10 Interchange of Power and Energy

10.1 INTRODUCTION

This chapter reviews the interchange of power and energy, primarily the practices in Canada and the United States where there are numerous, major electric utilities operating in parallel in three large AC interconnections. In many other parts of the world, simpler commercial structures of the electric power industry exist. Many countries have one to two major generation–transmission utilities with local distribution utilities. The industry structure is important in discussing the interchange of power and energy since the purchase and sale of power and energy is a commercial business where the parties to any transaction expect to enhance their own economic positions under nonemergency situations. In North America, the "market place" is large, geographically widespread, and the transmission networks in the major interconnections are owned and operated by multiple entities. This has led to the development of a number of common practices in the interchange of power and energy between electric utilities. Where the transmission network is (or was) owned by a single entity, the past and developing practices regarding transactions may be different than those in the United States and Canada. We will confine the discussions of the commercial aspects of the electric energy markets to the practices in North America, circa early 1995.

The market structures for electric energy and power are changing. In the past, interconnected electric utility systems dealt only with each other to buy and sell power and energy. Only occasionally did nonutility entities become involved, and these were usually large industrial organizations with their own generation. Many of these industrial firms had a need for process heat or steam and developed internal generation (i.e., *cogeneration plants*) to supply steam and electric power. Some developed electric power beyond the internal needs of the plant so that they could arrange for sale of the excess to the local utility system. The earlier markets only involved "wholesale transactions", the sale and purchase of electric energy to utilities for ultimate delivery to the consumer. With the exception of industrial cogenerators, all aspects of the interchange arrangements were made between interconnected utilities.

In more recent times, there has been an opening of the market to facilitate the involvement of more nonutility organizations, consumers as well as generators. Throughout the world there has been a movement towards deregulation of the electric utility industry and an opening of the market to nonutility entities, mainly nonutility generating firms. There is agitation to open the use of the transmission system to all utilities and nonutility generators by providing "open transmission access." Because of the multiple ownership of the transmission systems in North America and the absence of a single entity charged with the control of the entire (or even regional) bulk power system, there are many unresolved issues (as of July 1995). These concern generation control, control of flows on the transmission system circuits, and establishment of schemes for setting "fair and equitable" rates for the use of the transmission network by parties beyond the utility owner of the local network. This last factor is an important issue since it is the very transmission interconnections that make the commercial market physically feasible. The discussions involve concerns over monopoly practices by, and the property rights of, the owners of the various parts of the network.

Nevertheless, the movement towards more nonutility participation continues and more entities are becoming involved in the operation of the interconnected systems. Most all of the nonutility participants are involved in supplying power and energy to utilities or large industrial firms. The use of a transmission system by parties other than its owner may involve "wheeling" arrangements (that is, an arrangement to use the transmission system owned by another party to deliver power). There have been wheeling arrangements as long as there have been interconnections between more than two utilities. In most cases, the development of transmission service (i.e., wheeling) rates has been based on simplified physical models designed to facilitate commercial arrangements. As long as the market was restricted to a few parties, these arrangements were usually mutually satisfactory. With the introduction of nonutility participants, there is a need for the development of rate structures based on more realistic models of the power system.

The growth of the number and size of energy transactions has emphasized the need for intersystem agreements on power flows over "parallel" transmission circuits. Two neighboring utilities may engage in the purchase–sale of a large block of power. They may have more than enough unused transmission capacity in the direct interconnections between the systems to carry the power. But, since the systems are interconnected in an AC network that includes a large number of utilities, when the transaction takes place, a large portion of the power may actually flow over circuits owned by other systems. The flow pattern is determined by physical laws, not commercial arrangements. The problems caused by these parallel path flows have been handled (at least in North America) by mutual agreements between interconnected utility systems. In the past, there was a general, if unspoken, agreement to attempt to accommodate the transactions. But, as the numbers and sizes of the transactions have increased, there have been more incidences of local circuit overloads caused by remote transactions.

We emphasize these points because in other parts of the world they do not exist in the same form. Many of the problems associated with transmission system use, transmission access, and parallel path issues, are a consequence of multiple ownership of the transmission network. They are *structural problems*, not physical problems. On the other hand, when a formerly nationalized grid is deregulated and turned into a single, privatized network there are problems, but they are not the problems that arise from the need to treat multiple transmission owners on a fair and equitable basis.

Interutility transfers of energy are easily accomplished. Recall the computation of the *area control error*, ACE, in the chapter on generation control. A major component of ACE is the *scheduled net interchange*. To arrange for the sale of energy between two interconnected systems, the seller increases its net interchange by the amount of the sale, and the purchaser decreases its net interchange by a similar amount. (We ignore losses.) The AGC systems in the two utilities will adjust the total generation accordingly and the energy will be transferred from the selling system to the purchaser. With normal controls, the power will flow over the transmission network in a pattern determined by the loads, generation, control settings, and network impedances and configuration. (Notice that *network ownership* is not a factor.)

The AGC scheme of Chapter 9 develops an autonomous, local control based upon ACE. It is predicated (implicitly, at least) on the existence of a well-defined control area that usually corresponds to the geographical and electrical boundaries of one or more utilities. Interchanges are presumed to be scheduled between utility control centers so that the net interchange schedule is well defined and relatively stable over time. With many participants engaged in transactions and, perhaps, private generators selling power to entities beyond the local control area, the interchange schedule may be subject to more frequent changes and some local loads may no longer be the primary responsibility of the local utility. AGC systems may have to become more complex with more information being supplied in real time on all local generation, load substations, and all transactions. New arrangements may be needed to assign responsibility for control actions and frequency regulation. Utilities have done these tasks in the past out of their own self-interest. A new incentive may be needed as the need for frequency and tie-line control becomes a marketplace concern; not just the concern of the utility.

This chapter reviews the practices that have evolved in all-utility interchange arrangements. This leads to a brief discussion of power pools and other commercial arrangements designed to facilitate economic interchange. Many of the issues raised by the use of the transmission system are unresolved issues that await the full and mature development of new patterns for coordinating bulk power system operations and defining, packaging and pricing transmission services. We can only discuss possible outcomes.

There are evolving market structures that include nonutility participants. These may include organizations that have generation resources, distributing utilities, and consumers, usually larger industrial firms. In these areas, we must venture into questions involving *price*. No transactions take place without involving prices, even those between utilities. Disputes naturally arise over what are *fair price levels.* (Price and fairness, like beauty, are in the eye of the beholder. The price level wanted for an older automobile may seem very fair to me as the seller and outrageous to the purchaser. We may both be correct and no sale will take place. Or one, or both, of us may be willing to change our views so that we do consummate a sale; in which case, the price agreed upon is "fair," by definition.)

In areas where there is regulation of utility charges to consumers, prices are usually based on costs. (In most markets in capitalistic economies, prices are based on market action rather than being administered by governments.) There is usually a stated principal that utilities may recover no more than a given margin above "cost." There may be some dispute over what costs should be included and how they should be allocated to each consumer class, but, generally, the notion of cost-based pricing is firmly established. Where utilities are dealing with each other or with nonutility entities, there may, or may not, be an obligation to base prices on costs. In many situations, market forces will set price levels. Transactions will be negotiated when both parties can agree upon terms that each considers advantageous, or at least satisfactory.

This chapter also introduces the concept of *wheeling*, the delivery of power and energy over a transmission system (or systems) not owned by controlled by the generating entity or the purchasing entity. At the center of the idea of selling transmission capacity to others is the definition and measurement of the available transmission capacity for transferring power. This is not an easy quantity to define since it depends upon acceptable notions of reliable, or *secure* system operating practices, a very subjective issue. In the communication network areas such as telephone systems, data transmission networks, and so on, the path capacities are more readily definable. Signals may be rerouted when a channel is fully loaded and the party desiring communication service will receive a "busy signal" if there is no capacity currently available. This does not carry over into interconnected AC power systems. Certainly, there are definable physical limits to the current that may be carried by each portion of the system without causing permanent physical damage. There is a need to reduce these absolute limits to provide some margin for the inability to predict the loading levels with certainty. There must also be some margin, or reserve, retained to permit the system to survive forced outages of circuit elements and generators. Voltage magnitudes in the system must be kept within controllable ranges. It is here where art, experience, and opinion enter and make the exact definition of available transmission capacity difficult. Thus, in any commercial arrangement for energy transactions, the question of available transmission capacity may arise and need to be settled.

Outside of North America, a major shift in the structure of the electric utility industries that has taken place in the past decade is that of splitting up formerly integrated, government utility organizations. This has usually involved the *privatization* of governmentally sponsored utilities and the separation of the original utility into separate and independent, private organizations owned by shareholders. Some of the resuting entities may be generation companies, others distribution utilities with the responsibility for the distribution of power to the ultimate consumer, and one organization that has control of the transmission network and is responsible for establishing a market for, and scheduling of, generation. Where this has happened, it has led to the development of a market structure involving a few large organizations that were formerly part of the state system, plus nonutility generators. These are markets that tend to be dominated by a few large participants.

In the United States, the electric utility industry is very diverse, with 200 to 400 major utilities (depending upon the precise definition used), plus a few thousand other organizations that are also classified as utilities. Many are investors-owned. Some are governmentally sponsored organizations at both state and federal levels. Still others are consumer-owned utilities. Given this diversity, the new market structures that may evolve under *deregulation* in the United States are apt to be different than those in countries where state systems have been privatized.

The discussions of these issues and their resolutions in this text has to be tentative, and, we trust, unbiased. Any change in a long-standing industry naturally meets with opposition, objections, and controversy, as well as enthusiastic advocacy.

10.2 ECONOMY INTERCHANGE BETWEEN INTERCONNECTED UTILITIES

Electric power systems interconnect because the interconnected system is more reliable, it is a better system to operate, and it may be operated at less cost than if left as separate parts. We saw in a previous chapter that interconnected systems have better regulating characteristics. A load change in any of the sytems is taken care of by all units in the interconnection, not just the units in the control area where the load change occurred. This fact also makes interconnections more reliable since the loss of a generating unit in one of them can be made up from spinning reserve among units throughout the interconnection. Thus, if a unit is lost in one control area, governing action from units in all connected areas will increase generation outputs to make up the deficit until standby units can be brought on-line. If a power system were to run in isolation and lose a large unit, the chance of the other units in that isolated system being able to make up the deficit are greatly reduced. Extra units would have to be run as spinning reserve, and this would mean less-economic operation. Furthermore, a generation system will generally require a smaller installed generation capacity reserve if it is planned as part of an interconnected system.

One of the most important reasons for interconnecting with neighboring systems centers on the better economics of operation that can be attained when utilities are interconnected. This opportunity to improve the operating economics arises any time two power systems are operating with different incremental costs. As Example 10A will show, if there is a sufficient difference in the incremental cost between the systems, it will pay both systems to exchange power at an equitable price. To see how this can happen, one need merely reason as follows. Given the following situation:

- Utility A is generating at a lower incremental cost than utility B.
- If utility B were to buy the next megawatt of power for its load from utility A at a price less than if it generated that megawatt from its own generation, it would save money in supplying that increment of load.
- Utility A would benefit economically from selling power to utility B, as long as utility B is willing to pay a price that is greater than utility A's cost of generating that block of power.

The key to achieving a mutually beneficial transaction is in establishing a "fair" price for the economy interchange sale.

There are other, longer-term interchange transactions that are economically advantageous to interconnected utilities. One system may have a surplus of power and energy and may wish to sell it to an interconnected company on a long-term firm-supply basis. It may, in other circumstances, wish to arrange to see this excess only on a "when, and if available" basis. The purchaser would probably agree to pay more for a firm supply (the first case) than for the interruptable supply of the second case.

In all these transactions, the question of a "fair and equitable price" enters into the arrangement. The economy interchange examples that follow are all based on an equal division of the operating costs that are saved by the utilities involved in the interchange. This is not always the case since "fair and equitable" is a very subjective concept; what is fair and equitable to one party may appear as grossly unfair and inequitable to the other. The 50–50 split of savings in the examples in this chapter should not be taken as advocacy of this particular price schedule. It is used since it has been quite common in interchange practices in the United States. Pricing arrangements for long-term interchange between vary widely and may include "take-or-pay" contracts, split savings, or fixed price schedules.

Before we look at the pricing of interchange power, we will present an example showing how the interchange power affects production costs.

EXAMPLE 10A

Two utility operating areas are shown in Figure 10.1. Data giving the heat rates and fuel costs for each unit in both areas are given here.

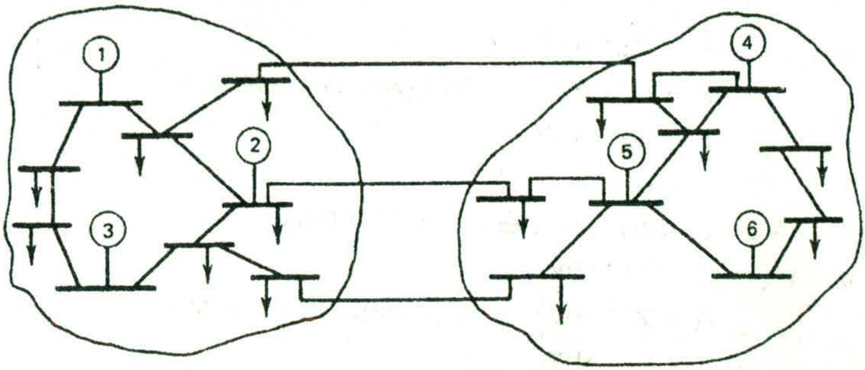

FIG. 10.1 Interconnected areas for Example 10A.

Unit data:

$$F_i(P_i) = f_i(a_i + b_iP_i + c_iP_i^2)$$
$$P_i^{\min} \le P_i \le P_i^{\max}$$

Unit No.	Fuel Cost f_i (R/MBtu)	Cost Coefficients a_i	b_i	c_i	Unit Limits $P_i^{\min}$ (MW)	$P_i^{\max}$ (MW)
1	2.0	561	7.92	0.001562	150	600
2	2.0	310	7.85	0.00194	100	400
3	2.0	78	7.97	0.00482	50	200
4	1.9	500	7.06	0.00139	140	590
5	1.9	295	7.46	0.00184	110	440
6	1.9	295	7.46	0.00184	110	440

Area 1:

Load = 700 MW

Max total generation = 1200 MW

Min total generation = 300 MW

Area 2:

Load = 1100 MW

Max total generation = 1470 MW

Min total generation = 360 MW

First, we will assume that each area operates independently; that is, each will supply its own load from its own generation. This will necessitate performing a separate economic dispatch calculation for each area. The results of an independent economic dispatch are given here.

Area 1: $\left.\begin{aligned} P_1 &= 322.7 \text{ MW} \\ P_2 &= 277.9 \text{ MW} \\ P_3 &= 99.4 \text{ MW} \end{aligned}\right\}$ Total generation = 700 MW

$\lambda = 17.856$ R/MWh

Operating cost, area 1 = 13,677.21 R/h

Area 1: $\left.\begin{aligned} P_4 &= 524.7 \text{ MW} \\ P_5 &= 287.7 \text{ MW} \\ P_6 &= 287.7 \text{ MW} \end{aligned}\right\}$ Total generation = 1100 MW

$\lambda = 16.185$ R/MWh

Operating cost, area 2 = 18,569.23 R/h

Total operating cost for both areas = 13,677.21 + 18,569.23
= 32,246.44 R/h

Now suppose the two areas are interconnected by several transmission circuits such that the two areas may be thought of, and operated, as one system. If we now dispatch them as one system, considering the loads in each area to be the same as just shown, we get a different dispatch for the units.

$\left.\begin{aligned} P_1 &= 184.0 \text{ MW} \\ P_2 &= 166.2 \text{ MW} \\ P_3 &= 54.4 \text{ MW} \end{aligned}\right\}$ Total generation in area 1 = 404.6 MW

$\left.\begin{aligned} P_4 &= 590.0 \text{ MW} \\ P_5 &= 402.7 \text{ MW} \\ P_6 &= 402.7 \text{ MW} \end{aligned}\right\}$ Total generation in area 2 = 1395.4 MW

Total generation for entire system = 1800.0 MW

$\lambda = 16.990$ R/h

Operating cost, area 1 = 8530.93 R/h

Operating cost, area 2 = 23,453.89 R/h

Total operating cost = 31,984.82 R/h

Interchange power = 295.4 MW from area 2 to area 1

Note that area 1 is now generating less than when it was isolated, and area 2 is generating more. If we ignore losses, we can see that the change in generation in each area corresponds to the net power flow over the interconnecting circuits. This is called the *interchange power*. Note also that the overall cost of operating both systems is now less than the sum of the costs to operate the areas when each supplied its own load.

Example 10A has shown that interconnecting two power systems can have a marked economic advantage when power can be interchanged. If we look at the net change in operating cost for each area, we will discover that area 1 had a decrease in operating cost while area 2 had an increase. Obviously, area 1 should pay area 2 for the power transmitted over the interconnection, but how much should be paid? This question can be, and is, approached differently by each party.

Assume that the two systems did interchange the 295.4 MW for 1 h. Analyzing the effects of this interchange gives the following.

Area 1 costs:	without the interchange	13,677.21 R
	with the interchange	8530.93
	Savings	5146.28 R
Area 2 costs:	without interchange	18,569.23 R
	with interchange	23,453.89
	Increased cost	4884.66 R
	Combined, net savings	261.62 R

Area 1: Area 1 can argue that area 2 had a net operating cost increase of 4884.66 R and therefore area 1 ought to pay area 2 this 4884.66 R. Note that if this were agreed to, area 1 should reduce its net operating cost by $13{,}677.21 - (8530.93 + 4884.66) = 261.62$ R when the cost of the purchase is included.

Area 2: Area 2 can argue that area 1 had a net decrease in operating cost of 5146.28 R and therefore area 1 ought to pay area 2 this 5146.28 R. Note that if this were agreed to, area 2 would have a net decrease in its operating costs when the revenues from the sale are included: $18{,}569.23 - 23{,}453.89 + 5146.28 = 261.62$ R.

The problem with each of these approaches is, of course, that there is no agreement concerning a mutually acceptable "fair" price. In both cases, one party to the transaction gets all the economic benefits while the other gains nothing. A common practice in such cases is to price the sale at the cost of generation plus one-half the savings in operating costs of the purchaser. This splits the savings equally between the two operating areas. This means that area 1 would pay area 2 the amount of 5015.47 R and each area would have 130.81 R reduction in operating costs.

Such transactions are usually not carried out if the net savings are very small. In such a case, the errors in measuring interchange flows might cause the transaction to be uneconomic. The transaction may also appear to be uneconomic to a potential seller if the utility is concerned with conserving its fuel resources to serve its own customers.

10.3 INTERUTILITY ECONOMY ENERGY EVALUATION

In example 10A, we saw how two power systems could operate interconnected for less money than if they operated separately. We obtained a dispatch of the interconnected systems by assuming that we had all the information necessary (input–output curves, fuel costs, unit limits, on-line status, etc.) in one location and could calculate the overall dispatch as if the areas were part of the same system. However, unless the two power systems have formed a power pool or transmit this information to each other, or a third party, who will arrange the transaction; this assumption is incorrect. The most common situation involves system operations personnel, located in offices within each of the control areas, who can talk to each other by telephone. We can assume that each office has the data and computation equipment needed to perform an economic dispatch calculation for its own power system and that all information about the neighboring system must come over the telephone (or some other communications network). How should the two operations offices coordinate their operations to obtain best economic operation of both systems?

The simplest way to coordinate the operations of the two power systems is to note that if someone were performing an economic dispatch for both systems combined, the most economic way to operate would require the incremental cost to be the same at each generating plant, assuming that losses are ignored. The two operations offices can achieve the same result by taking the following steps.

1. Assume there is no interchange power being transmitted between the two systems.
2. Each system operations office runs an economic dispatch calculation for its own system.
3. By talking over the telephone, the offices can determine which system has the lower incremental cost. The operations office in the system with lower incremental cost then runs a series of economic dispatch calculations, each one having a greater total demand (that is, the total load is increased at each step). Similarly, the operations office in the system having higher incremental cost runs a series of economic dispatch calculations, each having a lower total demand.
4. Each increase in total demand on the system with lower incremental cost will tend to raise its incremental cost, and each decrease in demand on the high incremental cost system will tend to lower its incremental cost. By running the economic dispatch steps and conversing over the telephone, the two operations offices can determine the level of interchange energy that will bring the two systems toward most economic operation.

Under idealized "free market" conditions where both utilities are attempting to minimize their respective operating costs, and assuming no physical limitations on the transfer, their power negotiations (or bartering) will lead to the same economic results as a pool dispatch performed on a single area basis. These assumptions, however, are critical. In many practical situations, there are both physical and institutional constraints that prevent interconnected utility systems from achieving optimum economic dispatch.

EXAMPLE 10B

Starting from the "no interchange" conditions of Example 10A, we will find the most economic operation by carrying out the steps outlined earlier. Since area 2 has a lower incremental cost before the transaction, we will run a series of economic dispatch calculations with increasing load steps of 50 MW, and an identical series on area 1 with decreasing load steps of 50 MW.

Area 1:

Step	Demand (MW)	Area 1 Incremental Cost (RMWh)	Assumed Interchange from Area 2 (MW)
1	700	17.856	0
2	650	17.710	50
3	600	17.563	100
4	550	17.416	150
5	500	17.270	200
6	450	17.123	250
7	400	16.976	300
8	350	16.816	350

Area 2:

Step	Demand (MW)	Area 2 Incremental Cost (RMWh)	Assumed Interchange from Area 1 (MW)
1	1100	16.185	0
2	1150	16.291	50
3	1200	16.395	100
4	1250	16.501	150
5	1300	16.656	200
6	1350	16.831	250
7	1400	17.006	300
8	1450	17.181	350

Note that at step 6, area 1's incremental cost is just slightly above area 2's incremental cost, but that the relationship then changes at step 7. Thus, for minimum total operating costs, the two systems ought to be interchanging between 250 and 300 MW interchange.

This procedure can be repeated with smaller steps between 250 and 300 MW, if desired.

10.4 INTERCHANGE EVALUATION WITH UNIT COMMITMENT

In Examples 10A and 10B, there was an implicit assumption that conditions remained constant on the two power systems as the interchange was evaluated. Usually, this assumption is a good one if the interchange is to take place for a period of up to 1 h. However, there may be good economic reasons to transmit interchange power for periods extending from several hours to several days. Obviously, when studying such extended periods, we will have to take into account many more factors than just the relative incremental costs of the two systems.

Extended interchange transactions require that a model of the load to be served in each system (i.e., the expected load levels as a function of time) be included, as well as the unit commitment schedule for each. The procedure for studying interchange of power over extended periods of time is as follows.

1. Each system must run a base-unit commitment study extending over the length of the period in question. These base-unit commitment studies are run without the interchange, each system serving its own load as given by a load forecast extending over the entire time period.
2. Each system then runs another unit commitment, one system having an increase in load, the other a decrease in load over the time the interchange is to take place.
3. Each system then calculates a total production cost for the base-unit commitment and for the unit commitment reflecting the effect of the interchange. The difference in cost for each system represents the cost of the interchange power (a positive change in cost for the selling sytem and a negative change in cost for the buying system). The price for the interchange can then be negotiated. If the agreed-on pricing policy is to "split the savings," the price will be set by splitting the savings of the purchaser and adding the change in the cost for the selling system. If the savings are negative, it obviously would not pay to carry out the interchange contract.

The unit commitment calculation allows the system to adjust for the start-up and shut-down times to take more effective advantage of the interchange power.

It may pay for one system to leave an uneconomical unit off-line entirely during a peak in load and buy the necessary interchange power instead.

10.5 MULTIPLE-UTILITY INTERCHANGE TRANSACTIONS

Most power systems are interconnected with all their immediate neighboring systems. This may mean that one system will have interchange power being bought and sold simultaneously with several neighbors. In this case, the price for the interchange must be set while taking account of the other interchanges. For example, if one system were to sell interchange power to two neighbouring systems in sequence, it would probably quote a higher price for the second sale, since the first sale would have raised its incremental cost. On the other hand, if the selling utility was a member of a power pool, the sale price might be set by the power and energy pricing portions of the pool agreement to be at a level such that the seller receives the cost of the generation for the sale plus one-half the total savings of all the purchasers. In this case, assuming that a pool control center exists, the sale prices would be computed by this center and would differ from the prices under multiple interchange contracts. The order in which the interchange transaction agreements are made is very important in costing the interchange where there is no central pool dispatching office.

Another phenomenon that can take place with multiple neighbors is called "wheeling." This occurs when a system's transmission system is simply being used to transmit power from one neighbor, through an intermediate system, to a third system. The intermediate system's AGC will keep net interchange to a specified value, regardless of the power being passed through it. The power being passed through will change the transmission losses incurred in the intermediate system. When the losses are increased, this can represent an unfair burden on the intermediate system, since if it is not part of the interchange agreement, the increased losses will be supplied by the intermediate system's generation. As a result, systems often assess a "wheeling" charge for such power passed through its transmission network.

The determination of an appropriate (i.e., "acceptable") wheeling charge involves both engineering and economics. Utilities providing a wheeling service to other utilities are enlarging the scope of the market for interchange transactions. Past practices amongst utilities have been established by mutual agreement amongst interconnected systems in a region. A transaction between two utilities that are not directly interconnected may also be arranged by having *each intermediate utility* purchase and resell the power until it goes from the original generator of the sale power to the utility ultimately purchasing it. This is known (in the United States, at least) as *displacement*.

For example, consider a three-party transaction. A locates power and energy in C and makes an arrangement with an intervening system B for transmission. Then C sells to B and B sells to A. The price level to A may be set as the cost of C's generation plus the wheeling charges of B plus one-half of A's savings. It may also be set at B's net costs plus one-half of A's savings. Price is a matter of negotiation in this type of transaction, when prior agreements on pricing policies are absent.

Often, utility companies will enter into interchange agreements that give the amount and schedule of the interchange power but leave the final price out. Instead of agreeing on the price, the contract specifies that the systems will operate with the interchange and then decide on its cost after it has taken place. By doing so, the systems can use the actual load on the systems and the actual unit commitment schedules rather than the predicted load and commitment schedules. Even when the price has been negotiated prior to the interchange, utilities will many times wish to verify the economic gains projected by performing after-the-fact production costs.

Power systems are often interconnected with many neighboring systems and interchange may be carried out with each one. When carrying out the after-the-face production costs, the operations offices must be careful to duplicate the order of the interchange agreements. This is illustrated in Example 10C.

EXAMPLE 10C

Suppose area 1 of Example 10A was interconnected with a third system, here designated area 3, and that interchange agreements were entered into as follows.

Interchange agreement A: area 1 buys 300 MW from area 2

Interchange agreement B: area 1 sells 100 MW to area 3

Data for area 1 and area 2 will be the same as in Example 10A. For this example, we assume that area 3 will not reduce its own generation below 450 MW for reasons that might include unit commitment or spinning-reserve requirements. The area 3 cost characteristics are as follows.

Total Demand (MW)	Area 3 Incremental Cost (R/MWh)	Area 3 Total Production Cost (R/h)
450	18.125	8220.00
550	18.400	10042.00

First, let us see what the cost would be under a split-savings pricing policy if the interchange agreements were made with agreement A first, then agreement B.

	Area 1 Gen. (MW)	Area 1 Cost (R/h)	Area 2 Gen. (MW)	Area 2 Cost (R/h)	Area 3 Gen. (MW)	Area 3 Cost (R/h)
Start	700	13677.21	1100	18569.23	550	10042.00
After agreement A	400	8452.27	1400	23532.25	550	10042.00
After agreement B	500	10164.57	1400	23532.25	450	8220.00

Agreement A: Saves area 1 5224.94 R
Costs area 2 4963.02 R

After splitting savings, area 1 pays area 2 5093.98 R

Agreement B: Costs area 1 1712.30 R
Saves area 3 1822.00 R

After splitting savings, area 3 pays area 1 1767.15 R

Summary of payments:

Area 1 pays a net 3326.83 R
Area 2 receives 5093.98 R
Area 3 pays 1767.15 R

Now let the transactions be costed assuming the same split-savings pricing policy but with the interchange agreements made with agreement B first, then agreement A.

	Area 1 Gen. (MW)	Area 1 Cost (R/h)	Area 2 Gen. (MW)	Area 2 Cost (R/h)	Area 3 Gen. (MW)	Area 3 Cost (R/h)
Start	700	13677.21	1100	18569.23	550	10042.00
After agreement B:	800	15477.55	1100	18569.23	450	8220.00
After agreement A:	500	10164.57	1400	23532.25	450	8220.00

Agreement B: Costs area 1 1800.34 R
Saves area 3 1822.00 R

After splitting savings, area 3 pays area 1 1811.17 R

Agreement A: Saves area 1 5312.98 R
Costs area 2 4963.02 R

After splitting savings, area 1 pays area 2 5138.00 R

Summary of payments:

Area 1 pays a net 3326.83 R
Area 2 receives 5138.00 R
Area 3 pays 1811.17 R

Except for area 1, the payments for the interchanged power are different, depending on the order in which the agreements were carried out. If agreement A were carried out first, area 2 would be selling power to area 1 at a lower incremental cost than if agreement B were carried out first. Obviously, it would be to a seller's (area 2 in this case) advantage to sell when the buyer's (area 1) incremental cost is high, and, conversely, it is to a buyer's (area 3) advantage to buy from a seller (area 1) whose incremental cost is low.

When several two-party interchange agreements are made, the pricing must follow the proper sequence. In this example, the utility supplying the energy receives more than its incremental production costs no matter which transaction is costed initially. The rate that the other two areas pay per MWh are different and depend on the order of evaluation. These differences may be summarized as follows in terms of R/MWh.

	Cost Rates (R/MWh)	
Area	A Costed First	B Costed First
1 pays	16.634	16.634
2 receives	16.980	17.127
3 pays	17.673	18.112

The central dispatch of a pool can avoid this problem by developing a single cost rate for every transaction that takes place in a given interval.

10.6 OTHER TYPES OF INTERCHANGE

There are other reasons for interchanging power than simply obtaining economic benefits. Arrangements are usually made between power companies to interconnect for a variety of reasons. Ultimately, of course, economics plays the dominant role.

10.6.1 Capacity Interchange

Normally, a power system will add generation to make sure that the available capacity of the units it has equals its predicted peak load plus a reserve to cover unit outages. If for some reason this criterion cannot be met, the system may enter into a capacity agreement with a neighboring system, provided that neighboring system has surplus capacity beyond what it needs to supply its own peak load and maintain its own reserves. In selling capacity, the system that has a surplus agrees to cover the reserve needs of the other system. This may require running an extra unit during certain hours, which represents a cost to the selling system. The advantage of such agreements is to let each system schedule generation additions at longer intervals by buying capacity when it is short and selling capacity when a large unit has just been brought on-line and it has a surplus. Pure capacity reserve interchange agreements do not entitle the purchaser to any energy other than emergency energy requirements.

10.6.2 Diversity Interchange

Daily diversity interchange arrangements may be made between two large systems covering operating areas that span different time zones. Under such circumstances, one system may experience its peak load at a different time of the day than the other system simply because the second system is 1 h behind. If the two systems experience such a phenomenon, they can help each other by interchanging power during the peak. The system that peaked first would buy power from the other and then pay it back when the other system reached its peak load.

This type of interchange can also occur between systems that peak at different seasons of the year. Typically, one system will peak in the summer due to air-conditioning load and the other will peak in winter due to heating load. The winter-peaking system would buy power during the winter months from the summer-peaking system whose system load is presumably lower at that time of year. Then in the summer, the situation is reversed and the summer-peaking system buys power from the winter-peaking system.

10.6.3 Energy Banking

Energy-banking agreements usually occur when a predominantly hydro system is interconnected to a predominantly thermal system. During high water runoff periods, the hydro system may have energy to spare and will sell it to the thermal system. Conversely, the hydro system may also need to import energy during periods of low runoff. The prices for such arrangements are usually set by negotiations between the specific systems involved in the agreement.

Instead of accounting for the interchange and charging each other for the transactions on the basis of hour-by-hour operating costs, it is common practice in some areas for utilities to agree to a banking arrangement, whereby one of the systems acts as a bank and the other acts as a depositor. The depositor would "deposit" energy whenever it had a surplus and only the MWh "deposited" would be accounted for. Then, whenever the depositor needed energy, it would simply withdraw the energy up to MWh it had in the account with the other system. Which system is "banker" or "depositor" depends on the exchange contract. It may be that the roles are reversed as a function of the time of year.

10.6.4 Emergency Power Interchange

It is very likely that at some future time a power system will have a series of generation failures that require it to import power or shed load. Under such emergencies, it is useful to have agreements with neighboring systems that commit them to supply power so that there will be time to shed load. This may occur at times that are not convenient or economical from an incremental cost point of view. Therefore, such agreements often stipulate that emergency power be priced very high.

10.6.5 Inadvertent Power Exchange

The AGC systems of utilities are not perfect devices with the result that there are regularly occurring instances where the error in controlling interchange results in a significant, accumulated amount of energy. This is known as *inadvertent interchange*. Under normal circumstances, system operators will "pay back" the accumulated inadvertent interchange energy megawatt-hour for megawatt-hour, usually during similar time periods in the next week. Differences in cost rates are ignored.

Occasionally, utilities will suffer prolonged shortages of fuel or water, and the inadvertent interchange energy may grow beyond normal practice. If done deliberately, this is known as "leaning on the ties." When this occurs, systems will normally agree to pay back the inadvertent energy at the same time of day that the errors occurred. This tends to equalize the economic transfer. In severe fuel shortage situations, interconnected utilities may agree to compensate each other by paying for the inadvertent interchange at price levels that reflect the real cost of generating the exchange energy.

10.7 POWER POOLS

Interchange of power between systems can be economically advantageous, as has been demonstrated previously. However, when a system is interconnected with many neighbors, the process of setting up one transaction at a time with each neighbor can become very time consuming and will rarely result in the optimum production cost. To overcome this burden, several utilities may form a power pool that incorporates a central dispatch office. The power pool is administered from a central location that has responsibility for setting up interchange between members, as well as other administrative tasks. The pool members relinquish certain responsibilities to the pool operating office in return for greater economies in operation.

The agreement the pool members sign is usually very complex. The complexity arises because the members of the pool are attempting to gain greater benefits from the pool operation and to allocate these benefits equitably among the members. In addition to maximizing the economic benefits of interchange between the pool members, pools help member companies by coordinating unit commitment and maintenance scheduling, providing a centralized assessment of system security at the pool office, calculating better hydro-schedules for member companies, and so forth. Pools provide increased reliability by allowing members to draw energy from the pool transmission grid during emergencies as well as covering each others' reserves when units are down for maintenance or on forced outage.

Some of the difficulties in setting up a power pool involving nonaffiliated companies or systems arise because the member companies are independently owned and for the most part independently operated. Therefore, one cannot just make the assumption that the pool is exactly the same entity as a system under one ownership. If one member's transmission system is heavily loaded with power flows that chiefly benefit that member's neighbors, then the system that owns the transmission is entitled to a reimbursement for the use of the transmission facilities. If one member is directed to commit a unit to cover a reserve deficiency in a neighboring system, that system is also likewise entitled to a reimbursement.

These reimbursement arrangements are built into the agreement that the members sign when forming the pool. The more the members try to push for maximum economic operation, the more complicated such agreements become. Nevertheless, the savings obtainable are quite significant and have led many interconnected utility systems throughout the world to form centrally dispatched power pools when feasible.

A list of operating advantages for centrally dispatched power pools, ordered by greatest expected economic advantage, might look as follows:

1. Minimize operating costs (maximize operating efficiency).
2. Perform a system-wide unit commitment.
3. Minimize the reserves being carried throughout the system.
4. Coordinate maintenance scheduling to minimize costs and maximize reliability by sharing reserves during maintenance periods.
5. Maximize the benefits of emergency procedures.

There are disadvantages that must be weighed against these operating and economic advantages. Although it is generally true that power pools with centralized dispatch offices will reduce overall operating costs, some of the individual utilities may perceive the pool requirements and disciplines as disadvantageous. Factors that have been cited include.

1. The complexity of the pool agreement and the continuing costs of supporting the interutility structure required to manage and administer the pool.
2. The operating and investment costs associated with the central dispatch office and the needed communication and computation facilities.
3. The relinquishing of the right to engage in independent transactions outside of the pool by the individual companies to the pool office and the requirement that any outside transactions be priced on a split-saving basis based on pool members' costs.

4. The additional complexity that may result in dealing with regulatory agencies if the pool operates in more than one state.
5. The feeling on the part of the management of some utilities that the pool structure is displacing some of an individual system's management responsibilities and restricting some of the freedom of independent action possible to serve the needs of its own customers.

Power pools without central dispatch control centers can be administered through a central office that simply acts as a brokerage house to arrange transactions among members. In the opposite extreme, the pool can have a fully staffed central office with real-time data telemetered to central computers that calculate the best pool-wide economic dispatch and provide control signals to the member companies.

By far the most difficult task of pool operation is to decide who will pay what to whom for all the economic transactions and special reimbursements built into the pool agreement. There are several ways to solve this problem, and some will be illustrated in Section 10.7.2.

10.7.1 The Energy-Broker System

As with sales and purchases of various commodities or financial instruments (e.g., stock), it is often advantageous for interconnected power systems to deal through a broker who sets up sales and purchases of energy instead of dealing directly with each other. The advantage of this arrangement is that the broker can observe all the buy and sell offers at one time and achieve better economy of operation. When utilities negotiate exchanges of power and energy in pairs, the "market place" is somewhat haphazard like a bazaar. The introduction of a central broker to accept quotations to sell and quotations to purchase creates an orderly marketplace where supply, demand, and prices are known simultaneously.

The simplest form of "broker" scheme is the "bulletin board." In this type of scheme, the utility members post offers to buy or sell power and energy at regular, frequent intervals. Members are free to access the bulletin board (via some sort of data exchange network) at all times. Members finding attractive offers are free to contact those posting the offers and make direct arrangements for the transaction. Like any such informally structured market, many transactions will be made outside the marketplace. More complex brokers are those set up to arrange the matching of buyers and sellers directly, and, perhaps, to set transaction prices.

In one power broker scheme in use, the companies that are members of the broker system send hourly buy and sell offers for energy to the broker who matches them according to certain rules. Hourly, each member transmits an incremental cost and the number of megawatt-hours it is willing to sell or its decremental cost and the number of megawatt-hours it will buy. The broker sets up the transactions by matching the lowest cost seller with the highest cost buyer, proceeding in this manner until all offers are processed. The matched buyers and sellers will price the transaction on the basis of rules established in setting up the power broker scheme. A common arrangement is to compensate the seller for this incremental generation costs and split the savings of the buyer equally with the seller. The pricing formula for this arrangement is as follows. Let

F'_s = incremental cost of the selling utility (₽/MWh)

F'_b = decremental cost of the buying utility (₽/MWh)

F_c = cost rate of the transaction (₽/MWh)

Then,

$$F_c = F'_s + \frac{1}{2}(F'_b - F'_s) \\ = \frac{1}{2}(F'_s + F'_b) \tag{10.1}$$

In words, the transaction's cost rate is the average of the seller's incremental cost and the purchaser's decremental cost. In this text, decremental cost is the reduction in operating cost when the generation is reduced a small amount. Example 10D illustrates the power brokerage process.

EXAMPLE 10D

In this example, four power systems have sent their buy/sell offers to the broker. In the table that follows, these are tabulated and the maximum pool savings possible is calculated.

Utilities Selling Energy	Incremental Cost (₽/MWh)	MWh for Sale	Seller's Total* Increase in Cost (₽)
A	25	100	2500
B	30	100	3000
Utilities Buying Energy	**Decremental Cost (₽/MWh)**	**MWh for Purchase**	**Buyer's Total Decrease in Cost (₽)**
C	35	50	1750
D	45	150	6750

$$\text{Net pool savings} = (1750\text{ ₽} + 6750\text{ ₽}) - (2500\text{ ₽} + 3000\text{ ₽}) \\ = 8500\text{ ₽} - 5500\text{ ₽} = 3000\text{ ₽}$$

The broker sets up transactions as shown in the following table.

Transaction	Savings Computation		Total Transaction Savings (R)
1. A sells 100 MWh to D	100 MWh (45 – 25) R/MWh	=	2000
2. B sells 50 MWh to D	50 MWh (45 – 30) R/MWh	=	750
3. B sells 50 MWh to C	50 MWh (35 – 30) R/MWh	=	250
		Total	3000

The rates and total payments are easily computed under the split-savings arrangement. These are shown in the following table.

Transaction	Price (R/MWh)	Total cost (R)
1. A sells 100 MWh to D	35.0	3500
2. B sells 50 MWh to D	37.5	1875
3. B sells 50 MWh to C	32.5	1625
	Total	7000

A receives 3500 R from D; B receives 3500 R from D and C. Note that each participant benefits: A receives 1000 R above its costs; B receives 500 R above its costs; C saves 125 R; and D saves 1375 R.

The chief advantage of a broker system is its simplicity. All that is required to get a broker system into operation is a communications circuit to each member's operations office and some means of setting up the transactions. The transactions can be set up manually or, in the case of more modern brokerage arrangements, by a computer program that is given all the buy/sell offers and automatically sets up the transactions. With this type of broker, the quoting systems are commonly only informed of the "match" suggested by the broker and are free to enter into the transaction or not as each see fit.

Economists have sometimes argued that the broker pricing scheme should set one single "clearing price" for energy each time period. The logic behind this is that the market-determined price level should be based on the participants' needs and willingness to buy or sell. This removes the absolute need for quoting cost-based prices. Utilities would be free to quote offers at whatever price level they wished, but would be (under most rules that have been suggested) obligated to deliver or purchase the energy quoted at the market clearing price. The transactions market would be similar to the stock exchange. Objections raised have been that in times of shortage, price levels could rise dramatically and uncontrollably.

Power broker schemes can be extended to handle long-term economy interchange and to arrange capacity sales. This enables brokers to assist in minimizing costs for spinning reserves and coordinate unit commitments in interconnected systems.

10.7.2 Allocating Pool Savings

All methods of allocating the savings achieved by a central pool dispatch are based on the premise that no pool member should have higher generation production expenses than it could achieve by dispatching its own generation to meet its own load.

We saw previously in the pool broker system that one of the ways to allocate pool savings is simply to split them in proportion each system's net interchange during the interval. In the broker method of matching buyers and sellers based on their incremental and decremental costs, calculations of savings are relatively easy to make since the agreed incremental costs and amounts of energy must be transmitted to the broker at the start. When a central economic dispatch is used, it is easier to act as if the power were sold to the pool by the selling systems and then bought from the pool by the buying systems. In addition, allowances may be made for the fact that one system's transmission system is being used more than others in carrying out the pool transactions.

There are two general types of allocation schemes that have been used in U.S. pool control centers. One, illustrated in Example 10E, may be performed in a real-time mode with cost and savings allocations made periodically using the incremental and decremental costs of the systems. In this scheme, power is sold to and purchased from the pool and participants' accounts are updated currently. In the other approach, illustrated in Example 10F, the allocation of costs and savings is done after the fact using total production costs. Example 10E shows a scheme using incremental costs similar to one used by a U.S. pool made up of several member systems.

EXAMPLE 10E

Assume that the same four systems as given in Example 10D were scheduled to transact energy by a central dispatching scheme. Also, assume that 10% of the gross system's savings was to be set aside to compensate those systems that provided transmission facilities to the pool. The first table shows the calculation of the net system savings.

Utilities Selling Energy	Incremental Cost (R/MWh)	MWh for Sale	Seller's Total Increase in Cost (R)
A	25	100	2500
B	30	100	3000

Utilities Buying Energy	Decremental Cost (₨/MWh)	MWh for Purchase	Buyer's Total Decrease in Cost (₨)
C	35	50	1750
D	45	150	6750
		Pool savings	3000
Savings withheld for transmission compensation[a]			300
		Net savings	2700

[a] 10% savings withheld for transmission compensation.

Next, the weighted average incremental costs for selling and buying power are calculated.

Seller's weighted average incremental cost

$$= \left[\frac{(25\ ₨/\text{MWh} \times 100\ \text{MWh}) + (30\ ₨/\text{MWh} \times 100\ \text{MWh})}{100\ \text{MWh} + 100\ \text{MWh}}\right] = 27.50\ ₨/\text{MWh}$$

Buyer's weighted average decremental cost

$$= \left[\frac{(35\ ₨/\text{MWh} \times 50\ \text{MWh}) + (45\ ₨/\text{MWh} \times 150\ \text{MWh})}{50\ \text{MWh} + 150\ \text{MWh}}\right] = 42.50\ ₨/\text{MWh}$$

Finally, the individual transactions savings are calculated.

1. A sells 100 MWh to pool:

$$100\ \text{MWh}\ \frac{42.50 - 25\ ₨/\text{MWh}}{2} \times 0.9 = 787.50\ ₨$$

2. B sells 100 MWh to pool:

$$100\ \text{MWh}\ \frac{42.50 - 30\ ₨/\text{MWh}}{2} \times 0.9 = 562.50\ ₨$$

3. C buys 50 MWh from pool:

$$50\ \text{MWh}\ \frac{35 - 27.50\ ₨/\text{MWh}}{2} \times 0.9 = 168.75\ ₨$$

4. D buys 150 MWh from pool:

$$150\ \text{MWh}\ \frac{45 - 27.50\ ₨/\text{MWh}}{2} \times 0.9 = 1181.25\ ₨$$

$$2700.00\ ₨\ \text{Net savings}$$

The total transfers for this hour are then:

C buys 50 MWh for $42.5 \times 50 - 168.75 =$	1956.23 ₱
D buys 150 MWh for $42.5 \times 150 - 1181.25 =$	5193.75 ₱
Total	7150.00 ₱
A sells 100 MWh for $27.5 \times 100 + 787.5 =$	3537.50 ₱
B sells 100 MWh for $27.5 \times 100 + 562.5 =$	3312.50 ₱
	6850.00 ₱
Total transmission charge	300.00 ₱
Total	7150.00 ₱

The 300 ₱ that was set aside for transmission compensation would be split up among the four systems according to some agreed-upon rule reflecting each system's contribution to the pool transmission network.

The second type of savings allocation method is based on after-the-fact computations of individual pool member costs as if each were operating strictly so as to serve their own individual load. In this type of calculation, the unit commitment, hydro-schedules, and economic dispatch of each individual pool member are recomputed for an interval after the pool load has been served. This "own load dispatch" is performed with each individual system's generating capacity, including any portions of jointly owned units, to achieve maximum operating economy for the individual system.

The costs for these computed individual production costs are then summed and the total pool savings are computed as the difference between this cost and the actual cost determined by the central pool dispatch.

These savings are then allocated among the members of the pool according to the specific rules established in the pool agreement. One method could be based on rules similar to those illustrated previously. That is, any interval for which savings are being distributed, buyers and sellers will split the savings equally.

Specific computational procedures may vary from pool to pool. Those members of the pool supplying energy in excess of the needs of their own loads will be compensated for their increased production expenses and receive a portion of the overall savings due to a pool-wide dispatch. The process is complicated because of the need to perform individual system production cost calculations. Pool agreements may contain provisions for compensation to members supplying capacity reserves as well as energy to the pool. A logical questions that requires resolution by the pool members involves the fairness of comparing an after-the-fact production cost analysis that utilizes a known load pattern with a pool dispatch that was forced to use load forecasts. With the load pattern known with certainty, the internal unit commitment may be optimized to a greater extent that was feasible by the pool control center. Example 10F illustrates this type of procedure for the three systems of Example 10C for one period. In this example, only the effects of the economic dispatch are shown since the unit commitment process is not involved.

EXAMPLE 10F

The three areas and load levels are identical to those in Example 10C. (Generation data are in Examples 10A and 10B as well.) In this case, the three areas are assumed to be members of a centrally dispatched power pool. The pool's rules for pricing pool interchange are as follows.

1. Each area delivering power and energy to the pool in excess of its own load will receive compensation for its increased production costs.
2. The total pool savings will be computed as the difference between the sum of the production costs of the individual areas (each computed on the basis that it supplied its own load) and the pool-wide production cost.
3. These savings will be split equally between the suppliers of pool capacity or energy and the areas receiving pool-supplied capacity or energy.
4. In each interval where savings are allocated (usually a week, but in this example only 1 h), the cost rate for pricing the interchange will be one-half the sum of the total pool savings plus the cost of generating the pool energy divided by the total pool energy. The total pool energy is the sum of the energies in the interval supplied by all areas, each generating energy in excess of its own load.

The pool production costs are as follows.

Area	Area Load (MW or MWh)	Own-Load Production Cost (₨/h)
1	700	13677.21
2	1100	18569.23
3	550	10042.00
Total	2350	42288.44

Under the pool dispatch, areas 1 and 2 are dispatched at an incremental cost of 17.149 ₨/MWh to generate a total of 1900 MW. Area 3 is limited to supplying 450 MW of its own load at an incremental cost of 18.125 ₨/MWh. The generation and costs of the three areas and the pool under pool dispatch are given in the following table.

Area	Area Generation (MW or MWh)	Production Cost (₨/h)	Incremental Cost (₨/MWh)
1	458.9	9458.74	17.149
2	1441.1	24232.66	17.149
3	450.0	8220.00	18.125
Pool	2350.0	41911.40	17.149

Therefore, the total savings due to the pool dispatch for this 1 h are

$$42{,}288.44\ \text{Ʀ} - 41{,}911.40\ \text{Ʀ} = 377.04\ \text{Ʀ}$$

In this example, area 2 is supplying 341.1 MWh in excess of its own load to the pool. This is the total pool energy. Therefore, the price rate for allocating savings is computed as follows.

Cost of pool energy:

$$\begin{aligned} &\text{Cost of energy supplied to the pool by area 2} \\ &\qquad = 24{,}232.66\ \text{Ʀ} - 18{,}569.23\ \text{Ʀ} = 5663.43\ \text{Ʀ} \\ &\qquad + 1/2 \text{ pool savings} = \underline{188.52\ \text{Ʀ}} \\ &\qquad \text{Total} \quad 5851.95\ \text{Ʀ} \end{aligned}$$

$$\text{Interchange price rate} = \frac{5851.95}{341.1} = 17.156\ \text{Ʀ/MWh}$$

The final outcome for each area is shown in the following table.

Area	Pool Energy Received (MWh)	Interchange Cost (Ʀ)	Production Cost (Ʀ)	Net Cost (Ʀ)
1	+241.1	4136.34	9458.74	13595.08
2	−341.1	−5851.95	24232.66	18380.71
3	+100	1715.61	8220.00	9935.61
Pool		0	41911.40	41911.40

Note that each area's net production costs are reduced as compared with what they would have been under isolated dispatch. Furthermore, the ambiguity involved in pricing different transactions in alternative sequences has been avoided.

Example 10F is based on only a single load level so that after-the-fact unit commitment and production costing is not required. It could have been done on a real-time basis, in fact. This example also illustrates the complete transaction allocation that must be done for savings allocation schemes.

Complete own-load dispatch computations for cost and savings allocations are usually performed for a weekly period. The implementation may be complex since hourly loads and unit status data are required. An on-line, real-time allocation scheme avoids these complications.

No matter how these savings allocations are performed, you should appreciate that any estimates of "savings" involves finding the difference between actual, known costs and *costs as they might have been*. There is a great deal of room for disagreement about how to estimate these second, hypothetical costs.

10.8 TRANSMISSION EFFECTS AND ISSUES

This topic involves both technical and structural considerations. There are some technical issues that transcend the organizational market structure issues, but many of these arise only because of the multiple ownership of interconnected power transmission networks. There are basic technical issues of defining a network's capability to transfer power that involve physical capacity to handle power flows reliably (or *securely*). Even here (or is it *especially here*?), nontechnical matters are involved in defining *acceptable levels of network unreliability*. In an economic environment where capital and financing is available to develop multiple parallel paths in a transmission network, transmission capability may be restricted by the desire of the utilities and involved governmental agencies to insure very high levels of system security. Widespread blackouts and prolonged power shortages are to be avoided. Networks are designed with large capacity margins so that elements tend to be loaded conservatively. Normal failures of single major elements will not cause loss of load. Even simultaneous occurrences of two failures of major elements will not cause load curtailment. In most foreseeable circumstances, there will not be cascading outages that spread across the interconnected system. *Cascading outages* can occur where the loss of a transmission circuit, due to a prolonged fault, would result in the overloading of parallel circuits. These, in turn, might be opened in time by the action of protective relaying systems. Thus, the single event could cascade into a regional series of events that could result in a blackout.

In economic climates where capital and financing are difficult to obtain, and in areas where environmental restrictions prevent adding transmission capacity, power transmission networks may be designed using less-stringent reliability standards and operated in a fashion such that loads are expected to be curtailed when major transmission elements suffer outages. Security and reliability standards may be similar to the previous situation, with the exception that controlled load disconnection is not considered to be a "failure" event. Even in systems where "defensive operational scheduling" practices are normally followed (i.e., loss of single or two major system elements does not result in cascading outages), there are occasions where it is more economic to resort to using special system controls. These might drop load automatically when a remote generation source loses one of its transmission links to the system. This is a simple example; there are more complex arrangements that have been used. When a variety of specialized system control schemes are used, it is necessary to keep track of the various systems and keep every interconnected system abreast of changes and new developments.

