log_a (a^x) = \log_a (b^y) = \log_a (c^z) = \log_a (d^w) = x$$

$(\because \ \log_a a^x = x \log_a a = x)$

$\therefore \quad \log_a (b^y) = y \log_a b = x \quad \Rightarrow \quad \log_a b = \frac{x}{y}$

$\log_a (c^z) = z \log_a c = x \quad \Rightarrow \quad \log_a c = \frac{x}{z}$

and $\quad \log_a (d^w) = w \log_a d = x \quad \Rightarrow \quad \log_a d = \frac{x}{w}$

$$\text{L.H.S.} = \log_a (bcd) = \log_a b + \log_a c + \log_a d = \frac{x}{y} + \frac{x}{z} + \frac{x}{w}$$

$$= x\left[\frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right] = \text{R.H.S.}$$

Example 19. *Solve the following equations :*

(*a*) $\log_x 3 + \log_x 9 + \log_x 729 = 9,$

(*b*) $\log_{10} (x-9) + \log_{10} x = 1.$

Solution. (*a*) $\log_x 3 + \log_x 9 + \log_x 729 = 9$

$\Rightarrow \quad \log_x (3 \times 9 \times 729) = 9$

$\Rightarrow \quad \log_x (3^9) = 9$

$\Rightarrow \quad x^9 = 3^9$

$\Rightarrow \quad x = 3$

(*b*) $\quad \log_{10} (x-9) + \log_{10} x = 1$

$\Rightarrow \quad \log_{10} (x-9)\, x = 1$

$\Rightarrow \quad x(x-9) = 10^1$ (by definition of logarithm)

$\Rightarrow \quad x^2 - 9x - 10 = 0$

$\Rightarrow \quad x = \frac{9 \pm \sqrt{81+40}}{2} = \frac{9 \pm 11}{2}$

$\therefore \qquad x=10 \quad \text{or} \quad x=-1$

But logarithms of negative numbers are not defined, therefore $x \neq -1$. Hence $x=10$.

EXERCISE (I)

1. Change into logarithmic form :

(*i*) $6^{-1}=\frac{1}{6}$, (*ii*) $2^4=16$, (*iii*) $\sqrt[3]{8}=2$, (*iv*) $a^0=1$.

2. Change into exponential form :

(*i*) $\log_4 64=3$, (*ii*) $\log_5 \frac{1}{625}=-4$, (*iii*) $\log_{\sqrt{2}} 16=8$.

3. Find the value of x, if

(i) $\log_5 x=3$, (*ii*) $\log_a x=0$.

4. (*a*) Simplify

(*i*) $\frac{\log_{10} 1000}{\log_{10} 100}$, $\frac{\log_2 32}{\log_2 4}$; (*ii*) $\log (y^2)-\log y$

(*iii*) $\log 44-\log 176$, and (*iv*) $\log_4 256 \div \log_4 1124$.

(*b*) Prove that

(*i*) $\frac{\log_9 11}{\log_5 13} \div \frac{\log_3 11}{\log_5 13}=\frac{1}{2}$

(*ii*) $(\log a)^2-(\log b)^2=\log (ab) \log (a/b)$.

5. (*a*) Find the values of (*i*) $\log_4 256$, (*ii*) $\log_2 64$.

(*b*) Find the logarithm of

(*i*) 784 to the base $2\sqrt{7}$, (*ii*) 19683 to the base $3\sqrt{3}$.

6. (*a*) Given that $u=\log_9 x$, find in terms of u

(*i*) x, (*ii*) $\log_9 (3x)$, (*iii*) $\log_x (81)$

(*b*) Find x in the following cases :

(*i*) $\log_{\sqrt{x}} x=-\frac{2}{3}$, (*ii*) $\log_x\left(\frac{1}{9}\right)=4$, (*iii*) $\log_x 125=3$.

7. (*a*) If $\log_a x=m$, $\log_a y=n$, what are the values of

(*i*) $\log_a\left(\frac{x}{y}\right)$, (*ii*) $\log_a\left(\frac{x^2}{y}\right)$, (*iii*) $\log_a\left(\frac{x}{y^2}\right)$, (*iv*) $\log_a\left(\frac{x^3}{y}\right)$

(*b*) Correct the following :

(*i*) If $\log_a N=x$, then $x^a=N$

(*ii*) $\log_a (mn) = \log_a m \times \log_a n$, (*iii*) $\log_a \left(\frac{m}{n}\right) = \frac{\log_a m}{\log_a n}$

(*iv*) $\log_a (m^n) = \log_a m \cdot \log_a m \ldots\ldots n$ times.

(*c*) Given that $\log_{10} y = 2 - \log_{10} x$, express y in the form ax^n.

(*d*) If $3 + \log_{10} x = 2 \log_{10} y$, express x in terms of y.

8. (*a*) Prove that

(*i*) $\log_y x \cdot \log_z y \cdot \log_x z = 1$

(*ii*) $\log_y (x^3) \cdot \log_z (y^3) \cdot \log_x (z^3) = 27$

(*iii*) $\log_y (\sqrt{x}) \cdot \log_z (y^3) \cdot \log_x (\sqrt[3]{z^2}) = 1$

(*b*) Prove that

(*i*) $x^{\log y/z} \cdot y^{\log z/x} \cdot z^{\log x/y} = 1$

(*ii*) $(yz)^{\log y/z} \cdot (zx)^{\log z/x} \cdot (xy)^{\log x/y} = 1$

9. Show that

(*i*) $\log \frac{x^n}{y^n} + \log \frac{y^n}{z^n} + \log \frac{z^n}{x^n} = 0$

(*ii*) $\log\left(\frac{x^2}{yz}\right) + \log\left(\frac{y^2}{zx}\right) + \log\left(\frac{z^2}{xy}\right) = 0.$

10. Show that

$$\frac{\log 343}{1 + \frac{1}{2} \log \left(\frac{49}{4}\right) + \frac{1}{3} \log\left(\frac{1}{125}\right)} = 3.$$

11. (*a*) Without using tables, evaluate

$$\log \frac{41}{35} + \log 70 - \log \frac{41}{2} + 2 \log 5$$

(*b*) Prove that

$$\log \frac{81}{8} - 2 \log \frac{3}{2} + 3 \log \frac{2}{3} + \log \frac{3}{4} = 0$$

(*c*) Show that

(*i*) $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} + \log \frac{1}{2} = 0$

(*ii*) $\log \frac{75}{16} - 2 \log \frac{5}{9} + \log \frac{32}{243} = \log 2$

(*iii*) $\log \frac{384}{5} + \log \frac{81}{32} + 3 \log \frac{5}{3} - \log 9 = 2$

12. (*a*) Find the value of $\frac{1}{6} \cdot \frac{\sqrt{3 \log 1728}}{1 + \frac{1}{2} \log 0{\cdot}36 + \frac{1}{3} \log 8}$ without reference to log tables.

[**Hint.** $\log 1728 = \log (2^6 \times 3^3) = 6 \log 2 + 3 \log 3$

$$\log 0{\cdot}36 = \log \frac{36}{100} = \log 6^2 - \log 10^2 = 2 \log 6 - 2 \log 10$$
$$= 2 \log 2 + 2 \log 3 - 2 \log 10$$
$$\log 8 = \log 2^3 = \log 2$$

Substituting these values in the given expression.]

(*b*) Find the value of

$$\frac{1}{6}\sqrt{\frac{3 \log 1728}{\frac{1}{2} \log 36 + \frac{1}{3} \log 8}}$$

13. (*a*) If $\log \frac{x+y}{3} = \frac{1}{2} (\log x + \log y)$, show that

$$\frac{x}{y} + \frac{y}{x} = 7$$

(*b*) If $x^2 + y^2 = 7\,xy$, prove that

$$\log \{\tfrac{1}{3}(x+y)\} = \tfrac{1}{2} (\log x + \log y)$$

(*c*) If $x^3 + y^3 = 0$ and $x + y \neq 0$, prove that

$$\log (x+y) = \tfrac{1}{2} (\log x + \log y + \log 3)$$

(*d*) If $a = \log_{24} 12$, $b = \log_{36} 24$ and $c = \log_{48} 36$, then prove that

$$1 + abc = 2bc$$

[**Hint.** Here $24^a = 12$, $36^b = 24$, $48^c = 36$

$\therefore$ $12 = 24^a = (36)^{ab} = 48^{abc}$

$\therefore$ $12 \times 48 = 48^{abc+1} \Rightarrow (24)^2 = 48^{abc+1}$ ...(*)

$\Rightarrow$ $(24)^2 = (36)^{2b} = (48)^{2bc} = 48^{1+abc}$ [From (*)]

Hence $2bc = 1 + abc$]

14. (*a*) If $\frac{\log a}{q-r} = \frac{\log b}{r-p} = \frac{\log c}{p-q}$, prove that

(*i*) $abc = 1$,

(*ii*) $a^{q+r} \,.\, b^{r+p} \,.\, c^{p+q} = 1$

(*b*) If $\frac{\log x}{1} = \frac{\log y}{2} = \frac{\log z}{5}$, show that $x^4 y^3 z^{-2} = 1$

(*c*) If $\frac{\log x}{2} = \frac{\log y}{3} = \frac{\log z}{5}$, show that $x^4 = yz$

(*d*) If $\frac{\log_2 x}{4} = \frac{\log_2 y}{6} = \frac{\log_2 z}{3k}$, and $x^3 y^2 z = 1$, show that value of k is -8.

