8

Equations : Linear, Quadratic, Cubic and Higher Orders

STRUCTURE

OBJECTIVES

After studying this chapter, you should be able to understand

- equations, identities and inequalities
- to solve quadratic, cubic and bi-quadratic equations
- to solve simultaneous linear and quadratic equations
- nature of roots
- to form an equation

8·1. EQUATIONS

Equations signify relation between two algebraic expressions symbolised by the sign of equality (=). However, the equality is true only for certain value or values of the variable or the variables symbolised generally by x, y, z. For example the equation

$3x+5=2x+7$ is true only for $x=2$ and not for $x=3$.

Since when $x=2$, the equation is $3\,(2)+5=(2)\,(2)+7$ or $11=11$ and when $x=3$ the equation is

$$(3)\,(3)+5=(2)\,(3)+7$$

or $$14=13 \text{ which is not true.}$$

Thus, the above equality is true for the value of x variable as 2.

But in an equation with two variables $x+y=5$, the equality holds true for several sets of values such as (0, 5) (1, 4) (2, 3), (3, 2), (4, 1), (5, 0) etc., and not for any values assigned to them. It is only in the case of identities that the relation of equality holds true whatever value is put on the variable.

8· . IDENTITIES

When equalities hold true whatever be the value of the variables, they are called identities. For example

$$(a+b)^2=a^2+2ab+b^2$$

The above identity is derived as follows :

$$\begin{aligned}(a+b)^2&=(a+b)\,(a+b)\\&=(a+b)a+(a+b)b\\&=a^2+ab+ab+b^2\\&=a^2+2ab+b^2\end{aligned}$$

We can prove that identities hold true whatever be the values of the variables by substituting say (*i*) $a=2$ and $b=3$, and (*ii*) $a=-2$ and $b=-3$. First by substituting the values of $a=2$ and $b=3$, we have

$$(2+3)^2=(2)^2+2(2)(3)+(3)^2$$

$$\Rightarrow \quad (5)^2=(2)^2+12+(3)^2$$

$$\Rightarrow \quad 25=4+12+9$$

$$\Rightarrow \quad 25=25$$

Now by substituting the values of $a=-2$ and $b=-3$, we have

$$\{(-2)+(-3)\}^2=(-2)^2+2(-2)(-3)+(-3)^2$$

$$(-5)^2=(-2)^2+12+(-3)^2$$

$$25=4+12+9$$

$$25=25$$

Thus, identities hold true whatever value is put for variables.

The following identities can be expressed as simple binomial expan-ions (c.f. Chapter X) :

$$(1+x)^2=1+2x+x^2$$

$$(x+3)^2=x^2+6x+9$$

$$(\sqrt{x}+\sqrt{y})^2=x+2\sqrt{xy}+y$$

$$(x+\tfrac{1}{2})^2=x^2+x+\tfrac{1}{4}$$

$$\left(x+\frac{1}{x}\right)^2=x^2+2+\frac{1}{x^2}$$

There are some other identities as follows :

(*i*) $(a-b)^2=a^2-2ab+b^2$

$(\sqrt{x}-\sqrt{y})^2=x-2\sqrt{xy}+y$

(*ii*) $(a+b)(a-b)=a^2-b^2$

$(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})=x-a$

(*iii*) $(a+b)(c+d)=ac+ad+bc+bd$

$(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$

(*iv*) $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$

(*v*) $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab\,(a+b)$

(*vi*) $(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab\,(a-b)$

(*vii*) $a^3+b^3=(a+b)(a^2-ab+b^2)$

(*viii*) $a^3-b^3=(a-b)(a^2+ab+b^2)$

However, it may be noticed that

$$(a+b)^2\neq a^2+b^2,\ (a-b)^2\neq a^2-b^2$$

$$(a+b)^3\neq a^3+b^3,\ (a-b)^3\neq a^3-b^3$$

Derived Identities

These are the identities derived by transposing the values in the basic identities and are very useful in tackling some problems in mathematics. For example

(*i*) $(a+b)^2=a^2+2ab+b^2$

$\Rightarrow$ $a^2+b^2=(a+b)^2-2ab$ and $2ab=(a+b)^2-(a^2+b^2)$

(*ii*) $(a-b)^2=a^2-2ab+b^2$

$\Rightarrow$ $a^2+b^2=(a-b)^2+2ab$ and $2ab=a^2+b^2-(a-b)^2$

By adding (*i*) and (*ii*),

$(a+b)^2+(a-b)^2=2(a^2+b^2)$

By subtracting (*ii*) from (*i*), we get

$$(a+b)^2-(a-b)^2=4ab$$

By dividing both (*i*) and (*ii*) by 4 and then subtracting (*ii*) from (*i*),

$$\left[\frac{(a+b)^2}{4}\right]-\left[\frac{(a-b)^2}{4}\right]=ab$$

Other identities derived from the above are

$$(a+b)^2-4ab=(a-b)^2$$

$$\Rightarrow \quad (a+b)^2=(a-b)^2+4ab$$

(*iii*) $a^2-b^2=(a+b)(a-b) \Rightarrow \dfrac{a^2-b^2}{a+b}=a-b$

(*iv*) $(a+b)^3=a^3+b^3+3ab(a+b) \Rightarrow a^3+b^3=(a+b)^3-3ab(a+b)$

(*v*) $(a-b)^3=a^3-b^3-3ab(a-b) \Rightarrow a^3-b^3=(a-b)^3+3ab(a-b)$

$$a^3+b^3=(a+b)(a^2-ab+b^2) \Rightarrow \frac{a^3+b^3}{a+b}=a^2-ab+b^2$$

$$a^3-b^3=(a-b)(a^2+ab+b^2) \Rightarrow \frac{a^3-b^3}{a-b}=a^2+ab+b^2$$

8 3. INEQUALITIES

In addition to the relation of equality we have a new relation called order relation denoted by the symbol "$<$". The statement "$a<b$" is read as "*a* is less than *b*". It can be stated also as "$b>a$" read as "*b* is greater than *a*". (Note that large part of the sign is closest to the larger value.) The statement $a>b$ is true only when $a-b$ is positive and $a<b$ is true only when $a-b$ is negative. For example, when $8>5$ then $8-5=3$ which is positive and $5<8$ then $5-8=-3$, which is negative. Some expressions of inequalities are as follows :

$a>b$	"*a* is greater than *b*"
$a<b$	"*a* is less than *b*"
$a \not> b$	"*a* is not greater than *b*"
$a \leqslant b$	"*a* is less than or equal to *b*"
$a \not< b$	"*a* is not less than *b*"
$a \geqslant b$	"*a* is greater than or equal to *b*"

Properties :

Order axioms : If *a* and *b* are any elements, then

(*i*) one and only one of the following is true :

$a=b$, $a<b$, $b<a$ (Trichotomy Law)

(*ii*) If $a<b$ and $b<c$, then $a<c$. (Transitivity Law)

(*iii*) If $a<b$ then $a+c<b+c$. (Monotone Property of Addition)

(*iv*) If $a<b$ and $a<c$, then $a.c<bc$

(Monotone Property of Multiplication)

(*v*) Since "$a>b$" and "$b<a$" are the same statements, all the above axioms can be rephased in terms of "$a>b$"

As shown earlier sometimes equality signs are combined with inequality signs :

$a \leqslant b$ means $a < b$ or $a = b$

$a \nless b$ means a is not less than b which means $a = b$ or $b < a$

$a \nless b$ means $b \leqslant a$

We also say that a is positive when $a \geqslant 0$ and a is negative, when $a < 0$.

Operation axioms. (*i*) On addition or subtraction of any number from both sides of an inequality, the inequality is preserved. For example, if

$$2x - 3 < 7$$

We may add 3 to both sides :

$$2x - 3 + 3 < 7 + 3$$
$$2x < 10$$

Any term in an inequality can be moved from one side to the other provided that its sign is changed. For example, if

$$a - c < b, \text{ then } a < b + c$$

This, in other words, is the transposing of a term from one side of the inequality to the other.

(*ii*) If we multiply or divide both sides of an inequality by a positive (non-zero) number, the inequality does not change. For example

$$3x > 5$$

multiply the inequality by 3, then

$$3(3x) > 3(5) \quad \Rightarrow \quad 9x > 15$$

If, say, $x = 2$ then the inequality holds true in both of the above cases.

(*iii*) If we multiply or divide both sides by a negative number, the direction of the inequality is reversed.

For example

$$5x > 10, \text{ where } x = 3$$

By multiplying both sides of the inequality by -5, we have

$$-5(5u) < -5(10)$$
$$-25x < -50$$

This is because when $x = 3$, the inequality -75 is less than -50.

(*iv*) An inequality can be converted into an equation :

If $a > b$ then $a = b + p$.

where p is positive real number (*i.e.*, $p > 0$)

If $c > d$, then we write

$$c = d + q, \text{ where } q > 0$$

Also $\quad a.c = (b + p)(d + q) = bd + pd + bq + pq$

Now p and q are positive. If in addition b and d are positive, then every term on the right-hand side is also positive so that

$$a.c > b.d$$

(v) If the two sides of an inequality, each having the same sign, be inverted (*i.e.*, turned upside down) then the sign must be reversed.

If $\frac{a}{c} > \frac{b}{d}$ then, $\frac{c}{a} < \frac{d}{b}$

and in particular if $a > b$ then $\frac{1}{a} < \frac{1}{b}$.

(vi) If the signs of all the terms on both sides of the inequality are changed, the inequality is reversed.

If $a > b$ then $-a < -b$

(vii) Now if $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3 \ldots a_n > b_n$

then $a_1 + a_2 + a_3 + \ldots + a_n > b_1 + b_2 + b_3 + \ldots + b_n$

and $a_1.a_2.a_3 \ldots a_n > b_1.b_2.b_3 \ldots b_n$

($viii$) If $a > b$ and $n > 0$ then $a^n > b^n$

and $\frac{1}{a^n} < \frac{1}{b^n}$

(ix) Arithmetic mean (A) of two positive numbers say a and b is greater than or equal to their geometric means (G) (c.f. Chapter XII). This is proved as follows. We know that

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$= \tfrac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geqslant 0$$

$$\therefore \quad A \geqslant G$$

Example 1. *Solve the inequality* $x+3<7 \; \forall \; x \in N$

Solution. $x+3-3<7-3$ (Substract 3 from both sides)

$\Rightarrow$ $x<4$

Example 2. *Solve the inequality* $-6x>24 \; \forall \; x \in N$

Solution. $\frac{1}{6}(-6x) > \frac{24}{6}$

$\Rightarrow$ $-x>4$

$\Rightarrow$ $x<-4$ (inequality reversed due to change of sign on both sides)

$\Rightarrow$ $x=-5, -6$ and so on

Example 3. *Solve the inequality* $2(x+1)-3(x-\frac{4}{3})>7x \; \forall \; x \in Q$

Solution. $2(x+1)-3(x-\frac{4}{3})>7x \; \forall \; x \in Q$

$\Rightarrow \quad 2x+2-3x+4>7x$

$\Rightarrow \quad -x+6>7x$

$\Rightarrow \quad x-x+6>7x+x$ (adding x on both sides)

$\Rightarrow \quad 6>8x$, *i.e.*, $8x<6$ or $x<\frac{3}{4}$.

Example 4. *Show that the following inequality is consistent.*

$$\frac{1}{3}(3x+15)>x+5$$

Solution. $\frac{1}{3}(3x+15)>x+5$

$$x+5>x+5$$

The equality is inconsistent.

8·4. GROUPING SYMBOLS

Before coming to the solution of equation, we consider this because often in equations we come across grouping symbols like parentheses ( ), braces { } and the brackets [ ]. They signify that the numbers or symbols contained therein are one unit and therefore should be treated as such in solving equations.

The normal procedure is to remove the grouping symbols by working from the inside out. In other words, we must start from the innermost pair of the parentheses and remove them before we take up the braces. The brackets are taken up in the end. For example

$$2x-3[x+2\{y-3(x+2y)-2(2-y)\}+1]$$
$$=2x-3[x+2\{y-3x-6y-4+2y\}+1]$$
$$=2x-3[x+2y-6x-12y-8+4y+1]$$
$$=2x-3[-5x-6y-7]$$
$$=2x+15x+18y+21=17x+18y+21$$

Further it should be borne in mind that the parentheses preceded by a + sign may be removed without changing the signs of inner numbers but if the parentheses is preceded by − sign, the parentheses can be removed by changing − sign to −1 and applying the distributive law as shown below :

$$2x-(4y-8)=2x-1(4y-8)=5x-1(4y)-1\,(-8)$$
$$=2x-4y+8$$

8·5. GENERAL SOLUTIONS

The particular value or values of the variable or the variables which satisfy the relationship given in the equation is called the solution of the equation. It is also known as the root of the equation. In a linear equation. with one variable, there is only one root or one solution to the equality. For example, in an equation $2x-10=4$, it is the $x=7$ which satisfies the relationship and therefore is the solution to the equation.

To find a solution to a simple e ̦uation we may use simplification techniques and the axioms of equality to transform that into the form $x=b$, indicating that b is the solution to the equation. In this process we should justify each step by any of the axioms of equation. Our main objective is to get each term involving an unknown variable to one side and all unattached numbers to the other side. For example, in an equation

$$3x+5=-x+13,$$

to get all the x's on the left hand side, we may employ the addition axiom and add x to both the sides, *i.e.*,

$$\begin{array}{r} 3x+5=-x+13 \\ +x=+x \\ \hline 3x+5+x=-x+13+ \end{array}$$

$$\therefore \quad 4x+5=13$$

Now to bring 5 to the right hand side, we employ the subtraction axiom and subtract 5 from both the sides. The equation now becomes

$$\begin{array}{r} 4x+5=13 \\ -5=-5 \\ \hline 4x=8 \end{array}$$

Then divide both sides by 4, we get $x=2$.

If we substitute 2 for x in the original equation, the truth of the statement can be proved.

$$3x+5=-x+13, \text{ if } x=2$$

then $$(3)(2)+5=(-2)+13$$

$$\therefore \quad 6+5=11$$

The above equation is a linear equation in one variable. Now, let us consider two linear equations in two variables :

$$x+y=3 \qquad ...(1)$$

$$3x+2y=7 \qquad ...(2)$$

Such equations can be satisfied by large number of sets of related values of x and y in individual equations some of which are

$$(0,3), (1,2), (-1,4) \qquad ...(i)$$

$$(0,\tfrac{7}{2}), (1,2), (-2, \tfrac{13}{2}) \qquad ...(ii)$$

It can be shown that the common set (1,2) simultaneously satisfies both the equations. This shows that two simultaneous equations are necessary when where are two variables in order we can find a unique solution which would satisfy both the equations. Likewise for linear equations in three variables there should be three simultaneous equations to enable us to get the solutions for all the three variables satisfying those equations.

8·6. DEGREE OF AN EQUATION

The degree of an equation is denoted by the highest index of the variable in any equation. An equation with the highest index or powre as 1 (as in the equation $x+5=7$) is of the first degree. It is also called a linear equation since its graph represents a straight line.

The higher degree equations are also called higher degree polynomials or polynomial equations. An equation having its highest index as 2 is called the quadratic equation. For example

$$x^2+5x+6=0$$

is quadratic equation in one variable. But the equations

$$x^2+y^2=25 \text{ and } x^2+xy+y^2=8$$

are quadratic equations in two variables.

Further, higher order equations are cubic with highest index of the variable 3 and biquadratic with the highest index of the variable 4. For example

$$x^3+6x^2+12x+7=0$$

is a cubic equation in one variable. There can be a cubic equation in two or more variables also.

Similarly $x^4+8x^2+7x=16$ is a biquadratic equation in one variable which can also have two are more variables.

Use of Equations

The practical use of the equations is in evolving certain relations and finding out the value of the unknown. Sometimes complicated verbal statements when translated into equations or inequalities can be solved with great ease. A few illustrations will make the point clear.

Example 5. (*i*) *In the two consecutive numbers one-fourth of the smaller one exceeds the one-fifth of the larger one by 3. Find the numbers.*

Solution. Let the two consecutive numbers be x and $x+1$. Now one-fourth of the smaller is $\frac{x}{4}$ and one-fifth of the larger is $\frac{x+1}{5}$. If the first exceeds the second by 3, we can express this in the equation form as

$$\frac{x}{4}-\frac{x+1}{5}=3.$$

$\Rightarrow \quad 5x-4x-4=60 \quad$ [by multiplying both the sides by 20]

$\Rightarrow \quad x=64$

$\therefore$ The two numbers are 64 and 65.

We can check, $\frac{64}{4}=16$ and $\frac{65}{5}=13$, the difference is of 3.

(*ii*) *A father is 28 years older than the son. In 5 years the father's age will be 7 years more than twice that of the son Find their present ages.*

Solution. We normally suppose what we have to find, let the present age of son be x then the age of fatner will be $x+28$. Now, after

5 years their ages will be $x+5$ and $x+28+5$ respectively. If the age of the father then will be 7 years more than twice that of the son, we can represent this in the equation form as

$$x+28+5=2(x+5)+7$$

$$\Rightarrow \quad x+33=2x+10+7$$

$$\Rightarrow \quad x-2x=-33+10+7$$

$$\Rightarrow \quad x=+33-10-7$$

$$\therefore \quad x=16$$

$\therefore$ The son's present age is 16 and the father's present age is 44.