In any interconnected system, there is a need to define in quantitative terms the maximum amount of power that may be transferred without violating whatever system reliability and security criteria are in place. Therefore, it is necessary to consider the types of operating limitations that exist in AC power networks. These include thermal limits sets by the capability of the lines and apparatus to absorb and dissipate the heat created by the current flowing in the various elements. These limits are usually expressed as a maximum allowable temperature rise above specified ambient conditions. The intent is to prevent the extreme, sustained temperatures that might cause lines to sag and equipment to be damaged. Even with these straightforward thermal limitations, there are variable ambient conditions that make actual danger points occur in the summer at lower power transfers than in the colder months. Next are limits set by the interplay of system limitations, equipment limitations, economics, and service reliability ("security") standards. These include voltage–VAR-related conditions and stability considerations.

Voltage and VAR conditions arise because voltage magnitudes within the system must remain within a bandwidth that is set by the voltage tolerances of both system and consumer equipment. Large high-voltage equipment and consumer equipment (motors, transformers, etc.) are generally limited to excursions of about $\pm 5\%$ of their rated voltage. The voltage magnitude bandwidth tolerance on the system is affected (and generally enlarged) by the ability of various voltage-correcting devices to restore voltages to a bandwidth acceptable to the apparatus. Key control devices include tapchanging transformers and various types of VAR-supply devices. At shorter transmission distances (say 50 miles or 100 km), the thermal limits and voltage–VAR limitations generally are the restricting system conditions. Of course, it is theoretically possible to add additional circuits and VAR-support equipment, but economic considerations generally set a practical limit on what is done to increase transmission capacity.

Transmission capability limits can be imposed by voltage instability, steady-state stability, and transient stability. In all cases, the network has to be able to survive possible conditions that can lead to unstable situations. These instability-inducing conditions usually become more intense as the system loading increases. The need to avoid these operating regimes then places a practical limit on the power that can be transmitted. At longer distances it is usually transient stability that sets the limits. The various limits are found by testing the network under increasingly heavy loading conditions and seeking ways to alleviate or prevent the instabilities. At some point, it becomes impossible or uneconomic to increase the limits further. Besides economic considerations, the actual power transfer limitations found will depend upon the testing criteria utilized. Is it sufficient to test the network's ability to survive a single-phase fault that is successfully cleared and the line reclosed, or should the network be tested using a bolted three-phase fault that requires switching a line segment?

10.8.1 Transfer Limitations

The operators in an interconnected AC system are interested in the limits to the amount of power that may be transferred between various systems or buses. The amount of power transfer capability available at any given time is a function of the system-wide pattern of loads, generation and circuit availability. This has led the United States systems to establish definitions of "incremental transfer capability." These definitions depend upon testing the network to meet selected security constraints (one or two simultaneous outages) under various sets of operating conditions to determine the added ("incremental") power that maybe transferred safely. This requires the cooperative efforts of a number of utilities in a region and only provides general guidelines concerning the transfer capability limits.

All of these tests and limitations depend a great deal upon the use of subjective criteria, definitions, and procedures that are a result of mutual agreement amongst the utilities. Practices differ. As an example, take the matter of determining the ability of an interconnected system to transmit an additional block of 500 MW between two systems separated by one or more intervening systems. If the operators test the systems' capability under the existing and planned optimal generation schedules, the network's loading criteria are violated. However, by shifting generation by a fairly small amount, the transfer would satisfy all of the systems' criteria. Should the transfer take place? In the systems in North America the answer would generally be "yes," with the added proviso that the cost for the transfer would include the recovery of the added generation cost of the systems that shifted generation off of an optimal economic dispatch.

Transfer limits can be determined for relatively simple interconnections where DC approximations are satisfactory to establish network flows. Sometimes these techniques may be used to study incremental flows. But, in most cases, it requires an AC power flow of some sort to investigate transfer limits and answer questions similar to the one in the previous paragraph.

This leads to what has been termed the "busy-signal problem." When I attempt to place a call that would require the use of an already-loaded communication channel, the system controls attempt to reroute my call, and if they are unsuccessful, I receive a busy signal. In present AC power systems, if a request is made in initiate a new transaction over a transmission system that is loaded to near maximum capability, it is feasible to do a moderate amount of "rerouting" of power flows by shifting generation and perhaps some switching of circuits. But if these measures are unsuccessful, or precluded by current operating practices, I will only find out some time after the request has been made, and, unless I am conversant in power system operating practices, I may not understand why the particular answer was given.

This is the point in the discussion where institutional problems become quite important. As long as the parties that are interested are interconnected electric utilities and other technically competent organizations that all can agree with each other about the operating rules, definitions of transfer capability, and the various assumptions used in establishing limitations, there is not a serious problem. Suppose, however, that all these parties do not agree. Suppose that some are satisfied with the present arrangements while others are eager to expand the network capability for the marketing, or purchasing, of power over a wider geographic area. They would like a concrete definition of network transfer capacity that did incorporate so many variable and ambiguous factors. The lack of a simple "busy signal" becomes even more pressing when nonutility entities are permitted access to the transmission system to make sales and purchases.

The situation is similar when measures to relieve local constraints are required in order to facilitate the use of the interconnected system by nonlocal parties. Who should pay for these measures? How should the costs be allocated? These are all real concerns when the interconnected system is owned by many individual utilities and serves the needs of even more individual organizations.

10.8.2 Wheeling

The term "wheeling" has a number of definitions; we will stay with a simple one. Wheeling is the use of some party's (or parties') transmission system(s) for the benefit of other parties. Wheeling occurs on an AC interconnection that contains more than two utilities (more properly, two parties) whenever a transaction takes place. (If there are only two parties, there is no third party to perform wheeling.) As used here, the term "parties" includes both utility and nonutility organizations.

Consider the six interconnected control areas shown in Figure 10.2. Suppose areas A and C negotiate the sale of 100 MW by A to C. Area A will increase its

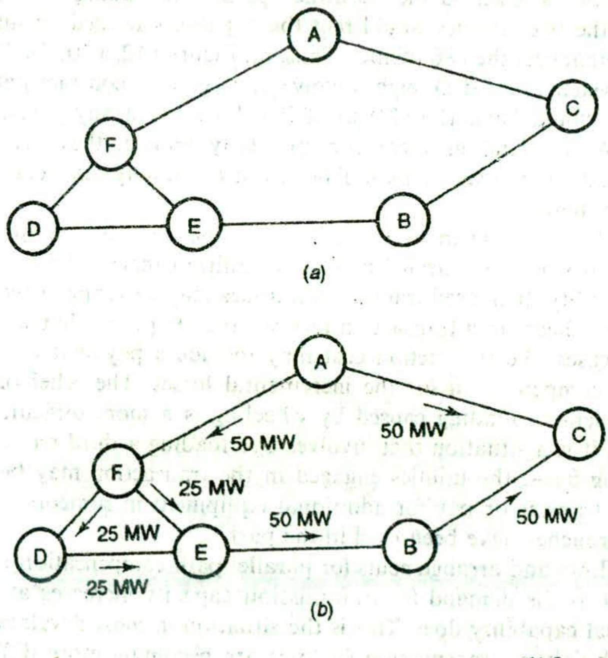

FIG. 10.2 Six interconnected control areas. (*a*) Configuration; (*b*) Incremental power flows when area A sells 100 MW to area C.

scheduled net interchange by 100 MW and C will reduce its net interchange schedule by the same amount. (We ignore losses.) The generation in A will increase by the 100 MW sale and that in C will decrease by the 100 MW purchase. Figure 10.2*b* shows the resulting *changes* in power flows, obtained by finding the difference between power flows before and after the transaction. Note that not all of the transaction flows over the direct interconnections between the two systems. The other systems are all wheeling some amount of the transaction. (In the United States, these are called "*parallel path or loop flows.*")

The number of possibilities for transactions is very large, and the power flow pattern that results depends on the configuration and the purchase–sale combination plus the schedules in all of the systems. In the United States, various arrangements have been worked out between the utilities in different regions to facilitate interutility transactions that involve wheeling. These past arrangements would generally ignore flows over parallel paths were the two systems were contiguous and owned sufficient transmission capacity to permit the transfer. (This capacity is usually calculated on the basis of nominal or nameplate ratings) In that case, wheeling was *not* taking place, by mutual agreement. The extension of this arrangement to noncontiguous utilities led to the artifice known as the "contract path." In making arrangements for wheeling, the two utilities would rent the capability needed on any path that would interconnect the two utilities. Thus, on Figure 10.2, a 100-MW transaction between systems A and D might involve arranging a "contract path" between them that would have 100 MW available. Flows over any parallel paths are ignored. As artificial as these concepts may appear, they are commercial arrangements that have the merit of facilitating mutually beneficial transactions between systems.

Difficulties arise when wheeling increases power losses in the intervening systems and when the parallel path flows utilize capacity that is needed by a wheeling utility. Increased transmission losses may be supplied by the seller so that the purchaser in a transaction receives the net power that was purchased. In other cases, the transaction cost may include a payment to the wheeling utility to compensate it for the incremental losses. The relief of third-party network element loading caused by wheeling is a more difficult problem to resolve. If it is a situation that involves overloading a third party's system on a recurring basis, the utilities engaged in the transaction may be required to cease the transfer or pay for additional equipment in someone else's system. Both approaches have been used in the past.

Loop flows and arrangements for parallel path compensation become more important as the demand for transmission capacity increases at a faster rate than actual capability does. This is the situation in most developed countries. New, high-voltage transmission facilities are becoming more difficult to construct. Another unresolved issue has to do with the participation of organizations that are basically consumers. Should they be allowed access to the power transmission network so that they may arrange for energy supplies from nonlocal resources? In the deregulated natural gas industry in the United States, this has been done.

10.8.3 Rates for Transmission Services in Multiparty Utility Transactions

Rates for transmission service have a great deal of influence on transactions when wheeling is involved. We have previously considered energy transaction prices based on split-savings concepts. Where wheeling services are involved, this same idea might be carried over so that the selling and wheeling utilities would split the savings with the purchaser on some agreed-upon basis. Both the seller and wheeling systems would want to recover their costs and would wish to receive a profit by splitting the savings of the purchaser. Some (many economists) would argue that transmission services should be offered on the basis of a "cost plus" price. A split-savings arrangement involving four or five utility systems might lose its economic attractiveness to the buyer by the time the potential savings were redistributed.

The notion of selling transmission service is not new. A number of different pricing schemes have been proposed and used. Most are based upon simplified models that allow such fictions as the "contract path." Some are based on an attempt to mimic a power flow, in that they would base prices on incremental power flows determined in some cases by using DC power flow models. The very simplest rates are a charge per MWh transferred, and ignore any path considerations.

More complex schemes are based on the "marginal cost" of transmission which is based on the use of bus incremental costs (BIC). The numerical evaluation of BIC is straightforward for a system in economic dispatch. In that case, the bus penalty factor times the incremental cost of power at the bus is equal to the system λ, except for generator buses that are at upper or lower limits. This is true for load buses as well as generator buses. (We will treat this situation in more detail in Chapter 13 on the optimal power flow.)

Consider any power system in economic dispatch.

1. If we have a single generator, then the cost to deliver an additional small increment of power at the generator bus is equal to the incremental cost of power for that generator.
2. If we have more than one generator attached to a bus and this is the only source of power, and the generators have been dispatched economically (i.e., equal λ), then the cost to deliver an additional small increment of power at this bus is equal to λ.
3. If there are multiple generators at different buses throughout the power system, and they have been dispatched economically, i.e., accurate penalty factors have been calculated and used in the economic dispatch—then the cost of delivery of an addition small increment of power at any individual generator bus will be that generator's own incremental cost. This cost will

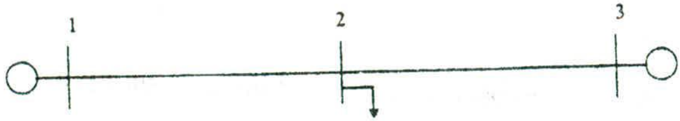

FIG. 10.3 Three-bus system.

not be equal across the system due to the fact that each generator's incremental cost is multiplied by its penalty factor.

It is important to stress that we are talking of an "additional small increment" of power at a bus and not a large increment. If the power increase is very small, the three statements above hold. If we are talking of a large increment in power delivered anywhere, the optimal dispatch must be recalculated and the cost is not equal to the incremental cost in any of the three cases above.

If we have the case shown in Figure 10.3, the power is all delivered to a load bus that is separated from either generator by a transmission line. In this case, the incremental cost of delivery of power to the load is not equal to the incremental cost of delivery at either generator bus. The exact value of the incremental cost at the load bus can be calculated, however, using the techniques developed in Chapter 13 (see Section 13.7). The incremental cost to deliver power at a bus is called the bus incremental cost (BIC) and plays a very important role in the operation of modern power systems. For a power system without any transmission limitations, the BIC at any bus in the system will usually be fairly close to the BIC at other buses. However, when there is a transmission constraint, this no longer holds.

Suppose the following situation were to arise in the system in Figure 10.3.

1. Generator 1 has high incremental cost and is at its low limit.
2. Generator 3 has low incremental cost and is not at either limit.

In such a case, the BIC at the load bus will be very close to the low incremental cost of the generator at bus 3.

Now let there be a limit to the power flowing on the transmission line from bus 3 to bus 2 so that no further power can be generated at bus 3. When the load is increased at bus 2, the increase must come entirely from the generator at bus 1 and its BIC will be much higher, reflecting the incremental cost of the bus 1 generator. Thus, the BICs are very useful to show when loading of the transmission system shifts the cost of delivery at certain buses in the network.

Next, let us consider how the bus incremental costs can be used to calculate the *short run marginal costs* (SRMC) of wheeling. Figure 10.4 shows three systems, A, B, and C, with A selling P_W MW to system C and system B wheeling that amount. The figure shows a single point for injecting the power (bus 1) and a single point for delivery to system C (bus 2). The operators of the wheeling

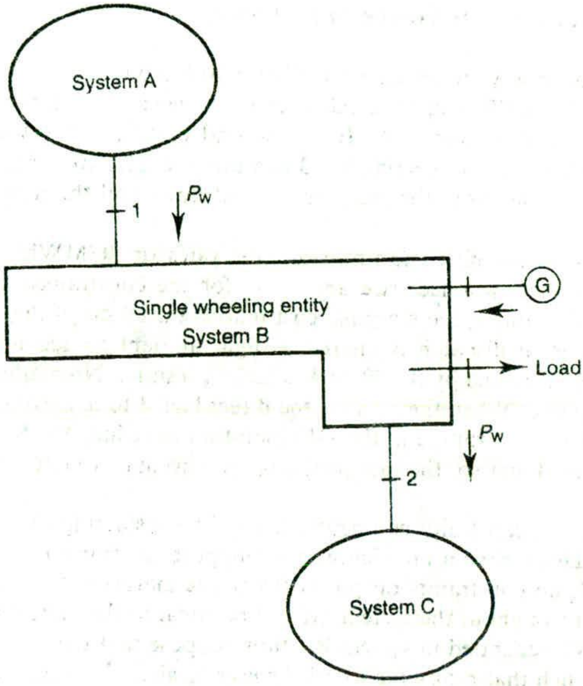

FIG. 10.4 Simple wheeling example.

system, B, can determine the incremental cost of power at both buses by using an optimal power flow (OPF). If these operators were to purchase the block of wheeled power at bus 1 at the incremental cost there, and sell it to system C at the incremental cost of power at bus 2, they would recover their (short run) marginal cost of transmission. Many engineers and economists have suggested that transmission service prices should be based upon these marginal costs since they include the cost of incremental transmission losses and network constraints. The equation to determine this marginal cost is,

$$\Delta F = (\partial F/\partial P_W)\Delta P_W = [\partial F/\partial P_i - \partial F/\partial P_j]\Delta P_W \tag{10.2}$$

where the power ΔP_W is injected at bus i and withdrawn at bus j. Various implementations of the OPF may be programmed to determine the rate-of-change of the objective function with respect to independent variables and constraints. These computations may be used to evaluate the marginal transmission cost directly.

The six-bus case used previously in Chapter 4 may be used to illustrate these ideas. Two separate wheeling examples were run. In both examples, 50 MW were injected at bus 3 and withdrawn at bus 6. In the first case, no flow limits were imposed on any circuit element. Figure 10.5 shows the power flow that results when the OPF is used to schedule the base case using the generation cost data given in Example 4E. In the second case, a 100-MVA limit was imposed on the circuit connecting bus 3 and bus 6. Figure 10.6 shows the OPF results for this case. Note the redistribution of flows and the new generation schedule.

The short-run marginal transmission cost rates (in R/MWh) found were 0.522 for the unconstrained case and 3.096 for the constrained case. In the unconstrained example, the marginal cost reflects the effects of the incremental losses. The system dispatch is altered a slight amount to accommodate the additional losses caused by the 50-MW wheeling transfer. No major generation shifts are required. When the flow on the direct line, 3 to 6, is constrained, the generation pattern is shifted in the OPF solution to reduce the MVA flow on that circuit. In doing so, the marginal cost of wheeling is increased to reflect that change.

The effect of a constraint can be illustrated by considering the three-system wheeling situation shown on Figure 10.4. Suppose the transmission system is lossless. With no constraints on power flows, the marginal cost of power will be the same throughout the system. (It will be equal to the incremental cost of the next MWh generated in system B.) Now suppose that there is a constraint in system B such that *before the wheeled power is injected*, no more power may flow from the area near bus 1 to loads near bus 2. (See Figure 10.7 which shows a cut labeled "Transmission bottleneck.") Then, when the power to be wheeled is injected at bus 1 and withdrawn at bus 2, the schedule in system B will be adjusted so that the delivered power is absorbed near bus 1 and generated by units near bus 2. The difference in marginal costs will now increase, reflecting the marginal cost of the constraint. With no constraint violations, marginal costs of wheeling rise gradually to reflect incremental losses. When constraints are reached, the marginal wheeling costs are more volatile and change rapidly.

Marginal cost-based pricing for transmission services has a theoretical appeal. Not everyone is in agreement that transmission services should be priced this way. If the entire transaction is priced at the marginal cost rate after the transaction is in place, the wheeling utility may over- or under-recover its changes in operating costs. Perhaps more importantly, short-run marginal operating costs do not reflect the revenue required to pay the costs related to the investments in the wheeling system's facilities. These facilities make it possible to wheel the power. (It is quite possible that short-run marginal wheeling costs could be negative if a transaction were to result in incremental power flows that reduced the losses in the wheeling system.) Any pricing structure for transmission service needs to incorporate some means of generating the funds required to install and support any new facilities that are needed in order to accommodate growing demands for service. These are the long-run marginal costs. If the transmission network is to be treated as a separate entity, the price structure for transmission service needs to include the long-run costs as well as short-run operating costs.

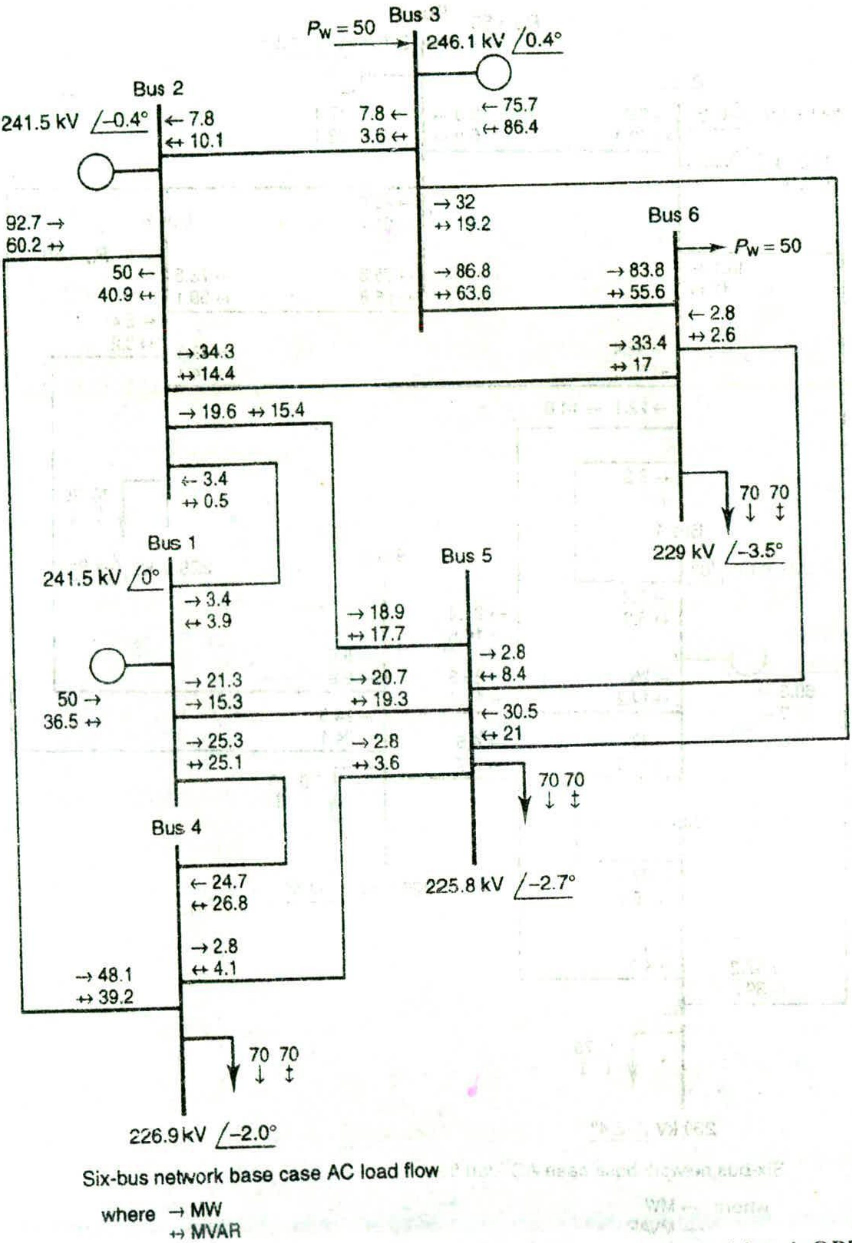

FIG. 10.5 Six-bus case with 50 MW being wheeled between bus 3 and bus 6. OPF schedule with no line-flow constraint.

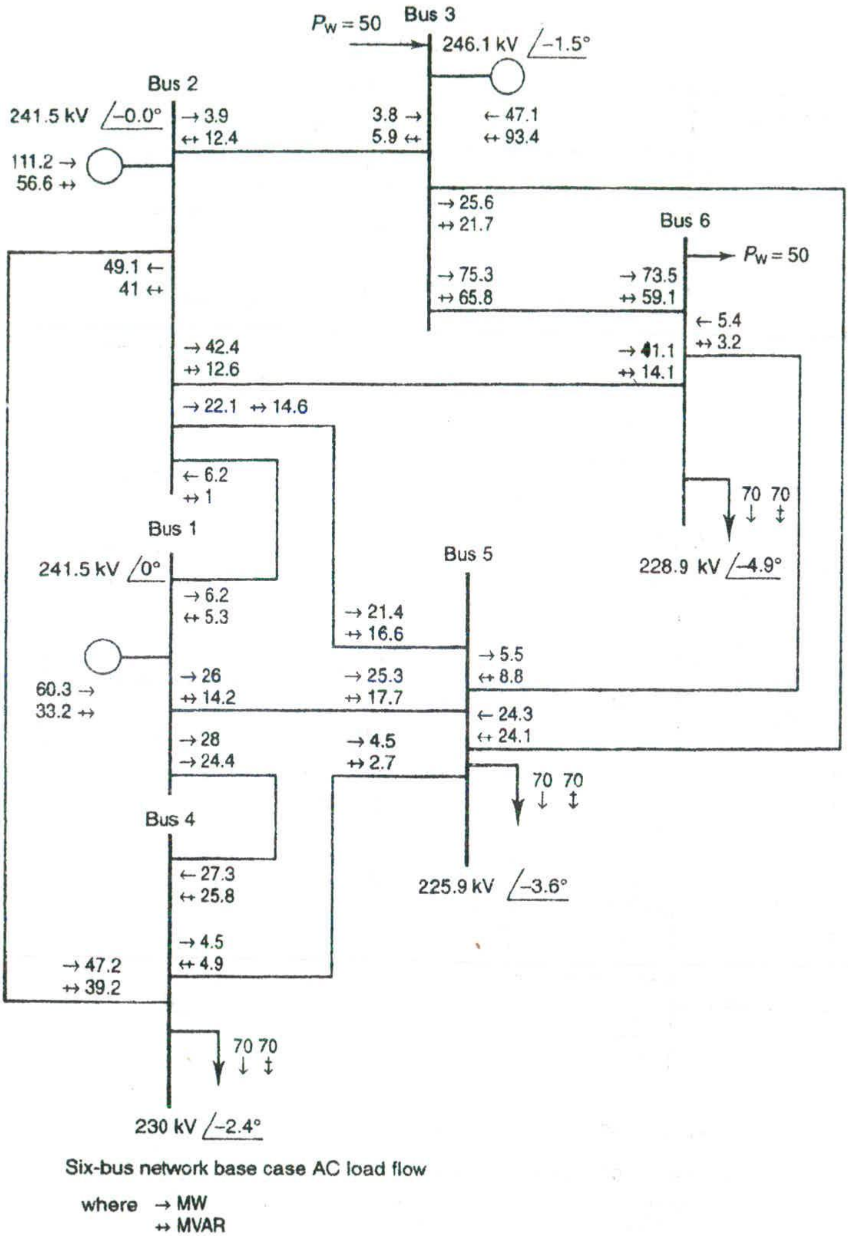

FIG. 10.6 Six-bus case with $P_W = 50$ MW and 100-MVA flow constraint imposed on lines 3–6.

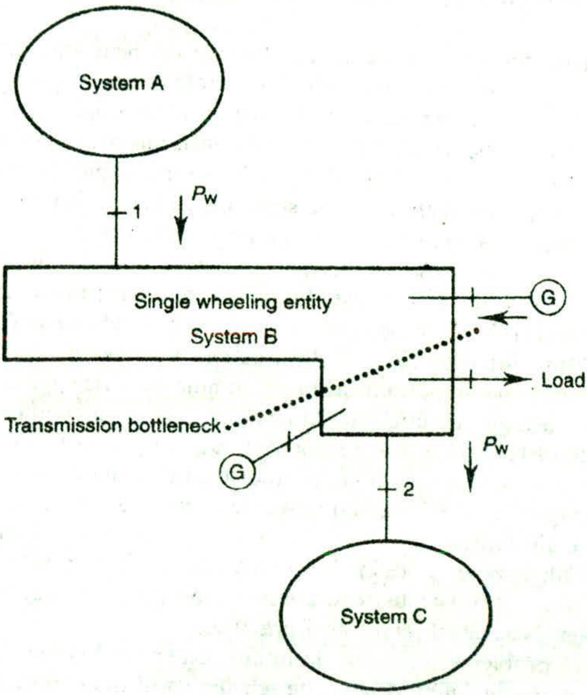

FIG. 10.7 Simple wheeling example with a "transmission bottleneck."

10.8.4 Some Observations

The nature of the electric utility business is changing. In the United Kingdom, the nationwide system was split into several generating companies and 11 regional distribution companies. The former state-owned system was privatized and a market set up to permit the introduction of independent generating companies. Similar developments have taken place in South America and the Philippines. In North America, these types of developments may result in changes in the scheduling and operation of electric power systems. It is conceivable that regional control centers may have the primary function of scheduling the use of the transmission system. Generation dispatch within any organization could still be based on minimizing operating costs, but constraints might be imposed by the transmission system dispatch and the scheduling of transactions could become the primary task of the regional control centers. It is too early (July 1995 at the time of writing) to tell if this will happen and exactly how it might happen.

10.9 TRANSACTIONS INVOLVING NONUTILITY PARTIES

Transactions involving nonutility organizations are increasing. A growing number of larger nonutility generators are being developed. Some of these are large industrial firms that have a need for process heat and steam and can generate electric energy for sale to others at very favorable costs. In some areas, nonutility generating companies have been created to supply some of the needs for new capacity in the region. These are established as profit-making organizations and not as regulated utilities. They must operate in parallel with the utility system and, therefore, there must be some coordination between the groups. The type of relationship and specific operating rules vary.

The customers of these nonutility generators may be utilities or retail consumers. Utilities may purchase the power for resale; this is classified as a "*wholesale market*." Where sales are made directly to consumers (certain large industrial firms, for example), the transaction is a "*retail*" transaction. The distinction may be important from a commercial viewpoint because the transactions usually require the utilization of the interconnected utilities transmission systems, as well as the load's local utility transmission system. The same distinction between a wholesale and retail transaction would be made if the generating party to a transaction was an electric utility that was making a sale to a retail consumer located in the service territory of another, interconnected utility. When wheeling is involved, the distinction between wholesale and retail transactions tends to become more significant, particularly in the United States because of established practices.

Technical problems involving nonutility generators primarily involve coordination and scheduling issues. The scheduling of the nonutility generator's level of output may be handled in different fashions. It could have a fixed output contract, it might be scheduled by the local utility's control center, or it could be dispatched to meet the load(s) of the buyer of its power. In a market structured like the scheme developed in the United Kingdom, the schedule for some suppliers is determined by a posted price level.

The next four figures illustrate some of the control area configurations that can occur with nonutility parties involved in transactions. In each figure, the nonutility generator is denoted by "G". In Figure 10.8, the generator G is supplying power to the local utility, a wholesale transaction. The dispatch of G

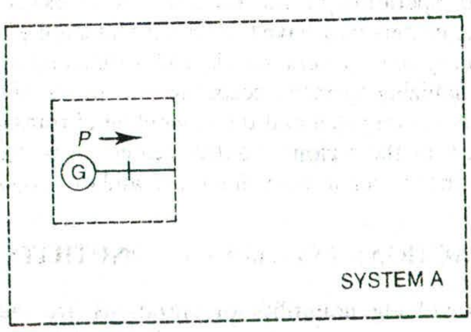

FIG. 10.8 Nonutility generator G delivering P MW to local system A.

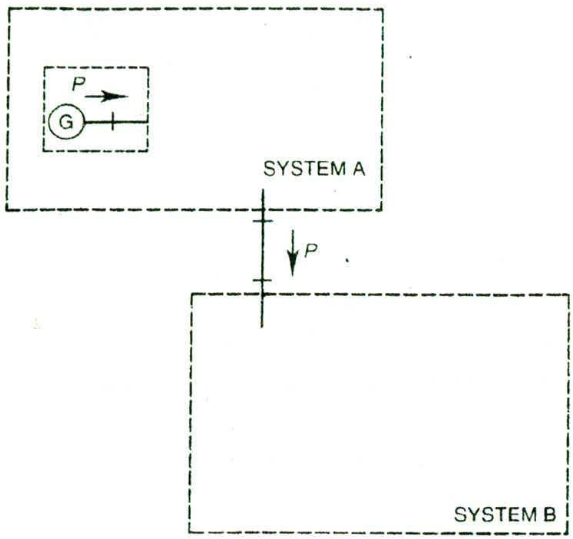

FIG. 10.9 Nonutility generator G delivering P MW to system B.

might be fixed, under control of the local utility, or be based upon a posted purchase price for energy and power. The utility AGC system could treat the generator as a local source or as part of scheduled interchange. In Figure 10.9, the generator is supplying power to a remote utility, and wholesale wheeling is involved. The output of G would be treated as scheduled interchange by both systems.

Suppose the generator G were to sell his output to a retail customer located within the service territory of the local utility. This is illustrated on Figure 10.10. This transaction requires retail wheeling by system A. The unit G could be scheduled in a variety of different fashions depending upon the agreement with system A. It might follow the load demands of the customer, in which case the

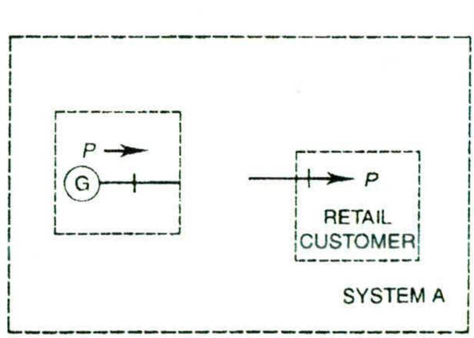

FIG. 10.10 Nonutility generator G delivering power to a retail customer in system A.

utility might treat the output and load as an interchange in its AGC system. If G were contracted to supply a fixed output level, utility A could treat it as a must-run unit and include both the load and the unit in its AGC system.

When this type of transaction involves a retail customer located in an interconnected system, such as shown in Figure 10.11, the situation is more complicated. One alternative would be for system A to treat the output of G as part of a scheduled interchange, with all of the output being delivered to B. System B could then treat the interchange as a schedule between A and the retail customer. The possible arrangements are many. The same type of arrangements would be required if the source were not the nonutility generator G but a third utility, say system C, that was supplying the customer in system B. Further, the "retail customer" could be a distribution utility; in which case "wholesale wheeling" is involved even though the physical situations are identical.

There has been a general movement towards the development of a nonutility generation industry. In many areas, the utilities (particularly those that face a shortage of generation capacity) encourage the installation of unregulated generation resources, and, in some instances, the utilities themselves have become involved in this industry. The movements towards privatization and deregulation encourage this trend. The situation with regard to allowing retail customers access to the transmission system is more contentious. There are a number of larger industrial firms where the cost of electric energy is a significant portion of their cost of production. Many of these organizations would like to obtain access to energy from sources other than the local utility. The issues are unresolved as yet.

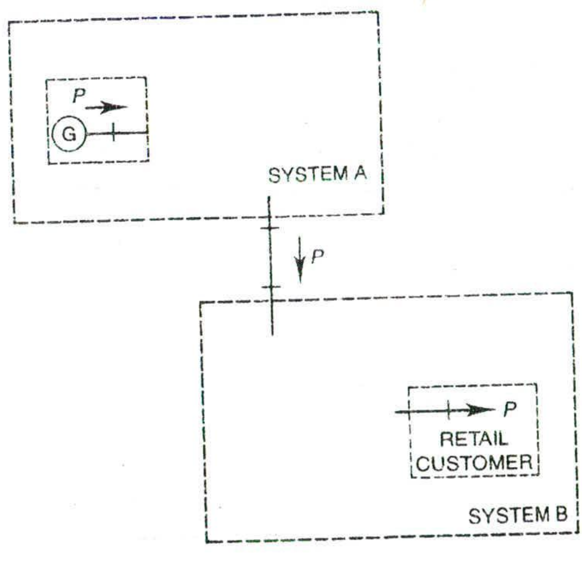

FIG. 10.11 Nonutility generator G delivering power to a retail customer in system B.

In countries where former integrated government power systems have been broken up and privatized, the industry structure seems to be headed for one where the bulk power transmission systems and central dispatch system remain as regulated monopolies. They have the responsibility to provide a market for the purchase and sale of generation and to schedule the operation of the power system to accommodate the generating utilities, the private generating organizations, and the distribution utilities. Furthermore, they may have to coordinate the operation of the system to facilitate the implementation of supply–purchase contracts made directly.

On the other hand, the trend in the United States seems to be less uniform. Some larger transmission-owning utilities favor a system based upon the centrally dispatched power pool. In this concept, the central dispatch office would be responsible for controlling generation and the transmission network. Contracts between buyers and sellers could be made separately, but the actual generation would be the result of an economic schedule of all of the units. Pool agreements would be structured similarly to existing power pool agreements, where no generating entity would have an operating cost higher than the one that it would have had, absent the pool control. This type of arrangement preserves the technical control of the system in the utility, while theoretically permitting any sort of transaction to take place. The "devil is in the details;" prices for transmission and generation services would require careful definition and, perhaps, continued regulatory supervision.

Other transmission-owning utilities appear to favor a more loosely structured market where transactions could be made between various parties, subject to the availability of transmission capacity. Transmission use would then become a separately priced item. This would, it is claimed, allow third-party brokers to make a more efficient (economic) marketplace. Here, the sticking points are apt to be the control and availability of transmission services, as well as their pricing. Technical problems may require more utility control than is deemed acceptable by "free marketers."

Utilities without extensive transmission want access to the networks of others in order to avail themselves of the generation markets. Large industrial concerns with significant electrical consumption are in the same camp. These groups advocate open transmission access with continued regulatory supervision of transmission rates and control, but market-determined pricing for power and energy.

PROBLEMS

10.1 Four areas are interconnected as shown in Figure 10.12. Each area has a total generation capacity of 700 MW currently on-line. The minimum loading of these units is 140 MW in each area. Area loads for a given hour are as shown in Figure 10.12. The transmission lines are each sufficient to transfer any amount of power required.

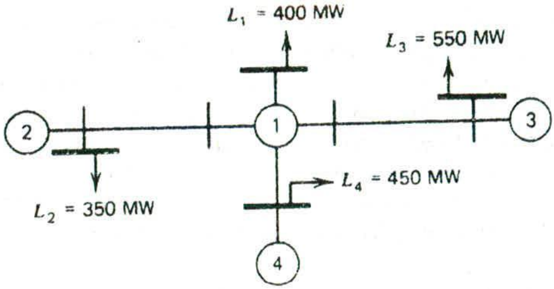

FIG. 10.12 Four-area system for Problem 10.1.

The composite input–output production cost characteristics of each area are as follows:

$$F_1 = 200 + 2P_1 + 0.005P_1^2 \qquad (\text{R}/\text{h})$$
$$F_2 = 325 + 3P_2 + 0.002143P_2^2 \quad (\text{R}/\text{h})$$
$$F_3 = 275 + 2.6P_3 + 0.003091P_3^2 \ (\text{R}/\text{h})$$
$$F_4 = 190 + 3.1P_4 + 0.00233P_4^2 \quad (\text{R}/\text{h})$$

In all cases, $140 \leq P_i \leq 700$ MW. Find the cost of each area if each independently supplies its own load, and the total cost for all four areas.

10.2 Assume that area 1 of Problem 10.1 engages in two transactions.

a. Area 1 buys 190 MW from area 2.

b. Area 1 sells 120 MW to area 3.

For each of these transactions, the price is based upon a 50–50 split-savings agreement. Find the price of each transaction, the net generation costs for each area including the sum it pays or receives under the split-savings agreement, with the order of the transactions (as given above) being as follows.

i. a then b.

ii. b then a.

In both instances, find the total cost for the four-area pool.

10.3 Assume that the four areas of Problem 10.1 are centrally dispatched by a pool control center.

a. Find the generation and production cost in each area.

b. Assume a split-savings pool agreement such that each area exporting receives its increased costs of production plus its proportionate share of 50% of the pool savings. Find the cost per MWh of transfer energy (i.e., "pool energy") and the net production cost of each area.

10.4 Assume that the four areas of Problem 10.1 are members of a "power broker." Previous to the hour shown in Problem 10.1, each area submits quotations to the broker to sell successive blocks of 25 or 50 MW and bids to purchase blocks of 25 or 50 MW. In furnishing these data to the broker, assume that the prices quoted are the average incremental costs for the block. The broker matching rules are as follows.

Rule 1. Quotations to sell and bids to buy are matched only wherever there is a direct connection between the quoting and bidding company.

Rule 2. Transactions are arranged in a priority order where the lowest remaining incremental cost for the quoting area is matched with the highest decremental cost for the bidding areas. [That is, lowest available incremental cost energy available for sale is matched with the area with the greatest available potential incremental cost savings (= decremental cost).]

Rule 3. "Matches" may be made for all or part of a block. The remainder of the block must be used, if possible, before the next block is utilized. Matching will cease when the absolute value of the difference between the incremental and decremental cost drops below 0.33 R/MWh.

Rule 4. No area may be both a buyer and a seller in any given hour.

Rule 5. The price per MWh for each matched transaction is one-half the sum of the absolute values of the incremental and decremental costs.

For this problem, assume that quotes and bids are supplied to the broker by each area as follows.

Area	Quotes to Sell	Quotes to Buy
1	100 MW in 25 MW blocks	100 MW in 25 MW blocks
2	200 MW in 50 MW blocks	None
3	None	200 MW in 50 MW blocks
4	25 MW	25 MW

a. Set up the power broker matching system and establish the transactions that can take place and the price of each.

b. Assume that all feasible transactions take place and find the net production cost to each area and the pool.

10.5 Repeat Problem 10.4 with the following assumptions simultaneously taken in place of those in Problem 10.4.

a. Each area is interconnected with every other area and transfers may take place directly between all pairs of areas.

b. The matched transactions will proceed until the difference between decremental costs is zero instead of 0.33 ₱/MWh.

10.6 Repeat Problem 10.5 with one "clearing price" that applies to all transactions and is equal to the price determined for the last matched transaction.

10.7 Use the cost data for the six-bus base case in Chapter 4, and the power flow and generator output data presented in Figures 10.5 and 10.6 that illustrate the wheeling of 50 MW between bus 3 and bus 6. We want to compute an estimate of the utility's net costs under all three cases. Let

Net cost = total production cost for all generators − charges for wheeling

Produce a table that shows the power generation for each unit and the total system operating cost in ₱/h for the three cases: the base case and the two wheeling cases. The generation data for an optimal power flow calculation of the base case to minimize operating costs with no line flow limits shows the following:

$$P_1 = 50.00 \text{ MW}$$

$$P_2 = 89.63 \text{ MW}$$

$$P_3 = 77.07 \text{ MW}$$

$$P_{\text{loss}} = 6.70 \text{ MW}$$

and a cost rate of 3126.36 ₱/h. For the two cases with 50 MW being wheeled, compute the charges for wheeling as (50 MW × the SRMC) for wheeling given in the chapter. These are 0.522 ₱/MWh and 3.096 ₱/MWh for the two wheeling cases. These charges represent income to the utility and reduce the total operating cost. (The question is really: "Does the use of the SRMC for wheeling only recover additional operating costs for the wheeling, or does it make an added profit for the utility?" Remember, this is only one example.)

FURTHER READING

References 1–3 provide a good historical look at the techniques that have gone into power pooling. Reference 4 is an excellent summary of the state-of-the-art (1980) of power brokering and pooling, and reviews the practices of most major U.S. power pools. The list of possible references dealing with the issues of utility deregulation, utility privatization and transmission access is very long. Only a few suggestions are given as a starting point. The references listed at the end of Chapter 13 are also relevant to the topics of this chapter; especially the treatment of SRMC for transmission. A great deal of original work was done by the late Professor Fred Schweppe and his associates at MIT and Harvard. This is summarized in the "*Spot Pricing of Electricity*" book, reference 5. Reference 6 is one example of an approach to establishing wheeling rates. The last three references discuss experiences in the United Kingdom and South America.

1. Mochon, H. H. Jr. "Practices of the New England Power Exchange," *Proceedings of the American Power Conference*, Vol. 34, 1972, pp. 911–925.
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11 Power System Security

11.1 INTRODUCTION

Up until now we have been mainly concerned with minimizing the cost of operating a power system. An overriding factor in the operation of a power system is the desire to maintain system security. System security involves practices designed to keep the system operating when components fail. For example, a generating unit may have to be taken off-line because of auxiliary equipment failure. By maintaining proper amounts of spinning reserve, the remaining units on the system can make up the deficit without too low a frequency drop or need to shed any load. Similarly, a transmission line may be damaged by a storm and taken out by automatic relaying. If, in committing and dispatching generation, proper regard for transmission flows is maintained, the remaining transmission lines can take the increased loading and still remain within limit.

Because the specific times at which initiating events that cause components to fail are unpredictable, the system must be operated at all times in such a way that the system will not be left in a dangerous condition should any credible initiating event occur. Since power system equipment is designed to be operated within certain limits, most pieces of equipment are protected by automatic devices that can cause equipment to be switched out of the system if these limits are violated. If any event occurs on a system that leaves it operating with limits violated, the event may be followed by a series of further actions that switch other equipment out of service. If this process of cascading failures continues, the entire system or large parts of it may completely collapse. This is usually referred to as a *system blackout*.

An example of the type of event sequence that can cause a blackout might start with a single line being opened due to an insulation failure; the remaining transmission circuits in the system will take up the flow that was flowing on the now-opened line. If one of the remaining lines is now too heavily loaded, it may open due to relay action, thereby causing even more load on the remaining lines. This type of process is often termed a *cascading outage*. Most power systems are operated such that any single initial failure event will not leave other components heavily overloaded, specifically to avoid cascading failures.

Most large power systems install equipment to allow operations personnel to monitor and operate the system in a reliable manner. This chapter will deal with the techniques and equipment used in these systems. We will lump these under the commonly used title *system security*.

Systems security can be broken down into three major functions that are carried out in an operations control center:

1. System monitoring.
2. Contingency analysis.
3. Security-constrained optimal power flow.

System monitoring provides the operators of the power system with pertinent up-to-date information on the conditions on the power system. Generally speaking, it is the most important function of the three. From the time that utilities went beyond systems of one unit supplying a group of loads, effective operation of the system required that critical quantities be measured and the values of the measurements be transmitted to a central location. Such systems of measurement and data transmission, called *telemetry systems*, have evolved to schemes that can monitor voltages, currents, power flows, and the status of circuit breakers, and switches in every substation in a power system transmission network. In addition, other critical information such as frequency, generator unit outputs and transformer tap positions can also be telemetered. With so much information telemetered simultaneously, no human operator could hope to check all of it in a reasonable time frame. For this reason, digital computers are usually installed in operations control centers to gather the telemetered data, process them, and place them in a data base from which operators can display information on large display monitors. More importantly, the computer can check incoming information against prestored limits and alarm the operators in the event of an overload or out-of-limit voltage.

State estimation is often used in such systems to combine telemetered system data with system models to produce the best estimate (in a statistical sense) of the current power system conditions or "state." We will discuss some of the highlights of these techniques in Chapter 12.

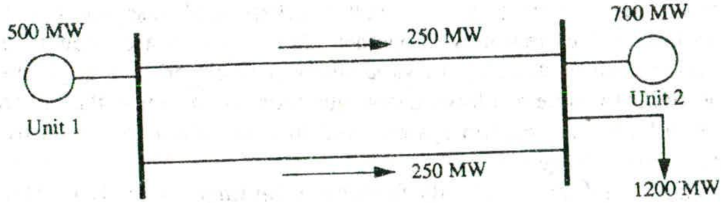
Such systems are usually combined with supervisory control systems that allow operators to control circuit breakers and disconnect switches and transformer taps remotely. Together, these systems are often referred to as *SCADA systems*, standing for supervisory control and data acquisition system. The SCADA system allows a few operators to monitor the generation and high-voltage transmission systems and to take action to correct overlords or out-of-limit voltages.

The second major security function is contingency analysis. The results of this type of analysis allow systems to be operated defensively. Many of the problems that occur on a power system can cause serious trouble within such a quick time period that the operator could not take action fast enough. This is often the case with cascading failures. Because of this aspect of systems operation, modern operations computers are equipped with contingency analysis programs that model possible systems troubles before they arise. These programs are based on a model of the power system and are used to study outage events and alarm the operators to any potential overlords or out-of-limit voltages. For example, the simplest form of contingency analysis can be put together with a standard power-flow program such as described in Chapter 4, together with procedures to set up the power-flow data for each outage to be studied by the power-flow program. Several variations of this type of contingency analysis scheme involve fast solution methods, automatic contingency event selection, and automatic initializing of the contingency power flows using actual system data and state estimation procedures.

The third major security function is security-constrained optimal power flow. In this function, a contingency analysis is combined with an optimal power flow which seeks to make changes to the optimal dispatch of generation, as well as other adjustments, so that when a security analysis is run, no contingencies result in violations. To show how this can be done, we shall divide the power system into four operating states.

- **Optimal dispatch:** this is the state that the power system is in prior to any contingency. It is optimal with respect to economic operation, but it may not be secure.
- **Post contingency:** is the state of the power system after a contingency has occurred. We shall assume here that this condition has a security violation (line or transformer beyond its flow limit, or a bus voltage outside the limit).
- **Secure dispatch:** is the state of the system with no contingency outages, but with corrections to the operating parameters to account for security violations.
- **Secure post-contingency:** is the state of the system when the contingency is applied to the base-operating condition—with corrections.

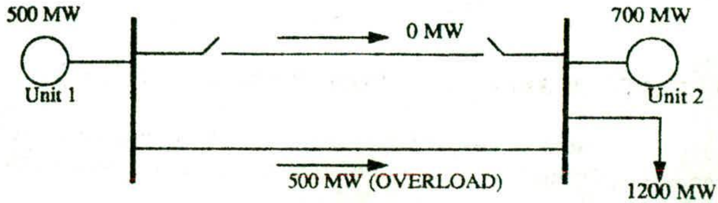
We shall illustrate the above with an example. Suppose the trivial power system consisting of two generators, a load, and a double circuit line, is to be operated with both generators supplying the load as shown below (ignore losses):



OPTIMAL DISPATCH

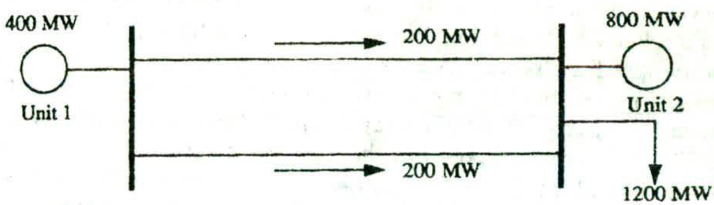
We assume that the system as shown is in economic dispatch, that is the 500 MW from unit 1 and the 700 MW from unit 2 is the optimum dispatch. Further, we assert that each circuit of the double circuit line can carry a maximum of 400 MW, so that there is no loading problem in the base-operating condition.

Now, we shall postulate that one of the two circuits making up the transmission line has been opened because of a failure. This results in



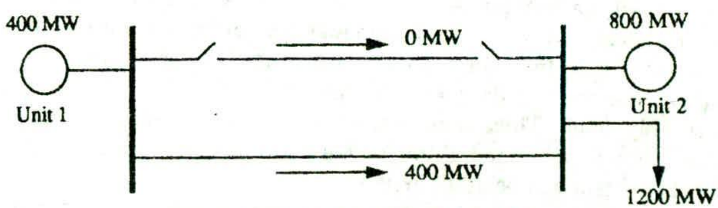
POST CONTINGENCY STATE

Now there is an overload on the remaining circuit. We shall assume for this example that we do not want this condition to arise and that we will correct the condition by lowering the generation on unit 1 to 400 MW. The secure dispatch is



SECURE DISPATCH

Now, if the same contingency analysis is done, the post-contingency condition is



SECURE POST CONTINGENCY STATE

By adjusting the generation on unit 1 and unit 2, we have prevented the post-contingency operating state from having an overload. This is the essence of what is called "security corrections." Programs which can make control adjustments to the base or pre-contingency operation to prevent violations in the post-contingency conditions are called "security-constrained optimal power flows" or SCOPF. These programs can take account of many contingencies and calculate adjustments to generator MW, generator voltages, transformer taps, interchange, etc. We shall show how the SCOPF is formed in Chapter 13.

Together, the functions of system monitoring, contingency analysis, and corrective action analysis comprise a very complex set of tools that can aid in the secure operation of a power system. This chapter concentrates on contingency analysis.

11.2 FACTORS AFFECTING POWER SYSTEM SECURITY

As a consequence of many widespread blackouts in interconnected power systems, the priorities for operation of modern power systems have evolved to the following.

- Operate the system in such a way that power is delivered reliably.
- Within the constraints placed on the system operation by reliability considerations, the system will be operated most economically.

The greater part of this book is devoted to developing methods to operate a power system to gain maximum economy. But what factors affect its operation from a reliability standpoint? We will assume that the engineering groups who have designed the power system's transmission and generation systems have done so with reliability in mind. This means that adequate generation has been installed to meet the load and that adequate transmission has been installed to deliver the generated power to the load. If the operation of the system went on without sudden failures or without experiencing unanticipated operating states, we would probably have no reliability problems. However, any piece of equipment in the system can fail, either due to internal causes or due to external causes such as lightning strikes, objects hitting transmission towers, or human errors in setting relays. It is highly uneconomical, if not impossible, to build a power system with so much redundancy (i.e., extra transmission lines, reserve generation, etc.) that failures never cause load to be dropped on a system. Rather, systems are designed so that the probability of dropping load is acceptably small. Thus, most power systems are designed to have sufficient redundancy to withstand all major failure events, but this does not guarantee that the system will be 100% reliable.

Within the design and economic limitations, it is the job of the operators to try to maximize the reliability of the system they have at any given time. Usually, a power system is never operated with all equipment "in" (i.e., connected) since failures occur or maintenance may require taking equipment out of service. Thus, the operators play a considerable role in seeing that the system is reliable.

In this chapter, we will not be concerned with all the events that can cause trouble on a power system. Instead, we will concentrate on the possible consequences and remedial actions required by two major types of failure events—transmission-line outages and generation-unit failures.

