9

Permutations and Combinations

STRUCTURE

9·0 INTRODUCTION
9·1 FUNDAMENTAL RULES OF COUNTING
9·2 PERMUTATIONS
9·3 FACTORIAL NOTATIONS
9·4 PERMUTATION OF *n* DIFFERENT THINGS
9·5 CIRCULAR PERMUTATION
9·6 PERMUTATION OF THINGS NOT ALL DIFFERENT
9·7 RESTRICTED PERMUTATIONS
9·8 COMBINATIONS
9·9 COMPLEMENTARY THEOREMS
9·10 RESTRICTED COMBINATIONS
9·11 COMBINATIONS OF THINGS NOT ALL DIFFERENT

OBJECTIVES

After studying this chapter, you should be able to understand :

- permutations, factorial notations, and problems involving permutations
- combinations and problems involving combinations
- difference between permutation and combination.

9·0. INTRODUCTION

Permutations refer to different arrangements of things from a given lot taken one or more at a time whereas combinations refer to different sets or groups made out of a given lot, without repeating an element, taking one or more of them at a time. The distinction will be clear from the following illustration of combinations and permutations made out of a set of three elements $\{a, b, c\}$.

Combinations	*Permutations*
(*i*) one at a time : $\{a\}, \{b\}, \{c\}$	$\{a\}, \{b\}, \{c\}$
(*ii*) two at a time : $\{a, b\}\ \{b, c\}\ \{a, c\}$	$\{a, b\}\ \{b, a\}$
	$\{b, c\}\ \{c, b\}$
	$\{a, c\}\ \{c, a\}$
(*iii*) three at a time : $\{a, b, c\}$	$\{a, b, c\}\ \{a, c, b\}$
	$\{b, c, a\}\ \{b, a, c\}$
	$\{c, a, b\}\ \{c, b, a\}$.

It may be noticed that on the left above, every set has different combination whereas on the right above, there are sets with different arrangements wherever possible of the same group. However no element appears twice in any set, *e.g.*, $\{a, a\}$, $\{b, b\}$, $\{c, c\}$, $\{a, b, b\}$, $\{c, c, c\}$ etc.

9·1. FUNDAMENTAL RULES OF COUNTING

There are two fundamental rules of counting or selection based on the simple principles of multiplication and addition, the former when events occur independently one after another, and the latter when either of the events can occur. At times we have to combine the two, depending on the nature of the problem. We can state the principle as follows :

If one thing can be done in m ways and when it has been done in any of the m ways, a second thing can be done in n ways, then the two things together can be done in $m \times n$ ways.

Proof. Let $a_1, a_2, \ldots, a_m$ be the m ways of doing the first thing and $b_1, b_2, \ldots, b_n$ be the n ways of doing the second thing independently of the first. Then, the two things can be done simultaneously in the following ways :

$$\begin{array}{l} a_1b_1\ ;\ a_1b_2\ ;\ a_1b_3\ ;\ \ldots\ ;\ a_1b_n \\ a_2b_1\ ;\ a_2b_2\ ;\ a_2b_3\ ;\ \ldots\ ;\ a_2b_n \\ \vdots \\ a_mb_1\ ;\ a_mb_2\ ;\ a_mb_3\ ;\ \ldots\ ;\ a_mb_n \end{array}$$

These are $m \times n$ number of ways of selecting both the things simultaneously.

	1	2	3	4	5	6
1	1,1	1,2	1.3	1 4	1 5	1,6
2	2,1	2,2	2.3	2,4	2,5	2,6
3	3,1	3 2	3.3	3,4	3,5	3,6
4	4,1	4,2	4.3	4,4	4,5	4,6
5	5,1	5,2	5.3	5 4	5 5	5,6
6	6 1	6 2	6,3	6,4	6 5	6,6

We can illustrate the idea by the shown on page 300 diagram which indicates how two dice with number 1, 2, 3, 4, 5 and 6 on its six sides can combine in 6^2, *i.e.*, 6×6 or 36 ways.

Therefore, from the fundamental principle of counting, if repetitions are allowed all the N elements taken together can occur in N^N ways. If, however, only r of the N numbers are taken at a time, the possible ways are N^r or 6^2 in the above case. If repetitions are not allowed then the diagonal comprising {1, 1}, {2,2} etc., is avoided and the total choices. are $6 \times 5 = 30$ or $n \times (n-1)$ only.

Example 1. *There are five routes for journey from station A to station B. In how many different ways can a man go from A to B and return, if for returning*

(*i*) *any of the routes is taken,*

(*ii*) *the same route is taken,*

(*iii*) *the same route is not taken.*

Solution. (*i*) The man can go from A to B in 5 different ways, for he may take any one of the five routes. When he has done so in any of the 5 ways, he may return in 5 different ways, *i.e*, there are 5 different ways of returing.

$\therefore$ The total number of different ways are $5 \times 5 = 25$.

(*ii*) In case there is only one way of returning, then the total number of different ways are $5 \times 1 = 5$.

(*iii*) If there are 4 different ways of returning, then the total number of different ways are $5 \times 4 = 20$.

Example 2. *How many telephone connections can be allotted with 5 and 6 digits from the natural numbers 1 to 9 inclusive ?*

Solution. As per the rules of counting, the total number of telephone connections can be

$$N^r = 9^5 = 59{,}049$$
$$= 9^6 = 5{,}31{,}441$$

Example 3. *In how many ways can a chairman and a vice-chairman of a board of 6 members can occupy their seats ?*

Solution. Whoever is chosen first, he would be seated in 6 ways and having seated, the other one can be seated in 5 ways because one person cannot hold both the seats. Therefore, both the chairman and the vice-chairman can be seated in $6 \times 5 = 30$ ways.

Example 4. (*a*) *In how many different ways, 3 rings of a lock can combine when each ring has 10 digits 0 to 9 ? If the lock opens in only one combination of 3 digits how many unsuccessful events are possible ?*

(*b*) *An automobile dealer provides motor cycles and scooters in 2 body patterns and 5 different colours each. Indicate the number of choices open to a customer visiting him.*

Solution. (*a*) Since the lock opens in one of the combination of 3 given digits, the unsuccessful attempts can be $1000-1=999$.

(*b*) With 2 body patterns and 5 different colours, a choice of each of the motor cycle and scooter can be made in $2\times5=10$ ways. Now he has to decide whether to buy a motor cycle or a scooter so that the total number of options becomes

$$2\times5+2\times5=20.$$

Example 5. *Three persons go into a railway carriage, where there are 8 seats. In how many ways can they seat themselves ?*

Solution. Since there are eight vacant seats, the first man can choose any one of these 8 seats. There are thus 8 ways of filling the first seat, when that one is occupied 7 seats are left, therefore, the second man can occupy any one of the 7 seats. The last man can now seat himself in one of the remaining 6 seats.

$\therefore$ Number of ways in which three persons can occupy 8 seats is

$$8\times7\times6=336$$

9·2. PERMUTATIONS

In the rules of counting we have considered the possible choices of r different objects from N different objects or events, with or without repetition.

In permutations we have different arrangements of certain number of objects, say r at a time taken from n different objects without repetition of any given object in any one set more than once. To illustrate there are many permutations of *ABC* but none will be like *ABB* or *AAA*, thus the objects in each set are different and there will be as many sets as are the arrangements possible from a given number of objects For example a bookseller has received three new books *A*, *B*, *C*. He can place them in his showcase in any of the following 6 ways :

$$ABC, ACB, BAC, BCA, CAB, CBA$$

There are thus 6 ways of arranging three distinct objects when each arrangement is of all the 3 objects. No repetition has been allowed in any one arrangement, each element appears only once. Mathematically, we can say that three distinct objects can be arranged in $3.2.1=6$ ways. We can reason out this as follows : "There are three places to be filled, the first can be filled in 3 ways, the second in 2 ways while for the third there is only 1 way. Hence, there are 3.2.1 ways in all.

9·3. KRAMP'S FACTORIAL NOTATION

The product of the first n natural numbers, *viz*., $1, 2, 3, \ldots, n$, is called *factorial n* or *n factorial* and is written as $\lfloor n$ or $n\,!$

Thus $$n\,! = 1\times 2\times 3\times \ldots \times (n-1)\times n$$

From this it follows that

$$\begin{aligned} n\,! &= n\,.\,(n-1)\,! \\ &= n\,.\,(n-1)\,.\,(n-2)\,! \\ &\ \vdots \qquad\qquad \vdots \\ &= n(n-1)(n-2)\ldots(n-r+1)\,\{(n-r)\,!\} \end{aligned}$$

Illustrations 1. *Show that* $\dfrac{10\,!}{8\,!} = 90$

$$\frac{10\,!}{8\,!} = \frac{10\,.\,9\,.\,8\,!}{8\,!} = 90$$

2. *Show that* $\dfrac{(n+1)\,!}{(n-2)\,!} = n^3 - n.$

$$\begin{aligned} \text{L.H.S.} &= \frac{(n+1)\,!}{(n-2)\,!} = \frac{(n+1)n(n-1)\,.\,(n-)2\,!}{(n-2)\,!} \\ &= (n+1)n(n-1) = n^3 - n. \end{aligned}$$

3. *Show that* $30\,! = 2^{15}\,.\,15\,!\,.\,(1\,.\,3\,.\,5\ldots 29)$.

$$\begin{aligned} 30\,! &= 1.2.3.4.5\ldots 29.30 \\ &= (1.3.5\ldots 29)(2.4.6\ldots 30) \\ &= (1.3.5\ldots 29)[(2.1).(2.2).(2.3)\ldots(2.15)] \\ &= (1.3.5\ldots 29)2^{15}(1.2.3\ldots 15) \\ &= (1.3.5\ldots 29)2^{15}.15\,! \end{aligned}$$

9·4 PERMUTATIONS OF n DIFFERENT THINGS

Permutations of n different things taken r at a time, where $r<n$ are $n(n-1)(n-2)\ldots(n-r+1)$.