(*iii*) *A person receives a total return of Rs. 402 from an investment of Rs. 8001 in two debenture issues of a company. The first one carrying an interest of 6% p.a. was bought for Rs. 110 each and the other one carrying an interest rate of 5% p.a. were bought at Rs. 105 each. Find the sum invested in each type of debentures.*

Solution. Let the sum x be invested in the first category, therefore, Rs. $8001-x$ must have been invested in the second category. The return on each for the year will be

$$x\times\frac{6}{110} \text{ and } (8001-x)\times\frac{5}{105}$$

The total return is

$$\frac{6x}{110}+\left[(8001-x)\times\frac{5}{105}\right]=402$$

$$\Rightarrow \quad \frac{3x}{55}+\frac{8001}{21}-\frac{x}{21}=402$$

$$\Rightarrow \quad \frac{3x}{55}-\frac{x}{21}=402-381$$

$$\Rightarrow \quad 63x-55x=1155(21)$$

$$\Rightarrow \quad 8x=24255$$

$$\Rightarrow \quad x=3{,}032 \text{ (approx.)}$$

$\therefore$ The sum invested in each type of debentures is Rs. 3,032 and Rs. 4,969 respectively.

(*iv*) *The speed of a boat in still water is 10 km per hour. If it can travel 24 km down stream and 14 km in the upstream in equal time, indicate the speed of the flow of stream.*

Solution. Let the speed of the flow of water be x, then the speed of the boat in the downstream and upstream will be $10+x$ and $10-x$ respectively then the time taken in going 24 km downstream will be $\frac{24}{10+x}$ and 14 km upstream will be $\frac{14}{10-x}$. Now the time taken both way in the form of an equation can be written as

$$\frac{24}{10+x}=\frac{14}{10-x}$$

$$\Rightarrow \quad 24(10-x)=14(10+x)$$
$$\Rightarrow \quad 240-24x=140+14x$$
$$\Rightarrow \quad -24x-14x=-240+140$$
$$\Rightarrow \quad -38x=-100$$
$$38x=100$$
$$\Rightarrow \quad x=100/38$$

$\therefore$ The stream is flowing at a speed of 100/38 km per hour.

(v) *Mr. Ray buys 100 units of the Unit Trust of India at Rs. 10·30 per unit. He purchases another lot of 200 at Rs. 10·40 per unit. At Rs. 10·50 per unit, he takes up another lot of 400 and a further lot of 300 at Rs. 10·80 per unit. He watches as the price goes down and desires to take up as many units at Rs. 10·25 per unit as would make the average cost of his holding to Rs. 10·50 per unit. Assuming that Mr. Ray always buys units in multiples of 100, find the number of units he purchases at the lowest price of Rs. 10·25 per unit.*

Solution. Let x be the number of units purchased at Rs. 10·25. Total number of units purchased at an average price per unit of Rs. 10·50

$$=100+200+400+300+x$$
$$=1000+x$$

$\therefore$ Value of units $=$ Rs. $10{\cdot}50\times(1000+x)=10{,}500+10{\cdot}50x$ ...(1)

But the value of units held by him

$$=\text{Rs. } 100\times 10{\cdot}30+\text{Rs. } 200\times 10{\cdot}40+\text{Rs. } 400 \times 10{\cdot}50+\text{Rs. } 300\times 10{\cdot}80+\text{Rs. } x\times 10{\cdot}25$$
$$=\text{Rs. } 10550+\text{Rs. } 10{\cdot}25x \qquad ...(2)$$

From given data, we have

$$10500+10{\cdot}50x=10550+10{\cdot}25x \quad [(1)=(2)]$$
$$\Rightarrow \quad 10{\cdot}50x-10{\cdot}25x=10550-10500$$
$$\Rightarrow \quad 0{\cdot}25x=50$$
$$\therefore \quad x=\frac{50}{0{\cdot}25}=200$$

Hence he purchases 200 units at Rs. 10·25.

8·7. SIMULTANEOUS LINEAR EQUATIONS

A system of simultaneous equations is helpful for finding unique values for the unknowns. The number of equations should be equal to the number of unknowns. However, the equations can be of varying degrees. First we take two linear equations in two unknowns which are in the following form

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0.$$

Now each equation individually has an unlimited number of solutions (x, y) corresponding to the unlimited number of points on the locus (straight line) which the equation represents. Our problem is to find all solutions common to the two equations or the co-ordinates of all points common to the two lines. There can be three possible situations in this.

(*i*) The equations will be consistent and independent if there is only one solution, *i.e.*, the two lines have only one common point as shown in Fig. 1.

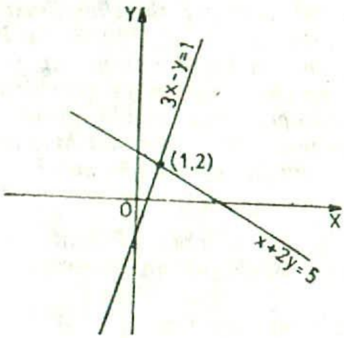


Fig 1.

(*ii*) The two lines are coincident. The equations are consistent but dependent as shown in Fig. 2.

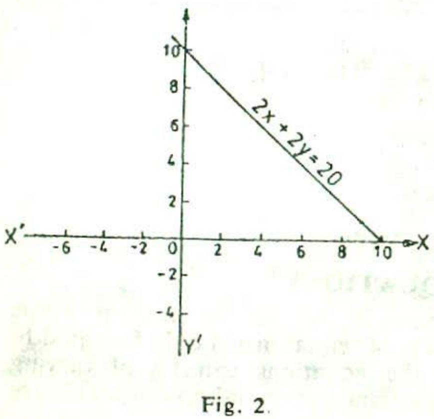


Fig. 2.

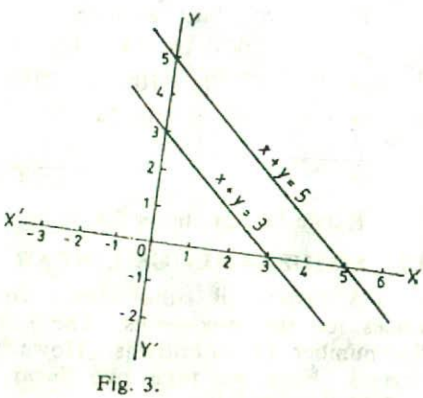


Fig. 3.

(*iii*) The system has no solution when two lines are parallel and distinct. The equations are inconsistent as shown in Fig. 3.

There can then be graphic solutions as well as algebraic solutions of equations, the former of course are not precise but easy to use in some cases. A graphic solution to three linear equations has been shown in Fig: 4.

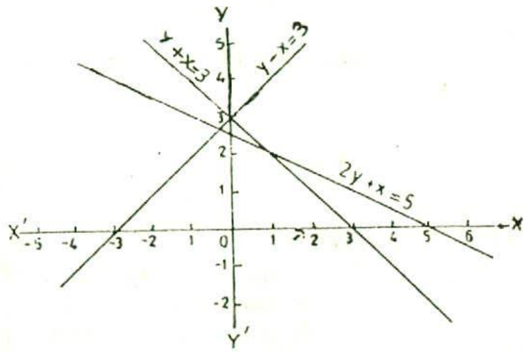


Fig. 4.

Algebraic Solutions

We are now illustrating the algebraic method of solving three linear simultaneous equations. The other methods we shall discuss in the chapter on Matrices.

Illustration. *Solve the system of the following three consistent and independent equations in the three unknowns ;*

$$2x+3y-4z=1 \quad \ldots(i)$$

$$3x-y-2z=4 \quad \ldots(ii)$$

$$4x-7y-6z=-7 \quad \ldots(iii)$$

Solution. Let us first eliminate y.

We rewrite (i) : $2x+3y-4z=1$

$3\times(ii)$: $9x-3y-6z=12$

Add : $11x \quad -10z=13 \quad \ldots(iv)$

Now, we rewrite (iii) :

$4x-7y-6z=-7$

$-7\times(ii)$: $-21x+7y+14z=-28$

Add : $-17x \quad +8z=-35 \quad \ldots(v)$

Let us now eliminate z from (iv) and (v), *i.e.*

$4\times(iv)$: $44x-40z=52$

$5\times(v)$: $-85x+40z=-175$

Add : $-41x \quad =-123$

$x=3$

Now, substitute the value of x in (*iv*) to get the value of z, *i.e.*,

$$11(3)-10z=13$$

$$\Rightarrow \quad 33-10z=13$$

$$\Rightarrow \quad -10z=-33+13=-20$$

$$\Rightarrow \quad z=2$$

Now, substitute the values of x and z in (*i*), we have

$$2(3)+3y-4(2)=1$$

$$\Rightarrow \quad 6+3y-8=1$$

$$\Rightarrow \quad 3y=+8-6+1=3$$

$$\Rightarrow \quad y=1$$

8·8. QUADRATIC EQUATIONS

An equation which when reduced to the rational integral form contains the square of the unknown quantity and no higher power is called a quadratic equation or an equation of the second degree.

An equation which contains only the square of the unknown and not the first power is called a pure quadratic equation, *e.g.*,

$$5x^2=21$$

But an equation which contains the square as well as the first power of the unknown is called an "adfactor" or complete quadratic equation, *e.g.*,

$$3x^2-5x+2=0 \quad \text{or} \quad ax^2+bx+c=0$$

where x is the unknown and a, b, c represent the constants of the equation. However, sometimes the fact is not obvious from the observation whether the equation is a quadratic as in the following case

$$\sqrt{3x-2}+\sqrt{x}=2 \quad \Rightarrow \quad \sqrt{3x-2}=2-\sqrt{x}$$

Squaring it, we get

$$3x-2=4+x-4\sqrt{x} \quad \text{or} \quad 2x-6=-4\sqrt{x}$$

Squaring it again and taking it in proper form, we have

$$4x^2-24x+36=16x$$

$$\Rightarrow \quad 4x^2-40x+36=0$$

$$\Rightarrow \quad x^2-10x+9=0$$

It is now in a rational integral form. The general form of a quadratic equation is

$$ax^2+bx+c=0$$

where a, b and c are any real numbers and $a \neq 0$. This is because if $a=0$ then the expression ax^2 becomes equal to zero and the equation becomes a linear one.

The graphic presentation of a quadratic equation takes the form of a parabolic (chapter XV) which is a smooth and more or less a cup-shaped curve. This may open upwards or downwards depending upon whether 'a' which is the coefficient of x^2 in the above equation has a plus or a minus sign.

8·9. SOLUTIONS TO QUADRATIC EQUATIONS

There can be both graphic and algebraic solutions to the quadratic equations. The following three figures show three possible situations of a quadratic equations.

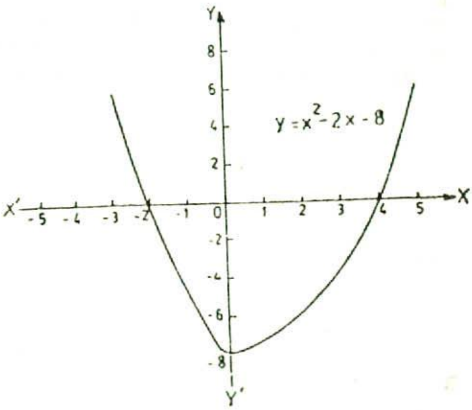


Fig. 5.

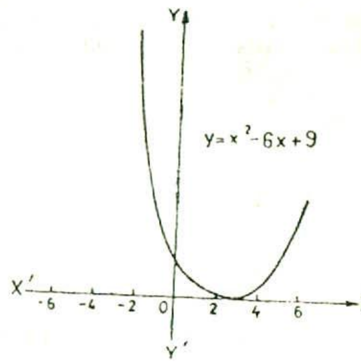


Fig. 6.

In the first one there are two real solutions at the points where the curve intersects the x-axis. In the second one there is only one real solution where the curve touches the x-axis. In the third case there are no real solutions.

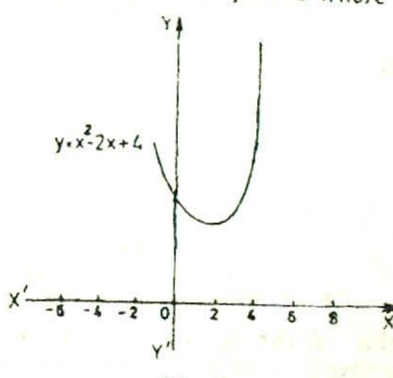


Fig. 7.

For graphic solution of a quadratic equation in the form of $ax^2+bx+c=0$, we have to obtain table values of x, using a suitable sequence of values. It is suggested that the value of $-\frac{b}{2a}$ may be taken as the central value and a few greate and lower values may then be chosen Let us take an illustration.

Equation : $y = x^2 - 2x - 3$

Table values :

	x	x^2	$-2x$	-3	$=y$
	4	16	-8	-3	$=5$
	3	9	-6	-3	$=0$
	2	4	-4	-3	-3
$\frac{b}{2a} = -\frac{-2}{2} = 1$		1	-2	-3	$=-4$
	0	0	0	-3	$=-3$
	-1	1	$+2$	-3	$=0$
	-2	4	$+4$	-3	$=5$

It may be noted that if $a>0$ the parabola opens upwards. The value of $x=-\frac{b}{2a}=1$ is the folding point or the axis of symmetry of the parabola. The points where the parabola crosses the x-axis are the values where $y=0$. In the Fig. 8 where $x=-1$ and $x=3$ then $y=0$, therefore, these are the two roots or the solutions of the equation.

Use of parabola is very common in economics. It represents the behaviour of average cost and marginal cost functions. Also to represent output and revenue it is used, however, in which case the parabola will open downwards as illustrated in Fig. 9 for the equation $y=4x-x^2$

Table of values :

if	$x=0,\ y=0$	if	$x=1,\ y=3$
if	$x=2,\ y=4$	if	$x=3,\ y=3$
if	$x=4,\ y=0$		

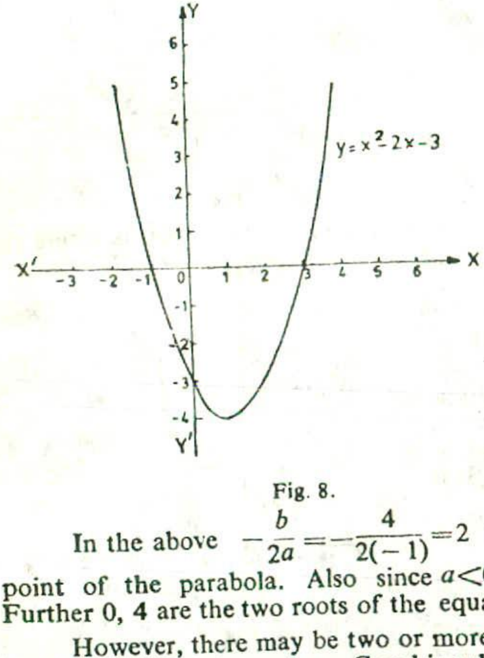


Fig. 8.

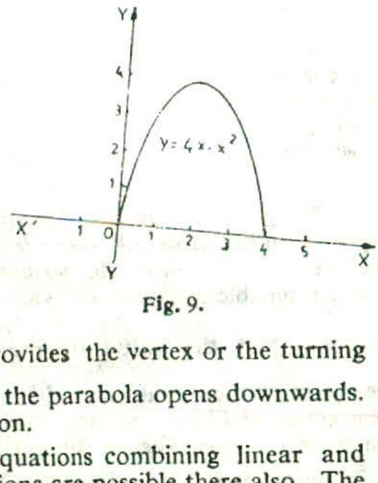


Fig. 9.

In the above $-\frac{b}{2a}=-\frac{4}{2(-1)}=2$ provides the vertex or the turning point of the parabola. Also since $a<0$, the parabola opens downwards. Further 0, 4 are the two roots of the equation.

However, there may be two or more equations combining linear and quadratic forms of equations. Graphic solutions are possible there also. The points of intersection of the various lines or curves as the case may be will give a solution. The same we derive algebraically by solving simultaneous equations. Among other things the smooth graphs of functions will give approximate values of a function with several intermediate values of a dependent variable where precise mathematical calculation may be difficult and tedious.

ALGEBRAIC SOLUTIONS

First we take general methods of solving a quadratic equation and then some special methods for quadratics involving radicals etc.

(*i*) Method of factorisation,

(*ii*) Method of completing a perfect square.

(*i*) **Method of factorisation.** This method is used where the quadratic expression can easily be resolved into linear factors.

Example 6. *Solve* (*a*) $4x^2=25$, (*b*) $x^2-(a+b)x+ab=0$.

Solution. (*a*) We have, by transposition

$$4x^2-25=0$$

$$\Rightarrow \quad (2x)^2-5^2=0$$

$$\Rightarrow \quad (2x-5)(2x+5)=0$$

$$\Rightarrow \quad \text{either } 2x+5=0, \text{ i.e., } x=-\tfrac{5}{2}$$

or $$2x-5=0, \text{ i.e., } x=\tfrac{5}{2}$$

Hence the roots are $-\frac{5}{2}, \frac{5}{2}$.