Transmission-line failures cause changes in the flows and voltages on the transmission equipment remaining connected to the system. Therefore, the analysis of transmission failures requires methods to predict these flows and voltages so as to be sure they are within their respective limits. Generation failures can also cause flows and voltages to change in the transmission system, with the addition of dynamic problems involving system frequency and generator output.

11.3 CONTINGENCY ANALYSIS: DETECTION OF NETWORK PROBLEMS

We will briefly illustrate the kind of problems we have been describing by use of the six-bus network used in Chapter 4. The base-case power flow results for Example 4A are shown in Figure 11.1 and indicate a flow of 43.8 MW and 60.7 MVAR on the line from bus 3 to bus 6. The limit on this line can be expressed in MW or in MVA. For the purpose of this discussion, assume that we are only interested in the MW loading on the line. Now let us ask what will happen if the transmission line from bus 3 to bus 5 were to open. The resulting flows and voltages are shown in Figure 11.2. Note that the flow on the line from bus 3 to bus 6 has increased to 54.9 MW and that most of the other transmission lines also experienced changes in flow. Note also that the bus voltage magnitudes changed, particularly at bus 5, which is now almost 5% below nominal. Figures 11.3 and 11.4 are examples of generator outages and serve to illustrate the fact that generation outages can also result in changes in flows and voltages on a transmission network. In the example shown in Figure 11.3, all the generation lost from bus 3 is picked up on the generator at bus 1. Figure 11.4 shows the case when the loss of generation on bus 3 is made up by an increase in generation at buses 1 and 2. Clearly, the differences in flows and voltages show that how the lost generation is picked up by the remaining units is imporant.

If the system being modeled is part of a large interconnected network, the lost generation will be picked up by a large number of generating units outside the system's immediate control area. When this happens, the pickup in generation is seen as an increase in flow over the tie lines to the neighboring systems. To model this, we can build a network model of our own system plus an equivalent network of our neighbor's system and place the swing bus or reference bus in the equivalent system. A generator outage is then modeled so that all lost generation is picked up on the swing bus, which then appears as an increase on the tie flows, thus approximately modeling the generation loss when interconnected. If, however, the system of interest is not interconnected, then the loss of generation must be shown as a pickup in output on the other generation units within the system. An approximate method of doing this is shown in Section 11.3.2.

Operations personnel must know which line or generation outages will cause flows or voltages to fall outside limits. To predict the effects of outages,

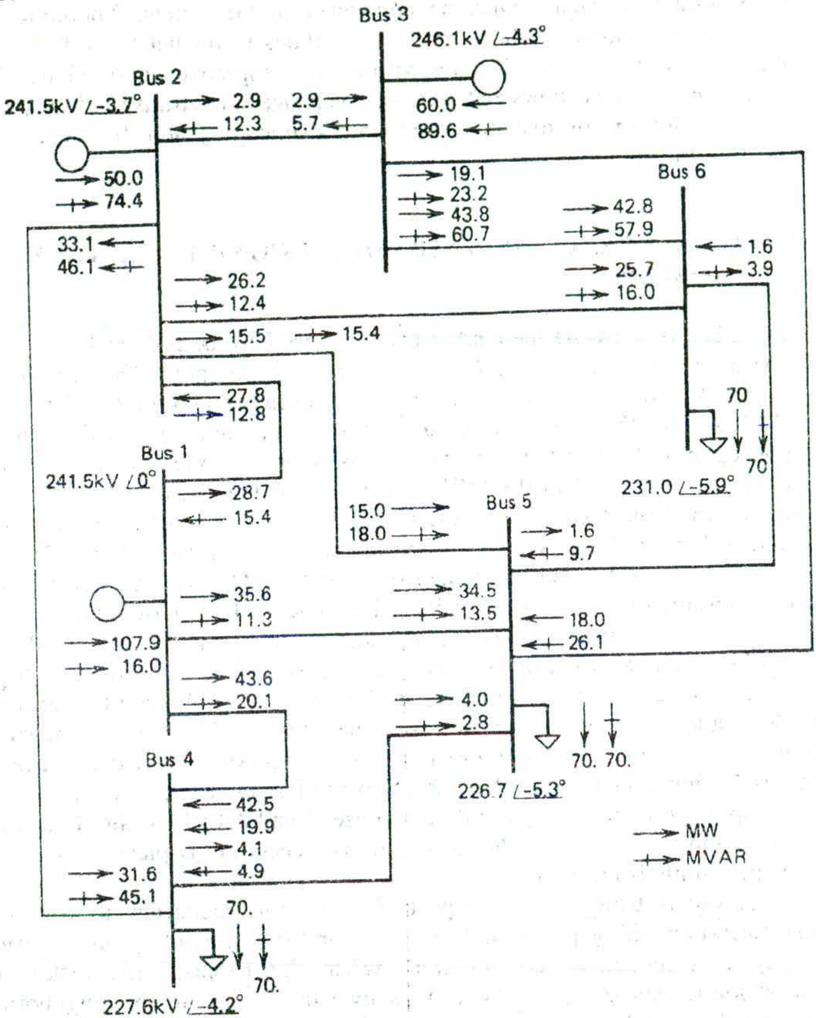


FIG. 11.1 Six-bus network base case AC power flow (see Example 4A).

contingency analysis techniques are used. Contingency analysis procedures model single failure events (i.e., one-line outage or one-generator outage) or multiple equipment failure events (i.e., two transmission lines, one transmission line plus one generator, etc.), one after another in sequence until "all credible outages" have been studied. For each outage tested, the contingency analysis procedure checks all lines and voltages in the network against their respective limits. The simplest form of such a contingency analysis technique is shown in Figure 11.5.

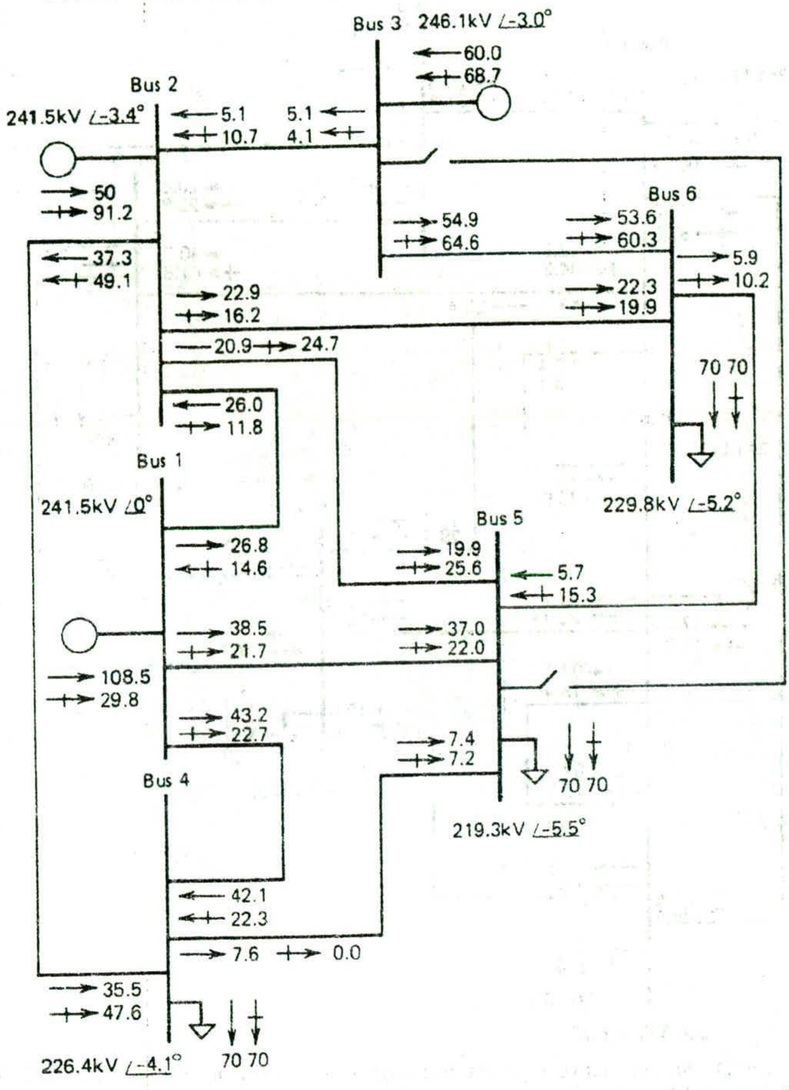


FIG. 11.2 Six-bus network line outage case; line from bus 3 to bus 5 opened.

The most difficult methodological problem to cope with in contingency analysis is the speed of solution of the model used. The most difficult logical problem is the selection of "all credible outages." If each outage case studied were to solve in 1 sec and several thousand outages were of concern, it would take close to 1 h before all cases could be reported. This would be useful if the system conditions did not change over that period of time. However, power

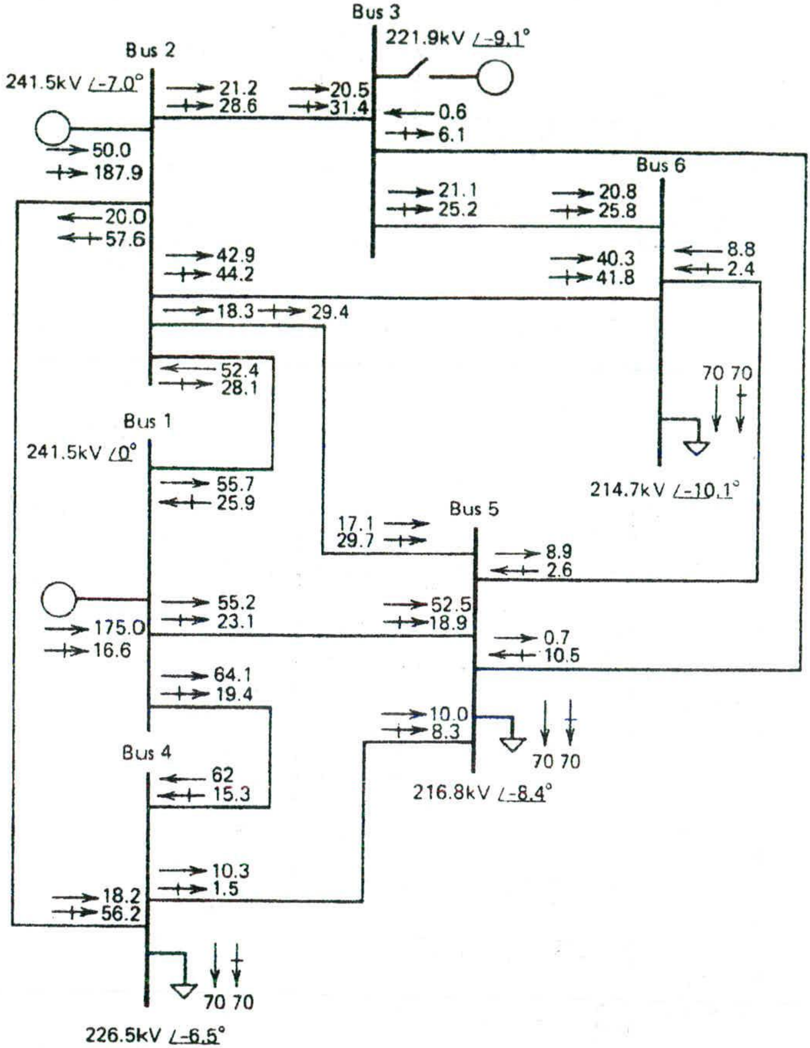


FIG. 11.3 Six-bus network generator outage case. Outage of generator on bus 3; lost generation picked up on generator 1.

systems are constantly undergoing changes and the operators usually need to know if the present operation of the system is safe, without waiting too long for the answer. Contingency analysis execution times of less than 1 min for several thousand outage cases are typical of computer and analytical technology as of 1995.

One way to gain speed of solution in a contingency analysis procedure is to

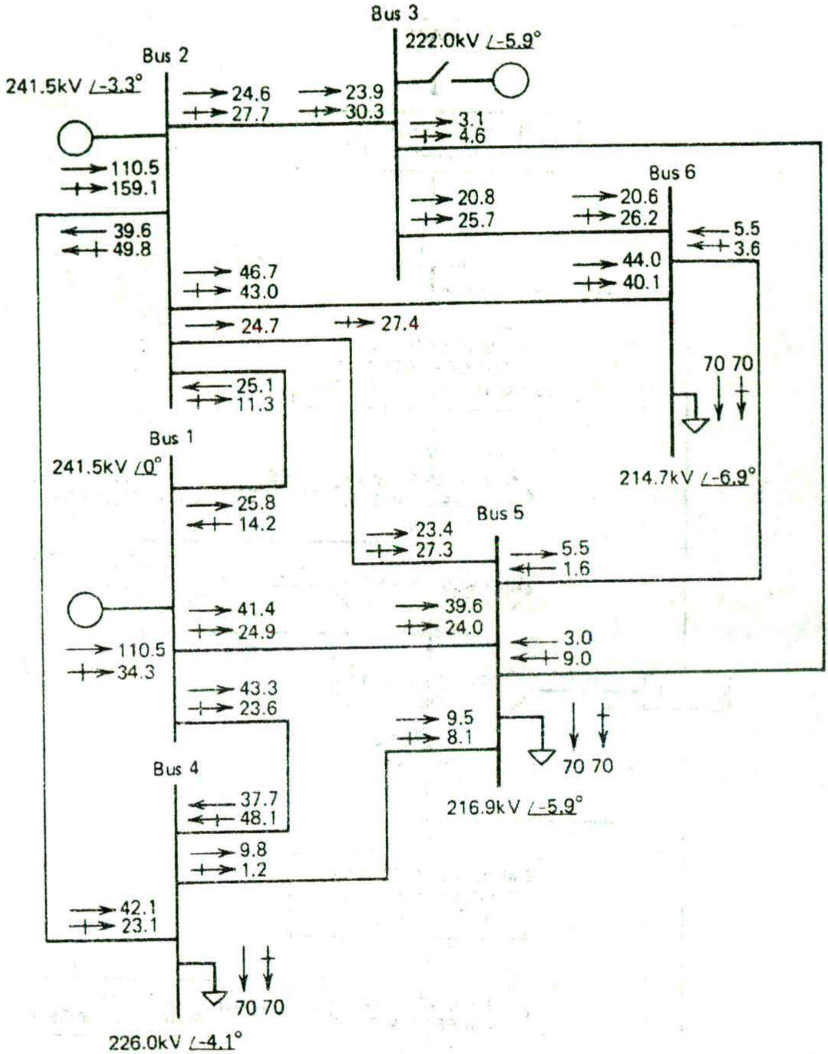


FIG. 11.4 Six-bus network generator outage case. Outage of generator on bus 3; lost generation picked up on generator 1 and generator 2.

use an approximate model of the power system. For many systems, the use of DC load flow models provides adequate capability. In such systems, the voltage magnitudes may not be of great concern and the DC load flow provides sufficient accuracy with respect to the megawatt flows. For other systems, voltage is a concern and full AC load flow analysis is required.

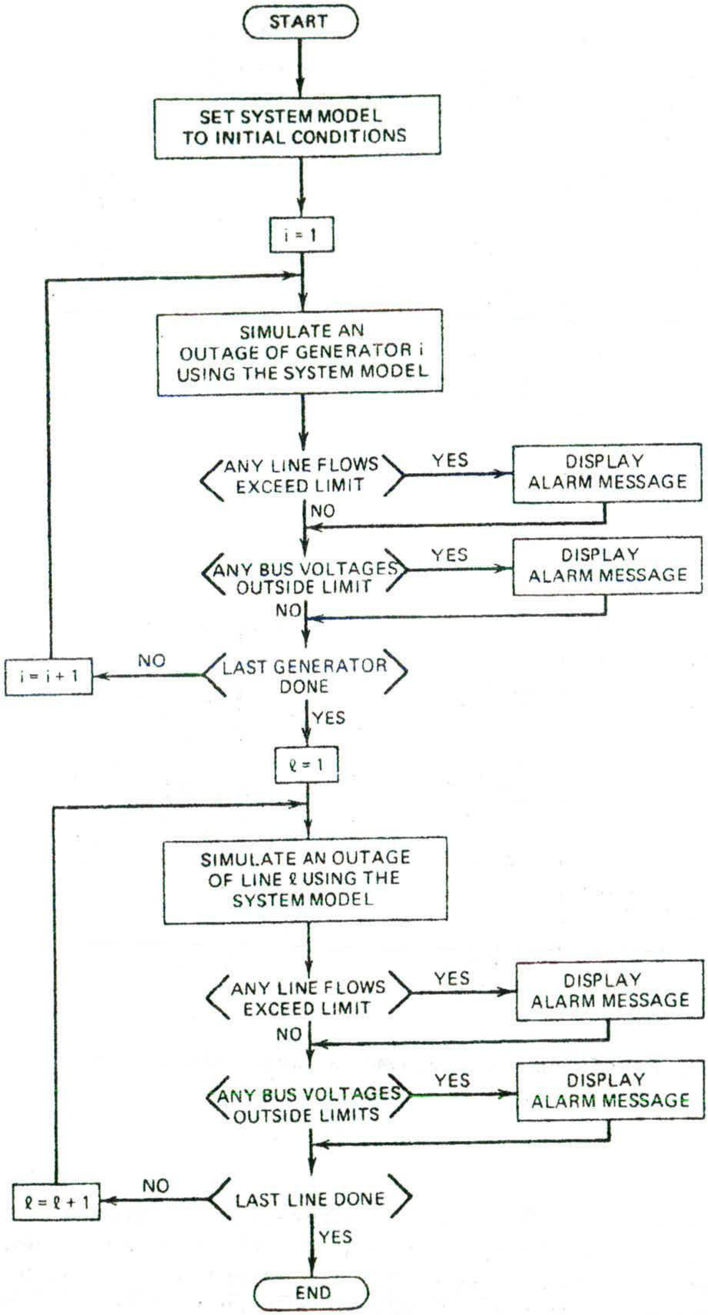


FIG. 11.5 Contingency analysis procedure.

11.3.1 An Overview of Security Analysis

A security analysis study which is run in an operations center must be executed very quickly in order to be of any use to operators. There are three basic ways to accomplish this.

- Study the power system with approximate but very fast algorithms.
- Select only the important cases for detailed analysis.
- Use a computer system made up of multiple processors or vector processors to gain speed.

The first method has been in use for many years and goes under various names such as "D factor methods," "linear sensitivity methods," "DC power flow methods," etc. This approach is useful if one only desires an approximate analysis of the effect of each outage. This text presents these methods under the name linear sensitivity factors and uses the same derivation as was presented in Chapter 4 under the DC power flow methods. It has all the limitations attributed to the DC power flow; that is, only branch MW flows are calculated and these are only within about 5% accuracy. There is no knowledge of MVAR flows or bus voltage magnitudes. Linear sensitivity factors are presented in Section 11.3.2.

If it is necessary to know a power system's MVA flows and bus voltage magnitudes after a contingency outage, then some form of complete AC power flow must be used. This presents a great deal of difficulty when thousands of cases must be checked. It is simply impossible, even on the fastest processors in existence today (1995) to execute thousands of complete AC power flows quickly enough. Fortunately, this need not be done as most of the cases result in power flow results which do not have flow or voltage limit violations. What is needed are ways to eliminate all or most of the nonviolation cases and only run complete power flows on the "critical" cases. These techniques go under the names of "contingency selection" or "contingency screening" and are introduced in Section 11.3.4.

Last of all, it must be mentioned that there are ways of running thousands of contingency power flows if special computing facilities are used. These facilities involve the use of many processors running separate cases in parallel, or vector processors which achieve parallel operation by "unwinding" the looping instruction sets in the computer code used. As of the writing of this edition (1995), such techniques are still in the research stage.

11.3.2 Linear Sensitivity Factors

The problem of studying thousands of possible outages becomes very difficult to solve if it is desired to present the results quickly. One of the easiest ways to provide a quick calculation of possible overloads is to use *linear sensitivity factors*. These factors show the approximate change in line flows for changes in generation on the network configuration and are derived from the DC load flow presented in Chapter 4. These factors can be derived in a variety of ways and basically come down to two types:

1. Generation shift factors.
2. Line outage distribution factors.

Here, we shall describe how these factors are used. The derivation of sensitivity factors is given in Appendix 11A.

The generation shift factors are designated $a_{\ell i}$ and have the following definition:

$$a_{\ell i} = \frac{\Delta f_\ell}{\Delta P_i} \tag{11.1}$$

where

ℓ = line index

i = bus index

Δf_ℓ = change in megawatt power flow on line ℓ when a change in generation, ΔP_i, occurs at bus i

ΔP_i = change in generation at bus i

It is assumed in this definition that the change in generation, ΔP_i, is exactly compensated by an opposite change in generation at the reference bus, and that all other generators remain fixed. The $a_{\ell i}$ factor then represents the sensitivity of the flow on line ℓ to a change in generation at bus i. Suppose one wanted to study the outage of a large generating unit and it was assumed that all the generation lost would be made up by the reference generation (we will deal with the case where the generation is picked up by many machines shortly). If the generator in question was generating P_i^0 MW and it was lost, we would represent ΔP_i as

$$\Delta P_i = -P_i^0 \tag{11.2}$$

and the new power flow on each line in the network could be calculated using a precalculated set of "a" factors as follows:

$$\hat{f}_\ell = f_\ell^0 + a_{\ell i}\Delta P_i \quad \text{for } \ell = 1 \ldots L \tag{11.3}$$

where

$\hat{f}_\ell$ = flow on line ℓ after the generator on bus i fails

f_ℓ^0 = flow before the failure

The "outage flow," $\hat{f}_\ell$, on each line can be compared to its limit and those exceeding their limit flagged for alarming. This would tell the operations personnel that the loss of the generator on bus i would result in an overload on line

The generation shift sensitivity factors are linear estimates of the change in flow with a change in power at a bus. Therefore, the effects of simultaneous changes on several generating buses can be calculated using superposition. Suppose, for example, that the loss of the generator on bus i were compensated by governor action on machines throughout the interconnected system. One frequently used method assumes that the remaining generators pick up in proportion to their maximum MW rating. Thus, the proportion of generation pickup from unit j ($j \neq i$) would be

$$\gamma_{ji} = \frac{P_j^{\max}}{\sum\limits_{\substack{k \\ k \neq i}} P_k^{\max}} \tag{11.4}$$

where

$P_k^{\max}$ = maximum MW rating for generator k

γ_{ji} = proportionality factor for pickup on generating unit j when unit i fails

Then, to test for the flow on line ℓ, under the assumption that all the generators in the interconnection participate in making up the loss, use the following:

$$\hat{f}_\ell = f_\ell^0 + a_{\ell i}\Delta P_i - \sum_{j \neq i} [a_{\ell j}\gamma_{ji}\Delta P_i] \tag{11.5}$$

Note that this assumes that no unit will actually hit its maximum. If this is apt to be the case, a more detailed generation pickup algorithm that took account of generation limits would be required.

The line outage distribution factors are used in a similar manner, only they apply to the testing for overloads when transmission circuits are lost. By definition, the line outage distribution factor has the following meaning:

$$d_{\ell,k} = \frac{\Delta f_\ell}{f_k^0} \tag{11.6}$$

where

$d_{\ell,k}$ = line outage distribution factor when monitoring line ℓ after an outage on line k

Δf_ℓ = change in MW flow on line ℓ

f_k^0 = original flow on line k before it was outaged (opened)

If one knows the power on line ℓ and line k, the flow on line ℓ with line k out can be determined using "d" factors.

$$\hat{f}_\ell = f_\ell^0 + d_{\ell,k} f_k^0 \tag{11.7}$$

where

$$f_\ell^0, f_k^0 = \text{preoutage flows on lines } \ell \text{ and } k\text{, respectively}$$

$$\hat{f}_\ell = \text{flow on line } \ell \text{ with line } k \text{ out}$$

By precalculating the line outage distribution factors, a very fast procedure can be set up to test all lines in the network for overload for the outage of a particular line. Furthermore, this procedure can be repeated for the outage of each line in turn, with overloads reported to the operations personnel in the form of alarm messages.

Using the generator and line outage procedures described earlier, one can program a digital computer to execute a contingency analysis study of the power system as shown in Figure 11.6. Note that a line flow can be positive or negative so that, as shown in Figure 11.6, we must check f against $-f_\ell^{max}$ as well as f_ℓ^{max}. This figure makes several assumptions; first, it assumes that the generator output for each of the generators in the system is available and that the line flow for each transmission line in the network is also available. Second, it assumes that the sensitivity factors have been calculated and stored, and that they are correct. The assumption that the generation and line flow MWs are available can be satisfied with telemetry systems or with state estimation techniques. The assumption that the sensitivity factors are correct is valid as long as the transmission network has not undergone any significant switching operations that would change its structure. For this reason, control systems that use sensitivity factors must have provision for updating the factors when the network is switched. A third assumption is that all generation pickup will be made on the reference bus. If this is not the case, substitute Eq. 11.5 in the generator outage loop.

EXAMPLE 11A

The $[X]$ matrix for our six-bus sample network is shown in Figure 11.7, together with the generation shift distribution factors and the line outage distribution factors.

The generation shift distribution factors that give the fraction of generation shift that is picked up on a transmission line are designated $a_{\ell i}$. The a factor is obtained by finding line ℓ along the rows and then finding the generator to be shifted along the columns. For instance, the shift factor for a change in the flow on line 1-4 when making a shift in generation on bus 3 is found in the second row, third column.

The line outage distribution factors are stored such that each row and column corresponds to one line in the network. The distribution factor $d_{\ell,k}$ is obtained by finding line ℓ along the rows and then finding line k along that row in the appropriate column. For instance, the line outage distribution factor that gives the fraction of flow picked up on line 3-5 for an outage on line 3-6

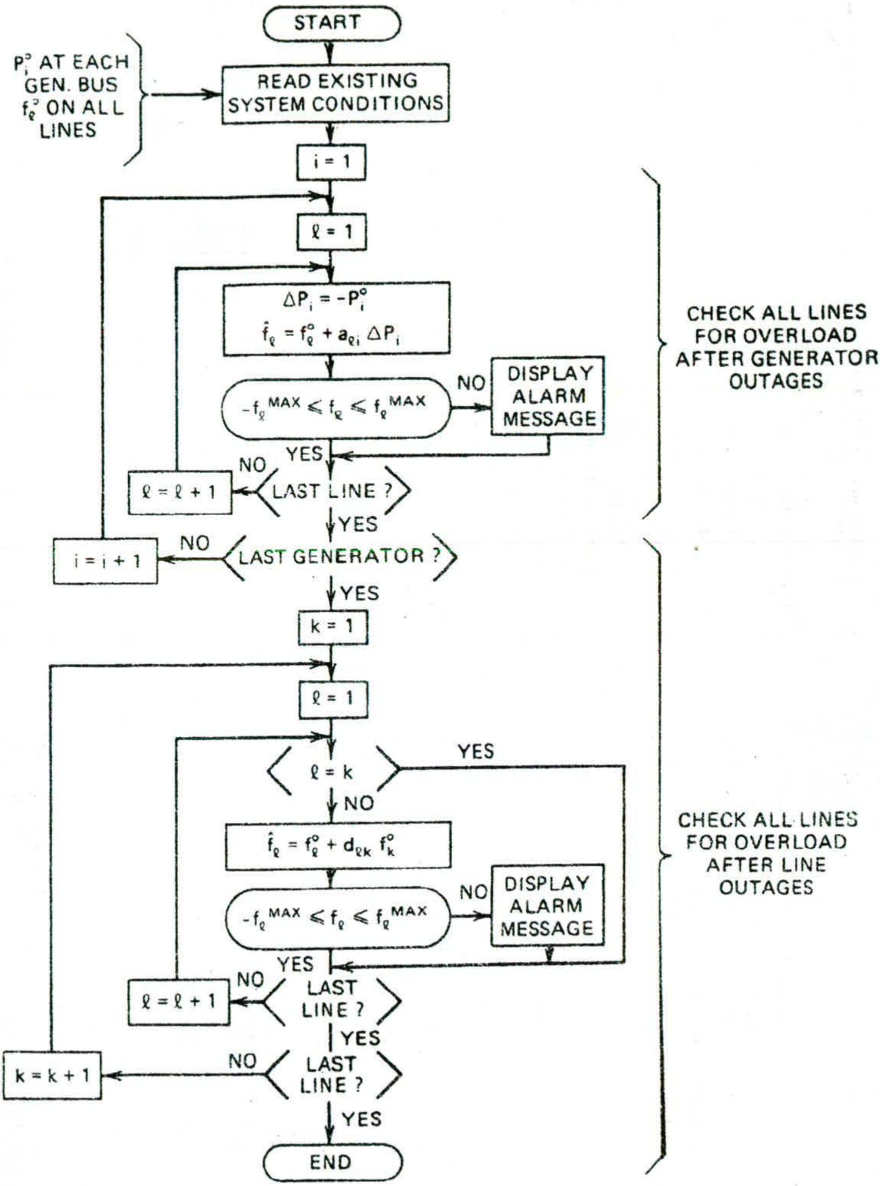


FIG. 11.6 Contingency analysis using sensitivity factors.

X Matrix for Six-Bus Sample System (Reference at Bus 1)

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.09412 & 0.08051 & 0.06298 & 0.06435 & 0.08129 \\
0 & 0.08051 & 0.16590 & 0.05897 & 0.09077 & 0.12895 \\
0 & 0.06298 & 0.05897 & 0.10088 & 0.05422 & 0.05920 \\
0 & 0.06435 & 0.09077 & 0.05422 & 0.12215 & 0.08927 \\
0 & 0.08129 & 0.12895 & 0.05920 & 0.08927 & 0.16328
\end{bmatrix}
$$

Generation Shift Factors For Six-Bus Sample System

	Bus 1	Bus 2	Bus 3
$\ell = 1$ (line 1-2)	0	−0.47	−0.40
$\ell = 2$ (line 1-4)	0	−0.31	−0.29
$\ell = 3$ (line 1-5)	0	−0.21	−0.30
$\ell = 4$ (line 2-3)	0	0.05	−0.34
$\ell = 5$ (line 2-4)	0	0.31	0.22
$\ell = 6$ (line 2-5)	0	0.10	−0.03
$\ell = 7$ (line 2-6)	0	0.06	−0.24
$\ell = 8$ (line 3-5)	0	0.06	0.29
$\ell = 9$ (line 3-6)	0	−0.01	0.37
$\ell = 10$ (line 4-5)	0	0	−0.08
$\ell = 11$ (line 5-6)	0	−0.06	−0.13

Line Outage Distribution Factors for Six-Bus Sample System

	$k=1$ (Line 1-2)	$k=2$ (Line 1-4)	$k=3$ (Line 1-5)	$k=4$ (Line 2-3)	$k=5$ (Line 2-4)	$k=6$ (Line 2-5)	$k=7$ (Line 2-6)	$k=8$ (Line 3-5)	$k=9$ (Line 3-6)	$k=10$ (Line 4-5)	$k=11$ (Line 5-6)
$\ell = 1$ (line 1-2)		0.64	0.54	−0.11	−0.50	−0.21	−0.12	−0.14	0.01	0.01	0.13
$\ell = 2$ (line 1-4)	0.59		0.46	−0.03	0.61	−0.06	−0.04	−0.04	0	−0.33	0.04
$\ell = 3$ (line 1-5)	0.41	0.36		0.15	−0.11	0.27	0.16	0.18	−0.02	0.32	−0.17
$\ell = 4$ (line 2-3)	−0.10	−0.03	0.18		0.12	0.23	0.47	−0.40	−0.53	0.17	0.13
$\ell = 5$ (line 2-4)	−0.59	0.76	−0.17	0.16		0.30	0.17	0.19	−0.02	−0.67	−0.19
$\ell = 6$ (line 2-5)	−0.19	−0.06	0.33	0.22	0.23		0.24	0.27	−0.03	0.31	−0.26
$\ell = 7$ (line 2-6)	−0.12	−0.04	0.21	0.51	0.15	0.27		−0.20	0.58	0.20	0.44
$\ell = 8$ (line 3-5)	−0.12	−0.04	0.20	−0.38	0.14	0.27	−0.17		0.47	0.19	−0.42
$\ell = 9$ (line 3-6)	0.01	0	−0.03	−0.62	−0.02	−0.03	0.64	0.60		−0.02	0.56
$\ell = 10$ (line 4-5)	0.01	−0.24	0.29	0.13	−0.39	0.24	0.14	0.15	−0.02		−0.15
$\ell = 11$ (line 5-6)	0.11	0.03	−0.18	0.12	−0.13	−0.23	0.36	−0.40	0.42	−0.18	

FIG. 11.7 Outage factors for a six-bus system.

is found in the eighth row and ninth column. Figure 11.3 shows an outage of the generator on bus 3 with all pickup of lost generation coming on the generator at bus 1. To calculate the flow on line 1-4 after the outage of the generator on bus 3, we need (see Figure 11.1):

$$\text{Base-case flow on line 1-4} = 43.6 \text{ MW}$$

$$\text{Base-case generation on bus 3} = 60 \text{ MW}$$

$$\text{Generation shift distribution factor} = a_{1\text{-}4,3} = -0.29$$

Then the flow on line 1-4 after generator outage is = base-case $\text{flow}_{1\text{-}4}$ + $a_{1\text{-}4,3}\Delta P_{\text{gen}_3} = 43.6 + (-0.29)(-60 \text{ MW}) = 61 \text{ MW}$.

To show how the line outage and generation shift factors are used, calculate some flows for the outages shown in Figures 11.2 and 11.3. Figure 11.2 shows an outage of line 3-5. If we wish to calculate the power flowing on line 3-6 with line 3-5 opened, we would need the following.

$$\text{Base-case flow on line 3-5} = 19.1 \text{ MW}$$

$$\text{Base-case flow on line 3-6} = 43.8 \text{ MW}$$

$$\text{Line outage distribution factor: } d_{3\text{-}6,3\text{-}5} = 0.60$$

Then the flow on 3-6 after the outage is = base $\text{flow}_{3\text{-}6} + d_{3\text{-}6,3\text{-}5} \times$ base $\text{flow}_{3\text{-}5} = 43.8 + (0.60) \times (19.1) = 55.26 \text{ MW}$.

In both outage cases, the flows calculated by the sensitivity methods are reasonably close to the values calculated by the full AC load flows as shown in Figures 11.2 and 11.3.

11.3.3 AC Power Flow Methods

The calculations made by network sensitivity methods are faster than those made by AC power flow methods and therefore find wide use in operations control systems. However, there are many power systems where voltage magnitudes are the critical factor in assessing contingencies. In addition, there are some systems where VAR flows predominate on some circuits, such as underground cables, and an analysis of only the MW flows will not be adequate to indicate overloads. When such situations present themselves, the network sensitivity methods may not be adequate and the operations control system will have to incorporate a full AC power flow for contingency analysis.

When an AC power flow is to be used to study each contingency case, the speed of solution and the number of cases to be studied are critical. To repeat what was said before, if the contingency alarms come too late for operators to act, they are worthless. Most operations control centers that use an AC power flow program for contingency analysis use either a Newton–Raphson or the decoupled power flow. These solution algorithms are used because of their speed of solution and the fact that they are reasonably reliable in convergence when solving difficult cases. The decoupled load flow has the further advantage that a matrix alteration formula can be incorporated into it to simulate the outage of transmission lines without reinverting the system Jacobian matrix at each iteration.

The simplest AC security analysis procedure consists of running an AC power flow analysis for each possible generator, transmission line, and transformer outage as shown in Figure 11.8. This procedure will determine the overloads and voltage limit violations accurately (at least within the accuracy of the power flow program, the accuracy of the model data, and the accuracy with which we have obtained the initial conditions for the power flow). It does suffer a major drawback, however, and that concerns the time such a program takes to execute. If the list of outages has several thousand entries, then the total time to test for all of the outages can be too long.

We are thus confronted with a dilemma. Fast, but inaccurate, methods involving the *a* and *d* factors can be used to give rapid analysis of the system, but they cannot give information about MVAR flows and voltages. Slower, full AC power flow methods give full accuracy but take too long.

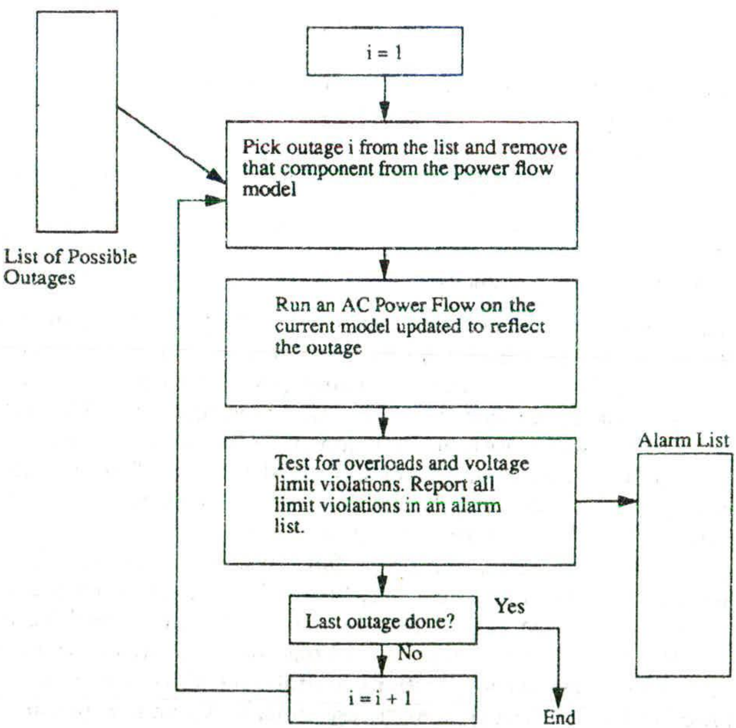


FIG. 11.8 AC power flow security analysis.

Fortunately, there is a way out of this dilemma. Because of the way the power system is designed and operated, very few of the outages will actually cause trouble. That is, most of the time spent running AC power flows will go for solutions of the power flow model that discover that there are no problems. Only a few of the power flow solutions will, in fact, conclude that an overload or voltage violation exists.

The solution to this dilemma is to find a way to select contingencies in such a way that only those that are likely to result in an overload or voltage limit violation will actually be studied in detail and the other cases will go unanalyzed. A flowchart for a process like this appears in Figure 11.9. Selecting

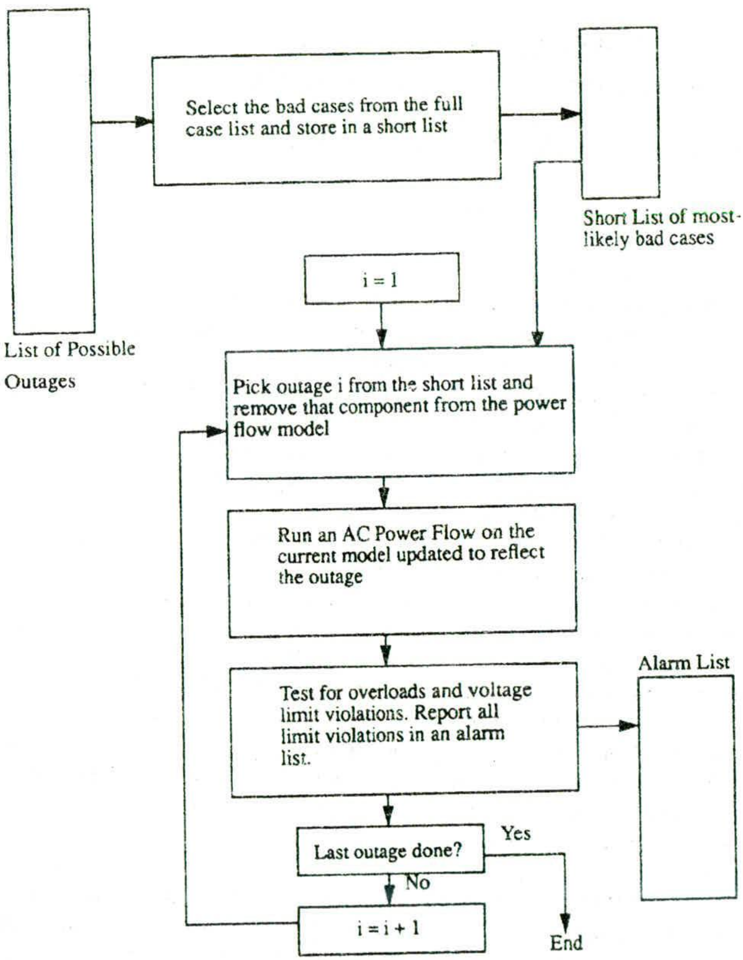


FIG. 11.9 AC power flow security analysis with contingency case selection.

the bad or likely trouble cases from the full outage case list is not an exact procedure and has been the subject of intense research for the past 15 years. Two sources of error can arise.

1. **Placing too many cases on the short list:** this is essentially the "conservative" approach and simply leads to longer run times for the security analysis procedure to execute.
2. **Skipping cases:** here, a case that would have shown a problem is not placed on the short list and results in possibly having that outage take place and cause trouble without the operators being warned.

11.3.4 Contingency Selection

We would like to get some measure as to how much a particular outage might affect the power system. The idea of a performance index seems to fulfill this need. The definition for the overload performance index (PI) is as follows:

$$\mathrm{PI} = \sum_{\substack{\text{all branches} \\ l}} \left(\frac{P_{\text{flow } l}}{P_l^{\max}} \right)^{2n} \tag{11.8}$$

If n is a large number, the PI will be a small number if all flows are within limit, and it will be large if one or more lines are overloaded. The problem then is how to use this performance index.

Various techniques have been tried to obtain the value of PI when a branch is taken out. These calculations can be made exactly if $n = 1$; that is, a table of PI values, one for each line in the network, can be calculated quite quickly. The selection procedure then involves ordering the PI table from largest value to least. The lines corresponding to the top of the list are then the candidates for the short list. One procedure simply ordered the PI table and then picked the top N_c entries from this list and placed them on the short list (see reference 8).

However when $n = 1$, the PI does not snap from near zero to near infinity as the branch exceeds its limit. Instead, it rises as a quadratic function. A line that is just below its limit contributes to PI almost equal to one that is just over its limit. The result is a PI that may be large when many lines are loaded just below their limit. Thus the PI's ability to distinguish or detect bad cases is limited when $n = 1$. Ordering the PI values when $n = 1$ usually results in a list that is not at all representative of one with the truly bad cases at the top. Trying to develop an algorithm that can quickly calculate PI when $n = 2$ or larger has proven extremely difficult.

One way to perform an outage case selection is to perform what has been called the *1P1Q method* (see references 9 and 10). Here, a decoupled power flow is used. As shown in Figure 11.10, the solution procedure is interrupted after one iteration (one $P - \theta$ calculation and one $Q - V$ calculation; thus, the name 1P1Q). With this procedure, the PI can use as large an n value as desired, say $n = 5$. There appears to be sufficient information in the solution at the end of

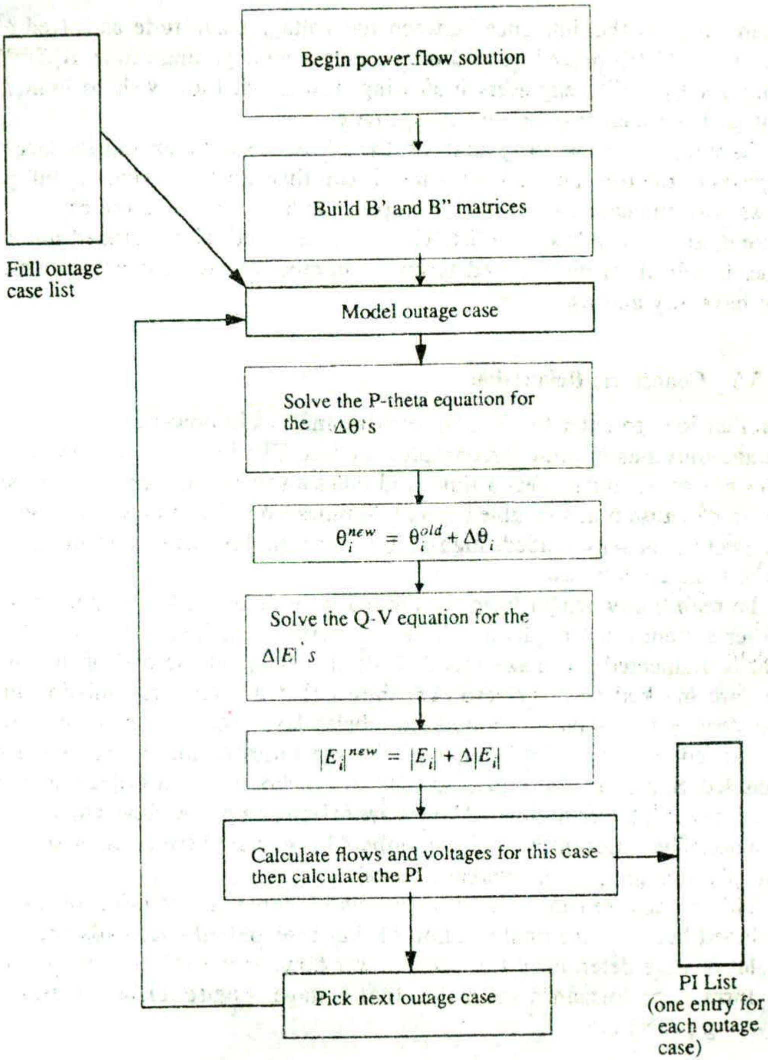


FIG. 11.10 The 1P1Q contingency selection procedure.

the first iteration of the decoupled power flow to give a reasonable PI. Another advantage to this procedure is the fact that the voltages can also be included in the PI. Thus, a different PI can be used, such as:

$$PI = \sum_{\substack{\text{all branches} \\ l}} \left(\frac{P_{\text{flow } l}}{P_l^{\max}} \right)^{2n} + \sum_{\substack{\text{all buses} \\ i}} \left(\frac{\Delta |E_i|}{\Delta |E|^{\max}} \right)^{2m} \tag{11.9}$$

where $\Delta|E_i|$ is the difference between the voltage magnitude as solved at the end of the 1P1Q procedure and the base-case voltage magnitude. $\Delta|E|^{max}$ is a value set by utility engineers indicating how much they wish to limit a bus voltage from changing on one outage case.

To complete the security analysis, the PI list is sorted so that the largest PI appears at the top. The security analysis can then start by executing full power flows with the case which is at the top of the list, then solve the case which is second, and so on down the list. This continues until either a fixed number of cases is solved, or until a predetermined number of cases are solved which do not have any alarms.

11.3.5 Concentric Relaxation

Another idea to enter the field of security analysis in power systems is that an outage only has a limited geographical effect. The loss of a transmission line does not cause much effect a thousand miles away; in fact, we might hope that it doesn't cause much trouble beyond 20 miles from the outage, although if the line were a heavily loaded, high-voltage line, its loss will most likely be felt more than 20 miles away.

To realize any benefit from the limited geographical effect of an outage, the power system must be divided into two parts: the affected part and the part that is unaffected. To make this division, the buses at the end of the outaged line are marked as layer zero. The buses that are one transmission line or transformer from layer zero are then labeled layer one. This same process can be carried out, layer by layer, until all the buses in the entire network are included. Some arbitrary number of layers is chosen and all buses included in that layer and lower-numbered layers are solved as a power flow with the outage in place. The buses in the higher-numbered layers are kept as constant voltage and phase angle (i.e., as reference buses).

This procedure can be used in two ways: either the solution of the layers included becomes the final solution of that case and all overloads and voltage violations are determined from this power flow, or the solution simply is used to form a performance index for that outage. Figure 11.11 illustrates this layering procedure.

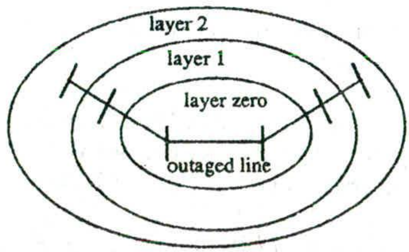


FIG. 11.11 Layering of outage effects.

The concentric relaxation procedure was originally proposed by Zaborsky (see reference 13). The trouble with the concentric relaxation technique is that it requires more layers for circuits whose influence is felt further from the outage.

11.3.6 Bounding

A paper by Brandwajn (reference 11) solves at least one of the problems in using the concentric relaxation method. Namely, it uses an adjustable region around the outage to solve for the outage case overloads. In reference 11, this is applied only to the linear (DC) power flow; it has subsequently been extended for AC network analysis.

To perform the analysis in the bounding technique we define three subsystems of the power system as follows:

N1 = the subsystem immediately surrounding the outaged line

N2 = the external subsystem that we shall not solve in detail

N3 = the set of boundary buses that separate N1 and N2

The subsystems appear as shown in Figure 11.12. The bounding method is based on the fact that we can make certain assumptions about the phase angle spread across the lines in N2, given the injections in N1 and the maximum phase angle appearing across any two buses in N3. In Appendix 11A of this chapter we show how to calculate the ΔP_k and the ΔP_m injections that will make the phase angles on buses k and m simulate the outage of line k–m.

If we are given a transmission line in N2 with flow f_{pq}^0, then there is a maximum amount that the flow on pq can shift. That is, it can increase from

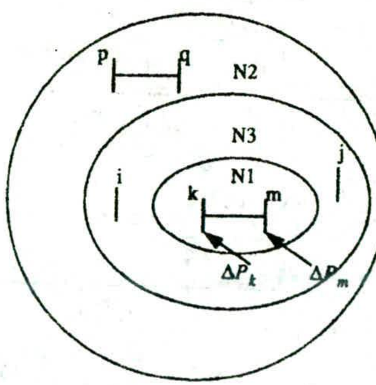


FIG. 11.12 Layers used in bounding analysis.

f^0_{pq} to its upper limit or it can decrease to its lower limit. Then,

$$\Delta f^{\max}_{pq} = \text{smaller of } [(f^+_{pq} - f^0_{pq}), (f^0_{pq} - f^-_{pq})] \tag{11.10}$$

Further, we can translate this into a maximum change in phase angle difference as follows:

$$f_{pq} = \frac{1}{x_{pq}}(\theta_p - \theta_q) \tag{11.11}$$

or

$$\Delta f_{pq} = \frac{1}{x_{pq}}(\Delta\theta_p - \Delta\theta_q) \tag{11.12}$$

and finally:

$$(\Delta\theta_p - \Delta\theta_q)^{\max} = \Delta f^{\max}_{pq} x_{pq} \tag{11.13}$$

Thus, we can define the maximum change in the phase angle difference across pq. Reference 11 develops the theorem that:

$$|\Delta\theta_p - \Delta\theta_q| < |\Delta\theta_i - \Delta\theta_j| \tag{11.14}$$

where i and j are any pair of buses in N3, $\Delta\theta_i$ is the largest $\Delta\theta$ in N3, and $\Delta\theta_j$ is the smallest $\Delta\theta$ in N3 (see Appendix 11B).

Equation 11.14 is interpreted as follows: the right-hand side, $|\Delta\theta_i - \Delta\theta_j|$, provides an upper limit to the maximum change in angular spread across any circuit in N2. Thus, it provides us with a limit as to how far any of the N2 circuits can change their flow. By combining Eqs. 11.13 and 11.14 we obtain:

$$\Delta f^{\max}_{pq} x_{pq} < |\Delta\theta_i - \Delta\theta_j| \tag{11.15}$$

Figure 11.13 shows a graphical interpretation of the bounding process. There are two cases represented in Figure 11.13: a circuit on the top of the figure that

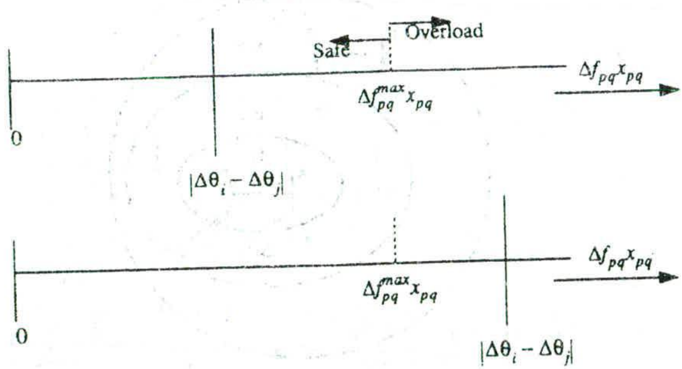


FIG. 11.13 Interpretation of bounding.

cannot go over limit, while that on the bottom could. In each case, the horizontal line represents the change in flow on circuit pq times its reactance, $\Delta f_{pq} x_{pq}$; the dotted line, labeled $\Delta f_{pq}^{\max} x_{pq}$, represents the point where circuit pq will go into overload and is determined as explained previously. Any value of $\Delta f_{pq} x_{pq}$ to the right of the dotted line represents an overload.

The solid line labeled $|\Delta\theta_i - \Delta\theta_j|$ represents the upper limit on $\Delta f_{pq} x_{pq}$. Thus, if the solid line is below (to the left) of the dotted line, then the circuit theory upper limit predicts that the circuit cannot go into overload; if on the other hand, the solid line is above (to the right of) the dotted line, the circuit may be shifted in flow due to the outage so as to violate a limit.