The number of permutations of n different things taken r at a time is the same as the number of different ways in which r places can be filled up by n things.

The first place can be filled up in n ways, for any one of the n things can be put in it.

$$\therefore \qquad {}^nP_1 = n$$

When the first place has been filled up in any one of the n ways, the second place can be filled up in $(n-1)$ different ways, for any one of the remaining $n-1$ things can be put in it. Since each way of filling up the first place can be associated with each way of filling uo the second place, the first two places can be filled up in $n(n-1)$ ways.

$$\therefore \qquad {}^nP_2 = n(n-1)$$

When the first two places have been filled up in any one of the $n(n-1)$ ways, the third place can be filled up in $(n-2)$ ways.

$\therefore$ The first three places can be filled up in $n(n-1)(n-2)$ ways, *i.e.*,

$$^nP_3 = n(n-1)(n-2)$$

Proceeding in the same way and noticing that the number of factors are same as the number of places to be filled up and that each factor is less than the former by 1, we have the total number of ways in which r places can be filled up.

$$= n(n-1)(n-2)\ldots\text{to } r \text{ factors}$$
$$= n(n-1)(n-2)\ldots(n-\overline{r-1})$$
$$= n(n-1)(n-2)\ldots(n-r+1)$$

i.e., $$^nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

Remarks 1. The number of permutations of n different things taken all at a time is

$$^nP_n = n(n-1)(n-2)\ldots3.2.1 = n\,!$$

2. $$^nP_{n-1} = n(n-1)(n-2)\ldots3.2.1. = {}^nP_n$$

3. $$^nP_r = n(n-1)(n-2)\ldots(n-r+1)$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)\,\{(n-r)\}\,!}{(n-r)\,!}$$
$$= \frac{n\,!}{(n-r)\,!}$$

4. We have $^nP_n = n\,!$

Also $$^nP_n = \frac{n\,!}{(n-n)\,!} = \frac{n\,!}{0\,!}$$

$\Rightarrow$ $$n\,! = \frac{n\,!}{0\,!}$$

$\therefore$ $$0\,! = \frac{n\,!}{n\,!} = 1$$

According to the definition, 0 ! is meaningless. But when used as a symbol, its value is 1.

5. The number of permutations of n different things taken r at a time in which each thing is repeated r times in any arrangement is n^r.

Example 6. *Find how many four-letter words can be formed out of the word LOGARITHMS. (The words may not have any meaning.)*

Solution. There are 10 different letters, therefore, n is equal to 10 and since we have to find four-letter words, r is 4. Hence the required number of words are

$$^{10}P_4 = \frac{10\,!}{(10-4)\,!} = \frac{10\times9\times8\times\ldots\times2\times1}{6\times4\times\ldots\times2\times1}$$
$$= 10\times9\times8\times7 = 5{,}040$$

Example 7. *Indicate how many 4 digit numbers greater than 7,000 can be formed from the digits 3, 5, 7, 8, 9.*

Solution. If the digits are to be greater than 7000, then the first digit can be any one of the 7, 8 and 9.

Now the first digit can be chosen in 3 ways ($\because\ {}^3P_1=3$) and the remaining three digits can be any of the four digits left, which can be chosen in ${}^4P_3$ ways. Therefore, the total number of ways

$$=3\times{}^4P_3=3\times4\times3\times2=72$$

Example 8. *In how many ways can 5 Telugu, 3 English and 3 Tamil books be arranged if the books of each different language are are kept together.*

Solution. The each language book amongst themselves can be arranged in the following ways :

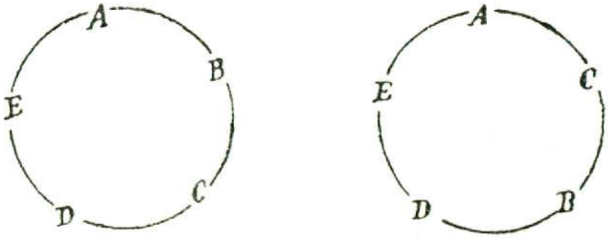
Telugu :	5 books in ${}^5P_5$, *i.e.*, 5 ! ways
English :	3 books in ${}^3P_3$, *i.e.*, 3 ! ways
Tamil :	3 books in ${}^3P_3$, *i.e.*, 3 ! ways

Also arrangement of these groups can be made in ${}^3P_3$ or 3 ! ways, hence by the fundamental theorem, the required arrangements are

$$5\,!\times3\,!\times3\,!\times3\,!=25{,}920$$

9·5 CIRCULAR PERMUTATIONS

These are related with arrangement of objects as in the case of a sitting arrangement of members in a round-table conference. Here the arrangement does not change unless the order changes. Let us consider the following two arrangements of 5 members :



It may be seen that the above two arrangements are the same. But it is not so in the following cases where the order changes :

Therefore, in the circular arrangement, the relative position of the other objects depends on the position of the object placed first, it is only then the arrangement of the remaining objects is made. Therefore, the circular arrangement of n objects will be in $(n-1)$! ways and not n !

ways, when all the objects are considered for the purpose. Thus the circular arrangement of 5 persons will be in 4 ! ways, *i.e.*, 4.3.2.1 = 24 ways. We now make use of the above principle in a slightly complex situation.

Example 9. *In how many ways can 5 boys and 5 girls be seated around a table so that no 2 boys are adjacent.*

Solution. Let the girls be seated first. They can sit in 4 ! ways according to the rule indicated above. Now since the places for the boys in between girls are fixed. the option is there for the boys to occupy the remaining 5 places. There are 5 ! ways for the boys to fill up the 5 places in between 5 girls seated around a table already. Thus, the total number of ways in which both girls and boys can be seated such that no 2 boys are adjacent are $4! \times 5! = 2880$ ways.

Remark. In the circular arrangements, the clockwise and anti-clockwise arrangement do not make any difference because mere turning of a given arrangement will make it otherwise. However, if the neighbourhood of one or more is restricted, the arrangement will get restricted to that extent. It there is a question of arrangement of n different objects in such a way that no two similar things are close to each other then the number of ways will be $\frac{1}{2}(n-1)!$.

For example, if 7 persons are seated around a table so that all of arrangements have not the same neighbours, then the required number of ways will be $\frac{1}{2}(n-1)!$ or $\frac{1}{2}(6.5.4.3.2.1) = 360$.

Example 10. *In how many ways can 4 Indians and 4 Pakistanis be seated at a round table so that no two Indians may be together ?*

Solution. Put one of the Pakistani in a fixed position and then arrange the remaining three Pakistanis in all possible ways. Thus the number of ways in which the four Pakistanis be seated at a round table is 3 !. After they have taken their seats in any one way, there are four seats for the Indians. each between two Pakistanis. Therefore, the Indians can be seated in 4 ! ways corresponding to one way of seating the Pakistani.

$\therefore$ Total number of arrangements is $4! \times 3! = 144$

Example 11. *The chief ministers of 18 States in India meet to discuss the problem of unemployment. In how many ways can they seat themselves at a round table if the Punjab and Bengal chief ministers choose to sit together ?*

Solution. (*a*) Since the chief ministers are to sit at a round table, we shall have to fix the position of one of the chief ministers and then make the other 17 chief ministers take their seats. Since the Punjab and Bengal chief ministers are to sit together, consider them as one. These 16 can now be arranged among themselves in 16 ! ways. Further the Punjab and Bengal chief ministers can be arranged in 2 ! ways.

Hence the required number of ways is $16! \times 2!$

9·6. PERMUTATIONS OF THINGS NOT ALL DIFFERENT

The number of permutations of n things of which p things are of one kind, q things are of a second kind, r things are of a third kind and all the rest are different is given by

$$x=\frac{n\,!}{p\,!\times q\,!\times r\,!}$$

Let the n things be represented by n letters and suppose p number of them are each similar to a, q of them are each similar to b, r of them are each similar to c and the rest all different.

Let x be the required number of permutations. If the p number of letters a be replaced by p new letters, different from each other and different from the rest, then without changing the position of any other letter, they would produce p ! permutations.

$\therefore$ x number of permutations would produce $x\times p$! permutations, *i.e.*, the total number of permutations would become $x\times p$!

Again, if the q number of letters 'b' be replaced by q new letters different from each other and different from the rest, then the total number of permutations would become $x\times p\,!\times q\,!$.

Again, if the r number of letters 'c' be replaced by r new letters differernt from each other and different from the rest, the total number of permutations would become $x\times p\,!\times q\,!\times r\,!$.

But now the n things are all different and the permutations of n different things taken all at a time is n !

$$\therefore \qquad x\times p\,!\times q\,!\times r\,!=n\,!$$

$$\Rightarrow \qquad x=\frac{n\,!}{p\,!\times q\,!\times r\,!}$$

The above principle can easily be generalised.

Example 12. (*a*) *Find the number of permutations of the word ACCOUNTANT.*

(*b*) *Find the number of permutations of letters in the word ENGINEERING.*

Solution. (*a*) The word ACCOUNTANT has 10 letters, of which 2 are *A*s, 2 are *C*s, 2 are *N*s and 2 are *T*s, the rest are different. Therefore, the number of permutations is

$$\frac{10\,!}{2\,!\,2\,!\,2\,!\,2\,!}=\frac{10.9.8.7.6.5.4.3.2.1}{2.2.2.2}$$
$$=2{,}26{,}800$$

(*b*) Since the word ENGINEERING consists of 11 letters, in which there are 3 *Es*, 3 *Ns*, 2 *Gs*, 2 *Is* and one *R*, the total number of permutations is

$$\frac{11\,!}{3\,!\;3\,!\,2\,!\,2\,!}$$

Example 13. *Find the number of arrangements that can be made out of the letters of the word "ASSASSINATION"*

Solution. There are 13 letters in the word of which *A* occurs thrice, *S* occurs four times, *I* occurs twice and *N* occurs twice and the rest are all different. Hence, the required number of arrangements is

$$\frac{13\,!}{3\,!\,4\,!\,2\,!\,2\,!}$$

Example 14. *How many numbers greater than a million can be formed with the digits 4, 5, 5, 0, 4, 5, 3 ?*

Solution. Each number must consist of 7 or more digits. There are 7 digits in all, of which there are 2 fours, 3 fives and the rest different.