(*b*) $$x^2-(a+b)x+ab=0$$

$$\Rightarrow \quad x^2-ax-bx+ab=0$$

$$\Rightarrow \quad x(x-a)-b(x-a)=0, \text{ i.e., } (x-a)(x-b)=0.$$

Hence $$x=a, b$$

Example 7. *Solve* $x^2-6x+8=0$.

Solution. We have

$$x^2-(4+2)x+8=0 \Rightarrow x(x-4)-2(x-4)=0$$

$$\Rightarrow \quad (x-4)(x-2)=0$$

$\Rightarrow$ either $x-4=0$, *i.e.*, $x=4$

or $x-2=0$, *i.e.*, $x=2$

Hence the roots are 4, 2.

Example 8. *Solve* $\dfrac{x}{b}+\dfrac{b}{x}=\dfrac{a}{b}+\dfrac{b}{a}$

Solution. By transposition, we get

$$\frac{x}{b}-\frac{a}{b}=\frac{b}{a}-\frac{b}{x}$$

$$\Rightarrow \quad \frac{x-a}{b}=\frac{b(x-a)}{ax}$$

$$\Rightarrow \quad \frac{1}{b}(x-a)=\frac{b}{ax}(x-a)$$

$$\Rightarrow \quad \text{either } x-a=0, \text{ i.e., } x=a$$

$$\text{or } \frac{1}{b}=\frac{b}{ax}, \quad \text{i.e., } x=\frac{b^2}{a}$$

Thus the roots are $a, \dfrac{b^2}{a}$.

Example 9. *Solve* $\dfrac{9x-2}{3}-\dfrac{4x^2-7}{4x^2+3}=\dfrac{6x-1}{2}$

Solution. By transposition, we get

$$\frac{4x^2-7}{4x^2+3}=\frac{6x-1}{2}-\frac{9x-2}{3}=\frac{1}{6}$$

$\Rightarrow \quad 6(4x^2-7)=4x^2+3$

$\Rightarrow \quad 24x^2-4x^2=3+42$

$\Rightarrow \quad 4x^2=9$, *i.e.* $x^2=\frac{9}{4}$

Hence $\quad x=\pm\frac{3}{2}$

(*ii*) Method of Completing Square :

Example 10. *Solve* $3x^2-14x+8=0$.

Solution. Dividing both sides by 3, we get

$$x^2-\frac{14}{3}x+\frac{8}{3}=0$$

Now we add on both sides, the square of half the coefficient of x to make the L.H.S. a perfect square.

Adding $\frac{49}{9}$ to both sides, we get

$$x-\frac{14}{3}x+\frac{49}{9}=\frac{49}{9}-\frac{8}{3}$$

$$\Rightarrow \quad \left(x-\frac{7}{3}\right)^2=\frac{25}{9}$$

$$\Rightarrow \quad x-\frac{7}{3}=\pm\frac{5}{3}$$

$$\Rightarrow \quad x=\frac{7}{3}\pm\frac{5}{3}=\frac{12}{3},\ \frac{2}{3}=4,\ \frac{2}{3}$$

General method of Completing the Square. Let the general quadratic equation be $ax^2+bx+c=0$.

By transposition, we have $ax^2+bx=-c$

Dividing both sides by a, the coefficient of x^2, we have

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Now half the coefficient of x is $\frac{b}{2a}$ and its square is $\frac{b^2}{4a^2}$.

$\therefore$ Adding $\frac{b^2}{4a^2}$ to both sides, we get

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a^2}\right)^2=\frac{b^2-4ac}{4a^2}$$

Extracting square root of both sides, we have

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

∴ The two roots of $ax^2+bx+c=0$ are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

The following method of completing the square is due to the great Hindu Mathematician Sridhar Acharyya.

Since $ax^2+bx+c=0$, we have by transposition

$$ax^2+bx=-c$$

Multiplying both sides by $4a$, *i.e.*, 4 times the coefficient of x^2, we have

$$4a^2x^2+4abx=-4ac$$

Adding b^2, *i.e.*, square of half the coefficient of x, to both sides, we get

$$4a^2x^2+4abx+b^2=b^2-4ac$$

$$\Rightarrow \quad (2ax+b)^2=b^2-4ac$$

$$\Rightarrow \quad (2ax+b)=\pm\sqrt{b^2-4ac} \quad \text{(extracting the square root)}$$

$$\Rightarrow \quad 2ax=-b\pm\sqrt{b^2-4ac}$$

$$\Rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \ldots(*)$$

It may be verified that the solutions given by (*) above satisfy the given quadratic equation, *i.e.*, if we put the values of x found in (*) in the L.H.S. of the given quadratic equation, we will get zero which is the R.H.S. The two roots given by (*) are generally denoted by the Greek letters α and β. Thus

$$\alpha, \beta=\frac{-\text{coeffi. of } x\pm\sqrt{(\text{coeffi. of } x)^2-4\,(\text{coeffi. of } x^2)(\text{constant term})}}{2\,(\text{coeffi. of } x^2)}$$

or

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

A quadratic equation has thus exactly two roots.

Sum of the two roots :

We take $\alpha+\beta$=sum of the two roots

$$=\left\{\frac{-b+\sqrt{b^2-4ac}}{2a}\right\}+\left\{\frac{-b-\sqrt{b^2-4ac}}{2a}\right\}$$

$$=\frac{-2b}{2a}=-\frac{b}{a}$$

Product of the two roots :

Similarly, $\alpha\,.\,\beta$=Product of the roots

$$=\left\{\frac{-b+\sqrt{b^2-4ac}}{2a}\right\}\left\{\frac{-b-\sqrt{b^2-4ac}}{2a}\right\}$$

$$=\frac{1}{4a^2}\left\{(-b)+\sqrt{b^2-4ac}\right\}\left\{(-b)-\sqrt{b^2-4ac}\right\}$$

$$=\frac{1}{4a^2}[(-b)^2-\{\sqrt{(b^2-4ac)}\}^2]$$

$$=\frac{1}{4a^2}[b^2-b^2+4ac]=\frac{4ac}{4a^2}=\frac{c}{a}$$

Thus we have shown that

$$\alpha+\beta=\text{Sum of the roots}=-\frac{b}{a}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \quad ...(*)$$

$$\alpha\beta=\text{Product of the roots}=\frac{c}{a}=\frac{\text{Constant term}}{\text{Coefficient of } x^2} \quad ...(**)$$

(*) and (**), in fact, express the *Relations between Roots and Coefficients* of a quadratic equation.

Illustration 1. *Solve the equation* $2x^2-10x+5=0$.

Here $a=2,\ b=-10,\ c=5$

$\therefore$ The roots are $=\dfrac{-(-10)\pm\sqrt{(-10)^2-4.2.5}}{2.2}$

$$=\frac{10\pm2\sqrt{15}}{4}=\frac{5\pm\sqrt{15}}{2}$$

2. *Solve* $(b-c)x^2+(c-a)x+(a-b)=0$.

Here the roots are

$$x=\frac{-(c-a)\pm\sqrt{(c-a)^2-4(b-c)(a-b)}}{2\,(b-c)}$$

$$=\frac{-(c-a)\pm\sqrt{(a+c-2b)^2}}{2(b-c)}=\frac{-(c-a)\pm\{(a+c)-2b\}}{2(b-c)}$$

$$=\frac{2(a-b)}{2(b-c)},\ \frac{2(b-c)}{2(b-c)} \text{ or } \frac{a-b}{b-c},\ 1$$

Equations Adaptable to Quadratic Form. Sometimes we come across disguised quadratic equations or equations adaptable by suitable substitutions to quadratic form. In the following examples we shall consider a few simple cases of such types.

Example 11. *Solve the equation*

$$\sqrt{\frac{x}{1-x}}+\sqrt{\frac{1-x}{x}}=2\frac{1}{6}$$

Solution. Putting $\sqrt{\dfrac{x}{1-x}}=y$, the given equation becomes

$$y+\frac{1}{y}=\frac{13}{6}$$

After multiplying both sides by $6y$ and transposing, we have

$$6y^2-13y+6=0$$

$$\Rightarrow \quad 6y^2-9y-4y+6=0$$
$$\Rightarrow \quad 3y(2y-3)-2(2y-3)=0$$
$$\Rightarrow \quad (3y-2)(2y-3)=0, \text{ i.e., } y=\tfrac{2}{3}, \tfrac{3}{2}$$

Now

$$y=\frac{2}{3} \Rightarrow \sqrt{\frac{x}{1-x}}=\frac{2}{3}$$
$$\Rightarrow \frac{x}{1-x}=\frac{4}{9}$$
$$\Rightarrow 9x=4-4x$$
$$\Rightarrow x=\frac{4}{13}$$

$$y=\frac{3}{2} \Rightarrow \sqrt{\frac{x}{1-x}}=\frac{3}{2}$$
$$\Rightarrow \frac{x}{1-x}=\frac{9}{4}$$
$$\Rightarrow 4x=9-9x$$
$$\Rightarrow x=\frac{9}{13}$$

Hence the roots are $\frac{4}{13}, \frac{9}{13}$

Example 12. *Solve the equation :*

(*a*) $$x^2-6x+9=4\sqrt{x^2-6x+6}$$

(*b*) $$x^2-4x-12\sqrt{x^2-4x+19}+51=0$$

Solution. (*a*) The given equation may be written as

$$(x^2-6x+6)+3=4\sqrt{x^2-6x+6}$$

Putting $x^2-6x+6=y$, the equation reduces to

$$y+3=4\sqrt{y}$$
$$\Rightarrow \quad (y+3)^2=(4\sqrt{y})^2$$
$$\Rightarrow \quad y^2+6y+9=16y$$
$$\Rightarrow \quad y^2-10y+9=0$$
$$\Rightarrow \quad y^2-y-9y+9=0$$
$$\Rightarrow \quad y(y-1)-9(y-1)=0, \text{ i.e., } (y-9)(y-1)=0$$

Hence $$y=9 \text{ or } y=1$$

Now

$$y=9 \Rightarrow x^2-6x+6=9$$
$$\Rightarrow x^2-6x-3=0$$
$$\Rightarrow x=\frac{6\pm\sqrt{36+12}}{2}=3\pm2\sqrt{3}$$

$$y=1 \Rightarrow x^2-6x+6=1$$
$$\Rightarrow x^2-6x+5=0$$
$$\Rightarrow x=\frac{6\pm\sqrt{36-20}}{2}=5, 1$$

Hence the roots are $3\pm2\sqrt{3}$, 5, 1.

(*b*) Put $$x^2-4x=y$$

$$\therefore \quad y-12\sqrt{y+19}+51$$
$$\Rightarrow \quad y+51=12\sqrt{y+19}+51$$
$$\Rightarrow \quad y^2+102y+2601=144(y+19) \quad \text{[Square both sides]}$$

$\Rightarrow \quad y^2-42y-135=0$, *i.e.*, $(y+3)(y-45)=0$

$\therefore \quad y=-3$ and $y=45$

But $\quad x^2-4x=y$

(*i*) $x^2-4x+3=0$

$\Rightarrow \quad x=\dfrac{4\pm\sqrt{16-12}}{2}=1, 3$

(*ii*) $x^2-4x-45=0$

$\Rightarrow \quad x=\dfrac{4\pm\sqrt{16+180}}{2}=-5, 9$

Hence $x=1, 3, -5, 9$.

Example 13. *Solve*

$$\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}}=\frac{\sqrt{a}+1}{\sqrt{a}-1}$$

Solution. By componendo and dividendo, *i.e.*, if $\dfrac{a}{b}=\dfrac{c}{d}$

then $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$, we have

$$\frac{2x}{2\sqrt{12a-x}}=\frac{2\sqrt{a}}{2} \text{ or } \frac{x}{\sqrt{12a-x}}=\sqrt{a}$$

$$\Rightarrow \quad x=\sqrt{a}\,.\,\sqrt{12a-x}$$

Squaring both sides, we get

$$x^2=a(12a-x)=12a^2-ax$$

$$\Rightarrow \quad x^2+ax-12a^2=0$$

$$\Rightarrow \quad x^2+4ax-3ax-12a^2=0$$

$$\Rightarrow \quad x(x+4a)-3a(x+4a)=0, \text{ i.e., } (x-3a)(x+4a)=0$$

$$\therefore \quad x=3a \text{ or } -4a$$

Fxample 14. *Solve* $\quad (a+x)^{2/3}+(a-x)^{2/3}=4(a^2-x^2)^{1/3}$

Solution. Cubing both sides, we get

$$\{(a+x)^{2/3}\}^3+\{(a-x)^{2/3}\}^3+3(a+x)^{2/3}(a-x)^{2/3}\{(a+x)^{2/3}+(a-x)^{2/3}\}$$
$$=64\{(a^2-x^2)^{1/3}\}^3$$

[Formula : $(A+B)^3=A^3+B^3+3AB(A+B)$]

$$\Rightarrow \quad (a+x)^2+(a-x)^2+3(a^2-x^2)^{2/3}\{4(a^2-x^2)^{1/3}\}=64(a^2-x^2)$$

$$\Rightarrow \quad (a+x)^2+(a-x)^2+12(a^2-x^2)=64(a^2-x^2)$$

$$\Rightarrow \quad 2a^2+2x^2-52a^2+52x^2=0, \text{ i.e., } 54x^2=50a^2$$

$$\Rightarrow \quad x^2=\frac{50a^2}{54}=\frac{25}{27}a^2$$

Hence $\quad x=\pm\dfrac{5a}{3\sqrt{3}}$

Example 15. *Find the value of*

$$\sqrt{6+\sqrt{6+\sqrt{6+\ldots\infty}}}$$

Solution. Let

$$x=\sqrt{6+\sqrt{6+\sqrt{6+\ldots\infty}}}$$

Since the terms go on to infinity, the given quantity will not change if we omit the radical before $\sqrt{6}$, and those after the first one are taken to be equal to x. Hence we have

$$x=\sqrt{6+x}$$

Squaring both sides, we get

$$x^2=6+x$$

$\Rightarrow$ $x^2-x-6=0$, using the formula for finding roots, we have

$$\Rightarrow \quad x=\frac{1\pm\sqrt{1+24}}{2}=\frac{1\pm5}{2}=3,\ -2$$

But the given quantity is positive.

Hence $x=3$.

Example 16. *Solve the equations :*

(*a*) $x+\sqrt{x}=\frac{6}{25}$

(*b*) $x^{10}-33x^5+32=0$.

Solution. (*a*) Putting $\sqrt{x}=y$, the given equation reduces to

$$y^2+y=\frac{6}{25}$$

$$\Rightarrow \quad 25y^2+25y-6=0$$

$$\Rightarrow \quad y=\frac{-25\pm\sqrt{625+600}}{50}=\frac{-25\pm35}{50}=-\frac{6}{5},\ \frac{1}{5}$$

$\therefore$ Either $\sqrt{x}=-\frac{6}{5}$, *i.e.*, $x=\frac{36}{25}$ or $\sqrt{x}=\frac{1}{5}$, *i.e.*, $x=\frac{1}{25}$

(*b*) Putting $x^5=y$, the given equation reduces to

$$y^2-33y+32=0$$

$$\Rightarrow \quad y^2-(32+1)\,y+32=0$$

$$\Rightarrow \quad y\,(y-32)-(y-32)=0$$

$$\Rightarrow \quad (y-32)\,(y-1)=0$$

$\therefore$ Either $y-32=0 \Rightarrow y=32$, *i.e.*, $x^5=32$, whence $x=2$

or $y-1=0 \Rightarrow y=1$, *i.e.*, $x^5=1$, whence $x=1$

Example 17. *Solve* $(2x+3)\,(2x+5)\,(x-1)\,(x-2)=30$.

Solution. Multiplying together the first and third factors and the second and fourth factors of L.H.S., we have

$$\{(2x+3)\,(x-1)\}\,\{(2x+5)\,(x-2)\}=30$$

$$\Rightarrow \quad (2x^2+x-3)\,(2x^2+x-10)=30$$

Putting $2x^2+x=y$, we have

$(y-3)(y-10)=30 \Rightarrow y^2-13y=0$

$\therefore \quad y(y-13)=0$, *i.e.*, $y=0, 13$

(*i*) Taking $y=0$, $2x^2+x=0 \Rightarrow x(2x+1)=0$

$\therefore \quad x=0, -\frac{1}{2}$

(*ii*) Taking $y=13$, $2x^2+x=13 \Rightarrow 2x^2+x-13=0$

$$\therefore \quad x=\frac{-1\pm\sqrt{105}}{4}$$

Irrational equations reducible to quadratics :

Example 18. *Solve* $\sqrt{2x+1}+\sqrt{3x+4}=7$.

Solution. By transposition of one radical, we get

$$\sqrt{2x+1}=7-\sqrt{3x+4}$$

Squaring, we get

$$2x+1=49+(3x+4)-14\sqrt{3x+4}$$

$$\Rightarrow \quad x+52=14\sqrt{3x+4}$$

Squaring, we get $x^2+104x+2704=196(3x+4)$

$$\Rightarrow \quad x^2-484x+1920=0$$

$$\Rightarrow \quad x^2-4x-480x+1920=0$$

$$\Rightarrow \quad x(x-4)-480(x-4)=0$$

$$(x-4)(x-480)=0$$

$\therefore \quad$ Either $x=4$ or $x=480$

The root 4 is found to satisfy the given equation, the other root 480 does not satisfy it.

$\therefore$ $x=4$ is a root of the given equation, but $x=480$ is not a root.