A completely safe N2 region would be one in which the maximum $|\Delta\theta_i - \Delta\theta_j|$ upper limit is small enough to be less than all of the $\Delta f_{pq}^{\max} x_{pq}$ limits. In fact, as the N1 region is enlarged, the value of $|\Delta\theta_i - \Delta\theta_j|$ will become smaller and smaller. Therefore, the test to determine whether the N1 region encompasses all possible overloaded circuits should be as follows:

> All circuits in N2 are safe from overload if the value of $|\Delta\theta_i - \Delta\theta_j|$ is less than the smallest value of $\Delta f_{pq}^{\max} x_{pq}$ over all pairs pq, where pq corresponds to the buses at the ends of circuits in N2

If this condition fails, then we have to expand N1, calculate a new $|\Delta\theta_i - \Delta\theta_j|$ in N3, and rerun the test over the newly defined N2 region circuits. When an N2 is found which passes the test, we are done and only region N1 need be studied in detail.

References 10 and 12 extend this concept to screening for AC contingency effects. Such contingency selection/screening techniques form the foundation for many real-time computer security analysis algorithms.

EXAMPLE 11B

In this example, we shall take the six-bus sample system used previously and show how the bounding technique works so that not all of the circuits in the system need be analyzed. Note that this is a small system so that the net savings in computer time may not be that great. Nonetheless, it demonstrates the principles used in the bounding technique quite well.

We shall study the outage of transmission line 3-6. The DC power flow will be used throughout and the initial conditions will be those shown in Figure 4.12. The MW limits on the transmission lines are shown in the table at the top of the next page.

Line	MW Limit
1-2	30
1-4	50
1-5	40
2-3	20
2-4	40
2-5	20
2-6	30
3-5	20
3-6	60
4-5	20
5-6	20

In this example, we shall proceed in steps. Step A will analyze the system as if the N1 and N3 regions consist of only line 3-6 itself, as shown in Figure 11.14. If the bounding criteria is met, no other analysis need be done as it will establish that no overloads exist anywhere in the system. If the bounding criteria fails, we still proceed to step B. Step B expands the bounded region from line 3-6 to include all buses which are once removed from buses 3 and 6; that is, it includes buses 2, 3, 5, and 6 as shown in Figure 11.15, and in this case the boundary of the region, N3, consists of buses 2 and 5.

To start, we need to calculate $\Delta f_{pq}^{\max}$ and then $\Delta f_{pq}^{\max} x_{pq}$ as given in Eqs. 11.10 through 11.13. These values are given below where the flows and flow limits are all converted to per unit on a 100 MVA base. (The line reactances are found in the appendix to Chapter 4.)

Line	MW Limit (per unit)	f_{pq}^0 (per unit)	$\Delta f_{pq}^{\max}$	x_{pq}	$\Delta f_{pq}^{\max} x_{pq}$
1-2	0.30	0.253	0.047	0.20	0.0094
1-4	0.50	0.416	0.084	0.20	0.0168
1-5	0.40	0.331	0.069	0.30	0.0207
2-3	0.20	0.018	0.182	0.25	0.0455
2-4	0.40	0.325	0.075	0.10	0.0075
2-5	0.20	0.162	0.038	0.30	0.0114
2-6	0.30	0.248	0.052	0.20	0.0104
3-5	0.20	0.169	0.031	0.26	0.00806
3-6	0.60	0.449	—	—	—
4-5	0.20	0.041	0.159	0.40	0.0636
5-6	0.20	0.003	0.197	0.30	0.0591

For step A, we use Eq. 11A.13 from Appendix 11A to calculate $\delta_{3,36}$ and $\delta_{6,36}$ as

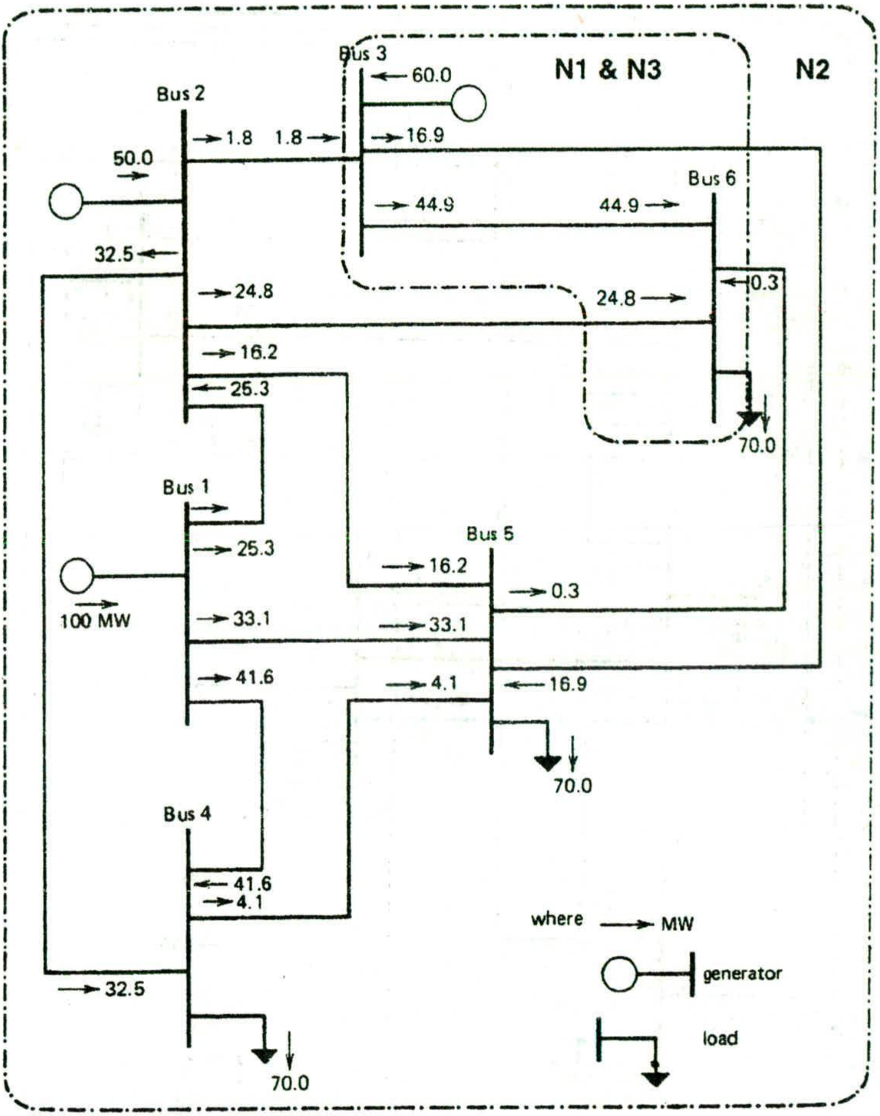


FIG. 11.14 Step A of Example 11B.

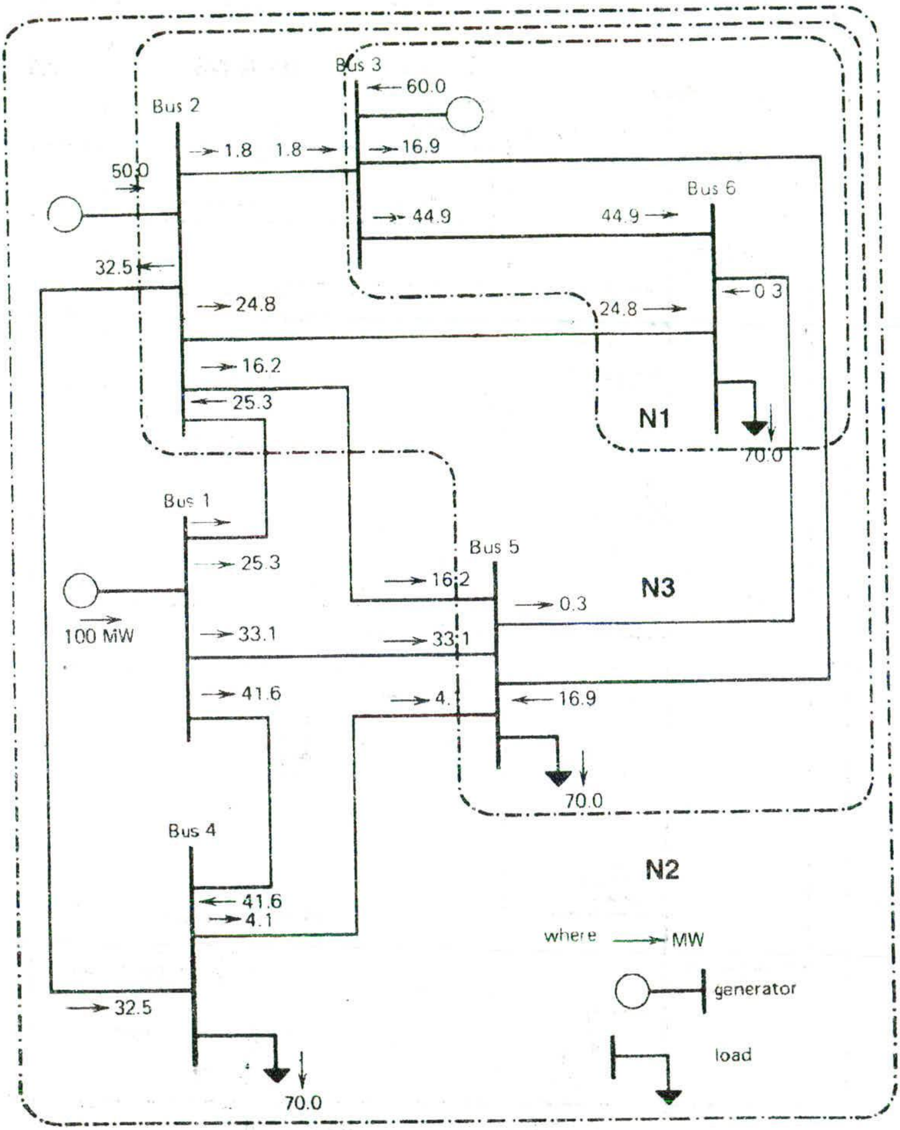


FIG. 11.15 Step B of Example 11B.

shown below.

$$\delta_{3,36} = \frac{(X_{33} - X_{36})x_{36}}{x_{36} - (X_{33} + X_{66} - 2X_{36})} = 0.12865$$

$$\delta_{6,36} = \frac{(X_{63} - X_{66})x_{36}}{x_{36} - (X_{ss} + X_{66} - X_{36})} = -0.11953$$

Then using Eq. 11A.11

$$|\Delta\theta_3 - \Delta\theta_6| = 0.111437$$

According to the criterion in Eq. 11.14, the value $|\Delta\theta_i - \Delta\theta_j|$ must be less that the smallest value of $|\Delta\theta_p - \Delta\theta_q|$ which equals $\Delta f_{pq}^{\max} x_{pq}$ and is found in the table above to be at line 2-4. Since $|\Delta\theta_3 - \Delta\theta_6| = 0.111437$ and the minimum $|\Delta\theta_i - \Delta\theta_j|$ is $|\Delta\theta_2 - \Delta\theta_4|$ which has a value of 0.0075, the criteria fails. We must proceed to step **B**.

Step **B** requires that we calculate $|\Delta\theta_i - \Delta\theta_j|$ for buses 2 and 5. This value is 0.003564 and the bounding criteria is satisfied.

If we had used the *d* factors for the six-bus system as shown in Example 11A, we could simply find all the line flows for the 3-6 outage as shown in the table below.

Line	MW Limit (per unit)	f_{pq}^{0} (per unit)	$f_{pq}^{3\text{-}6\text{ out}}$	
1-2	0.30	0.253	0.257	
1-4	0.50	0.416	0.416	
1-5	0.40	0.331	0.322	
2-3	0.20	0.018	−0.220	overload
2-4	0.40	0.325	0.316	
2-5	0.20	0.162	0.148	
2-6	0.30	0.248	0.508	overload
3-5	0.20	0.169	0.380	overload
3-6	0.60	0.449	—	
4-5	0.20	0.041	0.320	
5-6	0.20	0.003	0.191	

Note that three overloads exist on lines 2-3, 2-6, and 3-5, which are all within the bounded region N1 + N3 in Figure 11.15.

APPENDIX 11A
Calculation of Network Sensitivity Factors

First, we show how to derive the generation-shift sensitivity factors. To start, repeat Eq. 4.36.

$$\boldsymbol{\theta} = [X]\mathbf{P} \tag{11A.1}$$

This is the standard matrix calculation for the DC load flow. Since the DC power-flow model is a linear model, we may calculate perturbations about a given set of system conditions by use of the same model. Thus, if we are interested in the changes in bus phase angles, $\Delta\theta$, for a given set of changes in the bus power injections, $\Delta\mathbf{P}$, we can use the following calculation.

$$\Delta\boldsymbol{\theta} = [X]\Delta\mathbf{P} \tag{11A.2}$$

In Eq. 11A.1, it is assumed that the power on the swing bus is equal to the sum of the injections of all the other buses. Similarly, the net perturbation of the swing bus in Eq. 11A.2 is the sum of the perturbations on all the other buses.

Suppose that we are interested in calculating the generation shift sensitivity factors for the generator on bus i. To do this, we will set the perturbation on bus i to $+1$ and the perturbation on all the other buses to zero. We can then solve for the change in bus phase angles using the matrix calculation in Eq. 11A.3.

$$\Delta\boldsymbol{\theta} = [X]\begin{bmatrix} +1 \\ -1 \end{bmatrix}\begin{array}{l} \text{—row } i \\ \text{—ref row} \end{array} \tag{11A.3}$$

The vector of bus power injection perturbations in Eq. 11A.3 represents the situation when a 1 pu power increase is made at bus i and is compensated by a 1 pu decrease in power at the reference bus. The $\Delta\theta$ values in Eq. 11A.3 are thus equal to the derivative of the bus angles with respect to a change in power injection at bus i. Then, the required sensitivity factors are

$$\begin{aligned} a_{\ell i} &= \frac{\mathrm{d}f_\ell}{\mathrm{d}P_i} = \frac{\mathrm{d}}{\mathrm{d}P_i}\left[\frac{1}{x_\ell}(\theta_n - \theta_m)\right] \\ &= \frac{1}{x_\ell}\left(\frac{\mathrm{d}\theta_n}{\mathrm{d}P_i} - \frac{\mathrm{d}\theta_m}{\mathrm{d}P_i}\right) = \frac{1}{x_\ell}(X_{ni} - X_{mi}) \end{aligned} \tag{11A.4}$$

where

$$X_{ni} = \frac{\mathrm{d}\theta_n}{\mathrm{d}P_i} = n^{\text{th}} \text{ element from the } \Delta\boldsymbol{\theta} \text{ vector in Eq. 11A.3}$$

$$X_{mi} = \frac{\mathrm{d}\theta_m}{\mathrm{d}P_i} = m^{\text{th}} \text{ element from the } \Delta\boldsymbol{\theta} \text{ vector in Eq. 11A.3}$$

$$x_\ell = \text{line reactance for line } \ell$$

A line outage may be modeled by adding two power injections to a system, one at each end of the line to be dropped. The line is actually left in the system and the effects of its being dropped are modeled by injections. Suppose line k

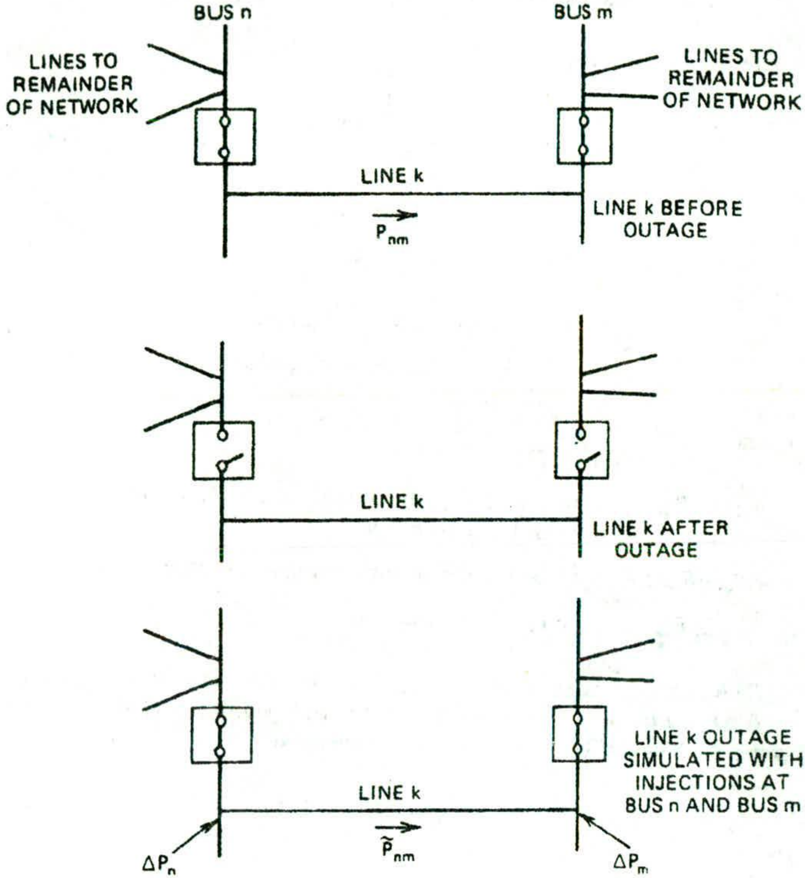


FIG. 11.16 Line outage modeling using injections.

from bus n to bus m were opened by circuit breakers as shown in Figure 11.16. Note that when the circuit breakers are opened, no current flows through them and the line is completely isolated from the remainder of the network. In the bottom part of Figure 11.16, the breakers are still closed but injections ΔP_n and ΔP_m have been added to bus n and bus m, respectively. If $\Delta P_n = \tilde{P}_{nm}$, where $\tilde{P}_{nm}$ is equal to the power flowing over the line, and $\Delta P_m = -\tilde{P}_{nm}$, we will still have no current flowing through the circuit breakers even though they are closed. As far as the remainder of the network is concerned, the line is disconnected.

Using Eq. 11A.2 relating to $\mathbf{\Delta\theta}$ and $\mathbf{\Delta P}$, we have

$$\mathbf{\Delta\theta} = [X]\mathbf{\Delta P}$$

where

$$\mathbf{\Delta P} = \begin{bmatrix} \vdots \\ \Delta P_n \\ \vdots \\ \Delta P_m \end{bmatrix}$$

so that

$$\begin{aligned} \Delta\theta_n &= X_{nn}\Delta P_n + X_{nm}\Delta P_m \\ \Delta\theta_m &= X_{mn}\Delta P_n + X_{mm}\Delta P_m \end{aligned} \tag{11A.5}$$

define

$\theta_n, \theta_m, P_{nm}$	to exist before the outage, where P_{nm} is the flow on line k from bus n to bus m
$\Delta\theta_n, \Delta\theta_m, \Delta P_{nm}$	to be the incremental changes resulting from the outage
$\tilde{\theta}_n, \tilde{\theta}_m, \tilde{P}_{nm}$	to exist after the outage

The outage modeling criteria requires that the incremental injections ΔP_n and ΔP_m equal the power flowing over the outaged line *after* the injections are imposed. Then, if we let the line reactance be x_k

$$\tilde{P}_{nm} = \Delta P_n = -\Delta P_m \tag{11A.6}$$

where

$$\tilde{P}_{nm} = \frac{1}{x_k}(\tilde{\theta}_n - \tilde{\theta}_m)$$

then

$$\begin{aligned} \Delta\theta_n &= (X_{nn} - X_{nm})\Delta P_n \\ \Delta\theta_m &= (X_{mm} - X_{mn})\Delta P_n \end{aligned} \tag{11A.7}$$

and

$$\begin{aligned} \tilde{\theta}_n &= \theta_n + \Delta\theta_n \\ \tilde{\theta}_m &= \theta_m + \Delta\theta_m \end{aligned} \tag{11A.8}$$

giving

$$\tilde{P}_{nm} = \frac{1}{x_k}(\tilde{\theta}_n - \tilde{\theta}_m) = \frac{1}{x_k}(\theta_n - \theta_m) + \frac{1}{x_k}(\Delta\theta_n - \Delta\theta_m)$$

or

$$\tilde{P}_{nm} = P_{nm} + \frac{1}{x_k}(X_{nn} + X_{mm} - 2X_{nm})\Delta P_n \tag{11A.9}$$

Then (using the fact that $\tilde{P}_{nm}$ is set to ΔP_n)

$$\Delta P_n = \left[\frac{1}{1 - \frac{1}{x_k}(X_{nn} + X_{mm} - 2X_{nm})} \right] P_{nm} \tag{11A.10}$$

Define a sensitivity factor δ as the ratio of the change in phase angle θ, anywhere in the system, to the original power P_{nm} flowing over a line nm before it was dropped. That is,

$$\delta_{i,nm} = \frac{\Delta\theta_i}{P_{nm}} \tag{11A.11}$$

If neither n or m is the system reference bus, two injections, ΔP_n and ΔP_m, are imposed at buses n and m, respectively. This gives a change in phase angle at bus i equal to

$$\Delta\theta_i = X_{in}\Delta P_n + X_{im}\Delta P_m \tag{11A.12}$$

Then using the relationship between ΔP_n and ΔP_m, the resulting δ factor is

$$\delta_{i,nm} = \frac{(X_{in} - X_{im})x_k}{x_k - (X_{nn} + X_{mm} - 2X_{nm})} \tag{11A.13}$$

If either n or m is the reference bus, only one injection is made. The resulting δ factors are

$$\begin{aligned} \delta_{i,nm} &= \frac{X_{in}x_k}{(x_k - X_{nn})} \qquad \text{for } m = \text{ref} \\ &= \frac{-X_{im}x_k}{(x_k - X_{mm})} \qquad \text{for } n = \text{ref} \end{aligned} \tag{11A.14}$$

If bus i itself is the reference bus, then $\delta_{i,nm} = 0$ since the reference bus angle is constant.

The expression for $d_{\ell,k}$ is

$$\begin{aligned} d_{\ell,k} = \frac{\Delta f_\ell}{f_k^0} &= \frac{\frac{1}{x_\ell}(\Delta\theta_i - \Delta\theta_j)}{f_k^0} \\ &= \frac{1}{x_\ell}\left(\frac{\Delta\theta_i}{P_{nm}} - \frac{\Delta\theta_j}{P_{nm}}\right) \\ &= \frac{1}{x_\ell}(\delta_{i,nm} - \delta_{j,nm}) \end{aligned} \tag{11A.15}$$

if neither i nor j is a reference bus

$$d_{\ell,k} = \frac{1}{x_\ell}\left(\frac{(X_{in} - X_{im})x_k - (X_{jn} - X_{jm})x_k}{x_k - (X_{nn} + X_{mm} - 2X_{nm})}\right)$$

$$= \frac{\dfrac{x_k}{x_\ell}(X_{in} - X_{jn} - X_{im} + X_{jm})}{x_k - (X_{nn} + X_{mm} - 2X_{nm})} \tag{11A.16}$$

The fact that the a and d factors are linear models of the power system allows us to use superposition to extend them. One very useful extension is to use the a and d factors to model the power system in its post-outage state; that is, to generate factors that model the system's sensitivity after a branch has been lost.

Suppose one desired to have the sensitivity factor between line ℓ and generator bus i when line k was opened. This is calculated by first assuming that the change in generation on bus i, ΔP_i, has a direct effect on line ℓ and an indirect effect through its influence on the power flowing on line k, which, in turn, influences line ℓ when line k is out. Then

$$\Delta f_\ell = a_{\ell i}\Delta P_i + d_{\ell,k}\Delta f_k \tag{11A.17}$$

However, we know that

$$\Delta f_k = a_{ki}\Delta P_i \tag{11A.18}$$

therefore,

$$\Delta f_\ell = a_{\ell i}\Delta P_i + d_{\ell,k}a_{ki}\Delta P_i = (a_{\ell i} + d_{\ell,k}a_{ki})\Delta P_i \tag{11A.19}$$

We can refer to $a_{\ell i} + d_{\ell,k}a_{ki}$ as the "compensated generation shift sensitivity."

The compensated sensitivity factors are useful in finding corrections to the generation dispatch that will make the post-contingency state of the system secure from overloads. This will be dealt with in Chapter 13 under the topic of "security-constrained optimal power flow."

APPENDIX 11B
Derivation of Equation 11.14

Equation 11.14, repeated here as Eq. 11B.1:

$$|\Delta\theta_p - \Delta\theta_q| < |\Delta\theta_i - \Delta\theta_j| \tag{11B.1}$$

is proved as shown in reference 11 (the proof is attributed to Moslehi).

Suppose that buses i and j have the highest and lowest values of $\Delta\theta$ in the N3 region. Then the following both hold:

$$\Delta\theta_i > \Delta\theta_f$$

and

$$\Delta\theta_j < \Delta\theta_f$$

for all buses f in N3. Taking any external bus in N2, call it bus e, we shall state that

$$\Delta\theta_e < \Delta\theta_i \tag{11B.2}$$

and

$$\Delta\theta_e > \Delta\theta_j \tag{11B.3}$$

Proof: Suppose Eq. 11B.2 is not true and there exists a bus e' such that

$$\Delta\theta_{e'} > \Delta\theta_i$$

and, further, suppose that

$$\Delta\theta_{e'} > \Delta\theta_e \tag{11B.4}$$

for all the buses in N3. This implies that Eq. 11B.4 holds for the union of buses in N2 and N3. If we now look at the network as a DC power flow network, with no impedances to ground, and only the two injections at buses k and m, then all incremental power flows leaving node e' must be positive, since the incremental flows leaving node e' are found from

$$\Delta f_{e'e} = \frac{1}{x_{e'e}}(\Delta\theta_{e'} - \Delta\theta_e) \tag{11B.5}$$

However, since the network in N2 and N3 is strictly passive, and there are no impedances to ground, this violates Kirchoff's current law, which requires all branch flows incident to a bus to sum to zero. The only way for this to be true would be if all flows were zero; that is, all incremental angle spreads were equal. We can continue this reasoning to the neighbor buses of e' until we reach node i and conclude that

$$\Delta\theta_{e'} = \Delta\theta_i \tag{11B.6}$$

which contradicts Eq. 11B.4; thus, Eq. 11B.2 is proved. Equation 11B.3 is proved in a similar fashion. Then, as a result, Eq. 11B.1 is also proved.

PROBLEMS

11.1 Figure 11.17 shows a four-bus power system. Also given below are the impedance data for the transmission lines of the system as well as the generation and load values.

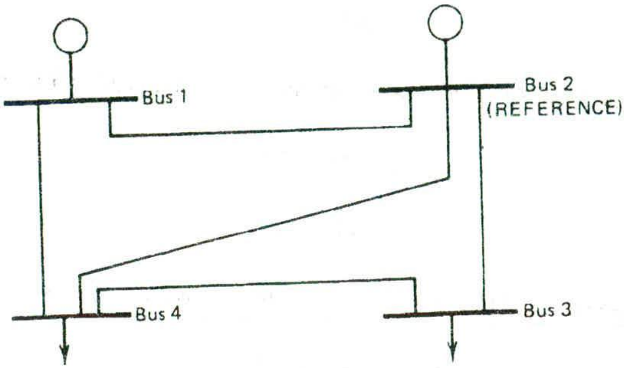


FIG. 11.17 Four-bus network for Problem 11.1.

Line	Line rectance (pu)
1-2	0.2
1-4	0.25
2-3	0.15
2-4	0.30
3-4	0.40

Bus	Load (MW)	Generation (MW)
1		150
2		350
3	220	
4	280	

a. Calculate the generation shift sensitivity coefficients for a shift in generation from bus 1 to bus 2.

b. Calculate the line outage sensitivity factors for outages on lines 1-2, 1-4, and 2-3.

11.2 In the system shown in Figure 11.18, three generators are serving a load of 1300 MW. The MW flow distribution, bus loads, and generator outputs are as shown. The generators have the following characteristics.

Generator No.	P_{min} (MW)	P_{max} (MW)
1	100	600
2	90	400
3	100	500

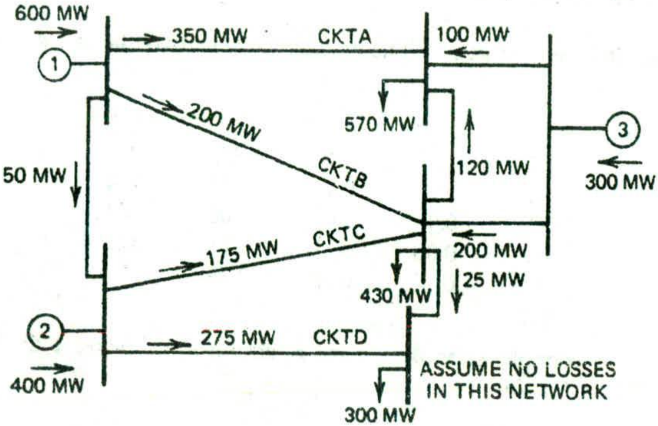


FIG. 11.18 Three-generator system for Problem 11.2.

The circuits have the following limits:

CKT A	600 MW max
CKT B	600 MW max
CKT C	450 MW max
CKT D	350 MW max

Throughout this problem we will only be concerned with flows on the circuit labeled A, B, C, and D. The generation shift sensitivity coefficients, $a_{\ell i}$, for circuits, A, B, C, and D are as follows.

CKT	Shift on Gen. 1	Shift on Gen. 2
A	0.7	0.08
B	0.2	0.02
C	0.06	0.54
D	0.04	0.36

Example: $$\Delta P_{\text{flow}_\ell} = a_{\ell,i} \times \Delta P_i$$

if

$$\ell = C \quad \text{and} \quad i = 2$$

$$\Delta P_{\text{flow}_c} = (0.54)\Delta P_2$$

Assume a shift on gen. 1 or gen. 2 will be compensated by an equal (opposite) shift on gen. 3. The line outage sensitivity factors $d_{\ell,k}$ are

$$\begin{array}{cc|cccc} & & & & & \xrightarrow{\quad\quad} k \\ & & \text{A} & \text{B} & \text{C} & \text{D} \\ & \text{A} & X & 0.8 & 0.21 & 0.14 \\ \ell & \text{B} & 0.9 & X & 0.06 & 0.04 \\ \downarrow & \text{C} & 0.06 & 0.12 & X & 0.82 \\ & \text{D} & 0.04 & 0.08 & 0.73 & X \end{array}$$

As an example, suppose the loss of circuit k will increase the loading on circuit ℓ as follows.

$$P_{\text{flow}_\ell} = P_{\text{flow}_\ell} \text{ (before outage)} + d_{\ell,k} \times P_{\text{flow}_k} \text{ (before outage)}$$

if

$$\ell = \text{A} \quad \text{and} \quad k = \text{B}$$

The new flow on ℓ would be

$$P_{\text{flow}_A} = P_{\text{flow}_A} + (0.8) P_{\text{flow}_B}$$

a. Find the contingency (outage) flow distribution on circuits A, B, C, and D for an outage on circuit A. Repeat for an outage on B, then on C, then on D. (Only one circuit is lost at one time.) Are there any overloads?

b. Can you shift generation from gen. 1 to gen. 3, or from gen. 2 to gen. 3, so that no overloads occur? If so, how much shift?

11.3 Given the three-bus network shown in Figure 11.19 (see Example 4B), where

$$x_{12} = 0.2 \text{ pu}$$

$$x_{13} = 0.4 \text{ pu}$$

$$x_{23} = 0.25 \text{ pu}$$

the $[X]$ matrix is

$$\begin{bmatrix} 0.2118 & 0.1177 & 0 \\ 0.1177 & 0.1765 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

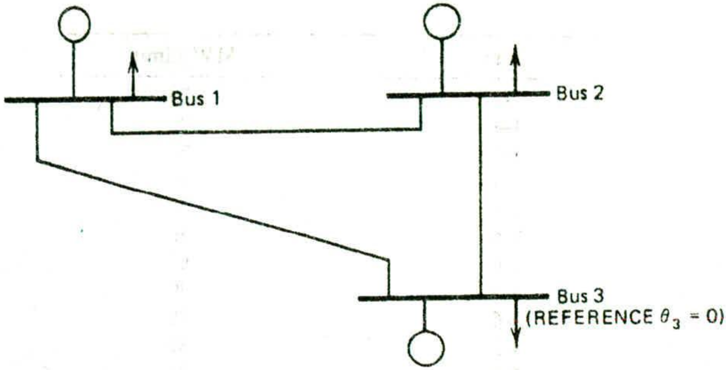


FIG. 11.19 Three-bus system for Problem 11.3.

Use a 100-MVA base. The base loads and generations are as follows.

Bus	Load (MW)	Gen. (MW)	Gen. min (MW)	Gen. max (MW)
1	100	150	50	250
2	300	180	60	250
3	100	170	60	300

a. Find base power flows on the transmission lines.

b. Calculate the generation shift factors for line 1-2. Calculate the shift in generation on bus 1 and 2 so as to force the flow on line 1-2 to zero MW. Assume for economic reasons that any shifts from base conditions are more expensive for shifts at the generator on bus 1 than for shifts on bus 2, and that the generator on bus 3 can be shifted without any penalty.

11.4 Using the system shown in Example 11B, find N1, N2 and N3 for the outage of the line from bus 2 to bus 4. Do you need to expand region N1? Where are the overloads, if any? (Use the same branch flow limits as shown in Example 11B.)

11.5 Using the data found in Figure 11.7, find the base-case bus phase angles and all line flows using the following bus loads and generators: all loads are 100 MW and all generators are also at 100 MW. Assume line flow limits as shown in the following table.

Line	MW Limit
1-2	70
1-4	90
1-5	70
2-3	20
2-4	50
2-5	40
2-6	60
3-5	30
3-6	70
4-5	30
5-6	20

For a line outage on line 1-4, find the change in phase angle across each of the remaining lines and see if the phase angle change across buses 1 and 4 meets the bounding criteria developed in the text.

11.6 Using the data from Problem 11.2, calculate the performance index, PI, for each outage case. Use a value of $n = 1$ and $n = 5$; that is for

$$\mathrm{PI} = \sum_{\text{all lines}} \left(\frac{\text{flow}_{ij}}{\text{flow max}_{ij}} \right)^{2n}$$

Which PI does a better job of predicting the case with the overload? Explain why.

FURTHER READING

The subject of power system security has received a great deal of attention in the engineering literature since the middle 1960s. The list of references presented here is therefore large but also quite limited nonetheless.

Reference 1 is a key paper on the topic of system security and energy control system philosophy. Reference 2 provides the basic theory for contingency assessment of power systems. Reference 3 covers contingency analysis using DC power flow methods. Reference 4 is a broad overview of security assessment and contains an excellent bibliography covering the literature on security assessment up to 1975.

The use of AC power flows in contingency analysis is possible with any AC load flow algorithm. However, the *fast-decoupled* power flow algorithm is generally recognized as the best for this purpose since its Jacobian matrix is constant and single-line outages can be modeled using the matrix inversion lemma. Reference 5 covers the fast-decoupled power flow algorithm and its application.

Correcting the generation dispatch by sensitivity methods is covered by reference 6. The use of linear programming to solve power systems problems is covered in reference 7.

References 8–12 cover some of the literature on contingency selection, and reference 13 gives a technique for solving the power flow using an approximation called concentric relaxation. References 14 and 15 give an indication of recent research on dynamic security assessment; that is, detecting fault cases that may cause dynamic or transient stability problems. Finally, reference 16 is concerned with the emerging area of voltage stability, which seeks to find contingencies which will cause such severe voltage problems as to bring on what is known as a "voltage collapse."

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12 An Introduction to State Estimation in Power Systems

12.1 INTRODUCTION

State estimation is the process of assigning a value to an unknown system state variable based on measurements from that system according to some criteria. Usually, the process involves imperfect measurements that are redundant and the process of estimating the system states is based on a statistical criterion that estimates the true value of the state variables to minimize or maximize the selected criterion. A commonly used and familiar criterion is that of minimizing the sum of the squares of the differences between the estimated and "true" (i.e., measured) values of a function.

The ideas of least-squares estimation have been known and used since the early part of the nineteenth century. The major developments in this area have taken place in the twentieth century in applications in the aerospace field. In these developments, the basic problems have involved the location of an aerospace vehicle (i.e., missile, airplane, or space vehicle) and the estimation of its trajectory given redundant and imperfect measurements of its position and velocity vector. In many applications, these measurements are based on optical observations and/or radar signals that may be contaminated with noise and may contain system measurement errors. State estimators may be both static and dynamic. Both types of estimators have been developed for power systems. This chapter will introduce the basic development of a static-state estimator.

In a power system, the state variables are the voltage magnitudes and relative phase angles at the system nodes. Measurements are required in order to estimate the system performance in real time for both system security control and constraints on economic dispatch. The inputs to an estimator are imperfect power system measurements of voltage magnitudes and power, VAR, or ampere-flow quantities. The estimator is designed to produce the "best estimate" of the system voltage and phase angles, recognizing that there are errors in the measured quantities and that there may be redundant measurements. The output data are then used in system control centers in the implementation of the security-constrained dispatch and control of the system as discussed in Chapters 11 and 13.

12.2 POWER SYSTEM STATE ESTIMATION

As introduced in Chapter 11, the problem of monitoring the power flows and voltages on a transmission system is very important in maintaining system security. By simply checking each measured value against its limit, the power system operators can tell where problems exist in the transmission system—and, it is hoped, they can take corrective actions to relieve overloaded lines or out-of-limit voltages.

Many problems are encountered in monitoring a transmission system. These problems come primarily from the nature of the measurement transducers and from communications problems in transmitting the measured values back to the operations control center.

Transducers from power system measurements, like any measurement device, will be subject to errors. If the errors are small, they may go undetected and can cause misinterpretation by those reading the measured values. In addition, transducers may have gross measurement errors that render their output useless. An example of such a gross error might involve having the transducer connected up backward; thus, giving the negative of the value being measured. Finally, the telemetry equipment often experiences periods when communications channels are completely out; thus, depriving the system operator of any information about some part of the power system network.

It is for these reasons that power system state estimation techniques have been developed. A state estimator, as we will see shortly, can "smooth out" small random errors in meter readings, detect and identify gross measurement errors, and "fill in" meter readings that have failed due to communications failures.

To begin, we will use a simple DC load flow example to illustrate the principles of state estimation. Suppose the three-bus DC load flow of Example 4B were operating with the load and generation shown in Figure 12.1. The only information we have about this system is provided by three MW power flow meters located as shown in Figure 12.2.

Only two of these meter readings are required to calculate the bus phase angles and all load and generation values fully. Suppose we use M_{13} and M_{32} and further suppose that M_{13} and M_{32} give us perfect readings of the flows on their respective transmission lines.

$$M_{13} = 5 \text{ MW} = 0.05 \text{ pu}$$

$$M_{32} = 40 \text{ MW} = 0.40 \text{ pu}$$

Then, the flows on lines 1-3 and 3-2 can be set equal to these meter readings.

$$f_{13} = \frac{1}{x_{13}}(\theta_1 - \theta_3) = M_{13} = 0.05 \text{ pu}$$

$$f_{32} = \frac{1}{x_{23}}(\theta_3 - \theta_2) = M_{32} = 0.40 \text{ pu}$$

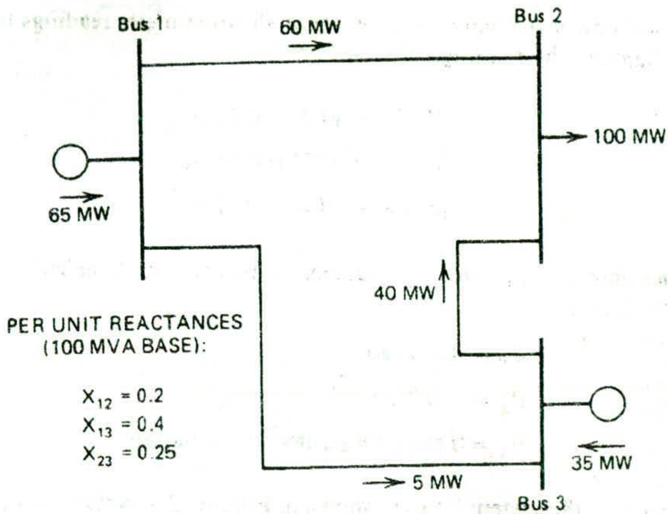

FIG. 12.1 Three-bus system from Example 4B.

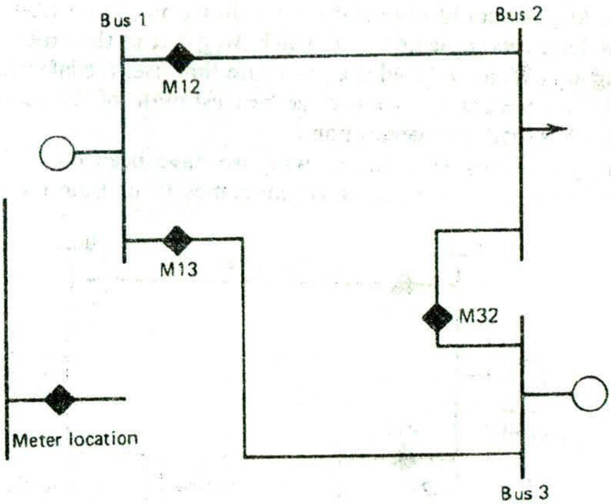

FIG. 12.2 Meter placement.

Since we know that $\theta_3 = 0$ rad, we can solve the f_{13} equation for θ_1, and the f_{32} equation for θ_2, resulting in

$$\theta_1 = 0.02 \text{ rad}$$

$$\theta_2 = -0.10 \text{ rad}$$

We will now investigate the case where all three meter readings have slight errors. Suppose the readings obtained are

$$M_{12} = 62 \text{ MW} = 0.62 \text{ pu}$$
$$M_{13} = 6 \text{ MW} = 0.06 \text{ pu}$$
$$M_{32} = 37 \text{ MW} = 0.37 \text{ pu}$$

If we use only the M_{13} and M_{32} readings, as before, we will calculate the phase angles as follows:

$$\theta_1 = 0.024 \text{ rad}$$
$$\theta_2 = -0.0925 \text{ rad}$$
$$\theta_3 = 0 \text{ rad (still assumed to equal zero)}$$

This results in the system flows as shown in Figure 12.3. Note that the predicted flows match at M_{13} and M_{32}, but the flow on line 1-2 does not match the reading of 62 MW from M_{12}. If we were to ignore the reading on M_{13} and use M_{12} and M_{32}, we could obtain the flows shown in Figure 12.4.

All we have accomplished is to match M_{12}, but at the expense of no longer matching M_{13}. What we need is a procedure that uses the information available from all three meters to produce the best estimate of the actual angles, line flows, and bus load and generations.

Before proceeding, let's discuss what we have been doing. Since the only thing we know about the power system comes to us from the measurements,

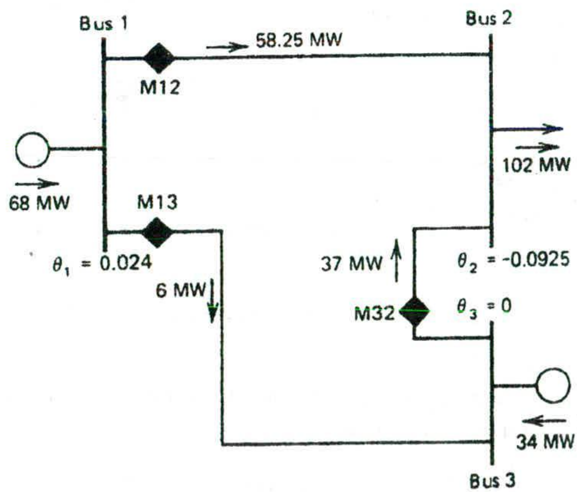

FIG. 12.3 Flows resulting from use of meters M_{13} and M_{32}.

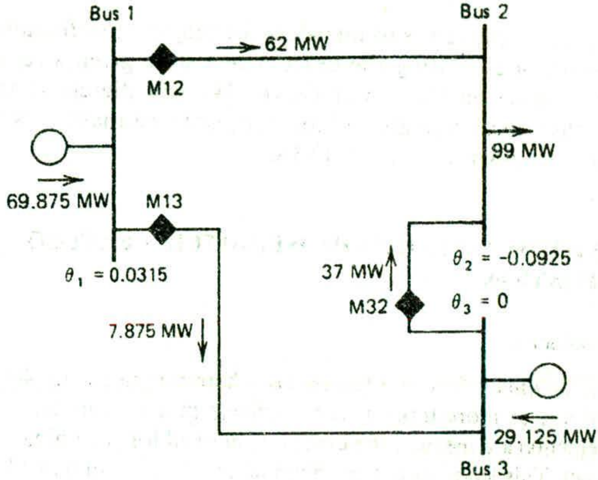

FIG. 12.4 Flows resulting from use of meters M_{12} and M_{32}.

we must use the measurements to estimate system conditions. Recall that in each instance the measurements were used to calculate the bus phase angles at bus 1 and 2. Once these phase angles were known, all unmeasured power flows, loads, and generations could be determined. We call θ_1 and θ_2 the *state variables* for the three-bus system since knowing them allows all other quantities to be calculated. In general, the state variables for a power system consist of the bus voltage magnitude at all buses and the phase angles at all but one bus. The swing or reference bus phase angle is usually assumed to be zero radians. Note that we could use real and imaginary components of bus voltage if desired. If we can use measurements to estimate the "states" (i.e., voltage magnitudes and phase angles) of the power system, then we can go on to calculate any power flows, generation, loads, and so forth that we desire. This presumes that the network configuration (i.e., breaker and disconnect switch statuses) is known and that the impedances in the network are also known. Automatic load tap changing autotransformers or phase angle regulators are often included in a network, and their tap positions may be telemetered to the control as a measured quantity. Strictly speaking, the transformer taps and phase angle regulator positions should also be considered as states since they must be known in order to calculate the flows through the transformers and regulators.

To return to the three-bus DC power flow model, we have three meters providing us with a set of redundant readings with which to estimate the two states θ_1 and θ_2. We say that the readings are redundant since, as we saw earlier, only two readings are necessary to calculate θ_1 and θ_2, the other reading is always "extra." However, the "extra" reading does carry useful information and ought not to be discarded summarily.

This simple example serves to introduce the subject of *static-state estimation*, which is the art of estimating the exact system state given a set of imperfect measurements made on the power system. We will digress at this point to develop the theoretical background for static-state estimation. We will return to our three-bus system in Section 12.3.4.

12.3 MAXIMUM LIKELIHOOD WEIGHTED LEAST-SQUARES ESTIMATION

12.3.1 Introduction

Statistical estimation refers to a procedure where one uses samples to calculate the value of one or more unknown parameters in a system. Since the samples (or measurements) are inexact, the estimate obtained for the unknown parameter is also inexact. This leads to the problem of how to formulate a "best"estimate of the unknown parameters given the available measurements.

The development of the notions of state estimation may proceed along several lines, depending on the statistical criterion selected. Of the many criteria that have been examined and used in various applications, the following three are perhaps the most commonly encountered.

1. *The maximum likelihood criterion*, where the objective is to maximize the probability that the estimate of the state variable, $\hat{\mathbf{x}}$, is the true value of the state variable vector, $\mathbf{x}$ (i.e., maximize $P(\hat{\mathbf{x}}) = \mathbf{x}$).
2. *The weighted least-squares criterion*, where the objective is to minimize the sum of the squares of the weighted deviations of the estimated measurements, $\hat{\mathbf{z}}$, from the actual measurements, $\mathbf{z}$.
3. *The minimum variance criterion*, where the object is to minimize the expected value of the sum of the squares of the deviations of the estimated components of the state variable vector from the corresponding components of the true state variable vector.

When normally distributed, unbiased meter error distributions are assumed, each of these approaches results in identical estimators. This chapter will utilize the maximum likelihood approach because the method introduces the measurement error weighting matrix $[R]$ in a straightforward manner.

The maximum likelihood procedure asks the following question: "What is the probability (or likelihood) that I will get the measurements I have obtained?" This probability depends on the random error in the measuring device (transducer) as well as the unknown parameters to be estimated. Therefore, a reasonable procedure would be one that simply chose the estimate as the value that maximizes this probability. As we will see shortly, the maximum likelihood estimator assumes that we know the probability density function (PDF) of the random errors in the measurement. Other estimation schemes could also be used. The "least-squares" estimator does not require that we know the probability density function for the sample or measurement errors. However, if we assume that the probability density function of sample or measurement error is a normal (Gaussian) distribution, we will end up with the same estimation formula. We will proceed to develop our estimation formula using the maximum likelihood criterion assuming normal distributions for measurement errors. The result will be a "least-squares" or more precisely a "weighted least-squares" estimation formula, even though we will develop the formulation using the maximum likelihood criteria. We will illustrate this method with a simple electrical circuit and show how the maximum likelihood estimate can be made.

First, we introduce the concept of *random measurement error*. Note that we have dropped the term "sample" since the concept of a measurement is much more appropriate to our discussion. The measurements are assumed to be in error: that is, the value obtained from the measurement device is close to the true value of the parameter being measured but differs by an unknown error. Mathematically, this can be modeled as follows.

Let z^{meas} be the value of a measurement as received from a measurement device. Let z^{true} be the true value of the quantity being measured. Finally, let η be the random measurement error. We can then represent our measured value as

$$z^{\text{meas}} = z^{\text{true}} + \eta \tag{12.1}$$

The random number, η, serves to model the uncertainty in the measurements. If the measurement error is unbiased, the probability density function of η is usually chosen as a normal distribution with zero mean. Note that other measurement probability density functions will also work in the maximum likelihood method as well. The probability density function of η is

$$\text{PDF}(\eta) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\eta^2/2\sigma^2) \tag{12.2}$$

where σ is called the standard deviation and σ^2 is called the variance of the random number. PDF(η) describes the behavior of η. A plot of PDF(η) is shown in Figure 12.5. Note that σ, the standard deviation, provides a way to model the seriousness of the random measurement error. If σ is large, the measurement is relatively inaccurate (i.e., a poor-quality measurement device), whereas a small value of σ denotes a small error spread (i.e., a higher-quality measurement device). The normal distribution is commonly used for modeling measurement errors since it is the distribution that will result when many factors contribute to the overall error.

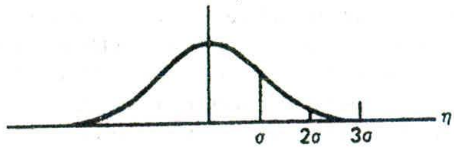

FIG. 12.5 The normal distribution.

12.3.2 Maximum Likelihood Concepts

The principle of maximum likelihood estimation is illustrated by using a simple DC circuit example as shown in Figure 12.6. In this example, we wish to estimate the value of the voltage source, x^{true}, using an ammeter with an error having a known standard deviation. The ammeter gives a reading of z_1^{meas}, which is equal to the sum of z_1^{true} (the true current flowing in our circuit) and η_1 (the error present in the ammeter). Then we can write

$$z_1^{\text{meas}} = z_1^{\text{true}} + \eta_1 \tag{12.3}$$

Since the mean value of η_1 is zero, we then know that the mean value of z_1^{meas} is equal to z_1^{true}. This allows us to write a probability density function for z_1^{meas} as

$$\text{PDF}(z_1^{\text{meas}}) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[\frac{-(z_1^{\text{meas}} - z_1^{\text{true}})^2}{2\sigma_1^2}\right] \tag{12.4}$$

where σ_1 is the standard deviation for the random error η_1. If we assume that the value of the resistance, r_1, in our circuit is known, then we can write

$$\text{PDF}(z_1^{\text{meas}}) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[\frac{-\left(z_1^{\text{meas}} - \dfrac{1}{r_1}x\right)^2}{2\sigma_1^2}\right] \tag{12.5}$$

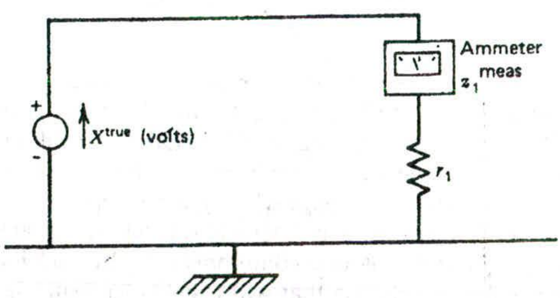

FIG. 12.6 Simple DC circuit with current measurement.