$\therefore$ The total numbers are $\dfrac{7\,!}{2\,!\,3\,!}=420$

Of these numbers, some begin with zero and are less than one million which must be rejected.

The numbers beginning with zero are $\dfrac{6\,!}{2\,!\,3\,!}=60$

$\therefore$ The required numbers are $420-60=360$

Example 15. *How many different words can be made out of the letters in the word ALLAHABAD. In how many of these will the vowels occupy the even places ?*

Solution. (*i*) The word 'ALLAHABAD' consists of 9 letter of which *A* is repeated four times, *L* is repeated twice and the rest all different. Hence the required number of words are

$$\frac{9\,!}{4\,!\,2\,!}=7560$$

(*ii*) Since the word ALLAHABAD consists of 9 letters, there are 4 even places which can be filled up by the 4 vowels in 1 way only, since all the vowels are similar. Further, the remaining 5 places can be filled up by the 5 consonants of which two are similar which can be filled in $\dfrac{5\,!}{2\,!}$ ways. Hence the required number of arrangements are $1\times\dfrac{5\,!}{2\,!}=60.$

Example 16. *How many arrangements can be made with the letters of the word MATHEMATICS and in how many of them vowels occurs together ?*

Solution. The word MATHEMATICS consists of 11 letters of which 2 are *As*, 2 *Ms*, 2 *Ts* and the rest all different.

$\therefore$ The total number of arrangements are

$$\frac{11\,!}{2\,!\,2\,!\,2\,!}$$

The word MATHEMATICS consists of 4 vowels A, A, E and I (two are similar). To find the number of arrangements in which the four vowels occur together, consider the four vowels as tied together and forming one letter. Thus we are left with 8 letters of which 2 are Ms, 2 are Ts, 1 is H, 1 is C, 1 is S and the vowels as 1 letter. These letters can be permuted in $\frac{8\,!}{2\,!\,2\,!}$ ways. The 4 vowels which are tied together can again be permuted among themselves in $\frac{4\,!}{2\,!}$ ways (since two of the vowels are similar). Hence the total number of arrangements are

$$\frac{8\,!}{2\,!\times 2\,!}\times\frac{4\,!}{2\,!}=120960$$

Example 17. *In how many ways can the letters of word 'ARRANGE' be arranged? How many of these arrangements are there in which*

(*i*) *the two Rs come together,*

(*ii*) *the two Rs do not come together,*

(*iii*) *the two Rs and the two As come together?*

Solution. The word ARRANGE consists of 7 letters of which two are As, two are Rs and the rest all different. Hence they can be arranged amongst themselves in $\frac{7\,!}{2\,!\,2\,!}=1260$ ways

(*i*) The number of arrangements in which the two Rs come together can be obtained by treating the two Rs as one letter. Thus there are 6 letters of which two (the two As) are similar and so the total number of arrangements$=\frac{6\,!}{2\,!}=360$.

(*ii*) The number of arrangements in which the two Rs do not come together can be obtained by subtracting from the total number of arrangements, the arrangements in which the two Rs come together. Thus the required number is $1260-360=900$.

(*iii*) The number of arrangements in which the two Rs and the two As come together can be obtained by treating the two Rs and the two As as a single letter. Thus there are 5 letters which are all different and so the number of arrangements is $5\,!=120$.

9·7. RESTRICTED PERMUTATIONS

(*i*) *The number of permutations of n different things taken r at a time in which p particular things do not occur is* ${}^{n-p}P_r$.

Keep aside the p particular things and fill up the r places with the remaining $n-p$ things at our disposal. The number of such ways is

$${}^{n-p}P_r$$

(*ii*) *The number of permutations of n different things taken r at a time in which p particular things are present is* ${}^{n-p}P_{r-p}\times{}^{r}P_r$.

Keep aside the p particular things and form the permutations of the remaining $n-p$ things taken $r-p$ at a time. The number of such permutations ${}^{n-p}P_{r-p}$.

In each of these permutations introduce the p particular things taken aside, one by one.

The first thing can be introduced in $r-p+1$ ways. The second thing can be introduced in $r-p+2$ ways and the pth thing in $r-p+p$ or r ways.

$\therefore$ The p things can be introduced in each permutation in $(r-p+1)(r-p+2)\ldots r$ ways which is clearly equal to ${}^{r}P_{p}$.

$\therefore$ The required number of ways are ${}^{n-r}P_{r-p}\times {}^{r}P_{p}$.

Example 18. *If* ${}^{n}P_{4}=12\ .\ {}^{n}P_{2}$, *find* n.

Solution. $${}^{n}P_{4}=\frac{n\,!}{(n-4)\,!} \text{ and } {}^{n}P_{2}=\frac{n\,!}{(n-2)\,!}$$

Now $$\frac{n\,!}{(n-4)\,!}=12\,.\,\frac{n\,!}{(n-2)\,!}$$

$\Rightarrow \quad 12(n-4)\,!=(n-2)\,!$

$\Rightarrow \quad 12(n-4)\,!=(n-2)(n-3)\,\{(n-4)\,!\}$

$\Rightarrow \quad 12=n^2-5n+6$

$\Rightarrow \quad n^2-5n-6=0$

$\Rightarrow \quad (n-6)(n+1)=0$

$\Rightarrow \quad n=6 \text{ or } n=-1$

Since n is positive integer, we reject the second value of n. Thus $n=6$.

Example 19. *Find the value of n if four times the number of permutations of n things taken 3 together is equal to 5 times the number of permutations of $(n-1)$ things taken 3 together.*

Solution. We are given that

$$4\times {}^{n}P_{3}=5\times {}^{n-1}P_{3}$$

$\Rightarrow \quad 4\times n(n-1)(n-2)=5\times (n-1)(n-2)(n-3)$

Dividing throughout by $(n-1)(n-2)$, we get

$4n=5(n-3)$, *i.e.*, $4n=5n-15$

$\Rightarrow \quad -n=-15$

$\therefore \quad n=15$

Example 20. *Prove that*

$${}^{n}P_{r}=n\times {}^{n-1}P_{r-1}$$

Solution. $\text{R.H.S.}=n\times {}^{n-1}P_{r-1}$

$$=n\times\frac{(n-1)\,!}{\{(n-1)-(r-1)\}\,!}=n\times\frac{(n-1)\,!}{(n-r)\,!}$$

$$=\frac{n\,!}{(n-r)\,!}={}^{n}P_{r}$$

Example 21. *Find the numbers less than 1000 and divisible by 5 which can be formed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 such that each digit does not occur more than once in each number.*

Solution. The required numbers may be of one digit, two digits or three digits and each of them must end in 5 or 0, except the number of one digit which must end with 5.

The number of one digit ending in 5 is 1

The number of two digits ending in 5 is ${}^9P_1-1$

($\because$ the number having 0 as the first figure is to be rejected)

The number of two digits ending in 0 is ${}^9P_1$

The number of three digits ending in 5 is ${}^9P_2-{}^8P_1$

($\because$ the numbers having 0 as the first figure are ${}^8P_1$)

The number of three digits ending in 0 is ${}^9P_2$

Hence the total number of required numbers is

$$=1+({}^9P_1-1)+{}^9P_1+({}^9P_2-{}^8P_1)+{}^9P_2$$
$$=154$$

Example 22. (*a*) *In how many different ways can 8 examination papers be arranged in a line so that the best and worst papers are never together ?*

Solution. The total number of arrangements that can be made of 8 papers is 8 ! . Now let the best and the worst papers be taken together. These taken as one and the remaining 6 can be arranged amongst themselves in 7 ! ways. In each of these arrangements the best and the worst papers can be arranged in 2 ! ways.

$\therefore$ The total number of arrangements in which the best and the worst papers can come together are $7!\times 2!$.

$\therefore$ The number of arrangements in which the two particular papers are not together are

$$8!-2\times 7!=40{,}320-10{,}080=30{,}240$$

(*b*) *Six papers are set in an examination, of which two are 'Statistics'. In how many different orders can the papers be arranged so that the two statistics papers are not together ?* [*I.C.W.A., December 1990*]

Solution. Number of ways in which six papers can be arranged

$$=6!$$

If two statistics papers are to be kept together then the six papers can be arranged in $5!\times 2!$ ways.

Hence the number of arrangements in which six papers can be arranged so that the two statistics papers are not together

$$=6!-5!\times 2!=5!\times 4=480.$$

Example 23. *There are 5 boys and 3 girls. In how many ways can they stand in a row so that no two girls are together ?*

Solution. Since no two girls are to be together, each girl must be placed between two boys. Place the 5 boys thus

$$\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times$$

In order that no two of the girls be together, they can only be placed in the places marked as $\times$. There are 6 such places and so the 3 girls can be placed in ${}^6P_3$ ways. Further, the 5 boys can be arranged among themselves in 5 ! ways.

Since, for each way of placing girls there are 5 ! ways of placing the boys, the total number of arragements

$$= {}^6P_3 \times 5\,! = \frac{6\,!}{3\,!} \times 5\,! = 14{,}400$$

Example 24. *How many numbers of six digits can be formed from the digit 4, 5, 6, 7, 8, 9; no digit being repeated. How many of them are not divisible by 5 ?*

Solution. The six digits being different, they can be arranged among themselves in 6 ! ways, all the digits being taken at a time.