The value $x=480$, which does not satisfy the original equation and as such is not a root of the equation is called an extraneous root.

Example 19. *Solve* $\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$

Solution. Factorising expressions under radicals, we get

$$\sqrt{(x-3)(x+7)}+\sqrt{(x-3)(x+2)}=\sqrt{(x-3)(6x+13)}$$

$$\Rightarrow \quad \sqrt{x-3}[\sqrt{x+7}+\sqrt{x+2}]=\sqrt{x-3}\sqrt{6x+13}$$

$\therefore$ Either $\sqrt{x-3}=0$, *i.e.*, $x=3$

or $\quad \sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

Squaring both sides, we have

$$(x+7)+(x+2)+2\sqrt{(x+7)(x+2)}=6x+13$$

$$\Rightarrow \quad 2\sqrt{(x+7)(x+2)}=4x+4$$

$$\Rightarrow \quad \sqrt{(x+7)(x+2)}=(2x+2)$$

Squaring again, we get

$$x^2+9x+14=4x^2+8x+4$$

$$\Rightarrow \quad 3x^2-x-10=0$$

$$\Rightarrow \quad x=\frac{1\pm\sqrt{1+120}}{6}=2 \text{ or } -\frac{5}{3}$$

Example 20. *Solve* $\sqrt{x^2+4ax+5}+\sqrt{x^2+4bx+5}=2(a-b)$.

Solution. Let $$\sqrt{x^2+4ax+5}=y \qquad ...(1)$$

and $$\sqrt{x^2+4bx+5}=z \qquad ...(2)$$

Then the given equation becomes

$$y+z=2(a-b) \qquad ...(3)$$

Squaring and subtracting (1) and (2), we get

$$y^2-z^2=(x^2+4ax+5)-(x^2+4bx+5)$$

$$\Rightarrow \quad (y+z)(y-z)=4(a-b)x \qquad ...(4)$$

Dividing (4) by (3), we get

$$y-z=2x \qquad ...(5)$$

Adding (3) and (5), we get

$$2y=2(a-b)+2x, \text{ i.e., } y=a-b+x$$

Substituting this value in (1), we get

$$\sqrt{x^2+4ax+5}=a-b+x=(a-b)+x$$

Squaring both sides, we have

$$x^2+4ax+5=(a-b)^2+2(a-b)x+x^2$$

$$\Rightarrow \quad 4ax-2(a-b)x=(a-b)^2-5$$

$$\Rightarrow \quad 2(a+b)x=(a-b)^2-5$$

$$\Rightarrow \quad x=\frac{(a-b)^2-5}{2(a+b)}$$

Subtracting (3) and (5), we get

$$2z=2(a-b)-2x, \text{ i.e., } z=(a-b)-x$$

Substituting this value in (2), we get

$$\sqrt{x^2+4bx+5}=(a-b)-x$$

Squaring both sides, we have

$$x^2+4bx+5=(a-b)^2-2(a-b)x+x^2$$

$$\Rightarrow \quad 4bx+2(a-b)x=(a-b)^2-5$$

$$\Rightarrow \quad 2(a+b)x=(a-b)^2-5$$

$$\Rightarrow \quad x=\frac{(a-b)^2-5}{2(a+b)}$$

Example 21. *Solve* $\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x-5.$

Solution. Let $\sqrt{3x^2-7x-30}=y$ ...(1)

and $\sqrt{2x^2-7x-5}=z$ ...(2)

Then the given equation becomes

$$y+z=x-5 \quad \text{...(3)}$$

Also squaring and subtracting (2) from (1), we get

$$y^2-z^2=(3x^2-7x-30)-(2x^2-7x-5)=x^2-25$$

$$\Rightarrow \quad (y+z)(y-z)=(x+5)(x-5)$$

Dividing this equation by (3), we get

$$y-z=x+5 \quad \text{...(4)}$$

Adding (3) and (4), we have

$$2y=2x \quad \Rightarrow \quad y=x$$

Substituting this value of y in (1), we have

$$\sqrt{3x^2-7x-30}=x \quad \Rightarrow \quad 3x^2-7x-30=x^2$$

$$\Rightarrow \quad 2x^2-7x-30=0$$

$$\Rightarrow \quad x=\frac{7\pm\sqrt{49+240}}{4}=6,\ -\frac{5}{2}$$

Subtracting (3) and (4), we get

$$2z=-10 \Rightarrow z=-5$$

Substituting this value of z in (2), we have

$$\sqrt{2x^2-7x-5}=-5$$

$$\Rightarrow \quad 2x^2-7x-5=25$$

$$\Rightarrow \quad 2x^2-7x-30=0$$

$$\Rightarrow \quad x=\frac{7\pm\sqrt{49+240}}{4}=6,\ -\frac{5}{2}$$

8·10. RECIPROCAL EQUATIONS

Example 22. *Solve* $10x^4+63x^3+52x^2-63x+10=0.$

Solution. Rearranging the terms, we have

$$10(x^4+1)+63(x^3-x)+52x^2=0$$

Dividing both sides by x^2, we have

$$10\left(x^2+\frac{1}{x^2}\right)+63\left(x-\frac{1}{x}\right)+52=0$$

$$\Rightarrow \quad 10\left\{\left(x-\frac{1}{x}\right)^2+2\right\}+63\left(x-\frac{1}{x}\right)+52=0$$

$$\left[\because \; x^2+\frac{1}{x^2}=x^2-2+\frac{1}{x^2}+2=\left(x-\frac{1}{x}\right)^2+2\right]$$

$$\Rightarrow \quad 10\left(x-\frac{1}{x}\right)^2+63\left(x-\frac{1}{x}\right)+72=0$$

Putting y for $x-\frac{1}{x}$, we have

$$10y^2+63y+72=0$$

$$\Rightarrow \quad 10y^2+15y+48y+72=0$$

$$\Rightarrow \quad 5y(2y+3)+24(2y+3)=0$$

$$\Rightarrow \quad (5y+24)(2y+3)=0$$

$$\therefore \quad y=-\frac{24}{5} \text{ or } -\frac{3}{2}$$

(*i*) when $y=-\frac{24}{5}$ then $x-\frac{1}{x}=-\frac{24}{5}$

$$\Rightarrow \quad 5x^2+24x-5=0$$

$$\Rightarrow \quad x=\frac{-24\pm\sqrt{576+100}}{10}=\frac{-24\pm26}{10}=-5, \frac{1}{5}$$

(*ii*) when $y=-\frac{3}{2}$ then $x-\frac{1}{x}=-\frac{3}{2}$

$$\Rightarrow \quad 2x^2+3x-2=0$$

$$\Rightarrow \quad x=\frac{-3\pm\sqrt{9+16}}{4}=\frac{-3\pm5}{4}=-2, \frac{1}{2}$$

EXERCISE (I)

Solve the following equations :

1. (*i*) $6\frac{1}{3}-\frac{x-7}{3}=\frac{4x-2}{5}$, (*ii*) $\frac{3}{x-2}+\frac{5}{x-6}=\frac{8}{x+3}$

(*iii*) $4x-\frac{x-1}{3}=x+\frac{2(x-1)}{5}+3$, (*iv*) $\frac{a-x}{a}+\frac{2a-x}{2a}=\frac{3a-x}{3a}$

(v) $\frac{x-bc}{b+c}+\frac{x-ca}{c+a}+\frac{x-ab}{a+b}=a+b+c$

2. (i) $25x^2=16$, (ii) $\frac{x}{x+2}=\frac{x+3}{5(x+11)}$

(iii) $\frac{2(45+2x^2)}{x^2+9}+\frac{3(x^2-9)}{x^2+3}=7$

(iv) $3x^2-14x+11=0$, (v) $x^2-(p+q)x+pq=0$

(vi) $\frac{x}{3}+\frac{3}{x}=\frac{10}{3}$, (vii) $\frac{x}{a}+\frac{a}{x}=\frac{a}{b}+\frac{b}{a}$

(viii) $x^2-2\sqrt{3}x+1=0$, (ix) $x^2-(\sqrt{3}+3)x+(\sqrt{3}+2)=0$

3. (i) $\frac{x+2}{x-2}+\frac{x+3}{x-3}=\frac{x-2}{x+2}+\frac{x-3}{x+3}$

(ii) $\frac{x-p}{q}+\frac{x-q}{p}=\frac{q}{x-p}+\frac{p}{x-q}$

4. (i) $\frac{\sqrt{12-x}}{5}=\frac{3}{2+\sqrt{12-x}}$

(ii) $\sqrt{\frac{x}{x+16}}+\sqrt{\frac{x+16}{x}}=\frac{25}{12}$

(iii) $\frac{x-a^2-b^2}{c^2}+\frac{c^2}{x-a^2-b^2}=2$

5. (i) $x^{\frac{2}{3}}+x^{\frac{1}{3}}-2=0$, (ii) $x^{\frac{2}{13}}+12=7x^{\frac{1}{13}}$,

(iii) $x^{-4}+4=5x^{-2}$, (iv) $6x^{\frac{3}{4}}+3x^{-\frac{1}{4}}=11x^{\frac{1}{4}}$

6. (i) $(x^2-3x)^2-8(x^2-3x)-20=0$, (ii) $(2x-7)(x^2-9)(2x+5)=91$.

7. (i) $x^2+x+10\sqrt{x^2+3x+16}=2(20-x)$

(ii) $3x^2-18+\sqrt{3x^2-4x-6}=4x$

8. (i) $\sqrt{1-5x}+\sqrt{1-3x}=2$

(ii) $\sqrt{3x+10}+\sqrt{9x+7}=9$

9. (i) $\sqrt{x+5}+\sqrt{x+12}=\sqrt{2x+41}$

(ii) $\sqrt{x^2-16}-(x-4)=\sqrt{x^2-5x+4}$

(iii) $\sqrt{x^2-8x+15}+\sqrt{x^2+2x-15}=\sqrt{4x^2-18x+18}$

10. (i) $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$

(*ii*) $\sqrt{x^2-7ax+10a^2}-\sqrt{x^2+ax-6a^2}=x-2a$

(*iii*) $\sqrt{3x^2-4x-5}+\sqrt{2x^2-4x-1}=x+2$

11. (*i*) $\left(x-\frac{1}{x}\right)^2-6\left(x+\frac{1}{x}\right)+12=0$

(*ii*) $\left(x-\frac{1}{x}\right)^2-10\left(x-\frac{1}{x}\right)+24=0$

(*iii*) $\left(x-\frac{1}{x}\right)^2+9=\frac{5}{2}\left(x+\frac{1}{x}+2\right)$

12. (*i*) $x^4+8x^2+1=5x(x^2+1)$

(*ii*) $x^4+2x^3-13x^2+2x+1=0$

(*iii*) $4x^4-16x^3+23x^2-16x+4=0$

13 $$\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=3$$

ANSWERS

1 (*i*) 8, (*ii*) 3, (*iii*) 1, (*iv*) $\frac{6a}{7}$, (*v*) $ab+bc+ca$.

2. (*i*) $x=\pm\frac{4}{5}$, (*ii*) $\frac{-25\pm\sqrt{649}}{4}$, (*iii*) ± 3, (*iv*) 1, $\frac{11}{3}$, (*v*) p, q,

(*vi*) 9, 1, (*vii*) b, $\frac{a^2}{b}$, (*viii*) $\sqrt{3}\pm\sqrt{2}$, (*ix*) $\sqrt{3}+2$, 1.

3. (*i*) 0, $\pm\sqrt{6}$, (*ii*) $\frac{p^2+q^2}{p+q}$, 0, $p+q$.

4. (*i*) 3, -3, (*ii*) $\frac{144}{7}$, $\frac{-256}{7}$; (*iii*) $a^2+b^2+c^2$

5. (*i*) -8, 1, (*ii*) 3^{13}, 4^{13}, (*iii*) ± 1, $\pm\frac{1}{2}$, (*iv*) $\frac{9}{4}$, $\frac{1}{9}$

6. (*i*) 1, 2, -2, 5, (*ii*) $\frac{1\pm\sqrt{65}}{4}$, 4, $\frac{-7}{2}$.

7. (*i*) 0, -3 (*ii*) $\frac{2\pm\sqrt{10}}{3}$, $\frac{2\pm i\sqrt{23}}{3}$

8. (*i*) 0, -16, (*ii*) 2. **9.** (*i*) 4, (*ii*) 4, 5, (*iii*) 3, $\frac{17}{3}$

10. (*i*) 0, 3, (*ii*) $2a$, $6a$, $\frac{-10a}{3}$, (*iii*) $\frac{2\pm\sqrt{14}}{2}$.

11. (*i*) 1, 1, $-2\pm\sqrt{3}$, (*ii*) $3\pm\sqrt{10}$, $2\pm\sqrt{5}$, (*iii*) $\pm i$, 2, $\frac{1}{2}$

12. (*i*) 1, 1, $\frac{3\pm\sqrt{5}}{2}$, (*ii*) $\frac{-5\pm\sqrt{21}}{2}$, $\frac{3\pm\sqrt{5}}{2}$, (*iii*) 2, $\frac{1}{2}$, $\frac{3\pm i\sqrt{7}}{4}$

13. $\pm\sqrt{\frac{3}{5}}$.

8·11. NATURE OF THE ROOTS

Since the roots of the quadratic equation $ax^2+bx+c=0$ are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

the nature of the roots shall depend on the numerical value of $\sqrt{b^2-4ac}$. The expression b^2-4ac which discriminates the nature of the roots is called *discriminant* of the equation $ax^2+bx+c=0$, and is denoted for brevity by the symbol $\triangle$, which is a greek letter pronounced as delta. Assuming that a, b, c are real and rational, we obtain the following results :

(*i*) (*a*) If $\triangle>0$ and is a perfect square, then $\sqrt{\triangle}$ is rational, *i.e.*, both the roots are rational and unequal.

(*b*) If $\triangle>0$, but not a perfect square, then $\sqrt{\triangle}$ is irrational, *i.e.*, both the roots are irrational and unequal.

(*ii*) If $\triangle=0$, then $\sqrt{\triangle}=0$ and both the roots are real and equal, each being equal to $-\frac{b}{2a}$. They will be rational or irrational according as $\frac{b}{a}$ is rational or irrational.

(*iii*) If $\triangle<0$, then $\sqrt{\triangle}$ is imaginary and both the roots are complex and unequal.

The reader should note the following points :

(*i*) If one root of a quadratic equation with rational coefficients is irrational, the other will also be irrational, called the *irrational conjugates*, *e.g.*, if one root of a quadratic equation with a rational coefficient is $2+\sqrt{3}$, the other one will be $2-\sqrt{3}$.

(*ii*) If one root of a quadratic equation with a real coefficient is imaginary, the other will also be imaginary, called the *imaginary conjugates*, *e.g.*, if one root of a quadratic equation is $2+3i$, the other will be $2-3i$, *i.e.*, imaginary roots occur in pairs.

Example 23. *Discuss the nature of the roots of the following equations :*

(*a*) $x^2+2x+3=0$, (*b*) $(x-a)(x-b)=h^2$.

Solution. (*a*) Here $a=1$, $b=2$, $c=3$

$\therefore \quad \Delta=b^2-4ac=4-12=-8<0$

$\therefore$ The roots are imaginary and unequal.

(*b*) The equation may be written as

$$x^2-ax-bx+ab=h^2$$
$$x^2-(a+b)x+(ab-h^2)=0$$
$$\Delta=\{-(a+b)\}^2-4.\ 1.\ (ab-h^2)$$
$$=\{(a+b)^2-4ab\}+4h^2$$
$$=(a-b)^2+4h^2>0 \qquad \text{(Sum of squares is always +ve)}$$

$\therefore$ The roots are real and unequal.

Example 24. *For what values of m will the equation*

$$(m+1)x^2+2(m+3)x+(2m+3)=0$$

have equal roots.

Solution. Since the discriminant for equal roots is zero, we have

$$0=4(m+3)^2-4(m+1)(2m+3)$$
$$\Rightarrow \quad m^2-m-6=0$$
$$\Rightarrow \quad m=3,-2$$

Example. 25. *If the roots of the equation* $(m-n)x^2+(n-l)x+l=m$ *are equal, show that l, m, n are in A.P.* *[I.C.W.A., June 1990]*

Solution. The roots of the equation

$$(m-n)x^2+(n-l)x+(l-m)=0$$

will be equal if

$$(n-l)^2-4(m-n)(l-m)=0$$
$$\Rightarrow \quad n^2-2nl+l^2=4(ml-m^2-nl+mn)$$
$$\Rightarrow \quad n^2-2nl+l^2=4ml-4m^2-4nl+4mn$$
$$\Rightarrow \quad n^2+2nl+l^2=4m(n+l)-4m^2$$
$$\Rightarrow \quad (n+l)^2-4m(n+l)+4m^2=0$$
$$\Rightarrow \quad (n+l-2m)^2=0$$
$$\Rightarrow \quad n+l=2m$$
$$\Rightarrow \quad m=\frac{l+n}{2}$$

$\Rightarrow$ l, m, n are in A.P.