Coming back to our definition of a maximum likelihood estimator, we now wish to find an estimate of x (called x^{est}) that maximizes the probability that the observed measurement z_1^{meas} would occur. Since we have the probability density function of z_1^{meas}, we can write

$$\begin{aligned}\text{prob}(z_1^{\text{meas}}) &= \int_{z_1^{\text{meas}}}^{z_1^{\text{meas}}+dz_1^{\text{meas}}} \text{PDF}(z_1^{\text{meas}})\, dz_1^{\text{meas}} \quad \text{as } dz_1^{\text{meas}} \to 0 \\ &= \text{PDF}(z_1^{\text{meas}})\, dz_1^{\text{meas}}\end{aligned} \tag{12.6}$$

The maximum likelihood procedure then requires that we maximize the value of prob(z_1^{meas}), which is a function of x. That is,

$$\max_x \text{prob}(z_1^{\text{meas}}) = \max_x \text{PDF}(z_1^{\text{meas}})\, dz_1^{\text{meas}} \tag{12.7}$$

One convenient transformation that can be used at this point is to maximize the natural logarithm of PDF(z_1^{meas}) since maximizing the Ln of PDF(z_1^{meas}) will also maximize PDF(z_1^{meas}). Then we wish to find

$$\max_x \text{Ln}[\text{PDF}(z_1^{\text{meas}})]$$

or

$$\max_x \left[-\text{Ln}(\sigma_1\sqrt{2\pi}) - \frac{\left(z_1^{\text{meas}} - \dfrac{1}{r_1}x\right)^2}{2\sigma_1^2} \right]$$

Since the first term is constant, it can be ignored. We can maximize the function in brackets by minimizing the second term since it has a negative coefficient, that is,

$$\max_x \left[-\text{Ln}(\sigma_1\sqrt{2\pi}) - \frac{\left(z_1^{\text{meas}} - \dfrac{1}{r_1}x\right)^2}{2\sigma_1^2} \right]$$

is the same as

$$\min_x \left[\frac{\left(z_1^{\text{meas}} - \dfrac{1}{r_1}x\right)^2}{2\sigma_1^2} \right] \tag{12.8}$$

The value of x that minimizes the right-hand term is found by simply taking the first derivative and setting the result to zero:

$$\frac{d}{dx}\left[\frac{\left(z_1^{\text{meas}} - \dfrac{1}{r_1}x\right)^2}{2\sigma_1^2} \right] = \frac{-\left(z_1^{\text{meas}} - \dfrac{1}{r_1}x\right)}{r_1\sigma_1^2} = 0 \tag{12.9}$$

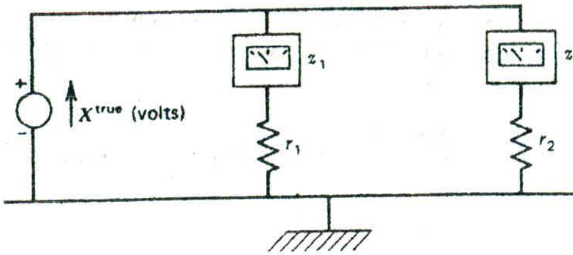

FIG. 12.7 DC circuit with two current measurements.

or

$$x^{est} = r_1 z_1^{meas}$$

To most readers this result was obvious from the beginning. All we have accomplished is to declare the maximum likelihood estimate of our voltage as simply the measured current times the known resistance. However, by adding a second measurement circuit, we have an entirely different situation in which the best estimate is not so obvious. Let us now add a second ammeter and resistance as shown in Figure 12.7.

Assume that both r_1 and r_2 are known. As before, model each meter reading as the sum of the true value and a random error:

$$\begin{aligned} z_1^{meas} &= z_1^{true} + \eta_1 \\ z_2^{meas} &= z_2^{true} + \eta_2 \end{aligned} \tag{12.10}$$

where the errors will be represented as independent zero mean, normally distributed random variables with probability density functions:

$$\begin{aligned} \mathrm{PDF}(\eta_1) &= \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(\frac{-(\eta_1)^2}{2\sigma_1^2}\right) \\ \mathrm{PDF}(\eta_2) &= \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(\frac{-(\eta_2)^2}{2\sigma_2^2}\right) \end{aligned} \tag{12.11}$$

and as before we can write the probability density functions of z_1^{meas} and z_2^{meas} as

$$\begin{aligned} \mathrm{PDF}(z_1^{meas}) &= \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[\frac{-\left(z_1^{meas} - \frac{1}{r_1}x\right)^2}{2\sigma_1^2}\right] \\ \mathrm{PDF}(z_2^{meas}) &= \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[\frac{-\left(z_2^{meas} - \frac{1}{r_2}x\right)^2}{2\sigma_2^2}\right] \end{aligned} \tag{12.12}$$

The likelihood function must be the probability of obtaining the measurements z_1^{meas} and z_2^{meas}. Since we are assuming that the random errors η_1 and η_2 are independent random variables, the probability of obtaining z_1^{meas} and z_2^{meas} is simply the product of the probability of obtaining z_1^{meas} and the probability of obtaining z_2^{meas}

$$\begin{aligned}
\text{prob}(z_1^{\text{meas}} \text{ and } z_2^{\text{meas}}) &= \text{prob}(z_1^{\text{meas}}) \times (\text{prob}(z_2^{\text{meas}}) \\
&= \text{PDF}(z_1^{\text{meas}})\,\text{PDF}(z_2^{\text{meas}})\,\mathrm{d}z_1^{\text{meas}}\,\mathrm{d}z_2^{\text{meas}} \\
&= \left[\frac{1}{\sigma_1\sqrt{2\pi}}\exp\left(\frac{-\left(z_1^{\text{meas}} - \frac{1}{r_1}x\right)^2}{2\sigma_1^2}\right)\right] \\
&\quad \times \left[\frac{1}{\sigma_2\sqrt{2\pi}}\exp\left(\frac{-\left(z_2^{\text{meas}} - \frac{1}{r_2}x\right)^2}{2\sigma_2^2}\right)\right]\mathrm{d}z_1^{\text{meas}}\,\mathrm{d}z_2^{\text{meas}}
\end{aligned} \tag{12.13}$$

To maximize the function we will again take its natural logarithm:

$$\begin{aligned}
&\max_x \text{prob}(z_1^{\text{meas}} \text{ and } z_2^{\text{meas}}) \\
&= \max_x\left[-\text{Ln}(\sigma_1\sqrt{2\pi}) - \frac{\left(z_1^{\text{meas}} - \frac{1}{r_1}x\right)^2}{2\sigma_1^2} - \text{Ln}(\sigma_2\sqrt{2\pi}) - \frac{\left(z_2^{\text{meas}} - \frac{1}{r_2}x\right)^2}{2\sigma_2^2}\right] \\
&= \min_x\left[\frac{\left(z_1^{\text{meas}} - \frac{1}{r_1}x\right)^2}{2\sigma_1^2} + \frac{\left(z_2^{\text{meas}} - \frac{1}{r_2}x\right)^2}{2\sigma_2^2}\right]
\end{aligned} \tag{12.14}$$

The minimum sought is found by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\left(z_1^{\text{meas}} - \frac{1}{r_1}x\right)^2}{2\sigma_1^2} + \frac{\left(z_2^{\text{meas}} - \frac{1}{r_2}x\right)^2}{2\sigma_2^2}\right] \\
&= \frac{-\left(z_1^{\text{meas}} - \frac{1}{r_1}x\right)}{r_1\sigma_1^2} - \frac{\left(z_2^{\text{meas}} - \frac{1}{r_2}x\right)}{r_2\sigma_2^2} = 0
\end{aligned}$$

giving

$$x^{\text{est}} = \frac{\left(\dfrac{z_1^{\text{meas}}}{r_1\sigma_1^2} + \dfrac{z_2^{\text{meas}}}{r_2\sigma_2^2}\right)}{\left(\dfrac{1}{r_1^2\sigma_1^2} + \dfrac{1}{r_2^2\sigma_2^2}\right)} \tag{12.15}$$

If one of the ammeters is of superior quality, its variance will be much smaller than that of the other meter. For example, if $\sigma_2^2 \ll \sigma_1^2$, then the equation for x^{est} becomes

$$x^{\text{est}} \simeq z_2^{\text{meas}} \times r_2$$

Thus, we see that the maximum likelihood method of estimating our unknown parameter gives us a way to weight the measurements properly according to their quality.

It should be obvious by now that we need not express our estimation problem as a maximum of the product of probability density functions. Instead, we can observe a direct way of writing what is needed by looking at Eqs. 12.8 and 12.14. In these equations, we see that the maximum likelihood estimate of our unknown parameter is always expressed as that value of the parameter that gives the minimum of the sum of the squares of the difference between each measured value and the true value being measured (expressed as a function of our unknown parameter) with each squared difference divided or "weighted" by the variance of the meter error. Thus, if we are estimating a single parameter, x, using N_m measurements, we would write the expression

$$\min_{x} J(x) = \sum_{i=1}^{N_m} \frac{[z_i^{\text{meas}} - \text{f}_i(x)]^2}{\sigma_i^2} \tag{12.16}$$

where

f_i = function that is used to calculate the value being measured by the i^{th} measurement

σ_i^2 = variance for the i^{th} measurement

$J(x)$ = measurement residual

N_m = number of independent measurements

z_i^{meas} = i^{th} measured quantity

Note that Eq. 12.16 may be expressed in per unit or in physical units such as MW, MVAR, or kV.

If we were to try to estimate N_s unknown parameters using N_m measurements, we would write

$$\min_{\{x_1, x_2, \ldots, x_{N_s}\}} J(x_1, x_2, \ldots, x_{N_s}) = \sum_{i=1}^{N_m} \frac{[z_i - \text{f}_i(x_1, x_2, \ldots, x_{N_s})]^2}{\sigma_i^2} \tag{12.17}$$

The estimation calculation shown in Eqs. 12.16 and 12.17 is known as a *weighted least-squares* estimator, which, as we have shown earlier, is equivalent to a maximum likelihood estimator if the measurement errors are modeled as random numbers having a normal distribution.

12.3.3 Matrix Formulation

If the functions $f_i(x_1, x_2, \ldots, x_{N_s})$ are linear functions, Eq. 12.17 has a closed-form solution. Let us write the function $f_i(x_1, x_2, \ldots, x_{N_s})$ as

$$f_i(x_1, x_2, \ldots, x_{N_s}) = f_i(\mathbf{x}) = h_{i1}x_1 + h_{i2}x_2 + \ldots + h_{iN_s}x_{N_s} \tag{12.18}$$

Then, if we place all the f_i functions in a vector, we may write

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_{N_m}(\mathbf{x}) \end{bmatrix} = [H]\mathbf{x} \tag{12.19}$$

where

$[H]$ = an N_m by N_s matrix containing the coefficients of the linear functions $f_i(\mathbf{x})$

N_m = number of measurements

N_s = number of unknown parameters being estimated

Placing the measurements in a vector:

$$\mathbf{z}^{\text{meas}} = \begin{bmatrix} z_1^{\text{meas}} \\ z_2^{\text{meas}} \\ \vdots \\ z_{N_m}^{\text{meas}} \end{bmatrix} \tag{12.20}$$

We may then write Eq. 12.17 in a very compact form.

$$\min_{\mathbf{x}} J(\mathbf{x}) = [\mathbf{z}^{\text{meas}} - \mathbf{f}(\mathbf{x})]^T[R^{-1}][\mathbf{z}^{\text{meas}} - \mathbf{f}(\mathbf{x})] \tag{12.21}$$

where

$$[R] = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_{N_m}^2 \end{bmatrix}$$

$[R]$ is called the *covariance matrix of measurement errors*. To obtain the general expression for the minimum in Eq. 12.21, expand the expression and substitute $[H]\mathbf{x}$ for $\mathbf{f}(\mathbf{x})$ from Eq. 12.19.

$$\min_{\mathbf{x}} J(\mathbf{x}) = \{\mathbf{z}^{\text{meas}^T}[R^{-1}]\mathbf{z}^{\text{meas}} - \mathbf{x}^T[H]^T[R^{-1}]\mathbf{z}^{\text{meas}} - \mathbf{z}^{\text{meas}^T}[R^{-1}][H]\mathbf{x} + \mathbf{x}^T[H]^T[R^{-1}][H]\mathbf{x}\} \qquad (12.22)$$

Similar to the procedures of Chapter 3, the minimum of $J(\mathbf{x})$ is found when $\partial J(\mathbf{x})/\partial x_i = 0$, for $i = 1, \ldots, N_s$; this is identical to stating that the gradient of $J(\mathbf{x})$, $\nabla J(\mathbf{x})$, is exactly zero.

The gradient of $J(\mathbf{x})$ is (see the appendix to this chapter)

$$\nabla J(\mathbf{x}) = -2[H]^T[R^{-1}]\mathbf{z}^{\text{meas}} + 2[H]^T[R^{-1}][H]\mathbf{x}$$

Then $\nabla J(\mathbf{x}) = \mathbf{0}$ gives

$$\mathbf{x}^{\text{est}} = [[H]^T[R^{-1}][H]]^{-1}[H]^T[R^{-1}]\mathbf{z}^{\text{meas}} \qquad (12.23)$$

Note that Eq. 12.23 holds when $N_s < N_m$; that is, when the number of parameters being estimated is less than the number of measurements being made.

When $N_s = N_m$, our estimation problem reduces to

$$\mathbf{x}^{\text{est}} = [H]^{-1}\mathbf{z}^{\text{meas}} \qquad (12.24)$$

There is also a closed-form solution to the problem when $N_s > N_m$, although in this case we are not estimating $\mathbf{x}$ to maximize a likelihood function since $N_s > N_m$ usually implies that many different values for $\mathbf{x}^{\text{est}}$ can be found that cause $f_i(\mathbf{x}^{\text{est}})$ to equal z_i^{meas} for all $i = 1, \ldots, N_m$ exactly. Rather, the objective is to find $\mathbf{x}^{\text{est}}$ such that the sum of the squares of x_i^{est} is minimized. That is,

$$\min_{\mathbf{x}} \sum_{i=1}^{N_s} x_i^2 = \mathbf{x}^T\mathbf{x} \qquad (12.25)$$

subject to the condition that $\mathbf{z}^{\text{meas}} = [H]\mathbf{x}$. The closed-form solution for this case is

$$\mathbf{x}^{\text{est}} = [H]^T[[H][H]^T]^{-1}\mathbf{z}^{\text{meas}} \qquad (12.26)$$

In power system state estimation, underdetermined problems (i.e., where $N_s > N_m$) are not solved, as shown in Eq. 12.26. Rather, "pseudo-measurements" are added to the measurement set to give a completely determined or overdetermined problem. We will discuss pseudo-measurements in Section 12.6.3. Table 12.1 summarizes the results for this section.

TABLE 12.1 Estimation Formulas

Case	Description	Solution	Comment
$N_s < N_m$	Overdetermined	$\mathbf{x}^{est} = [[H]^T[R^{-1}][H]]^{-1} \times \{H\}^T[R^{-1}]\mathbf{z}^{meas}$	$\mathbf{x}^{est}$ is the maximum likelihood estimate of $\mathbf{x}$ given the measurements $\mathbf{z}^{meas}$
$N_s = N_m$	Completely determined	$\mathbf{x}^{est} = [H]^{-1}\mathbf{z}^{meas}$	$\mathbf{x}^{est}$ fits the measured quantities to the measurements $\mathbf{z}^{meas}$ exactly
$N_s > N_m$	Underdetermined	$\mathbf{x}^{est} = [H]^T[[H][H]^T]^{-1}\mathbf{z}^{meas}$	$\mathbf{x}^{est}$ is the vector of minimum norm that fits the measured quantities to the measurements exactly. (The norm of a vector is equal to the sum of the squares of its components)

12.3.4 An Example of Weighted Least-Squares State Estimation

We now return to our three-bus example. Recall from Figure 12.2 that we have three measurements to determine θ_1 and θ_2, the phase angles at buses 1 and 2. From the development in the preceding section, we know that the states θ_1 and θ_2 can be estimated by minimizing a residual $J(\theta_1, \theta_2)$ where $J(\theta_1, \theta_2)$ is the sum of the squares of individual measurement residuals divided by the variance for each measurement.

To start, we will assume that all three meters have the following characteristics.

Meter full-scale value: 100 MW

Meter accuracy: ± 3 MW

This is interpreted to mean that the meters will give a reading within ± 3 MW of the true value being measured for approximately 99% of the time. Mathematically, we say that the errors are distributed according to a normal probability density function with a standard deviation, σ, as shown in Figure 12.8.

Notice that the probability of an error decreases as the error magnitude increases. By integrating the PDF between -3σ and $+3\sigma$ we come up with a value of approximately 0.99. We will assume that the meter's accuracy (in our case ± 3 MW) is being stated as equal to the 3σ points on the probability density function. Then ± 3 MW corresponds to a metering standard deviation of $\sigma = 1 \text{ MW} = 0.01$ pu.

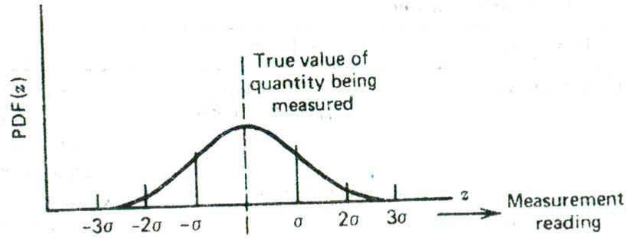

FIG. 12.8 Normal distribution of meter errors.

The formula developed in the last section for the weighted least-squares estimate is given in Eq. 12.23, which is repeated here.

$$\mathbf{x}^{\text{est}} = [[H]^T[R^{-1}][H]]^{-1}[H]^T[R^{-1}]\mathbf{z}^{\text{meas}}$$

where

$\mathbf{x}^{\text{est}}$ = vector of estimated state variables

$[H]$ = measurement function coefficient matrix

$[R]$ = measurement covariance matrix

$\mathbf{z}^{\text{meas}}$ = vector containing the measured values themselves

For the three-bus problem we have

$$\mathbf{x}^{\text{est}} = \begin{bmatrix} \theta_1^{\text{est}} \\ \theta_2^{\text{est}} \end{bmatrix} \tag{12.27}$$

To derive the $[H]$ matrix, we need to write the measurements as a function of the state variables θ_1 and θ_2. These functions are written in per unit as

$$\begin{aligned} M_{12} &= f_{12} = \frac{1}{0.2}(\theta_1 - \theta_2) = 5\theta_1 - 5\theta_2 \\ M_{13} &= f_{13} = \frac{1}{0.4}(\theta_1 - \theta_3) = 2.5\theta_1 \\ M_{32} &= f_{32} = \frac{1}{0.25}(\theta_3 - \theta_2) = -4\theta_2 \end{aligned} \tag{12.28}$$

The reference-bus phase angle, θ_3, is still assumed to be zero. Then

$$[H] = \begin{bmatrix} 5 & -5 \\ 2.5 & 0 \\ 0 & -4 \end{bmatrix}$$

The covariance matrix for the measurements, $[R]$, is

$$[R] = \begin{bmatrix} \sigma_{M12}^2 & & \\ & \sigma_{M13}^2 & \\ & & \sigma_{M32}^2 \end{bmatrix} = \begin{bmatrix} 0.0001 & & \\ & 0.0001 & \\ & & 0.0001 \end{bmatrix}$$

Note that since the coefficients of $[H]$ are in per unit we must also write $[R]$ and $\mathbf{z}^{\text{meas}}$ in per unit.

Our least-squares "best" estimate of θ_1 and θ_2 is then calculated as

$$\begin{bmatrix} \theta_1^{\text{est}} \\ \theta_2^{\text{est}} \end{bmatrix} = \left[\begin{bmatrix} 5 & 2.5 & 0 \\ -5 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0.0001 & & \\ & 0.0001 & \\ & & 0.0001 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -5 \\ 2.5 & 0 \\ 0 & -4 \end{bmatrix} \right]^{-1}$$

$$\times \begin{bmatrix} 5 & 2.5 & 0 \\ -5 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0.0001 & & \\ & 0.0001 & \\ & & 0.0001 \end{bmatrix}^{-1} \begin{bmatrix} 0.62 \\ 0.06 \\ 0.37 \end{bmatrix}$$

$$= \begin{bmatrix} 312500 & -250000 \\ -250000 & 410000 \end{bmatrix}^{-1} \begin{bmatrix} 32500 \\ -45800 \end{bmatrix}$$

$$= \begin{bmatrix} 0.028571 \\ -0.094286 \end{bmatrix}$$

where

$$\mathbf{z}^{\text{meas}} = \begin{bmatrix} 0.62 \\ 0.06 \\ 0.37 \end{bmatrix}$$

From the estimated phase angles, we can calculate the power flowing in each transmission line and the net generation or load at each bus. The results are shown in Figure 12.9. If we calculate the value of $J(\theta_1, \theta_2)$, the residual, we get

$$J(\theta_1, \theta_2) = \frac{[z_{12} - f_{12}(\theta_1, \theta_2)]^2}{\sigma_{12}^2} + \frac{[z_{13} - f_{13}(\theta_1, \theta_2)]^2}{\sigma_{13}^2} + \frac{[z_{32} - f_{32}(\theta_1, \theta_2)]^2}{\sigma_{32}^2}$$

$$= \frac{[0.62 - (5\theta_1 - 5\theta_2)]^2}{0.0001} + \frac{[0.06 - (2.5\theta_1)]^2}{0.0001} + \frac{[0.37 + (4\theta_2)]^2}{0.0001}$$

$$= 2.14 \tag{12.29}$$

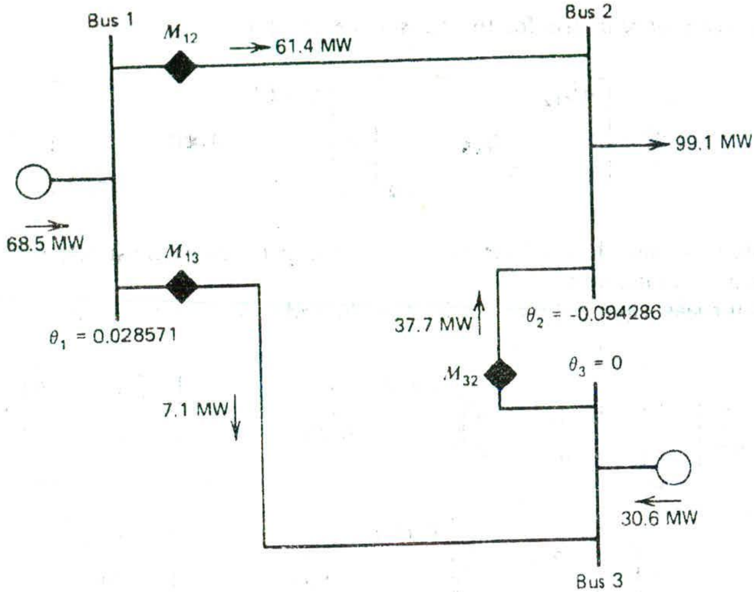

FIG. 12.9 Three-bus example with best estimates of θ_1 and θ_2.

Suppose the meter on the M_{13} transmission line was superior in quality to those on M_{12} and M_{32}. How will this affect the estimate of the states? Intuitively, we can reason that any measurement reading we get from M_{13} will be much closer to the true power flowing on line 1-3 than can be expected when comparing M_{12} and M_{32} to the flows on lines 1-2 and 3-2, respectively. Therefore, we would expect the results from the state estimator to reflect this if we set up the measurement data to reflect the fact that M_{13} is a superior measurement. To show this, we use the following metering data.

Meters M_{12} and M_{32}: 100 MW full scale
± 3 MW accuracy
($\sigma = 1$ MW $= 0.01$ pu)

Meter M_{13}: 100 MW full scale
± 0.3 MW accuracy
($\sigma = 0.1$ MW $= 0.001$ pu)

The covariance matrix to be used in the least-squares formula now becomes

$$[R] = \begin{bmatrix} \sigma^2_{M12} & & \\ & \sigma^2_{M13} & \\ & & \sigma^2_{M32} \end{bmatrix} = \begin{bmatrix} 1 \times 10^{-4} & & \\ & 1 \times 10^{-6} & \\ & & 1 \times 10^{-4} \end{bmatrix}$$

We now solve Eq. 12.23 again with the new $[R]$ matrix.

$$\begin{bmatrix}\theta_1\\\theta_2\end{bmatrix}=\left[\begin{bmatrix}5 & 2.5 & 0\\-5 & 0 & -4\end{bmatrix}\begin{bmatrix}1\times10^{-4} & & \\ & 1\times10^{-6} & \\ & & 1\times10^{-4}\end{bmatrix}^{-1}\begin{bmatrix}5 & -5\\2.5 & 0\\0 & -4\end{bmatrix}\right]^{-1}$$

$$\begin{bmatrix}5 & 2.5 & 0\\-5 & 0 & -4\end{bmatrix}\begin{bmatrix}1\times10^{-4} & & \\ & 1\times10^{-6} & \\ & & 1\times10^{-4}\end{bmatrix}^{-1}\begin{bmatrix}0.62\\0.06\\0.37\end{bmatrix}$$

$$=\begin{bmatrix}6.5\times10^{6} & -2.5\times10^{5}\\-2.5\times10^{5} & 4.1\times10^{5}\end{bmatrix}^{-1}\begin{bmatrix}1.81\times10^{5}\\-0.458\times10^{5}\end{bmatrix}$$

$$=\begin{bmatrix}0.024115\\-0.097003\end{bmatrix}$$

From these estimated phase angles, we obtain the network conditions shown in Figure 12.10. Compare the estimated flow on line 1-3, as just calculated, to the estimated flow calculated on line 1-3 in the previous least-squares estimate. Setting σ_{M13} to 0.1 MW has brought the estimated flow on line 1-3 much closer to the meter reading of 6.0 MW. Also, note that the estimates of flow on lines 1-2 and 3-2 are now further from the M_{12} and M_{32} meter readings, respectively, which is what we should have expected.

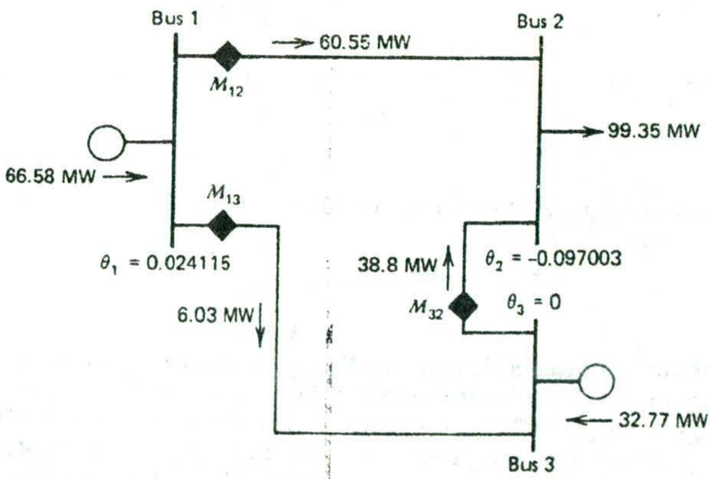

FIG. 12.10 Three-bus example with better meter at M_{13}.

12.4 STATE ESTIMATION OF AN AC NETWORK

12.4.1 Development of Method

We have demonstrated how the maximum likelihood estimation scheme developed in Section 12.3.2 led to a least-squares calculation for measurements from a linear system. In the least-squares calculation, we are trying to minimize the sum of measurement residuals:

$$\min_{\mathbf{x}} J(\mathbf{x}) = \sum_{i=1}^{N_m} \frac{[z_i - \mathrm{f}_i(\mathbf{x})]^2}{\sigma_i^2} \tag{12.30}$$

In the case of a linear system, the $\mathrm{f}_i(\mathbf{x})$ functions are themselves linear and we solve for the minimum of $J(\mathbf{x})$ directly. In an AC network, the measured quantities are MW, MVAR, MVA, amperes, transformer tap position, and voltage magnitude. The state variables are the voltage magnitude at each bus, the phase angles at all but the reference bus, and the transformer taps. The equation for power entering a bus is given in Eq. 4.21 and is clearly not a linear function of the voltage magnitude and phase angle at each bus. Therefore, the $\mathrm{f}_i(\mathbf{x})$ functions will be nonlinear functions, except for a voltage magnitude measurement where $\mathrm{f}_i(\mathbf{x})$ is simply unity times the particular x_i that corresponds to the voltage magnitude being measured. For MW and MVAR measurements on a transmission line from bus i to bus j we would have the following terms in $J(\mathbf{x})$:

$$\frac{\{\mathrm{MW}_{ij}^{\mathrm{meas}} - [|E_i|^2(G_{ij}) - |E_i|\,|E_j|(\cos(\theta_i - \theta_j)G_{ij} + \sin(\theta_i - \theta_j)B_{ij})]\}^2}{\sigma^2_{\mathrm{MW}_{ij}}} \tag{12.31}$$

and

$$\frac{\{\mathrm{MVAR}_{ij}^{\mathrm{meas}} - [-|E_i|^2(B_{\mathrm{cap}_{ij}} + B_{ij}) - |E_i|\,|E_j|(\sin(\theta_i - \theta_j)G_{ij} - \cos(\theta_i - \theta_j)B_{ij})]\}^2}{\sigma^2_{\mathrm{MVAR}_{ij}}} \tag{12.32}$$

A voltage magnitude measurement would result in the following term in $J(\mathbf{x})$:

$$\frac{(|E_i|^{\mathrm{meas}} - |E_i|)^2}{\sigma^2_{|E_i|}} \tag{12.33}$$

Similar functions can be derived for MVA or ampere measurements.

If we do not have a linear relationship between the states ($|E|$ values and θ values) and the power flows on a network, we will have to resort to an iterative technique to minimize $J(\mathbf{x})$. A commonly used technique for power system state estimation is to calculate the gradient of $J(\mathbf{x})$ and then force it to zero using Newton's method, as was done with the Newton load flow in Chapter 4. We will review how to use Newton's method on multidimensional problems before proceeding to the minimization of $J(\mathbf{x})$.

Given the functions $g_i(\mathbf{x})$, $i = 1, \ldots, n$, we wish to find $\mathbf{x}$ that gives $g_i(\mathbf{x}) = g_i^{\text{des}}$, for $i = 1, \ldots, n$. If we arrange the g_i functions in a vector we can write

$$\mathbf{g}^{\text{des}} - \mathbf{g}(\mathbf{x}) = 0 \tag{12.34}$$

by perturbing $\mathbf{x}$ we can write

$$\mathbf{g}^{\text{des}} - \mathbf{g}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{g}^{\text{des}} - \mathbf{g}(\mathbf{x}) - [\mathbf{g}'(\mathbf{x})]\Delta\mathbf{x} = 0 \tag{12.35}$$

where we have expanded $\mathbf{g}(\mathbf{x} + \Delta\mathbf{x})$ in a Taylor's series about $\mathbf{x}$ and ignored all higher-order terms. The $[\mathbf{g}'(\mathbf{x})]$ term is the Jacobian matrix of first derivatives of $\mathbf{g}(\mathbf{x})$. Then

$$\Delta\mathbf{x} = [\mathbf{g}'(\mathbf{x})]^{-1}[\mathbf{g}^{\text{des}} - \mathbf{g}(\mathbf{x})] \tag{12.36}$$

Note that if $\mathbf{g}^{\text{des}}$ is identically zero we have

$$\Delta\mathbf{x} = [\mathbf{g}'(\mathbf{x})]^{-1}[-\mathbf{g}(\mathbf{x})] \tag{12.37}$$

To solve for $\mathbf{g}^{\text{des}}$, we must solve for $\Delta\mathbf{x}$ using Eq. 12.36, then calculate $\mathbf{x}^{\text{new}} = \mathbf{x} + \Delta\mathbf{x}$ and reapply Eq. 12.36 until either $\Delta\mathbf{x}$ gets very small or $\mathbf{g}(\mathbf{x})$ comes close to $\mathbf{g}^{\text{des}}$.

Now let us return to the state estimation problem as given in Eq. 12.30:

$$\min_{\mathbf{x}} J(\mathbf{x}) = \sum_{i=1}^{N_m} \frac{[z_i - f_i(\mathbf{x})]^2}{\sigma_i^2}$$

We first form the gradient of $J(\mathbf{x})$ as

$$\nabla_x J(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial J(\mathbf{x})}{\partial x_1} \\[2ex] \dfrac{\partial J(\mathbf{x})}{\partial x_2} \\ \vdots \end{bmatrix}$$

$$= -2 \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_3}{\partial x_1} & \cdots \\[2ex] \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_2}{\partial x_2} & \dfrac{\partial f_3}{\partial x_2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sigma_1^2} & & \\ & \dfrac{1}{\sigma_2^2} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} [z_1 - f_1(\mathbf{x})] \\[2ex] [z_2 - f_2(\mathbf{x})] \\ \vdots \end{bmatrix} \tag{12.38}$$

If we put the $f_i(\mathbf{x})$ functions in a vector form $\mathbf{f}(\mathbf{x})$ and calculate the Jacobian of $\mathbf{f}(\mathbf{x})$, we would obtain

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{12.39}$$

We will call this matrix $[H]$. Then,

$$[H] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{12.40}$$

And its transpose is

$$[H]^T = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_3}{\partial x_1} & \cdots \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{12.41}$$

Further, we write

$$\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix} = [R] \tag{12.42}$$

Equation 12.38 can be written

$$\nabla_x \mathbf{J}(\mathbf{x}) = \left\{ -2[H]^T[R]^{-1} \begin{bmatrix} z_1 - f_1(\mathbf{x}) \\ z_2 - f_2(\mathbf{x}) \\ \vdots \end{bmatrix} \right\} \tag{12.43}$$

To make $\nabla_x \mathbf{J}(\mathbf{x})$ equal zero, we will apply Newton's method as in Eq. 12.37, then

$$\Delta \mathbf{x} = \left[\frac{\partial \nabla_x J(\mathbf{x})}{\partial \mathbf{x}} \right]^{-1} [-\nabla_x J(\mathbf{x})] \tag{12.44}$$

The Jacobian of $\nabla_x \mathbf{J}(\mathbf{x})$ is calculated by treating $[H]$ as a constant matrix:

$$\frac{\partial \nabla_x J(\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} \left\{ -2[H]^T[R]^{-1} \begin{bmatrix} z_1 - f_1(\mathbf{x}) \\ z_2 - f_2(\mathbf{x}) \\ \vdots \end{bmatrix} \right\}$$

$$= -2[H]^T[R]^{-1}[-H]$$

$$= 2[H]^T[R]^{-1}[H] \tag{12.45}$$

Then

$$\Delta \mathbf{x} = \frac{1}{2}[[H]^T[R]^{-1}[H]]^{-1}\left\{2[H]^T[R]^{-1}\begin{bmatrix} z_1 - f_1(\mathbf{x}) \\ \vdots \end{bmatrix}\right\}$$

$$= [[H]^T[R]^{-1}[H]]^{-1}[H]^T[R]^{-1} \begin{bmatrix} z_1 - f_1(\mathbf{x}) \\ z_2 - f_2(\mathbf{x}) \\ \vdots \end{bmatrix} \tag{12.46}$$

Equation 12.46 is obviously a close parallel to Eq. 12.23. To solve the AC state estimation problem, apply Eq. 12.46 iteratively as shown in Figure 12.11. Note that this is similar to the iterative process used in the Newton power flow solution.

12.4.2 Typical Results of State Estimation on an AC Network

Figure 12.12 shows our familiar six-bus system with $P + jQ$ measurements on each end of each transmission line and at each load and generator. Bus voltage is also measured at each system bus.

To demonstrate the use of state estimation on these measurements, the base-case conditions shown in Figure 11.1 were used together with a random number generating algorithm to produce measurements with random errors. The measurements were obtained by adding the random errors to the base-case flows, loads, generations, and bus-voltage magnitudes. The errors were generated so as to be representative of values drawn from a set of numbers having a normal probability density function with zero mean, and variance as specified for each measurement type. The measurement variances used were

$P + jQ$ measurements:	$\sigma = 5$ MW for the P measurement
	$\sigma = 5$ MVAR for the Q measurement
Voltage measurement:	$\sigma = 3.83$ kV

The base conditions and the measurements are shown in Table 12.2. The state estimation algorithm shown in Figure 12.11 was run to obtain estimates

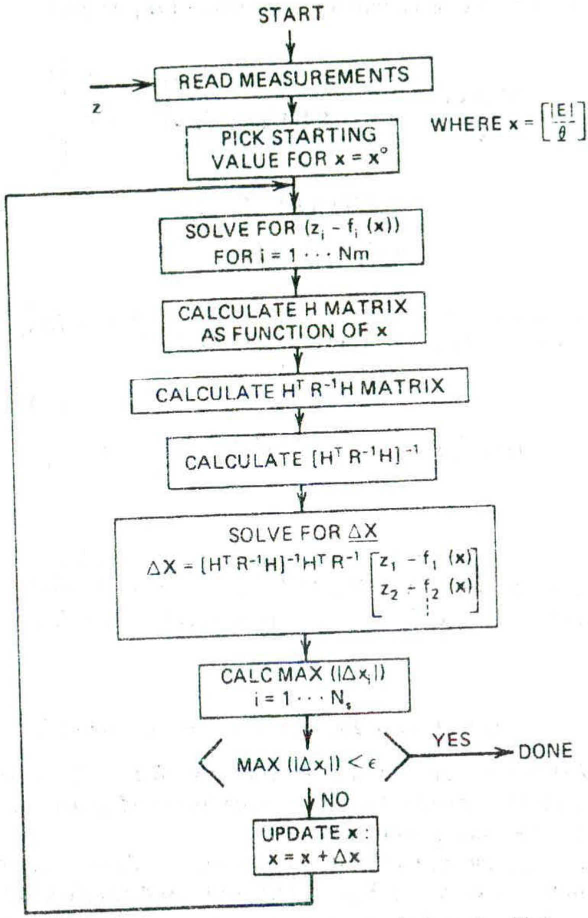

FIG. 12.11 State estimation solution algorithm.

for the bus-voltage magnitudes and phase angles given the measurements shown in Table 12.2. The procedure took three iterations with $\mathbf{x}^0$ initially being set to 1.0 pu and 0 rad for the voltage magnitude and phase angle at each bus, respectively. At the beginning of each iteration, the sum of the measurement residuals, $J(\mathbf{x})$ (see Eq. 12.30), is calculated and displayed. At the end of each iteration, the maximum $\Delta|E|$ and the maximum $\Delta\theta$ are calculated and displayed. The iterative steps for the six-bus system used here produced the results given in Table 12.3.

The value of $J(\mathbf{x})$ at the end of the iterative procedure would be zero if all measurements were without error or if there were no redundancy in the measurements. When there are redundant measurements with errors, the value of $J(\mathbf{x})$ will not normally go to zero. Its value represents a measure of the overall

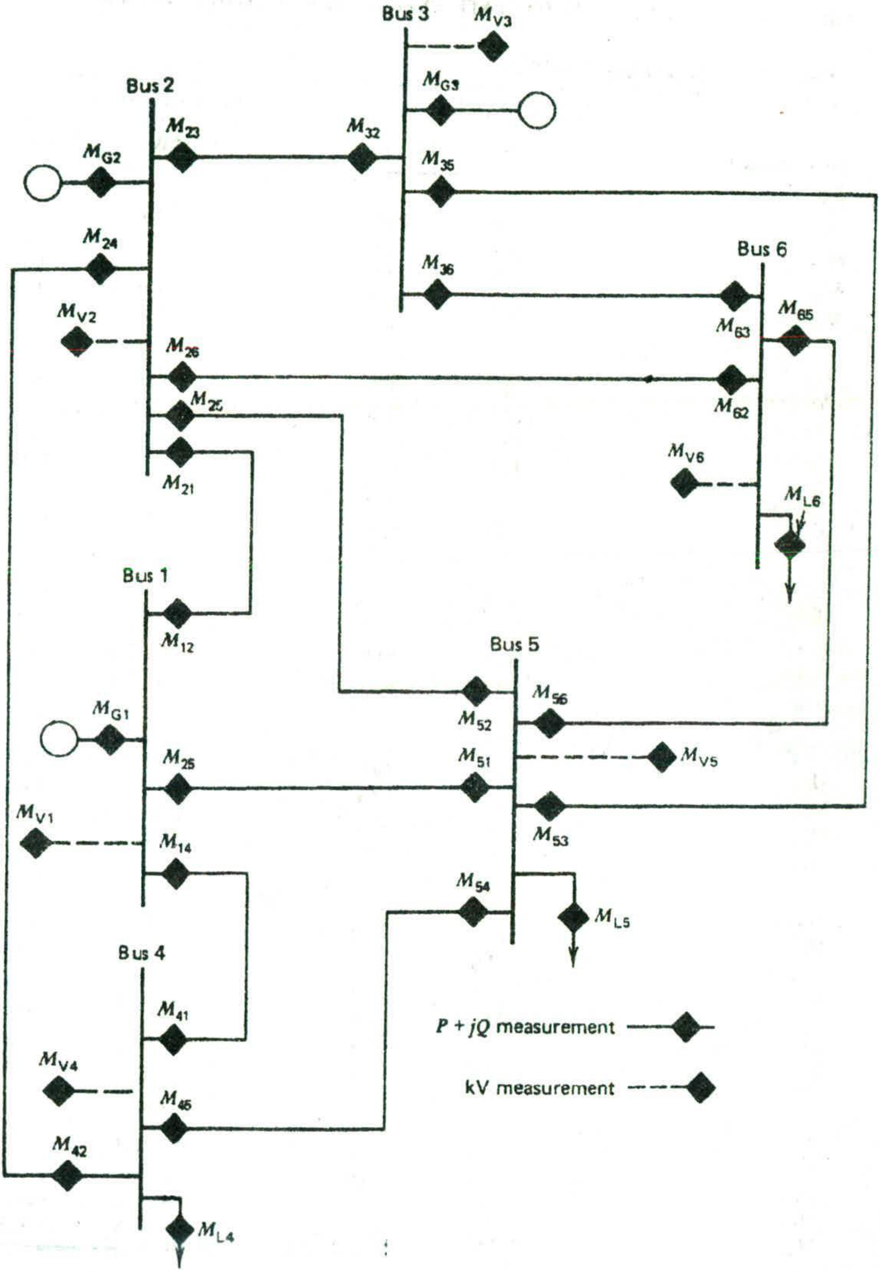

FIG. 12.12 Six-bus system with measurements.

TABLE 12.2 Base-Case Conditions

	Base-Case Value			Measured Value		
Measurement	kV	MW	MVAR	kV	MW	MVAR
M_{V1}	241.5			238.4		
M_{G1}		107.9	16.0		113.1	20.2
M_{12}		28.7	−15.4		31.5	−13.2
M_{14}		43.6	20.1		38.9	21.2
M_{15}		35.6	11.3		35.7	9.4
M_{V2}	241.5			237.8		
M_{G2}		50.0	74.4		48.4	71.9
M_{21}		−27.8	12.8		−34.9	9.7
M_{24}		33.1	46.1		32.8	38.3
M_{25}		15.5	15.4		17.4	22.0
M_{26}		26.2	12.4		22.3	15.0
M_{23}		2.9	−12.3		8.6	−11.9
M_{V3}	246.1			250.7		
M_{G3}		60.0	89.6		55.1	90.6
M_{32}		−2.9	5.7		−2.1	10.2
M_{35}		19.1	23.2		17.7	23.9
M_{36}		43.8	60.7		43.3	58.3
M_{V4}	227.6			225.7		
M_{L4}		70.0	70.0		71.8	71.9
M_{41}		−42.5	−19.9		−40.1	−14.3
M_{42}		−31.6	−45.1		−29.8	−44.3
M_{45}		4.1	−4.9		0.7	−17.4
M_{V5}	226.7			225.2		
M_{L5}		70.0	70.0		72.0	67.7
M_{54}		−4.0	−2.8		−2.1	−1.5
M_{51}		−34.5	−13.5		−36.6	−17.5
M_{52}		−15.0	−18.0		−11.7	−22.2
M_{53}		−18.0	−26.1		−25.1	−29.9
M_{56}		1.6	−9.7		−2.1	−0.8
M_{V6}	231.0			228.9		
M_{L6}		70.0	70.0		72.3	60.9
M_{65}		−1.6	3.9		1.0	2.9
M_{62}		−25.7	−16.0		−19.6	−22.3
M_{63}		−42.8	−57.9		−46.8	−51.1

TABLE 12.3 Iterative Results of State Estimator Solution

Iteration	$J(\mathbf{x})$ at Beginning of Iteration (pu)	Largest $\Delta\lvert E\rvert$ at End of Iteration (pu V)	Largest $\Delta\theta$ at End of Iteration (rad)
1	3696.86	0.1123	0.06422
2	43.67	0.004866	0.0017
3	40.33	0.0000146	0.0000227

fit of the estimated values to the measurement values. The value of $J(\mathbf{x})$ can, in fact, be used to detect the presence of bad measurements.

The estimated values from the state estimator are shown in Table 12.4, together with the base-case values and the measured values. Notice that, in general, the estimated values do a good job of calculating the true (base-case) conditions from which the measurements were made. For example, measurement M_{23} shows a P flow of 8.6 MW whereas the true flow is 2.9 MW and the estimator predicts a flow of 3.0 MW.

The example shown here started from a base case or "true" state that was shown in Table 12.2. In actual practice, we only have the measurements and the resulting estimate of the state, we never know the "true" state exactly and can only compare measurements with estimates. In the presentations to follow, however, we will leave the base-case or "true" conditions in our illustrations to aid the reader.

The results in Table 12.4 show one of the advantages of using a state estimation algorithm in that, even with measurement errors, the estimation algorithm calculates quantities that are the "best" possible estimates of the true bus voltages and generator, load, and transmission line MW and MVAR values.

There are, however, other advantages to using a state estimation algorithm. First, is the ability of the state estimator to detect and identify bad measurements, and, second, is the ability to estimate quantities that are not measured and telemetered. These are introduced later in the chapter.

12.5 STATE ESTIMATION BY ORTHOGONAL DECOMPOSITION

One problem with the standard least-squares method presented earlier in the chapter is the numerical difficulties encountered with some special state estimation problems. One of these comes about when we wish to drive a state estimator solution to match its measurement almost exactly. This is the case when we have a circuit such as shown in Figure 12.13. All of the actual flows and injections are shown in Figure 12.13 along with the values assumed for the measurements.

In this sample system, the measurement of power at bus 1 will be assumed to be zero MW. If the value of zero is dictated by the fact that the bus has no

TABLE 12.4 State Estimation Solution

Measurement	Base-Case Value kV	Base-Case Value MW	Base-Case Value MVAR	Measured Value kV	Measured Value MW	Measured Value MVAR	Estimated Value kV	Estimated Value MW	Estimated Value MVAR
M_{V1}	241.5			238.4			240.6		
M_{G1}		107.9	16.0		113.1	20.2		111.9	18.7
M_{12}		28.7	−15.4		31.5	−13.2		30.4	−14.4
M_{14}		43.6	20.1		38.9	21.2		44.8	21.2
M_{15}		35.6	11.3		35.7	9.4		36.8	11.8
M_{V2}	241.5			237.8			239.9		
M_{G2}		50.0	74.4		48.4	71.9		47.5	70.3
M_{21}		−27.8	12.8		−34.9	9.7		−29.4	11.9
M_{24}		33.1	46.1		32.8	38.3		32.4	45.3
M_{25}		15.5	15.4		17.4	22.0		15.6	14.8
M_{26}		26.2	12.4		22.3	15.0		25.9	10.8
M_{23}		2.9	−12.3		8.6	−11.9		3.0	−12.6
M_{V3}	246.1			250.7			244.7		
M_{G3}		60.0	89.6		55.1	90.6		59.5	87.4
M_{32}		−2.9	5.7		−2.1	10.2		−3.0	6.2
M_{35}		19.1	23.2		17.7	23.9		19.2	22.9
M_{36}		43.8	60.7		43.3	58.3		43.3	58.3
M_{V4}	227.6			225.7			226.1		
M_{L4}		70.0	70.0		71.8	71.9		70.2	70.2
M_{41}		−42.5	−19.9		−40.1	−14.3		−43.6	−20.7
M_{42}		−31.6	−45.1		−29.8	−44.3		−30.9	−44.4
M_{45}		4.1	−4.9		0.7	−17.4		4.3	−5.1
M_{V5}	226.7			225.2			225.3		
M_{L5}		70.0	70.0		72.0	67.7		71.8	69.4
M_{54}		−4.0	−2.8		−2.1	−1.5		−4.2	−2.5
M_{51}		−34.5	−13.5		−36.6	−17.5		−35.6	−13.6
M_{52}		−15.0	−18.0		−11.7	−22.2		−15.1	−17.4
M_{53}		−18.0	−26.1		−25.1	−29.9		−18.1	−25.8
M_{56}		1.6	−9.7		−2.1	−0.8		1.3	−10.1
M_{V6}	231.0			228.9			230.1		
M_{L6}		70.0	70.0		72.3	60.9		68.9	65.8
M_{65}		−1.6	3.9		1.0	2.9		−1.2	4.4
M_{62}		−25.7	−16.0		−19.6	−22.3		−25.4	−14.5
M_{63}		−42.8	−57.9		−46.8	−51.1		−42.3	−55.7

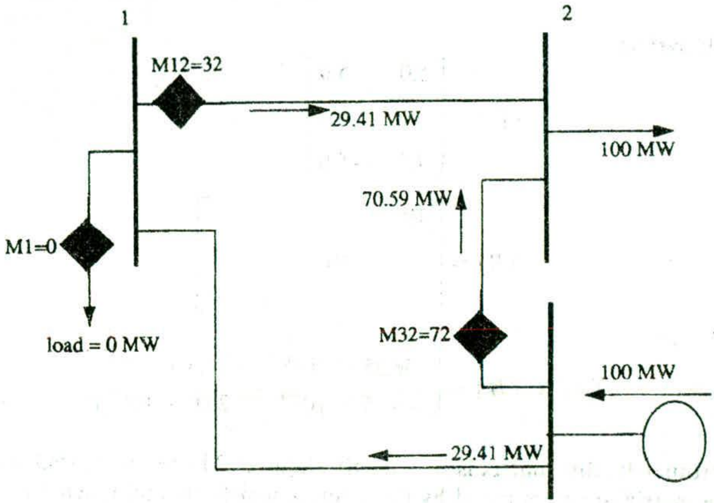

FIG. 12.13 Zero injection system example.

load or generation attached to it, then we know this value of zero MW with certainty and the concept of an error in its "measured" value is meaningless. Nonetheless, we proceed by setting up the standard state estimator equations and specifying the value of the measurement σ for M_1 as: $\sigma_{M1} = 10^{-2}$. This results in the following solution when using the state estimator equations as shown in Eq. 12.23:

$$P_{\text{flow}} \text{ estimate on line 1-2} = 30.76 \text{ MW}$$

$$P_{\text{flow}} \text{ estimate on line 3-2} = 72.52$$

$$\text{Injection estimate on bus 1} = 0.82$$

The estimator has not forced the bus injection to be exactly zero; instead, it reads 0.82 MW. This may not seem like such a big error. However, if there are many such buses (say 100) and they all have errors of this magnitude, then the estimator will have a large amount of load allocated to the buses that are known to be zero.

At first, the solution to this dilemma may seem to be simply forcing the σ value to a very small number for the zero injection buses and rerun the estimator. The problem with this is as follows. Suppose we had changed the zero injection σ to $\sigma_{M1} = 10^{-10}$. Hopefully, this would force the estimator to make the zero injection so dominant that it would result in the correct zero value coming out of the estimator calculation. In this case, the $[H^T R^{-1} H]$ matrix used in the standard least-squares method would look like this for the sample system:

$$[H] = \begin{bmatrix} 5.0 & -5.0 \\ 0 & -4.0 \\ 7.5 & -5.0 \end{bmatrix}$$

$$[R] = \begin{bmatrix} 10^{-4} & & \\ & 10^{-4} & \\ & & 10^{-20} \end{bmatrix}$$

then

$$[H^T R^{-1} H] = \begin{bmatrix} 56.25 \times 10^{20} & -37.5 \times 10^{20} \\ -37.5 \times 10^{20} & 25.0 \times 10^{20} \end{bmatrix}$$

Unfortunately, this matrix is very nearly singular. The reason is that the terms in the matrix are dominated by those terms which are multiplied by the 10^{20} terms from the inverse of the R matrix, and the other terms are so small by comparison that they are lost from the computer (unless one is using an extraordinarily long word length or extra double precision). When the above is presented to a standard matrix inversion routine or run into a Gaussian elimination solution routine, an error message results and garbage comes out of the estimator.

The solution to this dilemma is to use another algorithm for the least-squares solution. This algorithm is called the orthogonal decomposition algorithm and works as follows.