15. (*a*) Show that

(*i*) $\frac{1}{\log_x xy} + \frac{1}{\log_y xy} = 1$

(ii) $\frac{1}{\log_6 24}+\frac{1}{\log_{12} 24}+\frac{1}{\log_8 24}=2$

(iii) $\frac{1}{\log_a p}+\frac{1}{\log_b p}+\frac{1}{\log_c p}=\frac{1}{\log_x p}$, where $abc=x$

(b) If $x=1+\log_a bc$, $y=1+\log_b ca$, $z=1+\log_c ab$, prove that

$$xyz=xy+yz+zx$$

16. (a) If $u=v^2=w^3=z^4$, then prove that

$$\log_u(uvwz)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

(b) If $a^2=b^3=c^5=d^6$, that

$$\log_d(abc)=\frac{31}{5}$$

17. (a) Solve the following equations :

(i) $\log_x 4+\log_x 16+\log_x 64=12$,

(ii) If $\log_2 x+\log_4 x+\log_{16} x=\frac{21}{4}$, find x.

(iii) $\log_{10} x+\log_{10}(x-3)=1$,

(iv) $\log_{10}(x+3)^2-2=\log_{10}\frac{1}{x^2}$, and

(v) $\log_8 x+\log_4 x+\log_2 x=11$

(b) Solve the equations :

(i) $\log_{1/2}[\log_x(\log_4 32)]=2$

(ii) $\log_3[\log_2(\log_3 x)]=1$

18. If $p=\log_a bc$, $q=\log_b ac$, $r=\log_c ab$, show that

$$pqr=p+q+r+2$$

19. Prove that

$\log_a x+\log_{a^2} x^2+\log_{a^3} x^3+\ldots+\log_{a^n} x^n=\log_a x^n$

20. Show that

(i) $x=a^{1/\log_x a}$,

(ii) $x=a^{\log_a b.\ \log_b c.\ \log_c d.\ \log_d x}$

[**Hint.** (i) Let $x=a^y$ so that $y=\log_a x$

$\therefore\quad x=a^{\log_a x}=a^{1/\log_x a}$

(ii) We have $x=a^{\log_a x}$ ...(1)

Now $\log_a b\times\log_b c\times\log_c d\times\log_d x=\log_a x$.

Substitute in (1), we get the required result.]

21. If a,b,c are any three consecutive integers, prove that

$$\log(1+ac)=2\log b$$

22. If $x=\frac{e^y-e^{-y}}{e^y+e^{-y}}$, show that $y=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

[**Hint.** Apply componendo and dividendo, *viz.*,

$$\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{a+b}{a-b}=\frac{c+d}{c-d}\Big]$$

23. If $(3{\cdot}7)^x=(0{\cdot}37)^y=1000$, show that

$$x^{-1}-y^{-1}=3^{-1}$$

[**Hint.** $\because\quad (3{\cdot}7)^x=1000=10^3$

$\therefore\quad x\log 3{\cdot}7=3\log 10=3$

$\Rightarrow\quad \frac{1}{x}=\frac{1}{3}\log 3{\cdot}7$ ...(1)

and $\quad (0{\cdot}37)^y=1000=10^3$

$\therefore\quad y\log 0{\cdot}37=3\log 10=3$

$\Rightarrow\quad \frac{1}{y}=\frac{1}{3}\log 0{\cdot}37$ ...(2)

$\therefore\quad \frac{1}{x}-\frac{1}{y}=\frac{1}{3}\left[\log\frac{3{\cdot}7}{0{\cdot}37}\right]=\frac{1}{3}\log 10=\frac{1}{3}\Big]$

24. If $a^{2x-3}\,b^{2x}=a^{6-x}.\,b^{5x}$, prove that

$$3\log a=x\log\frac{a}{b}$$

(*C.A. Intermediate November 1981*)

25. Without the use of logarithm table, show that

$$\frac{2}{5}<\log_{10}3<\frac{1}{2}$$

ANSWERS

1. (*i*) $\log_6\frac{1}{6}=-1$, (*ii*) $\log_2 16=4$, (*iii*) $\log_8 2=\frac{1}{3}$, (*iv*) $\log_a 1=0$
2. (*i*) $4^3=64$, (*ii*) $5^{-4}=\frac{1}{625}$, (*iii*) $(\sqrt{2})^8=16$ 3. (*i*) 5^a, (*ii*) 1
4. (*i*) 3/2, 5/2, (*ii*) $\log y$, (*iii*) $-\log 4$, (*iv*) 4/5
5. (*a*) (*i*) 4, (*ii*) 6, (*b*) (*i*) 4, (*ii*) 6.
6. (*a*) (*i*) 9^u (*ii*) $\frac{1}{2}+u$, (*iii*) $2/u$.
   (*b*) (*i*) $\frac{1}{2}$, (*ii*) $1/\sqrt{3}$, (*iii*) 5, (*iv*) 25.
7. (*a*) (*i*) $m-n$, (*ii*) $2m-n$, (*iii*) $m-2n$, (*iv*) $3m-n$

(b) (i) $a^x = N$, (ii) $\log_a m + \log_a n$, (iii) $\log_a m - \log_a n$

(iv) $n \log_a m$, (c) $100x^{-1}$, (d) $x = \frac{y^2}{1000}$

11. (a) 2, **12.** (a) $\frac{3}{4}$ (b) $\frac{1}{2}$.

17. (a) (i) $x=2$, (ii) $x=8$, (iii) $x=5$, (iv) $x=2$ (v) $x=64$,
(b) (i) $x=625/16$, (ii) $x=6561$.

7·2. LOGARITHM TABLES

The logarithm of a number consists of two parts, the whole part or the integral part called the characteristic and the decimal part called the mantissa. Whereas the former can be known by mere inspection, the latter has to be obtained from the logarithm tables given at the end of the boŏk.

Characteristic. The characteristic of the logarithm of any number greater than 1 is positive and is one less than the number of digits to the left of the decimal point in the given number. The characteristic of the logarithm of any nnmber less than 1 is negative and numerically one more than the number of zeros to the right of the decimal point. If there is no zero than obviously it will be −1. The following table will illustrate :

Number	*Characteristic*	
37	1	One less than the number of digits to the left of the decimal point.
4623	3	
531·2	2	
6·21	0	
·8	−1	One more than the number of zeros on the right immediately after the decimal point.
·07	−2	
·00507	−3	
·000670	−4	

Zero or Positive Characteristic

When the numder under consideration is greater than unity :

Since	$10^0=1$	, $\log 1=0$
	$10^1=10$	, $\log 10=1$
	$10^2=100$	, $\log 100=2$
	$10^3=1000$	, $\log 1000=3$

It is clear that if a number is a positive integral power of 10, its logarithm is a positive integer. All numbers lying between 1 and 10, *i.e.*, numbers with 1 digit in the integral part have their logarithms lying between 0 and 1. Therefore, their integral parts are zero only.

All numbers lying between 10 and 100 have two digits in their integral parts. Their logarithms lie between 1 and 2. Therefore, numbers with two digits in their integral parts have 1 as characteristic.

In general, the logarithm of a number containing n digits only in its integral part is $(n-1)+a$ fraction.

Hence we deduce that *the characteristic of the logarithm of a number greater than unity is one less than the number of digits in its integral part and is positive.*

For example, the characteristics of log 75, log 79326, log 1·76 are 1, 4 and 0 respectively.

Negative Characteristics

Since $10^{-1}=\frac{1}{10}=0{\cdot}1, \quad \log\ 0{\cdot}1=-1$

$$10^{-2}=\frac{1}{100}=0{\cdot}01, \quad \log\ 0{\cdot}01=-2 \text{ etc.}$$

Obviously, if a number is a negative integral power of 10, the logarithm is a negative integer. All numbers lying between 1 and 0·1 have logarithms lying between 0 and −1, *i.e.*, greater than −1 and less than 0. Since the decimal part is always written positive, the characteristic is −1.

All numbers lying between 0·1 and 0·01 have their logarithms lying between −1 and −2. Therefore, such numbers have −2 as characteristic of their logarithms.

In general, the logarithm of a number having n zeros just after the decimal point is $-(n+1)+a$ fraction.

Hence, we deduce that the *characteristic of the logarithm of a number less than unity is one more than the number of zeros just after the decimal point, and is negative.*

Mantissa. The mantissa is the fractional part of the logarithm of a given number. It is same for a given set of figures in the same order and does not depend on the position of the decimal point.

Number	*Mantissa*	*Logarithm*
log 4597	=(... 6625)	=3·6625
log 459·7	=(...·6625)	=2·6625
log 45·94	=(...·6625)	=1·6625
log 4·594	=(...·6625)	=0·6695
log ·4594	=(...·6625)	=1·6625

Thus with the same figures there will be difference in the characteristic only. However, it can be noticed in the last logarithm that the minus sign of the characteristic has been placed on the top of it, which is read as bar 1 point 6625. This is to show that only characteristic has a negative value. It should be remembered, that the mantissa is always a positive quantity. The other way to indicate this is

$$\log\ {\cdot}004594=-3+{\cdot}6625=3{\cdot}6625$$

In the common system of logarithms, the characteristic of the logarithm may be positive or negative but the mantissa is always positive. A negative mantissa must be converted into a positive mantissa before reference to a logarithm table. For example

$$-3{\cdot}6872=-4+(4-3{\cdot}6872)=-4+0{\cdot}3128=\bar{4}{\cdot}3128$$

It may be noted that $4{\cdot}3128$ is different from $-4{\cdot}3128$.

How to consult a logarithm table ? For this purpose the first thing to remember is that we use all the digits irrespective of the decimal point (which is relevant for determining the value of the characteristic only). The second important thing is that we use only the significant digits thus we ignore zeros because they are there to determine the place of the significant digits only. But we do not ignore zeros which are in between the digits. For example the significant digits for consulting a log table with the following numbers would be –

Number	*Significant digits*
52	52
37500	375
·0163	163
5·012	5012

Now, the first two digits form the significant ones which are used for determining the row of the logarithm table. If there is only one digit then we add a zero. For example in case of 9 we will see the row which starts with 90 in the first column. We then proceed along that row till the column of our 3rd digit is reached. For further accuracy we use the 4th and 5th digits which we see in the same row in the outer or the difference column and add that difference to the value indicated by the 3rd digit in the main column of the relevant row.