8·12. SYMMETRICAL EXPRESSIONS

An expression in α and β is said to be symmetrical if it remains unchanged by the interchange of α and β. Thus $\alpha+\beta$ becomes $\beta+\alpha$ by the interchange of α and β. Therefore, $\alpha+\beta$ is symmetric in α and β. Other examples of symmetrical expressions can be given as

$$\alpha^2+\beta^2,\ \alpha^3+\beta^3,\ \alpha^4+\beta^4,\ \alpha\beta,\ \frac{1}{\alpha}+\frac{1}{\beta},\ \ \frac{1}{\alpha^2}+\frac{1}{\beta^2},\ \ldots\ldots$$

It may be noted that the expressions like $\alpha-\beta$, $\alpha^3+\beta^2$, $\alpha^2-\alpha\beta+\beta^2$ are not symmetrical, as their values are altered if α and β are interchanged. Such expressions are called asymmetric or skew expressions.

Example 26. *If α and β are the roots of the equation $ax^2+bx+c=0$, find the value of* (*i*) $\alpha-\beta$, (*ii*) $\alpha^2+\beta^2$, (*iii*) $\alpha^4\beta^7+\alpha^7\beta^4$, (*iv*) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$ *and* (*v*) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$.

Solution. Since α, β are the roots of the quadratic equation $ax^2+bx+c=0$, we have

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

Any symmetric expression in α, β can be expressed in terms of $\alpha+\beta$ and $\alpha\beta$ and hence can be evaluated in terms of the constants a, b, c of the equation.

(*i*) $$\alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\sqrt{\left(-\frac{b}{a}\right)^2-4\frac{c}{a}}=\frac{\sqrt{b^2-4ac}}{a}$$

(*ii*) $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{b}{a}\right)^2-2\frac{c}{a}=\frac{b^2-2ac}{a^2}$$

(*iii*) $$\alpha^7\beta^4+\alpha^4\beta^7=\alpha^4\beta^4(\alpha^3+\beta^3)=\alpha^4\beta^4[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)]$$

$$=\frac{c^4}{a^4}\left[-\frac{b^3}{a^3}-3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)\right]=\frac{bc^4(3ac-b^2)}{a^7}$$

(*iv*) $$\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2=\frac{(\alpha^2-\beta^2)^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2(\alpha-\beta)^2}{\alpha^2\beta^3}$$

$$=\frac{(\alpha+\beta)^2\,[(\alpha+\beta)^2-4\alpha\beta]}{\alpha^2\beta^2}$$

$$=\frac{b^2}{a^2}\left(\frac{b^2}{a^2}-4\,.\,\frac{c}{a}\right)\Big/\frac{c^2}{a^2}=\frac{b^2(b^2-4ac)}{a^2c^2}$$

(*v*) $$\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\alpha^3+\beta^3}{(\alpha\beta)^3}=\frac{(3abc-b^3)}{c^3}$$

8·13. FORMATION OF AN EQUATION

So far we were given a quadratic equation and were required to find the roots of the equation. We now study the converse problem, *i.e.*, to find the equation whose solution set is $\{\alpha, \beta\}$.

Let $ax^2+bx+c=0$ be the required equation.

The equation can be written as

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad \ldots(*)$$

We know $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

The equation (*) becomes $x^2-(\alpha+\beta)x+\alpha\beta=0$

This is the required equation whose roots are α and β, we may state the same result as follows :

If α and β, the roots of an equation are given, then the equation can be written as

$$x^2-x\text{ (sum of the roots)}+\text{product of the roots}=0.$$

Example 27. *Form the equation whose roots are (i) 6, 7, (ii) $5+\sqrt{3}$, $5-\sqrt{3}$.*

Solution. (*i*) The quadratic equation whose roots are 6, 7 is

$$x^2-x(6+7)+6.7=0 \Rightarrow x^2-13x+42=0$$

(*ii*) The quadratic equation whose roots are $5+\sqrt{3}$, $5-\sqrt{3}$ is

$$x^2-x(5+\sqrt{3}+5-\sqrt{3})+(5+\sqrt{3})(5-\sqrt{3})=0$$

$$\Rightarrow \qquad x^2+10x+22=0$$

Example 28. (*a*) *If α and β be the roots of $x^2+px+q=0$, find the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.* [*I.C.W.A., December 1990*]

(*b*) *If α and β be the roots of $x^2-px+q=0$, find the equation whose roots are α^2, β^2.*

Solution. Since α, β are the roots of $x^2+px+q=0$, therefore, we have

$$\alpha+\beta=-p$$

and

$$\alpha\beta=q$$

$$\text{Sum of the roots}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}$$

$$=\frac{p^2-2q}{q^2}$$

$$\text{Product of the roots}=\frac{1}{\alpha^2}\times\frac{1}{\beta^2}=\frac{1}{\alpha^2\beta^2}=\frac{1}{q^2}$$

$\therefore$ The equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ is

$$x^2-\left(\frac{p^2-2q}{q^2}\right)x+\frac{1}{q^2}=0$$

or

$$q^2x-(p^2-2q)x+1=0.$$

(*b*) Since α and β are the roots of $x^2-px+q=0$, therefore,

$$\alpha+\beta=p \qquad ...(1)$$

and $\alpha\beta=q$ ...(2)

The equadratic equation whose roots are α^2, β^2 is

$$x^2-x(\alpha^2+\beta^2)+\alpha^2\beta^2=0$$

$$\Rightarrow \quad x^2-x[(\alpha+\beta)^2-2\alpha\beta)]+(\alpha\beta)^2=0$$

$$\Rightarrow \quad x^2-x(p^2-2q)+q^2=0 \qquad \text{[from (1) and (2)]}$$

Example 29. *If α and β are the roots of $2x^2-4x+1=0$, from the equation whose roots are $\alpha^2+\beta$ and $\beta^2+\alpha$.*

Solution. Here $\alpha+\beta=-\left(-\frac{4}{2}\right)=2,\ \alpha\beta=\frac{1}{2}\cdot$

The required equation is

$$x^2-x\ (\text{sum of the roots})+\text{product of the roots}=0$$

Sum of the roots of the required equation

$$=(\alpha^2+\beta)+(\beta^2+\alpha)=\{(\alpha+\beta)^2-2\alpha\beta\}+(\alpha+\beta)$$

$$=(2)^2-2.\ \frac{1}{2}+2=5$$

Product of the roots$=(\alpha^2+\beta)\ (\beta^2+\alpha)$

$$=\alpha^2\beta^2+\alpha^3+\beta^3+\alpha\beta$$

$$=(\alpha\beta)^2+(\alpha+\beta)^3-3\alpha\beta\ (\alpha+\beta)+\alpha\beta$$

$$=\left(\frac{1}{2}\right)^2+(2)^3-3.\ \frac{1}{2}.2+\frac{1}{2}=\frac{23}{4}$$

$\therefore$ The required equation is

$$x^2-(5)x+\frac{23}{4}=0$$

$$\Rightarrow \quad 4x^2-20x+23=0$$

Example 30. *If α and β are the roots of the equation $ax^2+bx+c=0$, form the equation whose roots are*

(*i*) $\frac{\alpha}{\beta},\ \frac{\beta}{\alpha}$, (*ii*) $\frac{1}{a\alpha+b},\ \frac{1}{a\beta+b}$.

Solution. Since α and β are the roots of the equation $ax^2+bx+c=0$,

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

The required equation is

$$x^2-x\ (\text{sum of the roots})+\text{product of the roots}=0 \qquad ...(*)$$

(*i*) Sum of the roots$=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

$$=\frac{\frac{b^2}{a^2}-2\cdot\frac{c}{a}}{c/a}=\frac{b^2-2ac}{ac}$$

Product of the roots $=\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1$

∴ From (*), the required equation is

$$x^2-x\cdot\frac{b^2-2ac}{ac}+1=0 \Rightarrow acx^2-x(b^2-2ac)+ac=0$$

(*ii*) Sum of the roots $=\frac{1}{a\alpha+b}+\frac{1}{a\beta+b}=\frac{a(\alpha+\beta)+2b}{a^2\alpha\beta+ab(\alpha+\beta)+b^2}$

$$=\frac{a\left(-\frac{b}{a}\right)+2b}{a^2\cdot\frac{c}{a}+ab\left(-\frac{b}{a}\right)+b^2}=\frac{b}{ac}$$

Product of the roots $=\frac{1}{(a\alpha+b)(a\beta+b)}=\frac{1}{a^2\alpha\beta+ab(\alpha+\beta)+b^2}$

$$=\frac{1}{a^2\cdot\frac{c}{a}+ab\left(-\frac{b}{a}\right)+b^2}=\frac{1}{ac}$$

∴ From (*), the required equation is

$$x^2-x\left(\frac{b}{ac}\right)+\frac{1}{ac}=0$$

$$\Rightarrow \quad acx^2-bx+1=0.$$

Example 31. *If* α *and* β *are the roots of* $x^2-px+q=0$, *from the equation whose roots are* $(\alpha\beta+\alpha+\beta)$ *and* $(\alpha\beta-\alpha-\beta)$.

Solution. Here $\alpha+\beta=p$, $\alpha\beta=q$

The sum of the roots of the required equation

$$=(\alpha\beta+\alpha+\beta)+(\alpha\beta-\alpha-\beta)=2\alpha\beta=2q$$

The product of roots $=(\alpha\beta+\alpha+\beta)(\alpha\beta-\alpha-\beta)$

$$=(\alpha\beta)^2-(\alpha+\beta)^2=q^2-p^2$$

Now the required equation is $x^2-(\text{sum})\,x+\text{product}=0$

$$\therefore \quad x^2-2qx+(q^2-p^2)=0,$$

Example 32. *Find the condition that one root of* $ax^2+bx+c=0$ *shall be n times the other.* [*I.C.W.A., December 1989*]

Solution. Let one root of the equation be α then, the other will be $n\alpha$.

Sum of the roots $=\alpha(1+n)=-\frac{b}{a}$ ...(1)

Product of the roots $=\alpha^2 n=\frac{c}{a}$ ...(2)

Eliminating α between (1) and (2), the required condition is

$$\frac{b^2 n}{a^2(1+n)^2}=\frac{c}{a}$$

$$\Rightarrow \qquad b^2 n=ac(1+n)^2.$$

Example 33. *Find the condition that the roots of the equation $ax^2+bx+c=0$ may differ by 5.*

Solution. Let α and $\alpha+5$ be the two roots.

$\therefore$ Sum of the roots $=2\alpha+5=-\frac{b}{a}$ ...(1)

and the product of the roots $=\alpha^2+5\alpha=\frac{c}{a}$ ...(2)

The condition can be obtained by eliminating α in (1) and (2). We shall obtain the value of α from (1) and will substitute it in (2).

From (1), we have $\alpha=-\frac{b+5a}{2a}$

$\therefore$ Substituting in (2), we get

$$\left(-\frac{b+5a}{2a}\right)^2+5\left(-\frac{b+5a}{2a}\right)=\frac{c}{a}$$

$\Rightarrow$ $b^2+25a^2+10ab-10ab-50a^2=4ac$

$\Rightarrow$ $b^2-25a^2=4ac$ is the required condition.

Example 34. *If the roots of the equation $ax^2+bx+c=0$ may be in the ratio $m : n$, prove that*

$mnb^2=ac(m+n)^2$

Solution. Since the roots of the equation are in the ratio $m : n$, they can be taken as $m\alpha$ and $n\alpha$.

We then, have the sum of the roots

$$m\alpha+n\alpha=-\frac{b}{a}$$

i.e., $\alpha(m+n)=-\frac{b}{a}$ ...(1)

and the product of the roots is

$$mn\alpha^2=\frac{c}{a}$$ ...(2)

The required condition can be obtained by eliminating α between (1) and (2).

From (1), $\alpha = -\dfrac{b}{a(m+n)}$

and from (2), $\alpha^2 = \dfrac{c}{amn}$

$\therefore \quad \dfrac{b^2}{a^2(m+n)^2} = \dfrac{c}{amn}$

$\Rightarrow \quad mnb^2 = ac(m+n)^2$

which is the required result.

EXERCISE (II)

1. If α, β are the roots of $2x^2+3x+7=0$, find the values of
   (*i*) $\alpha^2+\beta^2$, (*ii*) $\alpha^3+\beta^3$, (*iii*) $\alpha^4+\beta^4$, (*iv*) $\alpha.\beta^{-1}+\beta.\alpha^{-1}$,
   (*v*) $(\alpha^2-\beta)^2+(\beta^2-\alpha)^2$, (*vi*) $\alpha^3-\beta^3$.
2. The roots of $x^2-px+q=0$ are α and β, prove that
   (*i*) $\dfrac{1}{\alpha^3}+\dfrac{1}{\beta^3}=\dfrac{p^3}{q^3}-3\dfrac{p}{q^2}$, (*ii*) $\dfrac{\alpha^2}{\beta^2}+\dfrac{\beta^2}{\alpha^2}=\dfrac{p^4}{q^2}-4\dfrac{p^2}{q}+2$.
3. Form the quadratic equation whose roots are
   (*i*) $4+i\sqrt{2}, 4-i\sqrt{2}$; (*ii*) $p+\sqrt{q}, p-\sqrt{q}$,
   (*iii*) $\dfrac{\sqrt{p}+\sqrt{q}}{\sqrt{p}-\sqrt{q}}, \dfrac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$.
4. If α, β are the roots of $x^2-2x+3=0$, form the quadratic equation whose roots are
   (*i*) $\alpha+3, \beta+3$, (*ii*) $2\alpha-3\beta, 3\alpha-2\beta$, (*iii*) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$, and
   (*iv*) $\dfrac{\alpha-1}{\alpha+1}, \dfrac{\beta-1}{\beta+1}$.
5. (*a*) If α, β be the roots of $ax^2+bx+c=0$, prove that the equation whose roots are $a\alpha+b\beta$, $b\alpha+a\beta$ is
   $$(ax+b^2)(x+b)+c(a-b)^2=0$$
   (*b*) If α and β are the roots of the equation $ax^2-bx+c=0$, form the equation whose roots are
   (*i*) $\dfrac{1}{\alpha+\beta}, \dfrac{1}{\alpha\beta}$, (*ii*) $\alpha+\dfrac{1}{\beta}, \beta+\dfrac{1}{\alpha}$, (*iii*) $\dfrac{1-\alpha}{1+\alpha}, \dfrac{1-\beta}{1+\beta}$.
6. If p, q be the roots of the equation $3x^2+6x+2=0$, show that the equation whose roots are $\dfrac{-p^2}{q}$ and $\dfrac{-q^2}{p}$ is $3x^2-18x+2=0$.
7. (*a*) If r be the ratio of the roots of the equation $ax^2+bx+c=0$,
   show that $\dfrac{(r+1)^2}{r}=\dfrac{b^2}{ac}$.

   (*b*) If the roots of the equation $ax^2+bx+c=0$ be in the ratio $p:q$, prove that $ac(p+q)^2=b^2pq$.

8. Find k, if

(*i*) the roots of $2x^2+3x+k=0$ are equal,
(*ii*) one of the roots of the equation $x^2-4x-k=0$ is $2(1+\sqrt{3})$,
(*iii*) one of the roots of the equation $x^2-6x+k=0$ is $3+i\sqrt{2}$,
(*iv*) one root of the equation $x^2-6x+k=0$ is double the other.

9. If the sum of the roots of a quadratic equation is 3 and the sum of their cubes is 7, find the equation.

ANSWERS

1. (*i*) $\frac{-19}{4}$, (*ii*) $\frac{99}{8}$, (*iii*) $\frac{-31}{16}$, (*iv*) $\frac{-19}{14}$
   (*v*) $\frac{61}{16}$, (*vi*) $-\frac{5}{8}\sqrt{-47}$.

3. (*i*) $x^2-8x+18=0$, (*ii*) $x^2-2px+p^2-q=0$,
   (*iii*) $(p-q)x^2-2(p+q)x+(p-q)=0$.

4. (*iii*) $3x^2+2x+3=0$, (*iv*) $3x^2-2x+1=0$.

5. (*b*) (*i*) $bcx^2-ax(b+c)+a^2=0$, (*ii*) $acx^2-bx(a+c)+(c+a)^2=0$,
   (*iii*) $(a+b+c)x^2-2x(a-c)+a-b+c=0$.

8. (*i*) $k=\frac{9}{8}$, (*ii*) $k=8$, (*iii*) $k=11$. (*iv*) $k=8$.

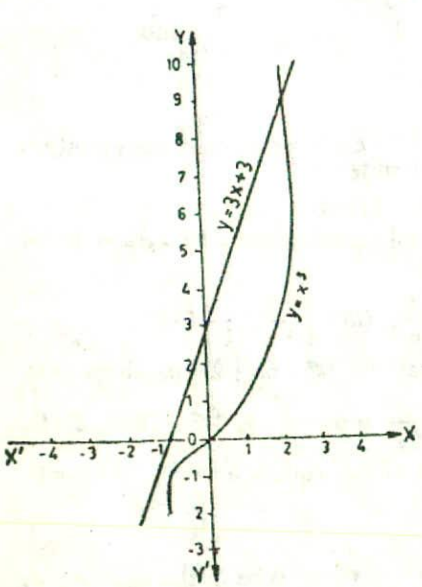
9. [**Hint.** Here $\alpha+\beta=3$ ...(1)

$$\alpha^3+\beta^3=7=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=27-9\alpha\beta$$

$$\Rightarrow \qquad 9\alpha\beta=20, \text{ i.e., } \alpha\beta=\frac{20}{9}$$

Hence the equation is $x^2-x(3)+\frac{20}{9}=0$.]