12.5.1 The Orthogonal Decomposition Algorithm

This algorithm goes under several different names in texts on linear algebra. It is often called the QR algorithm or the Gram–Schmidt decomposition. The idea is to take the state estimation least-squares equation, Eq. 12.23, and eliminate the R^{-1} matrix as follows: let

$$[R^{-1}] = R^{-1/2} R^{-1/2} \tag{12.47}$$

where

$$[R^{-1/2}] = \begin{bmatrix} \frac{1}{\sigma_{m1}} & & \\ & \frac{1}{\sigma_{m2}} & \\ & & \frac{1}{\sigma_{m3}} \end{bmatrix} \tag{12.48}$$

then

$$[H^T R^{-1} H]^{-1} = [H^T R^{-1/2} R^{-1/2} H]^{-1} = [H'^T H'] \tag{12.49}$$

with

$$[H'] = [R^{-1/2}][H] \tag{12.50}$$

Finally, Eq. 12.23 becomes

$$\mathbf{x}^{\text{est}} = [H'^T H']^{-1}[H'^T]\mathbf{z}'^{\text{meas}} \tag{12.51}$$

where

$$\mathbf{z}'^{\text{meas}} = [R^{-1/2}]\mathbf{z}^{\text{meas}} \tag{12.52}$$

The idea of the orthogonal decomposition algorithm is to find a matrix $[Q]$ such that:

$$[H'] = [Q][U] \tag{12.53}$$

(Note that in most linear algebra text books, this factorization would be written as $[H'] = [Q][R]$; however, we shall use $[Q][U]$ so as not to confuse the identity of the $[R]$ matrix.)

The matrix $[Q]$ has special properties. It is called an orthogonal matrix so that

$$[Q^T][Q] = [I] \tag{12.54}$$

where $[I]$ is the identity matrix, which is to say that the transpose of $[Q]$ is its inverse. The matrix $[U]$ is now upper triangular in structure, although, since the $[H]$ matrix may not be square, $[U]$ will not be square either. Thus,

$$[H'] = \begin{bmatrix} h'_{11} & h'_{12} \\ h'_{21} & h'_{22} \\ h'_{31} & h'_{32} \end{bmatrix} = [Q][U] = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \\ 0 & 0 \end{bmatrix} \tag{12.55}$$

Now, if we substitute $[Q][U]$ for $[H']$ in the state estimation equation:

$$\mathbf{x}^{\text{est}} = [U^T Q^T Q U]^{-1}[U^T][Q^T]\mathbf{z}' \tag{12.56}$$

or

$$\mathbf{x}^{\text{est}} = [U^T U]^{-1} U^T \hat{\mathbf{z}} \tag{12.57}$$

since

$$[Q^T Q] = I$$

and

$$\hat{\mathbf{z}} = [Q^T]\mathbf{z}' \tag{12.58}$$

Then, by rearranging we get

$$[U^T U]\mathbf{x}^{\text{est}} = [U^T]\hat{\mathbf{z}} \tag{12.59}$$

and we can eliminate U^T from both sides so that we are left with

$$[U]\mathbf{x}^{\text{est}} = \hat{\mathbf{z}} \tag{12.60}$$

or

$$\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{\text{est}} \\ x_2^{\text{est}} \end{bmatrix} = \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \end{bmatrix} \tag{12.61}$$

This can be solved directly since U is upper triangular:

$$x_2^{\text{est}} = \frac{\hat{z}_2}{u_{22}} \tag{12.62}$$

and

$$x_1^{\text{est}} = \frac{1}{u_{11}}(\hat{z}_1 - u_{12}x_2^{\text{est}}) \tag{12.63}$$

The Q matrix and the U matrix are obtained, for our simple two-state–three-measurement problem here, using the Givens rotation method as explained in reference 15.

For the Givens rotation method, we start out to define the steps necessary to solve:

$$[Q^T][H] = [U] \tag{12.64}$$

where $[H]$ is a 2×2 matrix:

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

and $[U]$ is

$$\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

The $[Q]$ matrix must be orthogonal, and when it is multiplied times $[H]$, it eliminates the h_{21} term. The terms in the $[Q]$ matrix are simply:

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

where

$$c = \frac{h_{11}}{\sqrt{h_{11}^2 + h_{21}^2}} \tag{12.65}$$

and

$$s = \frac{h_{21}}{\sqrt{h_{11}^2 + h_{21}^2}} \tag{12.66}$$

The reader can easily verify that the $[Q]$ matrix is indeed orthogonal and that:

$$\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & (ch_{12} + sh_{22}) \\ 0 & (-sh_{12} + ch_{22}) \end{bmatrix} \tag{12.67}$$

When we solve the 3×2 $[H]$ matrix in our three-measurement–two-state sample problem, we apply the Givens rotation three times to eliminate h_{21}, h_{31}, and h_{32}. That is, we need to solve

$$[Q^T]\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \\ 0 & 0 \end{bmatrix} \tag{12.68}$$

We will carry this out in three distinct steps, where each step can be represented as a Givens rotation. The result is that we represent $[Q^T]$ as the product of three matrices:

$$[Q^T] = [N_3][N_2][N_1] \tag{12.69}$$

These matrices are numbered as shown to indicate the order of application. In the case of the 3×2 $[H]$ matrix,

$$[N_1] = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.70}$$

where c and s are defined exactly as before. Next, $[N_2]$ must be calculated so as to eliminate the 31 term which results from $[N_1][H]$. The actual procedure loads $[H]$ into $[U]$ and then determines each $[N]$ based on the current contents of $[U]$. The $[N_2]$ matrix will have terms like

$$[N_2] = \begin{bmatrix} c' & 0 & s' \\ 0 & 1 & 0 \\ -s' & 0 & c' \end{bmatrix} \tag{12.71}$$

where c' and s' are determined from $[N_1][H]$. Similarly for $[N_3]$:

$$[N_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c'' & s'' \\ 0 & -s'' & c'' \end{bmatrix} \tag{12.72}$$

For our zero injection example, we start with the $[H]$ and $[R]$ matrices as shown before:

$$[H] = \begin{bmatrix} 5.0 & -5.0 \\ 0 & -4.0 \\ 7.5 & -5.0 \end{bmatrix}$$

and

$$[R] = \begin{bmatrix} 10^{-4} & & \\ & 10^{-4} & \\ & & 10^{-20} \end{bmatrix}$$

Then, the $[H']$ matrix is

$$[H'] = \begin{bmatrix} 5.0 \times 10^2 & -5.0 \times 10^2 \\ 0 & -4.0 \times 10^2 \\ 7.5 \times 10^{10} & -5.0 \times 10^{10} \end{bmatrix}$$

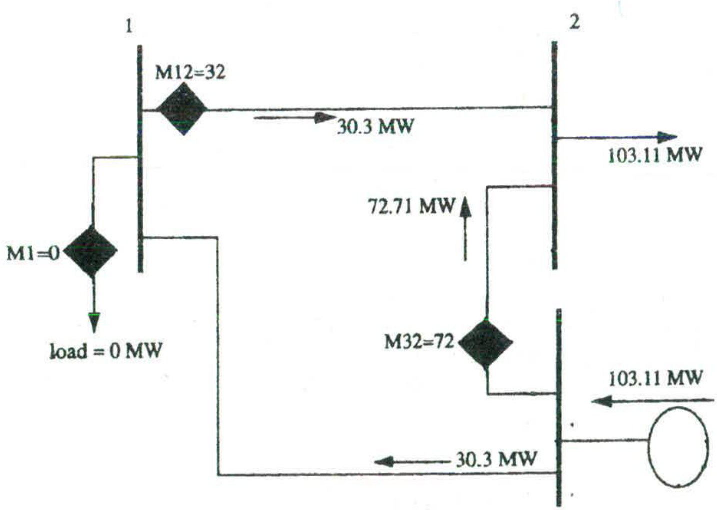

FIG. 12.14 State estimate resulting from orthogonal decomposition algorithm.

and the measurement vector is

$$\hat{\mathbf{z}} = \begin{bmatrix} 32 \\ 72 \\ 0 \end{bmatrix}.$$

The resulting state estimate is shown in Figure 12.14. Note particularly that the injection at bus 1 is estimated to be zero, as we desired.

The orthogonal decomposition algorithm has the advantage that measurement weights can be adjusted to extreme values as demonstrated by the numerical example shown. As such, its robust numerical advantages have made it a useful algorithm for power system state estimators.

12.6 AN INTRODUCTION TO ADVANCED TOPICS IN STATE ESTIMATION

12.6.1 Detection and Identification of Bad Measurements

The ability to detect and identify bad measurements is extremely valuable to a power system's operations department. Transducers may have been wired incorrectly or the transducer itself may be malfunctioning so that it simply no longer gives accurate readings. The statistical theory required to understand and anlayze bad measurement detection and identification is straightforward but lengthy. We are going to open the door to the subject in this chapter. The serious student who wishes to pursue this subject should start with the chapter references. For the rest, we present results of these theories and indicate application areas.

To detect the presence of bad measurements, we will rely on the intuitive notion that for a given configuration, the residual, $J(\mathbf{x})$, calculated after the state estimator algorithm converges, will be smallest if there are no bad measurements. When $J(\mathbf{x})$ is small, a vector $\mathbf{x}$ (i.e., voltages and phase angles) has been found that causes all calculated flows, loads, generations, and so forth to closely match all the measurements. Generally, the presence of a bad measurement value will cause the converged value of $J(\mathbf{x})$ to be larger than expected with $\mathbf{x} = \mathbf{x}^{\text{est}}$

What magnitude of $J(\mathbf{x})$ indicates the presence of bad measurements?

The measurement errors are random numbers so that the value of $J(\mathbf{x})$ is also a random number. If we assume that all the errors are described by their respective normal probability density functions, then we can show that $J(\mathbf{x})$

has a probability density function known as a *chi-squared distribution*, which is written as $\chi^2(K)$. The parameter K is called the degrees of freedom of the chi-squared distribution. This parameter is defined as follows:

$$K = N_m - N_s$$

where

N_m = number of measurements (note that a $P + jQ$ measurement counts as two measurements)

N_s = number of states = $(2n - 1)$

n = number of buses in the network

It can be shown that when $\mathbf{x} = \mathbf{x}^{est}$, the mean value of $J(\mathbf{x})$ equals K and the standard deviation, $\sigma_{J(\mathbf{x})}$, equals $\sqrt{2K}$.

When one or more measurements are bad, their errors are frequently much larger than the assumed $\pm 3\sigma$ error bound for the measurement. However, even under normal circumstances (i.e., all errors within $\pm 3\sigma$), $J(\mathbf{x})$ can get to be large—although the chance of this happening is small. If we simply set up a threshold for $J(\mathbf{x})$, which we will call t_J, we could declare that bad measurements are present when $J(\mathbf{x}) > t_J$. This threshold test might be wrong in one of two ways. If we set t_J to a small value, we would get many "false alarms." That is, the test would indicate the presence of bad measurements when, in fact, there were none. If we set t_J to be a large value, the test would often indicate that "all is well" when, in fact, bad measurements were present. This can be put on a formal basis by writing the following equation:

$$\text{prob}(J(\mathbf{x}) > t_J \mid J(\mathbf{x}) \text{ is a chi-squared with } K \text{ degrees of freedom}) = \alpha \tag{12.73}$$

This equation says that the probability that $J(\mathbf{x})$ is greater than t_J is equal to α, given that the probability density for $J(\mathbf{x})$ is chi-squared with K degrees of freedom.

This type of testing procedure is formally known as *hypothesis testing*, and the parameter α is called the *significance level* of the test. By choosing a value for the significance level α, we automatically know what threshold t_J to use in our test. When using a t_J derived in this manner, the probability of a "false alarm" is equal to α. By setting α to a small number, for example $\alpha = 0.01$, we would say that false alarms would occur in only 1% of the tests made. A plot of the probability function in Eq. 12.73 is shown in Figure 12.15.

In Table 12.3, we saw that the minimum value for $J(\mathbf{x})$ was 40.33. Looking at Figure 12.12 and counting all $P + jQ$ measurements as two measurements,

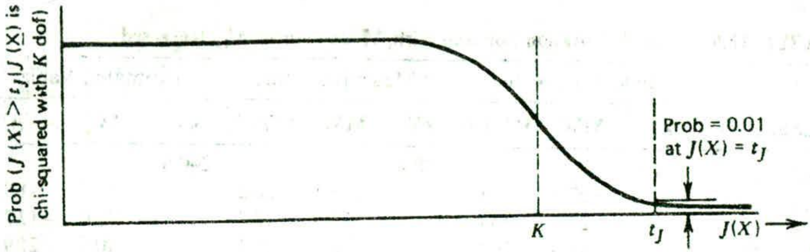

FIG. 12.15 Threshold test probability function.

we see that N_m is equal to 62. Therefore, the degrees of freedom for the chi-square distribution of $J(\mathbf{x})$ in our six-bus sample system is

$$K = N_m - N_s = N_m - (2n - 1) = 51$$

where

$$N_m = 62 \quad \text{and} \quad n = 6$$

If we set our significance level for this test to 0.01 (i.e., $\alpha = 0.01$ in Eq. 12.73), we get a t_J of 76.6.* Therefore, with a $J(\mathbf{x}) = 40.33$, it seems reasonable to assume that there are no "bad" measurements present.

Now let us assume that one of the measurements is truly bad. To simulate this situation, the state estimation algorithm was rerun with the M_{12} measurement reversed. Instead of $P = 31.5$ and $Q = -13.2$, it was set to $P = -31.5$ and $Q = 13.2$. The value of $J(\mathbf{x})$ and the maximum $\Delta|E|$ and $\Delta\theta$ for each iteration for this case are given in Table 12.5. The presence of bad data does not prevent the estimator from converging, but it will increase the value of the residual, $J(\mathbf{x})$.

The calculated flows and voltages for this situation are shown in Table 12.6. Note that the number of degrees of freedom is still 51 but $J(\mathbf{x})$ is now 207.94 at the end of our calculation. Since t_J is 76.6, we would immediately expect bad

TABLE 12.5 Iterative Results with Bad Measurement

Iteration	$J(\mathbf{x})$ at Beginning of Iteration (pu)	Largest $\Delta\|E\|$ at End of Iteration (pu V)	Largest $\Delta\theta$ at End of Iteration (rad)
1	3701.06	0.09851	0.06416
2	211.13	0.004674	0.001481
3	207.94	0.00002598	0.00004848

* Standard tables of $\chi^2(K)$ usually only go up to $K = 30$. For $K > 30$, a very close approximation to $\chi^2(K)$ using the normal distribution can be used. The student should consult any standard reference on probability and statistics to see how this is done.

TABLE 12.6 State Estimation Solution with Measurement M_{12} Reversed

	Base-Case Value			Measured Value			Estimated Value		
Measurement	kV	MW	MVAR	kV	MW	MVAR	kV	MW	MVAR
M_{V1}	241.5			238.4			240.6		
M_{G1}		107.9	16.0		113.1	20.2		99.3	21.9
M_{12}		28.7	−15.4		−31.5	+13.2		25.0	−12.2
M_{14}		43.6	20.1		38.9	21.2		40.6	21.9
M_{15}		35.6	11.3		35.7	9.4		33.7	12.3
M_{V2}	241.5			237.8			239.9		
M_{G2}		50.0	74.4		48.4	71.9		54.4	67.0
M_{21}		−27.8	12.8		−34.9	9.7		−24.4	9.2
M_{24}		33.1	46.1		32.8	38.3		35.0	44.1
M_{25}		15.5	15.4		17.4	22.0		16.3	14.7
M_{26}		26.2	12.4		22.3	15.0		25.1	11.3
M_{23}		2.9	−12.3		8.6	−11.9		2.3	−12.2
M_{V3}	246.1			250.7			244.6		
M_{G3}		60.0	89.6		55.1	90.6		61.4	86.3
M_{32}		−2.9	5.7		−2.1	10.2		−2.3	5.8
M_{35}		19.1	23.2		17.7	23.9		−20.5	22.2
M_{36}		43.8	60.7		43.3	58.3		43.2	58.2
M_{V4}	227.6			225.7			226.1		
M_{L4}		70.0	70.0		71.8	71.9		69.0	70.0
M_{41}		−42.5	−19.9		−40.1	−14.3		−39.6	−21.9
M_{42}		−31.6	−45.1		−29.8	−44.3		−33.5	−43.1
M_{45}		4.1	−4.9		0.7	−17.4		4.1	−5.0
M_{V5}	226.7			225.2			225.3		
M_{L5}		70.0	70.0		72.0	67.7		71.8	69.3
M_{54}		−4.0	−2.8		−2.1	−1.5		−4.1	−2.6
M_{51}		−34.5	−13.5		−36.6	−17.5		−32.7	−14.7
M_{52}		−15.0	−18.0		−11.7	−22.2		−15.8	−17.2
M_{53}		−18.0	−26.1		−25.1	−29.9		−19.3	−25.1
M_{56}		1.6	−9.7		−2.1	−0.8		0.1	−9.6
M_{V6}	231.0			228.9			230.0		
M_{L6}		70.0	70.0		72.3	60.9		66.9	66.7
M_{65}		−1.6	3.9		1.0	2.9		−0.1	3.9
M_{62}		−25.7	−16.0		−19.6	−22.3		−24.6	−15.0
M_{63}		−42.8	−57.9		−46.8	−51.1		−42.3	−55.6

measurements at our 0.01 significance level. If we had not known ahead of running the estimation algorithm that a bad measurement was present, we would certainly have had good reason to suspect its presence when so large a $J(\mathbf{x})$ resulted.

So far, we can say that by looking at $J(\mathbf{x})$, we can detect the presence of bad measurements. But if bad measurements are present, how can one tell which measurements are bad? Without going into the statistical theory, we give the following explanation of how this is accomplished.

Suppose we are interested in the measurement of megawatt flow on a particular line. Call this measured value z_i. In Figure 12.16(*a*) we have a plot of the normal probability density function of z_i. Since we assume that the error in z_i is normally distributed with zero mean value, the probability density function is centered on the true value of z_i. Since the errors on all the measurements are assumed normal, we will assume that the estimate, $\mathbf{x}^{est}$ is approximately normally distributed and that any quantity that is a function of $\mathbf{x}^{est}$ is also an approximately normally distributed quantity. In Figure 12.16(*b*), we show the probability density function for the calculated megawatt flow, f_i, which is a function of the estimated state, $\mathbf{x}^{est}$. We have drawn the density function of f_i as having a smaller deviation from its mean than the measurement z_i to indicate that, due to redundancy in measurements, the estimate is more accurate.

The difference between the estimate, f_i, and the measurement, z_i, is called the *measurement residual* and is designated y_i. The probability density function for y_i is also normal and is shown in Figure 12.16(*c*) as having a zero mean and a standard deviation of σ_{y_i}. If we divide the difference between the estimate f_i and the measurement z_i by σ_{y_i}, we obtain what is called a *normalized measurement residual*. The normalized measurement residual is designated y_i^{norm} and is shown in Figure 12.16(*d*) along with its probability density function, which is normal and has a standard deviation of unity. If the absolute value of y_i^{norm} is greater than 3, we have good reason to suspect that z_i is a bad measurement value. The usual procedure in identifying bad measurements is to calculate all f_i values for the N_m measurements once $\mathbf{x}^{est}$ is available from the state estimator. Using the z_i values that were used in the estimator and the f_i values, a measurement residual y_i can be calculated for each measurement. Also, using information from the state estimator, we can calculate σ_{y_i} (see references for details of this calculation). Using y_i and σ_{y_i}, we can calculate a normalized residual for each measurement. Measurements having the largest absolute normalized residual are labeled as prime suspects. These prime suspects are removed from the state estimator calculation one at a time, starting with the measurement having the largest normalized residual. After a measurement has been removed, the state estimation calculation (see Figure 12.11) is rerun. This results in a different $\mathbf{x}^{est}$ and therefore a different $J(\mathbf{x})$. The chi-squared probability density function for $J(\mathbf{x})$ will have to be recalculated, assuming that we use the same significance level for our test. If the new $J(\mathbf{x})$ is now less than the new value for t_J, we can say that the measurement that

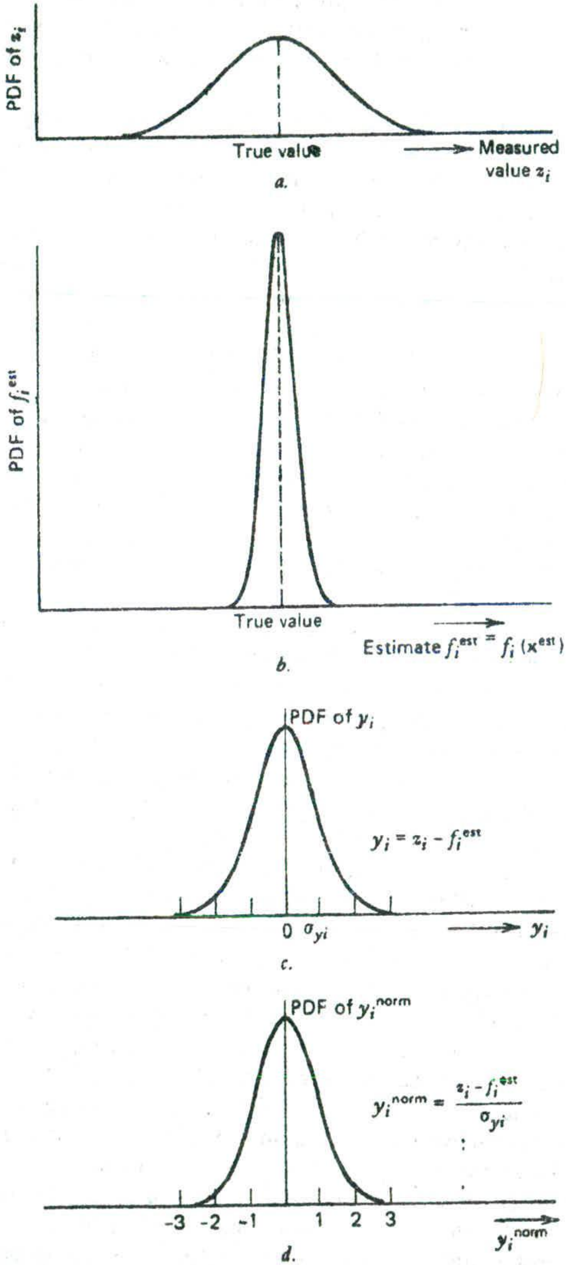

FIG. 12.16 Probability density function of the normalized measurement residual.

was removed has been identified as bad. If, however, the new $J(\mathbf{x})$ is greater than the new t_J, we must proceed to recalculate $f_i(\mathbf{x}^{est})$, σ_{y_i}, and then y_i^{norm} for each of the remaining measurements. The measurement with the largest absolute y_i^{norm} is then again removed and the entire procedure repeated successively until $J(\mathbf{x})$ is less than t_J. The references at the end of this chapter discuss a problem that the identification process may encounter, wherein several measurements may need to be removed to eliminate one "bad" measurement. That is, the identification procedure often cannot pinpoint a single bad measurement but instead identifies a group of measurements, one of which is bad. In such cases, the groups must be eliminated to eliminate the bad measurement.

The ability to detect (using the chi-squared statistic) and identify (using normalized residuals) are extremely useful features of a state estimator. Without the state estimator calculation using the system measurement data, those measurements whose values are not obviously wrong have little chance of being detected and identified. With the state estimator, the operations personnel have a greater assurance that quantities being displayed are not grossly in error.

12.6.2 Estimation of Quantities Not Being Measured

The other useful feature of a state estimator calculation is the ability to calculate (or estimate) quantities not being telemetered. This is most useful in cases of failure of communication channels connecting operations centers to remote data-gathering equipment or when the remote data-gathering equipment fails. Often data from some network substations are simply unavailable because no transducers or data-gathering equipment were ever installed.

An example of this might be the failure of all telemetry from buses 3, 4, 5, and 6 in our six-bus system. Even with the loss of these measurements, we can run the state estimation algorithm on the remaining measurements at buses 1 and 2, calculate the bus voltage magnitudes and phase angles at all six buses, and then calculate all network generations, loads, and flows. The results of such a calculation are given in Table 12.7. Notice that the estimate of quantities at the untelemetered buses are not as close to the base case as when using the full set of measurements (i.e., compare Table 12.7 to Table 12.4).

12.6.3 Network Observability and Pseudo-measurements

What happens if we continue to lose telemetry so that fewer and fewer measurements are available? Eventually, the state estimation procedure breaks down completely. Mathematically, the matrix

$$[[H]^T[R^{-1}][H]]$$

in Eq. 12.46 becomes singular and cannot be inverted. There is also a very interesting engineering interpretation of this phenomenon that allows us to alter the situation so that the state estimation procedure is not completely disabled.

TABLE 12.7 State Estimation Solution with Measurement at Buses 1 and 2 Only

	Base-Case Value			Measured Value			Estimated Value		
Measurement	kV	MW	MVAR	kV	MW	MVAR	kV	MW	MVAR
M_{V1}	241.5			238.4			238.8		
M_{G1}		107.9	16.0		113.1	20.2		112.4	20.5
M_{12}		28.7	−15.4		31.5	−13.2		30.6	−13.4
M_{14}		43.6	20.1		38.9	21.2		44.7	19.4
M_{15}		35.6	11.3		35.7	9.4		37.1	14.6
M_{V2}	241.5			237.8			237.6		
M_{G2}		50.0	74.4		48.4	71.9		48.2	71.7
M_{21}		−27.8	12.8		−34.9	9.7		−29.6	11.1
M_{24}		33.1	46.1		32.8	38.3		30.5	40.2
M_{25}		15.5	15.4		17.4	22.0		16.1	16.8
M_{26}		26.2	12.4		22.3	15.0		22.4	15.2
M_{23}		2.9	−12.3		8.6	−11.9		8.8	−11.7
M_{V3}	246.1						241.4		
M_{G3}		60.0	89.6					27.2	94.9
M_{32}		−2.9	5.7					−8.7	5.5
M_{35}		19.1	23.2					15.1	25.3
M_{36}		43.8	60.7					20.9	64.0
M_{V4}	227.6						225.0		
M_{L4}		70.0	70.0					67.6	61.2
M_{41}		−42.5	−19.9					−43.6	−18.9
M_{42}		−31.6	−45.1					−29.3	−39.7
M_{45}		4.1	−4.9					5.3	−2.6
M_{V5}	226.7						221.4		
M_{L5}		70.0	70.0					71.9	76.7
M_{54}		−4.0	−2.8					−5.2	−4.8
M_{51}		−34.5	−13.5					−35.9	−15.9
M_{52}		−15.0	−18.0					−15.5	−19.0
M_{53}		−18.0	−26.1					−14.0	−28.0
M_{56}		1.6	−9.7					−1.4	−9.0
M_{V6}	231.0						226.2		
M_{L6}		70.0	70.0					40.5	77.2
M_{65}		−1.6	3.9					1.4	3.4
M_{62}		−25.7	−16.0					−21.9	−18.8
M_{63}		−42.8	−57.9					−20.0	−61.8

If we take the three-bus example used in the beginning of Section 12.2, we note that when all three measurements are used, we have a redundant set and we can use a least-squares fit to the measurement values. If one of the measurements is lost, we have just enough measurements to calculate the states. If, however, two measurements are lost, we are in trouble. For example, suppose M_{13} and M_{32} were lost leaving only M_{12}. If we now apply Eq. 12.23 in a straightforward manner, we get

$$M_{12} = f_{12} = \frac{1}{0.2}(\theta_1 - \theta_2) = 5\theta_1 - 5\theta_2$$

Then

$$[H] = [5 \quad -5]$$

$$[R] = [\sigma^2_{M12}] = [0.0001]$$

and

$$\begin{bmatrix} \theta_1^{\text{est}} \\ \theta_2^{\text{est}} \end{bmatrix} = \left[\begin{bmatrix} 5 \\ -5 \end{bmatrix}[0.0001]^{-1}[5 \quad -5]\right]^{-1}[5 \quad -5][0.0001]^{-1}(0.55)$$

$$= \begin{bmatrix} 2500 & -2500 \\ -2500 & 2500 \end{bmatrix}^{-1}[5 \quad -5][0.0001]^{-1}(0.55) \tag{12.74}$$

The matrix to be inverted in Eq. 12.74 is clearly singular and, therefore, we have no way of solving for θ_1^{est} and θ_2^{est}. Why is this? The reasons become quite obvious when we look at the one-line diagram of this network as shown in Figure 12.17. With only M_{12} available, all we can say about the network is that

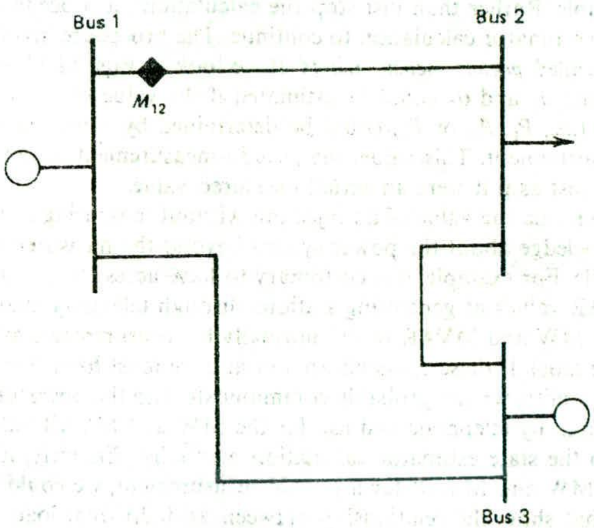

FIG. 12.17 "Unobservable" measurement set.

the phase angle across line 1-2 must be 0.11 rad, but with no other information available, we cannot tell what relationship θ_1 or θ_2 has to θ_3, which is assumed to be 0 rad. If we write down the equations for the net injected power at bus 1 and bus 2, we have

$$\begin{aligned} P_1 &= 7.5\theta_1 - 5\theta_2 \\ P_2 &= -5\theta_1 + 9\theta_2 \end{aligned} \tag{12.75}$$

If measurement M_{12} is reading 55 MW (0.55 pu), we have

$$\theta_1 - \theta_2 = 0.11 \tag{12.76}$$

and by substituting Eq. 12.75 into Eq. 12.76 and eliminating θ_1, we obtain

$$P_2 = 1.6P_1 - 1.87 \tag{12.77}$$

Furthermore,

$$P_3 = -P_1 - P_2 = -0.6P_1 + 1.87 \tag{12.78}$$

Equations 12.77 and 12.78 give a relationship between P_1, P_2, and P_3, but we still do not know their correct values. The technical term for this phenomenon is to say that the network is *unobservable*; that is, with only M_{12} available, we cannot observe (calculate) the state of the system.

It is very desirable to be able to circumvent this problem. Often a large power-system network will have missing data that render the network unobservable. Rather than just stop the calculations, a procedure is used that allows the estimator calculation to continue. The procedure involves the use of what are called *pseudo-measurements*. If we look at Eqs. 12.77 and 12.78, it is obvious that θ_1 and θ_2 could be estimated if the value of any one of the bus injections (i.e., P_1, P_2, or P_3) could be determined by some means other than direct measurement. This value, the pseudo-measurement, is used in the state estimator just as if it were an actual measured value.

To determine the value of an injection without measuring it, we must have some knowledge about the power system beyond the measurements currently being made. For example, it is customary to have access to the generated MW and MVAR values at generating stations through telemetry channels (i.e., the generated MW and MVAR would normally be measurements available to the state estimator). If these channels are out and we must have this measurement for observability, we can probably communicate with the operators in the plant control room by telephone and ask for the MW and MVAR values and enter them into the state estimator calculation manually. Similarly, if we needed a load-bus MW and MVAR for a pseudo-measurement, we could use historical records that show the relationship between an individual load and the total system load. We can estimate the total system load fairly accurately by knowing the total power being generated and estimating the network losses. Finally, if we have just experienced a telemetry failure, we could use the most recently

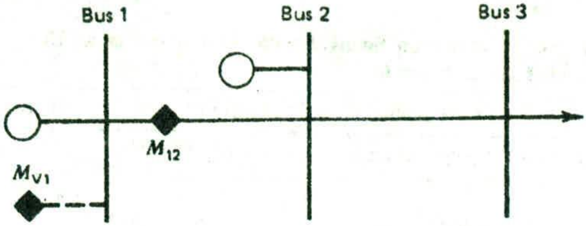

FIG. 12.18 Unobservable system showing importance of location of pseudo-measurements.

estimated values from the estimator (assuming that it is run periodically) as pseudo-measurements. Therefore, if needed, we can provide the state estimator with a reasonable value to use as a pseudo-measurement at any bus in the system.

The three-bus sample system in Figure 12.18 requires one pseudo-measurement. Measurement M_{12} allows us to estimate the voltage magnitude and phase angle at bus 2 (bus 1's voltage magnitude is measured and its phase angle is assumed to be zero). But without knowing the generation output at the generator unit on bus 2 or the load on bus 3, we cannot tell what voltage magnitude and phase angle to place on bus 3; hence, the network is unobservable. We can make this three-bus system observable by adding a pseudo-measurement of the net bus injected MW and MVAR at bus 2 or bus 3, but not at bus 1. That is, a pseudo-measurement at bus 1 will do no good at all because it tells nothing about the relationship of the phase angles between bus 2 and bus 3.

When adding a pseudo-measurement to a network, we simply write the equation for the pseudo-measurement injected power as a function of bus voltage magnitudes and phase angles as if it were actually measured. However, we do not wish to have the estimator treat the pseudo-measurement the same as a legitimate measurement, since it is often quite inaccurate and is little better than a guess. To circumvent this difficulty, we assign a large standard deviation to this measurement. The large standard deviation allows the estimator algorithm to treat the pseudo-measurement as if it were a measurement from a very poor-quality metering device.

To demonstrate the use of pseudo-measurements on our six-bus test system, all measurements were removed from buses 2, 3, 4, 5, and 6 so that bus 1 had all remaining measurements. This rendered the network unobservable and required adding pseudo-measurements at buses 2, 3, and 6. In the case, the pseudo-measurements were just taken from our base-case power flow. The results are shown in Table 12.8. Notice that the resulting estimates are quite close to the measured values for bus 1 but that the remaining buses have large measurement residuals. The net injections at buses 2, 3, and 6 do not closely match the pseudo-measurements since the pseudo-measurements were weighted much less than the legitimate measurements.

TABLE 12.8 State Estimation Solution with Measurements at Bus 1 and Pseudo-measurements at Buses 2, 3, and 6

	Base-Case Value			Measured Value			Estimated Value		
Measurement	kV	MW	MVAR	kV	MW	MVAR	kV	MW	MVAR
M_{V1}	241.5			238.4			238.4		
M_{G1}		107.9	16.0		113.1	20.2		111.4	19.5
M_{12}		28.7	−15.4		31.5	−13.2		33.3	−12.5
M_{14}		43.6	20.1		38.9	21.2		40.7	21.9
M_{15}		35.6	11.3		35.7	9.4		37.4	10.1
M_{V2}	241.5						236.2		
M_{G2}		50.0	74.4	Pseudo:	50.0	74.4		37.5	67.7
M_{21}		−27.8	12.8					−32.1	10.5
M_{24}		33.1	46.1					19.5	44.9
M_{25}		15.5	15.4					14.1	11.5
M_{26}		26.2	12.4					30.0	12.7
M_{23}		2.9	−12.3					6.0	−11.9
M_{V3}	246.1						240.5		
M_{G3}		60.0	89.6	Pseudo:	60.0	89.6		52.6	86.6
M_{32}		−2.9	5.7					−6.0	5.7
M_{35}		19.1	23.2					14.3	19.5
M_{36}		43.8	60.7					44.2	61.4
M_{V4}	227.6						223.8		
M_{L4}		70.0	70.0					51.9	73.3
M_{41}		−42.5	−19.9					−39.6	−21.8
M_{42}		−31.6	−45.1					−18.3	−44.6
M_{45}		4.1	−4.9					6.0	−6.9
M_{V5}	226.7						224.0		
M_{L5}		70.0	70.0					63.9	55.5
M_{54}		−4.0	−2.8					−5.9	−0.4
M_{51}		−34.5	−13.5					−36.3	−11.8
M_{52}		−15.0	−18.0					−13.7	−14.4
M_{53}		−18.0	−26.1					−13.6	−22.9
M_{56}		1.6	−9.7					5.5	−5.9
M_{V6}	231.0						224.9		
M_{L6}		70.0	70.0	Pseudo:	70.0	70.0		77.9	73.4
M_{65}		−1.6	3.9					−5.5	0.3
M_{62}		−25.7	−16.0					−29.3	−15.6
M_{63}		−42.8	−57.9					−43.2	−58.1

12.7 APPLICATION OF POWER SYSTEMS STATE ESTIMATION

In this last section, we will try to present the "big picture" showing how state estimation, contingency analysis, and generator corrective action fit together in a modern operations control center. Figure 12.19 is a schematic diagram showing the information flow between the various functions to be performed in an operations control center computer system. The system gets its information about the power system from remote terminal units that encode measurement transducer outputs and opened/closed status information into digital signals that are transmitted to the operations center over communications circuits. In addition, the control center can transmit control information such as raise/lower commands to generators and open/close commands to circuit breakers and switches. We have broken down the information coming into the control center as breaker/switch status indications and analog measurements. The analog measurements of generator output must be used directly by the AGC program (see Chapter 9), whereas all other data will be processed by the state estimator before being used by other programs.

In order to run the state estimator, we must know how the transmission lines are connected to the load and generation buses. We call this information the *network topology*. Since the breakers and switches in any substation can cause the network topology to change, a program must be provided that reads the telemetered breaker/switch status indications and restructures the electrical model of the system. An example of this is shown in Figure 12.20, where the opening of four breakers requires two electrical buses to represent the substation instead of one electrical bus. We have labeled the program that reconfigures the electrical model as the *network topology program.** The network topology program must have a complete description of each substation and how the transmission lines are attached to the substation equipment. Bus sections that are connected to other bus sections through closed breakers or switches are designated as belonging to the same electrical bus. Thus, the number of electrical buses and the manner in which they are interconnected can be changed in the model to reflect breaker and switch status changes on the power system itself.

As seen in Figure 12.20, the electrical model of the power system's transmission system is sent to the state estimator program together with the analog measurements. The output of the state estimator consists of all bus voltage magnitudes and phase angles, transmission line MW and MVAR flows calculated from the bus voltage magnitude and phase angles, and bus loads and generations calculated from the line flows. These quantities, together with the electrical model developed by the network topology program, provide the basis for the economic dispatch program, congtingency analysis program, and generation corrective action program. Note that since the complete electrical model of the transmission system is available, we can directly calculate bus penalty factors as shown in Chapter 4.

* Alternative names that are often used for this program are "system status processor" and "network configurator."

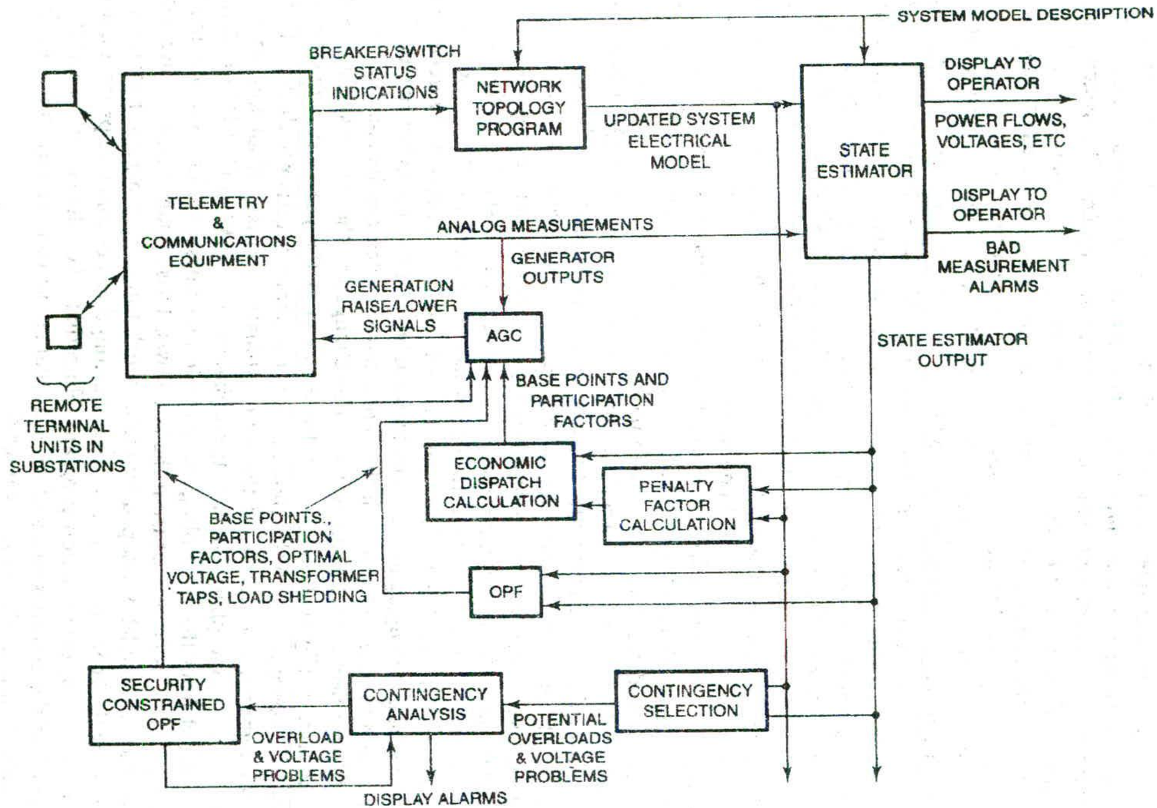

FIG. 12.19 Energy control center system security schematic.

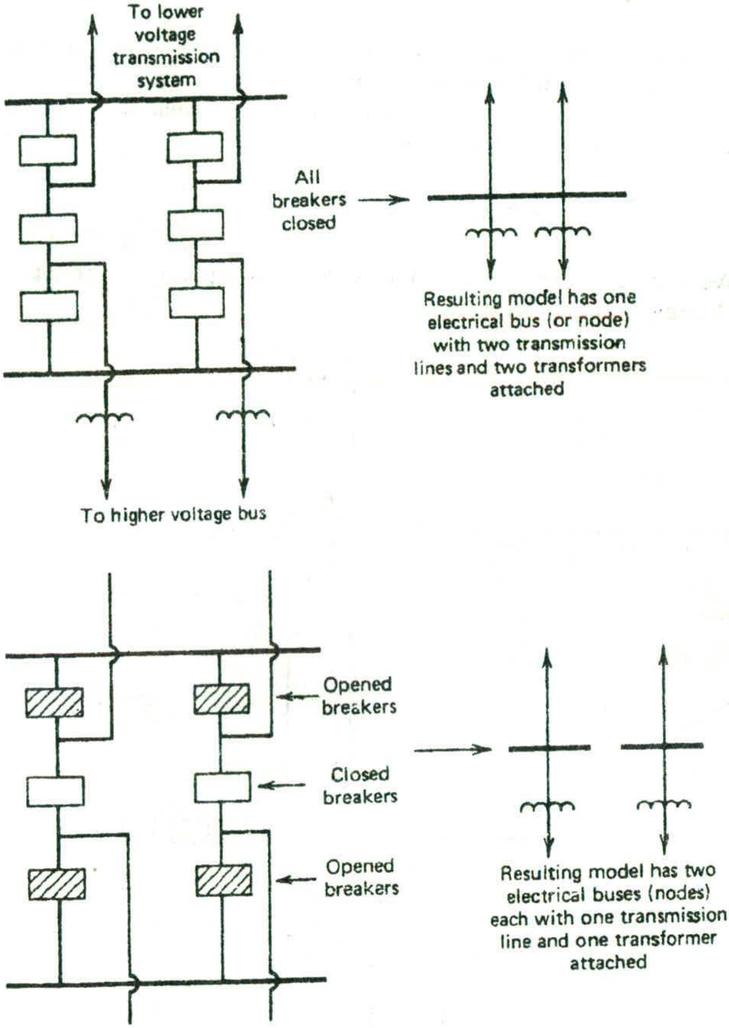

FIG. 12.20 Example of network topology updating.

APPENDIX
Derivation of Least-Squares Equations

One is often confronted with problems wherein data have been obtained by making measurements or taking samples on a process. Furthermore, the quantities being measured are themselves functions of other variables that we wish to estimate. These other variables will be called the state variables and designated $\mathbf{x}$, where the number of state variables is N_s. The measurement values will be called $\mathbf{z}$. We will assume here that the process we are interested in can be modeled using a linear model. Then we say that each measurement z_i is a linear function of the states x_i; that is,

$$z_i = h_i(\mathbf{x}) = h_{i1}x_1 + h_{i2}x_2 + \ldots + h_{iN_s}x_{N_s} \tag{12A.1}$$

We can also write this equation as a vector equation if we place the h_{ij} coefficients into a vector $\mathbf{h}$; that is,

$$\mathbf{h}_i = \begin{bmatrix} h_{i1} \\ h_{i2} \\ \vdots \\ h_{iN_s} \end{bmatrix} \tag{12A.2}$$

Then Eq. 12A.1 becomes

$$z_i = \mathbf{h}_i^T \mathbf{x} \tag{12A.3}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_s} \end{bmatrix}$$

Finally, we can write all the measurement equations in a compact form

$$\mathbf{z} = [H]\mathbf{x} \tag{12A.4}$$

where

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N_m} \end{bmatrix}$$

$$[H] = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1N_s} \\ h_{21} & h_{22} & \cdots & \\ \vdots & & & \\ h_{N_m1} & & \cdots & h_{N_mN_s} \end{bmatrix}$$

where row i of $[H]$ is equal to vector $\mathbf{h}_i^T$ (see Eq. 12A.2).

With N_m measurements we can have three possible cases to solve. That is,

N_s, the number of states, is either less than N_m, equal to N_m, or greater than N_m. We will deal with each case separately.

The Overdetermined Case ($N_m > N_s$)

In this case, we have more measurements or samples than state variables; therefore, we can write more equations, $h_i(\mathbf{x})$, than we have unknowns x_j. One way to estimate the x_i values is to minimize the sum of the squares of difference between the measurement values z_i and the estimate of z_i that is, in turn, a function of the estimates of x_i. That is, we wish to minimize

$$J(\mathbf{x}) = \sum_{i=1}^{N_m} [z_i - h_i(x_1, x_2, \ldots, x_{N_s})]^2 \tag{12A.5}$$

Equation 12A.5 can be written as

$$J(\mathbf{x}) = \sum_{i=1}^{N_m} (z_i - \mathbf{h}_i^T\mathbf{x})^2 \tag{12A.6}$$

and this can be written in a still more compact form as

$$J(\mathbf{x}) = (\mathbf{z} - [H]\mathbf{x})^T(\mathbf{z} - [H]\mathbf{x}) \tag{12A.7}$$

If we wish to find the value of $\mathbf{x}$ that minimizes $J(\mathbf{x})$, we can take the first derivative of $J(\mathbf{x})$ with respect to each x_j ($j = 1, \ldots, N_s$) and set these derivatives to zero. That is,

$$\frac{\partial J(\mathbf{x})}{\partial x_j} = 0 \quad \text{for } j = 1 \ldots N_s \tag{12A.8}$$

If we place these derivatives into a vector, we have what is called the gradient of $J(\mathbf{x})$, which is written $\nabla_x \mathbf{J}(\mathbf{x})$. Then,

$$\nabla_x \mathbf{J}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial J(\mathbf{x})}{\partial x_1} \\ \dfrac{\partial J(\mathbf{x})}{\partial x_2} \\ \vdots \end{bmatrix} \tag{12A.9}$$

Then the goal of forcing each derivative to zero can be written as

$$\nabla_x \mathbf{J}(\mathbf{x}) = \mathbf{0} \tag{12A.10}$$

where **0** is a vector of N_s elements, each of which is zero. To solve this problem, we will first expand Eq. 12A.7:

$$J(\mathbf{x}) = (\mathbf{z} - [H]\mathbf{x})^T(\mathbf{z} - [H]\mathbf{x})$$
$$= \mathbf{z}^T\mathbf{z} - \mathbf{x}^T[H]^T\mathbf{z} - \mathbf{z}^T[H]\mathbf{x} + \mathbf{x}^T[H]^T[H]\mathbf{x} \qquad (12A.11)$$

The second and third term in Eq. 12.A.11 are identical, so that we can write

$$J(\mathbf{x}) = \mathbf{z}^T\mathbf{z} - 2\mathbf{z}^T[H]\mathbf{x} + \mathbf{x}^T[H]^T[H]\mathbf{x} \qquad (12A.12)$$

Before proceeding, we will derive a few simple relationships.

The gradient is always a vector of first derivatives of a scalar function that is itself a function of a vector. Thus, if we define $F(\mathbf{y})$ to be a scalar function, then its gradient $\nabla_y \mathbf{F}$ is:

$$\nabla_y \mathbf{F} = \begin{bmatrix} \dfrac{\partial F}{\partial y_1} \\ \dfrac{\partial F}{\partial y_2} \\ \vdots \\ \dfrac{\partial F}{\partial y_n} \end{bmatrix} \qquad (12A.13)$$

Then, if we define F as follows:

$$F = \mathbf{y}^T\mathbf{b} = [y_1 \quad y_2 \quad \cdots] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} \qquad (12A.14)$$

where **b** is a vector of constants b_i, $i = 1, \ldots, n$, then, F can be expanded as

$$F = y_1 b_1 + y_2 b_2 + y_3 b_3 + \cdots \qquad (12A.15)$$

and the gradient of F is

$$\nabla_y \mathbf{F} = \begin{bmatrix} \dfrac{\partial F}{\partial y_1} \\ \dfrac{\partial F}{\partial y_2} \\ \vdots \\ \dfrac{\partial F}{\partial y_n} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{b} \qquad (12A.16)$$

It ought to be obvious that writing F with $\mathbf{y}$ and $\mathbf{b}$ reversed makes no difference. That is,

$$F = \mathbf{b}^T\mathbf{y} = \mathbf{y}^T\mathbf{b} \tag{12A.17}$$

and, therefore, $\nabla_y(\mathbf{b}^T\mathbf{y}) = \mathbf{b}$.

Suppose we now write the vector $\mathbf{b}$ as the product of a matrix $[A]$ and a vector $\mathbf{u}$.

$$\mathbf{b} = [A]\mathbf{u} \tag{12A.18}$$

Then, if we take F as shown in Eq. 12A.14,

$$F = \mathbf{y}^T\mathbf{b} = \mathbf{y}^T[A]\mathbf{u} \tag{12A.19}$$

we can say that

$$\nabla_y \mathbf{F} = [A]\mathbf{u} \tag{12A.20}$$

Similarly, we can define

$$\mathbf{b}^T = \mathbf{u}^T[A] \tag{12A.21}$$

If we can take F as shown in Eq. 12A.17,

$$F = \mathbf{b}^T\mathbf{y} = \mathbf{u}^T[A]\mathbf{y}$$

then

$$\nabla_y \mathbf{F} = [A]^T\mathbf{u} \tag{12A.22}$$

Finally, we will look at a scalar function F that is quadratic, namely,

$$\begin{aligned} F &= \mathbf{y}^T[A]\mathbf{y} \\ &= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} y_i a_{ij} y_j \end{aligned} \tag{12A.23}$$

Then

$$\begin{aligned} \nabla_y \mathbf{F} &= \begin{bmatrix} \dfrac{\partial F}{\partial y_1} \\ \dfrac{\partial F}{\partial y_2} \\ \vdots \\ \dfrac{\partial F}{\partial y_n} \end{bmatrix} = \begin{bmatrix} 2a_{11}y_1 + 2a_{12}y_2 + \cdots \\ 2a_{21}y_1 + 2a_{22}y_2 + \cdots \\ \vdots \end{bmatrix} \\ &= 2[A]\mathbf{y} \end{aligned} \tag{12A.24}$$

Then, in summary:

$$
\boxed{
\begin{array}{ll}
1.\ F = \mathbf{y}^T\mathbf{b} & \nabla_y \mathbf{F} = \mathbf{b} \\
2.\ F = \mathbf{b}^T\mathbf{y} & \nabla_y \mathbf{F} = \mathbf{b} \\
3.\ F = \mathbf{y}^T[A]\mathbf{u} & \nabla_y \mathbf{F} = [A]\mathbf{u} \\
4.\ F = \mathbf{u}^T[A]\mathbf{y} & \nabla_y \mathbf{F} = [A]^T\mathbf{u} \\
5.\ F = \mathbf{y}^T[A]\mathbf{y} & \nabla_y \mathbf{F} = 2[A]\mathbf{y}
\end{array}
}
\tag{12A.25}
$$

We will now use Eq. 12A.25 to derive the gradient of $J(\mathbf{x})$, that is $\nabla_x \mathbf{J}$, where $J(\mathbf{x})$ is shown in Eq. 12A.12. The first term, $\mathbf{z}^T\mathbf{z}$ is not a function of $\mathbf{x}$, so we can discard it. The second term is of the same form as (4) in Eq. 12A.25, so that,

$$\nabla_x(-2\mathbf{z}^T[H]\mathbf{x}) = -2[H]^T\mathbf{z} \tag{12A.26}$$

The third term is the same as (5) in Eq. 12A.25 with $[H]^T[H]$ replacing $[A]$; then,

$$\nabla_x(\mathbf{x}^T[H]^T[H]\mathbf{x}) = 2[H]^T[H]\mathbf{x} \tag{12A.27}$$

Then from Eqs. 12A.26 and 12A.27 we have

$$\nabla_x \mathbf{J} = -2[H]^T\mathbf{z} + 2[H]^T[H]\mathbf{x} \tag{12A.28}$$

But, as stated in Eq. A.10, we wish to force $\nabla_x \mathbf{J}$ to zero. Then

$$-2[H]^T\mathbf{z} + 2[H]^T[H]\mathbf{x} = 0$$

or

$$\mathbf{x} = [[H]^T[H]]^{-1}[H]^T\mathbf{z} \tag{12A.29}$$

If we had wanted to put a different weight, w_i, on each measurement, we could have written Eq. 12A.6 as

$$J(\mathbf{x}) = \sum_{i=1}^{N_m} w_i(z_i - \mathbf{h}_i^T\mathbf{x})^2 \tag{12A.30}$$

which can be written as

$$J(\mathbf{x}) = (\mathbf{z} - [H]\mathbf{x})^T[W](\mathbf{z} - [H]\mathbf{x})$$

where $[W]$ is a diagonal matrix. Then

$$J(\mathbf{x}) = \mathbf{z}^T[W]\mathbf{z} - \mathbf{x}^T[H]^T[W]\mathbf{z} - \mathbf{z}^T[W][H]\mathbf{x} + \mathbf{x}^T[H]^T[W][H]\mathbf{x}$$

If we once again use Eq. 12A.25, we would obtain

$$\nabla_x \mathbf{J} = -2[H]^T[W]\mathbf{z} + 2[H]^T[W][H]\mathbf{x}$$

and

$$\nabla_x \mathbf{J} = 0$$

gives

$$\mathbf{x} = ([H]^T[W][H])^{-1}[H]^T[W]\mathbf{z} \tag{12A.31}$$

The Fully-Determined Case ($N_m = N_s$)

In this case, the number of measurements is equal to the number of state variables and we can solve for $\mathbf{x}$ directly by inverting $[H]$.

$$\mathbf{x} = [H]^{-1}\mathbf{z} \tag{12A.32}$$

The Underdetermined Case ($N_m < N_s$)

In this case, we have fewer measurements than state variables. In such a case, it is possible to solve for many solutions $\mathbf{x}^{est}$ that cause $J(\mathbf{x})$ to equal zero. The usual solution technique is to find $\mathbf{x}^{est}$ that minimizes the sum of the squares of the solution values. That is, we find a solution such that

$$\sum_{j=1}^{N_s} x_j^2 \tag{12A.33}$$

is minimized while meeting the condition that the measurements will be solved for exactly. To do this, we treat the problem as a constrained minimization problem and use Lagrange multipliers as shown in Appendix 3A.