Let us find the digits divisible by 5, such digits can be obtained when 5 occurs in the unit place. The position of 5 being fixed, the remaining 5 digits can be arranged among themselves in 5 ! ways. So the numbers divisible by 5 are 5 !

Hence the numbers not divisible by 5 are $6\,! - 5\,! = 600$

Example 25. *In how many ways can 3 boys and 5 girls be arranged in a row so that all the 3 boys are together.*

Solution. Consider the 3 boys as one unit. Now there are 6 units and they can be arranged among themselves in $\underline{6}$ ways. In each of such arrangements, the 3 boys can be arranged among themselves in $\underline{|\,3}$ ways.

$\therefore$ Total number of arrangements in which the boys are together $= \underline{|\,6} \times \underline{|\,3} = 720 \times 6 = 4320$

Example 26. *In how many ways can the letters of the word FAILURE be arranged so that the consonants may occupy only odd positions ?*

Solution. There are 7 letters of which 3 (*ELR*) are consonants and 4 (*AIUE*) are vowels.

The 4 positions to be filled up with consonants are indicated below :

1	2	3	4	5	6	7
(*F*)		(*L*)		(*R*)		( )

The 3 consonants can be placed only in the 3 out of the 4 positions marked 1,3,5,7 The total number of ways in which this can be done is

$${}^4P_3 = 4.3.2 = 24$$

One such arrangement is shown on page 312. For this arrangement the 4 vowels can occupy the four remaining positions such as 2, 4, 6, 7, *i.e.*, positions not occupied by consonants is 4 ! or 24 ways.

$\therefore$ Total number of arrangements are $24\times 24=576$.

Example 27. *In how many ways can the letters of the word STRANGE be arranged so that*

(*i*) *the vowels are never separated,*

(*ii*) *the vowels never come together, and*

(*iii*) *the vowels occupy only the odd places.*

Solution. (*i*) There are 7 letters. Since the vowels are not to be separated we may regard them as forming one letter so there are six letters S, T, R, N, G and AE. They can be arranged among themselves in 6 ! ways. The two vowels can again be arranged in 2 ! ways.

$\therefore$ The total number of arrangements $=6\,!\times 2\,!=1440$

(*ii*) The number of arrangements in which the vowels do not come together can be obtained by subtracting from the total number of arrangements, the number of arrangements in which the vowels come together.

Since the total number of arrangements is 7 ! and the number of arrangements in which the vowels come together is $6\,!\times 2\,!$. Therefore, the number of arrangements in which the vowels do not come together

$$=7\,!-6\,!\times 2\,!=6\,!\times 5=3600.$$

(*iii*) Since the number of letters in the word STRANGE are 7, the total number of places are 7, and the number of odd places are 4(1, 3, 5, 7). The two vowels A and E are to occupy two of these four odd places which they can occupy in ${}^4P_2$ ways.

When the vowels have been placed in one way, there remain five places to be filled up by the remaining 5 consonants which can be done in ${}^5P_5$ ways. Now each of the ${}^4P_2$ ways of arranging the vowels can be associated with each of the ${}^5P_5$ ways of arranging the consonants.

$\therefore$ The total number of arrangements $={}^4P_2\times {}^5P_5=12\times 120=1440$.

Example 28. *A number of four different digits is formed by using the digits 1, 2, 3, 4, 5, 6, 7 in all possible ways. Find* (*i*) *how many such numbers can be formed, and* (*ii*) *how many of them are greater than 3400* ?

Solution. (*i*) With the seven digits 1, 2, 3, 4, 5, 6, 7, the numbers of four different digits that can be formed are ${}^7P_4=7\times 6\times 5\times 4=840$.

(*ii*) Now we want to find the numbers out of 840 that are greater than 3400.

If the number is greater than 3400, then the first left hand digit in the four digit number cannot be 1 or 2.

The left hand digit must, therefore, be either 3, 4, 5, 6 or 7. If the first left hand digit is 3, then the second left hand digit can be filled in 4 ways, *i.e.*, with 4, 5, 6 or 7 and the third digit can be filled in 5 ways, *i.e.*, with the numbers except those chosen for the first two digits and the fourth digit can be filled in 4 ways, *i.e.*, with the digits except those used for the first three digits. Thus the numbers starting with 3 and greater than 3400 are

$$4\times5\times4=80$$

Now if the first left hand digit is filled in 4 ways, *i.e.*, with either 4, 5, 6 or 7 then the second digit can be filled in 6 ways, *i.e.*, with any of the given digits except the one used for the first digit, the third similarly can be filled in 5 and the fourth digit can be filled in 4 ways. Thus the total four digit numbers greater than 3400 are

$$4\times6\times5\times4=480$$

Hence the required four digit numbers are $80+480=560$.

Example 29. *The letters of the word ZENITH are written in all possible orders. How many words are possible if all these words are written out as in a dictionary ? What is the rank of the word ZENITH ?*

Solution. The total number of possible words will be $^6P_6=720$ since there are 6 different alphabets.

The number of words beginning with E will be $^5P_5=120$

Thus, the total number of words beginning with E, N, I, T and H will be

$$=5\times{}^5P_5=600.$$

The words beginning with Z will have their rank between 600 and 720. Of these 120 words, the number of words with E in the second place, can be

$$120/5=24$$

Thus, the rank of the words beginning with ZE will be 601 to 624.

Now, taking into account three letters we have the following rank orders :

ZEH	601 – 606
ZEI	607 – 612
ZEN	613 – 618
ZET	619 – 624

The total words in the dictionary beginning with *ZEN* are :

ZENHIT	613
ZENHTI	614
ZENIHT	615
ZENITH	616
ZENTHI	617
ZENTIH	618

The required rank is, therefore, 616.

Example 30. *If the letters of the word "WOMAN" be permutated and the words so formed be arranged as in a dictionary, what will be the rank of the word "woman"?* (*C.A. Entrance December, 1983*)

Solution. The total number of possible words will be ${}^5P_5=120$ since there are 5 alphabets.

The number of words beginning with A will be $={}^4P_4=24$. Thus the number of words beginning with A, M, N and O are $4\times 24=96$. The words beginning with W will have their ranks from 97 to 120.

The words beginning with W and having A, M and N in the second place are $3\times {}^3P_3=18$.

$\therefore$ The words beginning with W, O and A will be ${}^2P_2=2$.

$\therefore$ The words beginning with W and O will have their ranks from $97+18=115$ onwards.

$\therefore$ The words beginning with W, O and M will have their ranks from $115+2=117$ onwards.

$\therefore$ We have

WOMAN	117
WOMNA	118

Hence the rank of the word "woman" is 117.

EXERCISE (I)

1. (*a*) Evaluate (*i*) ${}^{15}P_3$, (*ii*) ${}^4P_3$, (*iii*) ${}^6P_6$, (*iv*) ${}^9P_5+5$, (*v*) ${}^9P_4$.

(*b*) Find n if (*i*) ${}^{n-1}P_3 : {}^{n+1}P_3=5:12$, (*ii*) ${}^{n+3}P_6 : {}^{n+2}P_4=14:1$

(*c*) Find r if ${}^7P_r=60.{}^7P_{r-3}$.

2, There are four routes for going from A to B and five routes for going from B to C. In how many different ways can a man go from A to C via B.

3. There are 8 vacant chairs in a room. In how many ways can 5 persons take their seats?

4. There are 50 stations on a railway line. How many different kinds of single first class tickets must be printed so as to enable a passenger to go from one station to another?

5. How many different numbers of six digits can be formed with the digits 3, 1, 7, 0, 9, 5, ? How many of these have 0 in ten's place?

6. (*a*) How many different words can be formed with the letters of the word SUNDAY? How many of the words begin with N? How many begin with N and end in Y?

(*b*) How many different arrangements can be made by using all the letters of the word (*i*) MONDAY; (*ii*) ORIENTAL? How many of these arrangements begin with A and end with N?

(*c*) In how many ways can a consonant and a vowel be chosen out of the letters of each of the words (*i*) LOGARITHM, (*ii*) EQUATION?

7. How many different words containing all the letters of the word TRIANGLE can be formed ? How many of them

(*i*) begin with T, (*ii*) begin with E, (*iii*) begin with T and end with E ?, (*iv*) have T and E in the end places, (*v*) when consonants are never together, (*vi*) when no two vowels are together, (*vii*) when consonants and vowels are both always together, (*viii*) vowels occupy odd places ? (*ix*) the relative positions of the vowels and consonants remain unaltered ? (*x*) vowels occupy the second, third and fourth places ?

[**Hint.** $8\,!$, (*i*) $7\,!$, (*ii*) $7\,!$, (*iii*) $6\,!$, (*iv*) $2\,! \times 6\,!$,

(*v*) $8\,! - 4\,! \times 5\,!$, (*vi*) ${}^6P_3 \times 5\,!$, (*vii*) $2\,! \times 5\,! \times 3\,!$,

(*viii*) ${}^4P_3 \times 5\,!$, (*ix*) $5\,! \times 3\,!$.]

8. Find how many words can be formed of the letters of the word 'FAILURE', the four vowels always coming together.

9. In how many ways 10 examination papers be arranged so that the best and worst papers never come together.

10. Find the number of ways in which n books can be arranged on a shelf so that two particular books are not together.

11. (*a*) In how many ways can 3 books on Commercial Mathematics and 5 books on Secretarial Practice be placed on a shelf so that books on the same subject always remain together ? (no two books are identical).

(*b*) Six papers are set in an examination, of which two are mathematical. In how many different orders can the papers be arranged so that (*i*) the two mathematical papers are together and (*ii*) the two mathematical papers are not consecutive ?

12. (*a*) Find the number of permutations of the letters of the word 'SIGNAL' such that the vowels may occupy only odd positions.