Illustrations. 1. *Add $\bar{4}{\cdot}74628$ and $3{\cdot}42367$*

Solution.

$$\begin{array}{r} -4+{\cdot}74628 \\ 3+{\cdot}42367 \\ \hline -1+1{\cdot}16995=0{\cdot}16995 \end{array}$$

2. *Subtract $\bar{5}{\cdot}62493$ from $\bar{3}{\cdot}24567$*

Solution.

$$\begin{array}{r} -3+{\cdot}24567 \\ -5+{\cdot}62493 \\ + \\ \hline +2-1+{\cdot}62074 \end{array} \quad =1{\cdot}62074$$

3. *Multiply $\bar{3}{\cdot}77815$ by 5*

Solution.

$$\begin{array}{r} -3+{\cdot}77815 \\ 5 \\ \hline -15+3{\cdot}89075 \end{array} \quad =\overline{12}{\cdot}89075$$

In all these examples we have followed the ordinary rules of calculation except that we kept apart the initial operations of the negative characteristic from the positive mantissa and then got the final result. The same thing we are going to do in division with little variation in the characteristic.

4. *Divide $\overline{13}$·15836 by 5*

Solution. Here before performing division we raise the negative characteristic to a value which is divisible by 5 as follows :

$$\overline{15}+2·15836\div5=\overline{3}·43167$$

5. *Find log 34.*

Solution. The number has two digits. Its characteristic is $2-1=1$. To find mantissa move along 34 horizontally and note down the number under 0 (34 or 34·0). The number found is 5315.

$$\therefore \qquad \log 34=1·5315$$

6. *Find log 347.*

Solution. The number has three digits. Its characteristic is $3-1$ or 2.

To find mantissa move along 34 horizontally and note the number under 7. It is 5403.

$$\therefore \qquad \log 347=2·5403$$

Antilogarithms. If x is the logarithm of a given number n, with a given base then n is called the antilogarithm (antilog) of x to that very base. This can be expressed as follows :

If $\log_a n=x$ then $n=$antilog x

For example, if log 61720=4·7904 then 61720=antilog 4·7904

The process of finding the antilog of a given logarithm is just the reverse of the procedure adopted for finding the logarithm of a given number. Anti-log tables are available but log tables can also be used if anti-log tables are not available.

For finding out the anti-log of a log, we use only the decimal part or the mantissa in consulting the table, *e.g.*, in 1·9072 we will look ·90 in the first column of the anti-log table to locate the relevant row and then use the 3^{rd}, 4^{th} and 5^{th} digits as in the case of finding the logarithm. Now, the characteristic 1 in the above case will be used for placing the decimal point as indicated in the table below :

Number	*Logarithms*	*Antilogarithms*
206	2·3139	206·0
20·6	1·3139	20·60
2·06	0·3139	2·060
·206	$\overline{1}$·3139	·2060
·0206	$\overline{2}$·3139	·02060

This can be verified from the log and anti-log tables of 4 digits.

Illustrations. 1. *Find the value of log 5 if log 2 is equal to ·3010.*

Solution. $\log 5=\log\left(\frac{10}{2}\right)=\log 10-\log 2=1-\cdot 3010$

$=\cdot 6990$

Remark. Remember that $\frac{\log 1000}{\log 100}$ is not the same as

$\log\left(\frac{1000}{100}\right)$. Only the latter can be written as log 1000 − log 100.

2. *Find the number whose logarithm is 2·4678.*

Solution. From the antilog table,

for mantissa ·467, the number = 2931

for mean difference 8, the number = 5

∴ For mantissa ·4678, the number = 2936

The characteristic is 2, therefore the number must have 3 digits in the integral part.

Hence antilog 2·4678 = 293·6

3. *Find the number whose logarithm is −2·4678.*

$-2\cdot 4678=-3+3-2\cdot 4678=-3+\cdot 5322=\bar{3}\cdot 5322$

For mantissa ·532, the number = 3404

For mean difference 2, the number = 2

∴ For mantissa ·5322, the number = 3406

The characteristic is −3, therefore the number is less than one and there must be two zeros just after the decimal point.

∴ Antilog (−2·4678) = 0·003406

7·3. OPERATIONS WITH LOGARITHMS

1. Multiplication. *To multiply numbers, add their logarithms.* The sum of the logarithms is the logarithm of the product.

Illustration. *Find the value of 628·24 × 93·536.*

Solution. Let $x=628\cdot 24\times 93\cdot 536$

Then $\log x=\log 628\cdot 24+\log 93\cdot 536$

$=2\cdot 7981+1\cdot 9710=4\cdot 7691$

(The characteristic **4** determines the position of decimal point after 5 digits from the left).

∴ $x=58760$

The scheme of work may be shown as below :

$$\log 628{\cdot}24 = 2{\cdot}7981$$
$$\log 93{\cdot}536 = 1{\cdot}9710$$
$$4{\cdot}7691$$

$$\text{anti-log } 4{\cdot}7691 = 58760$$

II Division. *To divide one number by another, subtract the logarithm of the latter from the logarithm of the former.*

Illustration. *Find the value of* $628{\cdot}24 \div 93{\cdot}536$.

Solution. Let $x = 628{\cdot}24 \div 93{\cdot}536$

Then $\log x = \log 628{\cdot}24 - \log 93{\cdot}536 = 2{\cdot}7981 - 1{\cdot}9710$

$= 0{\cdot}8271$

$\therefore$ $= \text{anti-log } (0{\cdot}8271) = 6{\cdot}761$

III. Involution. *To raise a given number to any power, multiply he logarithm of the number by the index of the power.* The product is the logarithm of the power.

Illustration. *Find the value of* $(3{\cdot}786)^6$

Solution. Let $x = (3{\cdot}786)^6$

Then $\log x = 6 \log 3{\cdot}785 = 6 \times 0{\cdot}5782 = 3{\cdot}4692$

$\therefore$ $x = \text{anti-log } (3{\cdot}4692) = 2945$

IV. Evolution. *To extract any root of a given number, divide the logarithm of the number by the index of the root.* The quotient is the logarithm of the root.

Illustration. *Find the value of* $(789{\cdot}45)^{1/8}$

Solution. Let $x = (789{\cdot}45)^{1/8}$

Then $\log x = \frac{1}{8} \log (789{\cdot}45) = \frac{1}{8} \times 2{\cdot}89733 = 0{\cdot}36217$

$\therefore$ $x = \text{anti-log } (0{\cdot}36217) = 2{\cdot}3023$

Example 20. *If* $\log 3 = 0{\cdot}4771$, *find the number of digits in* 3^{43}.

Solution. Let $x = 3^{43}$

$\therefore$ $\log x = \log 3^{43} = 43 \log 3 = 43 \times 0{\cdot}4771 = 20{\cdot}5153$

$\therefore$ $x = \text{anti-log } (20{\cdot}5153)$

As the characteristic is 20, the number of digits in the integral part of x is $20 + 1 = 21$.

Hence number of digits in $321.^{43}$ is

Example 21. *If log 3 = ·47712, find the position of the first significant figure in* 3^{-20}.

Solution. Let $x = 3^{-20}$

$$\therefore \quad \log x = \log 3^{-20} = -20 \log 3$$
$$= -20 \times {\cdot}47712 = -9{\cdot}54240$$
$$= -10 + (10 - 9{\cdot}54240)$$
$$= \overline{10}{\cdot}4576$$

∴ The characteristic of the logarithm of $3^{-20} = -10$ which is numerically one more than the number of zeros immediately after the decimal point in 3^{-20}

∴ The number of zeros immediately after the decimal point in $3^{-20} = 10 - 1 = 9$.

Hence first significant figure is in the 10th place of decimal.

Example 22. *Show that*

$$\left(1\frac{1}{20}\right)^{100} > 100$$

Given that log 2 = 0·30103, log 3 = ·47712, log 7 = ·8450980

Solution. Let $x = \left(1\frac{1}{20}\right)^{100} = \left(\frac{21}{20}\right)^{100}$

$$\therefore \quad \log x = \log\left(\frac{21}{20}\right)^{100} = 100 \log\left(\frac{21}{20}\right)$$
$$= 100 \log\left(\frac{3 \times 7}{2 \times 10}\right)$$
$$= 100[\log(3 \times 7) - \log(2 \times 10)]$$
$$= 100[\log 3 + \log 7 - \log 2 - \log 10]$$
$$= 100[{\cdot}4771213 + {\cdot}8450980 - {\cdot}3010300 - 1]$$
$$= 100[1{\cdot}3222193 - 1{\cdot}3010300]$$
$$= 100[{\cdot}0211893] = 2{\cdot}11893 > 2$$

$$\therefore \quad \log\left(\frac{21}{20}\right)^{100} > 2 = 2\,.\,(1) = 2 \log 10 = \log 10^2 = \log 100$$

$$\log\left(\frac{21}{20}\right)^{100} > \log 100$$

$$\Rightarrow \quad \left(1\frac{1}{20}\right)^{100} > 100$$

Example 23. *Simplify* : $6253\left(1+\frac{5}{100}\right)\left(1+\frac{5}{400}\right)-6253$, *you can use logarithm tables.*