8·14. SOLUTION OF SIMULTANEOUS EQUATIONS



As indicated earlier there can be both graphic as well as algebraic methods of solving simultaneous equations. Graphic method at time is very handy in solving such equations and, therefore, employed quite often in Linear and Non-linear Programming. The graphic method is also employed in case of inqualities. We illustrate below its use when the simultaneous equations consist of both linearand non-linear equations.

The point of intersection gives a common solution to the two equations, one of which is linear ($y=3x+3$) and the other one cubic ($y=x^3$). The value of $x=2{\cdot}1$ gives an approximate solution which is good enough for business decision-making.

We, now discuss the algebraic methods of solving such equations. The use of matrices for the same shall be dealt in relevant chapter. Various methods are indicated depending on the combination of linear with linear or non-linear equations.

(A) When both equations are linear :

There are generally three methods to solve such equations. These are :

(*i*) Method of substitution.

(*ii*) Method of elimination.

(*iii*) Method of cross multiplication.

We will explain below these methods by taking the two equations in standard form as

$$a_1x+b_1y+c_1=0 \quad \ldots(1)$$

$$a_2x+b_2y+c_2=0 \quad \ldots(2)$$

(*i*) *Method of substitution.* From (1), by transposition

$$a_1x=-(b_1y+c_1)$$

$$\Rightarrow \quad x=-\frac{b_1y+c_1}{a_1} \quad \ldots(3)$$

Substituting this value of x in (2), we get

$$a_2\left(-\frac{b_1y+c_1}{a_1}\right)+b_2y+c_2=0$$

$$\Rightarrow \quad -a_2b_1y-a_2c_1+a_1b_2y+a_1c_2=0$$

$$\Rightarrow \quad (a_1b_2-a_2b_1)y=-(a_1c_2-a_2c_1)$$

$$\Rightarrow \quad y=-\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$$

From (3), we get

$$x=\frac{b_1\times\dfrac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}-c_1}{a_1}=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$$

Illustration. *Solve* $5x+2y=8 \quad \ldots(1)$

$9x-5y=23 \quad \ldots(2)$

Solution. From (1), we have $2y=8-5x$

$$\Rightarrow \quad y=\frac{8-5x}{2} \quad \ldots(3)$$

Substituting this value of y in (2), we get

$$9x-5\left(\frac{8-5x}{2}\right)=23$$

$$\Rightarrow \quad 18x-40+25x=46$$

$$\Rightarrow \quad 43x=86, \text{ i.e., } x=2$$

$\therefore$ From (3), we have $y=\dfrac{8-5\times 2}{2}=-1$

Hence $x=2, y=-1$ is the required solution.

(*ii*) *Method of elimination.* Under this method, the two equations are transformed to equivalent equations such that coefficients of any of the variables in both the transformed equations become numerically equal. Thereafter by addition or subtraction of these equations, that variable can be eliminated, so that the resulting equation becomes a simple equation. The solution for the variable of the simple equation can be determined by methods already discussed. The method of elimination can be repeated for the other variable or the solution for the other variable can be determined by method of substitution.

The two general equations already considered are :

$$a_1x+b_1y+c_1=0 \qquad ...(1)$$

$$a_2x+b_2y+c_2=0 \qquad ...(2)$$

Let equations (1) and (2) be transformed to equivalent equations having equal coefficient of x. The L.C.M. of the coefficients of x in the two equations, *viz.*, a_1a_2 will be the coefficient of x in the equivalent equations. Accordingly, (1) is to be multiplied by a_2 and (2) by a_1. Thus the transformed equations are :

$$a_1a_2x+a_2b_1y+a_2c_1=0 \qquad ...(3)$$

$$a_1a_2x+a_1b_2y+a_1c_2=0 \qquad ...(4)$$

Subtracting (4) from (3), we have

$$(a_2b_1-a_1b_2)y+(a_2c_1-a_1c_2)=0$$

or $$y=-\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}=-\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$$

By substituting the value of y in (1) or (2), we shall get

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$$

It may be noted that the method of elimination is preferable when none of the variables can be expressed in simple form in terms of the other from the equations.

Let us illustrate this method with equations having numerical coefficients. Let the equations be

$$5x+2y=8 \qquad ...(1)$$

$$9x-5y=23 \qquad ...(2)$$

L.C.M. of the coefficients of x, *viz.*, 5 and 9 is 45.

$\therefore$ Multiplying (1) by 9 and (2) by 5, we get

$$45x+18y=72 \quad ...(3)$$

$$45x-25y=115 \quad ...(4)$$

Subtracting (3) from (4), we get

$$43y=-43 \quad \Rightarrow \quad y=-1$$

$\therefore$ From (1), we have $5x-2\times 1=8$

$\Rightarrow$ $x=2$

(*iii*) *Method of Cross Multiplication* :

The two general equations already considered are :

$$a_1x+b_1y+c_1=0 \quad ...(1)$$

$$a_2x+b_2y+c_2=0 \quad ...(2)$$

Multiplying the first equation by a_2 and second equation by a_1, we have

$$a_1a_2x+a_2b_1y+a_2c_1=0$$

$$a_1a_2x+a_1b_2y+a_1c_2=0$$

By subtraction, we have $(a_2b_1-a_1b_2)y+(a_2c_1-a_1c_2)=0$

$$\Rightarrow \quad \frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1} \quad ...(1)$$

Similarly multiplying the first equation by b_2 and the second equation by b_1, we have by subtraction

$$\frac{x}{b_1c_2-b_2c_1}=\frac{1}{a_1b_2-a_2b_1} \quad ...(2)$$

$\therefore$ From (1) and (2), we get

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

This is called *Rule of cross multiplication.*

Illustration. *Solve* $5x+2y=8,\ 9x-5y=23$

Solution. The equations must be arranged in the standard form as

$$5x+2y-8=0$$
$$9x-5y-23=0$$

By the method of cross multiplication, we have

$$\frac{x}{2(-23)-(-5)(-8)}=\frac{y}{(-8)(9)-(5)(-23)}=\frac{1}{5(-5)-9(2)}$$

$$\Rightarrow \qquad \frac{x}{-46-40}=\frac{y}{-72+115}=\frac{1}{-25-18}$$

$$\Rightarrow \qquad \frac{x}{2}=\frac{y}{-1}=1$$

$$\Rightarrow \qquad x=2 \text{ and } y=-1$$

(B) When one equation is linear and the other one is quadratic. The general method of solution consists in (*i*) expressing one unknown, say x in terms of another, say y, from the linear equation, (*ii*) substituting the value of y in the quadratic equation and obtaining values of y and then (*iii*) finding corresponding values of x.

Example 35. *Solve* $x^2+y^2=29$ ...(1)

$$x-y=3 \qquad ...(2)$$

Solution. From (2), $x=3+y$ ...(3)

Substituting this value in (1), we have

$$(3+y)^2+y^2=29$$

$$\Rightarrow \qquad y^2+3y-10=0$$

$$\Rightarrow \qquad y=\frac{-3\pm\sqrt{9+40}}{2}=-5 \text{ or } 2$$

∴ From (3), we have $x=-2$ or 5

∴ The roots are $x=-2, y=-5$ or $x=5, y=2$

Example 36. (*a*) *Solve*

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2} \qquad ...(1)$$

$$x+y=10 \qquad ...(2)$$

Solution. On simplification (1) reduces to

$$\frac{x+y}{\sqrt{xy}}=\frac{5}{2}$$

$$\Rightarrow \qquad \frac{10}{\sqrt{xy}}=\frac{5}{2} \qquad \text{[from (2)]}$$

$$\Rightarrow \qquad \sqrt{xy}=4 \quad \text{and} \quad xy=16 \qquad ...(3)$$

From (2), we have $x=10-y$ ...(4)

Substituting this value of x in (3), we get

$$(10-y)y=16$$

$\Rightarrow \qquad y^2-10y+16=0$

$\Rightarrow \qquad (y-8)(y-2)=0$

$\Rightarrow \qquad y=8 \text{ or } 2$

$\therefore$ From (4), we get $\quad x=2$ or 8

Hence the roots are $\quad x=2, y=8$ or $x=8, y=2$

Example 36. (*b*) *Solve* $\frac{1}{x^2}+\frac{1}{y^2}=13, \quad \frac{1}{x}+\frac{1}{y}=5.$

Solution. Put $u=\frac{1}{x}$ $v=\frac{1}{y}$, then the equations reduces to

$$u^2+v^2=13 \qquad \ldots(1)$$

$$u+v=5 \qquad \ldots(2)$$

From (2), we have $\quad u=5-v \qquad \ldots(3)$

Substituting this value of u in (1), we have

$$(5-v)^2+v^2=13$$

$\Rightarrow \qquad v^2-5v+6=0$

$\Rightarrow \qquad (v-2)(v-3)=0$, *i.e.*, $v=2$ or 3

$\therefore$ From (3), we have $\quad u=3$ or 2

$$u=3, v=2 \text{ or } u=2, v=3$$

Hence the roots are $x=\frac{1}{3}, y=\frac{1}{2}$ or $x=\frac{1}{2}, y=\frac{1}{3}$.

(C) When both equations are quadratic :

If the equations are homogeneous and of the second degree, *i.e.*, if the sum of the indices of x and y in each term is 2, they may be solved by putting $y=mx$ as follows :

Example 37. *Solve* $x^2+xy=12$

$$xy-2y^2=1$$

Solution. Putting $y=mx$ in the equations, we have

$$x^2(1+m)=12 \qquad \ldots(1)$$

$$x^2(m-2m^2)=1 \qquad \ldots(2)$$

Dividing (1) by (2), we have $\quad \frac{1+m}{m-2m^2}=12$

$\Rightarrow \qquad 24m^2-11m+1=0$

$\Rightarrow \qquad (8m-1)(3m-1)=0$, *i.e.*, $m=\frac{1}{8}$ or $\frac{1}{3}$

(*i*) When $m=\frac{1}{8}$, we have $\frac{9}{8}x^2=12$ [from (1)]

or $x^2=\frac{32}{3} \Rightarrow x=\pm 4\sqrt{\frac{2}{3}}$

and $y=mx=\pm\frac{1}{8}\cdot 4\sqrt{\frac{2}{3}}=\pm\frac{1}{2}\sqrt{\frac{2}{3}}$

(*ii*) $m=\frac{1}{3}$, we have $\frac{4}{3}x^2=12$ [from (1)]

or $x^2=9 \Rightarrow x=\pm 3$

and $y=mx=\pm\frac{1}{3}.\ 3=\pm 1$

Thus the four pairs of roots are

$$\left.\begin{matrix} x=4\sqrt{\frac{2}{3}} \\ y=\frac{1}{2}\sqrt{\frac{2}{3}}\end{matrix}\right\},\quad \left.\begin{matrix} x=-4\sqrt{\frac{2}{3}} \\ y=-\frac{1}{2}\sqrt{\frac{2}{3}}\end{matrix}\right\},\quad \left.\begin{matrix} x=3 \\ y=1\end{matrix}\right\},\quad \left.\begin{matrix} x=-3 \\ y=-1\end{matrix}\right\}$$

Example 38. *Solve* $x^2+xy+y^2=19,\ 3xy+2y^2=36.$

Solution. Putting $y=mx$ in the equations, we have

$$x^2(1+m+m^2)=19 \quad \text{...(1)}$$

$$x^2(3m+2m^2)=36 \quad \text{...(2)}$$

Dividing (1) by (2), we get $\frac{1+m+m^2}{3m+2m^2}=\frac{19}{36}$

$\therefore \quad 57m+38m^2=36+36m+36m^2$

$\Rightarrow \quad 2m^2+21m-36=0$

$\Rightarrow \quad (m+12)(2m-3)=0$, *i.e.*, $m=-12$ or $\frac{3}{2}$

(*i*) Taking $m=-12$, from (1), we get

$$x^2\times 133=19 \Rightarrow x=\pm\frac{1}{\sqrt{7}}=\pm\frac{1}{7}\sqrt{7}$$

$$\therefore \quad y=-12\times\pm\frac{1}{\sqrt{7}}=\mp\frac{12}{\sqrt{7}}=\mp\frac{12}{7}\sqrt{7}$$

(*ii*) Taking $m=\frac{3}{2}$, from (1); we get

$$x^2\times\frac{19}{4}=19 \Rightarrow x=\pm 2$$

$$\therefore \quad y=\frac{3}{2}\times(\pm 2)=\pm 3$$

Thus the roots are

$$\left.\begin{array}{l} x=\frac{1}{7}\sqrt{7} \\ y=-\frac{12}{7}\sqrt{7} \end{array}\right\} \quad \left.\begin{array}{l} x=-\frac{1}{7}\sqrt{7} \\ y=\frac{12}{7}\sqrt{7} \end{array}\right\}, \left.\begin{array}{l} x=2 \\ y=3 \end{array}\right\}, \left.\begin{array}{l} x=-2 \\ y=-3 \end{array}\right\}$$

Example 39. *Demand for goods of an industry is given by the equation $pq=100$, where p is the price and q is quantity, supply is given by the equation $20+3p=q$. What is the equilibrium price and quantity ?*

Solution. The demand equation is $pq=100$ ...(1)
and supply equation is $20+3p=q$ ...(2)

Substituting the value of q from (2) in (1), we get

$$p(20+3p)=100$$

$$\Rightarrow \quad 3p^2+20p-100=0$$

$$\Rightarrow \quad p=\frac{-20\pm\sqrt{400+1200}}{6}=-10,\ \frac{10}{3}$$

But $p\neq-10$, therefore $p=\frac{10}{3}$

$\therefore$ From (2), we have $q=20+3\,.\,\frac{10}{3}=30$

$\therefore$ Equilibrium price$=\frac{10}{3}$, Quantity exchanged$=30$.

Example 40. *Solve* $3^x=9^y$ ...(1)

$5^{x+y+1}=25^{xy}$ ...(2)

Solution. From (1), we have $3^x=(3^2)^y=3^{2y} \Rightarrow x=2y$ ...(3)

From (2), $5^{x+y+1}=(5^2)^{xy}=5^{2xy} \Rightarrow x+y+1=2xy$ ...(4)

Substituting the value of x from (3) in (4), we get

$$2y+y+1=2\times2y\times y$$

$$4y^2-3y-1=0$$

$$\Rightarrow \quad y=\frac{3\pm\sqrt{9+16}}{8}=\frac{3\pm5}{8}=1 \text{ or } -\frac{1}{4}$$

When $y=1$, (3) gives $x=2$

and when $y=-\frac{1}{4}$, (3) gives $x=2\times(-\frac{1}{4})=-\frac{1}{2}$.

Hence solutions are : $(2, 1)$ and $(-\frac{1}{2}, -\frac{1}{4})$

Example 41. *Solve the simultaneous equations*

$$4^x.2^x=128 \quad \ldots(1)$$

and

$$3^{3x+2y}=9^{xy} \quad \ldots(2)$$

for x and y.

Solution. Equation (1) may be re-written as

$$2^{2x+y}=2^7$$

$$\Rightarrow \quad 2x+y=7$$

$$\Rightarrow \quad y=7-2x \quad \ldots(3)$$

(2) may be re-written as

$$3^{3x+2y}=(3^2)^{xy}=3^{2xy}$$

$$\Rightarrow \quad 3x+2y=2xy \quad \ldots(4)$$

Substituting (3) in (4), we get

$$3x+2(7-2x)=2x(7-2x)$$

$$\Rightarrow \quad 3x+14-4x=14x-4x^2$$

$$\Rightarrow \quad 4x^2-15x+14=0$$

$$\Rightarrow \quad x=\frac{15\pm\sqrt{225-224}}{8}=\frac{15\pm1}{8}=2,\ \frac{7}{4}$$

From (3), we get $y=3,\ \frac{7}{2}$

Hence $x=\frac{7}{4},\ y=\frac{7}{2}$ or $x=2,\ y=3.$

Example 42. *Solve the equations*

$$9x+3y-4z=35 \quad \ldots(1)$$

$$x+y-z=4 \quad \ldots(2)$$

$$2x-5y-4z+48=0 \quad \ldots(3)$$

Solution. Multiplying (2) by 9, $9x+9y-9z=36$ ...(4)

Subtracting (4) from (1), $-6y+5z+1=0$ ...(5)

Multiplying (2) by 2, $2x+2y-2z=8$ ...(6)

Subtracting (6) from (3), $-7y-2z+56=0$ ...(7)

From (5) and (7), by cross-multiplication,

$$\frac{y}{5\times(56)-(1)(-2)}=\frac{-z}{(-6)\times(56)-(1)\times(-7)}$$

$$=\frac{1}{(-6)\times(-2)-(5)\times(-7)}$$

$$\therefore \quad \frac{y}{282}=\frac{-z}{-329}=\frac{1}{47}$$

$\therefore \quad y=\frac{282}{47}=6,\ z=\frac{329}{47}=7$

Substituting these values of y and z in (2), we get

$$x+6-7=4 \Rightarrow x=5$$

Thus $\quad x=5, y=6, z=7$

Example 43. *Solve the equations* $x+2y+2z=0$ ...(1)

$3x-4y+z=0$ ...(2)

$x^2+3y^2+z^2=11$ ...(3)

Solution. From (1) and (2), by cross multiplication

$$\frac{x}{2\times1-2\times(-4)}=\frac{y}{2\times3-1\times1}=\frac{z}{1\times(-4)-2\times3}$$

$$\therefore \quad \frac{x}{10}=\frac{y}{5}=\frac{z}{-10}$$

$$\therefore \quad \frac{x}{2}=\frac{y}{1}=\frac{z}{-2}=k, \text{ say}$$

$$\therefore \quad x=2k,\ y=k,\ z=-2k \qquad ...(4)$$

Substituting these values of x, y, z in (3), we have

$$(2k)^2+3k^2+(-2k)^2=11$$

$$\Rightarrow \quad 11k^2=11, \text{ i.e., } k=\pm1$$

When $k=1$, from (4), we have $x=2, y=1, z=-2$

and when $k=-1$, from (4), we have $x=-2, y=-1, z=2$.