(*b*) In how many ways can the letters of the word VIOLENT be arranged so that the vowels I, O, E. occupy even places only.

13. (*a*) How many numbers between 1000 and 10,000 can be formed with the digits 1. 2, 3, 4, 5, 6, 7, 8, 9 ? How many of them are odd ?

(*b*) How many numbers between 3000 and 4000 can be formed with the digits 1, 2, 3, 4, 5, 6 ?

14. The figures 1, 2, 3, 4, 5 are written in every possible order.

How many of the numbers so formed will be greater than 23000 ?

15. A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of three books and single copies of 8 books. In how many ways can all the books be arranged ?

16. (*a*) How many permutations can be made out of the letters of the following words taken all together ?

(*i*) PERMUTATION, (*ii*) EXAMINATION,
(*iii*) MISSISSIPPI, (*iv*) COLLEGE.

(*b*) How many permutations can be made out of the letters of the word INDEPENDENCE ? In how many of them the vowels occur together ?

17. How many numbers greater than a million can be formed with the digits, 1, 7, 1, 0, 7, 3, 7 ?

18. (*a*) Find the number of all possible different words into which the word 'INTERFERENCE' can be converted by change of place of the letters, it being given that no two consonants are to be together.

(*b*) How many different words can be formed with the letters of HARYANA ? In how many of these

(*i*) H and N are together,

(*ii*) begin with H and end with N ?

19. A telegraph post has five arms and each arm is capable of 4 distinct positions including the positions of rest. What is the total number of signals that can be made ?

20. In how many ways can 10 letters be posted in 5 letter boxes ?

21. (*a*) In how many ways can 8 different beads be strung on a necklace ?

(*b*) In how many ways can 8 boys form a ring ?

22. (*a*) In how many ways 6 men can sit at a round table so that all shall not have the same neighbours in any two arrangements ?

(*b*) In how many ways can 7 Indians and 6 Pakistanis sit down a round table so that no two Indians are together ?

23. In how many ways can 4 men and 3 ladies be arranged at a round table if the 3 ladies (*i*) never sit together, (*ii*) always sit together ?

24. A guard of 15 men is formed from a group of m soldiers in all possible ways. Find

(*i*) the number of times three particular soldiers A, B and C are together on guard, and

(*ii*) the number of times two particular soldiers D and E are together on guard.

Also find m if it is found that D and E are three times as often together on guard as A, B and C are.

25. A family consisting of an old man, 6 adults and 4 children, is to be seated in a row for dinner. The children wish to occupy the two seats at each end and the old man refuses to have a child on either side of him. In how many ways can the seating arrangement be made for the dinner ?

26. Seven persons sit in a row. Find the total number of seating arrangements, if

(*i*) three persons A, B, C sit together in a particular order (*ii*) A, B, C sit together (in any order)

(*iii*) B and C occupy the end seats

(*iv*) C always occupies the middle seat.

27. If all the permutations of the letters of the word CHALK be written down as in a dictionary, what is the rank of this word ?

28. There are six students of whom 2 are Indians, 2 Americans and the remaining 2 are Russians. They have to stand in a line so that the two Indians are together, the two Americans are together and so also the two Russians. Show that there are 48 different ways of arranging the students.

ANSWERS

1. (*b*) (*i*) $n=8$, (*ii*) $n=4$, (*c*) $r=5$, **2.** 20, **3.** 6720, **4.** 2450, **5.** 600, 120, **6.** (*a*) 720, 120, 24, (*b*) (*i*) 720, (*ii*) 40, 320 and (*i*) 24, (*ii*) 720, (*c*) (*i*) 18, (*ii*) 15, **8.** 576, **9.** 2903040, **10.** $(n-2)\{(n-1)!\}$, **11.** (*a*) 1440, (*b*) 240, 480, **12.** (*a*) 144, (*b*) 144, **13.** (*a*) 3024, (*b*) 60, **14.** 78, **15.** $(39)!/5! \times (4!)^2 (6!)^3$. **16.** (*a*) (*i*) $\frac{1}{2}(11)!$ (*ii*) 4989600, (*iii*) 34650, (*iv*) 1260, (*v*) 1663200, 16800, **17.** 360, **18.** (*a*) 240, (*b*) 840, 240, 20, **19.** 1023, **20.** 5^{10}, **21.** (*a*) 2520, (*b*) 5040. **22.** (*a*) 60, (*b*) $(7!)^2$, **23.** (*i*) 4320, (*ii*) 720, **25.** $4! \times 5! \times 7!$, **26.** (*i*) 120, (*ii*) 720, (*iii*) 240, (*iv*) 720,

27. $(4!+3!+1!+1)^{th}$, *i.e.*, 32nd.

9·8 COMBINATIONS

In permutations the objects are based on the order of the arrangements where each change in order constitutes a different arrangement. But, if order is not of any consequence then it is a problem of combination. Combinations, therefore, are the groups which can be made by taking some or all of things at a time.

Thus in combination order does not matter, it is simply the identity of items in the selection that matters. A change in any one item will constitute a new combination. For example, the number of different ways in which 6 people can be arranged in a queue is a question of permutations where order matters. The number of different ways in which 6 people can sit in a committee of 3 is a question of combinations where order does not matter, but the constituents of each selection do matter. However, repetition in each combination is not allowed except otherwise stated. Thus combination of 3 objects taken all at a time is only one but permutations of 3 objects taken all at a time are 3 ! or 6 as was, in the previous example for arrangements of three books.

The mathematical formula for finding out combination requires a slight modification in the formula used for permutations. This is as follows :

For permutation, $\quad {}^nP_r = \dfrac{n!}{(n-r)!}$

For combination, ${}^nC_r$ or nC_r or $C(n, r)$ or $\dbinom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$

or $$\frac{{}^nP_r}{r\,!}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}$$

For example, a manager of a shop of readymade garments wants to display 4 combinations out of the total 6 colours of ladies garments received in his store, he can display in the following ways :

$$ {}^6C_4 \text{ or } \binom{6}{4}=\frac{6\,!}{(6-4)\,!\,4\,!}$$
$$=\frac{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}{(2\,.\,1)\,.\,(4\,.\,3\,.\,2\,.\,1)}=15$$

Theorem. *The number of combinations of n different things taken r at a time are given by*

$$ {}^nC_r=\frac{n\,!}{r\,!\,(n-r)\,!},\ \textit{where}\ (r\leqslant n)$$

Proof. Let ${}^nC_r$ denote the required number of combinations of n different things taken r at a time.

Each of these combinations has r different things.

∴ If the r different things be arranged among themselves in all possible ways, each combination would produce $r\,!$ permutations.

∴ ${}^nC_r$ combinations would produce ${}^nC_r\times r\,!$ permutations. But this number is clearly equal to the number of permutations of n different things taken r at a time.

Hence $${}^nC_r\times r\,!={}^nP_r$$

$$\Rightarrow\quad {}^nC_r=\frac{{}^nP_r}{r\,!}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}$$
$$=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!}\cdot\frac{(n-r)\,!}{(n-r)\,!}=\frac{n\,!}{r\,!\,(n-r)\,!}$$

Example 31. *In an examination in paper on Advanced Accounts, 10 questions are set. In how many different ways can an examinee choose 7 questions.*

Solution. The number of different choices is evidently equal to the number of ways in which 7 places can be filled up by 10 different things.

∴ the required number of ways

$$={}^{10}C_7={}^{10}C_3=\frac{10\times9\times8}{1\times2\times3}=120.$$

Example 32. *In how many ways can 4 white and 3 black balls be selected from a box containing 20 white and 15 black balls.*

Solution. This problem involves merely selection and hence, is a problem of combinations. 4 out of 20 white balls can be selected in ${}^{20}C_4$, *i.e.*, $\frac{20\times19\times18\times17}{4\times3\times2\times1}=4845$ ways. Call this process as the first process. 3 out of 15 black balls can be selected in ${}^{15}C_3$, *i.e.*, $\frac{15\times14\times13}{3\times2\times1}=455$ ways. Call this process as the second process. The two processes can be carried out together in $4845\times455=2,204,475$ ways.

Example 33. *From 6 boys and 4 girls, 5 are to be selected for admission for a particular course. In how many ways can this be done if there must be exactly 2 girls ?*

Solution. Since there has to be exactly 2 girls, there should be 3 boys, the possible combinations would, therefore be

$$ {}^{4}C_2\times{}^{6}C_3=\frac{4\times3}{2\times1}\times\frac{6\times5\times4}{3\times2\times1}=120 \text{ ways}$$

Example 34. *In a mercantile firm 4 posts fall vacant and 35 candidates apply for the posts. In how many ways can a selection be made,*

(*i*) *if one particular candidate is always included,*

(*ii*) *if one particular candidate is always excluded ?*

Solution. (*i*) Since a particular person is always to be selected, we must select the remaining 3 candidates out of the remaining 34.

$\therefore$ The required number of selections

$$={}^{34}C_3=\frac{34\times33\times32}{1\times2\times3}=5984$$

(*ii*) Since a particular person is always to be excluded, the choice is restricted to 4 candidates out of the remaining 34.

$\therefore$ The required number of selections

$$={}^{24}C_4=\frac{34\times33\times32\times31}{1\times2\times3\times4}=46,376$$

Example 35. *A father takes 8 children, three at a time to the Zoo, as often as he can without taking the same three together more than once, (i) how often will each child go ? (ii) how often will he go ?*

Solution. (*i*) A particular child goes as often as that child can be included in the combinations of 8 children taken 3 at a time.

Let us select one child first of all, then we have to select only 2 from the remaining 7. This can be done in ${}^{7}C_2$ ways.

$\therefore$ The number of times the particular child will go is

$${}^{7}C_2=\frac{7.6}{1.2}=21$$

(*ii*) The father goes as often as the combinations of 8 children taken 3 at a time.