Solution. Let

$$x=6253\left(1+\frac{5}{100}\right)^8\left(1+\frac{5}{400}\right)$$

$$=6253\left(\frac{105}{100}\right)^8\left(\frac{405}{400}\right)$$

$$\log x=\log 6253+\log\left(\frac{105}{100}\right)^8+\log\left(\frac{405}{400}\right)$$

$$=\log 6253+8\,[\log 105-\log 100]+(\log 405-\log 400)$$

$$=3{\cdot}7961+8(2{\cdot}0212-2{\cdot}000)+(2{\cdot}6075-2{\cdot}6021)=3{\cdot}9711$$

$$\therefore \quad x=\text{anti-log }(3{\cdot}9711)=9356$$

Hence $6253\left(1+\frac{5}{400}\right)^8\left(1+\frac{5}{100}\right)-6253=9356-6253=3103$

Example 24. *Evaluate* $\frac{24{\cdot}395\times(3{\cdot}16)^3}{8{\cdot}79}$

Solution. Let $x=\frac{24{\cdot}395\times(3{\cdot}16)^3}{8{\cdot}79}$

$$\log x=\log 24{\cdot}395+3\log 3{\cdot}16-\log 8{\cdot}79$$

$$=1{\cdot}3874+3\times 0{\cdot}4997-0{\cdot}9440$$

$$=1{\cdot}3874+1{\cdot}4991-0{\cdot}9440=1{\cdot}9425$$

$$\therefore \quad x=\text{anti-log }(1{\cdot}9425)=87{\cdot}60$$

Example 25 *Find the value of* $\frac{0{\cdot}0357\times\sqrt{0{\cdot}235}}{\sqrt[3]{0{\cdot}0637}}$ *using logarithm tables.*

Solution. $\log\left\{\frac{0{\cdot}0357\times\sqrt{0{\cdot}235}}{\sqrt[3]{0{\cdot}0637}}\right\}=\log 0{\cdot}0357+\frac{1}{2}\log 0{\cdot}235-\frac{1}{3}\log 0{\cdot}0637$

$$=2{\cdot}5527+\frac{1}{2}\,(\bar{1}{\cdot}3711)-\frac{1}{3}(\bar{2}{\cdot}8041)$$

Now in $\bar{1}{\cdot}3711$, the characteristic is negative but the mantissa ·3711 is positive. To divide $\bar{1}{\cdot}3711$ by 2, we write the logarithm as $(\bar{2}+1{\cdot}3711)$ and divide each term separately by 2.

$$\therefore \quad \tfrac{1}{2}(\bar{1}{\cdot}3711)=\tfrac{1}{2}(\bar{2}+1{\cdot}3711)=\bar{1}+{\cdot}6856=\bar{1}{\cdot}6856$$

Similarly $\frac{1}{3}(\bar{2}{\cdot}8041)=\frac{1}{3}(\bar{3}+1{\cdot}8041)=\bar{1}+{\cdot}6014=\bar{1}{\cdot}6014$

Hence if the characteristic is negative, increase its numerical value equal to the divisor or its nearest multiple so that the characteristic remains integral and the mantissa, positive and fractional. For example

$$\frac{\bar{5}\cdot9746}{2}=\tfrac{1}{2}(\bar{6}+1\cdot9746)=\bar{3}\cdot9873$$

$$\therefore \log\left\{\frac{0\cdot0357\times\sqrt{0\cdot235}}{\sqrt[3]{0\cdot0637}}\right\}$$

$$=\bar{2}\cdot5527+\bar{1}\cdot6856-\bar{1}\cdot6014$$

$$=\{(-2)+(-1)-(-1)\}+(\cdot5527+\cdot6856-\cdot6014)$$

$$=-2+\cdot6369=\bar{2}\cdot6369$$

Here again the characteristics are separately added.

$\therefore$ Antilog $\bar{2}\cdot6369=0\cdot04334$

Hence $\dfrac{0\cdot0357\times\sqrt{0\cdot235}}{\sqrt[3]{0\cdot0637}}=0\cdot04334$

Example 26. *Calculate, with the help of log tables, the value of*

$$\sqrt[7]{\frac{1}{0\cdot8176\times36\cdot21}}$$

Solution. Let $x=\left[\dfrac{1}{0\cdot8176\times36\cdot21}\right]^{\frac{1}{7}}$

$$\log x=\tfrac{1}{7}[\log 1-\log 0\cdot8176-\log 36\cdot21]$$

$$=\tfrac{1}{7}[0-\bar{1}\cdot9125-1\cdot5588]$$

$$=\tfrac{1}{7}(1-0\cdot9125-1\cdot5588)]=\tfrac{1}{7}(-1\cdot4713)$$

$$=-0\cdot2102=-1+1-0\cdot2102=\bar{1}.7898$$

$$x=\text{antilog }(\bar{1}\cdot7898)=0\cdot6103$$

Example 27. (*a*) *Simplify*

$$\frac{(63)^{\frac{1}{2}}\times(0\cdot0781)^{\frac{1}{4}}\times(46)^{\frac{1}{8}}}{0\cdot0032\times(24\cdot08)^{-\frac{1}{3}}}$$

Solution. Let $x=\dfrac{N}{D}$, where N and D denote the numerator and denominator of the given fraction,

Then $\log N=\tfrac{1}{2}\log 63+\tfrac{1}{4}\log 0\cdot0781+\tfrac{1}{8}\log 46$

$=\frac{1}{2}(1{\cdot}7993)+\frac{1}{4}(\bar{2}{\cdot}8927)+\frac{1}{6}(1{\cdot}6628)$

$=0{\cdot}8996+\frac{1}{4}(-2+0{\cdot}8927)+0{\cdot}2771$

$=0{\cdot}8996+(-0{\cdot}5+0{\cdot}2232)+0{\cdot}2771$

$=0{\cdot}8996-0{\cdot}5+0{\cdot}2232+0{\cdot}2771=0{\cdot}8999$

Similarly $\log D=\log 0{\cdot}0032-\frac{1}{3}\log 24{\cdot}08$

$=\bar{3}{\cdot}5051-\frac{1}{3}(1{\cdot}3816)$

$=(-3+0{\cdot}5051)-0{\cdot}4605$

$=-3+0{\cdot}5051-0{\cdot}4605=-2{\cdot}9554$

Now $\log x=\log N-\log D=0{\cdot}8999-(-2{\cdot}9554)$

$=0{\cdot}8999+2{\cdot}9554=3{\cdot}8553$

$\therefore$ $x=\text{antilog}\,(3{\cdot}8553)=7166$

Example 27 (*b*) *Evaluate*

$$\frac{(17{\cdot}5)^{\frac{1}{2}}+(15{\cdot}2)^{-\frac{1}{3}}}{(56{\cdot}3)^{\frac{3}{5}}-(12{\cdot}4)^{\frac{1}{4}}}$$

Solution. Let the expression $=\dfrac{x+y}{u-v}$

Now $x=(17{\cdot}5)^{\frac{1}{2}}$

$\therefore$ $\log x=\frac{1}{2}\log 17{\cdot}5=\frac{1}{2}(1{\cdot}2430)=0{\cdot}6215$

$\Rightarrow$ $x=4{\cdot}183$

$y=(15{\cdot}2)^{-\frac{1}{3}}$

$\log y=-\frac{1}{3}\log 15{\cdot}2=-\frac{1}{3}\times 1{\cdot}1818=-0{\cdot}3939=\bar{1}{\cdot}6061$

$\Rightarrow$ $y=0{\cdot}4037$

Also $u=(56{\cdot}3)^{\frac{3}{5}}$

$\log u=\frac{3}{5}\log 56{\cdot}3=\frac{3}{5}(1{\cdot}7505)=\dfrac{5{\cdot}2515}{5}=1{\cdot}0503$

$\Rightarrow$ $u=11{\cdot}23$

Now $v=(12{\cdot}4)^{\frac{1}{4}}$

$\therefore$ $\log v=\frac{1}{4}\log 12{\cdot}4=\frac{1}{4}(1{\cdot}0934)=0{\cdot}27135$

$\Rightarrow$ $v=1{\cdot}877$

$\therefore$ The expression $\dfrac{4{\cdot}183+0{\cdot}4037}{11{\cdot}23-1{\cdot}877}=\dfrac{4{\cdot}5867}{9{\cdot}353}=0{\cdot}489$

(On using logarithms again)

Example 28. *Using log tables, where necessary, calculate the value of*

$$\{\sqrt[5]{3\cdot219} \div (0\cdot0624)^7\} + \frac{(1\cdot78)^{-\frac{3}{4}}}{\sqrt{2\cdot13}}$$

Solution. Let $x = \{\sqrt[5]{3\cdot219} \div (0\cdot0624)^7\} + \frac{(1\cdot78)^{-\frac{3}{4}}}{\sqrt{2\cdot13}}$

Assuming $u = \sqrt[5]{3\cdot219} \div (0\cdot0624)^7$

$$\log u = \tfrac{1}{5}\log(3\cdot219) - 7\log(\cdot0624)$$
$$= \tfrac{1}{5}[0\cdot5077] - 7[\bar{2}\cdot7952]$$
$$= 0\cdot10154 - 7(-2 + 0\cdot7952)$$
$$= 0\cdot10154 + 14 - 5\cdot5664 = 8\cdot5351$$

$\Rightarrow$ $u = \text{antilog}(8\cdot5351) = 342900000$

$$u = \frac{(1\cdot78)^{-\frac{3}{4}}}{\sqrt{2\ 13}}$$

$\therefore$ $\log v = -\tfrac{3}{4}\log 1\cdot78 - \tfrac{1}{2}\log 2\cdot13$
$$= -\tfrac{3}{4} \times 0\cdot2504 - \tfrac{1}{2} \times 0\cdot3284$$
$$= -0\cdot1878 - 0\cdot1642 = -0\cdot3520$$
$$= -1 + 1 - 0\ 3520 = \bar{1}\cdot648$$