EXERCISE (III)

Solve the following simultaneous equations.

1. (*i*) $x-2y=1,\ 2x+y=-3$, (*ii*) $42x+33y=117,\ 48x+27y=123$,

(*iii*) $\frac{x-1}{2}+\frac{2y+1}{3}=0,\ \frac{x+4}{3}-\frac{y-1}{2}=1$

(*iv*) $\frac{x}{6}+\frac{y}{15}=4,\ \frac{x}{3}-\frac{y}{12}=\frac{19}{4}$

2. (*i*) $\frac{2}{x}+y=3,\ \frac{1}{2x}-\frac{2y}{3}=\frac{1}{6}$

(*ii*) $\frac{2}{x}+\frac{3}{y}=5,\ \frac{1}{x}-\frac{1}{2y}=\frac{1}{2}$

(*iii*) $8y-2x=3xy,\ \frac{10}{x}+\frac{1}{y}=2$

3. (*i*) $x^2+y^2=25,\ x+y=7$, (*ii*) $x^2+y^2=185,\ x-y=3$
(*iii*) $3x^2-7xy+4y^2=0,\ 4x+3y=5$,
(*iv*) $x^2+2y^2+3xy=0,\ x+3y=2$

4. (*i*) $\frac{1}{x}+\frac{1}{y}=\frac{5}{6},\ x+y=5$ (*ii*) $\frac{1}{x}+\frac{1}{y}=\frac{7}{12},\ xy=12.$

5. (*i*) $\frac{x}{y}+\frac{y}{x}=5\frac{1}{5},\ x+y=6$,
(*ii*) $\frac{x}{2}+\frac{y}{5}=5,\ \frac{2}{x}+\frac{5}{y}=\frac{5}{6}$

6. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{13}{6},\ xy=36.$

7. (*i*) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{4},\ \frac{1}{x}+\frac{1}{y}=\frac{3}{2}$
(*ii*) $x+y=a+b,\frac{a}{x}+\frac{b}{y}=2$, (*iii*) $\frac{a}{x}+\frac{b}{y}=\frac{a^2}{x^2}+\frac{b^2}{y^2}=2$

8. $x^2-y^2=1,\ x^4-2y^4=1$

9. (*i*) $\frac{x^2}{y}+\frac{y^2}{x}=\frac{9}{2},\ x+y=3$, (*iii*) $\frac{x+y}{1-xy}=3,\ \frac{x-y}{1+xy}=\frac{1}{3}$

10. $\frac{2}{x-1}+\frac{3}{y-1}=2,\ \frac{48}{x-1}+\frac{32}{y-1}=13$

11. (*i*) $x^2+y^2-3=3xy,\ 2x^2-6+y^2=0$
(*ii*) $2x^2+3xy=26,\ 3y^2+2xy=39$

12. $x^2+xy+y^2=19,\ x^2-xy+y^2=7$

13. $x^2-7xy+12y^2=0,\ x^2+5xy-8y^2=64$

14. $y^2-5xy+6x^2=0,\ x^2+y^2=45$

15. By selling a table for Rs. 56, gain is as much per cent as it costs in rupees. What is the cost price ?

16. A horse and a cow were sold for Rs. 3040 making a profit of 25% on the horse and 10% on the cow. By selling them for Rs. 3070, the profit realised would have been 10% on the horse and 25% on the cow. Find the cost price of each.

17. In a prefect competition, the demand curve of a commodity is $D=20-3p-p^2$ and the supply curve is $S=5p-1$, where p is price, D is demand and S is supply. Find the equilibrium price and the quantity exchanged.

18. If the demand and supply laws are respectively given by the equations

$$4q+9p=48 \text{ and } p=\frac{q}{9}+2$$

Find the equilibrium price and quantity.

19. Demand for goods of an industry is given by the equation $pq=100$, where p is price and q is quantity and supply is given by the equation $20+3p=q$. Find the equilibrium price and quantity.

20. The sum of the pay of two lecturers is Rs. 1600 per month. If the pay of one lecturer be decreased by 9% and the pay of the second be increased by 17%, their pays become equal. Find the pay of each lecturer.

21. The demand and supply equations are $2p^2+q^2=11$ and $p+2q=7$. Find the equilibrium price and quantity, where p stands for price and q for quantity.

22 A commodity is produced by using 3 units of labour and 2 units of capital. The total cost comes to 62. If the commodity is produced by using 4 units of labour and 1 unit of capital, the cost comes to 56.

What is the cost per unit of labour and capita.

23. A man's income from interest and wages is Rs. 500. He doubles his investment and also gets an increase of 50% in wages and his income increases to Rs. 800. What was his original income separately in terms of interest (I) and wages (W) ?

24. If there are two commodities X and Y with prices p_1 and p_2, demand D_1, D_2 and supplies S_1, S_2 and we have the demand and supply schedules.

$$D_1=10-p_1+p_2,\ S_1=6+p_1+2p_2$$
$$D_2=12+2p_1-p_2,\ D_2=19+3p_1+5p_2$$

(*i*) Find the equilibrium prices. (*ii*) Determine the equilibrium quantities exchanged in the market.

25. Solve the following simultaneous equations :

(*i*) $(27)^x=9^y,\ (81)^y=243.3^x$

(*ii*) $4^x . 8^y=128,\ 9^x \div 27^y=3$

(*iii*) $\dfrac{9^x}{3^{x+y}}=27$ and $\dfrac{4^x}{32^y}=1$

(*iv*) $4^x . 2^y=128$ and $3^{3x+2y}=9^{xy}$

ANSWERS

1. (*i*) $x=-1$, $y=-1$ (*ii*) $x=2$, $y=1$ (*iii*) $x=-1$, $y=1$ (*iv*) $x=18$, $y=15$

2 (*i*) $x=-1$, $y=\frac{7}{11}$ (*ii*) $x=1$, $y=1$ (*iii*) $x=4$, $y=-2$

3. (*i*) $x=3, 4$; $y=4, 3$ (*ii*) $x=11, -8$; $y=8, -11$ (*iii*) $x=\frac{4}{5}, \frac{6}{7}$; $y=\frac{3}{5}, \frac{5}{7}$

(*iv*) $x=-1, -4$; $y=1, 2$

4. (*i*) $x=2, 3$; $y=3, 2$ (*ii*) $x=4, 3$; $y=3, 4$

5. (*i*) $x=5, 1$; $y=1, 5$ (*ii*) $x=6, 4$; $y=10, 15$

6. $x=4, 9$; $y=9, 4$

7. (*i*) $x=1, 2$; $y=2, 1$ (*ii*) $x=a, \frac{1}{2}(a+b)$; $y=b, \frac{1}{2}(a+b)$ (*iii*) $x=a$; $y=b$

8. [**Hint.** Put $x^2=u$, $y^2=v$] ; $x=\pm 1, \pm\sqrt{3}$, $y=0, \pm\sqrt{2}$

9. (*i*) $x=1, 2$; $y=2, 1$ (*ii*) $x=+1, -1$; $y=\frac{1}{2}, -2$

10. $x=\frac{-11}{5}$, $y=\frac{15}{7}$

11. (*i*) $x=\pm\sqrt{3}, \pm\sqrt{\frac{3}{19}}$; $y=0, \pm 6\sqrt{\frac{3}{19}}$

(*ii*) $x=\pm 2$, $y=\pm 3$

12. $x=\pm 2, \pm 3$; $y=\pm 3, \pm 2$

13. $x=\pm\frac{16}{\sqrt{7}}, \pm 6$; $y=\pm\frac{4}{\sqrt{7}}, \pm 2$

14. $x=3, -3, \pm\frac{3}{\sqrt{2}}$; $y=6, -6, \pm\frac{9}{\sqrt{2}}$

15. 40

16. Rs. 1200, Rs. 1400

17. $p=-4\pm\sqrt{37}$

18. $p=\frac{8}{3}$, $q=6$

19. $p=\frac{10}{3}$, $q=30$

20. Rs. 900, Rs. 700

21. $p=\frac{5}{9}, 1$; $q=\frac{29}{9}, 3$

22. 10, 16

23. **Hint.** $I+W=500$, $2I+W=800$.

24. **Hint.** $D_1=S_1$ gives $2p_1+p_2=4$ and $D_2=S_2$ gives $p_1+6p_2+7=0$. Solving the two equations, we get the equilibrium prices and substituting these values of p_1, p_2 in D_1 and D_2, we get equilibrium quantities.

25. (*i*) $x=1, y=\frac{3}{2}$, (*ii*) $x=2, y=1$,

(*iii*) $x=5, y=2$, and (*iv*) $x=2, y=3$; $x=\frac{7}{4}, y=\frac{7}{2}$.

8·15. CUBIC AND BIQUADRATIC EQUATIONS

A cubic equation is an equation in which the highest power of the unknown is three. The general form of the cubic equation is

$$ax^3+bx^2+cx+d=0,\ a\neq 0;\ a,b,c,d \in R.$$

A graph of a cubic equation will have two turning points as against one turning point in the case of a quadratic equation discussed earlier. A graphic solution on the same pattern can be found for a cubic equation. The three points at which the curve of cubic equation intersects the x-axis will give the three solutions to the equation as shown in Fig. 11.

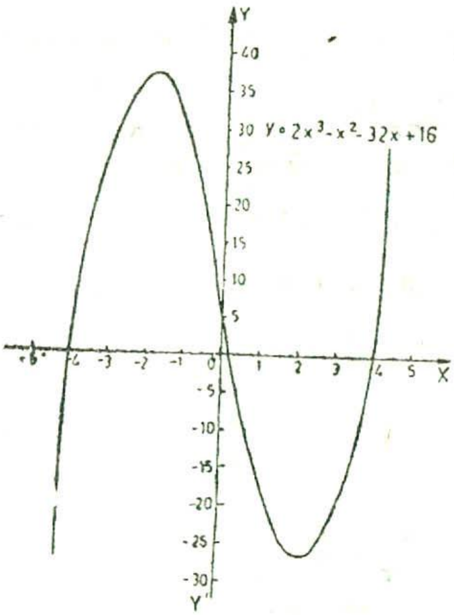


Fig. 11.

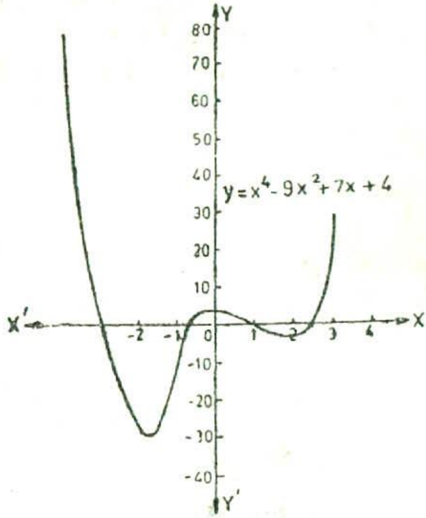


Fig. 12.

On the same analogy a biquadratic or a quartic equation with highest power equal to 4 will have three turning points and 4 solutions and can be found by graphic method in the same manner given in Fig. 12.

We now work out algebraically a cubic equation considering its general form.

$$ax^3+bx^2+cx+d=0 \qquad \ldots(1)$$

On dividing the equation (1) by a, we get

$$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0 \qquad \ldots(2)$$

The equation whose roots are α, β, γ can be written as

$$(x-\alpha)(x-\beta)(x-\gamma)=0$$

$$\Rightarrow \quad x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0 \quad ...(3)$$

Since equations (2) and (3) are identical, by comparing their coefficients, we have

$$S_1 \equiv \text{Sum of the roots}=\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{\text{coeff. of } x^2}{\text{coeff. of } x^3}$$

$S_2 \equiv$ Sum of the product of roots taken two at a time

$$=\alpha\beta+\beta\gamma+\gamma\alpha=+\frac{c}{a}=+\frac{\text{coeff. of } x}{\text{coeff. of } x^3}$$

$$S_3 \equiv \text{Product of the roots}=\alpha\beta\gamma=-\frac{d}{a}=-\frac{\text{constant term}}{\text{coeff. of } x^3}$$

A bi-quadratic equation is an equation in which the highest power of the unknown is four.

The general form of the biquadratic equation is

$$ax^4+bx^3+cx^2+dx+e=0;\ a\neq 0 \text{ and } a, b, c, d, e \in R$$

If α, β, γ and δ are the roots of the above equation, we have

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\gamma\delta=+\frac{c}{a}$$

$$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\beta\gamma\delta=-\frac{d}{a}$$

$$\alpha\beta\gamma\delta=+\frac{e}{a}$$

Thus, we get

$$S_1=\text{Sum of the roots} \quad =-\frac{\text{coeff. of } x^3}{\text{coeff. of } x^4}$$

$S_2=$ Sum of the products of the roots taken two at a time

$$=+\frac{\text{coeff of } x^2}{\text{coeff. of } x^4}$$

$S_3=$ Sum of the products of the roots taken three at a time

$$=-\frac{\text{coeff. of } x}{\text{coeff. of } x^4}$$

$$S_4=\text{Product of the roots}=+\frac{\text{constant term}}{\text{coeff. of } x^4}$$

Example 44. *Find all the roots of the equation*

$$x^3+9x^2-x-9=0$$

Solution. By inspection, we find $x=+1$ is the root.

$\therefore$ $x-1$ is a factor of the equation.

Hence we divide L.H.S. of the equation by $x-1$ as follows :

1	9	-1	-9	(1
	1	10	9	
1	10	9	0	

This division gives quotient$=x^2+10x+9$

$\therefore$ The depressed equation is $x^2+10x+9=0$

Solving this, we get $x=\dfrac{-10\pm\sqrt{100-36}}{2}=-1,\ -9$

Hence the required roots are $-1,\ +1,\ -9$.

Example 45. *Given that -6 is a root of the equation*

$$x^3+2x^2-17x+42=0$$

Solve the equation.

Solution. Since -6 is a root, $x+6$ is a factor of the L.H.S. of the equation. Now we divide the L.H.S. by $x+6$, *viz.*,

1	2	-17	42	$(-6$	...(1)
	-6	24	-42		.. (2)
1	-4	7	0		...(3)

we get quotient$=x^2-4x+7$ ...(4)

$\therefore$ The depressed equation is $x^2-4x+7=0$

(This 'depressed equation' is the one obtained on dividing L.H.S. of equation (*i*) by $x+6$ and gives the other roots of the equation.)

Solving (4), we get

$$x=\frac{4\pm\sqrt{-12}}{2}=2\pm i\sqrt{3}$$

Hence the required roots are $-6,\ 2\pm i\sqrt{3}$.

Explanation. Write down the coefficients of the given expression in line (1). Write nothing (or say, zero) below and add, getting 1. Multiply 1 by -6, write the product (-6) below 2 and add getting -4. Again multiply -4 by -6, write the product (24) below -17 and add getting 7 and so on. In line (3), 1 is the coefficient of the first term of quotient. The power of x in this term is 2 (one less than the degree of the given expression). The succeeding coefficients are -4, 7. The last number, *viz.*, 0 is the remainder.

Example 46. *Solve the equation $9x^3-36x^2+23x+12=0$, it being given that one of its roots is half the sum of the other two.*

Solution. Let the roots be α, β, γ.

$$\alpha+\beta+\gamma=\frac{36}{9}=4 \quad \text{...(1)}$$

But $$\alpha=\frac{\beta+\gamma}{2}\Rightarrow\beta+\gamma=2\alpha \qquad \ldots(2)$$

Substituting (2) in (1), we get

$$3\alpha=4 \quad \Rightarrow \quad \alpha=\frac{4}{3}.$$

$\therefore$ One root is $\frac{4}{3}$

$\therefore$ $(3x-4)$ is a factor of $9x^3-36x^2+23x+12$. Dividing, we obtain the other factor as $3x^2-8x-3$.

$$\therefore \quad (3x-4)(3x^2-8x-3)=0$$

$$\Rightarrow \quad (3x-4)(3x+1)(x-3)=0$$

Hence the roots are $\frac{4}{3}$, $-\frac{1}{3}$ and 3.

Example 47. *Solve the equation*

$$4x^3-24x^2+23x+18=0$$

given that the roots are in arithmetical progression.