$\therefore$ The number of times the father will go is

$${}^{8}C_3=\frac{8.7.6}{1.2.3}=56$$

Example 36. *At an election there are five candidates out of whom three are to be elected, and a voter is entitled to vote for any number of candidates not greater than the number to be elected. In how many ways may a voter choose to vote.*

Solution. The voter may vote for one, two or three candidates out of the 5 candidates. He can choose to vote for one candidate in ${}^5C_1$ ways, two candidates in ${}^5C_2$ ways and 3 candidates in ${}^5C_3$ ways. Hence the total number of ways in which the voter can choose to vote is

$${}^5C_1+{}^5C_2+{}^5C_3=5+10+10=25$$

Example 37. *The question paper of 'Cost Accounting and Income Tax' contains ten questions divided into two groups of five questions each. In how many ways can an examinee answer six questions taking at least two questions from each group ?*

Solution. The questions may be answered in the following ways :

(I) 2 questions from 1st group	+4 questions from 2nd group
(II) 3 ,, ,, ,,	+3 ,, ,, ,,
(III) 4 ,, ,, ,,	+2 ,, ,, ,,

(*i*) Two questions can be chosen from 1st group in ${}^5C_2$ ways and 4 questions from 2nd group in ${}^5C_4$ ways. Since each way of selecting questions from the 1st group can be associated with each way of selecting questions from the 2nd group, the total number of ways of selecting questions from both the groups is ${}^5C_2\times{}^5C_4$.

(*ii*) In like manner, 3 questions from the 1st group and 3 questions from the 2nd group may be selected in ${}^5C_3\times{}^5C_3$ ways.

(*iii*) Again, 4 questions from the first group and 2 questions from the second group may be selected in ${}^5C_4\times{}^5C_2$ ways.

∴ The total number of ways

$$={}^5C_2\times{}^5C_4+{}^5C_3\times{}^5C_3+{}^5C_4\times{}^5C_2$$
$$={}^5C_2\times{}^5C_1+{}^5C_2\times{}^5C_2+{}^5C_1\times{}^5C_2$$
$$=\frac{5\times4}{1\times2}\times5+\frac{5\times4}{1\times2}\times\frac{5\times4}{1\times2}+5\times\frac{5\times4}{1\times2}$$
$$=10\times5+10\times10+5\times10=200$$

Example 38. *For an examination, a candidate has to select 7 subjects from three different groups A, B and C. The three groups A, B, C contain 4, 5, 6 subjects respectively. In how many different ways can a candidate make his selection if he has to select at least 2 subjects from each group.*

Solution. The different ways in which a candidate can make a selection of 7 subjects so as to have at least 2 from each group can be as follows :

(*I*) 2 from *A*, 2 from *B* and 3 from *C*,

(*II*) 2 from *A*, 3 from *B* and 2 from *C*,

(*III*) 3 from *A*, 2 from *B* and 2 from *C*.

Now the selection of (I) can be done in ${}^4C_2 \times {}^5C_2 \times {}^6C_3$

$$=6 \times 10 \times 20=1200 \text{ ways}$$

Again the selection (II) can be done in ${}^4C_2 \times {}^5C_3 \times {}^6C_2$

$$=6 \times 10 \times 15=900 \text{ ways.}$$

Also the selection (III) can be done in ${}^4C_3 \times {}^5C_2 \times {}^6C_2$

$$=4 \times 10 \times 15=600 \text{ ways.}$$

$\therefore$ The required number of selections are $1200+900+600=2700$

Example 39. *Out of 10 consonants and 4 vowels, how many words can be formed each containing 6 consonants and 3 vowels ?*

Solution. 6 consonants can be chosen out of 10 in ${}^{10}C_6$ ways and 3 vowels can be chosen out of 4 in ${}^4C_3$ ways.

$\therefore$ Combining each way of selecting the consonants with each way of selecting the vowels, the number of selections having 6 consonants and 3 vowels$={}^{10}C_6 \times {}^4C_3$. Each of these selections contains 9 letters which can be arranged among themselves in 9 ! ways.

$\therefore$ The total number of words$={}^{10}C_6 \times {}^4C_3 \times 9\,!$

$$={}^{10}C_4 \times {}^4C_1 \times 9\,!$$

$$=\frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} \times 4 \times 362880=304{,}819{,}200$$

Example 40. *A boat's crew consist of 8 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.*

Solution. 4 men must row on each side. But on one side there are already three men and on the other side two. So one must be placed on the side of three men and two on the other side. This can be done in ${}^3C_1$ or 3 ways.

Again, 4 men on each side can be arranged among themselves in 4 ! ways. Hence the required number of ways

$$={}^3C_1 \times 4\,! \times 4\,!=3 \times 24 \times 24=1728$$

Example 41. *A party of 6 is to be formed from 10 boys and 7 girls so as to include 3 boys and 3 girls. In how many different ways can the party be formed if two particular girls refuse to join the same party ?*

Solution. If the two particular girls do not refuse to join the same party, then we can select 3 girls from 7 in ${}^7C_3$ ways and 3 boys from 10 in ${}^{10}C_3$ ways. Hence a party of 6 including 3 boys and 3 girls can be formed in ${}^7C_3 \times {}^{10}C_3=35 \times 120=4200$ ways. ...(1)

Now let us find the number of ways such that 2 girls who refuse to join the same party are included in the same party. For such arrangements we have to select only 1 girl from the remaining 5 and 3 boys from the ttoal of 10. This can be done in

$${}^5C_1 \times {}^{10}C_3=5 \times 120=600 \text{ ways} \qquad ..(2)$$

We notice that in arrangement (2) those two particular girls who refuse to join are included. Hence the required number of arrangements can be obtained by subtracting (2) from (1), *i.e.*, $4200-600=3600$.

Example 42. *A party of 3 ladies and 4 gentlemen is to be formed from 8 ladies and 7 gentlemen. In how many different ways can the party be formed if Mrs. X and Mr. Y refuse to join the same party ?*

[*I.C.W.A., June 1990*]

Solution. 3 ladies can be selected out of 8 ladies in ${}^8C_3$ ways and 4 gentlemen can be selected out of 7 gentlemen in ${}^7C_4$ ways.

$\therefore$ The number of ways of choosing the committee

$$={}^8C_3\times{}^7C_4=\frac{8\,!}{3\,!\,5\,!}\times\frac{7\,!}{4\,!\,3\,!}=1960.$$

If both Mrs. X and Mr. Y are members, there remain to be selected 2 ladies from 7 ladies and 3 gentlemen from 6 gentlemen. This can be done in

$${}^7C_2\times{}^6C_3=\frac{7\,!}{2\,!\,4\,!}\times\frac{6\,!}{3\,!\,3\,!}=420 \text{ ways.}$$

$\therefore$ The number of ways of forming the party in which Mrs. X and Mr. Y refuse to join

$$=1960-420=1540$$

Example 43 *A cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket-keepers. In how many different ways can a team be formed so that the team contains* (*a*) *exactly 3 bowlers and 1 wicket-keeper,* (*b*) *at least 3 bowlers and at least one wicket-keeper.*

Solution. (*a*) Here a cricket team of 11 is exactly to contain 3 bowlers and a wicket keeper. 3 bowlers can be selected out of 4 in ${}^4C_3$, *i.e.*, 4 ways. 1 wicket keeper can be selected out of 2 in ${}^2C_1$, *i.e.*, 2 ways. Now the remaining 7 players to complete the team can be selected from the remaining 10 players in ${}^{10}C_3$, *i.e.*, 120 ways. Hence by the fundamental principle, the total number of ways in which the team can be formed $=4\times2\times120=960$.

(*b*) In this case the cricket team of 11 can be formed in the following ways :

(I) 3 bowlers, 1 wicket keeper and 7 other players.

(II) 3 bowlers, 2 wicket keepers and 6 other players.

(III) 4 bowlers. 1 wicket keeper and 6 other players.

(IV) 4 bowlers, 2 wicket keepers and 5 other players.

We now consider all these 4 cases.

(*i*) 3 bowlers, 1 wicket keeper and 7 other players can be selected in ${}^4C_3\times{}^2C_1\times{}^{10}C_7=4\quad 2\times120=960$ ways

(*ii*) 3 bowlers, 2 wicket keepers and 6 other players can be selected ${}^4C_3\times{}^2C_2\times{}^{10}C_6=4\times1\times210=840$ ways.

(*iii*) 4 bowlers, 1 wicket keeper and 6 other players can be selected in ${}^4C_4 \times {}^2C_1 \times {}^{10}C_6 = 1 \times 2 \times 210 = 420$ ways

(*iv*) 4 bowlers, 2 wicket keepers and 5 other players can be selected in ${}^4C_4 \times {}^2C_2 \times {}^{10}C_5 = 1 \times 1 \times 252 = 252$ ways.

Hence the total number of different ways

$$= 960 + 840 + 420 + 252 = 2472$$

Example 44. *A guard of 12 men is formed from a group of n soldiers in all possible ways. Find (i) the number of times two particular soldiers A and B are together on guard and (ii) the number of times three particular soldiers C, D and E are together on guard.*

Also find n if it is found that A and B are three times as often together on guard as C, D and E are.

Solution. (*i*) A guard containing A and B will have 10 other men from the remaining $(n-2)$ soldiers. Hence the number of such guards in which A and B are together is ${}^{n-2}C_{10}$.

(*ii*) Similarly the guard with C, D, E will have 9 other men. This can be selected in ${}^{n-3}C_9$ ways. Hence C, D, E are together ${}^{n-3}C_9$ times.