We formulate the problem as

$$\begin{aligned} &\textbf{Minimize:} && \sum_{j=1}^{N_s} x_j^2 \\ &\textbf{Subject to:} && z_i = \sum_{j=1}^{N_s} h_{ij} x_j \quad \text{for } i = 1, \ldots, N_m \end{aligned} \tag{12A.34}$$

This optimization problem can be written in vector–matrix form as

$$\begin{aligned} &\min && \mathbf{x}^T\mathbf{x} \\ &\text{subject to} && \mathbf{z} = [H]\mathbf{x} \end{aligned} \tag{12A.35}$$

The Lagrangian for this problem is

$$\mathscr{L} = \mathbf{x}^T\mathbf{x} + \boldsymbol{\lambda}^T(\mathbf{z} - [H]\mathbf{x}) \tag{12A.36}$$

Following the rules set down in Appendix 3A we must find the gradient of $\mathscr{L}$ with respect to $\mathbf{x}$ and with respect to $\boldsymbol{\lambda}$. Using the identities found in Eq. 12A.25 we get

$$\nabla_x \mathscr{L} = 2\mathbf{x} - [H]^T \boldsymbol{\lambda} = \mathbf{0}$$

which gives

$$\mathbf{x} = \frac{1}{2}[H]^T \boldsymbol{\lambda}$$

and

$$\nabla_\lambda \mathscr{L} = \mathbf{z} - [H]\mathbf{x} = 0$$

which gives

$$\mathbf{z} = [H]\mathbf{x}$$

Then

$$\mathbf{z} = \frac{1}{2}[H][H]^T \boldsymbol{\lambda}$$

or

$$\boldsymbol{\lambda} = 2[[H][H]^T]^{-1}\mathbf{z}$$

and finally,

$$\mathbf{x} = [H]^T[[H][H]^T]^{-1}\mathbf{z} \tag{12A.37}$$

The reader should be aware that the matrix inversion shown in Eqs. 12A.29, 12A.32, and 12A.37 may not be possible. That is, the $[[H]^T[H]]$ matrix in Eq. 12A.29 may be singular, or $[H]$ may be singular in Eq. 12A.32, or $[[H][H]^T]$ may be singular in Eq. 12A.37. In the overdetermined case ($N_m > N_s$) whose solution is Eq. 12A.29, and the fully determined case ($N_m = N_s$) whose solution is Eq. 12A.32, the singularity implies what is known as an "unobservable" system. By unobservable we mean that the measurements do not provide sufficient information to allow a determination of the states of the system. In the case of the underdetermined case ($N_m < N_s$) whose solution is Eq. 12A.37, the singularity simply implies that there is no unique solution to the problem.

PROBLEMS

12.1 Using the three-bus sample system shown in Figure 12.1, assume that the three meters have the following characteristics.

Meter	Full Scale (MW)	Accuracy (MW)	σ (pu)
M_{12}	100	± 6	0.02
M_{13}	100	± 3	0.01
M_{32}	100	± 0.6	0.002

a. Calculate the best estimate for the phase angles θ_1 and θ_2 given the following measurements.

Meter	Measured Value (MW)
M_{12}	60.0
M_{13}	4.0
M_{32}	40.5

b. Calculate the residual $J(\mathbf{x})$. For a significance level, α, of 0.01, does $J(\mathbf{x})$ indicate the presence of bad data? Explain.

12.2 Given a single transmission line with a generator at one end and a load at the other, two measurements are available as shown in Figure 12.21. Assume that we can model this circuit with a DC load flow using the line reactance shown. Also, assume that the phase angle at bus 1 is 0 rad. Given the meter characteristics and meter readings telemetered from the meters, calculate the best estimate of the power flowing through the transmission line.

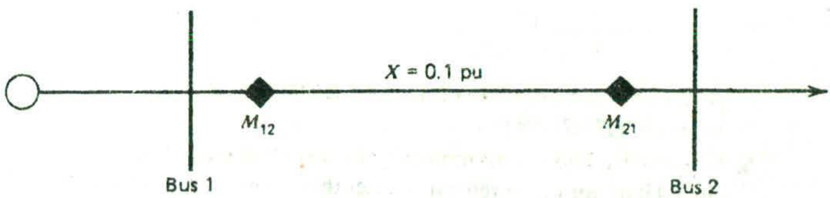

FIG. 12.21 Measurement configuration for Problem 12.2.

Meter	Full Scale (MW)	Meter Standard Deviation (σ) in Full Scale	Meter Reading (MW)
M_{12}	200	1	62
M_{21}	200	5	-52

Note: M_{12} measures power flowing from bus 1 to bus 2; M_{21} measures power flowing from bus 2 to bus 1.

Use 100 MVA as base.

12.3 You are given in the following network with meters at locations as shown in Figure 12.22.

Branch Impedances (pu)
$X_{12} = 0.25$
$X_{13} = 0.50$
$X_{24} = 0.40$
$X_{34} = 0.10$

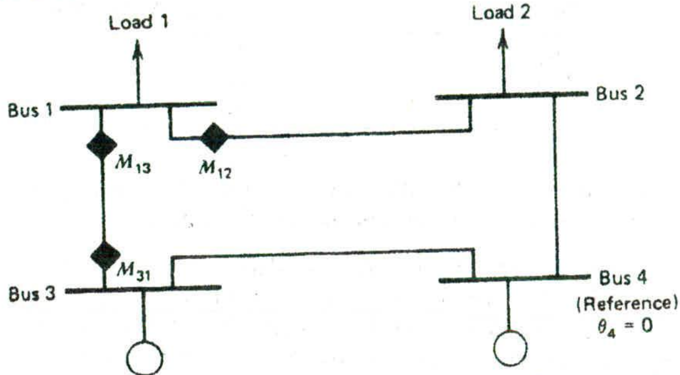

FIG. 12.22 Four-bus system with measurements for Problem 12.3.

Measurement Values	Measurement Errors
$M_{13} = -70.5$	$\sigma_{13} = 0.01$
$M_{31} = 72.1$	$\sigma_{31} = 0.01$
$M_{12} = 21.2$	$\sigma_{12} = 0.02$

a. Is this network observable? Set up the least-squares equations and try to invert $[H^T R^{-1} H]$.

b. Suppose we had a measurement of generation output at bus 3 and included it in our measurement set. Let this measurement be the following:

$$M_{3\,\text{gen}} = 92\ \text{MW} \quad \text{with } \sigma = 0.015$$

Repeat part a including this measurement.

12.4 Given the network shown in Figure 12.23, the network is to be modeled with a DC power flow with line reactances as follows (assume 100-MVA base):

$$x_{12} = 0.1\ \text{pu}$$
$$x_{23} = 0.25\ \text{pu}$$

The meters are all of the same type with a standard deviation of $\sigma = 0.01$ pu for each. The measured values are:

$$M_3 = 105\ \text{MW}$$
$$M_{32} = 98\ \text{MW}$$
$$M_{23} = -135\ \text{MW}$$
$$M_2 = 49\ \text{MW}$$
$$M_{21} = 148\ \text{MW}$$

a. Find the phase angles which result in a best fit to the measured values.

b. Find the value of the residual function J.

c. Calculate estimated generator output of each generator and the estimated power flow on each line.

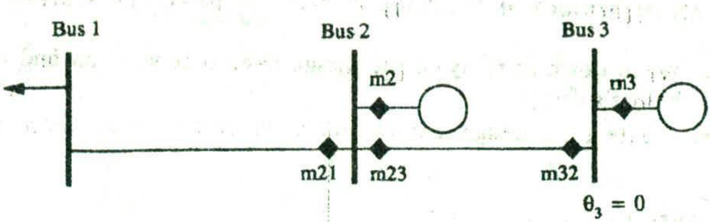

FIG. 12.23 Network for Problem 12.4.

d. Are there any errors in the measurements? If you think so, explain which meters are apt to be in error and why. Remove the suspected bad measurement and try to resolve the state estimator.

12.5 You are to purchase and install a set of programs that are to act as a monitor for the system security of a major utility company. You have solicited bids from major manufacturers of computer systems and are responsible for reviewing the technical contents of each bid. One of the manufacturers proposes to install a system with the flowchart and description given in Figure 12.24.

Bidders design:

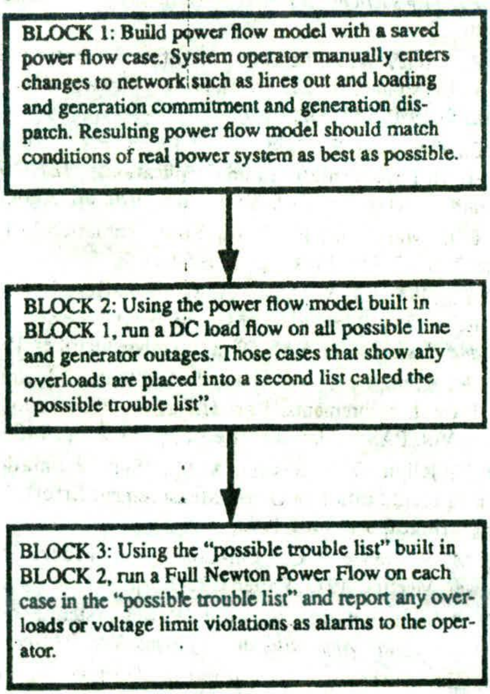

FIG. 12.24 Diagram for Problem 12.5.

a. Write down as many of the design flaws that you can find in this bidder's design.

b. Create a new design that you think will be a state-of-the-art system.

FURTHER READING

State estimation originated in the aerospace industry and only came to be of interest to power systems engineers in the late 1960s. Since then, state estimators have been installed on a regular basis in new energy control centers and have proved quite useful. References 1–4 provide a good introduction to this topic. Reference 4, in particular, is a carefully written overview with a good bibliography of literature up to 1974. References 5 and 6 show the variety of algorithms used to solve the state-estimation problem.

The remaining references cover some of the subtopics of state estimation. The use of the state estimator to detect bad measurements and model parameter errors is covered in references 7–10. Network observability determination is covered in references 11 and 12. Methods of automatically updating the network model topology to match switching status are covered in references 13 and 14. Finally, orthogonal decomposition methods are covered in references 15 and 16.

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13 Optimal Power Flow

13.1 INTRODUCTION

The optimal power flow of OPF has had a long history in its development. It was first discussed by Carpentier in 1962 (reference 1) and took a long time to become a successful algorithm that could be applied in everyday use. Current interest in the OPF centers around its ability to solve for the optimal solution that takes account of the security of the system.

In Chapter 3, we introduced the concept of economic dispatch. In the economic dispatch we had a single constraint which held the total generation to equal the total load plus losses. Thus, the statement of the economic dispatch problem results in a Lagrangian with just one constraint:

$$L = \sum F(P_i) + \lambda(P_{\text{load}} + P_{\text{losses}} - \sum P_i) \tag{13.1}$$

If we think about the single "generation equals load plus losses" constraint:

$$P_{\text{load}} + P_{\text{losses}} - \sum P_i = 0 \tag{13.2}$$

we realize that what it is actually saying is that the generation must obey the same conditions as expressed in a power flow—with the condition that the entire power flow is reduced to one simple equality constraint. There is good reason, as we shall see shortly, to state the economic dispatch calculation in terms of the generation costs, and the entire set of equations needed for the power flow itself as constraints. The power flow equations were introduced in Chapter 4. This formulation is called an optimal power flow.

We can solve the OPF for the minimum generation cost (as in Chapter 3) and require that the optimization calculation also balance the entire power flow—at the same time. Note also that the objective function can take different forms other than minimizing the generation cost. It is common to express the OPF as a minimization of the electrical losses in the transmission system, or to express it as the minimum shift of generation and other controls from an optimum operating point. We could even allow the adjustment of loads in order to determine the minimum load shedding schedule under emergency conditions. Regardless of the objective function, however, an OPF must solve so that the entire set of power constraints are present and satisfied at the solution.

Why set up the generation dispatch calculation as an OPF?.

1. If the entire set of power flow equations are solved simultaneously with the generation cost minimization, the representation of incremental losses is exact. Further, with an objective function that minimizes the losses themselves, the power flow equations are quite necessary.

2. The economic dispatch solutions in Chapter 3 only observed the generation limits $P_i^- \leq P_i \leq P_i^+$. With all of the power flow constraints included in the formulation, many more of the power system limits can be included. These include limits on the generator reactive power, $Q_i^- \leq Q_i \leq Q_i^+$, limits on the voltage magnitude at generation and load buses, $|E_i|^- \leq |E_i| \leq |E_i|^+$, and flows on transmission lines or transformers expressed in either MW, amperes or MVA (e.g. $\text{MVA}_{ij}^- \leq \text{MVA}_{ij} \leq \text{MVA}_{ij}^+$). This set of operating constraints now allows the user to guarantee that the dispatch of generation does not, in fact, force the transmission system into violating a limit, which might put it in danger of being damaged.

3. The OPF can also include constraints that represent operation of the system after contingency outages. These "security constraints" allow the OPF to dispatch the system in a defensive manner. That is, the OPF now forces the system to be operated so that if a contingency happened, the resulting voltages and flows would still be within limit. Thus, constraints such as the following might be incorporated:

$$|E_k|^- \leq |E_k| \text{ (with line } nm \text{ out)} \leq |E_k|^+ \tag{13.3}$$

$$\text{MVA}_{ij}^- \leq \text{MVA}_{ij} \text{ (with line } nm \text{ out)} \leq \text{MVA}_{ij}^+ \tag{13.4}$$

which implies that the OPF would prevent the post-contingency voltage on bus k or the post-contingency flow on line ij from exceeding their limits for an outage of line nm. This special type of OPF is called a "security-constrained OPF," or SCOPF.

4. In the dispatch calculation developed in Chapter 3, the only adjustable variables were the generator MW outputs themselves. In the OPF, there are many more adjustable or "control" variables that be be specified. A partial list of such variables would include:

- Generator voltage.
- LTC transformer tap position.
- Phase shift transformer tap position.
- Switched capacitor settings.
- Reactive injection for a static VAR compensator.
- Load shedding.
- DC line flow.

Thus, the OPF gives us a framework to have many control variables adjusted in the effort to optimize the operation of the transmission system.

5. The ability to use different objective functions provides a very flexible analytical tool.

Given this flexibility, the OPF has many applications including:

1. The calculation of the optimum generation pattern, as well as all control variables, to achieve the minimum cost of generation together with meeting the transmission system limitations.
2. Using either the current state of the power system or a short-term load forecast, the OPF can be set up to provide a "*preventative dispatch*" if security constraints are incorporated.
3. In an emergency, that is when some component of the system is overloaded or a bus is experiencing a voltage violation, the OPF can provide a "corrective dispatch" which tells the operators of the system what adjustments to make to relieve the overload or voltage violation.
4. The OPF can be used periodically to find the optimum setting for generation voltages, transformer taps and switched capacitors or static VAR compensators (sometimes called "voltage–VAR" optimization).
5. The OPF is routinely used in planning studies to determine the maximum stress that a planned transmission system can withstand. For example, the OPF can calculate the maximum power that can safely be transferred from one area of the network to another.
6. The OPF can be used in economic analyses of the power system by providing "*bus incremental costs*" (BICs). The BICs are useful to determine the marginal cost of power at any bus in the system. Similarly, the OPF can be used to calculate the incremental or marginal cost of transmitting power from one outside company—through its system—to another outside company.

13.2 SOLUTION OF THE OPTIMAL POWER FLOW

The optimal power flow is a very large and very difficult mathematical programming problem. Almost every mathematical programming approach that can be applied to this problem has been attempted and it has taken developers many decades to develop computer codes that will solve the OPF problem reliably.

Chapter 3 introduced the concept of the lambda-iteration methods, the gradient method and Newton's method. We shall review all of these here and introduce two new techniques, the linear programming (LP) method and the interior point method. The attributes of these methods are summarized next.

- **Lambda iteration method:** Losses may be represented by a $[B]$ matrix, or the penalty factors may be calculated outside by a power flow. This forms the basis of many standard on-line economic dispatch programs.
- **Gradient methods:** Gradient methods are slow in convergence and are difficult to solve in the presence of inequality constraints.
- **Newton's method:** Very fast convergence, but may give problems with inequality constraints.
- **Linear programming method (LPOPF):** One of the fully developed methods now in common use. Easily handles inequality constraints. Nonlinear objective functions and constraints handled by linearization.
- **Interior point method:** Another of the fully developed and widely used methods for OPF. Easily handles inequality constraints.

We introduced and analyzed the lambda-iteration method in Chapter 3. This method forms the basis of standard on-line economic dispatch codes. The technique works well and can be made to run very fast. It overlooks any constraints on the transmission system and does not produce a dispatch of the generation that will avoid overloads, voltage limit violations, or security constraint violations.

We shall derive the gradient method using the same mathematics used in Chapter 3, only with various advanced models of the transmission system instead of the "load plus losses equals generation" constraint used in Chapter 3. It is then a simple step to go on to develop the Newton's method applied with these same constraints. Finally, the LPOPF and interior point methods are presented.

The objective function in the OPF is usually minimized. In some cases, such as power transfers, it may be maximized. We shall designate the objective function as f. The equations that guarantee that the power flow constraints are met will be designated as

$$\mathbf{g}(\mathbf{z}) = 0 \tag{13.5}$$

Note that here we shall only be concerned with a variable vector $\mathbf{z}$. This vector contains the adjustable controls, the bus voltage magnitudes, and phase angles, as well as the fixed parameters of the system. Later, we shall break the variables up into sets of state variables, control variables, and fixed parameters.

The OPF can also solve for an optimal solution with inequality constraints on dependent variables, such as line MVA flows. These will be designated

$$\mathbf{h}^- \leq \mathbf{h}(\mathbf{z}) \leq \mathbf{h}^+ \tag{13.6}$$

In addition, limits may be placed directly on state variables or control variables:

$$\mathbf{z}^- \leq \mathbf{z} \leq \mathbf{z}^+ \tag{13.7}$$

The OPF problem then consists of minimizing (or maximizing) the objective function, subject to the equality constraints, the inequality constraints, and the state and control variable limits.

The developments and illustrative examples in this chapter concentrate (but not exclusively) on the LPOPF technique. The method is widely used and only requires an AC or DC power flow program, plus a suitable LP package for solving illustrative examples and (homework) problems.

13.2.1 The Gradient Method

In this section, we shall consider the objective function to be total cost of generation (later examples will demonstrate how other objectives can be used). The objective function to be minimized is:

$$\sum_{\text{all gen.}} F_i(P_i)$$

where the sum extends to all generators on the power system, including the generator at the reference bus.

We shall start out defining the unknown or state vector $\mathbf{x}$ as:

$$\mathbf{x} = \begin{bmatrix} \left.\begin{matrix} \theta_i \\ |E_i| \end{matrix}\right\} \text{on each } PQ \text{ bus} \\ \theta_i \quad \text{on each } PV \text{ bus} \end{bmatrix} \tag{13.8}$$

another vector, $\mathbf{y}$, is defined as:

$$\mathbf{y} = \begin{bmatrix} \left.\begin{matrix} \theta_k \\ |E_k| \end{matrix}\right\} \text{on the reference bus} \\ \left.\begin{matrix} P_k^{\text{net}} \\ Q_k^{\text{net}} \end{matrix}\right\} \text{on each } PQ \text{ bus} \\ \left.\begin{matrix} P_k^{\text{net}} \\ |E_k|^{\text{sch}} \end{matrix}\right\} \text{on each } QV \text{ bus} \end{bmatrix} \tag{13.9}$$

Note that the vector $\mathbf{y}$ is made up of all of the parameters that must be specified. Some of these parameters are adjustable (for example, the generator output, P_k^{net}, and the generator bus voltage). Some of the parameters are fixed, as far as the OPF calculation is concerned, such as the P and Q at each load bus. To make this distinction, we shall divide the $\mathbf{y}$ vector up into two parts, $\mathbf{u}$ and $\mathbf{p}$:

$$\mathbf{y} = \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} \tag{13.10}$$

where **u** represents the vector of control or adjustable variables, and **p** represents the fixed or constant variables. Note also that we are only representing equality constraints at this point.

Finally, we shall define a set of m equations that govern the power flow:

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} \left.\begin{matrix} P_i(|\mathbf{E}|, \theta) - P_i^{\text{net}} \\ Q_i(|\mathbf{E}|, \theta) - Q_i^{\text{net}} \end{matrix}\right\} \text{for each } PQ \text{ (load) bus } i \\ P_k(|\mathbf{E}|, \theta) - P_k^{\text{net}} \quad \text{for each } PV \text{ (gen.) bus } k, \text{ not including the reference bus} \end{bmatrix} \tag{13.11}$$

Note that these equations are the same bus equations as shown in Chapter 4 for the Newton power flow (Eq. 4.18).

We must recognize that the reference-bus power generation is not an independent variable. That is, the reference-bus generation always changes to balance the power flow; we cannot specify it at the beginning of the calculation. We wish to express the cost or objective function as a function of the control variables and of the state variables. We do this by dividing the cost function as follows:

$$\text{cost} = \sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}(P_{\text{ref}}) \tag{13.12}$$

where the first summation does not include the reference bus, The P_i are all independent, controlled variables whereas P_{ref} is a dependent variable. We say that the P_i are in the vector **u** and the P_{ref} is a function of the network voltages and angles:

$$P_{\text{ref}} = P_{\text{ref}}(|\mathbf{E}|, \theta) \tag{13.13}$$

then the cost function becomes:

$$\sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}[P_{\text{ref}}(|\mathbf{E}|, \theta)] = \mathrm{f}(\mathbf{x}, \mathbf{u}) \tag{13.14}$$

We can now set up the Lagrange equation for the OPF as follows:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathbf{f}(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^t \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{13.15}$$

where

$\mathbf{x}$ = vector of state variables

$\mathbf{u}$ = vector of control variables

$\mathbf{p}$ = vector of fixed parameters

$\boldsymbol{\lambda}$ = vector of Lagrange multipliers

$\mathbf{g}$ = set of equality constraints representing the power flow equations

f = the objective function

This Lagrange equation is perhaps better seen when written as:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}[P_{\text{ref}}(|\mathbf{E}|, \boldsymbol{\theta})] + [\lambda_1 \lambda_2, \ldots, \lambda_m] \begin{bmatrix} P_i(|\mathbf{E}|, \boldsymbol{\theta}) - P_i^{\text{net}} \\ Q_i(|\mathbf{E}|, \boldsymbol{\theta}) - Q_i^{\text{net}} \\ P_k(|\mathbf{E}|, \boldsymbol{\theta}) - P_k^{\text{net}} \\ \vdots \end{bmatrix} \tag{13.16}$$

We now have a Lagrange function that has a single objective function and m Lagrange multipliers, one for each of the m power flow equations.

To minimize the cost function, subject to the constraints, we set the gradient of the Lagrange function to zero:

$$\nabla \mathscr{L} = 0 \tag{13.17}$$

To do this, we break up the gradient vector into three parts corresponding to the variables $\mathbf{x}$, $\mathbf{u}$, and $\boldsymbol{\lambda}$:

$$\nabla \mathscr{L}_x = \frac{\partial \mathscr{L}}{\partial \mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} \tag{13.18}$$

$$\nabla \mathscr{L}_u = \frac{\partial \mathscr{L}}{\partial \mathbf{u}} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right]^T \boldsymbol{\lambda} \tag{13.19}$$

$$\nabla \mathscr{L}_\lambda = \frac{\partial \mathscr{L}}{\partial \boldsymbol{\lambda}} = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{13.20}$$

Some discussion of the three gradient equations above is in order. First, Eq. 13.18 consists of a vector of derivatives of the objective function with respect to the state variables, $\mathbf{x}$. Since the objective function itself is not a function of the state variable *except for the reference bus*, this becomes:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial}{\partial P_{\text{ref}}} F_{\text{ref}}(P_{\text{ref}}) \dfrac{\partial P_{\text{ref}}}{\partial \theta_1} \\ \dfrac{\partial}{\partial P_{\text{ref}}} F_{\text{ref}}(P_{\text{ref}}) \dfrac{\partial P_{\text{ref}}}{\partial |E_1|} \\ \vdots \end{bmatrix} \tag{13.21}$$

The $[\partial \mathbf{g}/\partial \mathbf{x}]$ term in Eq. 13.18 actually is the Jacobian matrix for the Newton power flow, which was developed in Chapter 4. That is:

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right] = \begin{bmatrix} \dfrac{\partial P_1}{\partial \theta_1} & \dfrac{\partial P_1}{\partial |E_1|} & \dfrac{\partial P_1}{\partial \theta_2} & \dfrac{\partial P_1}{\partial |E_2|} & \cdots \\ \dfrac{\partial Q_1}{\partial \theta_1} & \dfrac{\partial Q_1}{\partial |E_1|} & \dfrac{\partial Q_1}{\partial \theta_2} & \dfrac{\partial Q_1}{\partial |E_2|} & \cdots \\ \dfrac{\partial P_2}{\partial \theta_1} & \dfrac{\partial P_2}{\partial |E_1|} & & & \cdots \\ \dfrac{\partial Q_2}{\partial \theta_1} & \dfrac{\partial Q_2}{\partial |E_1|} & & & \\ \vdots & & & & \vdots \end{bmatrix} \tag{13.22}$$

Note that this matrix must be transposed for use in Eq. 13.18.

Equation 13.19 is the gradient of the Lagrange function with respect to the control variables. Here the vector $\partial \mathbf{f}/\partial \mathbf{u}$ is a vector of derivatives of the objective function with respect to the control variables:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial}{\partial P_1} F_1(P_1) \\ \dfrac{\partial}{\partial P_2} F_2(P_2) \\ \vdots \end{bmatrix} \tag{13.23}$$

The other term in Eq. 13. 19, $[\partial \mathbf{g}/\partial \mathbf{u}]$, actually consists of a matrix of all zeros with some -1 terms on the diagonals, which correspond to equations in $\mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p})$ where a control variable is present. Finally, Eq. 13.20 consists simply of the power flow equations themselves.

The solution of the gradient method of OPF is as follows:

1. Given a set of fixed parameters $\mathbf{p}$, assume a starting set of control variables $\mathbf{u}$.
2. Solve a power flow. This guarantees that Eq. 13.20 is satisfied.
3. Solve Eq. 13.19 for lambda;

$$\boldsymbol{\lambda} = -\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^{T-1} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{13.24}$$

4. Substitute λ into Eq. 13.18 to get the gradient of $\mathcal{L}$ with respect to the control variables.

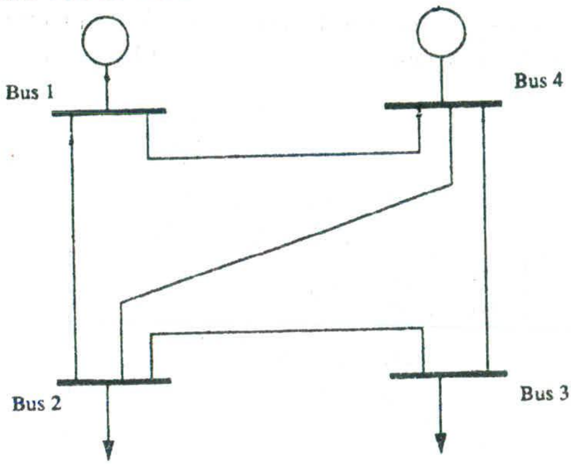

FIG. 13.1 Four-bus system for Example 13A.

The gradient will give the direction of maximum increase in the cost function as a function of the adjustments in each of the **u** variables. Since we wish to decrease the objective function, we shall move in the direction of the negative of the gradient. The gradient method gives no indication how far along the negative gradient direction we should move. Assuming that a distance is picked that reduces the objective, one must start at step 2 above, and repeat steps 2, 3, and 4 over and over until the gradient itself becomes sufficiently close to the zero vector, indicating that all conditions for the optimum have been reached.

EXAMPLE 13A

The following is a very simple example presented to show the meaning of each of the elements in the gradient equations. Example 13B will be a more practical example of the gradient method.

The four-bus system in Figure 13.1 will be modeled with a DC power flow. The following are known:

P_2, P_3, and $\theta_4 = 0$

Line reactances: x_{12}, x_{14}, x_{24}, x_{23}, and x_{34}

Cost functions: $F_1(P_1)$ and $F_4(P_4)$

All $|E|$ values are fixed at 1.0 per unit volts

The only independent control variable in this problem is the generator output P_1, or:

$$u = P_1 \tag{13.25}$$

The state variables are θ_1, θ_2, and θ_3, or:

$$\mathbf{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{13.26}$$

We wish to minimize the total generation cost while maintaining a solved DC power flow for the network. To do this with the gradient method we form the Lagrangian:

$$\mathscr{L} = F_1(P_1) + F_4[P_4(\theta_1 \dots \theta_4)] + [\lambda_1 \quad \lambda_2 \quad \lambda_3] \begin{bmatrix} P_1(\theta_1 \dots \theta_4) - P_1 \\ P_2(\theta_1 \dots \theta_4) - P_2 \\ P_3(\theta_1 \dots \theta_4) - P_3 \end{bmatrix} \tag{13.27}$$

In terms of the equations presented earlier:

$$\mathrm{f}(\mathbf{x}, \mathbf{u}) = F_1(P_1) + F_4[P_4(\theta_1 \dots \theta_4)] \tag{13.28}$$

$$g(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} P_1(\theta_1 \dots \theta_4) - P_1 \\ P_2(\theta_1 \dots \theta_4) - P_2 \\ P_3(\theta_1 \dots \theta_4) - P_3 \end{bmatrix} \tag{13.29}$$

Note that in $\mathbf{g}(\mathbf{x}, \mathbf{u})$, the P_1 is the control variable and P_2 and P_3 are fixed.

We shall now expand $\mathbf{g}(\mathbf{x}, \mathbf{u})$ as follows:

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} P_1(\theta_1 \dots \theta_4) - P_1 \\ P_2(\theta_1 \dots \theta_4) - P_2 \\ P_3(\theta_1 \dots \theta_4) - P_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{x_{12}}(\theta_1 - \theta_2) + \frac{1}{x_{14}}(\theta_1 - \theta_{24}) - P_1 \\ \vdots \\ \end{bmatrix} \tag{13.30}$$

The result is:

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = [B'] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} - \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{13.31}$$

and the Lagrange function becomes:

$$F_1(P_1) + F_4[P_4(\theta_1 \dots \theta_4)] + [\lambda_1 \quad \lambda_2 \quad \lambda_3] \left([B'] \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} - \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \right) \tag{13.32}$$

We now proceed to develop the three gradient components:

$$\nabla \mathscr{L}_{\lambda} = \mathbf{g}(\mathbf{x}, \mathbf{u}) = 0 \tag{13.33}$$

which simply says that we need to start by always maintaining the DC power flow:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = [B']^{-1} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{13.34}$$

The next component:

$$\Delta \mathscr{L}_x = \begin{bmatrix} \dfrac{\partial \mathscr{L}}{\partial \theta_1} \\[2ex] \dfrac{\partial \mathscr{L}}{\partial \theta_2} \\[2ex] \dfrac{\partial \mathscr{L}}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F_4}{\partial P_4} \dfrac{\partial P_4}{\partial \theta_1} \\[2ex] \dfrac{\partial F_4}{\partial P_4} \dfrac{\partial P_4}{\partial \theta_2} \\[2ex] \dfrac{\partial F_4}{\partial P_4} \dfrac{\partial P_4}{\partial \theta_3} \end{bmatrix} + [B']^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 0 \tag{13.35}$$

This can be used to solve the vector of Lagrange multipliers:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = (-1)[B']^{T^{-1}} \begin{bmatrix} \dfrac{\partial P_4}{\partial \theta_1} \\[2ex] \dfrac{\partial P_4}{\partial \theta_2} \\[2ex] \dfrac{\partial P_4}{\partial \theta_3} \end{bmatrix} \dfrac{\partial F_4}{\partial P_4} \tag{13.36}$$

where

$$\begin{bmatrix} \dfrac{\partial P_4}{\partial \theta_1} \\[2ex] \dfrac{\partial P_4}{\partial \theta_2} \\[2ex] \dfrac{\partial P_4}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{x_{14}} \\[2ex] -\dfrac{1}{x_{24}} \\[2ex] -\dfrac{1}{x_{34}} \end{bmatrix} \tag{13.37}$$

It can be easily demonstrated that:

$$[B']^{T-1}\begin{bmatrix}\dfrac{\partial P_4}{\partial \theta_1}\\[2ex]\dfrac{\partial P_4}{\partial \theta_2}\\[2ex]\dfrac{\partial P_4}{\partial \theta_3}\end{bmatrix}=\begin{bmatrix}-1\\-1\\-1\end{bmatrix} \tag{13.38}$$

so that

$$\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{\partial F_4}{\partial P_4} \tag{13.39}$$

Finally,

$$\frac{\partial \mathbf{g}}{\partial u}=\begin{bmatrix}-1\\0\\0\end{bmatrix} \tag{13.40}$$

and

$$\nabla\mathscr{L}_u=\frac{\partial F_1}{\partial P_1}+\frac{\partial \mathbf{g}^T}{\partial u}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\frac{\partial F_1}{\partial P_1}+[-1\quad 0\quad 0]\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{\partial F_4}{\partial P_4}\right)=\frac{\partial F_1}{\partial P_1}-\frac{\partial F_4}{\partial P_4} \tag{13.41}$$

the gradient with respect to the control variable is zero when the two incremental costs are equal, which is the common economic dispatch criterion (assuming neither generator is at a limit). Since the DC power flow represents a linear lossless system, the result simply confirms that the gradient method will produce a result that is the same as economic dispatch.

EXAMPLE 13B

In this example, we shall minimize the real power losses (MW losses) on the three-bus AC system in Figure 13.2. To work this example, the student must be able to run an AC power flow on the three-bus system. (This example is taken from reference 4.)

Given the three-bus network shown in Figure 13.2, where

$$P_3 + jQ_3 = 2.0 + j1.0 \text{ per unit}$$

and

$$P_2 = 1.7 \text{ per unit}$$

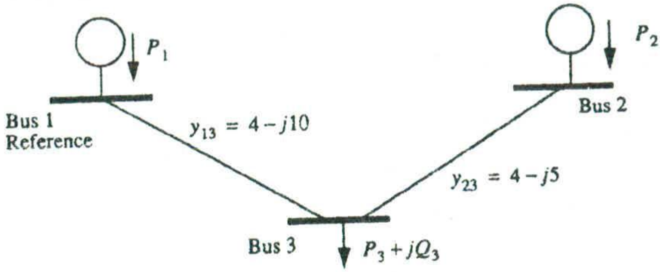

FIG. 13.2 Three-bus example for Example 13B.

In this problem, the generation at bus 2 will be fixed, *the only control variables will be the voltage magnitude at buses 1 and 2.* That is

$$\mathbf{u} = \begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix} \tag{13.42}$$

The state variables will be the phase angles at buses 2 and 3 and the voltage at bus 3:

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ |E_3| \end{bmatrix} \tag{13.43}$$

The fixed parameters are

$$\mathbf{p} = \begin{bmatrix} \theta_1 \\ P_2 \\ P_3 \\ Q_3 \end{bmatrix} \tag{13.44}$$

We shall solve for the minimum losses using the gradient method. This requires that we solve, repeatedly, the following:

$$\nabla \mathscr{L}_u = \begin{bmatrix} \dfrac{\partial P_{\text{losses}}}{\partial |E_1|} \\[2ex] \dfrac{\partial P_{\text{losses}}}{\partial |E_2|} \end{bmatrix} \tag{13.45}$$

Starting at an initial set of voltages:

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^0 = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix} \tag{13.46}$$

we proceed using

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^1 = \begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^0 + (-1)(\nabla \mathscr{L}_u)\alpha \tag{13.47}$$

where α was set to 0.03 after several trials.

As previously:

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} P_2(\mathbf{E}|, \boldsymbol{\theta}) - P_2 \\ P_3(|\mathbf{E}|, \boldsymbol{\theta}) - P_3 \\ Q_3(|\mathbf{E}|, \boldsymbol{\theta}) - Q_3 \end{bmatrix} \tag{13.48}$$

where the above represents the AC power flow equations as shown in Chapter 4. When we take the derivative,

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial P_2}{\partial \theta_2} & \dfrac{\partial P_2}{\partial \theta_3} & \dfrac{\partial P_2}{\partial |E_3|} \\ \dfrac{\partial P_3}{\partial \theta_2} & \dfrac{\partial P_3}{\partial \theta_3} & \dfrac{\partial P_3}{\partial |E_3|} \\ \dfrac{\partial Q_3}{\partial \theta_2} & \dfrac{\partial Q_3}{\partial \theta_3} & \dfrac{\partial Q_3}{\partial |E_3|} \end{bmatrix} \tag{13.49}$$

these derivatives are calculated as shown in Chapter 4, Eq. 4.22 and the above represents the Jacobian matrix that would be used in the Newton power flow solution to this network. Similarly:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial P_2}{\partial |E_1|} & \dfrac{\partial P_2}{\partial |E_2|} \\ \dfrac{\partial P_3}{\partial |E_1|} & \dfrac{\partial P_3}{\partial |E_2|} \\ \dfrac{\partial Q_3}{\partial |E_1|} & \dfrac{\partial Q_3}{\partial |E_2|} \end{bmatrix} \tag{13.50}$$

One special note, the objective function, P_{losses} can be expressed in two different ways. The first is simply to write out the losses as:

$$P_{\text{losses}} = \text{Re}\left(\sum_{\textit{both}\ \text{lines}} I^2 R \right) \tag{13.51}$$

or one can use the simple observation that since P_2 and P_3 are fixed, any change in the losses due to adjustments of V_1 and V_2 will be directly reflected in changes in P_1. That is, $\Delta P_{\text{losses}} = \Delta P_1$ and

$$\frac{\partial P_{\text{losses}}}{\partial \mathbf{x}} = \frac{\partial P_1}{\partial \mathbf{x}} \tag{13.52}$$

We shall use the second form of the objective so that

$$f = P_1 \tag{13.53}$$

and then:

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial P_1}{\partial \theta_2} \\ \dfrac{\partial P_1}{\partial \theta_3} \\ \dfrac{\partial P_1}{\partial |E_3|} \end{bmatrix} \tag{13.54}$$

The solution to the first AC power flow, with

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^0 = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix} \tag{13.55}$$

gives per unit losses of 0.3906 (39.06 MW losses on 100-MVA base). The reference-bus power, P_1, is 0.6906 per unit MW. Taking this solved power flow as the starting point, we have:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{bmatrix} 8.14 & 8.14 & 1.54 \\ 6.96 & 12.0 & 3.85 \\ -4.5 & -7.85 & 10.0 \end{bmatrix} \tag{13.56}$$

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 0 \\ 4.36 \\ 4.14 \end{bmatrix} \tag{13.57}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = (-1)\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^{T-1} \frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 0.743 \\ -0.98 \\ -0.154 \end{bmatrix} \tag{13.58}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial P_1}{\partial V_1} \\ \dfrac{\partial P_1}{\partial V_2} \end{bmatrix} = \begin{bmatrix} 5.533 \\ 0 \end{bmatrix} \tag{13.59}$$

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right]^T = \begin{bmatrix} 0 & 3.354 & 5.0 \\ 4.94 & 4.5 & 6.96 \end{bmatrix} \tag{13.60}$$

Then,

$$\Delta \mathscr{L}_u = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right]^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 2.25 \\ -1.78 \end{bmatrix} \tag{13.61}$$

and, with $\alpha = 0.03$, we obtain a new set of voltages:

$$\begin{bmatrix} |E_1| \\ |E_2| \end{bmatrix}^1 = \begin{bmatrix} 1.1 \\ 0.9 \end{bmatrix} - \begin{bmatrix} 2.25 \\ -1.787 \end{bmatrix} 0.03 = \begin{bmatrix} 0.95 \\ 1.03 \end{bmatrix} \tag{13.62}$$

This represents the new control variable settings that must be fed back to the AC power flow.

The new AC power flow, with the above new voltages, results in $P_{\text{losses}} = 0.2380$ per unit and the generation at the reference bus of $P_1 = 0.5380$. Another iteration of the gradient calculation yields $P_{\text{losses}} = 0.2680$ per unit for a controls setting of:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0.86 \\ 0.86 \end{bmatrix} \tag{13.63}$$

Note that for this simple problem, the gradient is able to find a reduction in losses after the first iteration, but the next iteration caused the losses to increase. Eventually, it will need tuning, in the form of additional adjustments to the value of α, so that it will not simply oscillate around a minimum. Further, we never specified any voltage limits for V_1 and V_2. As we reduce losses, we may very well run into voltage limits on buses 1 or 2, or both. Here, the gradient method loses whatever simplicity it has and tends to become unmanageable. This would further be the case if we were to place a limit on V_3, which would be a functional inequality and would be very difficult to express in the gradient formulation we have used.

13.2.2 Newton's Method

The problems with the gradient method lie mainly in the fact that the direction of the gradient must be changed quite often and this leads to a very slow convergence. To speed up this convergence, we can use Newton's method, where we take the derivative of the gradient with respect to $\mathbf{x}$, $\mathbf{u}$, and λ. Then, the optimal solution becomes:

$$\begin{bmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{u} \\ \Delta\lambda \end{bmatrix} = -\begin{bmatrix} \frac{\partial}{\partial \mathbf{x}}\nabla\mathcal{L}_x & \frac{\partial}{\partial \mathbf{u}}\nabla\mathcal{L}_x & \frac{\partial}{\partial \lambda}\nabla\mathcal{L}_x \\ \frac{\partial}{\partial \mathbf{x}}\nabla\mathcal{L}_u & \frac{\partial}{\partial \mathbf{u}}\nabla\mathcal{L}_u & \frac{\partial}{\partial \lambda}\nabla\mathcal{L}_u \\ \frac{\partial}{\partial \mathbf{x}}\nabla\mathcal{L}_\lambda & \frac{\partial}{\partial \mathbf{u}}\nabla\mathcal{L}_\lambda & \frac{\partial}{\partial \lambda}\nabla\mathcal{L}_\lambda \end{bmatrix}^{-1} \begin{bmatrix} \nabla\mathcal{L}_x \\ \nabla\mathcal{L}_u \\ \nabla\mathcal{L}_\lambda \end{bmatrix} \tag{13.64}$$

The form of Eq. 13.22 is essentially the same as that derived in Section 3.5 on Newton's method. This matrix equation is a very formidable undertaking to compute and manipulate. It is extremely sparse and requires special sparsity logic.

Handling inequality constraints is very difficult in either gradient or Newton approaches. The usual method is to form a constraint "penalty" function as follows. Suppose the voltage at a bus must meet limits:

$$|E_i|^{\min} \le |E_i| \le |E_i|^{\max} \tag{13.65}$$

It is possible to enforce this constraint by inventing the following exterior penalty functions:

$$h(|E_i|) = \begin{bmatrix} K(|E_i| - |E_i|^{\min})^2 \\ 0 \\ K(|E_i|^{\max} - |E_i|)^2 \end{bmatrix} \begin{array}{l} \text{for } |E_i| < |E_i|^{\min} \\ \text{for } E \text{ within limits} \\ \text{for } |E_i| > |E_i|^{\max} \end{array} \tag{13.66}$$

This penalty function is shown in Figure 13.3.

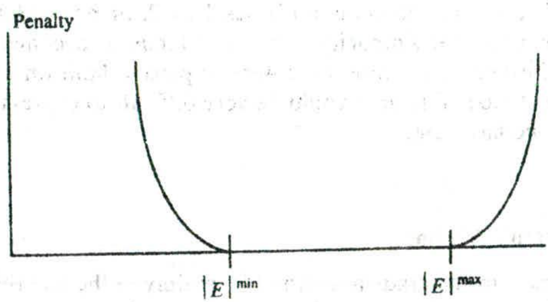

FIG. 13.3 Exterior penalty functions for voltage magnitude violations.

To solve the OPF with the voltage inequality constraint, we add the penalty function to the objective function, f. The resulting function will be large if the voltage is outside its limit, and thus the OPF will try to force it within its limits as it minimizes the objective.

Since Newton's method has the second derivative information built into it, it does not have great difficulty in converging and it can handle the inequality constraints as well. The difficulty with Newton's method arises in the fact that near the limit the penalty is small, so that the optimal solution will tend to allow the variable, a voltage in the example above, to float over its limit. The seemingly simple tuning procedure of raising the value of K may eventually cause the matrices to become ill-conditioned and the method fails. When there are few limits to be concerned with and the objective function is "shallow," that is, the variability of f with adjustments in the control variables is very low, Newton's method is the best method to use.

References 5–7 give examples of the development of Newton's method to solve the full AC OPF.

13.3 LINEAR SENSITIVITY ANALYSIS

Before continuing with the discussion of the linear programming and interior OPF methods, we shall develop the concept of linear sensitivity analysis. Linear sensitivity coefficients give an indication of the change in one system quantity (e.g., MW flow, MVA flow, bus voltage, etc.) as another quantity is varied (e.g., generator MW output, transformer tap position, etc.) These linear relationships are essential for the application of linear programming. Note that as the adjustable variable is changed, we assume th. the power system reacts so as to keep all of the power flow equations solved. As such, linear sensitivity coefficients can be expressed as partial derivatives for example:

$$\frac{\partial \text{MVA flow}_{ij}}{\partial \text{MW gen}_k}$$

shows the sensitivity of the flow (MVA) on line (i to j) with respect to the power generated at bus k.

Some sensitivity coefficients may change rapidly as the adjustment is made and the power flow conditions are updated. This is because some system quantities vary in a nonlinear relationship with the adjustment and resolution of the power flow equations. This is especially true for quantities that have to do with voltage and MVAR flows. Sensitivities such as the variation of MW flow with respect to a change in generator MW output are rather linear across a wide range of adjustments and lead to the usefulness of the DC power flow equations and the "a" and "d" factors introduced in Chapter 11.

For this reason, the value represented by a sensitivty coefficient is only good for small adjustments and the sensitivities must be recalculated often.

13.3.1 Sensitivity Coefficients of an AC Network Model

The following procedure is used to linearize the AC transmission system model for a power system. To start, we shall define two general equations giving the power injection at a bus. That is, the net power flowing into a transmission system from the bus. This function represents the power flowing into transmission lines and shunts at the bus:

$$
\begin{aligned}
P_i(|\mathbf{E}|,\boldsymbol{\theta}) &= Re\left[\left(\sum_j E_i[(E_i - t_{ij}E_j)y_{ij}]^*\right) + E_i\left(E_i \sum_i y_{\text{shunt}_i}\right)^*\right] \\
Q_i(|\mathbf{E}|,\boldsymbol{\theta}) &= Im\left[\left(\sum_j E_i[(E_i - t_{ij}E_j)y_{ij}]^*\right) + E_i\left(E_i \sum_i y_{\text{shunt}_i}\right)^*\right]
\end{aligned}
\qquad (13.67)
$$

where

$E_i = |E_i|\angle\theta_i$

t_{ij} = the transformer tap in branch ij

y_{ij} = the branch admittance

y_{shunt_i} = the sum of the branch and bus shunt admittances at bus i

Then, at each bus:

$$
\begin{aligned}
P_i(|\mathbf{E}|,\boldsymbol{\theta}) &= P_{\text{gen}_i} - P_{\text{load}_i} \\
Q_i(|\mathbf{E}|,\boldsymbol{\theta}) &= Q_{\text{gen}_i} - Q_{\text{load}_i}
\end{aligned}
\qquad (13.68)
$$

The set of equations that represents the first-order approximation of the AC network around the initial point is the same as generally used in the Newton power flow algorithm. That is:

$$
\begin{aligned}
\sum \frac{\partial P_i}{\partial |E_j|}\Delta|E_j| + \sum \frac{\partial P_i}{\partial \theta_j}\Delta\theta_j + \sum \frac{\partial P_i}{\partial j_{ij}}\Delta t_{ij} &= \Delta P_{\text{gen}_i} \\
\sum \frac{\partial Q_i}{\partial |E_j|}\Delta|E_j| + \sum \frac{\partial Q_i}{\partial \theta_j}\Delta\theta_j + \sum \frac{\partial Q_i}{\partial t_{ij}}\Delta t_{ij} &= \Delta Q_{\text{gen}_i}
\end{aligned}
\qquad (13.69)
$$

This can be placed in matrix form for easier manipulation:

$$
\begin{bmatrix}
\dfrac{\partial P_1}{\partial |E_1|} & \dfrac{\partial P_1}{\partial \theta_1} & \cdots \\
\dfrac{\partial Q_1}{\partial |E_1|} & \dfrac{\partial Q_1}{\partial \theta_1} & \cdots \\
 & \vdots &
\end{bmatrix}
\begin{bmatrix}
\Delta|E_1| \\ \Delta\theta_1 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
-\dfrac{\partial}{\partial t_{ij}}(P_i) & 1 & 0 \\
-\dfrac{\partial}{\partial t_{ij}}(Q_i) & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\Delta t_{ij} \\ \Delta P_{\text{gen}_i} \\ \Delta Q_{\text{gen}_i}
\end{bmatrix}
\qquad (13.70)
$$

This equation will be placed into a more compact format that uses the vectors $\mathbf{x}$ and $\mathbf{u}$, where $\mathbf{x}$ is the state vector of voltages and phase angles, and $\mathbf{u}$ is the vector of control variables. The control variables are the generator MW, transformer taps, and generator voltage magnitudes (or generator MVAR). Note that at any given generator bus we can control a voltage magnitude only within the limits of the unit VAR capacity. Therefore, there are times when the role of the state and control are reversed. Note that other controls can easily be added to this formulation. The compact form of Eq. 12.30 then is written:

$$[J_{px}]\Delta\mathbf{x} = [J_{pu}]\Delta\mathbf{u} \tag{13.71}$$

Now, we will assume that there are several transmission system dependent variables, $\mathbf{h}$, that represent, for example, MVA flows, load bus voltages, line amperes, etc., and we wish to find their sensitivity with respect to changes in the control variables. Each of these quantities can be expressed as a function of the state and control variables; that is, for example:

$$\mathbf{h} = \begin{bmatrix} \text{MVA flow}_{nm}(|\mathbf{E}|, \boldsymbol{\theta}) \\ |E_k| \end{bmatrix} \tag{13.72}$$

where $|E_k|$ represents only load bus voltage magnitude.