Solution. Let the roots be $\alpha-\beta$, α, $\alpha+\beta$

$\therefore$ Sum of the roots $=(\alpha-\beta)+\alpha+(\alpha+\beta)=\frac{24}{4}=6$

$$\Rightarrow \quad \alpha=2$$

Product of the roots $=(\alpha-\beta)\alpha(\alpha+\beta)=-\frac{18}{4}$

$$\Rightarrow \quad \alpha(\alpha^2-\beta^2)=-\frac{18}{4}$$

$$\Rightarrow \quad (4-\beta^2)=-\frac{9}{4}$$

$$\Rightarrow \quad \beta^2=\frac{25}{4}, \text{ i.e., } \beta=\pm\frac{5}{2}$$

Taking $\alpha=2$ and $\beta=+\frac{5}{2}$, we get the required roots as

$$2-\frac{5}{2},\ 2,\ 2+\frac{5}{2}, \text{ i.e., } -\frac{1}{2},\ 2,\ \frac{9}{2}$$

Example 48. *Solve* $8x^3-14x^2+7x-1=0$, *given that its roots are in G.P.*

Solution. Let $\frac{\alpha}{\beta}$, α, $\alpha\beta$ be the roots of the given equation, then

$$\text{Sum of the roots}=\frac{\alpha}{\beta}+\alpha+\alpha\beta=\frac{7}{4}$$

$$\Rightarrow \qquad \alpha\left(\frac{1+\beta+\beta^2}{\beta}\right)=\frac{7}{4} \qquad \ldots(1)$$

Product of the roots $=\frac{\alpha}{\beta}\cdot\alpha\cdot\alpha\beta=\frac{1}{8}$

$$\Rightarrow \qquad \alpha^3=\frac{1}{8},\ i.e.,\ \alpha=\frac{1}{2}$$

Substituting $\alpha=\frac{1}{2}$ in (1), we get

$$\frac{1+\beta+\beta^2}{\beta}=\frac{7}{2}$$

$$\Rightarrow \qquad 1+\beta+\beta^2=\frac{7}{2}\beta$$

$$\Rightarrow \qquad 2\beta^2-5\beta+2=0,\ i.e.,\ (2\beta-1)(\beta-2)=0$$

$$\therefore \qquad \beta=2 \quad \text{or} \quad \frac{1}{2}$$

Hence the roots are $\frac{1}{4}$, $\frac{1}{2}$, 1.

Example 49. *Solve the equation*

$$x^3-5x^2-2x+24=0$$

given that the product of the two roots is 12.

Solution. Let α, β, γ be the roots so that $\beta\gamma=12$ ...(*)

$$S_1' \equiv \alpha+\beta+\gamma=5 \qquad \ldots(1)$$

$$S_3 \equiv \alpha\beta\gamma=-24 \qquad \ldots(2)$$

From (2), we get $\alpha=-2$ ...(3)

Substituting (3) in (1), we get

$$\beta+\gamma=7 \qquad \ldots(4)$$

From (*) and (4), we find that β, γ are the roots of the equation

$$t^2-7t+12=0 \qquad \ldots(5)$$

Solving the above equation, we get

$$\beta, \gamma=4, 3$$

Hence the required roots are -2, 4, 3.

Example 50. *Solve the equation*

$$2x^3-x^2-22x-24=0,$$

two of its roots being in the ratio of 3 : 4.

Solution. Let the roots be 3α, 4α, β

$$S_1 \equiv 3\alpha+4\alpha+\beta=\tfrac{1}{2} \quad \Rightarrow \quad \beta=\tfrac{1}{2}-7\alpha \qquad \ldots(1)$$

$$S_2 \equiv 3\alpha \cdot 4\alpha + 3\alpha \cdot \beta + 4\alpha \cdot \beta = -11$$

$\Rightarrow$ $$12\alpha^2 + 7\alpha\beta = -11 \qquad \ldots(2)$$

$$S_3 \equiv 12\alpha^2\beta = 12$$

Substituting (1) in (2), we get

$$12\alpha^2 + 7\alpha\left(\frac{1}{2} - 7\alpha\right) = -11$$

$\Rightarrow$ $$74\alpha^2 - 7\alpha - 22 = 0$$

$\Rightarrow$ $$\alpha = \frac{7 \pm \sqrt{49 + 6512}}{148} = -\frac{1}{2}, \frac{22}{37}$$

The roots corresponding to $\alpha = \frac{22}{37}$ are discarded as they do not satisfy the condition $\alpha^2\beta = 1$.

Hence the roots are $-\frac{3}{2}$, -2, 4.

Example 51. *Find the condition that the roots of the equation*

$$ax^3 + 3bx^2 + 3cx + d = 0$$

may be (a) in A.P. (b) in G.P.

Solution. (*a*) Let the roots be $\alpha - \beta$, α and $\alpha + \beta$.

$\therefore$ Sum of the roots $= (\alpha - \beta) + \alpha + (\alpha + \beta) = -\frac{3b}{a}$

$\Rightarrow$ $$\alpha = -\frac{b}{a} \qquad \ldots(1)$$

Since α is a root of the given equation, we conclude

$$a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0 \qquad \ldots(2)$$

$\Rightarrow$ $$a\left(-\frac{b}{a}\right)^3 + 3b\left(-\frac{b}{a}\right)^2 + 3c\left(-\frac{b}{a}\right) + d = 0$$

$\Rightarrow$ $$-\frac{b^3}{a^2} + \frac{3b^3}{a^2} - \frac{3bc}{a} + d = 0$$

$\Rightarrow$ $2b^3 - 3abc + a^2d = 0$, is the required condition.

(*b*) Let the roots be $\frac{\alpha}{\beta}$, α and $\alpha\beta$

$\therefore$ Products of the roots $= \frac{\alpha}{\beta} \cdot \alpha \cdot \alpha\beta = -\frac{d}{a}$

$\Rightarrow$ $$\alpha^3 = -\frac{d}{a} \qquad \ldots(3)$$

Substituting the value of α^3 from (3) in (2), we get

$$a\left(-\frac{d}{a}\right) + 3b\alpha^2 + 3c\alpha + d = 0$$

$$\Rightarrow \quad 3b\alpha^2+3c\alpha=0$$

$$\Rightarrow \quad b\alpha=-c$$

$$\Rightarrow \quad b^3\alpha^3=-c^3$$

$$\Rightarrow \quad b^3\left(-\frac{d}{a}\right)=-c^3, \text{ i.e., } b^3d=c^3a$$

is the required condition.

Example 52. *Solve the equation*

$$x^4+2x^3-16x^2-22x+7=0$$

given that one root is $2+\sqrt{3}$.

Solution. Since one root is $2+\sqrt{3}$, the other root will be $2-\sqrt{3}$.

Let the other two roots be α and β.

$\therefore$ Sum of the roots $=(2+\sqrt{3})+(2-\sqrt{3})+\alpha+\beta=-2$

$$\Rightarrow \quad \alpha+\beta=-6 \qquad ...(1)$$

Also product of the roots $=(2+\sqrt{3})(2-\sqrt{3})\alpha\beta=7$

$$\Rightarrow \quad \alpha\beta=7 \qquad ...(2)$$

From (1) and (2), we conclude that α and β are the roots of

$$x^2+6x+7=0$$

$$\Rightarrow \quad x=\frac{-6\pm\sqrt{8}}{2}=-3\pm\sqrt{2}$$

Hence the roots are $-3\pm\sqrt{2}$, $2\pm\sqrt{3}$

Example 53. *Solve the equation*

$$x^4+4x^3+5x^2+2x-2=0$$

when it is given that $-1+i$ *is a root.*

Solution. Since imaginary roots occur in conjugate pairs, $-1-i$ is another root.

Let the remaining two roots be α and β.

$\therefore$ Sum of the roots $=(-1+i)+(-1-i)+\alpha+\beta=-4$

$$\Rightarrow \quad \alpha+\beta=-2$$

Also product of the roots $=(-1+i)(-1-i)\alpha\beta=-2$

$$\Rightarrow \quad \alpha\beta=-1$$

From (1) and (2), we conclude that α and β are the roots of

$$x^2+2x-1=0 \qquad \text{(Since } x^2-Sx+P=0)$$

$$\Rightarrow \quad x=\frac{-2\pm\sqrt{4+4}}{2}=-1\pm\sqrt{2}$$

Hence $-1\pm i$, $-1\pm\sqrt{2}$ are the required four roots of the equation.

Example 54. *The sum of two roots of*

$$x^4-8x^3+19x^2+4\lambda x+2=0$$

is equal to the sum of the other two roots. Find λ *and solve the equation.*

Solution. Let α, β, γ and δ be the required roots.

$$\therefore \quad \alpha+\beta+\gamma+\delta=8$$

$$\Rightarrow \quad \alpha+\beta=\gamma+\delta=4$$

$\therefore$ Quadratic factors corresponding to them are of the form

$$x^2-4x+a \quad \text{and} \quad x^2-4x+b \quad \ldots(*)$$

$$\therefore \quad (x^2-4x+a)(x^2-4x+b)\equiv x^4-8x^3+19x^2+4\lambda x+2$$

Equating coefficients of like powers, we have

$$a+b+16=19 \quad \Rightarrow \quad a+b=3 \quad \ldots(1)$$

$$-4a-4b=4\lambda \quad \Rightarrow \quad a+b=-\lambda \quad \ldots(2)$$

$$ab=2 \quad \ldots(3)$$

From (1) and (2), we get $\lambda=-3$

and from (1) and (3) we get $a=1, b=2$

$\therefore$ Quadratic factors in (*) become x^2-4x+1, x^2-4x+2

Solving the equations $x^2-4x+1=0$ and $x^2-4x+2=0$, we get the required roots as $2\pm\sqrt{3}$, $2\pm\sqrt{2}$

Example 55. *Solve the equation*

$$16x^4-64x^3+56x^2+16x-15=0$$

given that the roots are in arithmetic progression.

Solution. Let the roots be $\alpha-3\beta$, $\alpha-\beta$, $\alpha+\beta$, $\alpha+3\beta$.

$$\therefore S_1\equiv(\alpha-3\beta)+(\alpha-\beta)+(\alpha+\beta)+(\alpha+3\beta)=4$$

$$\Rightarrow \quad \alpha=1$$

$$S_2\equiv \alpha-3\beta)(\alpha-\beta)+(\alpha-3\beta)(\alpha+\beta)+(\alpha-3\beta)(\alpha+3\beta)+(\alpha-\beta)(\alpha+\beta)$$
$$+(\alpha-\beta)(\alpha+3\beta)+(\alpha+\beta)(\alpha+3\beta)=\frac{7}{2}$$

$$\Rightarrow \quad 6\alpha^2-10\beta^2=\frac{7}{2}$$

$$\Rightarrow \quad 10\beta^2=6-\frac{7}{2}=\frac{5}{2}$$

$$\Rightarrow \quad \beta=\pm\frac{1}{2}$$

Substituting the values of α and β, we get the required roots as

$$1-\frac{3}{2}, 1-\frac{1}{2}, +\frac{1}{2}, 1+\frac{3}{2}, \text{ i.e., } -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}.$$

Example 56. *The product of the two roots of the equation*

$$x^4-10x^3+42x^2-82x+65=0,$$

is 13. Solve the equation.

Solution. Let the roots of the equation be α, β, γ and δ. Then

$$S_1\equiv(\alpha+\beta)+(\gamma+\delta)=10 \quad \ldots(1)$$

$$S_2\equiv(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=42 \quad \ldots(2)$$

$$S_3 \equiv \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=82 \quad ...(3)$$

$$S_4 \equiv \alpha\beta\gamma\delta=65 \quad ...(4)$$

Since the product of the two roots is 13, we have

$$\alpha\beta=13 \quad ...(5)$$

From (4) and (5), we have

$$\gamma\delta=5 \quad ..(6)$$

From (2), (5) and (6), we have

$$(\alpha+\beta)(\gamma+\delta)=42-13-5=24 \quad ...(7)$$

From (1) and (7), we find that $\alpha+\beta$, $\gamma+\delta$ are the roots of the equation

$$t^2-10t+24=0$$

Solving the above equation, we have

$$\alpha+\beta=6, \gamma+\delta=4 \quad ...(8)$$

From (5) and (8), we find that the two of the numbers α, β, γ, δ are the roots of the equation

$$y^2-6y+13=0 \quad ..(9)$$

and (6) and (8) gives the equation for the two remaining roots as

$$y^2-4y+5=0 \quad ...(10)$$

Solving (9) and (10), we get the required roots of the given equation as $3\pm 2i$, $2\pm i$.

Example 57 *If α, β, γ are the roots of the equation*

$$x^3-px^2+qx-r=0$$

find the value of

(*a*) $\Sigma\alpha^2$, (*b*) $\Sigma\alpha^2\beta$, (*c*) $\Sigma\alpha^2\beta\gamma$, (*d*) $\Sigma\alpha^3$,

(*e*) $\sum \frac{1}{\alpha^2\beta^2}$, (*f*) $\sum\left(\frac{\beta}{\alpha}+\frac{\alpha}{\beta}\right)$, *and* (*g*) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$.

Solution. Here $\Sigma\alpha=\alpha+\beta+\gamma=p \quad ...(1)$

$$\Sigma\alpha\beta=\alpha\beta+\beta\gamma+\gamma\alpha=q \quad ...(2)$$

$$\alpha\beta\gamma=r \quad ...(3)$$

(*a*) $\Sigma\alpha^2=\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=p^2-2q$

(*b*) $\Sigma\alpha^2\beta=\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\beta$

Now $(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)=\Sigma\alpha^2\beta+3\alpha\beta\gamma$

$\Rightarrow$ $\Sigma\alpha^2\beta=pq-3r$

(*c*) $\Sigma\alpha^2\beta\gamma=\alpha^2\beta\gamma+\beta^2\gamma\alpha+\gamma^2\alpha\beta=\alpha\beta\gamma(\alpha+\beta+\gamma)=pr$

(*d*) $(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2)=\Sigma\alpha^3+\Sigma\alpha^2\beta$

$\Rightarrow$ $\Sigma\alpha^3=(\Sigma\alpha)(\Sigma\alpha^2)-\Sigma\alpha^2\beta$

$$=p(p^2-2q)-(pq-3r)=p^3-pq+3r$$

(*e*) $\sum\frac{1}{\alpha^2\beta^2}=\frac{1}{\alpha^2\beta^2}+\frac{1}{\beta^2\gamma^2}+\frac{1}{\gamma^2\alpha^2}=\frac{\gamma^2+\alpha^2+\beta^2}{\alpha^2\beta^2\gamma^2}=\frac{p^2-2q}{r^2}$

$$(f)\ \sum\left(\frac{\beta}{\gamma}+\frac{\gamma}{\beta}\right)=\sum\frac{\beta^2+\gamma^2}{\beta\gamma}=\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{\alpha\beta^2+\alpha\gamma^2+\beta\gamma^2+\beta\alpha^2+\gamma\alpha^2+\gamma\beta^2}{\alpha\beta\gamma}$$

$$=\frac{\Sigma\alpha^2\beta}{\alpha\beta\gamma}=\frac{pq-3r}{r}=\frac{pq}{r}-3$$

(g) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)=\Sigma\alpha^2\beta+2\alpha\beta\gamma=(pq-3r)+2r=pq-r$

EXERCISE (IV)

1. Find k if 2 is a root of the cubic equation $x^3-(k+1)x+k=0$. Also find the other roots.

2. (a) Solve the equation $x^3-4x^2-3x+18=0$, two of its roots being equal.

(b) Solve the equation $64x^3-104x^2=18x+45=0$, one root being double of the other.

3. Solve the equation $x^3-5x^2-16x+80=0$, sum of two roots being equal to zero.

4. (a) Solve the equation $32x^3-48x^2+22x-3=0$, the roots being in A.P.

(b) Solve the equation $27x^3+42x^2-28x-8=0$, roots are in geometric progression.

5. (a) Solve the equation $x^3-9x^2+14x+24=0$, two of the roots being in the ratio of 3:2. [*Lelhi Univ., B.A. (Hons.) Economics, 1981*]

(b) Solve the eqnation $11x^3+81x^2+121x+60=0$, one root being half the sum of the other two.

(c) Solve $2x^3+x^2-7x-6=0$, given that the difference of two of the roots is 3. [*Delhi Univ. B.A. (Hons.) Economics, 1982*]

6. Solve the equation $x^3-13x^2+15x+189=0$, it being given that one of the roots exceeds another by 2.

7. Solve $4x^4+8x^3+13x^2+2x+3=0$, given that the sum of the two roots is zero.

8. Solve $x^4-2x^3-21x^2+22x+40=0$, roots being in A.P.

9. Solve $6x^4-29x^3+40x^2-7x-12=0$, the product of the two roots being 2.

10. Solve the equation $x^4+2x^3-21x^2-22x+40=0$, the sum of the two roots is equal to the sum of other two.

ANSWERS

1. $k=6$; 2, 1, -3. 2. (a) 3, 3, -2, (b) $\frac{3}{4}$, $\frac{3}{2}$, $-\frac{5}{8}$. 3. 5, 4, 4 − 3
4. (a) $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, (b) $-\frac{2}{9}$, $\frac{2}{3}$, -2. 5. (a) 6, 4, -1 (b) $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{1}{3}$
(c) 2, -1, $-\frac{3}{2}$ 6. -3, 7, 9 7. $\pm i/2$, $-1\pm\sqrt{-2}$
8. -4, -1, 2, 5 9. $\frac{1}{3}$, $\frac{3}{2}$, $1\pm\sqrt{2}$ 10. -5, 4, -2, 1.