$\Rightarrow$ $v = \text{anti-log}(\bar{1}\cdot648) = 0\cdot4446$

Hence $x = u + v = 342900000 + 0\cdot4446 = 342900000\cdot4446$

Example 29. *Solve the equation*

$$11^{4x-5} \times 3^{2x} = 5^{3-x} \div 7^{-x}$$

Solution. The equation may be written as

$$11^{4x-5} \times 3^{2x} = 5^{3-x} \times 7^x$$

Taking logarithms of both sides, we get

$$\log 11^{4x-5} + \log 3^{2x} = \log .5^{3-x} + \log 7^x$$

$\Leftrightarrow$ $(4x-5)\log 11 + 2x\log 3 = (3-x)\log 5 + x\log 7$

$\Leftrightarrow$ $4x\log 11 - 5\log 11 + 2x\log 3 = 3\log 5 - x\log 5 + x\log 7$

$\Rightarrow$ $x(4\log 11 + 2\log 3 + \log 5 - \log 7) = 3\log 5 + 5\log 11$

$\Rightarrow$ $x(4 \times 1\cdot0414 + 2 \times 0\cdot4771 + 0\ 6990 - 0\cdot8451) = 3 \times 0\cdot6990 + 5 \times 1\cdot0414$

$\Rightarrow$ $4\cdot9737\ x = 7\cdot3040$

$\Rightarrow$ $x = \frac{7\cdot3040}{4\cdot9737}$

Again taking logs, we get

$$\log x = \log 7{\cdot}3040 - \log 4{\cdot}9737 = 0{\cdot}8635 - 0{\cdot}6968 = 0{\cdot}1667$$

$$\therefore \qquad x = \text{anti-log}\ (0{\cdot}1667) = 1{\cdot}468$$

EXERCISE (III)

1. Evaluate $\log_{43} 57$.

$$\left[\textbf{Hint.}\quad \log_{43} 57 = \frac{\log 57}{\log 43} = \frac{1{\cdot}7759}{1{\cdot}6335}\right]$$

2. Given $\log 3 = {\cdot}4771$, find

   (*i*) the number of digits in 3^{62},

   (*ii*) the position of the first significant figure in 3^{-65}.

   (*iii*) Find the number of zeros between the decimal point and the first significant figure in the value of $(0{\cdot}0504)^{12}$ (given that $\log 2 = {\cdot}301$, $\log 3 = {\cdot}477$, $\log 7 = {\cdot}845$, no log tables are to be used).

[**Hint.** Let

$$x = ({\cdot}0504)^{12} = (504 \times 10^{-4})^{12}$$
$$= (7 \times 3^2 \times 2^3 \times 10^{-4})^{12}$$

$$\therefore \quad \log x = 12(\log 7 + 3 \log 2 + 2 \log 3 - 4 \log 10)$$
$$= 12({\cdot}845 + {\cdot}903 + {\cdot}954 - 4{\cdot}00)$$
$$= 12(-1{\cdot}298) = 12(\bar{2}{\cdot}702) = \overline{16}{\cdot}424]$$

3. Given that $\log 2 = {\cdot}30103$ and $\log 3 = {\cdot}47712$, find the number of digits in 6^{20}.

4. Show that

$$\left(1\ \frac{1}{80}\right)^{1000} > 100{,}000$$

5. Find with the help of log tables, the value of

$$\frac{1}{5{\cdot}7002 \times 6{\cdot}0818 \div 69{\cdot}732}$$

6. Find the value of

$$\frac{(435)^3 - ({\cdot}056)^{\frac{1}{3}}}{(380)^4}$$

7. Find with the help of log tables, the value of

$$\frac{2{\cdot}389 \times 0{\cdot}004679}{0{\cdot}00556 \times 52{\cdot}14}$$

8. Evaluate $\dfrac{937{\cdot}6 \times (11{\cdot}059)^3 \times (0{\cdot}02097)}{\sqrt{6004} \times 10^3 \times 8{\cdot}06}$

9. Simplify by using log tables,

(*a*) $$\sqrt[6]{\frac{9268 \times 4{\cdot}573 \times 0{\cdot}0864}{87{\cdot}65 \times 0{\cdot}5432}}$$

(*b*) $$\frac{(6{\cdot}45)^3 \times (0{\cdot}00034)^{\frac{1}{3}} \times (981{\cdot}4)}{(9{\cdot}37)^2 \times (8{\cdot}93)^{\frac{1}{4}} \times (0{\cdot}0167)},$$

(*c*) $$\frac{({\cdot}0437)^{\frac{2}{3}} \times (1{\cdot}407)^2}{({\cdot}0015)^{\frac{1}{3}} \times (1{\cdot}235)^{\frac{1}{7}}}$$

(*d*) Using log tables, find the numerical value of x from the relation

$$2x = \log_{10} 26{\cdot}54 + \log_{10} 0{\cdot}004321 - \log_{10} 0{\cdot}00001357$$

and find the value of

$$\sqrt{\frac{26{\cdot}54 \times 0{\cdot}004321}{0{\cdot}00001357}},$$

correct to the nearest integer.

10. Simplify by using log tables,

$$\frac{\sqrt{85{\cdot}82} - \sqrt[4]{9172}}{\sqrt[5]{125{\cdot}7}}$$

11. Find the value of $30\{(1+0{\cdot}035)^{15} - 1\}$

12. Evaluate

(*i*) $\frac{400}{0{\cdot}06}\left[1 - \frac{1}{(1{\cdot}06)^4}\right]$, (*ii*) $\frac{45}{0{\cdot}04}\{(1{\cdot}05)^{10} - 1\}$

13. Find the square root of

$$\frac{\sqrt[3]{0{\cdot}0125} \times \sqrt{31{\cdot}15}}{0{\cdot}00081}$$

[**Hint.** We can write the above expression in the following form :

$x = \text{anti-log}\ [\frac{1}{6} \log 0{\cdot}0125 + \frac{1}{4} \log 31{\cdot}15 - \frac{1}{2} \log 0{\cdot}00081]$

$= \quad ,, \quad [\frac{1}{6} \times \bar{2}{\cdot}0969 + \frac{1}{4} \times 1{\cdot}4935 - \frac{1}{2} \times \bar{4}{\cdot}9085]$

$= \quad ,, \quad [\frac{1}{6} \times (\bar{6} + 4{\cdot}0969) + {\cdot}3734 - \bar{2}{\cdot}4643]$

$= \quad ,, \quad [(\bar{1}{\cdot}6828 + 0{\cdot}3734) - \bar{2}{\cdot}4543]$

$= \quad ,, \quad [0{\cdot}0562 - \bar{2}{\cdot}4543]$

$= \quad ,, \quad [1{\cdot}6019]$

$\therefore \quad x = 39{\cdot}98$]

14. Solve for x the equations :

(i) $2^x \cdot 3^{2x+1} = 7^{4x+3}$

(ii) $4^x \cdot 20^{2x-2} = 40^x \, 2^{3x-1}$

15. Pareto law of income for a certain place is

$$N = \frac{5 \times 10^{10}}{x^{1\cdot 2}},$$

where x is income level and N the number of persons earning incomes \$ x and over. Find the number of persons earning \$ 327500 and over. You can use logarithms tables.

16. Find the value of N when $x = 3362$ in the following :

$$N(2\cdot 5)^3 = \frac{0\cdot 00603 \times (4\cdot 6378)^{-2}}{x^{1\cdot 2}}$$

17. Use log-tables to find the value of x (correct to three places of decimal) if x satisfies the equation :

$$\frac{20}{14\cdot 7} = \left[\frac{0\cdot 0613}{x}\right]^{1\cdot 32}$$

ANSWERS

2. (i) 30, (ii) 32nd. 3. 16. 4. 87·57
5. 2·01 6. 0·0009342 7. 0·0382 8. 0·0426
9. (a) 2·063 (b) 1963 (c) 2·082 (d) 1·9635
10. 0·1986 11. 30{1·6764 – 1} = 20·292 12. (i) 1386, (ii) 707·6
14. (i) $x = -0\cdot 9672$, (ii) 2·301 15. 12040
16. 0·000002567 17. 0·04854.

7·4. COMPOUND INTEREST

The common logarithms can be conveniently used to solve problems on compound interest. Let P denote the principal, r the rate of interest per cent per annum, n the period in years and A the amount of P in n years.

The interest on P for the first year $= P \, \frac{r}{100}$

The amount at the end of the first year

$$= P + P \cdot \frac{r}{100} = P \left(1 + \frac{r}{100}\right)$$

Similarly the amount at the end of second year

$$= P\left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right) = P \left(1 + \frac{r}{100}\right)$$

Proceeding thus, amount at the end of n years is

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$A = P(1+i)^n, \quad \text{where } i = \frac{r}{100}$$

The formula for compound interest admits of logarithmic computation. Taking logarithm, we have

$$\log A = \log P + n \log(1+i)$$

The formula involves 4 quantities A, P, n and i. Given any three of them, the fourth can be determined. Thus

$$\log P = \log A - n \log(1+i)$$

$$\Rightarrow \quad \log(1+i) = \frac{\log A - \log P}{n}$$

$$\Rightarrow \quad n = \frac{\log A - \log P}{\log(1+i)}$$

If the interest is compounded half yearly $A = P\left(1+\frac{i}{2}\right)^{2n}$ and if the interest is compounded quarterly, $A = P\left(1+\frac{i}{4}\right)^{4n}$. The above formula in general may be written as $A = P\left(1+\frac{i}{q}\right)^{q \cdot n}$, where the interest is compounded q times.