(*iii*) A and B are together in ${}^{n-2}C_{10}$ times and C, D, E are together in ${}^{n-3}C_9$ times.

$$\therefore \qquad {}^{n-2}C_{10} = 3 \times {}^{n-3}C_9$$

$$\Rightarrow \qquad \frac{(n-2)\,!}{10\,!\,(n-12)\,!} = \frac{3 \times (n-3)\,!}{9\,!\,(n-12)\,!}$$

$$\Rightarrow \qquad \frac{(n-2)(n-3)\,!}{10 \times 9\,!\,(n-12)\,!} = \frac{3 \times (n-3)\,!}{9\,!\,(n-12)\,!}$$

$$\Rightarrow \qquad (n-2) = 3 \times 10$$

$$\therefore \qquad n = 32$$

Example 45. *Find the number of combinations that can be made by taking 4 letters of the word COMBINATION.*

Solution. There are 11 letters of 8 different kinds

$C, (O, O), M, B, (I, I), (N, N)\ A, T$

Thus two Os, two Is and $2N$s are alike. In all the required combinations some may contain all dissimilar letters, some may not contain all different letters. Following cases arise :

(*I*) All the four letters are different.

(*II*) 2 letters are alike, 2 are different,

(*III*) 2 letters are alike of one kind, 2 are alike of other kind.

(*i*) There are 8 different letters. The required number of combinations $= {}^8C_4$.

(*ii*) There are three pairs of alike letters, *viz.*, (O, O), (I, I), (N, N). One pair can be chosen in ${}^3C_1$ ways. Remaining 2 different letters can be selected from remaining 7 different letters in ${}^7C_2$ ways. Hence the number of combinations of this type is ${}^3C_1 \times {}^7C_2$.

(*iii*) Two pairs of similar letters can be chosen in ${}^3C_2$ ways.

Hence the total number of required combinations is

$$^5C_4+{}^3C_1\times{}^7C_2+{}^3C_2=136$$

Some Important Deductions :

I. $$^nC_0=\frac{n\,!}{0\,!\,(n-0)\,!}=\frac{n\,!}{0\,!\,n\,!}=\frac{n\,!}{1\times n\,!}=1$$

II. $^nC_1=n$; $^nC_2=\dfrac{n(n-1)}{2\,!}$ and $^nC_3=\dfrac{n(n-1)(n-2)}{3\,!}$

III. $$^nC_n=1$$

Proof : $$^nC_n=\frac{n\,!}{n\,!\,(n-n)\,!}=\frac{n\,!}{n\,!\,0\,!}=\frac{n\,!}{n\,!\times 1}=1$$

IV. $$^nC_{n-1}={}^nC_1=n$$

V. $$^nC_r=\frac{n}{r}\times{}^{n-1}C_{r-1}$$

These are important theorems and should be committed to the memory. We give below the proof of the last one which is quite important.

Let us find the number of combinations in which a particular letter say a_1 would occur. The number of such combinations is ${}^{n-1}C_{r-1}$ because we are to choose $(r-1)$ letters out of the remaining $(n-1)$ letters. Similarly the number of combinations containing a_2 is ${}^{n-1}C_{r-1}$, the number of combinations containing a_3 is also ${}^{n-1}C_{r-1}$ and so on. The total number of such letters is n, therefore the total number of letters written in these combinations is $n\times{}^{n-1}C_{r-1}$.

$$\therefore \quad r\times{}^nC_r=n\times{}^{n-1}C_{r-1} \quad \Rightarrow \quad {}^nC_r=\frac{n}{r}\times{}^{n-1}C_{r-1}$$

This result is true for all integral values of r and n.

Changing n to $n-1$, r to $r-1$, we get

$$^{n-1}C_{r-1}=\frac{n-1}{r-1}\times{}^{n-2}C_{r-2}$$

Similarly $$^{n-2}C_{r-2}=\frac{n-2}{r-2}\times{}^{n-3}C_{r-3}$$

$$\vdots \qquad\qquad \vdots$$

$$^{n-(r-2)}C_2=\frac{n-(r-2)}{r-(r-2)}\times{}^{n-(r-1)}C_1$$

Multiplying the corresponding sides and cancelling out the common factors, we get

$$^nC_r=\frac{n}{r}\cdot\frac{(n-1)}{(r-1)}\cdot\frac{(n-2)}{(r-2)}\cdots\frac{n-r+2}{2}\cdot{}^{n-r+1}C_1$$

$$=\frac{n(n-1)(n-2)..(n-r+2)(n-r+1)}{r(r-1)(r-2)...3.2.1}$$

$$=\frac{n(n-1)(n-2)...(n-r+2)(n-r+1)(n-r)...3.2.1}{[r(r-1)...2.1][(n-r)...3.2.1]}$$

$$=\frac{n!}{r!(n-r)!}$$

9·9. COMPLEMENTARY THEOREMS

The number of combinations of n different things taken r at a time, is same as the number of combinations of n different things taken $(n-r)$ at a time.

i.., $\quad {}^nC_r={}^nC_{n-r}$, where $0\leqslant r\leqslant n$

We have $\quad {}^nC_r=\dfrac{n!}{r!(n-r)!}$

and $\quad C_{n-r}=\dfrac{n!}{(n-r)!\{n-(n-r)\}!}=\dfrac{n!}{(n-r)!r!}$

$\therefore \quad {}^nC_r={}^nC_{n-r}$

Cor 1. If ${}^nC_r={}^nC_p$ then either $r=p$ or $r+p=n$ for ${}^nC_r={}^nC_{n-r}$ therefore, $n-r=p$ or $r+p=n$.

Cor 2. If in the formula ${}^nC_{n-r}={}^nC_r$, we put

(*i*) $r=n$, then ${}^nC_0={}^nC_n=1$

(*ii*) $r=n-1$, then ${}^nC_1={}^nC_{n-1}=n$, etc.

9·10. RESTRICTED COMBINATIONS

(*i*) *The number of combinations of n things taken r at a time in which p particular things always occur is* ${}^{n-p}C_{r-p}$.

If the p particular things are set aside, there remain $(n-p)$ things out of which $(r-p)$ things may be chosen in ${}^{n-p}C_{r-p}$ ways. With each of these groups we combine the p particular things so that we get all the combinations in each of which the p particular things will always occur.

$\therefore$ The required number of combinations$={}^{n-p}C_{r-p}$

(*ii*) *The number of combinations of n things taken r at a time in which p particular things never occur is* ${}^{n-p}C_r$.

Let the p particular things be set aside, then there will remain $(n-p)$ things out of which r things may be selected in ${}^{n-p}C_r$ ways.

In none of these groups p particular things will occur. Hence the required number of combinations $={}^{n-p}C_r$.

Example 46. *Prove that*

$${}^{n+1}C_r={}^nC_r+{}^nC_{r-1}$$

Solution We know that

$${}^nC_r=\frac{n!}{r!(n-r)!}$$

$$\therefore \quad {}^nC_r+{}^nC_{r-1}=\frac{n!}{r!\,(n-r)!}+\frac{n!}{(r-1)!\,(n-r+1)!}$$
$$=\frac{n!}{r(r-1)!\,(n-r)!}+\frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!}$$
$$=\frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r}+\frac{1}{(n-r+1)}\right]$$
$$=\frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$$
$$=\frac{(n+1)!}{r!\,(n-r+1)!}={}^{n+1}C_r$$

Example 47. *Find the value of r if* ${}^{18}C_r={}^{18}C_{r+2}$

Solution. Since ${}^nC_r={}^nC_{n-r}$, we have ${}^{18}C_r={}^{18}C_{18-r}$

But, we are given ${}^{18}C_{18-r}={}^{18}C_{r+2}$

$\Rightarrow \quad 18-r=r+2$

$\Rightarrow \quad 18-2=r+r$

$\Rightarrow \quad 16=2r$

$\Rightarrow \quad r=8.$

Example 48. *Find n,* ${}^nC_6:{}^{n-3}C_3=91:4.$

Solution. We know that

$${}^nC_6=\frac{n!}{6!\,(n-6)!} \text{ and } {}^{n-3}C_3=\frac{(n-3)!}{3!\,(n-3-3)!}$$

$$\therefore \quad \frac{{}^nC_6}{{}^{n-3}C_n}=\frac{n!}{6!\,(n-6)!}\times\frac{3!\,(n-6)!}{(n\quad 3)!}$$
$$=\frac{n!}{(n-3)!}\times\frac{1}{6.5.4}=\frac{n(n-1)\,(n-2)}{6.5.4}$$

Also we are given $\quad \dfrac{n(n-1)\,(n-2)}{6.5.4}=\dfrac{91}{4}$

$\therefore \quad n(n-1)\,(n-2)=5.6.91=5.6.7.13=15.14.13.$

Expressing the three consecutive integers in descending order, we get $n=15$.

9·11. COMBINATIONS OF THINGS NOT ALL DIFFERENT

We will here show that the total number of combinations of n different things taken some or all at a time is 2^n-1.

The first thing can be dealt within 2 ways, for it may either be left or taken. The second thing can also be dealt within 2 ways.

Since each way of dealing with the first can be associated with each way of dealing with the second, the total number of ways of dealing with the two things $=2\times2=2^2$.

Proceeding similarly, the total number of ways $=2^n$

But this number includes one case in which none of the things are taken.

$\therefore$ The required number of combinations $=2^n-1$

Now, we consider the number of combinations of n things not all different. The total number of combinations of $(p+q+r+\ldots)$ things, where p are of one kind, q of the second kind and r of the third kind and so on, taken any number at a time are

$$=(p+1)(q+1)(r+1)\ldots-1$$

Consider the p like things. The p things can be dealt with in $(p+1)$ ways, for we may take 1, or 2 or 3, ..., or p or none in any selection. Similarly the q like things can be dealt with in $(q+1)$ ways, r things in $(r+1)$ ways, etc. Associating each group of selections with the others, the total number of dealing with them is

$$(p+1)(q+1)(r+1)\ldots$$

But this number includes one case when all things are left. Therefore, the total number of ways

$$=(p+1)(q+1)(r+1)\ldots\ldots-1$$

Example 49. *In order to pass C A. (Intermediate) examination minimum marks have to be secured in each of the 7 subjects. In how many cases can a student fail ?*

Solution. Each subject can be dealt in two ways, the student may pass or fail in it. So the 7 subjects can be dealt in 2^7 ways. But this includes the case in which the student passes in all the 7 subjects. Excluding this, the number of ways in which the student can fail is $2^7-1=127$.