As before, we can write a linear version of these variables around the operating point

$$\Delta\mathbf{h} = \begin{bmatrix} \dfrac{\partial h_1}{\partial |E_1|} & \dfrac{\partial h_1}{\partial \theta_1} & \cdots \\ \dfrac{\partial h_2}{\partial |E_1|} & \dfrac{\partial h_2}{\partial \theta_1} & \cdots \\ & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \Delta|E_1| \\ \Delta\theta_1 \\ \vdots \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_1}{\partial u_1} & \dfrac{\partial h_1}{\partial u_2} & \cdots \\ \dfrac{\partial h_2}{\partial u_1} & \dfrac{\partial u_2}{\partial h_2} & \cdots \\ & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \end{bmatrix} \tag{13.73}$$

where

h_1 = the line nm MVA flow

h_2 = the bus k voltage magnitude

Again, we can put this into a compact format using the vectors $\mathbf{x}$ and $\mathbf{u}$ as before:

$$\Delta\mathbf{h} = [J_{hx}]\Delta\mathbf{x} + [J_{hu}]\Delta\mathbf{u} \tag{13.74}$$

We will now eliminate the $\Delta\mathbf{x}$ variables; that is:

$$\Delta\mathbf{x} = [J_{px}]^{-1}[J_{pu}]\Delta\mathbf{u} \tag{13.75}$$

Then, substituting:

$$\Delta\mathbf{h} = [J_{hx}][J_{px}]^{-1}[J_{pu}]\Delta\mathbf{u} + [J_{hu}]\Delta\mathbf{u} \tag{13.76}$$

This last equation gives the linear sensitivity coefficients between the transmission system quantities, **h**, and the control variables, **u**.

13.4 LINEAR PROGRAMMING METHODS

The gradient and Newton methods of solving an OPF suffer from the difficulty in handling inequality constraints. Linear programming, however, is very adept at handling inequality constraints, as long as the problem to be solved is such that it can be linearized without loss of accuracy.

Figure 13.4 shows the type of strategy used to create an OPF using linear programming. The power flow equations could be for the DC representation, the decoupled set of AC equations, or the full AC power flow equations. The choice will affect the difficulty of obtaining the linearized sensitivity coefficients and the convergence test used.

In the formulation below, we show how the OPF can be structured as an LP. First, we tackle the problem of expressing the nonlinear input–output or cost functions as a set of linear functions. This is similar to the treatment in Section 7.9 for hydro-units. Let the cost function be $F_i(P_i)$ as shown in Figure 13.5.

We can approximate this nonlinear function as a series of straight-line segments as shown in Figure 13.6. The three segments shown will be represented as P_{i1}, P_{i2}, P_{i3}, and each segment will have a slope designated:

$$s_{i1},\ s_{i2},\ s_{i3}$$

then the cost function itself is

$$F_i(P_i) = F_i(P_i^{\min}) + s_{i1}P_{i1} + s_{i2}P_{i2} + s_{i3}P_{i3} \tag{13.77}$$

and

$$0 \le P_{ik} \le P_{ik}^{+} \quad \text{for } k = 1, 2, 3 \tag{13.78}$$

and finally

$$P_i = P_i^{\min} + P_{i1} + P_{i2} + P_{i3} \tag{13.79}$$

The cost function is now made up of a linear expression in the P_{ik} values.

In the formulation of the OPF using linear programming, we only have the control variables in the problem. We do not attempt to place the state variables into the LP, nor all the power flow equations. Rather, constraints are set up in the LP that reflect the influence of changes in the control variables only. In the examples we present here, the control variables will be limited to generator real power, generator voltage magnitude, and transformer taps. The control variables will be designated as the u variables (see earlier in this chapter).

The next constraint to consider in an LPOPF are the constraints that

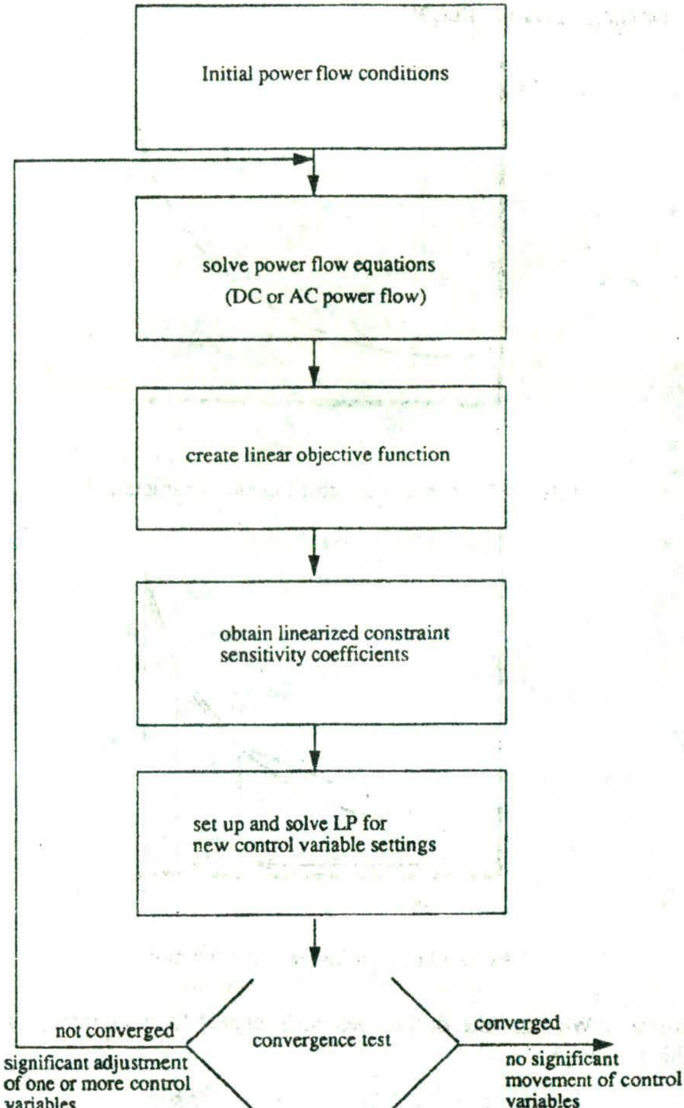

FIG. 13.4 Strategy for solution of the LPOPF.

represent the power balance between real and reactive power generated, and that consumed in the loads and losses. The real power balance equation is:

$$P_{gen} - P_{load} - P_{loss} = 0 \tag{13.80}$$

The loss term here represents the I^2R losses in the transmission lines and

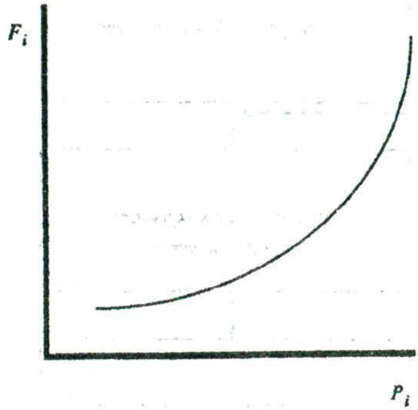

FIG. 13.5 A nonlinear cost function characteristic.

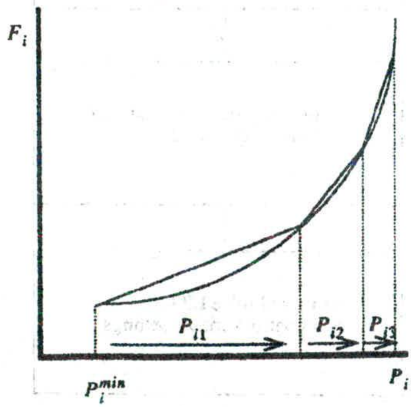

FIG. 13.6 A linearized cost function.

transformers. We can take derivatives with respect to the control variables, u, and this results in:

$$\sum_u \left(\frac{\partial P_{gen}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial P_{load}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial P_{loss}}{\partial u}\right)\Delta u = 0 \tag{13.81}$$

If we make the following substitution:

$$\Delta u = u - u^0 \tag{13.82}$$

then, the power balance equation becomes

$$\sum_u \left(\frac{\partial P_{gen}}{\partial u}\right)u - \sum_u \left(\frac{\partial P_{load}}{\partial u}\right)u - \sum_u \left(\frac{\partial P_{loss}}{\partial u}\right)u = K_p \tag{13.83}$$

where

$$K_p = \sum_u \frac{\partial P_{\text{gen}}}{\partial u} u^0 - \sum_u \frac{\partial P_{\text{load}}}{\partial u} u^0 - \sum_u \frac{\partial P_{\text{loss}}}{\partial u} u^0 \tag{13.84}$$

A similar equation can be written for the reactive power balance:

$$\sum_u \left(\frac{\partial Q_{\text{gen}}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial Q_{\text{load}}}{\partial u}\right)\Delta u - \sum_u \left(\frac{\partial Q_{\text{loss}}}{\partial u}\right)\Delta u = 0 \tag{13.85}$$

where the loss term is understood to include I^2X as well as the charging from line capacitors and shunt reactors. A substitution using $\Delta u = u - u^0$, as above, can also be done here.

The LP formulation, so far, would need to restrict control variables to move only within their respective limits, but it does not yet constrain the OPF to optimize cost within the limits of transmission flows and load bus voltages. To add the latter type constraints, we must add a new constraint to the LP. For example, say we wish to constrain the MVA flow on line nm to fall within an upper limit:

$$\text{MVA flow}_{nm} \leq \text{MVA flow}_{nm}^{\max} \tag{13.86}$$

We model this constraint by forming a Taylor's series expansion of this flow and only retaining the linear terms:

$$\text{MVA flow}_{nm} = \text{MVA flow}_{nm}^0 + \sum_u \left(\frac{\partial}{\partial u} \text{MVA flow}_{nm}\right)\Delta u \leq \text{MVA flow}_{nm}^{\max} \tag{13.87}$$

Again, we can substitute $\Delta u = u - u^0$ so we get:

$$\sum_u \left(\frac{\partial}{\partial u} \text{MVA flow}_{nm}\right) u \leq \text{MVA flow}_{nm}^{\max} - K_f \tag{13.88}$$

where

$$K_f = \text{MVA flow}_{nm}^0 + \sum_u \frac{\partial}{\partial u} \text{MVA flow}_{nm} u^0 \tag{13.89}$$

Other constraints such as voltage magnitude limits, branch MW limits, etc., can be added in a similar manner. We add as many constraints as necessary to constrain the power system to remain within its prescribed limits. Note, of course, that the derivatives of P_{loss} and MVA flow$_{nm}$ are obtained from the linear sensitivity coefficient calculations presented in the previous section.

13.4.1 Linear Programming Method with Only Real Power Variables

As an introduction to the LPOPF, we will set up and solve a power system example which only has generator real powers as control variables. Further, the model for the power system power balance constraint will assume that load is constant and that the losses are constant. Finally, since the entire model used in the LP is based on a MW-only formulation, we shall use the "a" and "d" factors derived in Chapter 11 to model the effect of changes in controls on the constraints. As indicated in Figure 13.4, we shall solve the LP and then make the adjustments to the control variables and solve a power flow in each main iteration. This guarantees that the total generation equals load plus losses, and that the MW flows are updated properly. The cost functions can be treated as before using multiple segmented "piecewise linear" approximations.

The "power balance" equation for this case is as follows:

$$P_1 + P_2 + \ldots + P_{\text{ref}} = P_{\text{load}} + P_{\text{losses}} = \text{constant} \tag{13.90}$$

To constrain the power system, we need the expansion of the constraints, such as MW flows, bus voltages, etc., as linear functions of the control variables. In this case, the linear control variables will be represented as a vector $\mathbf{u}$:

$$u = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{\text{ref}} \end{bmatrix} \tag{13.91}$$

This is done with the linear sensitivity approach, as derived in the previous section. The result is a set of constraints:

$$\mathbf{h}(\mathbf{u}) \le \mathbf{h}^+ \tag{13.92}$$

which is written as

$$\mathbf{h}(\mathbf{u}) = h(\mathbf{u}^0) + \frac{\partial h}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}^0) \le \mathbf{h}^+ \tag{13.93}$$

However, we shall observe that the derivatives $\partial h/\partial \mathbf{u}$ can be replaced with the "a" sensitivity coefficients developed in Chapter 11.

Thus, for a MW flow constraint on line rs we have:

$$\mathrm{MW}_{rs} = \mathrm{MW}_{rs}^0 + \sum_u a_{rs-u}(u - u^0) \le \mathrm{MW}_{rs}^{\max} \tag{13.94}$$

or

$$\mathrm{MW}_{rs} = \sum_u a_{rs-u}u \le \mathrm{MW}_{rs}^{\max} - \left(\mathrm{MW}_{rs}^0 - \sum_u a_{rs-u}u^0\right) \tag{13.95}$$

TABLE 13.1 Line Flows: Power Flow 0

Line	Limit	MW Flow
1-2	30	28.69
1-4	50	43.58
1-5	40	35.60
2-3	20	2.93
2-4	40	33.09
2-5	20	15.51
2-6	30	26.25
3-5	20	19.12
3-6	60	43.77
4-5	20	4.08
5-6	20	1.61

Similar constraints are added for any power system network quantity that is to held within its limit.

EXAMPLE 13C

We shall use the LPOPF reduced model method to solve an OPF problem. An LP and an AC power flow will be used to solve a series of dispatch problems. The transmission system will be the six bus system introduced in Chapter 4, the MW limits on the transmission lines will be those introduced in Example 11B and shown in Table 13.1. The generator cost functions are those found in Example 4E and linearized as shown below.

We shall solve a series of LP–AC power flow calculations as follows.

Step 0

Run a base AC power flow (this will be the AC power flow shown in Figure 4.8 and it will be designated as POWER Flow 0 in numbering the various power flow calculations in this example). Looking at Figure 4.8 and the limit set we are using from Example 11B, also shown below, we note that there are no overloads.

The generation values for this power flow are:

$P_1 = 107.87$ MW, $P_2 = 50$ MW, and $P_3 = 69$ MW *power flow* 0: *result*

The total cost for this initial dispatch is 3189.4 ₽/h.

Step 1

We now set up the LP to solve for the optimum cost with only the power balance equation in the LP constraint set. By the nature of the cost curve

TABLE 13.2 Generator Unit Break Point MWs

Unit	Break Point 1 (unit min)	Break Point 2	Break Point 3	Break Point 4 (unit max)
1	50	100	160	200
2	37.5	70	130	150
3	45	90	140	180

TABLE 13.3 Generator Cost Curve Segment Slope

Generator	s_{i1}	s_{i2}	s_{i3}
1	12.4685	13.0548	13.5875
2	11.2887	12.1110	12.8222
3	11.8333	12.5373	13.2042

segments, we also incorporate the limits on the generators. The generator cost functions are as follows:

Generator on bus 1: $F_1(P_1) = 213.1 + 11.669P_1 + 0.00533P_1^2$ R/h
with limits of: $50.0 \text{ MW} \leq P_1 \leq 200.0 \text{ MW}$

Generator on bus 2: $F_2(P_2) = 200.0 + 10.333P_2 + 0.00889P_2^2$ Rh
with limits of: $37.5 \text{ MW} \leq P_2 \leq 150.0 \text{ MW}$

Generator on bus 3: $F_3(P_3) = 240.0 + 10.833P_3 + 0.00741P_3^2$ R/h
with limits of: $45.0 \text{ MW} \leq P_3 \leq 180.0 \text{ MW}$

The LP will be run with the unit cost functions broken into three straight-line segments such that the break points are located as shown in Table 13.2. The generator cost function segment slopes are computed as follows:

$$s_{if} = \frac{F_i(P_{ij}^+) - F_i(P_{ij}^-)}{P_{ij}^+ - P_{ij}^-} \tag{13.96}$$

where P_{ij}^+ and P_{ij}^- are the values of P_i at the end of the j^{th} cost curve segment. The values are shown in Table 13.3. The segment limits are shown in Table 13.4.

The LP cost function is:

$$\begin{aligned} &[F_1(P_1^{min}) + 12.4685P_{11} + 13.0548P_{12} + 13.5878P_{13}] \\ &\quad + [F_2(P_2^{min}) + 11.2887P_{21} + 12.1110P_{22} + 12.8222P_{23}] \\ &\quad + [F_3(P_3^{min}) + 11.8333P_{31} + 12.5373P_{32} + 13.2042P_{33}] \end{aligned} \tag{13.97}$$

TABLE 13.4 Segment Limits

Segment	Min MW	Max MW
P_{11}	0	50
P_{12}	0	60
P_{13}	0	40
P_{21}	0	32.5
P_{22}	0	60
P_{23}	0	20
P_{31}	0	45
P_{32}	0	50
P_{33}	0	40

Since the $F_i(P_i^{\min})$ terms are constant, we can drop them in the LP. Then, the cost function becomes:

$$\begin{aligned} &12.4685P_{11} + 13.0548P_{12} + 13.5878P_{13} + 11.2887P_{21} \\ &\quad + 12.1110P_{22} + 12.8222P_{23} + 11.8333P_{31} \\ &\quad + 12.5373P_{32} + 13.2042P_{33} \end{aligned} \tag{13.98}$$

The generation, load, and losses equality constraint is

$$P_1 + P_2 + P_3 = P_{\text{load}} + P_{\text{losses}} \tag{13.99}$$

The load is 210 MW and the losses from the initial power flow are 7.87 MW. Substituting the equivalent expression for each generator's output in terms of its three linear segments, we obtain:

$$\begin{aligned} &P_1^{\min} + P_{11} + P_{12} + P_{13} + P_2^{\min} + P_{21} + P_{22} + P_{23} + P_3^{\min} \\ &\quad + P_{31} + P_{32} + P_{33} = P_{\text{load}} + P_{\text{losses}} \end{aligned} \tag{13.100}$$

This results in the following after the $P_i^{\min}$, P_{load}, and P_{loss} values are substituted:

$$\begin{aligned} &P_{11} + P_{12} + P_{13} + P_{21} + P_{23} + P_{33} + P_{31} + P_{32} + P_{33} \\ &\quad = 210 + 7.87 - 50 - 37.5 - 45 = 85.37 \end{aligned} \tag{13.101}$$

We now solve the LP with the cost function and equality constraint given above, and with the six variables representing the generator outputs. The solution to this LP is shown in Table 13.5.

TABLE 13.5 First LP Solution

Variable	Min MW	Solution MW	Max MW
P_{11}	0	0.0	50
P_{12}	0	0.0	60
P_{13}	0	0.0	40
P_{21}	0	32.5	32.5
P_{22}	0	7.87	60
P_{23}	0	0.0	20
P_{31}	0	45.0	45
P_{32}	0	0.0	50
P_{33}	0	0.0	40

The total generation on each generator is:

$$P_i = P_i^{\min} + P_{i1} + P_{i2} + P_{i3} \tag{13.102}$$

then the generator optimal outputs are

$$P_1 = 50 \text{ MW}, P_2 = 77.87 \text{ MW}, \quad \text{and} \quad P_3 = 90 \text{ MW} \textit{ LP 1: result}$$

Note that this solution of necessity will have only one of the variables not at a break point while the others will be at a break point. Note also that the output on bus 1 is at its low limit. When we substitute these values for the generation at buses 1, 2, and 3, and run the power flow, we get the following:

$$P_1 = 48.83 \text{ MW}, P_2 = 77.87 \text{ MW}, \quad \text{and} \quad P_3 = 90 \text{ MW} \textit{ power flow 1: result}$$

The total cost for this dispatch is 3129.1 R/h. This illustrates the fact that the LP uses a linear model of the power system and when we put its results into a nonlinear model, such as the power flow, there are bound to be differences. Since the losses have changed (to 6.70 MW), the power output of the reference bus must decrease to balance the power flow. However, the solution to the optimal LPOPF has the reference-bus power output below its minimum of 50 MW. To correct this condition we set up another LP solution with the same cost function but with a slightly different equality constraint that reflects the new value of losses. The result of this LP is:

$$P_1 = 50 \text{ MW}, P_2 = 76.7 \text{ MW}, \quad \text{and} \quad P_3 = 90 \text{ MW} \textit{ LP 1.1: result}$$

Once again, we enter these results into the power flow and obtain:

$$P_1 = 49.99 \text{ MW}, P_2 = 76.7 \text{ MW} \quad \text{and} \quad P_3 = 90 \text{ MW} \textit{ power flow 1.1 result}$$

TABLE 13.6 Line Flows: Power Flow 1.1

Line	Limit	MW Flow
1-2	30	4.28
1-4	50	25.60
1-5	40	20.11
2-3	20	−6.42
2-4	40	48.75[a]
2-5	20	17.75
2-6	30	20.88
3-5	20	28.91[a]
3-6	60	54.63
4-5	20	1.84
5-6	20	3.87

[a] Overloaded line.

The total cost for this dispatch is 3129.6 Ɍ/h and the losses are 6.7 MW. This represents the least cost dispatch that we shall obtain in this example. As constraints are added later to meet the flow limits, the cost will increase.

Note also that we have two overloads on the optimum cost dispatch as shown in Table 13.6.

Step 2

The LP and power flow executions in step 1 resulted in a less-costly dispatch than the original power flow, but in doing so we have overloaded two transmission lines. We shall refer to these overloads as $(n - 0)$ overloads. This notation means that there are n lines minus zero outages in the network at the time of the overload. [Later we shall use the notation $(n - 1)$ to indicate that there are n lines minus one line (that is, a single-line outage) in the network at the time of the overloads. This notation can be used for further levels of overload such as $(n - 2)$, $(n - 3)$, etc. However, many electric utility transmission operations departments only go as far as $(n - 1)$ in dispatching their systems.]

We must redispatch the power system at this point to remove the $(n - 0)$ overloads. To do this, we add two constraints to the LP, one for each overloaded line. The power flow constraint on line 2-4 is modeled as:

$$f_{2\text{-}4} = f^0_{2\text{-}4} + a_{2\text{-}4,1}(P_1 - P^0_1) + a_{2\text{-}4,2}(P_2 - P^0_2) + a_{2\text{-}4,3}(P_3 - P^0_3) \leq 40 \tag{13.103}$$

Substituting 48.75 for $f^0_{2\text{-}4}$, 76.7 for P^0_2, and 90 for P^0_3, we get the following for the constraint for line 2-4 (note that $a_{2\text{-}4,1} = 0$) and, finally, we expand P_2 and P_3 in terms of the segments:

$$\begin{aligned} 48.75 &+ 0.31(37.5 + P_{21} + P_{22} + P_{23} - 76.7) \\ &+ 0.22(45 + P_{31} + P_{32} + P_{33} - 90) \leq 40 \end{aligned} \tag{13.104}$$

or

$$0.31P_{21} + 0.31P_{22} + 0.31P_{23} + 0.22P_{31} + 0.22P_{32} + 0.22P_{33} \leq 13.302 \tag{13.105}$$

The constraint for line 3-5 is built similarly and results in:

$$0.06P_{21} + 0.06P_{22} + 0.06P_{23} + 0.29P_{31} + 0.29P_{32} + 0.29P_{33} \leq 6.492 \tag{13.106}$$

The solution to the LP gives:

$P_1 = 87.02$ MW, $P_2 = 70.0$ MW and $P_3 = 59.66$ MW *LP 2: result*

Also note that only the first transmission line constraint is binding in the LP, the remaining constraint is "slack," that is, it is not being forced up against its limit. When these values are put into the power flow we obtain:

$P_1 = 87.54$ MW, $P_2 = 70.0$ MW and $P_3 = 59.66$ MW *power flow 2: result*

The flows on the two constrained lines are:

$$f_{2\text{-}4} = 39.40 \text{ MW} \quad \text{and} \quad f_{3\text{-}5} = 20.36 \text{ MW}$$

The total operating cost has now increased to 3155.0 Ɽ/h.

We now run another complete LP–power flow iteration to account for changes in losses and to bring the constraints closer to their limits. The solution to the second-iteration LP gives:

$P_1 = 86.16$ MW, $P_2 = 73.3$ MW and $P_3 = 57.73$ MW *LP 2.1: result*

Both transmission line constraints are binding in the second LP. When these values are put into the power flow we obtain:

$P_1 = 86.16$ MW, $P_2 = 73.3$ MW and $P_3 = 57.73$ MW *power flow 2.1: result*

The flows on the two constrained lines are:

$$f_{2\text{-}4} = 39.99 \text{ MW} \quad \text{and} \quad f_{3\text{-}5} = 20.06 \text{ MW}$$

The total operating cost has now decreased slightly to 3153.3 Ɽ/h. There are no more $(n - 0)$ line overloads.

TABLE 13.7 Line Flows: Power Flow 2.1 (with Line 2-3 Out)

Line	Limit	MW Flow
1-2	30	18.1
1-4	50	36.37
1-5	40	31.74
2-3	20	—
2-4	40	40.73[a]
2-5	20	19.19
2-6	30	31.11[a]
3-5	20	18.26
3-6	60	39.47
4-5	20	4.59
5-6	20	1.17

[a] Overloaded line.

Step 3

We have now achieved an optimal dispatch with all $(n-0)$ overloads met. This dispatch will satisfy generation and all line flow limits; however, if we have a transmission line outage contingency, we may have overloads. By modeling the first contingency overloads, or the so-called $(n-1)$ overloads, we can guarantee that should the contingency outage take place, there would be no resulting overloads. This is the scheme involved in security-constrained OPF, or SCOPF, and is the subject of Section 13.5.

In this example, to make matters simple we shall only study the result of one contingency outage. In our sample system, we shall start from the result of power flow 2.1 and take out line 2-3. The flows that result from this contingency power flow are shown in Table 13.7.

We now must form a new LP that has the generation, load, losses equality constraint and the original two $(n-0)$ line flow constraints done in step 2, and two new constraints for each of the $(n-1)$ overloads (i.e., on line 2-4 and line 2-6). To model line 2-4 with line 2-3 removed, we use the following constraint, as derived in Appendix 11A of Chapter 11.

$$\Delta f_l^k = \sum_i (a_{li} + d_{l,k} a_{ki}) \Delta P_i \tag{13.107}$$

This now becomes:

$$f_l^k = \sum_i (a_{li} + d_{l,k} a_{ki})(P_i - P_i^0) + f_l^0 \le f_l^{\max} \tag{13.108}$$

The new LP has five constraints. The first result of this LP gives:

$P_1 = 91.39$ MW, $P_2 = 66.96$ MW, and $P_3 = 58.84$ MW *LP 3: result*

The $(n = 0)$ constraint on line 3-5 is binding and the $(n - 1)$ constraint on line 2-6 is binding. When these values are put into the power flow, we obtain (note that this power flow has all lines in):

$$P_1 = 91.52 \text{ MW}, P_2 = 66.96 \text{ MW}, \quad \text{and} \quad P_3 = 58.8 \text{ MW} \textit{ power flow 3: result}$$

The flows on the two $(n - 0)$ constrained lines are:

$$f_{2\text{-}4} = 38.23 \text{ MW} \quad \text{and} \quad f_{3\text{-}5} = 19.94 \text{ MW}$$

A second power flow with line 2-3 out is also run with the same generation values. The results of this power flow show that the two $(n - 1)$ flow constraints are:

$$f_{2\text{-}4}^{\text{contingency}} = 38.86 \text{ MW} \quad \text{and} \quad f_{2\text{-}6}^{\text{contingency}} = 30.00 \text{ MW}$$

The total operating cost has now increased to 3160.5 R/h. A complete second iteration of the LP and power flows is run and results in the following power flows:

$$P_1 = 90.53 \text{ MW}, P_2 = 67.92 \text{ MW}, \quad \text{and} \quad P_3 = 58.84 \text{ MW} \textit{ power flow 3.1: result}$$

The flows on the two $(n - 0)$ constrained lines are:

$$f_{2\text{-}4} = 38.54 \text{ MW} \quad \text{and} \quad f_{3\text{-}5} = 20.00 \text{ MW}$$

A second power flow with line 2-3 out is also run with the same generation values. The results of this power flow show that the two $(n - 1)$ flow constraints are:

$$f_{2\text{-}4}^{\text{contingency}} = 39.18 \text{ MW} \quad \text{and} \quad f_{2\text{-}6}^{\text{contingency}} = 30.09 \text{ MW}$$

The total operating cost has now increased to 3159.1 R/h.

13.4.2 Linear Programming with AC Power Flow Variables and Detailed Cost Functions

OPF programs that optimize the AC power flow of a power system go beyond the LPOPF introduced in the last section, in several respects.

First, they do not usually use fixed break points. Rather, the break points are added as needed as the solution progresses and can become close enough so that no error is perceptible between the piecewise linear approximation and the true nonlinear input–output curve of the generators. "Second, the AC quantities of voltage magnitude and perhaps phase angle become variables in the LP and the constraints are set up as linear functions using the sensitivity coefficients methods shown in Section 13.3. Usually, however, the nonlinear representations of the bus power and reactive power injections and the line or transformer MVA flows are not well represented as linear functions. To cope with the nonlinear nature of these constraints involves restricting the movement of each variable and then relinearlizing the equality and inequality constraints quite often. The result is an LP that "converges" on the optimal AC power flow, meeting all the power flow equality constraints and inequality constraints.

Reference 9 is an example of such an OPF code built around an LP.

13.5 SECURITY-CONSTRAINED OPTIMAL POWER FLOW

In Chapter 11, we introduced the concept of security analysis and the idea that a power system could be constrained to operate in a secure manner. Programs which can make control adjustments to the base or pre-contingency operation to prevent violations in the post-contingency conditions are called "security-constrained optimal power flows," or SCOPF.

We have seen previously that an OPF is distinguished from an economic dispatch by the fact that it constantly updates a power flow of the transmission system as it progresses toward the minimum of the objective function. One advantage of having the power flow updated is the fact that constraints can be added to the OPF that reflect the limits which must be respected in the transmission system. Thus, the OPF allows us to reach an optimum with limits on network components recognized.

An extension to this procedure is to add constraints that model the limits on components during contingency conditions. That is, these new "security constraints" or "contingency constraints" allow the OPF to meet precontingency limits as well as post-contingency limits. There is a price to pay, however, and that is the fact as we iterate the OPF with an AC power flow, we must also run power flows for all the contingency cases being observed. This is illustrated in Figure 13.7.

The SCOPF shown in Figure 13.7 starts by solving an OPF with $(n - 0)$ constraints only. Only when it has solved for the optimal, constrained conditions is the contingency analysis executed. In Figure 13.7, the contingency analysis starts by screening the power system and identifying the potential worst-contingency cases. As was pointed out in Chapter 11, not all of these cases are going to result in a post-contingency violation and it is important to limit the number of full power flows that are executed. This is especially important in the SCOPF, where each contingency power flow may result in new contingency constraints being added to the OPF. We assume here that only the M worst cases screened by the screening algorithm are added. It is possible to make $M = 1$, in which case only the worst potential contingency is added.

Next, all the $(n - 1)$ contingency cases that are under consideration must be solved by running a power flow with that contingency reflected in alterations to the power flow model. When the power flow results in a security violation,

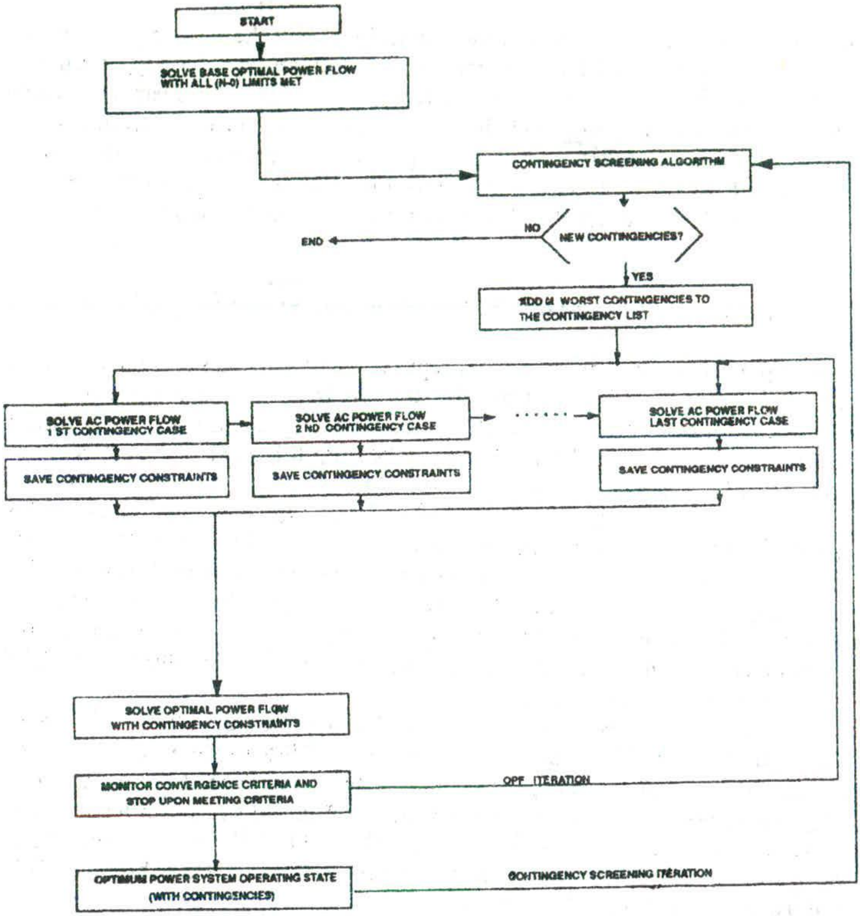

FIG. 13.7 Security-constrained optimal power flow.

the power system model is used to create a contingency constraint. In fact, what is done is to run a network sensitivity calculation (See Section 13.3) on the model with the contingency outage and save the resulting constraint sensitivities. When all contingency power flows are complete, all the contingency constraints are added to the OPF model and it is solved.

Note, in Figure 13.7, there are two main loops to be executed. The loop labeled "OPF Iteration" requires the OPF and each of the contingency power flows to be re-executed until the OPF has solved with all contingency constraints met. Next, the outer loop labeled "Contingency Screening Iteration" is tried. If the contingency screening algorithm does not pick up any new contingencies the SCOPF can end; if new contingencies are found, it must add them to the list and continue.

Why is all this necessary? The optimum operation conditions for a power system will often result in violation of system security. This is especially true when a large amount of interchange power is available at a favorable price. In this instance, the selling power system can be modeled in the OPF with its price of production set accordingly, and the OPF will then raise the interchange up to the point where transmission system components are limiting. Now, when the contingency analysis is run, there may be many cases which result in contingency violations and the OPF, with contingency constraints added, will have to back off the interchange power in order to meet the contingency limits.

It should also be noted that when some contingency constraints are added to the OPF, it will redispatch generation, and adjust voltages and transformers to meet these constraints. The process of adjustments may result in many new contingency violations when the screening algorithm and the power flows are run. The need to iterate between the OPF and the contingency screening represents an effort to find the "most constraining" contingencies.

SCOPF was introduced as step 3 in Example 13C and will also be illustrated in Example 13D, which follows.

EXAMPLE 13D

This example shows the results of running the same six-bus case used in Example 13C, with the same generator cost functions. However, we now are using a full AC OPF so that we will use line MVA limits and bus voltage limits as well. The MVA limits are shown in Table 13.8. The bus voltages are also limited, with bus 5 being the only one to hit its upper limit of 1.0 pu voltage magnitude.

The full AC OPF has six control variables: three generator outputs and three generator voltage magnitude schedules. In addition, the AC OPF can be used to minimize either MW losses, or to minimize operating cost. Table 13.9 summarizes these results.

TABLE 13.8 Line MVA Flows: Power Flow 0

Line	MVA Limit	MVA Flow
1-2	40	32.57
1-4	60	48
1-5	40	37.34
2-3	40	12.61
2-4	60	56.71
2-5	30	21.83
2-6	90	29.03
3-5	70	30.04
3-6	80	74.86
4-5	20	6.41
5-6	40	9.80

TABLE 13.9 Full AC OPF Results

Case	P_1	P_2	P_3	$\lvert E_1^{sched}\rvert$	$\lvert E_2^{sched}\rvert$	$\lvert E_3^{sched}\rvert$	MW Losses	Cost	Binding Constraints
Base case	107.9	50.0	60.0	1.05	1.0499	1.0429	7.87	3189.4	Generators 2 and 3 at max VAR limit
Min cost, adjust generator MW only	86.9	59.3	71.0	1.05	1.05	1.0458	7.14	3157.9	Line 2-4 MVA limit, generator 3 at max VAR limit
Min losses, adjust generator voltage only	107.1	50.0	60.0	1.05	1.0429	1.0499	7.1	3179.5	Bus 5 max voltage, line 1-5 MVA limit, generator 3 at max VAR limit
Min cost with generator MW adjustment, then min losses with adjustment of gen voltages	86.3	59.3	71.0	1.05	1.0429	1.0499	6.54	3150.3	Bus 5 max voltage, generator 3 at max VAR limit
Min cost with both generator MW and voltage adjustment	52.0	87.5	77.0	1.05	1.0429	1.0499	6.41	3124.6	Bus 5 max voltage, line 2-4 MVA limit, generator 3 at max VAR limit
Min cost, adjust generator MW with line 3-6 out	70.0	91.8	62.0	1.05	1.031	1.07	15.05	3219.7	Before, OPF line 1-5 has 12% MVA overload; after, line 2-4 at MVA limit, generator at max VAR limit

Note the variety of ways that a power system can be optimized using an OPF. For example, some power system operators may wish to simply reduce system losses through the adjustment of generator voltage schedules—this is often done with hydrosystems where the generator MW outputs must be kept on a fixed time schedule to meet hydro-requirements.

13.6 INTERIOR POINT ALGORITHM

In 1984, Karmarkar (refeence 10) presented a new solution algorithm for linear programming problems that did not solve for the optimal solution by following a series of points that were on the "constraint boundary" but, rather, followed a path through the interior of the constraints directly toward the optimal solution on the constraint boundary. This solution was much faster than conventional LP algorithms.

In 1986, Gill *et al.* (reference 11) showed the relationship between Karmarkar's algorithm and the so-called "logarithmic barrier function algorithm." This algorithm has become the basis for many OPF solution algorithms and is explained in reference 12.

In this derivation, no distinction is made between the control variables and the state variables; rather, all variables are considered in the $\mathbf{x}$ vector. The objective function will be $f(\mathbf{x})$. The constraints will be brokem into equality constraints and inequality constraints. The equality constraints are $\mathbf{g}(\mathbf{x}) = 0$ and the inequality constraints are

$$\mathbf{h}^- \leq h(\mathbf{x}) \leq \mathbf{h}^+ \tag{13.109}$$

where the $\mathbf{h}^-$ and $\mathbf{h}^+$ vectors are the lower and upper limits on the inequality constraints, respectively. Finally, we restrict the variables themselves to be within lower and upper bounds

$$\mathbf{x}^- \leq \mathbf{x} \leq \mathbf{x}^+ \tag{13.110}$$

The first step in transforming this problem is to add slack variables so that all the equations become equality constraints. We then obtain the following set of equations:

$$\begin{gathered}
\min f(\mathbf{x}) \\
g(\mathbf{x}) = 0 \\
h(\mathbf{x}) + \mathbf{s}_h = \mathbf{h}^+ \\
\mathbf{s}_h + \mathbf{s}_{sh} = \mathbf{h}^+ - \mathbf{h}^- \\
\mathbf{x} + \mathbf{s}_x = \mathbf{x}^+ \\
\mathbf{x} - \mathbf{x}^- \geq 0,\ \mathbf{s}_x,\ \mathbf{s}_h,\ \mathbf{s}_{sh} \geq 0
\end{gathered} \tag{13.111}$$

Note that we now have a set of equations with all equality constraints except the final consisting of nonnegativity conditions on $\mathbf{x} - \mathbf{x}^-$ and the slack variables. These nonnegativity conditions are handled by adding what is called a "logarithmic barrier function" to the objective. Basically, this is a form of penalty function which becomes very large as the function or variable gets close to zero. The new objective function then looks like:

$$f_\mu = f(\mathbf{x}) - \mu \sum_j \ln(x - x^-)_j - \mu \sum_j \ln(\mathbf{s}_x)_j - \mu \sum_i \ln(\mathbf{s}_h)_i - \mu \sum_i \ln(\mathbf{s}_{sh})_i \tag{13.112}$$

The parameter, μ, is called the "barrier parameter" and is a positive number that is forced to go to zero as the algorithm converges to the optimum. This then presents us with the Lagrange equation:

$$\begin{aligned}
\mathscr{L}_\mu = {} & f(\mathbf{x}) - \lambda^T g(\mathbf{x}) - \lambda_h^T[\mathbf{h}^+ - \mathbf{s}_h - h(\mathbf{x})] \\
& - \lambda_{sh}^T(\mathbf{h}^+ - \mathbf{h}^- - \mathbf{s}_h - \mathbf{s}_{sh}) - \lambda_x^T(\mathbf{x}^+ - \mathbf{x} - \mathbf{s}_x) \\
& - \mu \sum_j \ln(x - x^-)_j - \mu \sum_j \ln(\mathbf{s}_x)_j - \mu \sum_i \ln(\mathbf{s}_h)_i - \mu \sum_i \ln(\mathbf{s}_{sh})_i
\end{aligned} \tag{13.113}$$

The solution to this Lagrangian equation is obtained by setting its gradient to zero:

$$\begin{aligned}
\nabla_x \mathscr{L}_\mu &= \nabla f(\mathbf{x}) - \nabla_g(\mathbf{x})^T \lambda + \nabla h(\mathbf{x})^T \lambda_h + \lambda_x - \mu(\mathbf{x} - \mathbf{x}^-)^{-1} e = 0 \\
\nabla_{s_h} \mathscr{L}_\mu &= \lambda_h + \lambda_{sh} - \mu_h^{-1} e = 0 \\
\nabla_{sh} \mathscr{L}_\mu &= \lambda_{sh} - \mu s_{sh}^{-1} e = 0 \\
\nabla_{s_x} \mathscr{L}_\mu &= \lambda_x - \mu s_s^{-1} e = 0 \\
\nabla_\lambda \mathscr{L}_\mu &= -g(\mathbf{x}) \\
\nabla_{\lambda_{sh}} \mathscr{L}_\mu &= h(\mathbf{x}) + s_h - h^+ \\
\nabla_{\lambda_x} \mathscr{L}_\mu &= \mathbf{x} + s_x - x^+ \\
\nabla_{\lambda_h} \mathscr{L}_\mu &= s_h + s_{sh} - h^+ + h^-
\end{aligned} \tag{13.114}$$

These nonlinear equations are then solved iteratively by Newton's method, and the value of μ is adjusted toward zero.

The solution produces the values of the dual variables, some of which are the marginal costs for the real and reactive power at the buses. These bus incremental costs, BICs are the subject of the next section. Note that in Chapter 10, the BICs were calculated using an interior point OPF.

13.7 BUS INCREMENTAL COSTS

If we take the classical Lagrange equation for an optimal power flow:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathrm{f}(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^t\mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{p}) \tag{13.115}$$

and we asume that we have an optimal solution to this equation, then we can ask an interesting question: "What is the change in the optimal operating cost if we change one of the parameters **p**?" More specifically: 'What is the change in optimal operating cost if we change the power produced or consumed at a bus in the network?" Thus, what we want is the following derivative:

$$\frac{\partial \mathscr{L}}{\partial P_i}$$

If we expand the Lagrange equation as follows:

$$\mathscr{L}(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \sum_{\text{gen}} F_i(P_i) + F_{\text{ref}}[P_{\text{ref}}(/E|, \theta)] + [\lambda_1 \lambda_2, \ldots, \lambda_m] \begin{bmatrix} P_i^{\text{net}} - P_i(|\mathbf{E}|, \theta) \\ Q_i^{\text{net}} - Q_i(|\mathbf{E}|, \theta) \\ \vdots \end{bmatrix} \tag{13.116}$$

The derivative of $\mathscr{L}$ with respect to P_i is simple, since the parameters only appear in the second part of the Lagrange equation. The resulting derivative for bus i is:

$$\frac{\partial \mathscr{L}}{\partial P_i} = \lambda_i \tag{13.117}$$

We see that the interpretation of the vector of Lagrange multipliers is that they indicate the increment in optimal cost with respect to small changes in the parameters of the network. In the case of small change in power, either consumed or produced at a bus, the Lagrange multiplier for that bus then indicates the incremental cost that will be incurred as a result of this change. This cost has been given the name "bus incremental cost" or BIC and is the same incremental cost we dealt with in the beginning of the text, where we derived the incremental cost of delivery of power from a generator. A power system is in economic dispatch when the BIC for each generator matches the generator's own incremental cost for the power it is producing.

The BIC is a useful concept for nondispatched generator buses and for evaluating the marginal cost of wheeling. In some proposed schemes, this bus incremental cost is used to establish the spot market price for energy.

One point is worth noting before we leave this topic. The above discussion assumed that one has the vector of Lagrange multipliers for an optimal solution. However, depending on the method used to solve the OPF, this may not be the case. Certainly, in the case of the OPF that is based on linear programming, the λ values are not available unless a special formulation is used—yet we need the BICs for the buses.

The Lagrange equation at the optimal solution can be used to solve for the Lambda vector, even through it was not used in the OPF alogorithm. This is because, at the optimal solution to the OPF, the Lagrange equation is assumed to satisfy,

$$\nabla \mathscr{L} = 0 \tag{13.118}$$

or, for the state variable, $\mathbf{x}$, we have:

$$\nabla \mathscr{L}_x = \frac{\partial \mathscr{L}}{\partial \mathbf{x}} = \frac{\partial \mathrm{f}}{\partial \mathbf{x}} + \left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = 0 \tag{13.119}$$

which can be used to solve for $\boldsymbol{\lambda}$ as follows:

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = -\frac{\partial \mathrm{f}}{\partial \mathbf{x}} \tag{13.120}$$

The problem here is that the matrix

$$\left[\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right]^T \tag{13.121}$$

has N rows where N equals the number of state variables, and M columns corresponding to M binding constraints. We shall assume that $N \leq M$. The vector

$$\frac{\partial \mathrm{f}}{\partial \mathbf{x}}$$

has N elements and the lambda vector, $\boldsymbol{\lambda}$, has M elements. Thus, the equation which can be used to solve for the lambda vector is overdetermined; that is, there are more elements in the lambda vector than rows in the matrix or the right-hand side. This type of equation has many solutions for the lambda vector. The correct one is found by applying a least-squares technique, as explained in Chapter 12 on state estimation. Further, the usual method of solving for the lambda vector is to apply the QR algorithm (also explained in Chapter 12). Thus, we can use *any method* to solve for the optimal state vector for an OPF and then develop the matrix and right-hand side shown above and solve for the BIC vector.

EXAMPLE 13E

This example gives the bus incremental costs for the same six-bus sample used in Examples 13C and 13D. For the case where both generation MW and

TABLE 13.10 Bus Incremental Costs

Bus	R/MWh	R/MVARh
1	12.22	0
2	11.89	0
3	11.97	0.1
4	12.98	0.81
5	12.59	0.51
6	12.29	0.38

scheduled voltages are adjusted to obtain minimum cost, the bus incremental costs are given in Table 13.10.

There is a cost for increasing the MW delivered, as well as the MVAR delivered from or to any bus in the network. In Table 13.10, the bus incremental costs for delivering MW at buses 1, 2, and 3 are equal to the incremental costs of the generator cost functions at the optimal dispatch. The bus incremental cost to deliver MVAR at buses 1 and 2 is zero since these generators are not at their maximum VAR limit and can generate incremental MVAR for "free." The incremental cost to deliver more MVARs at bus 3 is nonzero since generator 3 is at maximum VAR limit and one would have to generate the extra VARs at buses 1 and 2. Finally, the delivery points have higher bus incremental costs since they require that all MW and MVAR consumed at these buses must be delivered via the transmission system, which will cost the system in MW and MVAR losses.

In addition to the bus incremental costs, the procedure outlined above can also be used to generate the cost of changing the limit at any binding constraint. In the case of the dispatch used in Table 13.10, line 2-4 is at an MVA linit and bus 5 is at maximum voltage. The incremental cost with respect to changing the MVA limit on line 2-4 is -1.01 R/MVAh, indicating that if the limit were increased the system operating cost would decrease. Last of all, the incremental cost of changing the bus 5 upper voltage limit -88.4 R/pu volt.

PROBLEMS

13.1 You are going to use a linear program and a power flow to solve an OPF. The linear program will be used to solve constrained dispatch problems and the power flow will confirm that you have done the correct thing. For each of the problems, you should use the power flow data for the six-bus problem found in Chapter 4.

The following data on unit cost functions applies to this problem:

Unit 1 (bus 1): $F(P) = 600.0 + 6.0P + 0.002P^2$

$P_{min} = 70$ MW

$P_{max} = 250.0$ MW

Unit 2 (bus 2): $F(P) = 220.0 + 7.3P + 0.003P^2$

$P_{min} = 55$ MW

$P_{max} = 135$ MW

Unit 3 (bus 3): $F(P) = 100.0 + 8.0P + 0.004P^2$

$P_{min} = 60$ MW

$P_{max} = 160$ MW

When setting up the LP you should use three straight-line segments with break points as below:

Unit 1, break points at: 70, 130, 180, and 250 MW

Unit 2, break points at: 55, 75, 95, and 135 MW

Unit 3, break points at: 60, 80, 120, and 160 MW

When using the LP for dispatching you should ignore losses.

Set up the power flow as follows:

Load = 300 MW

Generation on bus 2 = 100 MW

Generation on bus 3 = 100 MW

This should lead to a flow of about 67 MW on line 3-6.

Using the linear program, set up a minimum cost LP for the three units using the break points above and the generation shift (or "a") factors from Figure 11.7. You are to constrain the system so that the flow on line 3-6 is no greater than 50 MW.

When you obtain an answer from the LP, enter the values for P_2 and P_3 found in the LP into the load flow and see if, indeed, the flow on line 3-6 is close to the 30 MW desired. (Be sure the load is still set to 300 MW.)

13.2 Using the six-bus power flow example from Chapter 4 with load at 240 MW, try to adjust the MW generated on the three generators and the voltage on each generator to minimize transmission losses. Keep the generators within their economic limits and the voltages at the generators within 0.90 to 1.07 pu volts. Use the following as MVAR limits:

Bus 2 generator: 100 MVAR max

Bus 3 generator: 60 MVAR max

13.3 Using the six-bus power flow example from Chapter 4, set up the base case as in Problem 13.1 (300 MW load, 100 MW on generator buses 2 and 3). Solve the base conditions and note that the load voltages on buses 4, 5, and 6 are quite low. Now, drop the line from bus 2 to bus 3 and resolve the power flow. (Note that the VAR limits on buses 2 and 3 should be the same as in Problem 13.2.)

This results in a severe voltage drop at bus 6. Can you correct this voltage so it comes back into normal range (e.g., 0.90 per unit to 1.07 per unit)? Suggested options: Add fixed capacitance to ground at bus 6, raise the voltage at one or more of the generators, reduce the load MW and MVAR at bus 6, etc.

13.4 You are going to solve the following optimal power flow in two different ways. Given a power system with two generators, P_1 and P_2, with their corresponding cost functions $F_1(P_1)$ and $F_2(P_2)$. In addition, the voltage magnitudes on the generator buses are also to be scheduled.

The balance between load and generation will be assumed to be governed by a linear constraint:

$$\sum_i \beta_i P_i = \sum_i \beta_i P_i^0$$

In addition, two constraints have been identified and their sensitivities calculated. The first is a flow constraint where:

$$\Delta \text{flow}_{nm} = \sum_i af_i \Delta P_i + \sum_i av_i \Delta V_i$$

The second constraint involves a voltage magnitude at bus k which is assumed to be sensitive only to the generator voltages:

$$\Delta V_k = \sum_i \gamma_i \Delta V_i$$

a. Assume that the initial generator outputs are P_1^0 and P_2^0 and that the initial voltage magnitudes are V_1^0 and V_2^0 and that you have obtained the initial flow, flow_{nm}^0, and the initial voltage, V_k^0, from a power flow program.

Further assume that there are limits to be constrained flow and voltage: flow_{nm}^+ and flow_{nm}^- and for the voltage V_k^+ and V_k^-.

Express the flow on line *nm* and the voltage on bus *k* as linear functions of the four control variables: P_1, P_2, V_1, V_2.

b. Show how to obtain the minimum cost with the gradient method. In this case, you may assume that the flow constraint and the voltage constraint are equality constraints where we desire the constraints to be scheduled to the upper limit. Any matrices in this formulation should be shown with all terms; if the inverse is needed, just express it as an inverse matrix—do not try to show all the terms in the inverse itself.

c. Show the same minimun cost dispatch solution with an LP where we break each cost function into two segments.

FURTHER READING

Reference 1 is considered the classic paper that first introduced the concept of an optimal power flow. References 2 and 3 give a good overview of the techniques and methods of OPFs. Reference 4 is a good introduction to the basic mathematics of the gradient method, and references 5–7 cover the Newton OPF method.

Reference 8 shows how the bus incremental costs are calculated using a least-squares approach. Reference 9 is an excellent paper dealing with the application of linear programming to the OPF solution. References 10 and 11 introduce the concept of the interior point algorithm. References 12 and 13 deal with the application of the interior point algorithm to the OPF solution. References 14 and 15 talk extensively about how to incorporate security constraints into the OPF, while reference 16 shows some of the special AGC logic needed when an OPF is holding a line flow constraint.

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