Example 30. *Find the compound interest on Rs. 10,000 for 4 years at 5% per annum. What will be the simple interest in the above case ?*

Solution. Here $P = 10{,}000$, $n = 4$, $i = \frac{5}{100} = 0{\cdot}05$

We know $\quad A = P(1+i)^n$

$\therefore \quad A = 10{,}000\,(1+{\cdot}05)^4 = 10{,}000\,(1{\cdot}05)^4$

$= 10{,}000 \times 1{\cdot}215$

$= 12150$

C.I. = Compound Interest

$= A - P = \text{Rs. } 2150$

S.I. = Simple Interest

$= P.n.i. = 10{,}000 \times 4 \times \frac{5}{100}$

$= \text{Rs. } 2000$

$x = (1{\cdot}05)^4$

$\log x = 4 \log 1{\cdot}05$

$= 4 \times 0{\cdot}0212$

$= 0{\cdot}0848$

$x = \text{anti-log}\,(0{\cdot}0848)$

$= 1{\cdot}215$

Example 31. *Find the compound interest on Rs. 6950 for 3 years if interest is payable half yearly, the rate for the first two years being 6% p.a., and for the third year 9% p.a.*

Solution. We have $A=P\left(1+\frac{i}{2}\right)^{2n}$

Here $P=6950,\ n=2,\ i=\frac{6}{100}=0{\cdot}06$

$\therefore\quad A=6950\left(1+\frac{{\cdot}06}{2}\right)^{2\times 2}=6950\ (1{\cdot}03)^4$

$\log A=\log 6950+4\log 1{\cdot}03$

$=3{\cdot}8420+4\times 0{\cdot}0128$

$=3{\cdot}8932$

$\therefore\ A=$antilog $(3{\cdot}8932)=7820$, which is the principal for the third year.

Again $A=7820\left(1+\frac{{\cdot}09}{2}\right)^{2\times 1}$ $(\because\ n=1)$

or $A=7820(1{\cdot}045)^2$

$\log A=\log 7820+2\log 1{\cdot}045=3{\cdot}8932+2\times 0{\cdot}0191$

$=3{\cdot}9314$

$\therefore\quad A=$anti-log $(3{\cdot}9314)=8539$

$\therefore\quad$ C.I.$=8539-6950=$Rs. 1589

Example 32 *What is the present value of Rs. 10,000 due in 2 years at 8% p.a., C.I. according as the interest is paid (a) yearly or (b) half-yearly ?*

Solution. (*a*) $A=P(1+i)^n$

Here $A=10{,}000,\ i=\frac{8}{100}={\cdot}08,\ n=2$

$\therefore\quad P=\frac{A}{(1+i)^n}=\frac{10{,}000}{(1{\cdot}08)^2}$

$\Rightarrow\quad \log P=\log 10{,}000-2\log 1{\cdot}08=4-2\times 0{\cdot}0334$

$=3{\cdot}9332$

$\therefore\quad P=$anti-log $(3{\cdot}9332)=$Rs. 8574

(*b*) $A=P\left(1+\frac{i}{2}\right)^{2n}$

Here $A=10{,}000,\ \frac{i}{2}=\frac{0{\cdot}08}{2}=0{\cdot}04$

$\therefore\quad P=\frac{10{,}000}{(1{\cdot}04)^{2\times 2}}$

$$\Rightarrow \qquad \log P = \log 10{,}000 - 4 \log 1{\cdot}04 = 4 - 4 \times 0{\cdot}0170$$
$$= 3{\cdot}9320$$
$$\therefore \qquad P = \text{Rs. } 8551.$$

Example 33. *Mr. Mehta borrowed Rs. 20,000 from a money-lender but he could not repay any amount in a period of 4 years. Accordingly the money-lender demands now Rs. 26,500 from him. At what rate per cent per annum compound interest did the latter lend his money ?*

[*I.C.W.A., June, 1987*]

Solution. We have $A = P(1+i)^n$

Here $A = 26{,}500,\ P = 20{,}000,\ n = 4,\ i = ?$

$$\therefore \qquad 26{,}500 = 20{,}000(1+i)^4$$

Taking logarithms, we get

$$\log 26{,}500 = \log 20{,}000 + 4 \log (1+i)$$
$$\Rightarrow \qquad \log (1+i) = \frac{\log 26{,}500 - \log 20{,}000}{4}$$
$$= \frac{4{\cdot}4232 - 4{\cdot}3010}{4} = 0{\cdot}0305$$
$$\therefore \qquad (1+i) = \text{anti-log } (0{\cdot}0305) = 1{\cdot}073$$
$$\Rightarrow \qquad i = 1{\cdot}073 - 1 = 0{\cdot}073$$

Hence the required rate per cent

$$= 100 \times i = 100 \times 0{\cdot}073 = 7{\cdot}3$$

Example 34. (*a*) *Find the number of years and the fraction of a year in which a sum of money will treble itself at compound interest at 8 per cent per annum.*

(*b*) *In what time will a sum of Rs. 1234 amount to Rs. 5678 at 8% p.a. compound interest, payable quarterly ?*

Solution. (*a*) We have $A = P(1+i)^n$

Let $P = 100$, then $A = 300,\ i = \frac{8}{100} = 0{\cdot}08$

$$\Rightarrow \qquad 300 = 100(1{\cdot}08)^n,\ i.e.,\ 3 = (1{\cdot}08)^n$$
$$\Rightarrow \qquad n = \frac{\log 3}{\log 1{\cdot}08} = \frac{0{\cdot}4771}{0{\cdot}0334} = 14{\cdot}28 \text{ years}$$

(*b*) We have $A = P\left(1 + \frac{i}{4}\right)^{4n}$

$$\therefore \qquad 5678 = 1234\left(1 + \frac{{\cdot}08}{4}\right)^{4n} = 1234(1{\cdot}02)^{4n}$$

Taking log, we get

$$\log 5678=\log 1234+4n \log 1{\cdot}02$$

$$\Rightarrow \quad n=\frac{\log 5678-\log 1234}{4 \log 1{\cdot}02}$$

$$=\frac{3{\cdot}7542-3{\cdot}0913}{4\times 0{\cdot}0086}=\frac{0{\cdot}6629}{0{\cdot}0344}=19{\cdot}27 \text{ years}$$

Example 35. *A man borrows Rs. 750 from a money-lender and the bill is renewed after every half year at an increase of 21%. What time will elapse before it reaches Rs. 7,500 ? [You may use : $\log_{10} 121=2{\cdot}0828$]* [*I. C. W. A., June 1990*]

Solution. Let the time elapsed be n years. Since the bill is renewed every half year, so the number of half years is equal to $2n$.

∴ We have

$$7500=750\left(1+\frac{22}{100}\right)^{2n}$$

$$\Rightarrow \quad \frac{7500}{750}=\left(\frac{121}{100}\right)^{2n}$$

$$\Rightarrow \quad 10=\left(\frac{121}{100}\right)^{2n}$$

Taking logarithms of both sides, we get

$$\log 10=2n(\log 121-\log 100)$$

$$\Rightarrow \quad 1=2n(2{\cdot}0828-2)$$

$$\Rightarrow \quad n=\frac{1}{2\times 0{\cdot}0828}=6{\cdot}04$$

$$\therefore \quad n=6 \text{ years approximately}$$

Hence it will take about 6 years for Rs. 750 to reach Rs. 7,500.

7·5. DEPRECIATION

In case of depreciation, the principal value is diminished every year by a certain constant amount, and in the subsequent period the diminished value becomes the principal value. In case of uniform decrease or depreciation 'i' is to be substituted by '$-i$' and the formula is reduced to

$$A=P(1-i)^n$$

Example 36. *A machine, the life of which is estimated to be 10 years, costs Rs. 10,000. Calculate its scrap value at the end of its life, depreciation on the reducing instalment system being charged at 10% per annum.*

Solution. We have

$$A=P(1-i)^n$$

Here $\quad A=10{,}000,\ i=\frac{10}{100}=\frac{1}{10},\ n=10$

$\therefore \quad A = 10{,}000\left(1-\frac{1}{10}\right)^{10} = 10{,}000\left(\frac{9}{10}\right)^{10}$

Taking logarithms, we get

$$\begin{aligned}\log A &= \log 10{,}000 + 10(\log 9 - \log 10)\\ &= 4 + 10(0{\cdot}9542 - 1)\\ &= 4 + 9{\cdot}542 - 10 = 3{\cdot}542\end{aligned}$$

$\therefore \quad A = \text{antilog}\,(3{\cdot}542) = 3483$

Hence the scrap value of machine is Rs. 3483.

Example 37. *A machine is depreciated in such a way that the value of the machine at the end of any year is 90% of the value at the beginning of the year. The cost of the machine was Rs. 12,000 and it was sold eventually as waste metal for Rs. 200, find out the number of years during which the machine was in use.*

Solution. We have $A = P(1-i)^n$

Here $\quad P = 12{,}000,\ A = 200,\ i = \frac{10}{100},\ n = ?$

$$\therefore \quad 200 = 12{,}000\left(1-\frac{10}{100}\right)^n = 12{,}000\left(\frac{9}{10}\right)^n$$

$$\Rightarrow \quad \left(\frac{9}{10}\right)^n = \frac{200}{12{,}000}$$

Taking logarithms, we get

$$n(\log 9 - \log 10) = \log 200 - \log 12{,}000$$

$$\Rightarrow \quad n(0{\cdot}9542 - 1) = 2{\cdot}3010 - 4{\cdot}0792$$

$$\Rightarrow \quad n(-0{\cdot}0458) = -1{\cdot}7782$$

$$\Rightarrow \quad n = \frac{1{\cdot}7782}{0{\cdot}0458} = 39 \text{ years (approx).}$$

7·6 ANNUITIES

1. Annuity. An annuity is a series of payments, ordinarily of a fixed amount payable regularly at equal intervals. The intervals may be a year, a half-year, a month and so on.