Example 50. *A question paper contains 6 questions, each having an alternative. In how many ways can an examinee answer one or more questions ?*

Solution. The first question can be dealt with in 3 ways, for the question itself may be answered, or its alternative may be answered or none of them may be answered.

Similarly, the second question also can be dealt with in 3 ways. Hence the first two questions can be dealt with in 3×3 or 3^2 ways. Proceeding thus, all the 6 questions may be dealt with in 3^6 ways.

But this number includes one case in which none of the questions is answered.

$\therefore$ The required number of ways $=3^6-1=728$.

Example 51. *There are n points in a plane, no three of which are collinear (lying on the same straight line) with the exception of p points which are collinear. Find the number of*

(*i*) *different straight lines and*

(*ii*) *different triangles formed by joining them.*

Solution. (*i*) Any two points when joined give a straight line.

$\therefore$ The number of possible straight lines formed by joining n points in pairs $= {}^nC_2$.

But p of the points lie in the same straight line.

$\therefore$ ${}^pC_2$ straight lines are lost and instead we get only 1 straight line in which they lie.

$\therefore$ The required number of straight lines is

$$ {}^nC_2 - {}^pC_2 + 1 = \frac{n(n-1)}{2} - \frac{p(p-1)}{2} + 1. $$

(*ii*) Any 3 non-collinear points give a triangle.

$\therefore$ The number of triangles formed by joining n points taken three at a time $= {}^nC_3$.

Since p of the points are collinear, ${}^pC_3$ triangles are lost.

$\therefore$ The required number of triangles is

$$ {}^nC_3 - {}^pC_3 = \frac{n(n-1)(n-2)}{6} - \frac{p(p-1)(p-2)}{6} $$

EXERCISE (II)

1. Find the value of ${}^8C_3$, ${}^{21}C_{19}$, ${}^nC_{n-3}$

2. (*a*) If ${}^{2n}C_8 : {}^{2n}C_8 = 8$, find n.

   (*b*) If ${}^nC_3 = {}^nC_5$, find the value of ${}^{2n}C_2$.

   (*c*) If ${}^{10}P_r = 6{,}04{,}800$ and ${}^{10}C_r = 120$, show that $r = 7$.

3. A cricket team is to be formed consisting of 2 wicket keepers, 4 bowlers and 5 batsmen from a group of players containing 4 wicket keepers, 8 bowlers and 11 batsmen. Find the number of ways a cricket team can be constituted.

   [**Hint.** ${}^4C_2 \times {}^8C_4 \times {}^{11}C_5$]

4. In how many ways can you choose six out of nine questions? In how many of these ways the first question is always excluded? In how many ways the first and second questions are always included?

5. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further in how many of these committees

   (*i*) a particular professor is included?

   (*ii*) a particular student is excluded?

6. In how many ways can 21 white balls and 19 black balls be arranged in a row so that no two black balls may be together.

7. Find out the number of ways in which a cricket team consisting of 11 players can be selected from 14 players. Also find out how many of these (*i*) will include captain, (*ii*) will not include captain.

8. Out of 5 males and 6 females, a committee of 5 is to be formed. Find the number of ways in which it can be done so that among the persons chosen in the committee there are (*i*) 3 males and 2 females, (*ii*) 2 males, (*iii*) no females, (*iv*) at least one female, (*v*) not more than 3 males.

[Hint. (*i*) ${}^5C_3 \times {}^6C_2$, (*ii*) ${}^5C_2 \times {}^6C_3$, (*iii*) ${}^5C_5$,

(*iv*) ${}^6C_1 \times {}^5C_4 + {}^6C_2 \times {}^5C_3 + {}^6C_3 \times {}^5C_2 + {}^6C_4 \times {}^5C_1 + {}^6C_5 \times {}^5C_0 = 461$

(*v*) ${}^5C_0 \times {}^6C_5 + {}^5C_1 \times {}^6C_4 + {}^5C_2 \times {}^6C_3 + {}^5C_3 \times {}^6C_2 = 401$.]

9. From 7 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include *at least* one lady ?

10. The staff of a bank consists of the manager, the deputy manager and 10 other officers. A committee of 4 is to be selected. Find the number of ways in which this can be done so as to always include (*i*) the manager, (*ii*) the manager but not the deputy manager, (*iii*) neither the manager nor the deputy manager.

11. A council consists of 10 members, 6 belonging to the party A and 4 to the party B. In how many ways can a committee of 5 be selected so that the members of the party A are in a majority ?

12. A cricket club consists of 16 members of which only 6 can bowl. In how many ways can an elevan be chosen to include *at least* 4 bowlers ?

13. An examination paper which is divided into two groups consisting of 3 and 4 questions respectively, carries the note. "It is not necessary to answer all the questions. One question must be answered from each group." In how many ways can an examinee select the questions ?

14. (*a*) Out of 17 consonants and 5 vowels, how many different words can be formed each consisting of 3 consonants and 2 vowels ?

(*b*) Find the number of words which can be formed with two different consonants and one vowel, out of 7 different consonants and 3 different vowels, the vowel to lie between two consonants ?

15. There are 48 different books including 18 books on Advance Accounts, 16 books on Law and 14 books on Management. Find the number of different ways in which a selection of twelve books can be made so as to have 4 books from each group.

16. How many combinations can be formed of 8 counters marked 1, 2, 3, 4, 5, 6, 7, 8 taking them 4 at a time, there being at least one odd and one even counter, in each combination ?

[Hint. ${}^4C_1 \times {}^4C_3 + {}^4C_2 \times {}^4C_2 + {}^4C_3 \times {}^4C_1$].

17. There are 12 points in a plane of which 5 are in a line. Find (*i*) the maximum number of triangles that can be formed with vertices at these points, (*ii*) the maximum number of distinct straight lines that can be obtained by joining these points.

[Hint. (*i*) ${}^{12}C_3 - {}^5C_3$, (*ii*) ${}^{12}C_2 - {}^5C_2 + 1$].

18. A gentleman invites a party of 13 guests to a dinner and places 8 of them at one table and the remaining 5 at another, the tables being round. Find the number of ways in which he can arrange the guests.

[**Hint.** Total number of arrangements$={}^{13}C_8\times 7\,!\times 4\,!$].

19. Find the number of ways in which (*i*) a selection, (*ii*) an arrangement of 4 letters can be made from the letters of the word "MATHEMATICS"

[**Hint.** (*i*) Total number of selections

$$={}^8C_4+{}^3C_1\times{}^7C_2+{}^8C_2=136$$

(*ii*) Total number of permutations

$$={}^8C_4\times 4\,!+{}^3C_1\times{}^7C_2\times\frac{4\,!}{2\,!}+{}^3C_2\times\frac{4\,!}{2\,!\,2\,!}$$

$$=1680+756+18=2454]$$

20. In a crossword puzzle twenty words are to be guessed of which eight words have each an alternative solution also. Find the number of possible solution.

ANSWERS

1. 56, 210, $\{n(n-1)(n-2)\}\ 3\,!$ **2.** (*a*) 13, (*b*) 120. **3.** ${}^4C_2\times{}^8C_4\times{}^nC_5$, **4.** 84, 28, 35 **5.** ${}^{10}C_2\times{}^{20}C_3=51300$, (*i*) ${}^9C_1\times{}^{20}C_3=10260$, (*ii*) ${}^{10}C_2\times{}^{19}C_3=43605$. **6.** 1540. **7.** 364, (*i*) 286, (*ii*) 86. **9.** 441. **10.** (*i*) ${}^{11}C_3$, (*ii*) ${}^{10}C_3$, (*iii*) ${}^{10}C_4$. **11.** 186. **12.** 3096. **13.** 12 **14.** (*a*) 816000, (*b*) 126. **15.** ${}^8C_4\times{}^{16}C_4\times{}^{14}C_4$. **16.** 68. **17.** (*i*) 210, (*ii*) 57. **20.** 256.

10

Binomial Theorem

STRUCTURE

10·0 INTRODUCTION
10·1 BINOMIAL THEOREM
10·2 POSITION OF TERMS
10·3 BINOMIAL COEFFICIENTS
10·4 BINOMIAL THEOREM WITH ANY INDEX
10·5 SUMMATION OF SERIES

Objectives

After studying this chapter, you should be able to understand :

- *Binomial theorem, position of terms, binomial coefficients and its application.*
- *Binomial theorem with any index and calculation of square root, cube root etc. and simplification.*
- *Summation of series using binomial theorem.*

10·0 INTRODUCTION

A binomial expression in mathematics in one which has two terms, *e.g.*, $(a+b)$, $(4x+3y)$, $(x+a)$, etc. These terms are at time complementary when the expression is used for objects which are of dichotomous character, *i.e.*, success or failure, true or false, male or female, literate or illiterate. In business mathematics and statistics, there are various problems based on such classification where the theorem is found to be very useful.

From elementary algebra, we know

$$(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2$$
$$(a+b)^3=(a+b)(a^2+2ab+b^2)=a^3+3a^2b+3ab^2+b^3$$

These are quite simple but if the expansion is to a higher order or with negative and fractional indices the problem becomes quite complicated. It is here that the rule of expansion stated as the binomial theorem is very handy. It will be seen later that we can trace the individual terms of the expansion without writing the whole series.