Annuities may be divided into two classes—*Annuity Certain* and *Annuity Contingent.*

In *Annuity Certain* payments are to be made unconditionally, for a certain or fixed number of years.

In *Annuity Contingent* the payments are to be made till the happening of some contingent event such as the death of a person, the marriage of a girl, the education of a child reaching a specified age. Life Annuity is an example of Annuity Contingent. Annuity Certain may be divided into (*i*) Annuity Due, (*ii*) Annuity Immediate.

Annuity Due is one where the first payment falls due at the beginning of the first interval and so all payments are made at the beginning of successive intervals.

Immediate Annuity is one where the first payment falls due at the end of the first interval.

2. Present value. The present value of an annuity is the sum of the present values of its instalments. In finding the present value of an annuity it is always customary to reckon compound interest.

3. Present Value of an Immediate Annuity. Let A be the annuity, V the present value, i the rate of interest per unit per year and n the number of years to continue.

Then the present value of A due in 1 year $=\frac{A}{1+i}$

Present value of A due in 2 years $=\frac{A}{(1+i)^2}$

$\vdots \quad \vdots \quad \vdots$

Present value of A due in n years $=\frac{A}{(1+i)^n}$

$\therefore$ The sum of the present values of the different payments is given by

$$V=\frac{A}{1+i}+\frac{A}{(1+i)^2}+\dots+\frac{A}{(1+i)^n}, \text{ which is in G.P.}$$

$$=\frac{A}{1+i}\left\{\frac{1-\frac{1}{(1+i)^n}}{1-\frac{1}{1+i}}\right\}$$

$$=\frac{A}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

4. Present Value of an Annuity Due. An annuity due for n years is equivalent to an immediate payment of A plus an immediate annuity for $(n-1)$ years. Hence the present value of an annuity due is

$$V=A+\frac{A}{i}\left\{1-\frac{1}{(1+i)^{n-1}}\right\}$$

$$=\frac{A}{i}(1+i)\left\{1-\frac{1+i}{(1+i)^n}\right\}$$

5. Amount of an Immediate Annuity. Let A be set aside at the end of every year for n years. Then at the end of n years

the first payment will amount to $A(1+i)^{n-1}$

the second payment will amount to $A(1+i)^{n-2}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

the nth payment will amount to $A(1+i)^{n-n}$ or A

$\therefore$ The amount at the end of n years is given by

$$M = A(1+i)^{n-1} + A(1+i)^{n-2} + \dots + A, \text{ which is a G.P.}$$

$$= A\frac{(1+i)^n - 1}{(1+i) - 1} = \frac{A}{i}\{(1+i)^n - 1\}$$

6. Amount of an Annuity Due. Let A be the annual payment. Then each payment A is paid at the beginning of each year. The first payment earns interest for n years, the second for $(n-1)$ years...etc. and nth payment for 1 year. Hence

the first payment amounts to $A(1+i)^n$

the second payment amounts to $A(1+i)^{n-1}$

$\vdots \qquad \vdots \qquad \vdots$

the nth payment amounts to $A(1+i)$

$$M = A(1+i)^n + A(1+i)^{n-1} + \dots + A(1+i)$$

$$= A(1+i)\{(1+i)^{n-1} + (1+i)^{n-2} + \dots + 1\}$$

$$= A(1+i)\left\{\frac{(1+i)^n - 1}{(1+i) - 1}\right\}$$

$$= \frac{A}{i}(1+i)\{(1+i)^n - 1\}$$

Example 38. *A man borrows Rs. 20,000 at 4% C.I. and agrees to pay both the principal and the interest in 10 equal annual instalments at the end of each year, find the amount of these instalments.*

Solution. Using the formula for present value

$$V = \frac{A}{i}\left\{1 - \frac{1}{(1+i)^n}\right\}$$

where $V = 20{,}000$, $i = \frac{4}{100} = \cdot 04$, $n = 10$, we have

$$20{,}000 = \frac{A}{0 \cdot 04}\{1 - (1 \cdot 04)^{-10}\}$$

$$= \frac{A}{0 \cdot 04}(1 - 0 \cdot 6761)$$

$$= \frac{A}{0 \cdot 04}(0 \cdot 3239)$$

$$\Rightarrow \quad A = \frac{20{,}000 \times 0 \cdot 04}{0 \cdot 3239} = 2470$$

Let $x = (1 \cdot 04)^{-10}$

$$\Rightarrow \quad \log x = -10 \log 1 \cdot 04$$

$$= -10(\cdot 0170)$$

$$= -0 \cdot 1700$$

$$= -1 + 1 - 0 \cdot 1700$$

$$= \bar{1} \cdot 8300$$

$$\therefore \quad x = 0 \cdot 6761$$

Example 39. *A wagon is purchased on instalment basis, such that Rs. 5000 is to be paid on the signing of the contract and four yearly instalments of Rs. 3000 each payable at the end of the first, second, third and fourth year. If interest is charged at 5% p.a., what would be the cash down price ?*

Solution. To find the present value of four instalments, we use the formula

$V=\frac{A}{i}\{1-(1+i)^{-n}\}$, where

$A=3{,}000,\ i=\cdot05,\ n=4$

$V=\frac{3000}{\cdot05}\{1-(1+0\cdot05)^{-4}\}$

$=\frac{3000}{\cdot05}(1-0\cdot8226)$

$=\frac{3000}{\cdot05}\times\cdot1774$

$=10{,}644$

$x=(1\cdot05)^{-4}$

$\Rightarrow \log x=-4\log(1\cdot05)$

$=-(0\cdot0212)=-\cdot0848$

$=\bar{1}\cdot9152$

$x=0\cdot8226$

$\therefore$ Cash down price = Rs. 5,000 + Rs. 10,644 = Rs. 15,644.

Example 40. *A man retires at the age of 60 years and his employer gives him a pension of Rs. 1,200 a year paid in half-yearly instalments for the rest of his life. Reckoning his expectation of life to be 13 years and that interest is at 4% p.a. payable half-yearly, what single sum is equivalent to this pension ?*

Solution.

$V=\frac{A}{i/2}\left\{1-\left(1+\frac{i}{2}\right)^{-2n}\right\}$

$=\frac{600}{\cdot02}\{1-(1+\cdot02)^{-26}\}$

$=\frac{600}{\cdot02}(1-\cdot5975)$

$=\frac{600}{\cdot02}\times\cdot4025$

$=12{,}075$

$\therefore V$ Rs. 12,075

$A=600=\frac{1200}{2},\ i=\cdot04,\ n=13$

half-yearly instalment

$x=(1\cdot02)^{-26}$

$\log x=-26\log(1\cdot02)$

$=-26(\cdot0086)=-\cdot2236$

$=\bar{1}\cdot7764$

$x=\cdot5975$

Example 41. *A sinking fund is created for the redemption of debentures of Rs. 1,00,000/- at the end of 25 years. How much money should be provided out of profits each year for the sinking fund if the investment can earn interest @4% per annum ?*

Solution. Let A be the annual instalment. Then Rs. 1,00,000 is the amount of the annuity A to continue for 25 years. Using the formula

$$M=\frac{A}{i}\{(1+i)^n-1\}$$

where $M=1,00,000$, $i=\frac{4}{100}=\cdot 04$, $n=25$, we have

$$1,00,000=\frac{A}{\cdot 04}\left\{(1\cdot 04)^{25}-1\right\}$$
$$=\frac{A\times 100}{4}\left\{2\cdot 661-1\right\}$$
$$=A\times 25\times 1\cdot 661$$
$$\therefore \quad A=\frac{1,00,000\times 1000}{25\times 1661}$$
$$=\frac{4000\times 1000}{1661}=\text{Rs. } 2408\cdot 19$$

$$x=(1\cdot 04)^{25}$$
$$\Rightarrow \quad \log x=25\log 1\cdot 04$$
$$=25\times 0\cdot 0170$$
$$=0\cdot 4250$$
$$x=\text{antilog }(0\cdot 4250)$$
$$=2\cdot 661$$

Example 42. *A limited company intends to create a depreciation fund to replace at the end of the 25th year assets costing Rs. 1,00,000. Calculate the amount to be retained out of profits every year if the interest rate is 3%.*

Salution. Here

$$M=\frac{A}{i}\left\{(1+i)^n-1\right\}$$

where $M=1,00,000$, $i=\cdot 03$, $n=25$

$$1,00,000=\frac{A}{\cdot 03}\left\{(1+\cdot 03)^{25}-1\right\}$$
$$=\frac{A}{\cdot 03}\left\{(1\cdot 03)^{25}-1\right\}$$
$$=\frac{A}{03}\left(2\cdot 089-1\right)$$
$$=\frac{A}{\cdot 03}\left(1\cdot 089\right)$$

$$x=(1\cdot 03)^{25}$$
$$\Rightarrow \quad \log x=25\log(1\cdot 03)$$
$$=25(\cdot 0128)=\cdot 3200$$
$$x=\text{antilog }(\cdot 3200)$$
$$x=2\cdot 089$$

or $$A=\frac{1,00,000\times\cdot 03}{1\cdot 089}=\text{Rs. } 2755 \text{ (approx.)}$$

Example 43. *A machine costs the company Rs. 97,000 and its effective life is estimated to be 12 years. If the scrap realises Rs. 2,000 only, what amount should be retained out of profits at the end of each year to accumulate at compound interest at 5% per annum ?*

Solution. Let A be the annual instalment. Evidently the amount of the annuity A to continue for 12 years, *i.e.*, balance amount to be retained

$$=97,000-2,000=95,000$$

Hence from the formula

$$M=\frac{A}{i}\left\{(1+i)^n-1\right\}$$

where $M=95,000$, $i=\frac{5}{100}=0\cdot 05$, $n=12$, we have

$$95{,}000=\frac{A}{\cdot05}\left\{(1+0\cdot05)^{12}-1\right\}$$
$$=\frac{A}{\cdot05}\left(1\cdot797-1\right)$$
$$=\frac{A}{\cdot05}\times0\cdot797$$
$$\Rightarrow A=\frac{95{,}000\times0\cdot05}{0\cdot797}$$
$$=\text{Rs. } 5960 \text{ (approx).}$$

Let $x=(1\cdot05)^{12}$

$$\log x=12\log 1\cdot05$$
$$=12\times0\cdot1212$$
$$=0\cdot2544$$
$$x=\text{antilog }(0\cdot2544)$$
$$=1\cdot797$$

Example 44. *A man wishes to create an endowment fund to provide an annual prize of Rs. 500. If the fund is invested at 10% p.a. C.I., find the amount of this fund.* *[I.C.W.A., December 1989]*

Solution. The required amount of the endowment is the present value V of the perpetuity of Rs. 500.

We have $V=\frac{P}{i}$

Here $P=\text{Rs. } 500,\ i=\frac{10}{100}=0\cdot1$

$\therefore$ $V=\frac{500}{0\cdot1}=\text{Rs. } 5{,}000.$

EXERCISE (III)

1. (*a*) Find the compound interest on Rs. 1000 for 4 years at 5% per annum

(*b*) What will be the simple interest in the above case ?

2. Find the difference between simple and compound interest on Rs. 5,000 invested for 4 years at 5% per annum, interest payable yearly.

3. Find the compound interest on Rs. 6,950 for 3 years, if interest is payable half yearly, the rate for the first two years being 6%, and for the third year 9% p.a.

4. What is the present value of Rs. 1000 due in 2 years at 5% p.a. compound interest, according as the interest is paid (*a*) yearly or (*b*) half yearly.

5. Find the compound interest on Rs. 25,800 for 5 years if the rate of interest be 2% in the 1st year, $2\frac{1}{2}$% in the second year, 3% in the 3rd year and thereafter at 4% p.a.

6. A man left Rs. 18,000 with the direction that it should be divided in such a way that his 3 sons aged 9, 12 and 15 years should each receive the same amount when they reached the age of 25. If the rate of interest is $3\frac{1}{2}$% p.a., what should each son receive when he is 25 years old ?

7. *A* owes *B* Rs. 1600 but it is not due for payment till the end of 3 years from this date. How much should *A* pay *B* if he is willing to accept now in order to clear off the debt : (*a*) taking money to be worth 5% per annum simple interest (*b*) taking it to be worth 5% per annum compound interest, payable yearly ?

8. In what time will a sum of money double itself at 5% p.a., compound interest ?

9. In what time will a sum of money treble itself at 5% p.a. compound interest payable half-yearly ?

10. A machine depreciates at the rate of 10% of its value at the beginning of a year. The machine was purchased for Rs. 5810 and the scrap value realised when sold was Rs. 2250. Find the number of years that the machine was used.

11. A machinery in a factory is valued at Rs. 49,074 and it is decided to reduce the estimated value at the end of each year by 15 per cent of the value at the beginning of that year. When will the value be (*a*) Rs. 20,000, (*b*) 1/10th of the original value ?

12. Find the present value of an annuity of Rs. 1000 p.a. for 14 years following compound interest at 5% p.a.

13. Calculate the amount and present value of an annuity of Rs. 3000 for 15 years if the rate of interest be $4\frac{1}{4}$% p.a.

14. A man borrows Rs. 6,000 at 6% and promises to pay off the loan in 20 annual payments beginning at the end of the first year. What is the annual payment necessary ?

15. Calculate the amount and the present value of an annuity of Rs. 3000 for 15 years, if the rate of interest be $4\frac{1}{2}$% p.a.

16. Find the amount and present value of an annuity certain of Rs. 150 for 12 years, reckoning interest at $3\frac{1}{2}$% p.a., given $(1{\cdot}035)^{12} = 1{\cdot}511056$.

17. A man borrows Rs. 1500 promising to repay the sum borrowed and the proper interest by 10 equal yearly instalments, the first two falling due in 1 year's time. Reckoning C.I. at 5% p.a., find the value of the annual instalment, given $(1{\cdot}05)^{10} = 1{\cdot}629$.

18. A company buys a machine for Rs. 1,00,000. Its estimated life is 12 years and scrap value is Rs. 5,000. What amount is to be retained every year from the profit and allowed to accumulate at 5% C.I. for buying a new machine at the same price after 12 years ?

19. A man borrows Rs. 1000 on the understanding that it is to be paid back in four equal instalments at intervals of six months, the first payment to be made six months after the money was borrowed. Calculate the amount of each instalment, reckoning compound interest at $2\frac{1}{2}$% per half-year.

20. A loan of Rs. 40,000 is to be repaid in equal annual instalment consisting of principal and interest due in course of 30 years. Find the amount of each instalment reckoning interest at 4% p.a.

21. The annual subscription for the membership of a club is Rs. 25 and a person may become a life member by paying Rs. 1000 in a lump sum. Find the rate of interest charged.

22. What sum should be paid for an annuity of Rs. 2,400 for 20 years at $4\frac{1}{2}\%$ compound interest p.a. ?

(given log 1·045 = 0·0191 and log 4·150 = 0·6180).

23. A man wishes to create an endowment fund to provide an annual prize of Rs. 500 out of its income. If the fund is invested in $2\frac{1}{2}\%$ p.a., find the amount of this fund.

24. A machine costs the company Rs. 97,000 and its effective life is estimated to be 12 years. If the scrap realises Rs. 2,000 only, what amount should be retained out of profits at the end of each year to accumulate at compound interest at 5% per annum ?

25. A loan of Rs. 1000 is to be paid in 5 equal annual payments interest being at 6 per cent per annum compound interest and first payment being made after a year. Analyse the payment into those on account of interest and on account of amortisation of the principal.

26. On his 48th birthday a man decides to make a gift of Rs. 5,000 to a hospital. He decides to save this amount by making equal annual payments up to and including his 60th birthday to a fund which gives $3\frac{1}{2}$ per cent compound interest, the first payment being made at once. Calculate the amount of each annual payment. (Answer to the nearest paisa).

27. A machine costs a company Rs. 52,000 and its effective life is estimated to be 25 years. A sinking fund is created for replacing the machine by a new model at the end of its life time, when its scrap realizes a sum of Rs. 2,500 only. The price of the new model is estimated to be 25 per cent higher than the price of the present one. Find what amount should be set aside every year, out of the profits for the sinking fund, if it accumulates at $3\frac{1}{2}$ per cent per annum compound.

28. A man buys a car for Rs. 16,000. He estimates that its value will depreciate each year by 20 per cent of its value at the beginning of the year. Find the depreciated value (Rs. x, correct to the nearest rupee) of the car at the end of five years. If the man sets aside at the end of each of the five years a certain fixed sum (Rs. y) to accumulate at 4 per cent compound interest in order to be able to buy at the end of five years another car costing Rs. 22,000 (after allowing the above depreciated value Rs. x for the old car in part exchange), find to the nearest paisa, the value Rs. y of each payment.

29. A man aged 40 wishes his dependents to have Rs. 40,000 at his death. A banker agrees to pay this amount to his dependents on condition that the man makes equal annual payments of Rs. x to the bank commencing now and going on until his death. What should be the value of x, assuming that the bank pays interest at 3% p.a. compound ? From the table on the expectation of life it is found that the expectation of life of a man of 40 is 30 years.

30. A man borrows Rs. 750 from a money-lender and the bill li renewed after every half year at an increase of 21%. What time wi ls elapse before it reaches Rs. 7,500 ? [You may use $\log_{10} 121 = 2{\cdot}0828$].

[I.C.W.A., June, 1990]

[**Hint.** $7{,}500 = 750\left(1+\frac{21}{100}\right)^{2n}$ where n is the time elapsed]

31. The cost of a machine is Rs. 1,00,000 and its effective life is 12 years. If the scrap realises only Rs. 5,000, what amount should be retained out of profits at the end of each year to accumulate at C.I. at 5% p.a. ? (You can use $\log_{10} 1{\cdot}05 = 0{\cdot}0212$, $\log_{10} 1{\cdot}797 = 0{\cdot}2544$).

[I.C.W.A., December, 1990]

[**Hint.** If P is the amount provided every year, then

$$95{,}000 = P\left[\frac{\left(\frac{105}{100}\right)^{12} - 1}{\left(\frac{105}{100}\right) - 1}\right]$$
]

ANSWERS

1. (*a*) Rs. 215·51, (*b*) Rs. 200, **2.** 78, **3.** Rs. 1589, **4.** (*a*) Rs. 906·90, (*b*) Rs. 906·10, **5.** Rs. 4250, **6.** Rs. 9341, **7.** (*a*) Rs. 1391·30, (*b*) Rs. 1382, **8.** 14·2 years, **9.** 18·6 years, **10.** 9 years, **11.** 4·52 years, 11·6 years, **12.** Rs. 9899, **13.** Rs. 62183·33, 33070, **14.** Rs. 523·19, **15.** (*i*) Rs. 62166·67, (*ii*) Rs. 36644·44, **16.** (*i*) 2190·24, (*ii*) Rs. 1449·50, **18.** Rs. 1071·6, **19.** Rs. 266·5, **20.** Rs. 2315·48, **21.** Rs. 12075, **22.** Rs. 31, 203·32, **23.** Rs. 20,000, **24.** Rs. 5960, **25.** Rs. 237·40, **26.** 343·14, **27.** Rs. 1605, **28.** Rs. 3091·56, **29.** Rs. 844·48, **30.** 6 years, **31.** Rs. 5959·85.